Making Waves: Secondary Markets for Automobiles under Macro Shocks *

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Abstract

We develop a non-stationary version of the equilibrium model for trading automobiles in secondary market in Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022). By introducing aggregate macroeconomic shocks into the stationary dynamic equilibrium model of automobile markets we study the variations in quantity and prices of used cars over the business cycle. We calibrate the model using Danish data on car registrations and sales. Using the model we study the quantitive differences between the fully rational expectations solution and the much cheaper to compute temporary equilibrium solution. Our results show that the approximation errors from the simpler equilibrium concept are small when measured by ex-ante value functions, substantiating the use of temporary equilibrium in applications to durable goods markets.

Keywords: automobile markets, secondary markets, trade, macro shocks, business cycle, rational expectations equilibrium, temporary equilibrium

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1 Introduction

How do demand and supply shocks in durable good markets such as cars propogate and affect prices and quantities over time? Our paper answers this question using a dynamic equilibrium model with forward looking consumers and rational expectations. We show that our model generates empirically realistic "waves" in the distribution of ages of cars, but without correspondingly large waves in used car prices. We ascribe this differential impact to strong substitution effects, so that comparatively small price fluctations cause consumers to adapt to the "lumps and bumps" in the evolving age distribution of cars driven by procyclical waves of investment and disinvestment in new car purchases due to macro shocks. Cooper, Haltiwanger and Power (1999) were one of the first to use a machine replacement model to study the impact of the business cycle on *quantities* of durable goods that are not traded in secondary markets. Our contribution is to model the persistent impact of macro shocks on both *quantities and prices* in durable goods markets such as cars for which there is active trade in both the new and used markets.

Our model shows how even temporary shocks to new car demand in the primary (i.e. new car) market has substantial persistence and affects quantities and prices for many subsequent periods as demand shocks, which have their biggest impact on sales in the new car market, propogate through the economy via their effect on the evolution of the distribution of car holdings over time. Our model accounts for both intertemporal and cross-product substitution possibilities in the demand for cars. An obvious substitute for buying a new car when the economy enters a recession is to continue to hold one's existing used car, or to trade it for another used car. There is also substantial substitution into and out of the "outside good" i.e. consumers who do not own a car and use other modes of transportation. Due to the many avenues of substitution consumers have in the car market, prices do not react as strongly to macroeconomic shocks as do quantities and the persistence of quantity shocks is higher as well due to "vintage capital effects."

For example a temporary boom in new car sales this period results in a higher stock of 1 year old cars a year from now, and so forth. Thus booms in new car sales appear as a bulge (or wave crest) in the age distribution of car holdings that gradually dissipates over time. A shock that lowers demand in the new car market has the opposite effect, leading to a trough in the age distribution of cars that propogates over time and thus

has persistent effects. The waves in the age distribution of cars are similar to waves we observe in other aging "stocks" such as the age distribution of the population where the baby boom lead to a pronounced bulge in the population age distribution that propogates over time and also has persistent effects (including a looming crisis about age dependency ratios and how to fund Social Security systems around the world).

Modeling trade of cars in the presence of secondary markets and transactions costs is already challenging due the high dimensionality of the number of goods and prices. The demand for cars of any type or age depends in principle on the vector of prices for all makes and models and ages of cars, and these prices in turn depend on the quantities of all such cars carried over from the previous period as well as consumers' decisions on whether to trade or keep their existing cars (or switch to or from the no car state). In the absence of used car imports, the demand for new cars sold in the primary market sets an upper bound on the stock of cars that are t year old. Referring to the year a new car was sold as its *vintage* consider what happens to a given cohort of cars of a given vintage over time as they age. The total stock of age t cars will generally be lower than the number of new cars initially sold due to accidents and scrappage. However the fact that there is a fixed number of cars of each age t does not imply that the supply of such cars is inelastic. Consumers who own cars of age t will be less likely to scrap them and more likely to sell them in the used car market if the price P_t for such cars is higher. In an equilibrium of the car market the prices must be such as to set the difference between supply and demand for all types and ages of used cars to zero.

Our paper builds on the work of Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022) who developed a computationally tractable and empirically estimable large scale dynamic equilibrium model of the new and used car market. However their model assumes there are no macro level shocks to supply or demand for cars. They used the concept of *stationary equilibrium* of Rust (1985) to determine prices and quanities but extended to a framework where there are buy and sell side transactions costs that can dampen the level of trade in used cars. We can interpret transactions costs as costs of adjustment that dissuade consumers from more frequently trading their existing used car for their preferred make, model and age of replacement vehicle. Rust (1985) showed that when transactions costs are zero, then it is optimal to trade every period, so no consumers would choose to keep their vehicles, something completely at odds from what

we actually observe. Our model incorporates transactions costs to explain why, despite macro shocks that amplify consumers' desire to trade, the vast majority of consumers still choose to keep rather than trade their existing automobiles.

We extend the dynamic equilibrium model of car trading of Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022) to incorporate persistent macro shocks in the form of demand shocks that affect their willingness to pay for automobiles, both new and used. These shocks induce substitution into the outside good, but also disproportionally impact purchases of new cars which is a significantly more expensive option than either keeping their current car or trading it for some other used car. It is well known that sales of new cars is highly procyclical, and our simplified way of modeling demand shocks enables us to capture this variability.

Our model captures additional dynamics that amplify the cyclicality of new car purchases that we call *replacement cycles* and *rebound effects.* Replacement cycles occur because an aggregate demand shock that leads to a burst of new car purchases reduces the number of consumers in the economy with old cars that they would like to trade or scrap that reduces the demand for new cars in following periods. Even if the demand shock persisted for additional periods, demand for new cars will be less than the initial burst of sales at the start of the boom period since there are fewer consumers with older cars "at risk" for being traded or replaced. This is similar to a mechanism emphasized by Cooper, Haltiwanger and Power (1999), "the initial burst of investment is followed by a couple of periods of below-average investment." (p. 928).[1]

Rebound effects are short term amplifications in demand for new cars that reflect "pent up demand" due to the aging of the vehicle fleet during a long period of poor economic conditions and low demand for new cars. Conversely if a negative demand shock occurs after a sustained boom period, the rebound effect operates in reverse depressing the demand for new cars by more than would have occurred otherwise. This is due to the fact that a significant share of the vehicle stock has been recently replaced after a

[1] A similar dynamic also arises in the dynamic duopoly model of Goettler and Gordon (2011) who develop and innovative dynamic Markov perfect equilibrium model of competition between AMD and Intel. Even in the absence of macro shocks to demand due to strategic leapfrogging competition between these companies: "A short-term increase in innovation widens the quality gap between currently owned products and new offerings, boosting demand and raising prices and sales. After the upgrade boom, prices and sales fall as replacement demand drops. Firms must continue to innovate to rebuild replacement demand because microprocessors do not physically depreciate. We model this upgrade cycle and the timing of consumers' purchases given beliefs about future prices and innovation." (p. 1143).

long preceding boom period and this lowers the age of the vehicle stock and reduces the incentive to trade or replace one's current car, including new cars. We show that rebound effects are bigger but less frequent when macro demand shocks are more persistent.

We extend the definition of stationary equilibrium of Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022) to incorporate macro shocks in section 3, resulting in a nonstationary equilibrium concept that allows used car prices and the distribution of holdings to depend on the realized value of the macro state, and thus vary over time as the macroeconomy fluctuates. However the macro state alone is insufficient to determine equilibrium prices and quantities: our definition of a rational expectations equilibrium requires us to carry the entire holdings/age distribution of cars as an additional state variable in the model. The reason is similar to the one that motivates including the distribution of wealth in heterogeneous agent macro models such as Krusell and Smith (1998): the aggregate demand for cars next period depends on the age distribution of the fleet this period.[2]

In many cases in the macro literature, including the work of Krusell and Smith, this leads to severe computational issues including solving dynamic programming problems with an infinite dimensional state variable since rational consumers need to know the entire distribution of wealth in order to correctly predict equilibrium interest rates. In our case, consumers need to know the age distribution of the vehicle fleet which we we denote by q_t, to determine equilibrium prices at time t, p_t. Following Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022) we assume that the condition of a used car can be captured by its age and there is a maximum age of car that is allowed to be held in the economy (perhaps due to government safety regulations). This implies that the state variable q_t is an element of a finite dimensional simplex. This feature aids our ability to compute an *exact* rational expectations equilibrium whereas the fact that the space of all wealth distributions is infinite dimensional prompted Krusell and Smith (1998) to use an *ad hoc* approximation to a rational expectations equilibrium that assumes equilibrium interest rates are a function of a low dimensional set of sufficient statistics of the wealth distribution such as its mean and variance.[3]

[2] In Krusell and Smith (1998) consumer savings depend on wealth as well as the interest rate, and when there is heterogeneity in wealth, the equilibrium interest rate in the economy next period depends upon the distribution of wealth this period.

[3] Using numerical solutions that regressed the equilibrium interest rate on the mean of the wealth distribution, Krusell and Smith (1998) argued that interest rates could be predicted so well that it was unnecessary to use

The approximate solution approach of Krusell and Smith (1998) is unlikely to produce accurate approximations to the rational expectations equilibrium of our model because equilibrium prices are not well approximated as a simple function of the mean age of the vehicle stock. The reason is that the demand for cars depends on the entire age distribution of the existing car stock: we show that demand for cars can differ significantly for two different holdings distributions q_t with the same mean age. We employ the algorithm developed in Cao, Luo and Nie (2023) to compute the rational expectations solution to our model to a high degree of accuracy. Equilibrium prices in our model can be written as $p(q, m)$ where q is the current holdings distribution and m is the macro state. Even though q is finite-dimensional, the equilibrium price function $p(q, m)$ is an infinite-dimensional object and thus can only be approximated. Nevertheless, we show that the function p as well as consumers' value functions can be well approximated using the adaptive sparse grid method of Brumm and Scheidegger (2017).

Note that even though we refer to our concept of equilibrium as "nonstationary" since we allow prices and the distribution of car holdings to vary over time, in our equilibrium the variables $\{q_t, m_t\}$ actually evolve as a stationary Markov process, so perhaps the terminology "time and state varying" is a better descrption for our equilibrium that "nonstationary" since we can show the equilibrium process for $\{q_t, m_t\}$ is ergodic and has a stationary distribution. Thus, even though prices and quantities vary over time in response to macro shocks, we are able to compute stationary probability distributions for $\{q_t, m_t\}$ and an implied stationary distribution over used car prices $p(q_t, m_t)$.

We also compare the rational expectations equlibrium (RE) to the notion of *temporary equilibrium* (TE) introduced by Grandmont (1977). TE is a myopic equilibrium concept wherein each period consumers make their car trading strategy based on the assumption that the current equilibrium prices and macro state will remain at their current values in all future periods. Thus, TE consumers are continually "surprised" each successive period by the fact that current equilibrium prices are not the values that they expected to hold when made their trading decisions last period. We would expect TE to look very different from a RE where consumers solve a much more complicated dynamic pro-

additional moments of the wealth distribution or the entire distribution itself for agents in their model to make good savings decisions. However to our knowledge nobody has succeeded in providing a bound that compares how well their approximate equilibrium solution is to the exact rational expectations equilibrium solution, or even to a numerical approximation of the latter.

gramming problem to determine their car trading strategy and have rational expectations about how the state variables $\{q_t, m_t\}$ and implied used car prices $\{p(q_t, m_t)\}$ evolve over time.

Surprisingly, we show that TE and RE solutions can be close to each other and the loss in welfare from failiing to use all available information to predict the future results in negligible welfare losses. This is a fortunate result given the difficulties of predicting a high dimensional vector of used car prices into the future and can be viewed as an analog the approach of Krusell and Smith (1998) for predicting future interest rates, though one that does not rely on regressions of used car prices on sufficient statistics of the holdings distribution.

Section 2 introduces the notation and setup for our dynamic model of auto trading in the presence of persistent macro shocks and defines the rational expectations equilibrium. We prove several general results characterizing the equilibrium and trading strategies. Section 3 provides an illustration of the RE solution in the case where cars must be scrapped after 2 periods of life. In this case q_t is a member of the two-dimensional simplex and we can produce graphs providing intution for the form of the RE solution and characterize the ergodic distribution of $\{q_t, m_t\}$. Section 3 provides the results from a solution and simulation of a model where cars can live up to 10 periods. Here we compare the RE and TE solutions and illustrate the implications for the propogation of macro shocks over time, illustrating the replacement cycles and rebound effects implied by the model. Section 6 provides some concluding discussion and suggestions for other applications of our framework and results.

2 Stylized facts on automobile markets

This section presents some statistics from the automobile market in Denmark in the late 1990s and early 2000s.

Figure 1 shows Danish GDP growth rate and the number of registrations of new cars over the time span of 1992 to 2010. The emerging pattern indicates strong cyclicality in the new car purchases, with the number of registrations of new cars increasing during the economic upturns and decreasing during the downturns.

The initial humps and troughs in the inflow of the new cars to the country are reflected

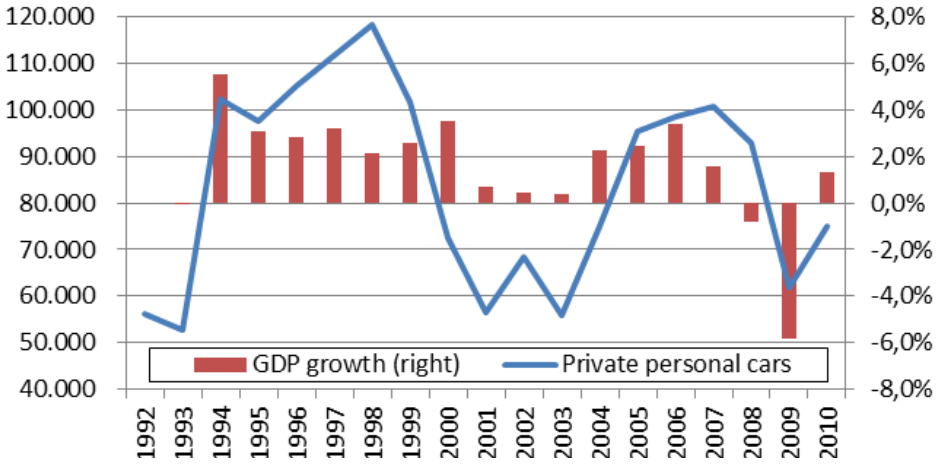


Figure 1: Cyclicality in registrations of new cars.

in the age distribution of the car fleet over time, as shown in Figure 2. Due to the fact that cars last over many years as any durable goods, when viewed over calendar time the car age distribution exhibits clear diagonal waves formed by cars of different cohorts.

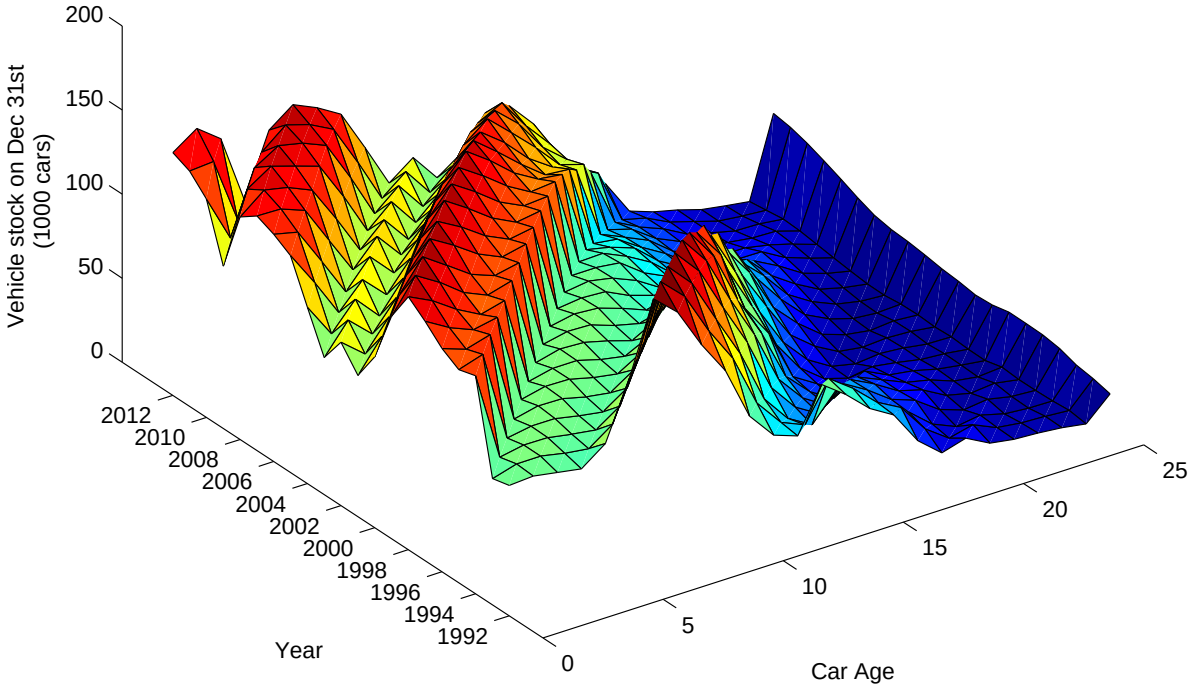


Figure 2: Fleet age distribution over calendar time 1992-2012.

The wave pattern in the age distribution is repeated in many different data slices characterizing the automobile market. Figure 3 shows such waves in the number of transactions on the primary and secondary market of cars of different ages over the

period from 1996 to 2008. Clearly, the strongest busyness cycle pattern shows up in the new car purchases, however the waves in sales and purchases are clearly visible through the whole distribution of car ages.

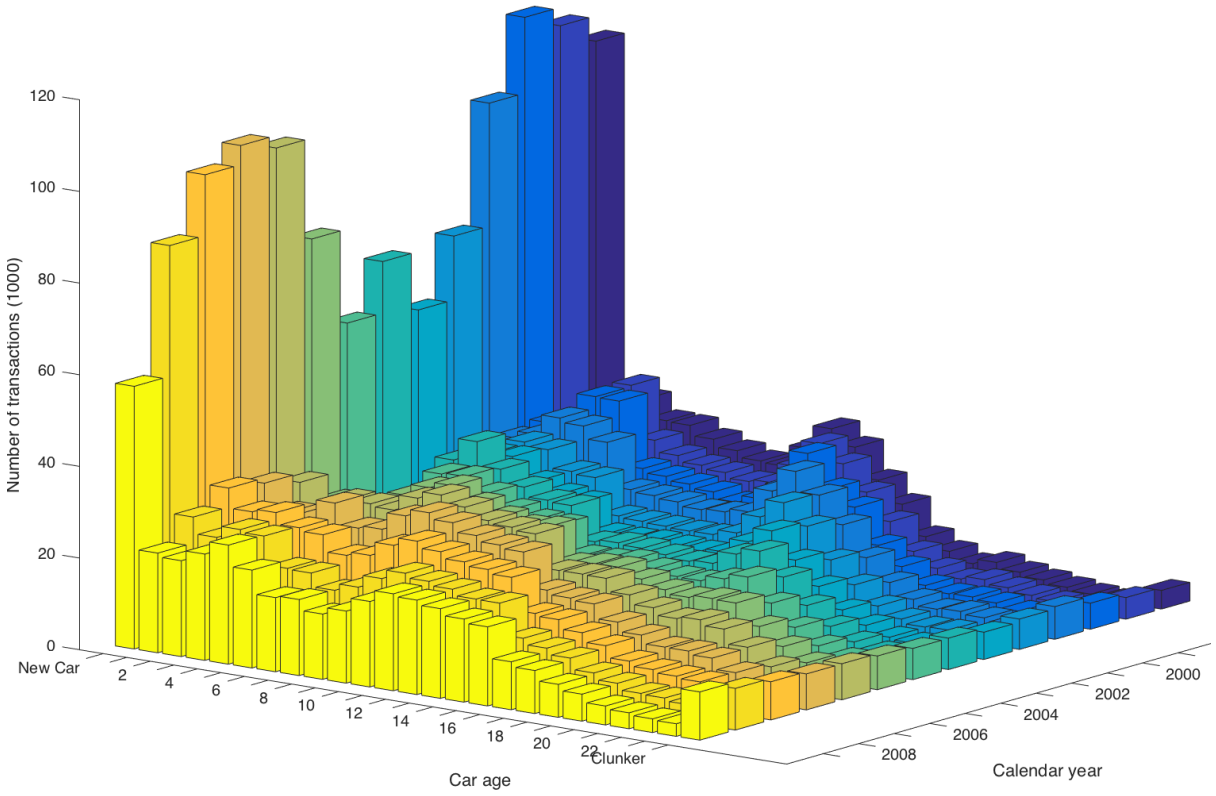


Figure 3: Transactions on the primary and secondary market by car age, 1996-2008.

3 Non-stationary equilibrium trading in automobiles

In this section we set up a non-stationary dynamic discrete choice model of the secondary market for the automobiles. We extends the stationary model of Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022) to allow an exogenous aggregate macro state variable to affect consumer preferences and thus create wave pattern in the new car purchases and equilibrium car ownership distribution.

3.1 Key assumptions

Consider the infinite horizon discrete time non-stationary equilibrium model of an automobile market where the new cars are initially supplied in the primary market, and

subsequently traded by consumers in the secondary market. Consumers make purchase, replacement, trading and scrapping decisions to maximize their expected utility which is discounted by with a common discount factor $\beta \in (0,1)$. The market operates in a stochastic macroeconomic environment which we describe by an exogenous Markov process that marks the periods of expansion and recession. The macro state $m \in \{0,1\}$ affects the marginal utility of money for the consumers as described below, and therefore influences the demand for the new cars as well as demand and supply on the market of used cars.

As discussed by Krusell and Smith (1998) and Cao (2020), the rational consumers in this framework form expectations about the aggregate future state of the economy in order to predict the future value of their cars and make optimal trading decisions. From the dynamic programming point of view, this requires including the whole *ownership distribution* into the state space of the model, posing an enormous computational challenge for the solution of the model. We adopt the recent global solution approach of Cao, Luo and Nie (2023) to solve for the rational expectations equilibrium in the homogenous version of the Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022) model.

In our model the new cars are supplied to the market infinitely elastically at a fixed price $\overline{P}$, and scrapped at the end of their life with fixed scrappage value $\underline{P}$. We make a simplifying assumption that the quality of the car is fully captured by its age $\{1,\dots,\bar{a}\}$ where $\bar{a}$ is the upper bound on the age of the car chosen exogenously to bound the model. Upon reaching the scrappage age $\bar{a}$ the cars can no longer be kept or sold on the market, and have to be disposed of. However, we allow for voluntary endogenous scrappage at any age, such that $\bar{a}$ can be chosen not to be binding in equilibrium.

The economy is populated by a unit mass of homogeneous consumers who differ only in the idiosyncratic taste shocks associated with the alternatives in the discrete choice they face. We assume that consumer preferences are characterized by a common quasi-linear utility function

$$U(\cdot) = u(a) - \mu(m)\big[\text{operating cost} + \text{trade and transaction cost}\big] + \text{switching cost}. \quad (1)$$

The term $u(a)$ captures the deteriorating car quality; in calibration we approximate this term by a second degree polynomial. The marginal utility of money $\mu(m)$ converts the

monetary cost of operating and trading the vehicles to the utility space, and is shifted between two levels depending on the macroeconomic state m. Trading costs contains the difference in prices (or scrap value) between the traded cars, with addition of freely specified transaction cost. Finally, the last term represent the psychological costs of replacing the automobile or changing car ownership state altogether. Both trading and psychological switching costs are zero if the existing car is kept. We do not consider the possibility of multiple car ownership, and instead assume that the consumers own at most one car at any time period. This assumption is supported well by the Danish data that is used to calibrate the model.

3.2 Consumer states and choices

In the beginning of each period consumer is characterized by the vector (a, ϵ) where $a \in \{1, ..., \bar{a}\} \cup \{\emptyset\}$ denotes their car ownership state including the outside option of not having a car $\emptyset$, and the vector of random components ϵ which incorporates the idiosyncratic heterogeneity in cars and consumers.

We assume that trade occurs instantaneously in the beginning of the period, and so if the existing car a is replaced by another car d purchased new or at the secondary market, it is this post-trade car that is utilized during the period. We denote the post-trade ownership state by $d \in \{0, ..., \bar{a} - 1\} \cup \{\emptyset\}$ which also includes the outside option. Those consumers who do not trade continue to use the car they own at the beginning of the period, in which case $d = a$.[4] The post-trade ownership state d becomes the beginning of the next period state, with cars ageing by one year, $a' = d+1$. We also model involuntary scrappage of cars due to loss in accidents by allowing transition to the terminal age state $\bar{a}$ with probability α_d. Thus, the evolion of the car ownership state is given by

$$a'(a, d) = \begin{cases} d + 1 & \text{with probablity } \alpha_d \text{ if } d \neq \emptyset, \\ \bar{a} & \text{with probablity } 1 - \alpha_d \text{ if } d \neq \emptyset, \\ \emptyset & \text{if } d = \emptyset. \end{cases}$$

In addition to the choice of the car to purchase, consumers who participate in trade have additional choice of whether to sell their existing car on the secondary market or

[4] We occasionally misuse the notation and denote this case by $d = \kappa$ which should be understood as a shorthand for $d = a$ under the "keep" choice.

to scrap it. We denote this decision as $s \in \{0_s, 1_s\}$ and model it as a nested discrete choice relevant for the consumers in states $a \in \{1, \bar{a}-1\}$. Consumers who own cars of the terminal age $\bar{a}$ at the beginning of the period do not have the option to keep it, nor do they have the option to sell it on the secondary market. Terminal age cars have to be scrapped, and their owners have to either buy a different car, or transition to the no car state $d = \emptyset$ at the trading stage.

Denote $C(a)$ the choice set of the consumer in ownership state a at the beginning of the period. It is given by a combination of trading and scrappage possibilities (we use empty set notation $\{\}$ when the latter is irrelevant). With the above assumptions the choice set is given by

$$C(a) = \begin{cases} \{0, 1, ..., \bar{a}-1\} \cup \{\emptyset\} \times \{\} & \text{if } a \in \{\emptyset, \bar{a}\} \\ \{0, 1, ..., \bar{a}-1\} \cup \{\emptyset, \kappa\} \times \{0_s, 1_s\} & \text{if } a < \bar{a}, \end{cases} \tag{2}$$

where the choice $\emptyset$ means "purge", implying that the consumer sells their car and goes into the no-car state, and the choice κ means that they keep the existing car.

Participation in the secondary market during the trading stage incurs trading costs which take the form of

$$\text{trade costs}(d, a) = T(a, d) + P(d, \cdot) - P(a, \cdot),$$

where $T(a, d)$ denotes flexible transactions cost which we fully specify at the model calibration stage. The market prices $P(a, \cdot)$ are determined in the non-stationary equilibrium defined below, and anticipated by the consumers when making their trading decisions. By convention $P(\emptyset, \cdot) = 0$ to extend the formula above to the outside option. We do allow for the same age car $a = d$ to be sold and bought back on the market, but the decision to keep the car achieves the same without incurring the trading costs.

3.3 Aggregate states

Along with the macro state of the economy m which evolves according to a fixed 2×2 Markov transition probability matrix $G(m'|m)$, the aggregate state of the economy is given by the market shares of each car traded in the secondary market, along with the outside option of not owning a car.

Let $0 \leq q_a \leq 1$ denote the fraction of the unit mass of households in car state a, namely those who own the car of age a at the start of the period before the trading phase. Similarly, let $0 \leq q_\emptyset \leq 1$ be the fraction of households without a car. The *ownership distribution* vector q summarizes the distribution of the unit mass of consumers in the economy over all possible car states:

$$q = \left(q_1, \ldots, q_{\bar{a}}, q_\emptyset\right) \in \mathbb{R}^{\bar{a}+1}. \tag{3}$$

The ownership distribution q is a proper probability vector (its elements sum up to 1), and thus belongs to the $J\bar{a}$-dimensional unit simplex.

Given the non-stationary nature of our model, both the market state and the ownership distribution have to be included in the state space of the model. Consumers form beliefs about the evolution of these states over time, and use these beliefs in their decision making process. The evolution of the ownership distribution itself is governed by the trading choices of the consumers, forming a closed loop system which equilibrium we formally define below.

As mentioned above, we accume that the macro state of the economy evoloves as an exogenous Markov chain with fixed transition matrix $G(m'|m)$. This transition probability matrix is calibrated to reflect the frequency of the business cycles in the Danish economy. In turn, the evolution of the ownwership distribution is governed by two processes: endogenous trading behavior of consumers, and the physical process of ageing and accidents in the cars.

To describe the two phases of the evolution of the ownership distribution we rely on the tools from Markov chain theory, and describe car ownership and state transitions using two transition probability matrices Q and $\Omega(P, m, q)$ defined below.

Denote $\Pi(d|a)$ the choice probability to trade the car a for car d, $\Pi(\kappa|a)$ the choice probability to keep the existing car a, and $\Pi(\emptyset|a)$ the choice probability to purge the car a and go into the no-car state. Implicitly, the choice probabilities are functions of the market prices $P = (\overline{P}, P(a, \cdot), \underline{P})$, and the aggregate states (m, q), and include the case of no-car state $a = \emptyset$. Denote $\Omega(P, m, q)$ the $(\bar{a}+1) \times (\bar{a}+1)$ *trade transition probability*

matrix given by

$$\begin{bmatrix} \Pi(1|1)+\Pi(\kappa|1) & \dots & \Pi(\bar{a}-1|1) & \Pi(0|1) & \Pi(\emptyset|1) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \Pi(1|\bar{a}-1) & \dots & \Pi(\bar{a}-1|\bar{a}-1)+\Pi(\kappa|\bar{a}-1) & \Pi(0|\bar{a}-1) & \Pi(\emptyset|\bar{a}-1) \\ \Pi(1|\bar{a}) & \dots & \Pi(\bar{a}-1|\bar{a}) & \Pi(0|\bar{a}) & \Pi(\emptyset|\bar{a}) \\ \Pi(1|\emptyset) & \dots & \Pi(\bar{a}-1|\emptyset) & \Pi(0|\emptyset) & \Pi(\emptyset|\emptyset) \end{bmatrix}, \quad (4)$$

which is constructed such that given the beginning of period ownership distribution q, the post-trade ownership distribution is given by $q\Omega(P, m, q)$, taking proper care of the fraction of consumers in no-car state. Note that next-to-last element in the resulting vector holds the fraction of consumers who have bought a new car in the current period.

During the rest of the period the households own and drive their cars and which are aging and having accidents. These transitions can be described by the $(\bar{a}+1) \times (\bar{a}+1)$ stochastic matrix that we refer to as *physical transition probability matrix* Q. It is given by

$$Q = \begin{bmatrix} 0 & 1-\alpha_1 & \dots & 0 & \alpha_1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & 1-\alpha_{\bar{a}-2} & \alpha_{\bar{a}-2} & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 1-\alpha_0 & 0 & \dots & 0 & \alpha_0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}. \quad (5)$$

The entries in the matrix Q map the elements in the post-trade ownership distribution $q\Omega(P, m, q)$ to the ownership distribution q' at the beginninng of the next period. The last row and column of Q reflect the fact that hte share of the population in the no-car state remains constant over time. The next-to-last row and column of Q convert the share of the new cars bought during the trading stage to the share of one year old cars (in case of no accident) and the share of new cars α_0 totalled during the period. The elements in the other $\bar{a}-1$ rows and columns describe the expected aging and accidents transitions.

With both phases of the transition expressed as transition probability matrices, the

next period ownership distribution is then given by

$$q' = q\Omega(P, m, q)Q, \tag{6}$$

which completes the description of the evolution of the aggregate states in the model.

3.4 Recursive formulation and Bellman equations

Denote $V(a, \epsilon, m, q)$ the value function equal to the net expected discounted utility over the infinite horizon. The optimal behavior of consumers is characterized by the solution to the following system of Bellman equations.

For a consumer who does not own a car ($a = \emptyset$) the value function is given by the best of the choices to remain in the no car state or buy any of the cars available in the market. The Bellman equation for this consumer is

$$V(\emptyset, \epsilon, m, q) = \max\left[v(\emptyset, \emptyset, m, q) + \sigma\epsilon_\emptyset; \max_{d\in\{0,1,\ldots,\bar{a}-1\}}\left[v(\emptyset, d, m, q) + \sigma\epsilon_d\right]\right], \tag{7}$$

where the *choice specific value functions* of remaining in the no car state and of leaving the no car state to buy a car of age d respectively are given by:

$$\begin{aligned} v(\emptyset, \emptyset, m, q) &= u(\emptyset) + \beta\sum_{m'} EV(\emptyset, m', q')G(m'|m), \qquad (8)\\ v(\emptyset, d, m, q) &= u(d) - \mu(m)\big[P(d, m, q) + T_b(d)\big] - T^{\text{psy}}(\emptyset, d)\\ &\quad + \beta\sum_{m'}\Big[\big(1-\alpha_d\big)EV(d+1, m', q') + \alpha_d EV(\bar{a}, m', q')\Big]G(m'|m), \end{aligned}$$

where $T_b(d)$ is monetary transaction cost on the buyer side and $T^{\text{psy}}(a, d)$ represents the psychological switching costs of transition from car a to car d. The *expected value functions* $EV(\emptyset, m, q)$ and $EV(a, m, q)$ provide the conditional expected values of starting the next period respectively without a car, and with car of age a, and are given by:

$$EV(a, m, q) = \int_\epsilon V(a, \epsilon, m, q)f(\epsilon|a)d\epsilon, \tag{9}$$

where $f(\epsilon|\cdot)$ is the probability density function of the distribution for the idiosyncratic shocks ϵ. We assume that this distribution belongs to the Generalized Extreme Value (GEV) family, allowing on one hand for the nested structure of trading and scrappage

choices, and on the other hand leading to close form expressions for the integral in (9) and conditional choice probabilities for trading and scrappage decisions. Implicit in these formulas is the assumption that idiosyncratic shocks affecting the consumer's choice are independent of their past realizations. This implies that the $EV(\cdot)$ functions only depend on the car that has been driven and aged during the current period and constitutes the car state in the beginning of the next period.

Under our assumption of GEV distribution of idiosyncratic shocks ϵ, the integral in (9) can be expressed in closed-form. The formulas are somewhat involved for the assumed nested structure of the trading and scrappage choices, but because there is no scrappage choice in no car state or when starting the period with the terminal age car, the formulas for the expected value functions are as follows:

$$\begin{aligned} EV(\emptyset, m, q) &= \sigma \log \left(\exp\{v(\emptyset, \emptyset, q)/\sigma\} + \sum_{d=0}^{\bar{a}-1} \exp\{v(\emptyset, d, m, q)/\sigma\} \right) \\ EV(\bar{a}, m, q) &= \sigma \log \left(\exp\{v(\bar{a}, \emptyset, m, q)/\sigma\} + \sum_{d=0}^{\bar{a}-1} \exp\{v(\bar{a}, d, m, q)/\sigma\} \right) \end{aligned} \tag{10}$$

where we define the choice specific value functions $v(\bar{a}, \emptyset, m, q)$ and $v(\bar{a}, d, m, q)$ below.

The value function for a consumer who starts the period with a car of terminal age $\bar{a}$ is characterized by the same choices and is therefore given by a similar expression:

$$V(\bar{a}, \epsilon, m, q) = \max \left[v(\bar{a}, \emptyset, m, q) + \sigma\epsilon_{\emptyset}; \max_{d \in \{0,1,\ldots,\bar{a}-1\}} \left[v(\bar{a}, d, m, q) + \sigma\epsilon_d \right] \right], \tag{11}$$

where again the first component corresponds to the decision to choose the outside option of not owning a car, and the second to the purchase of a different car of age d. Because at terminal age $\bar{a}$ the car is scrapped exogenously, the consumer does not have the option of keeping their car, and also does not have an additional choice of endogenous scrappage of the existing car ($\bar{a}$). This is why the the Bellman equation (11) looks very similar to that of the consumers who do not own a car (7). The difference, however, is in the net scrap value $\underline{P}$ the consumer receives from the scrap yard. Therefore the relevant choice

specific value functions are given by:

$$v(\bar{a},\emptyset,m,q) = u(\emptyset) + \mu(m)\underline{P} - T^{\text{psy}}(\bar{a},\emptyset) + \beta\sum_{m'} EV(\emptyset,m',q')G(m'|m), \tag{12}$$

$$\begin{aligned} v(\bar{a},d,m,q) = {} & u(d) - \mu(m)\big[P(d,m,q) - \underline{P} + T_b(d)\big] - T^{\text{psy}}(\bar{a},d) \\ & + \beta\sum_{m'}\Big[\big(1-\alpha_d\big)EV(d+1,m',q') + \alpha_d EV(\bar{a},m',q')\Big]G(m'|m). \end{aligned}$$

Finally, the optimal decisions of a consumer who starts the period with a car of age a are characterized by the following Bellman equation:

$$V(a,\epsilon,m,q) = \max\begin{bmatrix} v(a,\kappa,m,q) + \sigma\epsilon_\kappa; \\ \max_{s\in\{1_s,0_s\}}\big[v(a,\emptyset,s,m,q) + \sigma\epsilon_{\emptyset,s}\big]; \\ \max_{\substack{d\in\{0,1,\ldots,\bar{a}-1\},\\ s\in\{1_s,0_s\}}}\big[v(a,d,s,m,q) + \sigma\epsilon_{d,s}\big] \end{bmatrix}. \tag{13}$$

In this case the endogenous scrappage choice $s \in \{1_s, 0_s\}$ has to be accounted for, so the complete set of choice specific value functions which correspond to all the alternatives in the choice set $C(a)$ defined in (2), is given by:

$$v(a,\emptyset,1_s,m,q) = u(\emptyset) + \mu(m)\underline{P} - T^{\text{psy}}(a,\emptyset) + \beta\sum_{m'} EV(\emptyset,m',q')G(m'|m), \tag{14}$$

$$v(a,\emptyset,0_s,m,q) = u(\emptyset) + \mu(m)[P(a,m,q) - T_s(a)] - T^{\text{psy}}(a,\emptyset) + \beta\sum_{m'} EV(\emptyset,m',q')G(m'|m),$$

$$v(a,\kappa,m,q) = u(a) + \beta\sum_{m'}\Big[\big(1-\alpha_a\big)EV(a+1,m',q') + \alpha_a EV(\bar{a},m',q')\Big]G(m'|m),$$

$$\begin{aligned} v(a,d,1_s,m,q) = {} & u(d) - \mu(m)\big[P(d,m,q) - \underline{P} + T_b(d)\big] - T^{\text{psy}}(a,d) \\ & + \beta\sum_{m'}\Big[\big(1-\alpha_d\big)EV(d+1,m',q') + \alpha_d EV(\bar{a},m',q')\Big], \end{aligned}$$

$$\begin{aligned} v(a,d,0_s,m,q) = {} & u(d) - \mu(m)\big[P(d,m,q) - P(a,m,q) - T_s(a) + T_b(d)\big] - T^{\text{psy}}(a,d) \\ & + \beta\sum_{m'}\Big[\big(1-\alpha_d\big)EV(d+1,m',q') + \beta\alpha_d EV(\bar{a},m',q')\Big]G(m'|m). \end{aligned}$$

Here $v(a,\emptyset,0_s,q,m)$ is the value of selling the car on the market and not replacing it, $v(a,\emptyset,1_s,q,m)$ is the value of scrapping the car and not replacing it — in both of these cases the customer has no car to drive and ends up in the no car state in the next period.

Combined the value functions defined in (7), (11) and (13) cover all the possibilities

for the car ownership state $a \in \{\emptyset, 1, \ldots, \bar{a}\}$. The expected value function in the case where a consumer has an existing car of age $a < \bar{a}$ is given below

$$EV(a, m, q) = \tag{15}$$
$$\sigma \log \left(\exp\{v(a, \kappa, m, q)/\sigma\} + \exp\{I(a, \emptyset, m, q)/\sigma\} + \sum_{d=0}^{\bar{a}-1} \exp\{I(a, d, m, q)/\sigma\} \right),$$

where $I(a, \emptyset, m, q)$ and $I(a, d, m, q)$ are inclusive values (or log-sums, or expected maxima) associated with the decision to scrap or sell when the consumer chooses to purge their car (i.e. move to the no car state, $d = \emptyset$), or purchase a car of age d, respectively. For formulas for these inclusive values are given below

$$\begin{aligned} I(a, \emptyset, m, q) &= \sigma_s \log \left(\exp\{v(a, \emptyset, 1_s, m, q)/\sigma_s\} + \exp\{v(a, \emptyset, 0_s, m, q)/\sigma_s\} \right) \\ I(a, d, m, q) &= \sigma_s \log \left(\exp\{v(a, d, 1_s, m, q)/\sigma_s\} + \exp\{v(a, d, 0_s, m, q)/\sigma_s\} \right) \end{aligned} \tag{16}$$

where 1_s denotes the decision to scrap the currently held car of age a and 0_s denotes the decision to sell the car in the secondary market to another consumer.

Applying equations (9) for each element of vector EV, plugging in the expressions (7), (11), (13) and their corresponding choice specific value functions, we derive the system of $\bar{a} + 1$ nonlinear equations

$$EV = \Gamma(EV), \tag{17}$$

whose solution enables us to reconstruct V and characterize optimal trading behavior. Here Γ is the *smoothed Bellman operator* that constitutes a contraction mapping and hence has a unique fixed point.

3.5 Choice probabilities

In addition to already introduced *trade* choice probabilities $\Pi(d|a)$, $a \in \{1, ..., \bar{a}\} \cup \{\emptyset\}$, $d \in \{0, ..., \bar{a} - 1\} \cup \{\emptyset\}$, denote $\Pi(s|d, a)$ the choice probability of scrappage decision by the consumers who are in position to make this choice. Once gain, note that all the choice probabilities are implicitly dependent on the aggregate state variables (m, q) and the prices P.

Because the scrappage choice is nested (with scale parameter σ_s) we have

$$\Pi((d,s)|a) = \begin{cases} \Pi(d|a) & \text{if } a \in \{\emptyset, \bar{a}\}, \\ \Pi(d|a) \cdot \Pi(s|d,a) & \text{if } a < \bar{a} \text{ and } d \neq \kappa, \\ \Pi(\kappa|a) & \text{if } a < \bar{a} \text{ and } d = \kappa. \end{cases} \tag{18}$$

Probability of keeping does not involve a scrappage decision and is given by

$$\Pi(\kappa|a) = \frac{\exp(v(a,\kappa,m,q)/\sigma\}}{\sum_{d=0}^{\bar{a}-1} \exp\{I(a,\emptyset,m,q)/\sigma\} + \exp\{v(a,\kappa,m,q)/\sigma\} + \exp(I(a,d,m,q)/\sigma)}. \tag{19}$$

However the probability of purging the car does involve a scrappage decision and the expected maximum value of scrapping or selling (depending on the value of unobserved shocks associated with the decision to scrap or sell) is given by $I(a,\emptyset,m,q)$ so this probability is given by

$$\Pi(\emptyset|a) = \frac{\exp\{I(a,\emptyset,m,q)/\sigma\}}{\sum_{d=0}^{\bar{a}-1} \exp\{I(a,\emptyset,m,q)/\sigma\} + \exp\{v(a,\kappa,m,q)/\sigma\} + \exp(I(a,d,m,q)/\sigma)}. \tag{20}$$

Due to additively separable transaction costs, and provided that the scale parameter σ_s is the same in all nests of the choice tree where the decision is relevant, the choice between scrapping and selling the existing car is independent of the choice of the replacement car and has no implications for the future periods. This implies that the scrap/sell decision is static, so $\Pi(s|d,a) = \Pi(s|a)$. It is readily verified that the scrappage probability is the same for both the case of purging and replacing the current car a, so for all $a \in \{1,\ldots,\bar{a}-1\}$, it is given by

$$\Pi(1_s|a) = \left(1 + \exp\left(\tfrac{\mu(m)}{\sigma_s}[P_a(m,q) - T_s(a) - \underline{P}]\right)\right)^{-1}. \tag{21}$$

The remaining choice probabilities for all feasible alternatives in the choice sets (2) corresponding to owning a car of age a is given by

$$\Pi(d|a) = \frac{\exp(I(a,d,m,q)/\sigma\}}{\sum_{d=0}^{\bar{a}-1} \exp\{I(a,\emptyset,m,q)/\sigma\} + \exp\{v(a,\kappa,m,q)/\sigma\} + \exp(I(a,d,m,q)/\sigma)}. \tag{22}$$

where $I(a,d,m,q)$ is the inclusive value (16) associated with the scrapping decision for a consumer who decides to trade their existing age a car for a different car of age d.

3.6 Excess Demand

We assume that market clears in all macro states (m, q) so that aggregate excess demand which we denote by $ED(m, q)$ is equal to 0 for all (m, q). $ED(m, q)$ maps (m, q) into $R^{\bar{a}-1}$, and we denote by $ED_a(m, q)$ the excess demand for a used car of age a. We assume that new cars are supplied infinitely elastically to the market at price $\overline{P}$, and there is infinitely elastic demand for cars for their scrap value at the scrap price $\underline{P}$, so the market for new cars "automatically" clears (i.e. there is zero excess demand) in all states (q, m) and similarly the "scrap market" also clears at the price $\underline{P}$. Thus, it is only necessary to define the excess demand functions at the tradable ages of used cars, $a \in \{1, \ldots, \bar{a} - 1\}$. The demand for cars of age a is given by

$$D_a(m, q) = \Pi(a|\emptyset, m, q)q_\emptyset + \sum_{d=1}^{\bar{a}} \Pi(a|d, m, q)q_d, \tag{23}$$

where $q_\emptyset$ is the fraction of consumers who have no car and q_d is the fraction of consumers who own a car of age d. Note that at the start of each period before trading occurs, there are by definition no brand new cars in the population (i.e. $q_0 = 0$) since q represents the distribution of holdings of cars at the start of a period before any trading has occurred and thus, there cannot be in new cars in this distribution since any new cars enter the population only *after* trading has occurred, and clearly any new cars that were purchased last period and which were not involved in a car accident will be age $a = 1$ at the start of the current period.

The supply of cars of age a equals the fraction of cars of this age in the population that are not kept and not scrapped,

$$S_a(q, m) = (1 - \Pi(\kappa|a))(1 - \Pi(1_s|a))q_a. \tag{24}$$

The excess demand for cars of age a is then given by $ED_a(m, q) = D_a(m, q) - S_a(m, q)$.

3.7 Definition of Equilibrium

An rational expectations recursive equilibrium (REE) in this model is given by a price function $P(q, m)$, where $P_a(q, m)$ will denote the price of cars of age a, satisfying the following conditions:

1. Prices of new cars equal the supply price for new cars, $P_0(q,m) = \overline{P}$,
2. Prices of cars that must be scrapped equal the demand price for scrapped cars, $P_{\bar{a}}(q,m) = \underline{P}$,
3. Excess demand for all tradable used cars is zero, $ED(m,q) = 0$,
4. Consumers behave optimally so the Bellman equations defined in section 3.4 hold.
5. Consumers have rational expectations so they use $P(q,m)$ to solve their Bellman equations, and the true the law of motion of car holdings $q' = q\Omega(m,q)Q$.

A temporary recursive equilibrium (TE) is defined in the same way as REE except that in 4., the continuation value in the consumers' Bellman equation is replaced by the value in the stationary equilibria in which the macro state remains the same indefinitely. For example (12) is replaced by

$$\begin{aligned} v(\bar{a},\emptyset,m,q) &= u(\emptyset) + \mu(m)\underline{P} - T^{\text{psy}}(\bar{a},\emptyset) + \beta\overline{EV}(\emptyset,m), \\ v(\bar{a},d,m,q) &= u(d) - \mu(m)\big[P(d,m,q) - \underline{P} + T_b(d)\big] - T^{\text{psy}}(\bar{a},d) \\ &\quad + \beta\Big[\big(1-\alpha_d\big)\overline{EV}(d+1,m) + \alpha_d\overline{EV}(\bar{a},m)\Big], \end{aligned} \tag{25}$$

where $\overline{EV}(.,m)$ are the continuation values in the stationary equilibrium when the macro state is m.

Two useful results follow.

Proposition 1. *Independent of the equilibrium concept used, consumers starting the period with the terminal age car $a = \bar{a}$ and in the no-car state $a = \emptyset$ have identical choice probabilities for all available alternatives, provided that the psycological switching costs satisfy the following restriction:*

$$T^{psy}(\emptyset,d) = T^{psy}(\bar{a},d) - T^{psy}(\bar{a},\emptyset) \tag{26}$$

for all $d \in \{0,...,\bar{a}-1\} \cup \{\emptyset\}$.

Proof. Under the the restriction (26) consumers who start a period without a car and with a clunker ($a = \bar{a}$) have the same flow utility except for the monetary value from scrapping the car, $\mu(m)\underline{P}$ and the difference in the psychological transaction cost $\Delta T^{\text{psy}} = T^{\text{psy}}(\bar{a},d) - T^{\text{psy}}(\emptyset,d)$. Therefore, their value functions only differ by a constant which

varies with the exogenous aggregate state m but does not vary over the choice alternatives:

$$V(\bar{a}, \epsilon, m, q) = V(\emptyset, \epsilon, m, q) + L(m),$$

where $L(m)$ solves

$$L(m) = \mu(m)\underline{P} - \Delta T^{\text{psy}} + \beta \sum_{m'} L(m')G(m'|m).$$

The same result holds for choice specific value functions. This also implies that the transitional probabilities are the same for these consumers. Consequently, the dynamic equilibrium depends only on $\big(q_1, \ldots, q_{\bar{a}} + q_{\emptyset}\big)$ instead of the full vector $\big(q_1, \ldots, q_{\bar{a}}, q_{\emptyset}\big)$.

□

To evaluate and compare consumers' welfare across the the two equilibria, we use the proposition below which gives an analytical formula for valuation of the suboptimal policy functions in the REE equilibria under GEV assumption on the idiosyncratic shocks. Using this proportion, we can iterate over the consumers' Bellman equation using their equilibrium policy functions to calculate consumers' welfare without having to use costly numerical simulations.

Proposition 2. *The expected value function corresponding to an arbitrary decision rule (policy function) $\delta(a, m, q)$ expressed in a collection of choice probabilitis $\Pi_\delta(d|a)$ and $\Pi_\delta(s|d, a)$ can under REE equilibrium can be written as*

$$\begin{aligned} &EV_\delta(a, m, q) = \\ &= \textstyle\sum_{d,s \in C(\hat{a})} \Pi_\delta(d|\hat{a})\Pi_\delta(s|d, \hat{a})\Big[v(\hat{a}, d, s, m, q) + \sigma\mathbb{E}\{\epsilon_{d,s}|\delta, d, s, \hat{a}\}\Big] \\ &+ \textstyle\sum_{d \in C(\check{a})} \Pi_\delta(d|\check{a})\Big[v(\check{a}, d, m, q) + \sigma\mathbb{E}\{\epsilon_d|\delta, d, \check{a}\}\Big], \end{aligned} \tag{27}$$

where the first line corresponds to the ownership states $\hat{a}$ that have the nested scrappage decisions in the choiceset (2), and the second line corresponds to the ownership states $\check{a}$ that do not have the nested scrappage decisions, and the choice specific value functions $v(\cdot)$ are defined in (8), (12) and (14).

Moreover, the correction terms given by the conditional expectations of the idiosyn-

cratic taste shocks respectively for the nested and non-nested choices are given by

$$\begin{aligned}\mathbb{E}\{\epsilon_{d,s}|\delta,d,s,\hat{a}\} &= \gamma - \ln\big(\Pi_\sigma(d|\hat{a})\big) - \sigma_s \ln\big(\Pi_\sigma(s|d,\hat{a})\big),\\ \mathbb{E}\{\epsilon_d|\delta,d,\check{a}\} &= \gamma - \ln\big(\Pi_\sigma(d|\hat{a})\big),\end{aligned} \tag{28}$$

where $\gamma \approx 0.577$ *is Euler-Mascheroni constant and* σ_s *is the scale parameter corresponding to the nested scrappage decision.*

Proof. The proof is given in the Appendix A. □

4 Low dimensional numerical example

To illustrate the equilibrium concepts and equilibrium properties, we first consider a simple example in which $\bar{a} = 2$. The model parameters are given in Table 1. In particular, the marginal value of money, μ fluctuates between 0.10 in expansions ($m = e$) and 0.15 in recessions ($m = r$).

Parameters	**Values**
(u_0, u_1, u_n)	$(5, 1, 0)$
T	1.5
$(\overline{P}, \underline{P})$	$(50, 1)$
$\mu(m)$	$(0.10, 0.15)$
$G(e\|e), G(r\|r)$	$(0.75, 0.75)$

Table 1: Model parameters

From the equilibrium characterization in Proposition 1 (Condition (26) is automatically satisfied because the psychological transaction costs are set to zero), we just need to keep track of two endogenous state variables q_1 and $q_{\bar{a}} + q_\emptyset$. In addition

$$q_{\bar{a}} + q_\emptyset = 1 - q_1.$$

Therefore the only endogenous state variable we need to keep track of is q_1. In particular, all the policy functions (including prices and allocations) are functions of m and q_1

Figure 4 shows how the price of age-1 cars depends on the endogenous state variable and m. The red lines correspond to expansions ($\mu = 0.1$)and the blue lines correspond to recessions ($\mu = 0.15$). The solid lines are for rational expectations equilibrium and the dotted lines are for temporary equilibrium. The dashed lines are the price function when

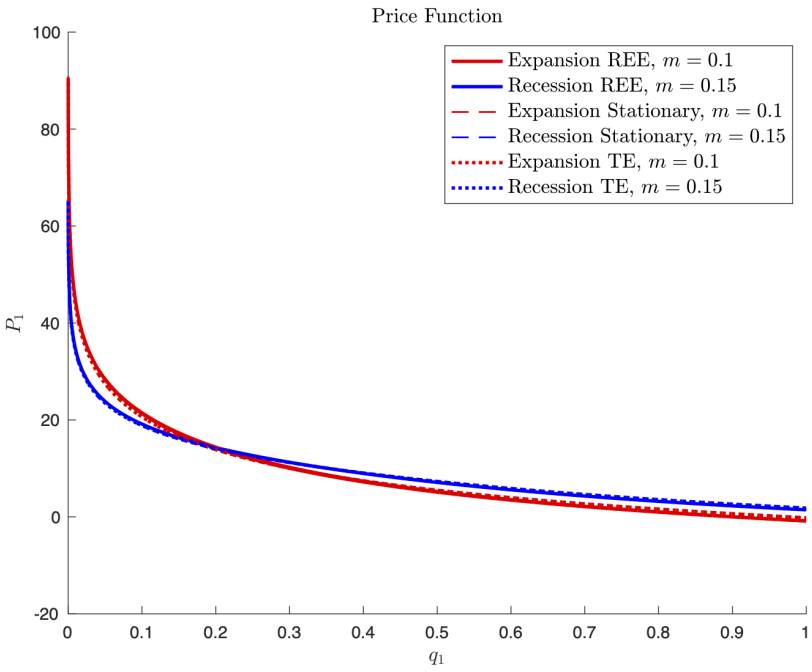


Figure 4: Price Function

there is no aggregate shock ($\mu \equiv 0.1$ or 0.15)

Notice that in all equilibria, when q_1 goes to zero, P_1 goes to ∞, exceeding the price of new car $\bar{P} = 50$. This is because the supply of age-1 cars given by (24) goes to zero while the demand given by (23) can only goes to zero if P_1 goes to ∞. The price functions in different equilibria are close to each other.

Figure 5 shows how the transition for the endogenous state variable q_1 depending on m. The intersection between the transition function for each value of the aggregate shock m and the 45-degree line corresponds to the long run value of the state variable q_1 if aggregate shock realizations are all equal to that value. The transition function in the temporary equilibrium and stationary equilibrium intersects on the 45-degree line for each aggregate state m. In the temporary equilibrium, the consumers assume that future periods values and prices are fixed at their stationary level when future aggregate states remain at the current aggregate state. This is an accurate forecast starting from the stationary distribution of car holding. Hence, at the stationary level of q_1, future values of q_1 remain at the stationary level.

When the aggregate state evolves over time according to the transition matrix $G(m'|m)$, q_1 also evolves according to the transition functions, depending on equilibria. In the sta-

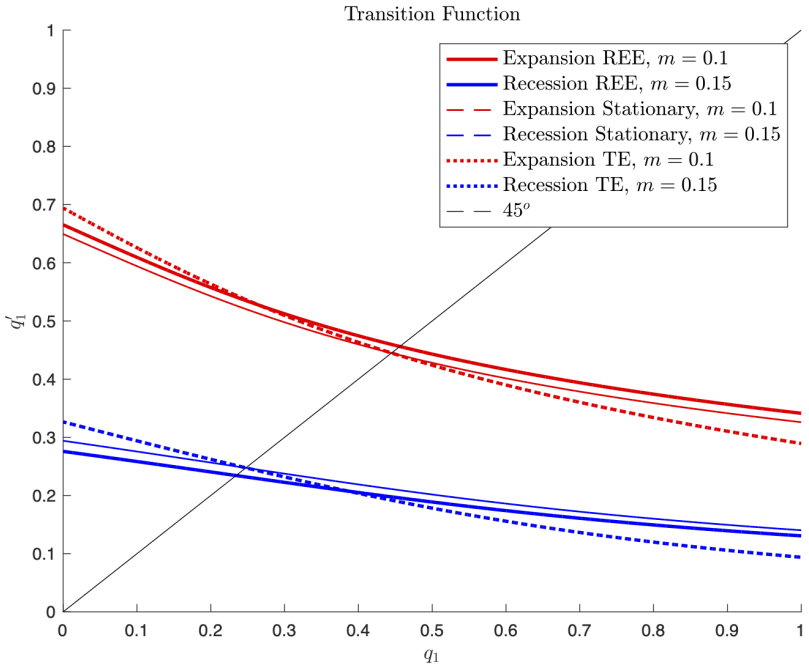


Figure 5: Transition Function

tionary equilibria, $m' \equiv m$, q_1 converges deterministically to the stationary level, i.e., the intersection of the transition function and the 45-degree line. With aggregate shocks, q_1 evolves stochastically as m changes over time, leading to a long run ergodic distribution. Figure 6 shows the long-run ergodic distribution of q_1 for the rational expectation equilibrium. The modes of the distributions are around the intersection of the transition functions with the 45-degree lines, for expansions and recessions respectively.

Figure 4 and Figure 5 show that the aggregate prices and quantities in rational expectations and temporary equilibria are very close to each other. In temporary equilibrium, consumers form simple (naive) expectations of future dynamics, including prices, based on permanent aggregate states and stationary equilibria. The more sophisticated expectations are complex, and hence costly to learn as they must take into account all others' decisions. However, consumers might not be willing to learn if the potential utility gain from doing so is small. To explore this issue, Figure 7 and Figure 8 plot the utility gains (dashed lines) from having sophisticated expectations in the temporary equilibrium, for the stationary economy and the stochastic economy respectively. The utility gains can be calculated using the choice probabilities, as described in Proposition 2, and are expressed as fractions of the price of new cars. Figure 7 is for the stationary economy.

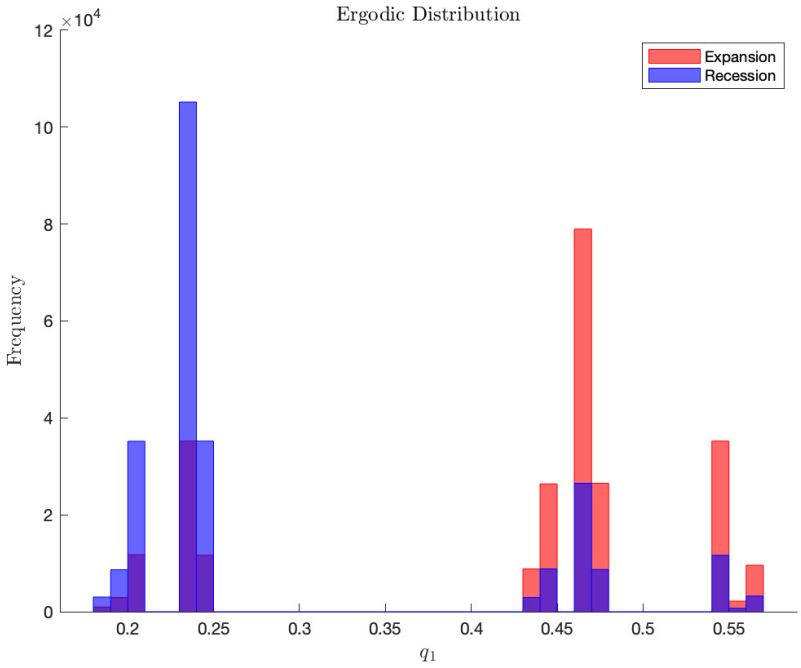


Figure 6: Ergodic Set

In this case, utility gain is exactly zero at the stationary value of q_1 because naive and sophisticated expectations are exactly the same. Outside of the stationary value, utility gain is less than 0.2% of the new car price. Therefore consumers have little incentive to acquire sophisticated expectations. Figure 8 is for the stochastic economy. The utility gain is larger but is still less than 1% of the new car price. These figures suggest that the temporary equilibrium is robust in the sense that consumers do not have significant incentive to adopt more sophisticated expectations, especially if that is costly.

On the flip side, Figure 7 and Figure 8 also plot the utility loss from not adopting sophisticated expectations, opting for the naive temporary equilibrium expectations instead. The utility losses are also small. This result suggests that the rational expectation equilibrium are not robust because agents do not loose much by not learning the complex equilibrium dynamics.

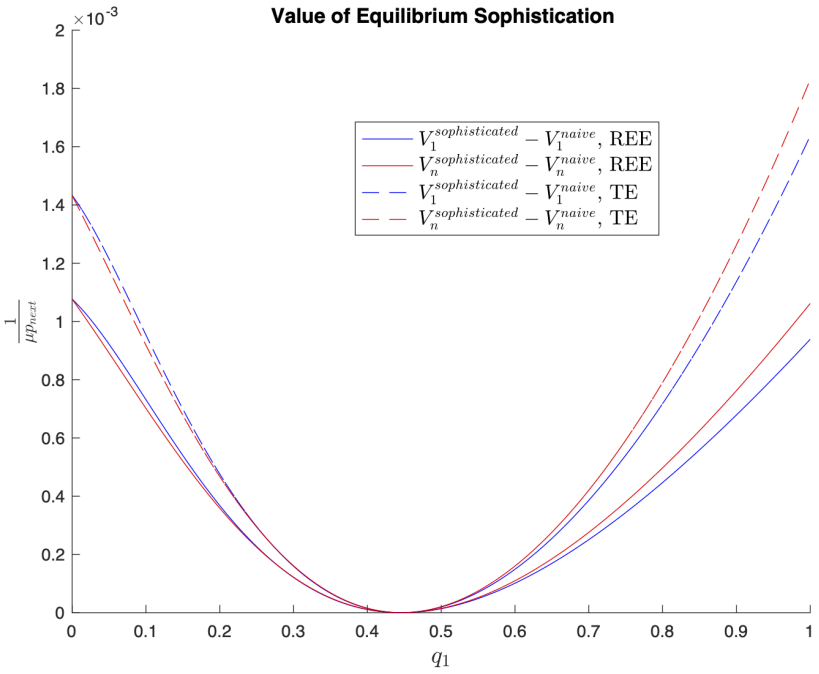


Figure 7: Value differences in the deterministic economy

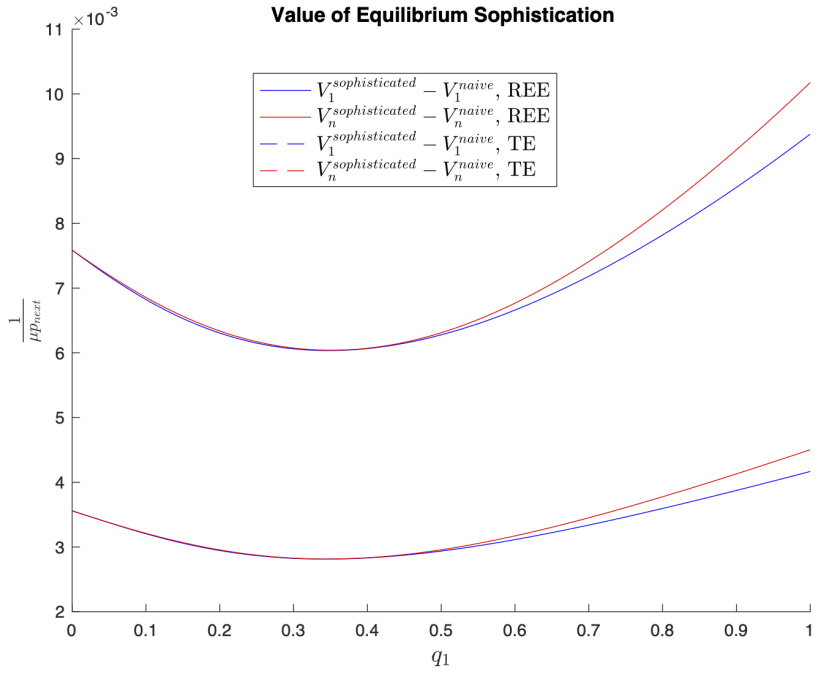


Figure 8: Value differences in the stochastic economy

5 A calibrated model

In this section, we consider a calibrated version of the model with 10 car ages, $\bar{a} = 10$. We assume that the flow utility has the following parametric form in a car's age a

$$u_a = u_0 + u_1 a + u_2 a^2$$

where the coefficients are estimated using the stationary model and Danish data. The calibrated parameters are given in Table 2. We allow μ varies to match the variation in new car purchases over the business cycles and the transition matrix is consistent with the frequency and duration of business cycles in Denmark.

Parameters	**Values**
(u_0, u_1, u_2)	$(2.8857, -0.2427, 0.0047)$
u_n	0
$T^s(P)$	0.0465P
switching cost	5.0830
switching cost n	5.0830 + 2.1604
$(\overline{P}, \underline{P})$	$(100, 10)$
$\mu(m)$	$(0.075, 0.085)$
$G(e\|e), G(r\|r)$	$(0.75, 0.75)$

Table 2: Model parameters

Again the equilibrium characterization in Proposition 1, we just need to keep track of nine endogenous state variables $\{q_1,\ q_2,\ \ldots,\ q_9\}$. Because of the large number of state variables, we use the Adaptive Sparse Grid from Brumm and Scheidegger (2017) implemented in GDSGE to solve the model. Figure 9 shows that the algorithm use more points to approximates the shares of older cars.

As the economy expands, consumers demand more new cars and these new cars age over time. The price of these cars also go down because of more supply.

Figure 10 presents the dynamics of quantities for a particular sequence of realizations of macro shocks in both rational expectation equilibrium and temporary equilibrium. The circles on top and in the bottom represent the values of the macro state (1 for expansions and 2 for recessions). When the economy switches from a recession to an expansion, new car purchases (magenta lines) increases and the fraction of users with no cars (green lines) decreases. These new cars age over time leading to a wave of increases in successively older cars ownership through time. When the economy switches back

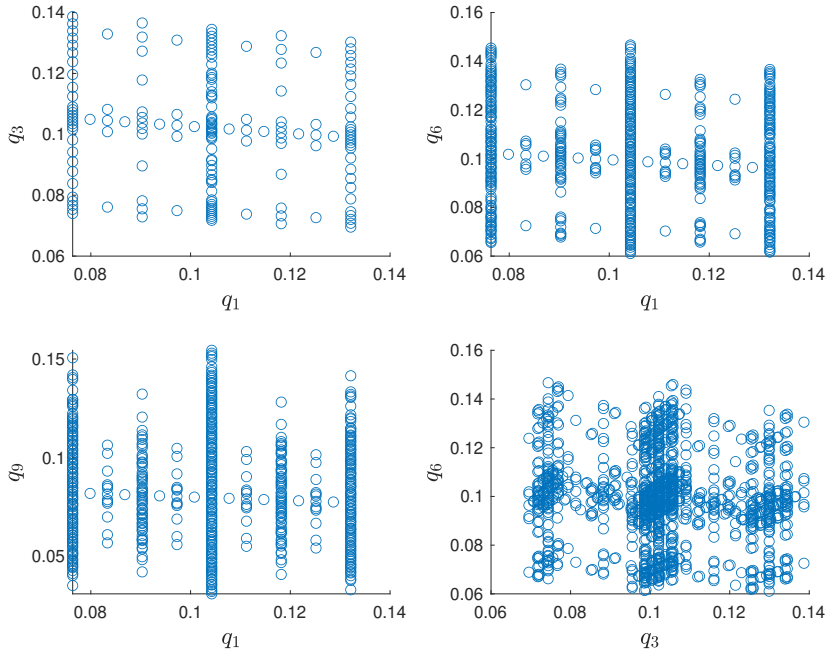


Figure 9: Adaptive Sparse Grid

to a recession, the fraction of consumers without a car increases and new car purchases decrease. Figure 11 presents the dynamics of prices over time. Prices fluctuate much less than quantities do but used car prices tend to increase slightly in recessions. Lastly, the rational expectations equilibrium features more variations in quantities and prices than the temporary equilibrium does.

Another way to examine the effects of business cycle shocks is to calculate the impulse responses by averaging the change in variables of interest across all possible future realizations after a particular shock (expansion or recession). Figure 12 and Figure 13 show the impulse responses of quantities and prices after an expansion. Figure 14 and Figure 15 show the impulse responses of quantities and prices after an expansion. Expansion and recession shocks have symmetric effects on prices and quantities. The propagation of the shocks over time is similar to the description above for a particular sequence of realizations of macro shocks.

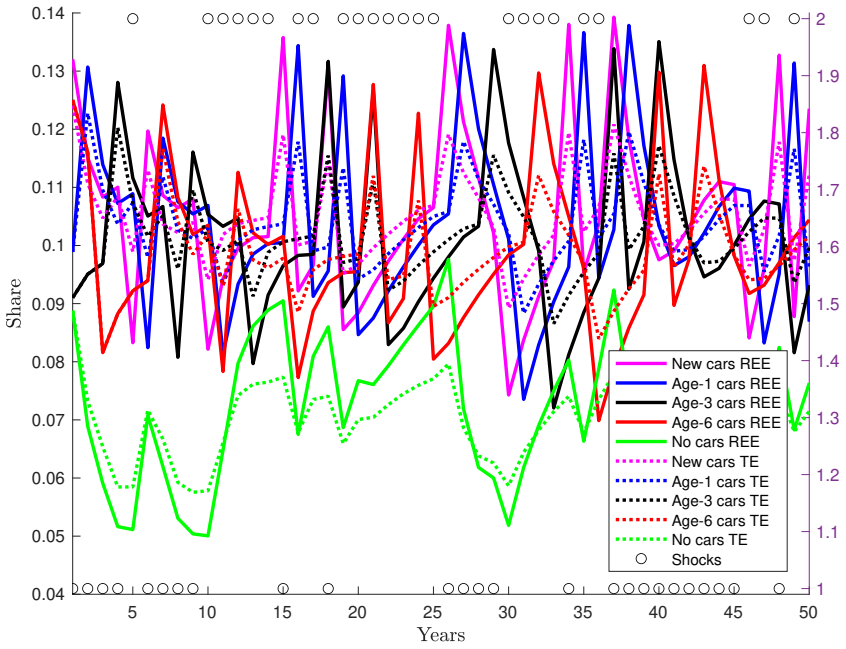


Figure 10: Quantities over business cycles

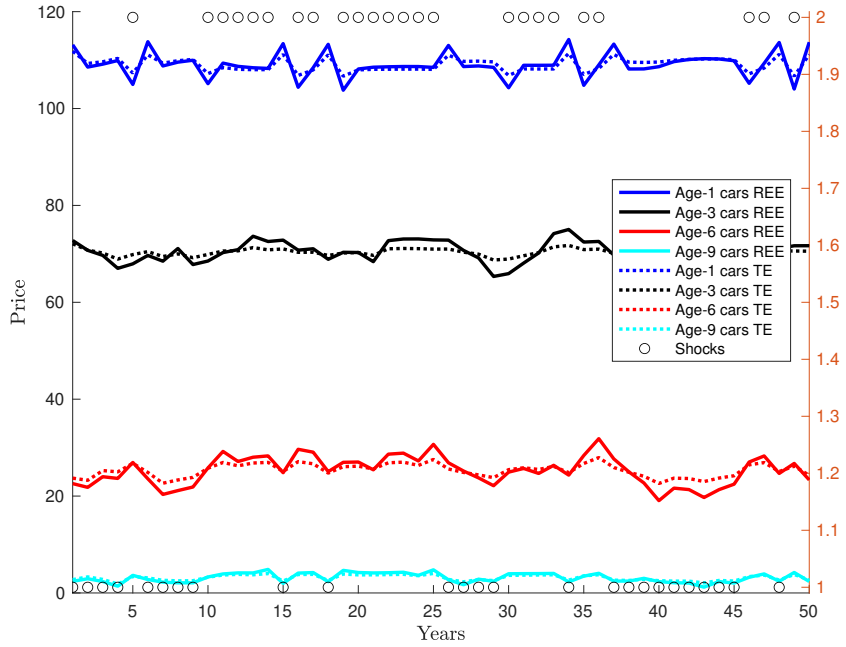


Figure 11: Prices over business cycles

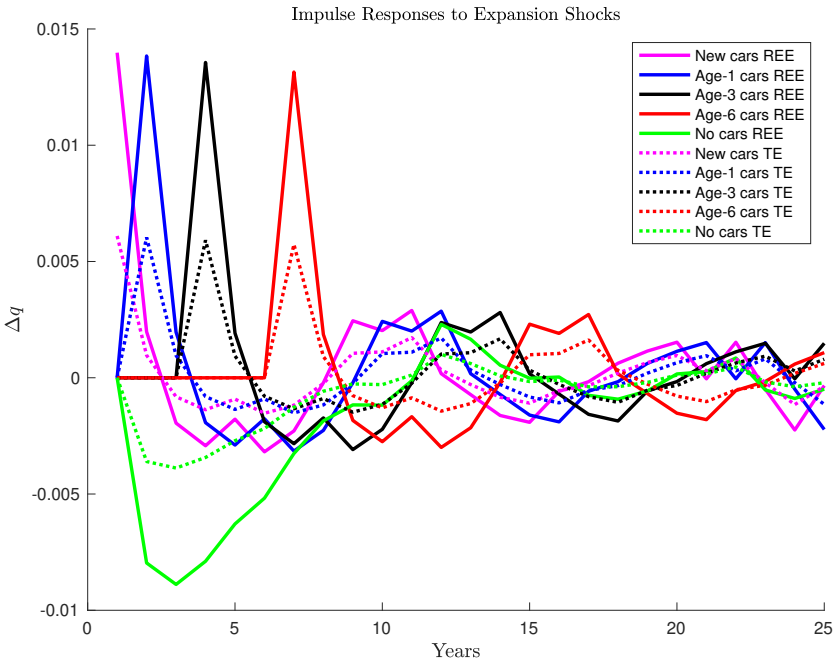


Figure 12: Quantity impulse responses

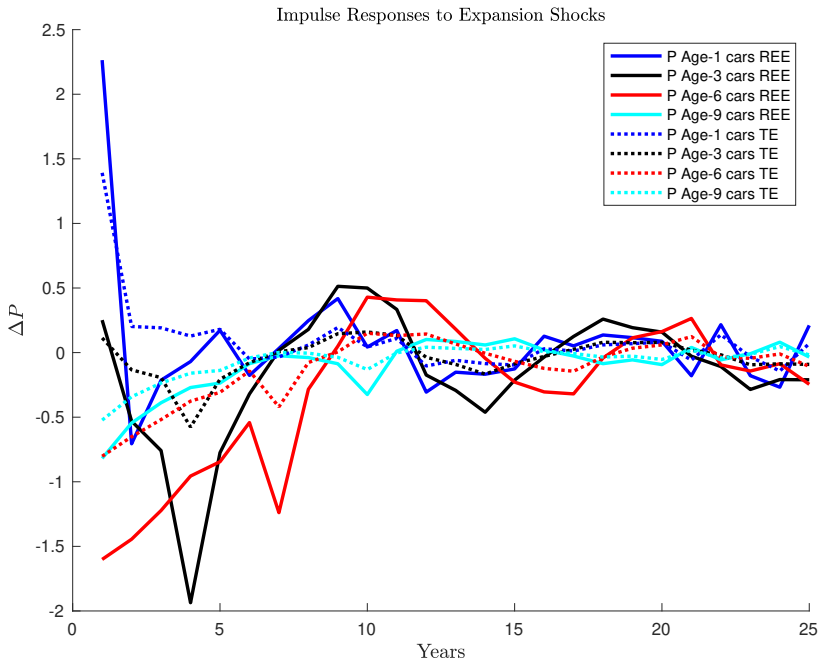


Figure 13: Price impulse responses

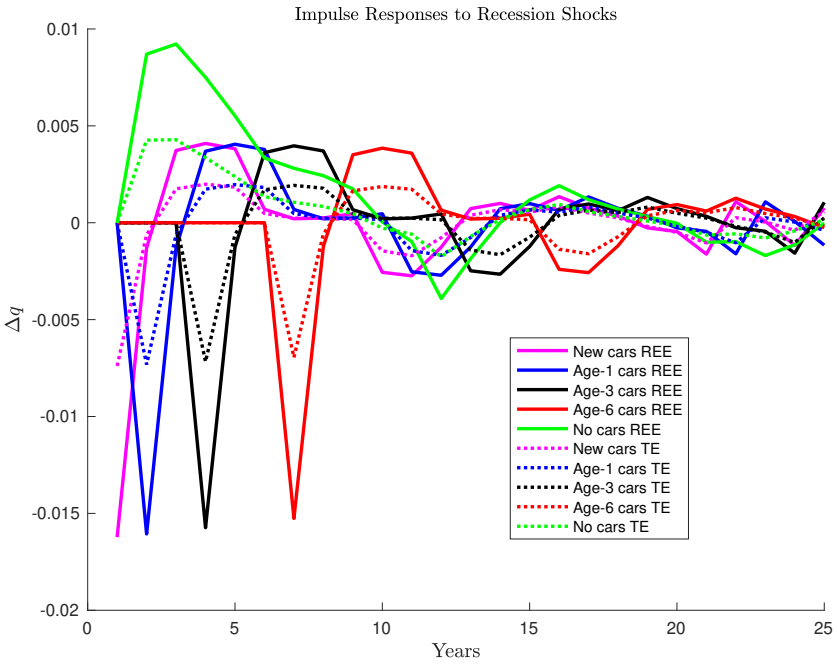


Figure 14: Quantity impulse responses

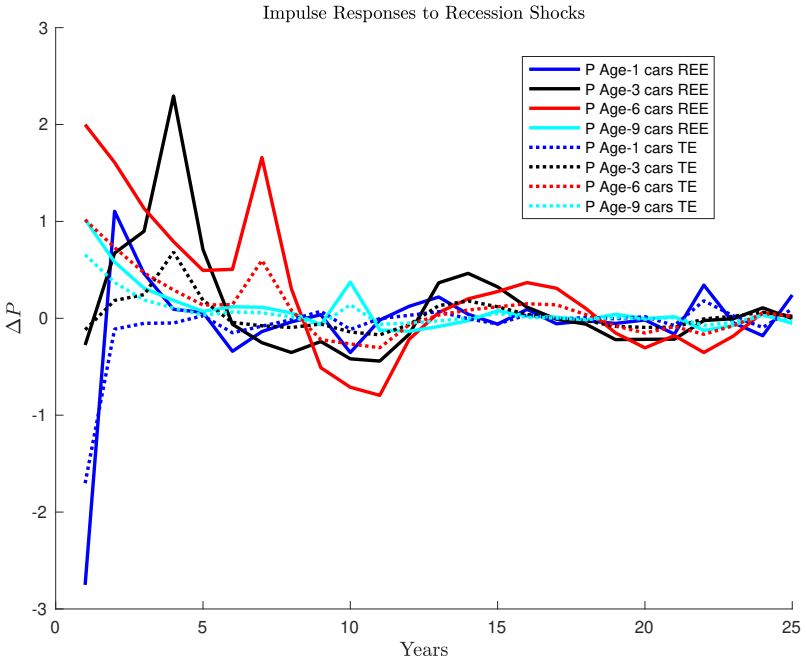


Figure 15: Price impulse responses

6 Conclusion

In this paper, we document stylized facts about the waves of used cars of different vintages over the business cycles in Denmark. We show that extending the stationary model of Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning (2022) to include aggregate shocks to marginal utilities can account for these patterns. The extended model allows us to examine the importance of consumers' expectations about future prices, which depend on the entire distribution of car holdings in the economy. Our findings indicate that consumers experience relatively low welfare losses even when following simple temporary equilibrium expectations. This result supports the use of low-computing-cost temporary equilibria in applications to durable goods markets.

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Appendix

A GEV correction terms

We are interested in computing the value EV_δ of an arbitrary policy δ given by a collection of choice probabilities $p(x)$ for all x. Below we rely on the representation of the Bellman equation given in Aguirregabiria and Mira (2002); Hotz and Miller (1993) to derive the *correction terms* that specify the discrepancy between the expected value of the optimal policy $\delta^\star$ and that of an arbitrary policy δ. Given these correction terms, it is possible to express the expected value as an infinite stream of corrected instantaneous payoffs, or as a solution to a system of linear equations.

For the purpose of this section denote $x = (a, m, q)$. With the nested scrappage choice, the structure of the choice set is given by (with slight abuse of notation):

$$C(x) = \begin{cases} \{0, 1, ..., \bar{a} - 1, \emptyset\} & \text{if } a \in \{\emptyset, \bar{a}\} \\ \Big\{(0, 0_s), (0, 1_s), \ldots, (\bar{a} - 1, 0_s), (\bar{a} - 1, 1_s), (\emptyset, 0_s), (\emptyset, 1_s), \kappa\Big\} & \text{if } a < \bar{a}. \end{cases} \tag{29}$$

Denote $x_\ominus = \big\{(a, m, q) : a \in \{\emptyset, \bar{a}\}\big\}$ the states in which available choices are not nested, and $x_\otimes = \big\{(a, m, q) : a < \bar{a}\big\}$ the states with nested scrappage choice (except choice to keep, κ).

Under the standard Rust (1987) assumptions of additive separability and conditional independence of the idiosyncratic the integrated (ex-ante) value function $EV(x)$ (the value function integrated over the distribution of the unobserved shocks ϵ) can be expressed as

$$EV(x) = \int_\epsilon \Big[\max_{d \in C(x)} \big\{ v\big(x, d, EV(x')\big) + \epsilon_d \big\} \Big] g(\epsilon|x) d\epsilon, \tag{30}$$

where $v\big(x, d, EV(x')\big)$ is defined in equations (8), (12), (14), and depends on $EV(x')$, incorporating the state transition from x to x' through:

- aging of the chosen car d (with stochastic accidents), unless no car state is chosen,
- evolution of the macro state from m to m' according to the exogenous Markov transition kernel G,
- deterministic evolution of the ownership distribution from q to q'.

We assumed that ϵ has a GEV distribution corresponding to the structure of choices $C(x)$, allowing for the integral in (30) to be expressed in closed form (9) (for $x_\ominus$ states) and (10) (for $x_\otimes$ states). The corresponding choice probabilities characterizing optimal policy $\delta^\star$ also have closed forms described in previous section.

Consider now an arbitrary policy δ given by the choice probabilities $p(x)$ for all x. Assume that these choice probabilities satisfy the feasibility conditions $p_j(x) > 0 \iff j \in C(x)$ for all x, and have the same structure as the choice probabilities in $\delta^\star$, except that arbitrary choice specific values $\tilde{v}(x, d)$ are used in place of true $v(x, d)$. For the states $x_\ominus$ where the scrappage choice is not nested, the choice probabilities are given by

$$\begin{aligned} p_j(x_\ominus) &= \textstyle\int_\epsilon \mathbf{1}\big\{\tilde{v}(x_\ominus, j) + \sigma\epsilon_j > \tilde{v}(x_\ominus, k) + \sigma\epsilon_k, \forall k \in C(x_\ominus)\backslash\{j\}\big\} g(\epsilon|x) d\epsilon \\ &= Pr\big\{\tilde{v}(x_\ominus, j) + \sigma\epsilon_j > \tilde{v}(x_\ominus, k) + \sigma\epsilon_k, \forall k \in C(x_\ominus)\backslash\{j\}\big\} \\ &= \exp\big(\tilde{v}(x_\ominus, j)/\sigma\big) \Big/ \textstyle\sum_{k \in C(x_\ominus)} \exp\big(\tilde{v}(x_\ominus, k)/\sigma\big). \end{aligned} \tag{31}$$

In states $x_{\otimes}$ where the choices are nested, the expressions for the choice probabilities are more involved, but follow the same nested structure as in the previous section with $\tilde{v}(x_{\otimes}, j)$ used in place of $v(x_{\otimes}, j)$.

Following Aguirregabiria and Mira (2002), we can express the integrated value from using policy δ as

$$\begin{array}{rcl} EV_{\delta}(x) & = & \sum_{j \in C(x)} p_j(x) \int_{\epsilon} \Big[v\big(x, j, EV_{\delta}(x')\big) + \sigma \epsilon_j \Big] g(\epsilon|x) d\epsilon \\ & = & \sum_{j \in C(x)} p_j(x) \Big[v\big(x, j, EV_{\delta}(x')\big) + \sigma \mathbb{E}\{\epsilon_j | j, p(x)\} \Big], \end{array} \tag{32}$$

where the correction term in the second line is a conditional expectation of the idiosyncratic shock ϵ_j given that choice j is made, thus imposing the conditions given in the calculation of the choice probability in (31) (in the states $x_{\ominus}$). Arcidiacono and Miller (2011, Lemma 1, p. 1834) prove that these correction terms exist for any $p(x)$ and only depend on this vector of the implied choice probabilities. In the non-nested case we have

$$\begin{array}{l} \mathbb{E}\{\epsilon_j | j, p(x_{\ominus})\} = \\ = \mathbb{E}\big\{ \epsilon_j | \tilde{v}(x_{\ominus}, j) + \epsilon_j > \tilde{v}(x_{\ominus}, k) + \epsilon_k, \forall k \in C(x_{\ominus}) \backslash \{j\} \big\} \\ = \frac{1}{p_j(x_{\ominus})} \int_{\epsilon} \epsilon_j \mathbf{1}\big\{ \tilde{v}(x_{\ominus}, j) + \epsilon_j > \tilde{v}(x_{\ominus}, k) + \epsilon_k, \forall k \in C(x_{\ominus}) \backslash \{j\} \big\} g(\epsilon|x) d\epsilon \end{array} \tag{33}$$

The last line is a multidimensional integral that simplifies to a one-dimensional integral over ϵ_j multiplied by one-dimensional EV1 distribution density function and multidimensional EV1 cumulative distribution function evaluated at the differences in choice specific values $\tilde{v}(x_{\ominus}, j) - \tilde{v}(x_{\ominus}, k), \forall k \in C(x_{\ominus}) \backslash \{j\}$. The general formula for this integral is given in Hotz and Miller (1993, equation 3.7, p. 501).

In application to a simple (non-nested) multinomial logit with scale parameter $\sigma = 1$ the integral takes the form (see Hotz and Miller (1993, equation 4.12, p. 504) or Aguirregabiria and Mira (2002, p. 1525))

$$\mathbb{E}\{\epsilon_j | j, p(x_{\ominus})\} = \gamma - \ln(p_j(x_{\ominus})), \tag{34}$$

where $\gamma \approx 0.57721$ is Euler-Mascheroni constant.

One subtlety that we have here is the fact that our model is written down for any scale parameter σ (which we however fix to 1 in estimation). However, according to the properties of the EV1 (Gumbel) distribution (Ben-Akiva and Lerman, 1985, p. 105), an EV1 random variable $\epsilon \in \mathbb{R}$ has the following cumulative distribution and probability density functions

$$\begin{array}{rcl} G(\epsilon) & = & \exp\Big(- \exp\big(- \mu(x - \eta)\big)\Big), \\ g(\epsilon) & = & \mu \exp\big(- \mu(x - \eta)\big) \exp\Big(- \exp\big(- \mu(x - \eta)\big)\Big), \end{array} \tag{35}$$

where η is the location parameter and μ is the scale parameter. The expectation and variance of an EV1(η, μ) random variable $\epsilon \in \mathbb{R}$ are given by $\mathbb{E}(\epsilon) = \eta + \gamma/\mu$ and $\mathbb{V}(\epsilon) = \pi^2/6\mu^2$. If $\epsilon \in \mathbb{R}$ has an EV1(η, μ) distribution, then $\sigma\epsilon + \rho$ has an EV1 distribution with location parameter $\sigma\eta + \rho$ and scale parameter $\frac{\mu}{\sigma}$. In the case when $\rho = 0$ the expected value is $\mathbb{E}(\sigma\epsilon) = \sigma\eta + \gamma/(\mu/\sigma) = \sigma\big(\eta + \gamma/\mu\big)$. In other words, when the EV1 shock is scaled with σ, the (unconditional) expectation is simply multiplied by the same factor. It is therefore natural that the application of (34) in (33) where the ϵ shocks are scaled

by σ results in a similar scaling of conditional expectation that is the correction term.

Now we turn to the nested choices in the states $x_\otimes$. The distribution of the idiosyncratic shocks ϵ in this case is given by the generalized extreme value (GEV) distribution (McFadden, 1978; Arcidiacono and Miller, 2011) with the cumulative probability function given by

$$G(\epsilon_1, \ldots, \epsilon_K) = \exp\Big(- \mathcal{H}\big(\exp(-\epsilon_1), \ldots, \exp(-\epsilon_K)\big)\Big), \tag{36}$$

where the multivariate function $\mathcal{H} : \mathbb{R}^K \to \mathbb{R}_+$ satisfies the properties posted by McFadden (1978). Note that this formula implicitly assumes that the vector of location parameters is zero, and the scale parameter is one.

Simple non-nested multinomial logit is a special case with $\mathcal{H}(x_1, \ldots, x_K) = \sum_{k=1}^K x_k$, and the nested logit is obtained with

$$\mathcal{H}(x_1, \ldots, x_K) = \sum_{m=1}^M \left(\sum_{j \in \mathcal{C}_m} x_j^{\frac{1}{\lambda_m}} \right)^{\lambda_m}, \tag{37}$$

where the choices are partitioned into mutually exclusive nests $\mathcal{C}_m$, $m \in \{1, \ldots, M\}$, $\mathcal{C}_m \subset \{1, \ldots, K\}$, and $\lambda_m \in [0, 1]$ are within-nest scale parameters. Clearly, the nested choice logit collapses to the simple multinomial logit when $\lambda_m = 1$ for all m. To represent a utility maximization problem the positive scale parameters in a nested logit model have to monotonically decline with the nesting level (McFadden, 1978). Therefore, assuming the top level scale parameter to be normalized to 1, values of λ_m are restricted to the unit interval.

In calculation of the correction factor for the nested choices in out model we rely on the result in Arcidiacono and Miller (2011, Lemma 3, p. 1841) stating that if the distribution $G(\epsilon)$ of the idiosyncratic shocks $\epsilon \in \mathbb{R}^K$ can be factored as

$$G(\epsilon) = G_0(\epsilon_{-\mathcal{J}}) \exp\left[- \left(\sum_{j \in \mathcal{J}} \exp(-\epsilon_j / \lambda) \right)^\lambda \right], \tag{38}$$

where a subset of choices $\mathcal{J}$ belongs to a nest, and subvector $\epsilon_{-\mathcal{J}}$ corresponding to other choices has a GEV distribution G_0, then the correction terms for the choices in the nest $j \in \mathcal{J}$ are given by

$$\mathbb{E}\{\epsilon_j | j, p\} = \gamma - \lambda \ln(p_j) - (1 - \lambda) \ln\Big(\sum_{j \in \mathcal{J}} p_j \Big), \tag{39}$$

Applying this result in our model sequentially for all nested choices in the states $x_\otimes$ we can express the correction terms for all nested choices in the choice set $C(x_\otimes)$, except the keep choice which is not nested. However, the previous formula (34) can be applied in this case, or, alternatively we can think of the keep choice to be in its own nest. In the later case, applying the formula (40) for any value of λ leads to the same result.

With this, we have the formula for all choices in the choice set $C(x_\otimes)$

$$\mathbb{E}\{\epsilon_j | j, p(x_\otimes)\} = \gamma - \sigma_s \ln(p_j(x_\otimes)) - (1 - \sigma_s) \ln\Big(\sum_{i \in \{\text{scrap,sell}\}} p_i(x_\otimes) \Big), \tag{40}$$

$$\mathbb{E}\{\epsilon_\kappa|\kappa, p(x_\otimes)\} = \gamma - \ln(p_\kappa(x_\otimes)). \tag{41}$$

Note that the choice probabilities $p_j(x_\otimes)$ here are thought of as combined probabilities of a trading choice and the corresponding scrappage choice.

Now, with all the correction terms defined in (34), (40), (41), we can return to the representation of the Bellman equation in (32). We can expand it further to obtained

$$\begin{array}{l} EV_\delta(x) = \\ = \sum_{j\in C(x)} p_j(x)\Big[v\big(x, j, EV_\delta(x')\big) + \sigma\mathbb{E}\{\epsilon_j|j, p(x)\}\Big] \\ = \sum_{j\in C(x)} p_j(x)\Big[u(x, j) + \beta\sum_{x'} EV_\delta(x')\big)f(x'|x, j) + \sigma\mathbb{E}\{\epsilon_j|j, p(x)\}\Big], \end{array} \tag{42}$$

where $u(x, j)$ contains all details of the immediate payoffs and $f(x'|x, j)$ represents the overall transition probability kernel for the state process. Taking into account that the state space is discretized, stacking the state space to form vectors from functions of x, and expressing the choice probability weighted transition probability matrix for the state as

$$F_\delta = \sum_j p_j(x)f(x'|x, j) \tag{43}$$

we can write the same equation "in matrix form" as

$$EV_\delta = \textstyle\sum_j p_j * \big[u(j) + \sigma\mathbb{E}\{\epsilon_j|j, p\}\big] + \beta F_\delta EV_\delta, \tag{44}$$

where $*$ denotes the Hadamard element-by-element product. Inverting the linear system leads to the final solution

$$EV_\delta = \Big[I - \beta F_\delta\Big]^{-1} \textstyle\sum_j p_j * \big[u(j) + \sigma\mathbb{E}\{\epsilon_j|j, p\}\big], \tag{45}$$

which gives the vector of values of the arbitrary decision rule δ in all points x of the state space.