Government Intervention in the Financial Market

Jiang Wang*

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Abstract

Government intervention in the financial market through its own trading fundamentally changes the market's structure, function, behavior and outcome. We develop a general equilibrium framework to study the impact of government trading on market outcome and investor welfare. We show that with incompleteness and asymmetric information, the market equilibrium is in general sub-optimal and government intervention can improve investor welfare even without any additional information. However, the welfare impact of government intervention is sensitive to its policy design and the economy's structural details. In addition, popular policy goals such as informational efficiency, price stability and market liquidity can have different welfare implications. Performance measures for government trades based on market prices can be misleading since these trades also affect prices and welfare.

JEL Codes: G11, G12, G18, D53, D61, H11

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*MIT Sloan School of Management and NBER (mail: E62-614, 100 Main Street, Cambridge, MA 02142, U.S.A.; tel: +1 617-253-2632; e-mail: wangjmit.edu). I thank Zhe Geng for excellent research assistance. Comments from seminar and conference participants at Georgia State University, INSEAD, People's Bank of China, Shanghai Advanced Institute of Finance, and Tsinghua University, are greatly appreciated. This draft is a substantial revision from earlier drafts dated February 12, 2025 and before.

1 Introduction

In addition to its conventional regulatory role, government also intervenes in the financial market as a direct participant.[1] The Federal Reserve's massive holdings of mortgage-backed securities, the European Central Bank's large positions in corporate bonds, and the Bank of Japan's extensive stock ETF purchase program are prominent examples of such interventions.[2] Government intervention through its own trading fundamentally changes the market's structure, function, behavior and outcome. Yet, in our conventional analysis of the financial market and asset pricing, the role of the government as a major market player has been mostly left out.

In this paper, we develop a parsimonious general equilibrium framework which includes the government as a unique market participant and examine the role it can potentially play. In addition to its market power and potential informational advantage, the government has two distinct features. First, it has the power of transfers and taxes, allowing it to directly influence the final allocation in the economy. Second, it may have objectives different in nature from those of other market participants. We show that with the most basic imperfections of the market such as incompleteness and information asymmetry, even without any additional information, the government can still improve investor welfare by trading in the market. When the government has additional information about the economy, its impact on market equilibrium and investor welfare are sensitive to the structural details of the economy as well as the design of the intervention policy. We further demonstrate that seemingly desirable policy goals such as informational efficiency, price stability and market liquidity can carry substantial welfare costs. Measures to assess the performance of government interventions relying on simple market gauges such as prices are incomplete and sometimes misleading.

We consider an economy in which investors face heterogeneous income risk, which they also have private information on. They can trade in the financial market to share/hedge their income risk and to speculate on their information. In addition, we assume that the government can directly trade in the market. Its gain/loss from trading is fully passed on to the investors as

[1]The market considered here is limited to non-government/private securities such as stocks, corporate bonds and asset-backed securities. For government securities like government bonds, the government is naturally an essential participant, hence will not be the aim of this paper. We also treat different government branches, including the central bank, as its integral parts.

[2]By the end of 2023, the Fed's MBS holdings exceeded 2.4 trillion dollars, over 22% of the overall market, ECB's corporate bond holdings were over 0.3 trillion euros, over 8% of the market, and BOJ's stock holdings reached 35.7 trillion yen, over 4% in market share.

required by its budget constraint, and its objective is to maximize investor welfare. Taking the government's trading policy as given, investors choose their optimal trading strategies, and the market equilibrium is achieved through market clearing. Given that the market is incomplete with respect to the investors' idiosyncratic risk, the equilibrium allocation is in general not Pareto optimal, which leaves room for government intervention. The government then chooses its intervention policy to improve investor welfare.

In our setting, investors' uninsured idiosyncratic risk gives rise to market incompleteness, the key market imperfection we allow in this paper. If this risk is correlated with the payoff of traded assets, say, the stock, the investors can then trade the stock to share/hedge their idiosyncratic risk in the market. In the benchmark case when there is no government intervention, we show that the market equilibrium fully reveals the underlying state of the economy, the aggregate income risk exposure, and allows the investors to achieve full risk sharing. However, despite its informational and allocational efficiency, such an equilibrium is in general not optimal. In particular, the investors' welfare level achieved through the market falls below its level in the first-best allocation, in which investors simply bear their fair share of the aggregate risk without going through the trading process.

The reason behind such an outcome is as follows. When investors trade in the market to share/hedge their idiosyncratic risk, it is not cost free. They have to pay a market risk premium, which depends on the aggregate risk. Given that the investors' idiosyncratic risk has a zero mean, the premium they pay for sharing it is zero *on average*. But, being risk averse, the risk of paying this premium lowers the investors' ex ante welfare. This inherent inefficiency in the market equilibrium then calls for government intervention. In particular, if the government can reduce the cost of risk sharing in the market though its own trading, it can potentially benefit the investors.

Government's intervention policy in general depends on the information it has. We consider three situations. The first is when the government only has public information such as market prices. The second situation is when it has full information on the underlying state of the economy, which we also refer to as *fundamental information*. The third situation is when the government has only partial information. In all three cases, we limit the government's information not to exceed the collective information of all investors.[3]

[3]Otherwise, the government can use its superior information to achieve outcomes that are not feasible to the

When the government has only public information, it can adopt a trading policy dependent on that information, such as the stock price. We show that in a rational expectations equilibrium, the government's trade will be known to the investors. As a result, they can adjust their own trades to fully offset the impact of the government's trade on their actual positions. Consequently, the government's trading, despite its market power, will have no impact on the market equilibrium and investor welfare. We believe that this "no-impact theorem" holds in general as long as the government's trade is known to the investors, who also bear the resulting gain/loss. This result is reminiscent of the notion of Ricardian equivalence in the current context.

This, however, does not mean that the government cannot do better, given that the market equilibrium only yields a sub-optimal outcome. In fact, we show that the government can improve investor welfare by acting as a noise trader. In particular, it can trade in the market on pure noise, unobservable to the investors. By doing so, the government creates an informational advantage for itself. The noise from its trading becomes a new and endogenous state variable that investors do not observe. In addition, this noise reduces the informativeness of the price about the fundamental state of the economy. The superior information in these two dimensions allows the government to device trading strategies that can shift the equilibrium outcome and investor welfare. In particular, its noise trading provides extra returns to the investors on their trades, lowering the risk premium they pay in the market. Such a price effect, within certain range, can actually improve investor welfare as it reduces the cost for their risk sharing.

We further observe that when the government trades on noise, it brings noise into the price. A noisy price reduces the market's informational efficiency, a seemingly undesirable outcome. Yet, it can improve investor welfare. Moreover, as the government's noise trading reduces the market risk premium, it tends to reduce price volatility and improve market liquidity. At the same time, it distorts investors' risk allocation and incurs welfare cost. These results imply that market properties often viewed as desirable in academic and policy analysis, such as price informativeness, price stability and market liquidity may well have different welfare implications.

When the government has full information on the underlying state of the economy, its intervention policy becomes dependent on the underlying state variable. Despite the informational advantage the government has, we find that such an intervention policy is in general ineffective. This is because, absent of any noise, trading on the precise fundamental information allows

investors on their own. More discussion on this matter is provided later in the paper.

the equilibrium price to fully reveal it. This also reveals the government's trade under rational expectations. As pointed out above, any of the government's trading that is known to the investors will have no impact on market outcome as it will be fully offset by investors' own trades. Given that in the benchmark case with no government intervention, the equilibrium fully reveals the underlying state of the economy, there is no additional benefit to bring the same information into the market.

When the government has partial information on the underlying state of the economy, the situation is like a combination of the previous two cases, trading on full fundamental information and trading on noise, respectively. When the government intervenes, its trade brings both fundamental information as well as noise into the market. The overall result then depends on how the two components impact the market individually, how they interact and which one ultimately dominates. As shown by the case of full information, trading on the fundamental information does not improve welfare. The only potential benefit comes from the government's trading on noise, which comes naturally when it trades on partial/noisy information. Hence, trading on partial information can improve welfare. The mechanism is the same as when trading on pure noise. Consequently, we show that with the proper choice of its trading policy, the government can improve investor welfare.

Moreover, while the actual policy does depend on the accuracy of the government's information, the magnitude of the optimal welfare gain does not. In fact, it is the same as when the government trades on pure noise. This result may seem surprising at first. But it follows simply from the fact that independent of the government's fundamental information, the only tool it has to influence welfare is its noise trading. The government's optimal policy based on noisy fundamental information with different accuracy really boils down to choosing the right amount of noise trading.

Despite the fact the optimal welfare level does not depend on the government's private information, the choice of the policy to achieve it does. More importantly, the actual welfare impact (except for the optimum) is in general quite sensitive to the structural details of the economy, such as different values of model parameters, and the design of the intervention policy. For example, small changes in the government's trading policy can lead to very different welfare impacts. This implies that while the government can improve welfare by intervening in the market, extra caution is needed in finding the right input and design for the policy.

Given the government's direct intervention/trading in the market, during crisis as well as during normal times, an important question is how to evaluate the performance of these interventions. Some heuristic measures have been used, such as realized payoffs, without or with risk adjustments.[4] These measures often rely on market prices of the assets involved. But, as our analysis illustrates, using market prices can be misleading when they are also influenced by the interventions. Our framework, however, provides a proper way to assess the intervention policies by looking at their welfare implications. Moreover, we can use it to examine the other heuristic measures, such as expected payoff or Sharpe ratio.

In particular, we consider two intervention policies as examples. One policy is "buy low and sell high." This corresponds to government interventions during financial crisis. The government may directly or indirectly buy assets when their prices are depressed and sell them back to the market when their prices recover. This strategy only relies on market prices and requires no additional information. It yields a positive expected payoff and a positive Sharpe ratio. But as discussed above, such a strategy has no impact on market equilibrium and investor welfare. This is simply because this strategy can also be used by the investors. In fact, it will be, but in the opposite direction to offset the impact of government's trade if it chooses to intervene. Another policy is "trade on noise," as discussed above. Given that the government's trading will move the price against itself, such a strategy will yield a negative expected payoff and a negative Sharpe ratio. However, as we show in the paper, this strategy, within a proper range, can actually improve investor welfare. These results show that extra caution is needed when using market prices to assess government intervention policies.

It is worth pointing out that the main purpose of this paper is to develop a simple general equilibrium framework to properly incorporate government as a unique participant in the financial market. It is not aimed at a specific intervention observed in the market, which often relies on more targeted and stylized assumptions on various forms of market imperfections. Hence, the model presented here is made as parsimonious as possible, allowing only the basic imperfections of the market, its incompleteness and the associated information asymmetry. As a result, the model sacrifices on various richness potentially useful in more practical applications. Hopefully, the underlying mechanisms we establish here remain useful in general and the model can be expanded for more complex situations. We discuss some of these extensions at the end of

[4]See, for example, Lucas (2019) and the references therein.

Section 2 following the model specification and in the concluding section. For the same reason, our analysis is streamlined to focus on the main implications of the model concerning investor welfare. More detailed results on other market outcomes are either only touched on when needed or simply left out for brevity.

Related Literature

As noted at the outset, this paper focuses on government intervention through its direct trading in non-government securities, a narrow notion of the financial market as well as the form of intervention. Thus, our discussion of the literature will be limited to the part that is closely related in this particular sense. In a more general sense, government intervention can also include other forms ranging from regulation and supervision/guidance, to guarantees/bailouts and targeted taxes/subsidies; the financial market can also include other parts of the financial system, in particular financial intermediaries and institutions. There is an extensive literature on different aspects of these types of interventions, which are beyond the scope of this paper.

It is well recognized that the financial market, when it is incomplete and contains information asymmetry, need not lead to optimal allocations (see, for example, Stiglitz (1982)). Greenwald and Stiglitz (1986) consider government actions through general taxes and transfers to improve welfare. They show that full re-distribution with taxes and transfers can improve allocational efficiency. Their analysis, however, does not examine the allocation and welfare impact of government intervention through its own trading in the market in a general equilibrium setting.

Market imperfections have been a main focus of research in finance, including the two basic imperfections we consider in this paper, market incompleteness and asymmetric information. For example, Mankiw (1986), Constantinides and Duffie (1996) and Heaton and Lucas (1996) explore how investors' uninsured income risks can influence the behavior of asset prices. Our analysis focus more on the welfare cost of market incompleteness and the potential role for government intervention.

There is also extensive work on the market's informational inefficiency when there is information asymmetry. They are mostly based on the classic models of Grossman and Stiglitz (1980) and Kyle (1985), in which investors trade on their private information about asset payoffs, either competitively or non-competitively.[5] In these models, trading for non-speculative

[5]See Hellwig (1980) and Diamond and Verrecchia (1981) for variants of the static Grossman-Stiglitz noisy

reasons is introduced as noise trading, to preserve information asymmetry in equilibrium. The noise trading itself is usually specified exogenously, leaving its origin, specification and welfare implications outside the model. For this reason, these models are often referred to as noisy rational-expectations-equilibrium models (Noisy-REE models). For tractability, our model shares some features in its specification, mainly concerning the information structure, but it departs in a fundamental dimension. It explicitly models the investors' non-speculative or risk-sharing trades. This is crucial for the purpose of this paper. In particular, our focus is how the government's intervention influences investor welfare. Hence, we need to properly capture all investors' trading motives, trading behavior, and mostly importantly their welfare.[6] In addition, the model includes the bond market in the equilibrium, without assuming an exogenous interest rate. Moreover, the model includes the government as a unique market participant with a closed budget. These specifications make the model a fully general equilibrium one to allow proper welfare analysis.

A number of studies have examined how government's trading in various markets can alter the market outcome such as price volatility, informational efficiency and market liquidity. For example, Naranjo and Nimalendran (2000) and Pasquariello (2007) analyze how under particular policy targets government intervention in the currency market influences its informational efficiency and liquidity. Pasquariello, Roush, and Vega (2020) consider Federal Reserve's trading in the U.S. Treasuries market. For emerging markets, government intervention is a more common phenomenon. In a study of Chinese government's intervention in the stock market, Brunnermeier, Sockin, and Xiong (2022) show how its trading against the noisy retail investors can influence price volatility and the market's informational efficiency. Huang, Qiu, Wang, and Wang (2022) consider the same problem, assuming the government trades strategically on its private information.

These papers all rely on the noisy rational expectations settings. While they provide useful insights on how government trading can influence different market outcomes, the results are partial equilibrium in nature. In addition to the fact the government's budget is often not closed,

rational expectations model and Wang (1993), among others, for its intertemporal extensions.

[6]From a theoretical perspective, it can be argued that trading for risk sharing is more fundamental than trading on private information. See, for example, Milgrom and Stokey (1982). See also Wang (1994), Huang and Wang (2010), and Vayanos and Wang (2012a), which fully endogenize the "noise" trading driven by investors' risk-sharing needs.

the exogenous nature of noise trading does not allow a full welfare analysis.

Our paper considers the government's market intervention in a broader context and focuses on its impact on investor welfare rather than particular market outcomes. Using a full equilibrium framework in which investors trade rationally for risk-sharing and speculation and the government has a closed budget, we examine the welfare implications of its intervention policy in a self-contained general equilibrium setting. As we demonstrate in the paper, proper characterization of the government's objective and budget constraint as well as investor welfare are critical in assessing government intervention policies.

Our analysis shows that policies targeted at seemingly desirable market outcomes, such as enhancing informational efficiency, stabilizing price and improving market liquidity, may carry substantial welfare costs. Yet, policies at the expense of these targets may actually improve investor welfare. Hopefully, our setting provides a more integrated and coherent basis to consolidate and extend the existing analyses. We can also modify the model to allow different objectives and budget conditions for the government.

On the empirical side, several papers have studied European Central Bank's purchase of corporate bonds, including Grosse-Rueschkamp, Steffen, and Streitz (2019), Torodov (2020) and Corell, Mota, and Papoutsi (2024). Barbon and Gianinazzi (2019), Charoenwong, Morck, and Wiwattanakantang (2021) and Li (2022) have examined Bank of Japan's stock purchase program. While these analyses provide informative empirical background for direct government interventions in the financial market, our study can help to supply a theoretical framework to support future empirical work in better understanding these interventions.

There is an extensive literature on government interventions in the financial system especially during crises, focusing primarily on the assistance or bailout of financial institutions.[7] The very existence of financial institutions and their potential need for government assistance rely on the presence of market imperfections in addition to those considered in this paper. From this perspective, our framework, which is at a more basic level with only two primitive imperfections, market incompleteness and information asymmetry, can provide a building block as well as a useful basis for the analysis of more complex situations.

The remainder of the paper is organized as follows. In Section 2, we develop the basic

[7]See, for example, Bianchi (2016), Bond and Goldstein (2015), Chari and Kehoe (2016), Lucas (2019), and Philippon and Schnabl (2013).

model for our analysis. Section 3 consider the market equilibrium when there is no government intervention. This case serves as a benchmark and illustrates some of the model's basic properties. Section 4 presents the solution to the model when there is government intervention. In Section 5, we examine the impact of government intervention on market equilibrium and investor welfare under different information assumptions. Section 6 discusses different performance measures for government's trading policies. Section 7 concludes. The appendix contains the proofs, technical details and additional results.

2 Model

The economy has two dates, 0 and 1, denoted by $t = 0, 1$.

2.1 Financial Market

There is a financial market consisting of two traded assets, a risk-free bond and a risky stock. The bond yields a sure payoff of 1 at date 1. The stock pays a dividend D at date 1, given by:

$$D = \bar{D} + u, \tag{2.1}$$

where $\bar{D}$ is a constant and u is a random variable with zero mean. Thus, $\bar{D}$ gives the stock's expected payoff and u represents its payoff risk.

Trading in the market is through a Walrasian auction. Using the bond as the numeraire, we denote the price of the stock at date 0 by P.

2.2 Investors

There is a continuum of investors, denote by $i \in [0, 1]$.

Endowment. All investors are endowed with 0 unit of the bond and 0 shares of the stock. Thus, their initial financial wealth, given by the total market value of their endowed asset holdings, is zero. In addition, each investor i receives a non-traded income N_i at date 1, given by:

$$N_i = Y_i\, u = (Y + y_i)\, u, \quad i \in [0, 1], \tag{2.2}$$

where Y and y_i are two independent random variables. Hence, u gives the income risk, while Y and y_i capture the investors' common and idiosyncratic exposures to the income risk, respectively. Here, non-traded income risk is assumed to be the same as the risk of stock payoff u, allowing the investors to use the stock to share/hedge their income risk.[8]

[8]All we need is that the income risk is correlated with the stock payoff risk. The correlation is set to 1 in

Information. At date 0, all investors observe the stock price P, which is public information. In particular, we define $\tilde{P}$ to be the variable or demeaned part of P as a public signal for the market. In addition, investor i also observes Y_i, which is private information. Thus, Y_i not only gives investor i's total income risk exposure, but also provides a private signal about the aggregate income risk exposure Y. Let I_i denote the information set of investor i at date 0. We then have $\{P, Y_i\} \in I_i$, $i \in [0, 1]$.

Trading. Investors trade competitively in the market at date 0. Let θ_i denote investor i's stock demand/holding at date 0. In general, it depends on her information at date 0. In absence of other income, the terminal wealth of investor i at date 1 is given by:

$$W_i = \theta_i(\bar{D} + u - P) + Y_i\, u, \quad i \in [0, 1]. \tag{2.3}$$

Preferences. The investors' preferences are given by an expected utility function over their terminal wealth. For simplicity, we assume that they all have constant absolute risk aversion with risk aversion parameter $\alpha > 0$. In particular, investor i has the following utility function:

$$u_i(W_i) = -e^{-\alpha W_i}, \quad i \in [0, 1], \tag{2.4}$$

where W_i is her terminal wealth at date 1.

It is important to note that there is perfect symmetry between the investors as defined above. This helps to simplify our analysis without losing the essence of the model. The heterogeneity among investors in both endowment and information gives rise to their need for trading in the market. The difference in endowment leads to the trading need for risk sharing and the difference in private information leads to the trading need for speculation.

2.3 Government

The government as a central planner can also participate in the market. We assume that the government has no additional endowment of assets or income of its own.

Let I_G denote the information set of the government at date 0. We assume that other than P, the stock price as a public signal, the government may also receive a private signal about Y, the aggregate state variable, given by:

$$S_{GY} = \beta_{GY} Y + \varepsilon_{GY}, \tag{2.5}$$

the paper merely for simplicity. The only exception is when the correlation is zero, in which case the financial market becomes useless for investors' risk sharing and yields an autarky equilibrium.

where ε_{GY} represents the noise in the signal. Hence, $\{P,\, S_{GY}\} \in I_G$.

The government can trade in the market. Let θ_G denote the stock holding of the government at date 0 on per capita basis, which can depend on the information it has. For tractability, we limit the government's trading policy to be linear in its signals:

$$\theta_G = h_{GP}\tilde{P} + h_{GY}S_{GY} = h_G\,[\tilde{P} \;\; S_{GY}]^\top, \quad h_G \equiv [h_{GP} \;\; h_{GY}], \tag{2.6}$$

where $[\cdot \;\; \cdots \;\; \cdot]$ denotes a row matrix and $[\cdot]^\top$ denotes its transpose.

Within the linear class, the government's trading policy is given by the two coefficients, h_{GP} and h_{GY}, or h_G. For brevity, we will use θ_G to denote both the government's trading policy and its actual stock position without further clarification when its meaning is clear in the context. In addition, the former is given by h_G in the case of linear policy, and is independent of the state of the economy. In general, the government's trading is not directly observable to the investors. This implies, as we show later, that government's trading in the market creates an informational advantage for itself and additional information asymmetry in the market.

Let W_G denote the the government's wealth at date 1, on per capital basis. Its budget constraint is then given by:

$$W_G = \theta_G\,(\bar{D} + u - P). \tag{2.7}$$

We assume that the government's wealth at date 1 will be fully transferred to the investors. Negative transfers correspond to taxes. Given the symmetry between the investors, we assume that government transfers will be equal among them. Thus, in the presence of government intervention, investors' terminal wealth becomes:

$$W_i = \theta_i(\bar{D} + u - P) + Y_i\,u + W_G = (\theta_i + \theta_G)(\bar{D} + u - P) + Y_i\,u, \quad i \in [0,1]. \tag{2.8}$$

The government's objective is assumed to be maximizing the investors' expected utility ex ante or welfare, which is given by:

$$\int_i \mathrm{E}\,[u_i(W_i)] = \int_i \mathrm{E}\,\left[-e^{-\alpha W_i}\right]. \tag{2.9}$$

2.4 Distributional Assumptions and Parameter Constraints

For tractability, we assume that all shocks to the economy, u, Y, y_i, for all $i \in [0,1]$, and ε_{GY} are independent normal with zero mean and variances σ_u^2, σ_Y^2, σ_y^2, and σ_{GY}^2, respectively. Unless otherwise stated, all variances are assumed to be strictly positive to avoid degeneracy.

Under normality, the investors' non-traded income is unbounded, which can yield extreme values for the utility function. In order to guarantee the expected utility to be well defined, we need the following parameter condition:

$$\alpha^2\sigma_u^2\left(\sigma_Y^2+\sigma_y^2\right)<1, \tag{2.10}$$

as shown in the appendix. In fact, under (2.10), investors' expected utility without trading is given by:

$$\mathrm{E}\left[u_i(W_i)\right]=-\frac{1}{\sqrt{1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)}}\equiv V^A. \tag{2.11}$$

Thus, V^A gives the level of their welfare under autarky. As expected, it is inversely related to the investors' total income risk, given by $\mathrm{V}\left[(Y+y_i)\,u\right]=(\sigma_Y^2+\sigma_y^2)\sigma_u^2$, and their risk aversion α.

2.5 Common Knowledge and Rational Expectations

The structure of the economy is assumed to be common knowledge and both the investors and the government are assumed to have rational expectations.

2.6 Market Equilibrium and Optimal Intervention

The notion of market equilibrium is the standard one of rational expectations defined as follows:

(1) Taking the government's trading policy θ_G and the stock price P as given, investors choose their trading policy θ_i to maximize their expected utility:

$$J(P,I_i,\theta_G)\equiv\max_{\theta_i}\ \mathrm{E}\left[u_i(W_i)|I_i,\theta_G\right]=\mathrm{E}\left[-e^{-\alpha W_i}\middle|I_i,\theta_G\right],\quad i\in[0,1], \tag{2.12}$$

where $J(P,I_i,\theta_G)$ gives the value function of investor i; and

(2) The stock market clears:

$$\int_i\theta_i+\theta_G=0, \tag{2.13}$$

which gives the stock price and investors' stock holdings and expected utility in equilibrium.

The government then chooses its trading policy θ_G (or h_G) to maximize the overall expected utility for all investors:

$$\max_{\theta_G}\ \mathrm{E}\left[\int_i J(P,I_i,\theta_G)\right]\ \equiv\ \max_{h_G}\ V(h_G). \tag{2.14}$$

The last equality simply states that given the linear form of the government's trading policy, the choice of θ_G reduces to the choice of h_G, to improve the investors' ex ante welfare $V(h_G)$.

2.7 Ex Ante First Best Allocation

In the economy defined above, investors face heterogeneous income risk, which give rise to their desire to trade and share risk. In addition, when they have private information, they also want to trade to speculate. In the absence of trading for risk-sharing, there will be no trading for speculation either (see, for example, Milgrom and Stokey (1982)). Thus, investors' primary trading need is risk sharing. In particular, they want to trade to share the risk from their idiosyncratic income exposure, given by y_i. If such a risk sharing is complete and costless, the investors will only be bearing their fair share of the aggregate income risk exposure, given by Y. In this case, their ex ante expected utility will be:

$$V^{FB} \equiv -\frac{1}{\sqrt{1-\alpha^2\sigma_u^2\sigma_Y^2}}, \tag{2.15}$$

which is independent of σ_y^2. V^{FB} gives the investors' welfare level under the first-best allocation. In fact, this outcome can be achieved by the central planner through pooling together the income from all investors, which eliminates all the idiosyncratic risk, and distribute the total, which only depends on Y, evenly among the investors.

However, when investors have to rely on their trading in the market to share their idiosyncratic income risk, which is correlated with the market risk, they have to pay the corresponding market risk premium. Consequently, they may not be able to achieve the first best welfare level. This then leaves room for the government to intervene in the market and improve investor welfare. We will demonstrate this mechanism in our analysis below.

2.8 Remarks on Model Specification

For tractability and clarity, we have made simplifying assumptions in the model defined above. Several remarks are in order regarding these simplifications, their economic implications and possible relaxations.

Investor heterogeneity is essential to the importance of financial market, which facilitates more efficient resource/risk allocations. In our model, we allow two forms of heterogeneity, heterogeneity in endowment and in information, respectively. The heterogeneity in endowment gives rise to investors' desire to trade in the market to share risk and the heterogeneity in information gives rise to their desire to trade to speculate on information. We have avoided heterogeneity in preferences, although our setting can be easily extended to allow differences in

investors' (absolute) risk aversion. In addition, for parsimony, the form of information asymmetry is assumed to be only about investors' income risk. In general, we can also allow information asymmetry about the stock payoff.

For the financial market, we allow only the riskless bond and the stock to be traded. Their payoffs only depend on aggregate shocks, such as Y and u, which contribute to aggregate consumption. Given that there exists uninsured idiosyncratic income risk, as captured by y_i, the financial market is incomplete. This is the basic imperfection in our setting, which prevents investors from achieving the Pareto optimal allocation and leaves room for government intervention.

We have also assumed that all investors are identical ex ante. This is mainly for simplicity. We can allow differences in investors' initial endowment of stock and bond holdings, their exposure to aggregate income risk, the amount of exposure to idiosyncratic income risk, and the precision of their private information. These extensions will of course enrich the structure of the model as well as its implications. They will also make the analysis more complex. In order to focus on the main issues of this paper aims to address, we consider the simplest structure and leave its extensions to more specific applications.

Regarding the role of the government, we assume that its objective is the same as the investors' and its budget is closed. These two assumptions are important to our analysis on the impact of government interventions. In principle, we can allow the government to have different objectives and budget allocations. This will complicate the welfare implications of its actions. Given the ex ante symmetry among the investors, we only consider the government's ex post transfer to be also symmetric. When the investors are heterogeneous ex ante, the government's intervention and transfer policies also need to be adapted accordingly.

In the model, investors receive noisy signals about the aggregate state variable Y. However, collectively their signals fully reveal Y. That is, $Y \in \bigcup_i I_i$. This property of the information structure is important for our analysis. In particular, when compared with the information of each individual investor, the government's private signal (S_{GY}) does contain additional information about Y, but it provides no new information beyond the information investors collectively have. Such a restriction is necessary to limit the government's potential role. If instead the government possesses information beyond the collective information of all investors, it can attempt to achieve outcomes that are not feasible for the investors on their own. We

avoid this situation in this paper as it falls outside our focus.

The joint assumption of normality for the underlying shocks and constant absolute risk aversion allows us to solve the model analytically.

Also, for tractability, we only consider the government's intervention (trading) policy within the linear class. Thus, any notion of optimality concerning the intervention policy is limited to this class. The desire for tractability forces us to consider linear equilibrium under linear policies for the government. Since the main goal of the paper is about conditions under which government intervention is warranted, we are less concerned about if a welfare-improving intervention is globally optimal or even optimal within the linear class.

For simplicity, we have also assumed that the total stock and bond holdings of all investors are zero. This can be easily relaxed without affecting the main results of the paper.

We will provide more discussions on the possible extensions of the basic model in Section 7.

3 Market Equilibrium with No Government Intervention

We first consider the market outcome when there is no government intervention. This case allows us to establish some basic properties of the model and the underlying mechanism. Since this is a special case of the general case with government intervention, we will simply state the results here. Their derivation follows from the general solution provided in the next section.

Proposition 1. *With no government intervention ($\theta_G = 0$), the market equilibrium is given as follows. Taking price P as given, the investors' optimal trading policy is given by:*

$$\theta_i = \frac{1}{\alpha\sigma_u^2}(\bar{D} - P) - Y_i. \tag{3.1}$$

In equilibrium, the stock price is:

$$P = \bar{D} - cY, \quad c = \alpha\sigma_u^2, \tag{3.2}$$

the investors' stock holdings are:

$$\theta_i = -y_i, \quad i \in [0,1], \tag{3.3}$$

and their ex ante welfare is given by:

$$V^0 \equiv V(0) = -\frac{1}{\sqrt{1 - \alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)}}. \tag{3.4}$$

The equilibrium as given in Proposition 1 exhibits several simple but important features.

First, investors' optimal trading policy consists of two components, as given by the two terms in (3.2), respectively. The first component is for speculation. It is proportional to the expected return on the stock, $\bar{D} - P$, and inversely proportional to the risk of the stock payoff σ_u^2 and the investor's risk aversion α. The second component is for risk-sharing. It is negatively proportional to the investor's exposure to the non-traded risk Y_i. In particular, given that her exposure to the non-traded risk u is the same as the stock payoff risk, the investor wants to take on a stock position, exactly $-Y_i$, to offset this risk, thereby share it in the market with other investors. Hence, our model succinctly captures the two basic trading motives for investors, speculation and risk sharing.

The investors' trading behavior leads to the stock price in equilibrium given in (3.2), which also has two parts. The first part, $\bar{D}$, equals the stock's expected payoff, which also gives the average price level. This simply reflects the fact that the equilibrium price level of the stock fully adjusts for its expected payoff.

The second part, $-cY$, represents the risk discount on the stock price. This discount arises from investors' risk-sharing trades. As they trade in the market to hedge/share their income risk, the price of the stock will adjust accordingly to compensate for the corresponding risk it now carries. Suppose, for example, investor i receives a positive Y_i. She now has a positive exposure to the market risk u. In order to share/hedge this risk in the market, she would like to short the stock, for exactly the amount Y_i. In order to attract other investors to take the opposite side of the trade and bear this risk, the stock price has to decrease accordingly, yielding a risk discount. This then offers a positive expected return for the other investors.

In equilibrium, the risk discount is proportional only to the total amount of income risk Y. The proportionality coefficient c, as given in (3.2), is the risk premium per unit of the aggregate risk. It is proportional to σ_u^2, the measure of aggregate risk, and α, the investors' risk aversion.[9]

In addition, the equilibrium stock price as given in (3.2) is linear in Y, the investors' common income risk exposure. This implies that the equilibrium is full-revealing about the aggregate state of the economy. That is, investors can infer Y precisely from the stock price. Hence, the

[9]It is worth noting that in our current setting, the number of stock shares endowed by the investors is set to zero. This implies that the stock's direct exposure to the aggregate risk u is zero. Nonetheless, it has an indirect exposure coming from the investors' use of the stock to share their non-traded risk, given by Y. If the number of endowed stock shares is positive, then there will be a constant risk discount on the stock price, proportional to the number of endowed stock shares. Since it won't influence our analysis in the paper, we have set the number of endowed shares to zero for simplicity.

equilibrium obtained here becomes the same as the one when all investors directly observe Y.[10] This outcome arises because investors' private information on their income risk exposure, given by $Y_i = Y + y_i$, is reflected in the stock price from their trading, and their information collectively fully reveals Y through the price as it efficiently aggregates the investor's private information. Following Grossman (1976), in this case, we refer to the market as being informationally efficient.

Moreover, in equilibrium investors manage to fully share their idiosyncratic income risk. This is shown clearly by their equilibrium stock holding, $\theta_i = -y_i$, which exactly offsets their idiosyncratic income risk exposure y_i, $i \in [0, 1]$. Hence, the market equilibrium achieves efficient risk sharing among the investors. This is facilitated by the fact the equilibrium price fully reveals the level of aggregate risk (Y) and properly compensates for it.

Finally, we observe that efficient risk sharing in equilibrium improves investor welfare from autarky, as expected, but does not necessarily lead to the first-best ex ante welfare level. In particular, from (2.11), (2.15) and (3.4), we have

$$V^A \leq V^0 \leq V^{FB}, \tag{3.5}$$

and the inequalities are strict when $\sigma_Y > 0$ and $\sigma_y > 0$. In particular, despite the fact that the investors fully share their idiosyncratic risk by trading in the market, their welfare still depends on the magnitude of idiosyncratic risk as measured by σ_y^2. As (3.4) shows, V^0 decreases with σ_y^2 and this dependence further depends on σ_Y^2 and σ_u^2, the total amount of aggregate risk. This result may seem a bit surprising, but is crucial to our analysis.

The reason behind the result is that sharing the idiosyncratic risk in the market is in general *not* cost free. Investors need to pay a premium for the market risk as capture by u, which is common to both the stock risk and the income risk. They pay this premium when they sell the stock and receive it when they buy. In particular, as (3.2) shows, the actual premium is $c = \alpha\sigma_u^2$ per unit of risk. Since the investors' idiosyncratic risk exposure y_i, $i \in [0, 1]$, has a zero mean, the premium they pay for sharing this risk is zero on average. However, ex ante they face the uncertainty of paying or receiving it. Being risk averse, this uncertainty reduces their ex ante welfare. The fact that $V^0 < V^{FB}$ (when $\sigma_Y > 0$ and $\sigma_y > 0$) clearly implies that the

[10] It is worth noting that "fully-revealing" here only means that aggregate state variables are fully revealed or become public information. Private information investors have about their own idiosyncratic risk exposures need not and may not be revealed in equilibrium. This information remains important in investors' trading decisions and welfare, but becomes only inconsequential in determining asset prices.

equilibrium allocation is in general not optimal in the Pareto sense.

The first-best welfare level is achieved in two special cases. The first is when $\sigma_y^2 = 0$. In this case, there is no idiosyncratic income risk. Every investor is exposed only to the aggregate income risk. Hence, no trading is need and autarky is optimal (first-best). The second case is when $\sigma_Y^2 = 0$. In this case, there is no aggregate income risk. All income risk is idiosyncratic. As a result, sharing these risks in the market is cost free as it requires no risk premium. Consequently, the market equilibrium achieves the first-best welfare level.

Hence, we can conclude that, in the current setting, in general, the market equilibrium on its own does not lead to the first-best welfare level. This suggests that government intervention has the potential to improve investor welfare.

4 Solution to Market Equilibrium with Government Intervention

We now turn to the situation when the government intervenes in the market. We first solve for the market equilibrium, taken as given the government's trading policy in the form of (2.6), and then consider the choice of its trading policy to improve investor welfare. We outline in this section the key steps and the main results in the general solution, leaving the derivations and proofs to the appendix. In the following section, we examine in more detail how the government's trading policy may influence the market outcome.

In order to proceed, we need to know the form of the stock price as a function of the underlying state variables. This will help to specify the informational role of market prices. We start with the conjecture that the stock price takes the following linear form in the aggregate state variables:

$$P = \bar{D} - c\,Y + d\,\varepsilon_{GY} = \bar{D} + \tilde{P}, \quad \tilde{P} \equiv -\,c\,Y + d\,\varepsilon_{GY}, \tag{4.1}$$

where c and d are constant price coefficients.[11] Since $\bar{D}$ only shifts the price by a constant, we will follow $\tilde{P}$, the variable part of P, in our analysis. Our results will be unchanged if we set $\bar{D} = 0$ and $P = \tilde{P}$. As noted before, P or $\tilde{P}$ also provides a public signal about the underlying state of the economy, common to all market participants including the government. Our task going forward in this section is to show that the stock price in the form of (4.1) indeed clears

[11] In general, the price function should also include a constant term. However, as we later confirm, this constant is zero in equilibrium, given that the mean exposure to the stock payoff risk is zero. Thus, it is omitted here to simplify exposition.

the market, yielding a market equilibrium.

4.1 Conditional State Variables

Given the linear price function (4.1), we can derive the distributions of the underlying state variables conditional on investors' signals, public and private. For convenience, we will express the relevant quantities in matrix form.

Denote the vector of signals for investor i by X_i. Thus, $X_i \in I_i$ and is normally distributed:

$$X_i = \begin{bmatrix} \tilde{P} \\ Y_i \end{bmatrix} \sim N\left(0, \Sigma_{XX}\right), \tag{4.2}$$

where the mean of zero in the normal distribution represents a (2×1) vector of zeros and Σ_{XX} is the covariance matrix of the two signals, given by (A.4a) in the appendix. Σ_{XX} is assumed to be positive definite, that is, $|\Sigma_{XX}| > 0$, where $|\cdot|$ denotes the determinant of a matrix.

Next, we derive the distribution of Y, u, and ε_{GY}, the three aggregate state variables, conditional on investor i's information X_i. With the stock price function in (4.1), these shocks are no longer independent conditional on $\tilde{P}$, which is in X_i. In particular, we have:

$$\varepsilon_{GY} = (1/d)(\tilde{P} + c\,Y), \tag{4.3}$$

when $d \neq 0$. (When $d = 0$, we will simply let $\varepsilon_{GY} = 0$.) Hence, we only need to include the two independent state variables going forward: Y and u. In particular, let:

$$Z = \begin{bmatrix} Y \\ u \end{bmatrix} \sim N\left(0, \Sigma_{ZZ}\right), \quad \Sigma_{ZZ} = \text{Diag}\left[\,\sigma_Y^2,\, \sigma_u^2\,\right], \tag{4.4}$$

where Σ_{ZZ} is the covariance matrix of Z and Diag$[\cdots]$ denotes a diagonal matrix of its elements. Given our distributional assumptions, Σ_{ZZ} reduces to a diagonal matrix with variances of Y and u, respectively.

Joint normality of Z and X_i implies that conditional on X_i, Z remains normal, as given in the lemma below:

Lemma 1. *Conditional on X_i, $i \in [0,1]$, Z has the following normal distribution:*

$$Z = \bar{Z}_i + \tilde{Z}_i, \quad \bar{Z}_i \equiv \mathrm{E}\left[Z|X_i\right] = \beta_{ZX} X_i, \quad \tilde{Z}_i \equiv Z - \bar{Z}_i \sim N\left(0, \Sigma_{ZZ|X}\right). \tag{4.5}$$

Here, $\bar{Z}_i$ is the mean of Z conditional on X_i and $\tilde{Z}_i$ is the remaining random component; β_{ZX} gives the matrix of projection coefficients of Z on X_i and $\Sigma_{ZZ|X}$ is the variance matrix of $\tilde{Z}_i$ conditional on X_i, given by (A.6) and (A.7), respectively, in the appendix.

To avoid degeneracy, we assume that Σ_{XX}, Σ_{ZZ} and $\Sigma_{ZZ|X}$ are all strictly positive definite.[12]

Next, we consider the government's trading in the market, which is not observable to the investors. But they can form expectations conditional on their signals. The following lemma gives the distribution of θ_G conditional on investors' signals:

Lemma 2. *From (2.6) and (4.3), θ_G has the following linear form:*

$$\theta_G = h'_{GP}\tilde{P} + h'_{GZ}Z, \tag{4.6}$$

where the coefficients h'_{GP} and h'_{GZ} are given by (A.10) in the appendix. From (4.6) and Lemma 1, we then have:

$$\theta_G = \bar{\theta}_{Gi} + \tilde{\theta}_{Gi}, \quad i \in [0,1], \tag{4.7}$$

where

$$\bar{\theta}_{Gi} \equiv \mathrm{E}\left[\theta_G | X_i\right] = h'_{GP}\tilde{P} + h'_{GZ}\bar{Z}_i = h'_{GX}X_i, \quad \tilde{\theta}_{Gi} \equiv \theta_G - \bar{\theta}_{Gi} = h'_{GZ}\tilde{Z}_i, \tag{4.8}$$

and $h'_{GX} \equiv h'_{GP}\left[\begin{array}{cc} 1 & 0 \end{array}\right] + h'_{GZ}\beta_{ZX}$.

Combining (4.7) and (4.8) gives:

$$\theta_G = h'_{GX}X_i + h'_{GZ}\tilde{Z}_i. \tag{4.9}$$

The first term in (4.9) gives investor i's expected value of θ_G based on her signal X_i, and the second term gives the unexpected part.

4.2 Investors' Optimal Trading Policy and Value Function

We now consider the investors' optimal trading policy, taken as given the government's trading policy and the price function. We can rewrite investor i's wealth as follows:

$$\begin{aligned} W_i &= (\theta_i + \bar{\theta}_{Gi} + \tilde{\theta}_{Gi})[(-\tilde{P}) + u] + Y_i\, u \\ &= (\theta_i + \bar{\theta}_{Gi})(-\tilde{P}) + (\theta_i + \bar{\theta}_{Gi} + Y_i)\, u + (-\tilde{P})h'_{GZ}\tilde{Z}_i + \tfrac{1}{2}\tilde{Z}_i^{\top} H_G \tilde{Z}_i, \end{aligned} \tag{4.10}$$

where

$$H_G \equiv h_{uZ}^{\top} h'_{GZ} + h_{GZ}^{\prime\top} h_{uZ}, \quad h_{uZ} = \left[\begin{array}{cc} 0 & 1 \end{array}\right]. \tag{4.11}$$

[12]This requires that the elements in X_i, Z and $Z|X_i$ are not degenerate (such as having a zero variance) or linearly dependent. When that is not true, we will remove the degenerate or linearly dependent elements from the corresponding vectors to maintain their covariance matrices to be positive definite.

All the terms in (4.10) are linear in $\tilde{Z}_i$ except the last term which is quadratic in $\tilde{Z}_i$. As shown in the appendix, this quadratic term leads to a modified variance matrix of $\tilde{Z}_i$ from $\Sigma_{ZZ|X}$ to $\Sigma_{ZZ|XG}$ in the computation of expected utility, given by:

$$\Sigma_{ZZ|XG}^{-1} \equiv \Sigma_{ZZ|X}^{-1} + \alpha H_G. \tag{4.12}$$

For the investors' expected utility to be well defined, $\Sigma_{ZZ|XG}$ needs to be positive definite, which is confirmed under certain parameter restrictions given in the appendix. Going forward, we will also use $\sigma_{uZ|XG}$ to denote the covariance between u and $Z = [Y \;\; u]^\top$ under $\Sigma_{ZZ|XG}$.

With $\Sigma_{ZZ|XG}$ as the modified variance matrix for $\tilde{Z}_i$, the investors' optimization problem becomes a mean-variance problem. We then have the following proposition:

Proposition 2. *For positive definite $\Sigma_{ZZ|XG}$, investor i's optimal trading policy is given by:*

$$\theta_i = -\bar{\theta}_{Gi} + \frac{1}{\alpha\sigma_u^2}(-\tilde{P}) - Y_i, \quad i \in [0,\, 1], \tag{4.13}$$

and her value function is:

$$J(W_t^i, X_i, \theta_G) = -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\, e^{-\alpha\hat{W}_i}, \quad \hat{W}_i = \tfrac{1}{2}\,\frac{1}{\alpha\sigma_u^2}\tilde{P}^2 + \tilde{P}Y_i. \tag{4.14}$$

Investors' optimal stock demand given in (4.13) has a simple interpretation. The first term reflects the investor i's desire to offset the expected distortion to her terminal wealth caused by the government's intervention. The next two terms take the similar form as in the case of no government trading, given in (3.1), the former reflects the investor's speculative trade based on her expected stock return and the latter represents her hedging trade to share her income risk.

4.3 Price and Stock Holdings in Equilibrium

Investors' optimal stock demand in (4.13) can also be rewritten in the following form:

$$\theta_i = -h_P\tilde{P} - h_Y Y_i = -h_P\tilde{P} - h_Y(Y + y_i), \quad i \in [0, 1], \tag{4.15}$$

where coefficients h_P and h_Y are given by (A.22) in the appendix. The market-clearing condition (2.13) now becomes:

$$0 = \int_i \theta_i + \theta_G = (-h_P\tilde{P} - h_Y Y) + (h_{GP}\tilde{P} + h_{GY}S_{GY}). \tag{4.16}$$

We then have the following proposition for the solution of the equilibrium stock price:

Proposition 3. *In the market equilibrium with the linear stock price function in the form of*

(4.1), the price coefficients are given by:

$$c = \frac{1}{h_P - h_{GP}}(h_Y - h_{GY}\beta_{GY}), \quad d = \frac{1}{h_P - h_{GP}}h_{GY}. \tag{4.17}$$

Under certain parameter conditions, c and d have a unique solution, given by Lemma A.3 in the appendix.

The above solution for the price coefficients confirms that the linear price function (4.1) can indeed yield a market equilibrium, with the coefficients given by (4.17) and no constant term.[13]

In equilibrium, the investors' stock holdings are:

$$\theta_i = -\theta_G - h_Y y_i, \quad i \in [0,1]. \tag{4.18}$$

It is important to note that (4.15) expresses θ_i in terms of investor i's signals (information) and stock price while (4.18) expresses it in terms of the underlying aggregate and idiosyncratic shocks. This difference comes from the fact that former describes investors' stock demand while the latter gives their holdings in equilibrium after substituting in the equilibrium stock price. Given the symmetry among investors, their equilibrium stock holdings do not depend on the aggregate shock Y except through θ_G, the government's intervention in the market. It also shows that in equilibrium investors end up with stock holdings θ_i, $i \in [0,1]$, which fully offset the distortion caused by the government's intervention θ_G.

4.4 Investor Welfare

Next, we consider the investors' ex ante welfare, given the government's trading policy. For convenience, define:

$$z_i \equiv [\, Y \;\; y_i \;\; \varepsilon_{GY} \,]^\top \sim N(0, \Sigma_{zz}), \quad \Sigma_{zz} \equiv \text{Diag}\left[\, \sigma_Y^2, \; \sigma_y^2, \; \sigma_{GY}^2 \,\right], \quad i \in [0,1]. \tag{4.19}$$

Note that while Z contains only aggregate shocks, z_i also includes the idiosyncratic shocks to the investors. From the equilibrium solution, we have the following corollary:

Corollary 4.1. *In the market equilibrium given above, the government's trading policy, the stock price, investor i' income risk exposure, stock holding and value function in equilibrium can be expressed as follows:*

$$\theta_G = B_G z_i, \quad \tilde{P} = B_P z_i, \quad Y_i = B_Y z_i, \quad \theta_i = -B_G z_i + B_\theta z_i, \quad \hat{W}_i = \tfrac{1}{2}\, z_i^\top B z_i, \tag{4.20}$$

[13]The constant term can become relevant if the market's unconditional exposure to the payoff risk is not zero. This occurs for example when the initial endowment of the stock shares or the mean of Y is not zero.

where B_G, B_P, B_Y, B_θ and B are given in (A.32).

Investor's welfare is then given by the following proposition:

Proposition 4. *Given the government's intervention policy (2.6), investors' ex ante expected utility is given by:*

$$V(\theta_G) = V(h_G) = -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}} \frac{|\Sigma_{zz}^{-1}|^{1/2}}{|\Sigma_{zzB}^{-1}|^{1/2}}, \quad \Sigma_{zzB}^{-1} = \Sigma_{zz}^{-1} + \alpha B, \quad i \in [0,1]. \tag{4.21}$$

Note that V is the same for all investors, hence it does not carry the index i.[14]

4.5 Optimal Intervention Policy

Under the linear intervention policy, the government's optimal choice is given by:

$$h_G^* = \arg\max\ V(h_G). \tag{4.22}$$

The first order condition is:

$$\nabla_{h_G} V(h_G) \ = \ 0, \quad \nabla_{h_G} \equiv \left[\frac{\partial}{\partial h_{GP}} \quad \frac{\partial}{\partial h_{GF}} \quad \frac{\partial}{\partial h_{GY}} \right]^\top. \tag{4.23}$$

Optimality is ensured by the second order condition, which requires the second order derivatives of $V(h_G)$ to be negative definite:

$$\left| \nabla_{h_G} \nabla_{h_G}^\top V(h_G) \right| < 0. \tag{4.24}$$

The solution to (4.23) will then yield the optimal trading policy within the linear class, when the second order condition is also met.

Instead of focusing solely on the optimal policy, we can also explore how small interventions, i.e., when $|h_G|$ is small, may improve welfare at the margin relative to no intervention. This entails to examining the derivatives of $V(h_G)$ at $h_G = 0$. Our analysis later will explore both approaches.

4.6 Additional Comments

We have shown that under the government's linear intervention policy (2.6) and the linear price function (4.1), the market equilibrium is fully characterized by Propositions 2 and 3. Its existence and uniqueness will depend on the parameter values, including h_G. In what follows, we will consider situations of interest under parameter conditions guaranteeing the existence and uniqueness of the equilibrium, as specified in the appendix.

[14] (4.21) requires $\Sigma_{zz}^{-1} + \alpha B$ to be positive definite, which can be guaranteed by proper parameter restrictions as shown in the appendix.

5 Government Intervention

With the general solution to the market equilibrium under a given government intervention policy, we now examine how different policy choices impact market outcome. Obviously, the government's intervention policy depends on the information it has. We consider three different situations. The first is when the government has only public information. The second situation is when the government has private and full information about the underlying state of the economy, and the third is when it only has partial information.

When examining the impact of government intervention, we focus on the major aspects of the market equilibrium, including the stock price, the risk allocation among investors, the informational efficiency of the market, and, especially, investor welfare. For brevity, we will leave out other properties of the equilibrium such as price volatility and market liquidity, which can be easily obtained from the general solution.

5.1 Intervention with Public Information

We start with the case when the government has *no* private information. In this case, the only information it has is the stock price P, which is public information. Thus, the government's trading in the market can be a function of P, that is, $\theta_G = \theta_G(P)$.[15] In this case, we do not need to assume θ_G to be linear in P.

Price function. We now assume that the price function takes the following form:

$$P = \bar{D} - c\,Y + d\,\theta_G(P) \equiv \bar{D} + \tilde{P} + d\,\theta(\tilde{P}), \quad \tilde{P} \equiv -\,c\,Y, \tag{5.1}$$

where c and d are constants. Given that θ_G is only a function of P or equivalently $\tilde{P}$, which is public information, it will be known to all investors. Thus, their information set is unaffected by the government's intervention in this case. In particular, $X_i = [\,\tilde{P} \;\; Y_i\,]^\top$.

Investors' optimal trading policy and value function. Investors' terminal wealth is now:

$$W_i = (\theta_i + \theta_G)(u - \tilde{P}) + Y_i\,u, \quad i \in [0,1], \tag{5.2}$$

same as (2.8). Given that θ_G is known to all investors, W_i is normal conditional on investor i's signal X_i. This corresponds to the situation in the general case when $\bar{\theta}_{Gi} = \theta_G$ and $\tilde{\theta}_{Gi} = 0$.

[15] In principle, the government can also intervene with pure noise with no information content. We consider this case in the next subsection.

Investor i's optimization problem is formally similar to the maximization problem with no government intervention, which yields the following solution:

$$\theta_i = -\theta_G + \frac{1}{\alpha\sigma_u^2}(-\tilde{P}) - Y_i, \quad i \in [0,1]. \tag{5.3}$$

(5.3) shows that investor i's optimal trading policy is to achieve her net optimal portfolio by fully offsetting the effect of any intervention from the government. This is achievable if the government's trading/position is public information. Consequently, investors' value function will be the same as with no government intervention.

Market equilibrium and optimal intervention policy. From the analysis above, we immediately conclude that when the government trades only on public information, any of its intervention will be fully offset by the investors through their own trading to achieve the net optimal portfolio. Thus, the market equilibrium and investor welfare will remain the same, independent of the government's trading. Consequently, there is no merit for the government to intervene in the market using only public information. We summarize these results in the following proposition:

Proposition 5. (No Impact Theorem) *When the government intervenes with only public information, its impact will be fully offset by the investors' own trading, leaving no net effect on the equilibrium price, investors' net stock holdings, and their welfare.*

This result is a form of Ricardian equivalence in the current context under the given conditions. Thus, in what follows we will simply let $h_{GP} = 0$ unless stated otherwise.

5.2 Intervention with Pure Noise

The result above, however, does not imply that the government cannot do better without private information. We now consider the situation when the government has no private information on the economy but chooses to intervene nonetheless, with pure noise. We assume that the government's trading/position is not public information (otherwise, we return to the previous case). In particular, we set $\beta_{GY} = 0$ in the general solution and let:

$$\theta_G = h_G\, \varepsilon_G \tag{5.4}$$

where $\varepsilon_G = \varepsilon_{GY}$ is pure noise. With no loss of generality, we let $\sigma_G^2 = 1$.

In this case, the government acts as a noise trader, adding random shocks to the market.

There are, however, two key differences between the government here and the noise traders in the noisy-rational-expectations models.[16] First, the government's gain/loss from its noise trading is passed on to the investors, not left outside the model. Second, the government's objective function is the investors' welfare, which it acts strategically to improve. Thus, our model remains closed and general equilibrium in nature even when the government trades on noise.

Given the government's noise trading, the stock price now takes the same linear from as in (4.1). Since the government knows ε_G and it can also infer Y from $\tilde{P}$, we now have $[Y \;\; \varepsilon_G] \in I_G$. This has three implications. First, the government can extract better information from the price about the aggregate state variables Y and ε_G than the investors. In other words, through its noise trading, the government creates an informational advantage for itself. Second, it also generates information asymmetry among investors as $\tilde{P}$ now no longer fully reveals Y, the underlying state of the economy. In addition to the price, each investor has to rely on her private signal Y_i, $i \in [0,1]$, to learn about Y. Third, the government can implement an intervention policy which also depends on Y, now its private information. Since our purpose here is to examine the impact of the government's pure noise trading, we will assume that it ignores the information $\tilde{P}$ reveals here about Y to simplify the analysis. In the next two subsections, we will consider in more detail the situation when the government has its own information on the state of the economy that individual investors do not have.

Market equilibrium. Applying the general solution in Section 4 with $\beta_{GY} = 0$, as shown in the appendix, we arrive at the solution to the market equilibrium for this case. The investors' optimal trading policy and value function take the same form as in the general solution.

In equilibrium, the price coefficients c and d satisfies the following solution:

$$0 = (\sigma_Y^2 + \sigma_y^2)k^3 - h_G\sigma_y^2 k^2 + \sigma_Y^2\sigma_y^2 k - h_G\sigma_Y^2\sigma_y^2, \quad k \equiv d/c, \tag{5.5}$$

which is a cubic equation whose general solution is given in the appendix. The form of (5.5) implies that k and d have the same sign as h_G. We then have the following solution for the price coefficients c and d:

$$c = (\alpha\sigma_u^2)\left[1 - \frac{\sigma_y^2}{\sigma_Y^2\sigma_y^2 + k^2(\sigma_Y^2+\sigma_y^2)} k\, h_G\right], \quad b = k\, c, \quad \textit{and} \quad k\, h_G \geq 0. \tag{5.6}$$

[16] See Grossman and Stiglitz (1980) and also Hellwig (1980), Diamond and Verrecchia (1981), among others.

The investors' equilibrium stock holdings are:

$$\theta_i = -h_G\,\varepsilon_G - \left(1 + |A|^{-1} c^2 \sigma_Y^2\, k\, h_G\right) y_i, \quad |A| = c^2\sigma_Y^2\sigma_y^2 + d^2(\sigma_Y^2+\sigma_y^2), \quad i \in [0,1]. \tag{5.7}$$

(5.6) and (5.7) reveal how the government's noise trading affect the equilibrium price and investors' equilibrium holdings. In particular, given that $k\,h_G \geq 0$, government's noise trading tends to decrease c, the market premium investors pay to share risk, as (5.6) clearly shows. This can be desirable for the investors. As discussed in Section 3, a lower risk premium reduces their ex ante uncertainty about paying it when sharing their risk in the market.

From (5.6), we further observe that with government's noise trading, investors tend to over trade on their idiosyncratic income risk. In particular, net of her trades to offset the government's trading, which is $\theta_G = h_G\varepsilon_G$, investor i's hedging trade is now $(1 + |A|^{-1} c^2\sigma_Y^2\, k\, h_G)\, y_i$, exceeding y_i, her idiosyncratic income risk exposure. Such a shift comes from the investor's speculative trading against the government, which is profitable. In particular, government's noise trading moves the price against it, leaving excess returns for its counter-parties in the trade.

We illustrate this mechanism using the following example. Consider the situation when investor i observes a large and positive $Y_i = Y + y_i$. She then infers that both Y and y_i are likely to be positive and large. A large and positive y_i leads her to hedge aggressively in the market by shoring the stock. A large and positive Y implies a negative pressure on the price. Conditional on the price, $\tilde{P} = -c\,Y + d\,\varepsilon_G = -c\,Y + (d/h_G)\,\theta_G$, however, this implies a large government stock purchase, i.e., $\theta_G > 0$, exerting a positive price pressure and balancing out the negative impact of Y. Such a price lift is purely caused by the government's purchase and unrelated to the stock's payoff. Hence, it is optimal for the investor to take the opposite side and sell the stock. This speculative trade is in the same direction as the investor's hedging trade, leading her net stock sale to be larger than y_i.

The following proposition summarizes the results above:

Proposition 6. *Government's noise trading in the form of (5.4) in general reduces the market risk premium and magnifies investors' hedging trades, which reflects their trades against the government.*

Given the solution to c and d, we can compute the investors' welfare from (4.21), which is given in the appendix. We can then examine in detail how the government's trading policy influences investor welfare.

Small interventions. What we consider next is when the government only intervenes in the market marginally, that is, when h_G is small. In this case, we can obtain solutions to the equilibrium price coefficients and investor welfare level to the lowest (non-trivial) order of h_G. The results are summarized in the following proposition:

Proposition 7. *For h_G small, the equilibrium price coefficients are given by:*

$$c = \alpha\sigma_u^2[1 - (1/\sigma_Y^2)h_G^2] + o\,(h_G^2), \quad d = \alpha\sigma_u^2\, h_G + o\,(h_G^2), \tag{5.8}$$

where $o\,(h_G^2)$ denotes terms of higher order of h_G^2. The investors' welfare is given by:

$$V(h_G) = -\frac{1}{\sqrt{1 - \alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)}}\left[1 - \frac{1}{2}\frac{(\alpha^2\sigma_u^2)^2\sigma_y^2(1+\alpha^2\sigma_u^2\sigma_Y^2)}{1 - \alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)}h_G^2\right] + o\,(h_G^2), \tag{5.9}$$

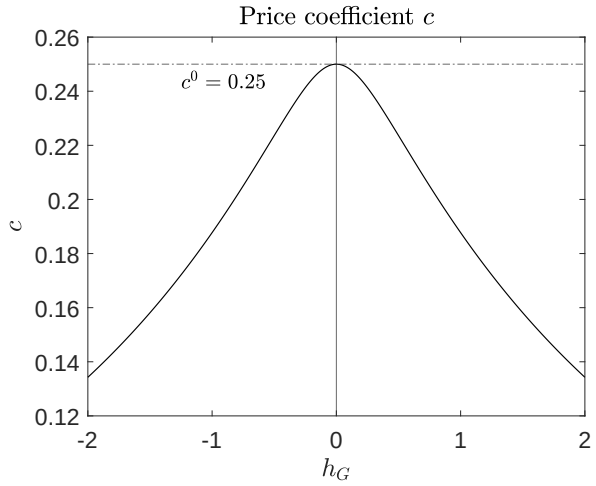
which is higher than V^0, their welfare level with no government intervention.

Proposition 7 clearly shows that for small values of h_G, the investors' welfare increases with h_G^2. It analytically confirms the mechanism we discussed earlier. The government's noise trading provides profitable trading opportunities for the investors in the market. This effectively lowers the market risk premium as captured by c. A lower risk premium lowers the cost of investors' risk sharing and improves their welfare. This welfare gain is despite the fact that the investors' total gain from trading with the government will be taxed away at the end.

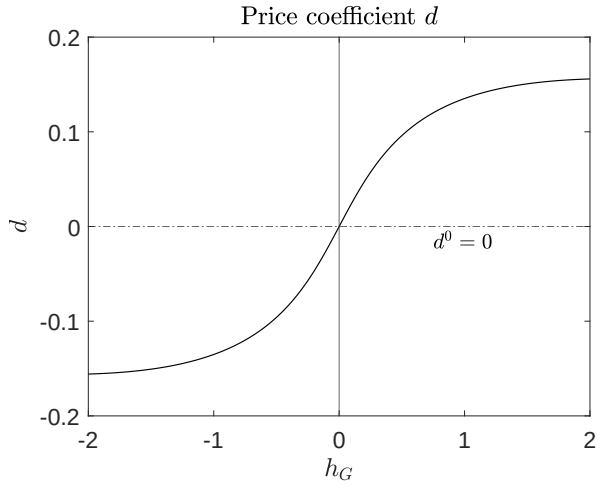
Optimal intervention. We now explore the optimal intervention policy when the government trades on noise. Instead of describing the results analytically, which is tedious and messy, we present the results numerically to illustrate their main features.

As mentioned before, we focus on how the government's intervention policy, as given by h_G, changes the market outcome, in particular, the equilibrium stock price, the market's informational efficiency and investor welfare. The parameter values are set as follows: $\alpha = 1.0$, $\sigma_u = 0.5$, $\sigma_Y = 1.25$, $\sigma_y = 1.0$, $\beta_{GY} = 0$, and $\sigma_G = 1.0$. Figure 1 plots the price coefficients c and d, the volatility of Y conditional on the investors' signals $\sigma_{Y|X}$, and the investors' ex ante welfare level V, for different values of h_G. Since the government trades on noise ($\theta_G = h_G\varepsilon_G$ with ε_G being a noise), the sign of h_G does not matter. Hence, all the plots will be symmetric with respect to $h_G = 0$.

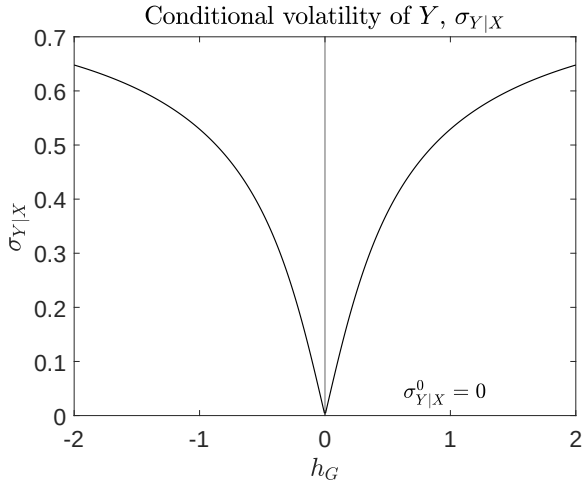
Panel (a) of Figure 1 plots the price coefficient c for different values of h_G around zero. As discussed earlier, c reflects the risk premium investors' pay for sharing their income risk in the



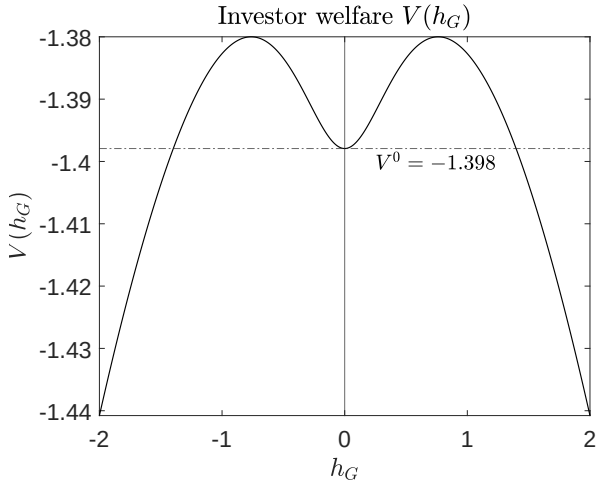
(a) Price coefficient c



(b) Price coefficient d



(c) Conditional standard error of Y



(d) Investor welfare V

Figure 1: Price coefficients c, d, conditional volatility of Y, $\sigma_{Y|X}$, and investor welfare V for different values of h_G under government intervention with pure noise. The parameter values are set at $\alpha = 1$, $\sigma_u = 0.5$, $\sigma_Y = 1.25$, $\sigma_y = 1.0$, $\beta_{GY} = 0$, and $\sigma_G = \sigma_{GY} = 1.0$. The dashed horizontal line in each plot represents the values of c, d, $\sigma_{Y|X}$ and V, respectively with no government intervention ($h_G = 0$). The maximum value of $V^* = -1.380$ is achieved at $h^*_G = \pm 0.772$.

market. The horizontal dash-dot line indicates the value of c with no government intervention, which is $c^0 = \alpha\sigma_u^2 = 0.25$. It is also the value of c at $h_G = 0$. The plot clearly shows that as h_G increases in magnitude, the value of c decreases. This is consistent with the analytic results we have in Proposition 6 and 7. The plot further shows that the negative impact of h_G on c is monotonic for a large range of h_G. We thus conclude that government's noise trading has the effect of reducing the risk premium investors pay for sharing their income risk in the market,

which can lead to their welfare gains.

In Panel (b) of Figure 1, we plot price coefficient d for different values of h_G. The value of d captures how much the noise in the government's trading shifts the price. Again, the horizontal dash-dot line gives the value of d when there is no government intervention, which is 0. Obviously, d has the same sign as h_G, as we proved earlier. In addition, its absolute value increases monotonically with the magnitude of h_G. This is intuitive. As the government trades more aggressively on noise, it will bring more noise into the equilibrium price. We further observe that d is proportional to h_G for small values of h_G, as shown in Proposition 7. However, when h_G becomes large in magnitude, its influence on d diminishes. This is because when the government's trading has a large impact on price, it also becomes more transparent to the investors. As we show in Section 5.1, any known intervention by the government will be countered by the investors' offsetting trades and thus have little effect on price.

Next, we examine how the government's noise trading affects the informational efficiency of the market, measured by the standard deviation of Y conditional each investor's signal X_i, $i \in [0, 1]$, given by $\sigma_{Y|X}$. As we saw in Section 3, when there is no government intervention ($h_G = 0$), the equilibrium price fully reveals Y and $\sigma_{Y|X} = 0$. This is represented by the dash-dot line in the plot, now overlapping with the x-axis. When $|h_G|$ becomes positive, i.e., when the government trades on noise, $\sigma_{Y|X}$ becomes positive and increases quickly with $|h_G|$. This is simply a reflection of the fact that the government's noise trading brings noise into the price and reduces its informational efficiency in revealing the underlying state of the economy. As the magnitude of h_G further increases, its impact on $\sigma_{Y|X}$ diminishes. The reason is the same as why its impact on d diminishes.

In considering the market's informational efficiency, we have focused on how informative the equilibrium is about Y, the fundamental state of the economy. When the government is trading on noise, it also brings the noise ε_G into the market, which is a new and endogenous state variable, not observable to the investors. Hence, we can also consider how informative the equilibrium is about ε_G, which can be measured by $\sigma_{\varepsilon_G|X}$. Given that $\tilde{P} = -c\,Y + d\,\varepsilon_G$ is in the investors' information set, we immediately have $\mathrm{V}\left[\tilde{P}|X_i\right] = 0$ (i.e., the variance of $\tilde{P}$ conditional on $\tilde{P}$ itself is zero). This then implies that conditional on $\tilde{P}$, Y and ε_G are perfectly

correlated. Consequently, when $\sigma_{Y|X}$ increases with $|h_G|$, $\sigma_{\varepsilon_G|X}$ decreases correspondingly.[17] For this reason, we only follow $\sigma_{Y|X}$ when gauging the market's informational efficiency.

Finally, we consider the investors' welfare. As we have shown in Proposition 7, small amount of noise trading by the government can improve investor welfare at the margin. Figure 1(d) plots the investors' welfare level for a range of h_G. Again, the dash-dot line marks the welfare level without intervention, V^0, which takes the value of $V^0 = -1.398$. With intervention, the welfare initially increases with $|h_G|$ quadratically, as shown in Proposition 7, levels off at some optimal magnitude, and then starts to decrease quickly. For large values of $|h_G|$, government's noise trading makes investors worse off than with no intervention.

Here, two factors are at work. On the one hand, government's noise trading lowers c as shown in Figure 1(a), which lowers the risk premium investors pay for hedging their income risk and improves welfare. On the other hand, government's noise trading makes the stock price less informative about the true state of the economy Y, which introduces additional risk in the investors' risk sharing and their terminal wealth/consumption, which lowers welfare. Figure 1(d) shows that the former dominates for small amount of noise trading by the government, and the latter dominates for large amount of noise trading.

For the parameter values given in Figure 1, the maximum welfare is achieved at $h_G = \pm 0.772$ at the level of $V(\pm 0.772) = -1.380$. This is higher than the welfare level of $V^0 = -1.398$ with no intervention. But it is still below the first best level, which is $V^{FB} = -1.281$ under the same parameter values.

Other policy goals. In the discussion of government interventions, commonly used policy goals include informational efficiency, price stability and market liquidity.[18] As we have learned from the analysis above, these goals also have important welfare implications. The results presented in Figiure 1 not only show that interventions that improve welfare may reduce the market's informational efficiency, but also demonstrate that interventions yielding improvements in these other policy goals may carry substantial welfare costs. In particular, the price volatility, given by $\sigma_P = \sqrt{c^2\sigma_Y^2 + d^2\sigma_G^2}$, decreases with c. As shown in Vayanos and Wang (2012b), various measures of market liquidity are inversely related to the value of c. Hence, large

[17] In fact, $\sigma^2_{Y|X} - 2k\,\sigma_{Y\varepsilon_G|X} + k^2\sigma^2_{\varepsilon_G|X} = 0$ and $\sigma^2_{\varepsilon_G} = 1$, which yields $\sigma_{\varepsilon_G|X} = \sigma_{Y|X}/k$.

[18] See, for example, Naranjo and Nimalendran (2000), Pasquariello (2007), Pasquariello, Roush, and Vega (2020), and Brunnermeier, Sockin, and Xiong (2022).

values of h_G (beyond the value of 1.0) tend to reduce price volatility and improve market liquidity, but they may substantially lower investor welfare. For example, for $h_G = 2.0$, we have $\sigma_P = 0.2290 < 0.3125 = \sigma_P^0$ and $c = 0.1343 < 0.25 = c^0$, showing improvements in both price stability and market liquidity relative to their levels with no intervention, and yet $V = -1.441 < -1.398 = V^0$, showing a large decline in welfare. For brevity, we omit a more detailed discussion of these results.

5.3 Intervention with Full Information

The next case we consider is when the government has perfect information about the aggregate state of the economy. In particular, we let $\beta_{GY} = 1.0$ and $\sigma_{GY} = 0$, hence, $S_{GY} = Y$, that is, the government observes Y directly and $[\tilde{P} \;\; Y] \in I_G$. The government's trading policy and the stock price then become:

$$\theta_G = h_G Y, \quad P = \bar{D} - c\,Y. \tag{5.10}$$

Hence, we have a fully-revealing equilibrium whenever $c \neq 0$, in which investors precisely learn about Y from the stock price. Moreover, they can also infer the government's trade. Consequently, we are back to the case when the government trades on public information, in which investors fully offset the impact of the government's trade, neutralizing their impact on the market. We thus end up with the following situation. Despite the fact the government has better information than each individual investor, it is not easy for the government to trade on this information and improves market efficiency. The reason is that the investors, if left on their own, can also achieve the same equilibrium which fully reveals the relevant aggregate state variable Y and achieves efficient risk allocation.

However, there is a special trading policy the government can adopt to avoid this ineffective situation. This is when the policy is such that it forces c to be zero. In this case, the stock price no longer reveals Y. Hence, the investors cannot undo the government's impact. In addition, the market risk premium becomes zero, which makes risk sharing in the market costless. As discussed earlier, lowering this cost can potentially improve welfare. Such a situation is indeed possible as given by the following proposition.

Proposition 8. *When the government has full information on Y and chooses the following*

trading policy:

$$\theta_G = h_G^* Y, \quad h_G^* \equiv 1 + \frac{\sigma_Y^2}{\sigma_y^2}, \tag{5.11}$$

the market equilibrium becomes:

$$P = \bar{D}, \quad \theta_i = -\left(1 + \frac{\sigma_Y^2}{\sigma_y^2}\right)(Y + y_i), \quad i \in [0,1], \tag{5.12}$$

and investors' welfare is given by:

$$V(h_G^*) = -\frac{1}{\sqrt{1 - \alpha^2\sigma_u^2\sigma_Y^2(\sigma_Y^2+\sigma_y^2)/\sigma_y^2}} > V^0 \quad when \quad \sigma_Y^2 < \alpha^2\sigma_u^2\sigma_y^4. \tag{5.13}$$

From Proposition 8, we see that when the government follows the trading policy in (5.11), $c = 0$ and the price becomes a constant, completely uninformative about the state of the economy. In this case, investors' can only rely on their private signal $Y_i = Y + y_i$ to guide their trades. In particular, $\bar{y}_i = \mathrm{E}[y_i|Y_i] = (1 + \sigma_Y^2/\sigma_y^2)Y_i$, which leads to her stock holding in equilibrium given in (5.12). In this case, by fully absorbing the aggregate risk, the government has made the equilibrium price constant. The cost for investors' risk sharing trades now drops to zero, which helps to improve investor welfare. At the same time, it completely removes the informational role of the price. This tends to reduce the information investors have on the amount of aggregate risk as captured by Y and leads to its inefficient allocation among the investors, lowering their welfare. (5.13) shows that the former effect dominates when the variability of Y, as measured by σ_Y, is relatively small. Hence, investors' welfare increases under h_G^*. When the variability of Y is relatively large, the latter effect dominates, and the trading policy h_G^* will actually decrease investor welfare.

We thus conclude that when the government has full information on Y, the aggregate state of the economy, and its variability is relatively small, as (5.13) specifies, it can find a unique strategy h_G^* as given in (5.11) that improves investor welfare. Otherwise, it cannot.

Given the analysis in Section 5.2, even without any non-public information, the government can improve welfare by trading on noise. It is natural to also allow the government to include a noise component in its trading policy here. The results from pure noise trading suggests that the government can find more trading policies to improve investor welfare. Since this case is similar to the case when the government trades on partial information, we will return to it in the next section.

When the government has its own private information, a related question is whether it can

alter the market outcome and improve investor welfare by simply making the information public. With the results we have obtained above, we can properly address this question in our setting. The answer is in general no, if the government's information does not go beyond the collective information of the investors. This is illustrated clearly by the case discussed above. If the government has full information on Y, it can simply announce it to the market. Assuming it is fully creditable, the investors will then learn about aggregate risk exposure (Y) from the announcement and their own idiosyncratic exposure (y_i, $i \in [0,1]$). We will then arrive at the same market equilibrium as in Section 3 with no government intervention. What we have learned above is that the government can do better under certain conditions by keeping its information and trading in the market following a special strategy.[19] In addition, with full information on Y, the government can combine it with noise in its trading policy to improve welfare, while revealing it will take away these opportunities.

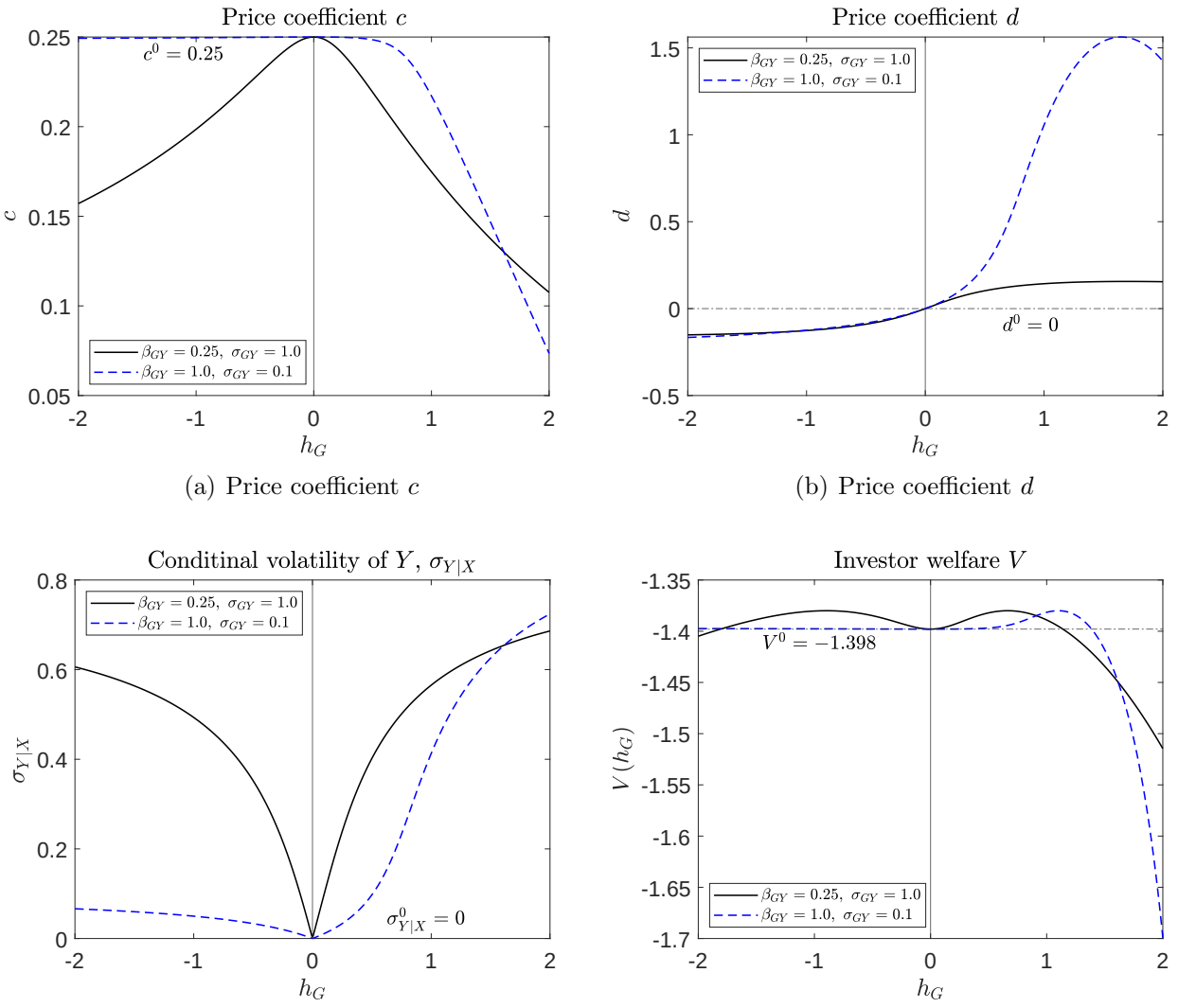
5.4 Intervention with Partial Information

The final case we consider is when the government's information about the underlying state of the economy is noisy. In particular, it receives a signal on Y in the form of (2.5) with $\beta_{GY} > 0$ and $\sigma_{GY} > 0$. Again, we examine how the government's trading policy as given by h_G impacts the market equilibrium and investor welfare.

For easy comparison, we hold the parameter values the same as in Figure 1, except for β_{GY} and σ_{GY}, which define how informative the government's signal is about Y. We then consider two cases corresponding to different levels informativeness of the government's signal S_{GY}. The first case is when $\beta_{GY} = 0.25$ and $\sigma_{GY} = 1.0$ and the signal is less informative, and the second case is when $\beta_{GY} = 1.0$ and $\sigma_{GY} = 0.1$ and the signal is more informative.

As in Figure 1, the four panels in Figure 2 plot the price coefficients c and d, the investors' conditional volatility of Y, and their welfare level, respectively, for different values of h_G, ranging from -2.0 to 2.0. As before, the dash-dot line in each plot (sometimes overlapping with the x-axis) represents the corresponding value of the variable with no intervention. The solid black lines correspond to the case when the government's signal is less informative, and the dashed

[19] One can also consider the possibility of government not being fully untruthful in its announcement. For example, it can throw some noise into the announcement. Under rational expectations, the investors can simply choose to ignore the government's announcements and achieve the marke equilibrium without the government. For brevity, we omit these extensions.



(a) Price coefficient c

(b) Price coefficient d

(c) Conditional standard error of Y

(d) Investor welfare V

Figure 2: Price coefficients c, d, conditional volatility of Y, $\sigma_{Y|X}$, and investor welfare V for different values of h_G under government intervention with partial information. The parameter values are set at $\alpha = 1$, $\sigma_u = 0.5$, $\sigma_Y = 1.25$, $\sigma_y = 1.0$. The dash-dot horizontal line in each plot represents the values of c, d, $\sigma_{Y|X}$ and V, respectively, with no government intervention ($h_G = 0$). The solid black line represents the case for $\beta_{GY} = 0.25$ and $\sigma_G = \sigma_{GY} = 1.0$, and the dashed blue line represents the case for $\beta_{GY} = 1.0$ and $\sigma_G = \sigma_{GY} = 0.1$. The maximum value of $V^* = -1.380$ is the same for both cases, reached at $h^*_G = 0.667$ and 1.098, respectively.

blue lines correspond to the case when the government's signal is more informative. For clarity, we will discuss the first case fully and then return to the second case for comparison.

In Figure 2(a), the solid black line shows the value of c for different values of h_G, when the government trades on a less informative signal. In this case, the government trading is mostly noise trading, which tends to reduce the value of c, the market risk premium. Hence, the overall

pattern of c as a function of h_G is quite similar to the pattern when the government trades on pure noise. That is, government trading in general lowers c.

However, given that the government's trading is also correlated with Y, although weakly, the magnitude of its impact on c becomes dependent on its direction. For a positive h_G, the government is buying the stock when it receives a positive signal about Y ($S_{GY} = 0.25\,Y + \varepsilon_{GY}$). This implies that when Y is positive, most of investors are shorting the stock to share their income risk, the government is on average buying. This helps to reduce the price impact of Y, yielding a lower c. Indeed, c decreases with h_G quite fast for $h_G > 0$. When h_G is negative, the government's trading policy is reversed. For the same S_{GY}, the government will sell the stock, which is in the same direction as the investors' trades. Consequently, it will further depress the price, implying a higher c.

Figure 2(b) plots the value of d as a function of h_G, with the solid black line. Its overall pattern is close to Figure 1(b), the case when the government trades on pure noise. This is not surprising given that a large part of the government's trading in this case is driven by noise. Since the remaining part is correlated with Y, the function is slightly asymmetric with respect to $h_G = 0$.

For the market's informational efficiency, in Figure 2(c) we plot $\sigma_{Y|X}$ for the same range of h_G, also shown by the solid black line. Clearly, absent of government intervention ($h_G = 0$), $\sigma_{Y|X} = 0$. When the government intervenes, a large part of its trading is noise, which then causes $\sigma_{Y|X}$ to increase sharply. This is very similar to the case when the government trades on pure noise. However, given that a small part of the government trading is informative about Y, its impact on $\sigma_{Y|X}$ also depends on the direction of its trade. As discussed above, for $h_G > 0$, the government is trading in the opposite direction as the investors, which softens the price impact of Y. This has the effect of reducing the price's informativeness and increasing $\sigma_{Y|X}$. For $h_G < 0$, we have the opposite effect, with slightly smaller magnitude.

Finally, we consider how the government's intervention policy influences investor welfare. In Figure 2(d), the solid black line shows the level of investor welfare for different choices of h_G. With a less informative signal, most part of the government's trade is on noise. Its welfare impact is similar to the results obtained in Section 5.2. Small interventions tend to increase investor welfare while large interventions tend to decrease investor welfare. Given that a small part of the trade is positively correlated with Y, the welfare impact now becomes dependent

on the sign of h_G. As discussed above, a positive h_G means that government is trading in the opposite direction as the investors. Relative to a negative h_G with the same magnitude, this reduces the market risk premium c more substantially, hence increases welfare, at least for small interventions.

Next, we consider the case when the government's private signal on the state of the economy Y is more informative ($\beta_{GY} = 1.0$ and $\sigma_{GY} = 0.1$). In this case, when the government trades on its signal, a large part of it will be correlated with the underlying state variable Y, and only a small part is noise. The results are represented by the dashed blue lines in the four plots in Figure 2. Depending on the sign of h_G, the impact of government trading becomes quite different.

Let us start with panel (a) of Figure 2. When the government trades positively on the signal, i.e., $h_G > 0$ and $\theta_G = h_G(Y + \varepsilon_{YG})$ with $\sigma_{GY} = 0.1$, it partially offsets the impact of Y on the stock price, leading to a smaller c, as before. When the government is trading negatively on the signal, however, its impact on c becomes minimal. This is because in this case, the government trades in the same direction as the investors, making the price more responsive to hence more informative about Y. This is further illustrated by the behavior of $\sigma_{Y|X}$ in panel (c) of Figure 2. In particular, for $h_G < 0$, $\sigma_{Y|X}$ remains very close to 0. With a more informative equilibrium, the impact of the government's trading on the equilibrium diminishes as we show in Section 5.3.

Price coefficient d in this case is shown in Figure 2(b). For $h_G < 0$, the price is mostly driven by Y, the coefficient d behaves very similarly as in the case of less informative signal for the government. For $h_G > 0$, the impact of Y on the price is dampened by the government's trading, the price becomes more responsive to its noise component. As a result, we see a sharp rise of d with h_G to much larger values than in the case of less informative signal. It only tapers off and starts to decrease when h_G becomes large.

The welfare impact of the government trading in this case is quite different from the case with less informative signal. For $h_G > 0$, the welfare improvement occurs when its magnitude becomes quite large (exceeding $h_G = 0.5$). As the government's trading intensity keeps increasing, investor welfare starts to improve. It reaches the maximum value of -1.380 at $h_G^* = 1.098$. Beyond this optimal trading intensity, the welfare level drops very quickly as h_G further increases. This suggests that, in this case with more informative signals, the welfare impact of the government's trading policy is very sensitive to its design. Small changes in its choice can have very different

welfare implications.

For $h_G < 0$, we see a very different situation. Investors' welfare level seems little affected, at least for a large range of h_G. This is consistent with the results on c and $\sigma_{Y|X}$. The government's trading, when it is in the same direction as the investors', largely reveals the underlying state of the economy (Y) and has little effect on the market risk premium. Consequently, it fails to improve investor welfare.

The discussion above about the impact of government trading when $h_G < 0$ is mostly for the completeness of our analysis. Our main results are for $h_G > 0$ when welfare improvement does occur.

Comparing the two cases we examined, when the government trades on less or more informative signals, respectively, we observe that the maximum welfare level achievable remains the same, which is $V^* = -1.380$. Moreover, it is also the same as in the case when the government trades on pure noise. This is no accident. The reason is as follows. As discussed in Section 5.2, when the government trades on pure noise, the price function takes the form $\tilde{P} = -c\,Y + d\,\varepsilon_G$. Knowing ε_G, the noise the government trades on, it can also infer Y from the price. In searching for the optimal intervention policy, the government is effectively using its precise information on both, Y and ε_G. When the government receives a noisy signal on Y, $S_{GY} = \beta_{GY} Y + \varepsilon_G$ (where $\varepsilon_G = \varepsilon_{GY}$), it also observes the stock price, $\tilde{P} = -c\,Y + d\,\varepsilon_G$. These two independent signals actually allow the government to infer both Y and ε_G precisely. Hence, in all the three cases with $\beta_{GY} = 0,\ 0.25,\ 1.0$, respectively, the government effectively has the same and full information about the economy. In addition, we have also learned from Section 5.3 that full information on Y in general does *not* help the government to improve investor welfare. The only effective tool the government has is trading on noise. Thus, in all three cases, it is choosing the right h_G, given its signals (including pure noise), that delivers the optimal amount of noise trading to achieve the same maximum welfare level for the investors.

5.5 Summary Comments

Our analysis in this section leads to a set of results from the model regarding the impact of government intervention. Under the assumption that the gain/loss from government trading is passed on to the investors, its influence on market equilibrium and investor welfare is in general independent of the information the government has. In particular, if it trades on public

information or full information on the underlying state of the economy, investors can infer the government's trade from prices and adjust their own trades accordingly to fully offset it. The government can shift the market outcome only when it trades on noise, either by adding pure noise to its trade or relying on the noise in its signal. In this case, the government's trade is not observable to the investors, hence won't be fully offset by their trades. By offering extra returns through its noise trading, the government is reducing the price cost of investors' risk sharing in the market, bringing welfare gains. At the same time, the government's noise trading also distorts the investors' net portfolio holdings and terminal wealth. Proper choices of moderate amount of noise trading can improve investor welfare. Nonetheless, the general welfare impact of government's intervention policy heavily depends on its inputs and design.

Our results also show that widely used policy targets based on market outcomes such as informational efficiency, price stability and market liquidity, can have substantial welfare implications in a general equilibrium context. In particular, welfare improving intervention policies may well cause these targets shift in seemingly undesirable directions. The reverse is true as well. We expand more on this topic in the next section.

6 Evaluating Government Intervention

Government often intervenes directly in the financial market during crisis. How to evaluate these interventions has always been a challenging question.[20] It requires a proper cost-benefit analysis with the right objective function. In this section, we consider some of the conventional assessment measures. We also develop a measure using our general equilibrium framework.

To demonstrate the key points, we consider two simple trading strategies for the government. The first is a contrarian strategy, buy low and sell high. Formally, it is given by:

$$\theta_G^{Contrarian} = -\,h_G\,\tilde{P}, \quad h_G > 0. \tag{6.1}$$

This strategy is reminiscent of government's interventions during market downturns such as buying distressed assets. The second is a noise trading strategy given in (5.4). Neither strategy requires any information advantage about the market for the government. The computational details are provided in the appendix. Below we only present the results.

Payoff. The first measure is the expected payoff on the government's trading strategy. Given

[20] See, for example, (Lucas, 2019) and the references there in.

the government's trading strategy θ_G, its terminal payoff is:

$$W_G = \theta_G(u - \tilde{P}). \tag{6.2}$$

For the contrarian strategy, its expected payoff is:

$$\mathrm{E}\left[W_G^{Contrarian}\right] = \mathrm{E}\left[\theta_G^{Contrarian}(u - \tilde{P})\right] = \mathrm{E}\left[-\,h_G\,\tilde{P}\,(u - \tilde{P})\right] = h_G\,c^2\sigma_Y^2 > 0. \tag{6.3}$$

Clearly, in this case, the average payoff on the contrarian strategy is also positive.

Next, we consider the noise-trading strategy. Its expected payoff is given by:

$$\mathrm{E}\left[W_G^{Noise}\right] = \mathrm{E}\left[\theta_G^{Noise}(u - P)\right] = \mathrm{E}\left[h_G\,\varepsilon_G\,(u - P)\right] = -\,h_G\,d\,\sigma_G^2 < 0, \tag{6.4}$$

which is always negative (since $h_G\,d \geq 0$), at least for small h_G.

The difference in the expected payoffs from these two strategies reflects important aspects of the government's trading in the market. In particular, in the case of contrarian strategy, the government trades only on public information. Thus, the investors can use their own trades to fully offset the government's trades. As a result, the government's trading has no impact on the price. As a contrarian, the government earns a positive expected payoff. However, this payoff is the same for any investor in the market who follows the same strategy. In other words, the government does not earn any extra payoff than what the market offers to all investors.

In the case of noise-trading strategy, the situation is very different. The government trades on noise, which is not public information. Thus, government's noise trading will move the market price, which cannot be fully offset by the investors' own trades, despite their efforts in doing so. Consequently, the government's price impact, which is in the same direction of its trades, will lead to negative expected payoffs.

The results above clearly demonstrate that we need to be cautious in assessing the government's trading in the market by simply using the observed market prices. These prices can be influenced by the government's trades. However, whether or not they are depends on how observable the government's trades are. More importantly, when the government's trading does move prices, using market prices to assess its merit can be misleading. As shown in the case of noise trading by the government, it always loses money but can actually improve welfare, as we have learned in Section 5.2.

Sharpe ratio. In addition to expected payoff of the government's trading strategy, we can also

include its risk in the assessment. A natural candidate is the Sharpe ratio (SR), given by:

$$SR = \frac{\mathrm{E}\,[W_G]}{\sigma[W_G]}, \tag{6.5}$$

where $\sigma[W_G]$ denotes the volatility of W_G. As shown in the appendix, simple computations yield the following:

$$SR^{\,Contrarian} = \frac{c\,\sigma_Y}{\sqrt{3\,c^2\sigma_Y^2 + \sigma_u^2}}, \qquad SR^{\,Noise} = -\frac{d\,\sigma_G}{\sqrt{c^2\sigma_Y^2 + 2\,d^2\sigma_G^2 + \sigma_u^2}}. \tag{6.6}$$

For reasons explained above, the Sharpe ratio for the government's contrarian strategy does not depend on its trading and is the same for all market participants. Keep in mind that the equilibrium price coefficients b and c here are not influenced by the government's trading strategy, as shown in Section 5.1. In the case of government's noise trading, however, the situation is different. Here, the government's noise trading, as characterized by h_G, will influence the price coefficients c and d. As shown in Section 5.2, for small values of h_G, d will be proportional to h_G. Thus, not only that the Sharpe ratio is negative, it also increases with the magnitude of the government's trading, at least when it is small.

Certainty equivalence of welfare gain. The ultimate measure of the government's intervention is the welfare gain or loss it generates for the investors. In particular, we define the certainty equivalence, CE, in terms of investor wealth as a measure of their welfare gain or loss. Specifically, we have:

$$V(W + CE(h_G), h_G = 0) = V(W, h_G), \tag{6.7}$$

where $V(W, h_G)$ is the investors' welfare level with wealth W and government trading policy h_G.

From the welfare results obtained in Section 5.1 and 5.2, we can easily compute CE for the two of the government's trading strategies under consideration. For the contrarian strategy, it yields no welfare gain. Hence, its CE will be zero. For the noise-trading strategy, however, there is a positive welfare gain, yielding a positive CE. Mathematically, we have:

$$CE^{\,Contrarian} = 0, \quad \text{for all } h_G; \tag{6.8a}$$

$$CE^{\,Noise} = \frac{1}{2}\alpha^3\sigma_u^4\sigma_y^2\frac{1+\alpha^2\sigma_u^2\sigma_Y^2}{1-\alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_Y^2)}h_G^2 + o\,(h_G^2), \quad \text{for small } h_G, \tag{6.8b}$$

where the second equation comes from (5.9). It is interesting to note that despite its positive expected payoff and Sharpe ratio, the contrarian strategy adds no value by the certainty equivalence measure. In contrast, the noise-trading strategy yields a positive certainty equivalence

measure despite its negative expected payoff and Sharpe ratio, which is proportional to the square of the trading magnitude h_G for small values of h_G.

The analysis above shows that the government's trading in the market cannot be simply evaluated using observed market prices. This is due to the fact that government trading will in general shift market prices, which influences investor welfare. When the government's trades are public information, they can be fully offset by investors' own trading and thus won't impact market prices. In this case, the government's trading can be evaluated using equilibrium market prices. However, in this case the intervention loses its value as it has no effect on investor welfare. When the government's trades are not public information, they will have non-trivial welfare implications. In this case, market prices will be influenced by the government's trading strategy and thus lose their validity in assessing the strategy's merits. A proper evaluation measure is based on the strategy's impact on investor welfare, which can be constructed in the context of a general equilibrium framework like ours.

7 Conclusion

In this paper, we have developed a general equilibrium model to examine the impact of government's direct trading in the financial market on market outcome and investor welfare. We show that when the market is incomplete and contains information asymmetry, the equilibrium does not lead to the first-best welfare level. While the market does allow full risk sharing, it is not cost free. In particular, investors face the risk of paying the market risk premium, which lowers their ex ante welfare. To the contrary, the first-best allocation yields the same perfect risk sharing without incurring the market cost.

Given the sub-optimality of the market equilibrium, the government's intervention through its own trading can improve welfare, even when it has no additional information. We show that any intervention based on public information does not alter market outcomes since investors can trade to offset the distortion caused by the government's trading. However, by trading on pure noise, the government is creating additional returns on the investors' trades. This effectively lowers the cost for investors to share/hedge risk in the market and can potentially improve their welfare.

When the government has additional information that individual investors do not have, the impact of its intervention policy is sensitive to the details of the economy and the design of the

policy. We also show that when the government plays a monopolistic role in the market, using market prices to assess the its intervention policies can be incomplete and misleading.

As stated in the introduction, the paper's main objective is to construct a parsimonious but comprehensive framework that allows us to examine the potential role for government intervention and its welfare implications. Thus, for tractability and clarity, we have made simplifying assumptions regarding the structure of the economy, the objective of the government and its information and intervention policies. As discussed in Section 2.8, many of these assumptions can be relaxed if the focus turns to specific imperfections of the market and the corresponding government intervention policies. We will leave these explorations for future research.

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Appendix

In this appendix, we provide the derivations and proofs for the main results in the paper.

A.1 Parameter constraint (2.10)

Under normality, the non-traded income given in (2.2) is unbounded, which can yield extreme values for the utility function. In particular,

$$W_i = Y_i\, u, \quad \mathrm{E}\left[u_i(W_i)|Y_i\right] = -e^{\frac{1}{2}\alpha^2 \mathrm{V}\left[Y_i u|Y_i\right]} = -e^{\frac{1}{2}\alpha^2\sigma_u^2 Y_i^2}, \tag{A.1}$$

which is unbounded when Y_i is large. The expected value of the utility function above (over Y_i) is bounded only if the condition (2.10) is satisfied. (2.11) follows when (2.10) holds.

A.2 Solution to Market Equilibrium with Government Intervention

Conditional state variables – Proof of Lemmas 1 and 2

The following lemma, which is a standard result about joint normal distributions, will be repeatedly used in our computations:

Lemma A.1. *Let x be an N-dimensional normal random variable:*

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right), \tag{A.2}$$

where $\bar{x}_1$, $\bar{x}_2$, Σ_{11}, Σ_{22} are the mean and covariance matrix of x_1 and x_2, respectively, and Σ_{12} is the covariance matrix between x_1 and x_2. Then, conditional on x_2, x_1 remains normal with:

$$\mathrm{E}\left[x_1|x_2\right] = \bar{x}_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \bar{x}_2), \quad \mathrm{V}\left[x_1, x_1|x_2\right] = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \equiv \Sigma_{11|2}, \tag{A.3}$$

where $\mathrm{V}\left[x_1, x_1|x_2\right]$ denotes the variance matrix of x_1 conditional on x_2.

Given the price function P in (4.1), signal vector X_i in (4.2) and state vector Z in (4.4), they are jointly normal. We immediately have:

$$\Sigma_{XX} \equiv A = \begin{bmatrix} c^2\sigma_Y^2 + d^2\sigma_{GY}^2 & -c\,\sigma_Y^2 \\ -c\,\sigma_Y^2 & \sigma_Y^2{+}\sigma_y^2 \end{bmatrix}, \quad |A| = c^2\sigma_Y^2\sigma_y^2 + d^2\sigma_{GY}^2(\sigma_Y^2{+}\sigma_y^2), \tag{A.4a}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} \sigma_Y^2{+}\sigma_y^2 & c\,\sigma_Y^2 \\ c\,\sigma_Y^2 & c^2\sigma_Y^2 + d^2\sigma_{GY}^2 \end{bmatrix}, \tag{A.4b}$$

and the joint distribution of Z and X_i, $i \in [0,1]$, is then given by:

$$\begin{bmatrix} Z \\ X_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{ZZ} & \Sigma_{ZX} \\ \Sigma_{XZ} & A \end{bmatrix}\right), \quad \Sigma_{ZX} = \begin{bmatrix} -c\,\sigma_Y^2 & \sigma_Y^2 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \sigma_{YX} \\ \sigma_{uX} \end{bmatrix}. \tag{A.5}$$

Lemma 1 immediately follows from Lemma A.1. For convenience, we define:

$$\beta_{ZX} = \Sigma_{ZX} A^{-1} = \begin{bmatrix} \sigma_{YX} A^{-1} \\ \sigma_{uX} A^{-1} \end{bmatrix} \equiv \begin{bmatrix} \beta_{YX} \\ \beta_{uX} \end{bmatrix} \equiv \begin{bmatrix} -c\,\sigma_Y^2 \sigma_y^2/|A| & d^2 \sigma_{GY}^2 \sigma_Y^2/|A| \\ 0 & 0 \end{bmatrix}, \tag{A.6}$$

since $\sigma_{uX} = 0$. In addition, we have:

$$\Sigma_{ZZ|X} = \Sigma_{ZZ} - \Sigma_{ZX} A^{-1} \Sigma_{XZ} \equiv \begin{bmatrix} \sigma_{Y|X}^2 & \sigma_{Yu|X} \\ \sigma_{Yu|X} & \sigma_{u|X}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{Y|X}^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}, \quad \sigma_{Y|X}^2 = d^2 \sigma_{GY}^2 \sigma_Y^2 \sigma_y^2/|A|, \tag{A.7}$$

which gives the variance and covariance of Y and u conditional on X_i, and:

$$\Sigma_{ZZ|X}^{-1} = \frac{1}{|\Sigma_{ZZ|X}|} \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_{Y|X}^2 \end{bmatrix}, \quad |\Sigma_{ZZ|X}| = \sigma_{Y|X}^2 \sigma_u^2. \tag{A.8}$$

Moreover,

$$\mathrm{E}\,[Y|X_i] = \bar{Y}_i = \beta_{YX} X_i, \quad \mathrm{E}\,[u|X_i] = \bar{u}_i = 0. \tag{A.9}$$

Given (2.5), (2.6) and (4.3), (4.6) is immediate with:

$$\theta_G = h'_{GP} \tilde{P} + h'_{GZ} Z, \quad h'_{GZ} = [\, h'_{GY} \quad h'_{Gu} \,], \tag{A.10}$$

where

$$h'_{GP} = h_{GP} + (1/d)\, h_{GY}, \quad h'_{GY} = (\beta_{GY} + c/d)\, h_{GY}, \quad h'_{Gu} = 0. \tag{A.11}$$

Lemma 2 then follows from Lemma A.1.

Investors' optimal trading policy – Proof of Proposition 2

Given investor i's wealth in (4.10), $i \in [0,1]$, for (4.11) we further have:

$$H_G \equiv h_{uZ}^\top h'_{GZ} + h_{GZ}'^\top h_{uZ} = \begin{bmatrix} 0 & h'_{GY} \\ h'_{GY} & 0 \end{bmatrix}. \tag{A.12}$$

We can now rewrite (4.10) as:

$$W_i = W'_i + \tfrac{1}{2} \tilde{Z}_i^\top H_G \tilde{Z}_i = \bar{W}'_i + \tilde{W}'_i + \tfrac{1}{2} \tilde{Z}_i^\top H_G \tilde{Z}_i, \quad i \in [0,1], \tag{A.13}$$

where

$$W'_i \equiv (\theta_i + \bar{\theta}_{Gi})(-\tilde{P}) + (\theta_i + \bar{\theta}_{Gi} + Y_i) h_{uZ} \tilde{Z}_i + (-\tilde{P}) h'_{GZ} \tilde{Z}_i \tag{A.14a}$$

$$\bar{W}'_i \equiv \mathrm{E}\,[W'_i|X_i] = (\theta_i + \bar{\theta}_{Gi})(-\tilde{P}), \tag{A.14b}$$

$$\tilde{W}'_i \equiv W'_i - \bar{W}'_i = \left[(\theta_i + \bar{\theta}_{Gi} + Y_i) h_{uZ} + (-\tilde{P}) h'_{GZ} \right] \tilde{Z}_i. \tag{A.14c}$$

Note that conditional on X_i, $\tilde{W}'_i$ is normal with mean zero.

We now solve for the investors' optimal trading policy. For $\Sigma_{ZZ|X}^{-1} + \alpha H_G$ positive definite, which will be guaranteed with proper parameter restrictions, we define $\Sigma_{ZZ|XG}$ as in (4.12), which is also positive definite (and symmetric). It can be interpreted as a modified covariance matrix for $\tilde{Z}_i$ from $\Sigma_{ZZ|X}$ with αH_G. In addition, we have:

$$\Sigma_{ZZ|XG}^{-1} \equiv \Sigma_{ZZ|X}^{-1} + \alpha H_G = \begin{bmatrix} \sigma_u^2/|\Sigma_{ZZ|X}| & \alpha\, h'_{GY} \\ \alpha\, h'_{GY} & \sigma_{Y|X}^2/|\Sigma_{ZZ|X}| \end{bmatrix}, \tag{A.15a}$$

$$|\Sigma_{ZZ|XG}^{-1}| = |\Sigma_{ZZ|X}|^{-1} - \alpha^2 h'^2_{GY} = |\Sigma_{ZZ|XG}|^{-1}, \tag{A.15b}$$

$$\Sigma_{ZZ|XG} \equiv \begin{bmatrix} \sigma_{Y|XG}^2 & \sigma_{Yu|XG} \\ \sigma_{Yu|XG} & \sigma_{u|XG}^2 \end{bmatrix} = |\Sigma_{ZZ|XG}| \begin{bmatrix} \sigma_{Y|X}^2/|\Sigma_{ZZ|X}| & -\alpha h'_{GY} \\ -\alpha h'_{GY} & \sigma_u^2/|\Sigma_{ZZ|X}| \end{bmatrix}, \tag{A.15c}$$

where $|\Sigma_{ZZ|X}|$ is given in (A.8).

The normality of $\tilde{W}'_i$ yields the following lemma:

Lemma A.2. *Given the trading policy θ_i of investor i, $i \in [0,1]$, her expected utility is:*

$$\begin{aligned} \mathrm{E}\left[-e^{-\alpha W_i}\middle|X_i\right] &= -\,\mathrm{E}\left[e^{-\alpha W'_i - \frac{1}{2}\alpha \tilde{Z}_i^\top H_G \tilde{Z}_i}\middle| X_i; \Sigma_{ZZ|X}\right] \\ &= -\,\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\,\mathrm{E}\left[e^{-\alpha \bar{W}'_i - \alpha \tilde{W}'_i}\middle| X_i; \Sigma_{ZZ|XG}\right] \\ &\equiv -\,\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\, e^{-\alpha \hat{W}_i}, \end{aligned} \tag{A.16}$$

where $\mathrm{E}[\cdot|X_i;\Sigma]$ denotes the expectation conditional on X_i with covariance matrix Σ, and

$$\hat{W}_i \equiv \bar{W}'_i - \tfrac{1}{2}\,\alpha \mathrm{V}\left[\tilde{W}'_i|X_i;\Sigma_{ZZ|XG}\right], \tag{A.17}$$

where $\mathrm{V}[\cdot|X_i;\Sigma]$ denotes the corresponding conditional variance.

From Lemma A.2, the investor's optimization problem new reduces to:

$$\begin{aligned} \max_{\theta_i}\ \hat{W}_i &= (\theta_i + \bar{\theta}_{Gi})(-\tilde{P}) \\ &\quad - \tfrac{1}{2}\,\alpha\,[(\theta_i + \bar{\theta}_{Gi} + Y_i)h_{uZ} + (-\tilde{P})h'_{GZ}]\Sigma_{ZZ|XG}[(\theta_i + \bar{\theta}_{Gi} + Y_i)h_{uZ} + (-\tilde{P})h'_{GZ}]^\top, \end{aligned} \tag{A.18}$$

which is a mean-variance problem. The first order condition is:

$$0 = (-\tilde{P}) - \alpha h_{uZ}\Sigma_{ZZ|XG}[(\theta_i + \bar{\theta}_{Gi} + Y_i)h_{uZ} + (-\tilde{P})h'_{GZ}]^\top. \tag{A.19}$$

Simple algebra leads to the following solution for the optimal trading policy:

$$\theta_i = -\,\bar{\theta}_{Gi} + \frac{1}{\alpha\sigma_u^2}(-\tilde{P}) - Y_i. \tag{A.20}$$

Substituting the optimal trading policy into $\hat{W}_i$, we have:

$$\begin{aligned}\hat{W}_i &= \frac{1}{\alpha\sigma_u^2}\left[1-\tfrac{1}{2}(1/\sigma_u^2)\left(h_{uZ}+\alpha\sigma_u^2 h'_{GZ}\right)\Sigma_{ZZ|XG}\left(h_{uZ}+\alpha\sigma_u^2 h'_{GZ}\right)^\top\right](\tilde{P})^2+\tilde{P}\,Y_i \\ &= \frac{1}{\alpha\sigma_u^2}\left[1-\tfrac{1}{2}(1/\sigma_u^2)\left(\sigma^2_{u|XG}+2\,\alpha\sigma_u^2 h'_{GY}\sigma_{Yu|XG}+\alpha^2\sigma_u^4 h'^2_{GY}\sigma^2_{Y|XG}\right)\right](\tilde{P})^2+\tilde{P}\,Y_i \\ &= \tfrac{1}{2}\,\frac{1}{\alpha\sigma_u^2}\tilde{P}^2+\tilde{P}Y_i, \end{aligned} \tag{A.21}$$

which gives the investor i's value function. The derivation of Proposition 2 is now complete.

Price and stock holdings in equilibrium – Proof of Proposition 3

From Lemma 1 and 2, we can rewrite the investors' stock demand in the form of (4.15) with:

$$h_P \equiv \frac{1}{\alpha\sigma_u^2}+h'_{GP}-\frac{c\,\sigma_Y^2\sigma_y^2}{|A|}h'_{GY},\quad h_Y \equiv 1+\frac{d^2\sigma^2_{GY}\sigma_Y^2}{|A|}h'_{GY}. \tag{A.22}$$

For equilibrium, the market-clearing condition (4.16) requires:

$$0=-[(-h_P+h_{GP})\,c+(h_Y-\beta_{GY}h_{GY})]\,Y+[(-h_P+h_{GP})\,d+h_{GY}]\,\varepsilon_{GY}. \tag{A.23}$$

It then leads to the following conditions for c and d:

$$0=(-h_P+h_{GP})\,c+(h_Y-\beta_{GY}h_{GY}),\quad 0=(-h_P+h_{GP})\,d+h_{GY}, \tag{A.24}$$

which gives (4.17). Moreover, combining the two equations in (2.10) yields the following equation:

$$k\equiv d/c=\frac{h_{GY}}{h_Y-\beta_{GY}h_{GY}}. \tag{A.25}$$

Some algebra yields the following equation for k:

$$a_3k^3+a_2k^2+a_1k+a_0=0, \tag{A.26}$$

where

$$a_3=\sigma^2_{GY}[(\sigma_Y^2+\sigma_y^2)-\sigma_y^2\beta_{GY}h_{GY}],\quad a_2=-\sigma^2_{GY}\sigma_y^2h_{GY}, \tag{A.27a}$$

$$a_1=\sigma_Y^2\sigma_y^2(1-\beta_{GY}h_{GY}),\quad a_0=-\sigma_Y^2\sigma_y^2h_{GY}. \tag{A.27b}$$

(A.26) is a cubic equation for k. Its solution is given by the following lemma:

Lemma A.3. *For a cubic equation of the following canonical form:*

$$a_3x^2+a_2x^2+a_1x+a_0=0, \tag{A.28}$$

define:

$$p\equiv\frac{a_1}{3a_3}-\frac{a_2^2}{9a_3^3},\quad q\equiv\frac{a_2a_1}{6a_3^2}-\frac{a_2^2}{27a_3^3}-\frac{a_0}{2a_3},\quad r\equiv-\frac{a_2}{3a_3},\quad \Delta\equiv q^2+p^3,\quad \omega\equiv\frac{-1+\sqrt{3}i}{2}, \tag{A.29}$$

where Δ is the discriminant of the cubic equation and ω is a complex cubic root of -1. (A.28) then has three roots given by:

$$x_1 = \sqrt[3]{-q+\sqrt{q^2+p^3}} + \sqrt[3]{-q-\sqrt{q^2+p^3}} + r\,, \tag{A.30a}$$

$$x_2 = \omega\sqrt[3]{-q+\sqrt{q^2+p^3}} + \omega^2\sqrt[3]{-q-\sqrt{q^2+p^3}} + r\,, \tag{A.30b}$$

$$x_3 = \omega^2\sqrt[3]{-q+\sqrt{q^2+p^3}} + \omega\sqrt[3]{-q-\sqrt{q^2+p^3}} + r\,. \tag{A.30c}$$

Moreover, when $\Delta > 0$, there is only one real root, when $\Delta < 0$, there are three distinct real roots, and when $\Delta = 0$, there are multiple real roots.

For $\Delta > 0$ with a_3, a_2, a_1 and a_0 given in (A.27), we have a unique solution for k. Substituting it into (A.24) gives the solution for c and d. In particular, we have:

$$c = \alpha\sigma_u^2\left[1 - \frac{(\beta_{GY}+k)\,k^2\sigma_{GY}^2\sigma_y^2}{\sigma_Y^2\sigma_y^2 + k^2\,(\sigma_Y^2+\sigma_y^2)}h_{GY}\right], \quad b = k\,c. \tag{A.31}$$

This completes the proof of Proposition 3.

Investor welfare – Proof of Corollary 4.1 and Proposition 4

With the state variables included in z_i for investor i, we can write the solution to the equilibrium in the form of (4.20) with:

$$B_Y \equiv [\,1 \;\; 1 \;\; 0\,], \quad B_G \equiv h_{GP}B_P + [\,\beta_{GY}h_{GY} \;\; 0 \;\; h_{GY}\,], \quad B_P \equiv [\,-c \;\; 0 \;\; d\,], \tag{A.32a}$$

$$B_\theta \equiv \frac{1}{\sigma_u^2}(-B_P) - B_Y, \quad B \equiv \frac{1}{\alpha\sigma_u^2}B_P^\top B_P + \left(B_P^\top B_Y + B_Y^\top B_P\right). \tag{A.32b}$$

This gives Corollary 4.1.

With the above, we can compute the investors' ex ante expected utility under government's intervention policy (2.6). In particular, we have:

$$\begin{aligned} V(\theta_G) = V(h_G) &= -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\,\mathrm{E}\left[e^{-\alpha\hat{W}_i}\right] = -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\,\mathrm{E}\left[e^{-\frac{1}{2}\alpha z_i^\top B z_i}\right] \\ &= -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\frac{|\Sigma_{zzB}|^{1/2}}{|\Sigma_{zz}|^{1/2}}, \quad i\in[0,1], \end{aligned} \tag{A.33}$$

where Σ_{zzB} is given in (4.21). This proves Proposition 4.

A.3 Equilibrium with No Government Intervention – Proof of Proposition 1

Market equilibrium with no government intervention and investor welfare can be obtained by simply letting $h_{GY} = 0$ in the general solution. We omit the details here for brevity. The main results are summarized in Proposition 1.

A.4 Government Intervention with Public Information – Proof of Proposition 5

In this case, the investors' optimization problem is similar to the maximization problem with no government intervention, which reduces to:

$$\max_{\theta_i} \ \hat{W}_i = (\theta_i + \theta_G)(-\tilde{P}) - \tfrac{1}{2}\,\alpha\sigma_u^2\,\left[(\theta_i + \theta_G)^2 + 2(\theta_i + \theta_G)Y_i + Y_i^2\right], \quad i \in [0,1]. \tag{A.34}$$

The the first order condition for θ_i is:

$$-\tilde{P} - \alpha\sigma_u^2 Y_i = \alpha\sigma_u^2(\theta_i + \theta_G), \tag{A.35}$$

which yields the solution to θ_i in (5.3). The rest of the results for this case follow easily from the general solution in Section 4.

A.5 Government Intervention with Pure Noise – Proof of Propositions 6 and 7

The solution in this case follows from the general solution in Section 4 by substituting in the government's particular trading policy. In particular, we only need to set $\beta_{GY} = 0$ and $\sigma_{GY}^2 = 1$ in the general solution. Details are provided below.

Market equilibrium. For (4.6) and (4.9), we now have:

$$h'_{GP} = (1/d)\,h_G,\ \ h'_{GY} = (c/d)\,h_G,\ \ h'_{GZ} = [(c/d)\,h_G\ \ 0],\ \ h'_{GX} = \frac{(c/d)d^2 h_G}{|A|}[\sigma_Y^2 + \sigma_y^2\ \ c\,\sigma_Y^2]. \tag{A.36}$$

Moreover,

$$H_G = \begin{bmatrix} 0 & (c/d)\,h_G \\ (c/d)\,h_G & 0 \end{bmatrix} \tag{A.37}$$

and

$$\Sigma_{ZZ|XG}^{-1} = \Sigma_{ZZ|X}^{-1} + \alpha H_G = \begin{bmatrix} |A|/(d^2\sigma_Y^2\sigma_y^2) & \alpha(c/d)\,h_G \\ \alpha(c/d)\,h_G & 1/\sigma_u^2 \end{bmatrix}, \tag{A.38a}$$

$$\Sigma_{ZZ|XG} = |\Sigma_{ZZ|XG}^{-1}| \begin{bmatrix} 1/\sigma_u^2 & -\alpha(c/d)\,h_G \\ -\alpha(c/d)\,h_G & |A|/(d^2\sigma_Y^2\sigma_y^2) \end{bmatrix}, \tag{A.38b}$$

where

$$|\Sigma_{ZZ|XG}^{-1}| = \frac{|A|}{d^2\sigma_Y^2\sigma_y^2\sigma_u^2} - \alpha^2(c/d)^2 h_G^2 = |\Sigma_{ZZ|XG}|^{-1}. \tag{A.39}$$

We then have:

$$\sigma_{Y|XG}^2 = |\Sigma_{ZZ|XG}^{-1}|\frac{1}{\sigma_u^2},\ \ \sigma_{u|XG}^2 = |\Sigma_{ZZ|XG}^{-1}|\frac{|A|}{d^2\sigma_Y^2\sigma_y^2},\ \ \sigma_{Yu|XG} = -|\Sigma_{ZZ|XG}^{-1}|\,\alpha(c/d)\,h_G. \tag{A.40}$$

Given investors' optimal trading policy, we now derive the equilibrium price coefficients, c

and d. From (A.22), we have:

$$h_P = \frac{1}{\alpha\sigma_u^2} + \frac{(1/d)d^2 h_G}{|A|}(\sigma_Y^2+\sigma_y^2), \quad h_Y = 1 + \frac{(c/d)d^2 h_G}{|A|}\sigma_Y^2. \tag{A.41}$$

From (4.17), we now have:

$$c = h_Y/h_P, \quad d = h_G/h_P, \quad d/c \equiv k = h_G/h_Y. \tag{A.42}$$

The last equation leads to the following cubic equation for k in the form of (A.26), which is:

$$(\sigma_Y^2+\sigma_y^2)k^3 - h_G\sigma_y^2 k^2 + \sigma_Y^2\sigma_y^2 k - h_G\sigma_Y^2\sigma_y^2 = 0. \tag{A.43}$$

Its solution is given in Lemma A.3, which is unique for a range of h_G. In particular, $k = 0$ when $h_G = 0$. Substituting the solution for k in (A.42) yields the solution to c and d:

$$c = (\alpha\sigma_u^2)\left[1 - \frac{k\,\sigma_y^2}{\sigma_Y^2\sigma_y^2 + k^2\,(\sigma_Y^2+\sigma_y^2)}h_G\right], \quad d = k\,c. \tag{A.44}$$

The investors' stock holdings in equilibrium is then given by:

$$\theta_i = -h_P\tilde{P} - h_Y Y_i = -h_G\,\varepsilon_G - \left(1 + |A|^{-1}c^2\sigma_Y^2 k\,h_G\right) y_i. \tag{A.45}$$

Welfare. We now compute the investors' welfare given the government's trading policy h_G. In this case, we have $z_i = [Y \;\; y_i \;\; \varepsilon_G]^\top$ and $\Sigma_{zz} = \text{Diag}[\sigma_Y^2, \sigma_y^2, 1]$. Consequently, (A.32) becomes:

$$B_Y = [1 \;\; 1 \;\; 0], \;\; B_G = [0 \;\; 0 \;\; h_G], \;\; B_P = [-c \;\; 0 \;\; d], \;\; B_\theta = [0 \;\; 1 + c^2\sigma_Y^2 k\,h_G/|A| \;\; h_G], \tag{A.46}$$

and B takes the same form as in (A.32). Hence,

$$B_P^\top B_P = c^2\begin{bmatrix} 1 & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & k^2 \end{bmatrix}, \quad B_P^\top B_Y + B_Y^\top B_P = -c\begin{bmatrix} 2 & 1 & -k \\ 1 & 0 & -k \\ -k & -k & 0 \end{bmatrix},$$

and

$$\alpha B = \begin{bmatrix} c^2/\sigma_u^2 - 2\alpha c & -\alpha c & -(c^2/\sigma_u^2 - \alpha c)k \\ -\alpha c & 0 & \alpha c\,k \\ -(c^2/\sigma_u^2 - \alpha c)k & \alpha c\,k & (c^2/\sigma_u^2)k^2 \end{bmatrix}.$$

From $\Sigma_{zzB}^{-1} = \Sigma_{zz}^{-1} + \alpha B$, we obtain:

$$|\Sigma_{zzB}^{-1}| = \left[\frac{1}{\sigma_y^2}\left(\frac{1}{\sigma_Y^2} + \frac{c^2}{\sigma_u^2} - 2\alpha c\right) - (\alpha c)^2\right] + \frac{c^2}{\sigma_Y^2\sigma_y^2\sigma_u^2}\left[1 - \alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)\right]k^2. \tag{A.47}$$

From (A.39) and (A.47) with $|\Sigma_{zz}| = \sigma_Y^2\sigma_y^2$, we can compute the investors' welfare from (4.21). This proves Proposition 6.

Small intervention. We now consider the situation when h_G is small. This entails to obtaining

approximate solution to the lowest (non-trivial) order of h_G. From (A.43), we have:

$$k = h_G + o\left(h_G^2\right)$$

where $o\left(h_G^2\right)$ denotes terms of higher orders of h_G^2. From (A.42), we have:

$$c = \left(\alpha\sigma_u^2\right)\left[1-(1/\sigma_Y^2)h_G^2\right] + o\left(h_G^2\right), \quad d = \left(\alpha\sigma_u^2\right)h_G + o\left(h_G^2\right),$$

where $c_0 = \alpha\sigma_u^2$ is the solution for c when $h_G = 0$. As a result, we have:

$$c^2 = \left(\alpha\sigma_u^2\right)^2\left[1-(2/\sigma_Y^2)h_G^2\right] + o\left(h_G^2\right).$$

From (A.39), we have:

$$\begin{aligned}\frac{|\Sigma_{ZZ|XG}|}{|\Sigma_{ZZ|X}|} &= \left[1-(\alpha c)^2\frac{d^2\sigma_Y^2\sigma_y^2\sigma_u^2}{|A|}(h_G/d)^2\right]^{-1} = \left[1-\frac{\alpha^2\sigma_Y^2\sigma_y^2\sigma_u^2}{\sigma_Y^2\sigma_y^2+(\sigma_Y^2+\sigma_y^2)k^2}h_G^2\right]^{-1} \\ &= 1+\alpha^2\sigma_u^2h_G^2 + o\left(h_G^2\right).\end{aligned}$$

From (A.47), we have:

$$\frac{|\Sigma_{zzB}|}{|\Sigma_{zz}|} = \left\{\left[1+\sigma_Y^2\left(\frac{c^2}{\sigma_u^2}-2\alpha c\right)-\sigma_Y^2\sigma_y^2(\alpha c)^2\right]+\left(\frac{c^2}{\sigma_u^2}\right)\left[1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)\right]k^2\right\}^{-1}. \quad \text{(A.48)}$$

Note that when $h_G = 0$, we have:

$$\frac{|\Sigma_{zzB}|_0}{|\Sigma_{zz}|_0} = \left[1-\alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)\right]^{-1} = Q^2, \quad Q \equiv 1/\sqrt{1-\alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)}.$$

Thus,

$$\frac{|\Sigma_{zzB}|}{|\Sigma_{zz}|} = Q^2\left\{1-Q^2\alpha^2\sigma_u^2\left[1-\alpha^2\sigma_u^2(\sigma_Y^2-\sigma_y^2)\right]h_G^2\right\} + o\left(h_G^2\right).$$

We then have:

$$\begin{aligned}\frac{|\Sigma_{ZZ|XG}|}{|\Sigma_{ZZ|X}|}\frac{|\Sigma_{zzB}|}{|\Sigma_{zz}|} &= Q^2\left\{1+\alpha^2\sigma_u^2h_G^2\right\}\left\{1-Q^2\alpha^2\sigma_u^2\left[1-\alpha^2\sigma_u^2(\sigma_Y^2-\sigma_y^2)\right]h_G^2\right\} + o\left(h_G^2\right) \\ &= Q^2\left\{1+\alpha^2\sigma_u^2h_G^2 - Q^2\alpha^2\sigma_u^2\left[1-\alpha^2\sigma_u^2(\sigma_Y^2-\sigma_y^2)\right]h_G^2\right\} + o\left(h_G^2\right) \\ &= Q^2\left\{1+Q^2(\alpha^2\sigma_u^2)\left[1/Q^2-1+(\alpha^2\sigma_u^2)(\sigma_Y^2-\sigma_y^2)\right]h_G^2\right\} + o\left(h_G^2\right) \\ &= Q^2\left[1-Q^2(\alpha^2\sigma_u^2)^2\sigma_y^2(1+\alpha^2\sigma_u^2\sigma_Y^2)\,h_G^2\right] + o\left(h_G^2\right)\end{aligned}$$

It then follows that:

$$V(h_G) = -\left\{\frac{|\Sigma_{ZZ|XG}|}{|\Sigma_{ZZ|X}|}\frac{|\Sigma_{zzB}|}{|\Sigma_{zz}|}\right\}^{1/2} = -Q\left[1-\frac{1}{2}Q^2(\alpha^2\sigma_u^2)^2\sigma_y^2(1+\alpha^2\sigma_u^2\sigma_Y^2)h_G^2\right] + o\left(h_G^2\right). \quad \text{(A.49)}$$

Obviously, $V(0) = V^0 = -Q$, $V'(0) = 0$, and $V''(0) = \frac{1}{2}Q^3(\alpha^2\sigma_u^2)^2\sigma_y^2(1+\alpha^2\sigma_u^2\sigma_Y^2) > 0$. This completes the proof of Proposition 7.

A.6 Government Intervention with Full Information – Proof of Proposition 8

Here, we consider the special case when the government has full information on the aggregate state variable Y and chooses to trade to fully remove its influence on the price.

In this case, we have $\theta_G = h_G Y$ and $\tilde{P} = 0$ under a particular value of h_G we seek to determine. Here, the state vector remains $Z = [Y \;\; u]^\top$ with $\Sigma_{ZZ} = \text{Diag}[\sigma_Y^2, \sigma_u^2]$. In addition, $X_i = Y + y_i$, $A = \Sigma_{XX} = \sigma_Y^2 + \sigma_y^2$, and

$$\Sigma_{ZX} = \begin{bmatrix} \sigma_Y^2 \\ 0 \end{bmatrix}, \quad \beta_{ZX} = \begin{bmatrix} \beta_{YX} \\ 0 \end{bmatrix}, \quad \Sigma_{ZZ|X} = \begin{bmatrix} \beta_{YX}\sigma_y^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}, \quad \beta_{YX} \equiv \frac{\sigma_Y^2}{\sigma_Y^2 + \sigma_y^2}.$$

Given that $\theta_G = h_G\, Y$, we have:

$$\theta_G = \bar{\theta}_{Gi} + \tilde{\theta}_{Gi}, \quad \bar{\theta}_{Gi} = h_G \beta_{YX} Y_i, \quad \tilde{\theta}_{Gi} = h_G \tilde{Y}_i, \quad \tilde{Y}_i \equiv Y - \beta_{YX} Y_i.$$

For the investors' optimization problem, we have:

$$W_i = (\theta_i + \bar{\theta}_{Gi} + Y_i)\, u + h_G\, \tilde{Y}_i u = (\theta_i + \bar{\theta}_{Gi} + Y_i)\, u + \tfrac{1}{2} \tilde{Z}_i^\top H_G \tilde{Z}_i$$

and

$$H_G = \begin{bmatrix} 0 & h_G \\ h_G & 0 \end{bmatrix}, \quad \tilde{Z}_i = \begin{bmatrix} \tilde{Y}_i \\ u \end{bmatrix}.$$

In addition,

$$\Sigma_{ZZ|X}^{-1} = \begin{bmatrix} 1/(\beta_{YX}\sigma_y^2) & 0 \\ 0 & 1/(\sigma_u^2) \end{bmatrix}, \quad \Sigma_{ZZ|XG}^{-1} = \begin{bmatrix} 1/(\beta_{YX}\sigma_y^2) & \alpha\, h_G \\ \alpha\, h_G & 1/(\sigma_u^2) \end{bmatrix}.$$

From Proposition 2, the investors' optimal trading policy is:

$$\theta_i = -\bar{\theta}_{Gi} - Y_i, \quad i \in [0, 1].$$

The market clearing condition with $\tilde{P} = 0$ requires:

$$h_G^* = \frac{\sigma_Y^2 + \sigma_y^2}{\sigma_y^2} = \frac{1}{1 - \beta_{YX}}.$$

This determines the government's trading policy in order to achieve $\tilde{P} = 0$. The investors' equilibrium stock holdings are then:

$$\theta_i = -\left(\frac{\sigma_Y^2}{\sigma_y^2} + 1\right)(Y + y_i), \quad i \in [0, 1].$$

For welfare, we have $z_i = [Y \;\; y_i]^\top$ and $\Sigma_{zz} = \text{Diag}[\sigma_Y^2, \sigma_y^2]$. From (A.32), we have $B_P = 0$,

$B = 0$, $\Sigma_{zzB} = \Sigma_{zz}$, and

$$|\Sigma_{ZZ|X}| = \beta_{YX}\sigma_u^2\sigma_y^2, \quad |\Sigma_{ZZ|XG}| = |\Sigma_{ZZ|X}|\frac{1}{1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)(\sigma_Y^2/\sigma_y^2)}.$$

Thus, Proposition 4 gives the investors' welfare under the specific intervention policy h_G^*:

$$V(h_G^*) = -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}} = -\frac{1}{\sqrt{1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)(\sigma_Y^2/\sigma_y^2)}},$$

which is (5.13). Hence, Proposition 8 follows.

A.7 Evaluation of Government Intervention Policy

Given the government's intervention policies: the simple contrarian trading strategy linear in stock price and the noise trading strategy linear in noise, their performance measures and welfare impact can be easily computed, as shown below.

Expected Payoff. For the linear contrarian strategy given in (6.1), it has no impact on the equilibrium stock price, which takes the form in (3.2). For convenience, we define the return/net payoff on one share of the stock by:

$$R \equiv u - \tilde{P}.$$

Then, the net payoff on the contrarian strategy is given by:

$$W_G = -\,h_G\tilde{P}\,(u-\tilde{P}) = -\,h_G B_P Z Z^\top B_R,$$

where $Z = [Y \;\; u]^\top$, $B_P = [-c \;\; 0]$ and $B_R = [c \;\; 1]$. Immediately, we have:

$$\mathrm{E}\,[W_G] = -\,h_G B_P \Sigma_{ZZ} B_R^\top = h_G c^2 \sigma_Y^2,$$

which gives (6.3).

For the noise-trading strategy, we have:

$$W_G = \theta_G\,(u-\tilde{P}) = h_G\varepsilon_G\,(u + c\,Y - d\,\varepsilon_G).$$

Let $Z = [Y \;\; u \;\; \varepsilon_G]^\top$, $B_N = [0 \;\; 0 \;\; 1]$, $B_P = [-c \;\; 0 \;\; d]$, and $B_R = [c \;\; 1 \;\; -d]$. We have:

$$W_G = h_G B_N Z Z^\top B_R^\top.$$

It then follows that:

$$\mathrm{E}\,[W_G] = h_G B_N \mathrm{E}\,[ZZ^\top] B_R^\top = h_G B_N \Sigma_{ZZ} B_R^\top = -\,h_G\,d\,\sigma_G^2,$$

which is negative, at least for h small.

Sharpe ratio. In order to compute the Sharpe ratio of the two strategies, we need to compute the variance of their payoffs, which are the products of two normal variables. For this purpose, we need the following lemma:

Lemma A.4. *Let z_1 and z_2 be standard normal with correlation ρ. Then,*

$$\mathrm{V}\left[z_1 z_2\right] = 1 + 2\,\rho^2. \tag{A.50}$$

Using the lemma, we can now compute the payoff variance for the two strategies under consideration. In general, we can write their payoffs as:

$$W_G = \pm\, h_G (B_I Z)(B_R Z)^\top,$$

wher B_I denotes the corresponding B for the two investment strategies. Let σ_I and σ_R denote the volatility of $B_I Z$ and $B_R Z$, respectively, and ρ_{IR} their correlation. Then, we have:

$$\sigma_I^2 = B_I \Sigma_{ZZ} B_R^\top, \quad \sigma_R^2 = B_R \Sigma_{ZZ} B_R^\top, \quad \mathrm{C}\left[B_I Z, B_R Z\right] = B_I \Sigma_{ZZ} B_R^\top, \quad \rho_{IR} = \frac{B_I \Sigma_{ZZ} B_R^\top}{\sigma_I \sigma_R}.$$

For the contrarian trading strategy, $Z = [Y \;\; u]^\top$, $B_I = B_P = [-c \;\; 1]$ and $\sigma_I = \sigma_P$. We have:

$$\mathrm{V}\left[PR\right] = c^2 \sigma_Y^2 (3\, c^2 \sigma_Y^2 + \sigma_u^2).$$

For the noise trading strategy, $Z = [Y \;\; u \;\; \varepsilon_G]^\top$, $B_I = [0 \;\; 0 \;\; d]$, and we have:

$$\mathrm{V}\left[(B_I Z)(B_R Z)\right] = h_G^2 \sigma_G^2 (c^2 \sigma_Y^2 + 2\, d^2 \sigma_G^2 + \sigma_u^2).$$

Combine the results above, we obtain the Sharpe ratios for the two strategies given in (6.5).

Certainty equivalence. Given the form of the value function (4.14), welfare function (4.21) and the definition of certainty equivalence (6.7), we have:

$$V(W, h_G = 0) e^{-\,\alpha\, CE\,(h_G)} = V(W, h_G).$$

In the case of contrarian strategy, $V(W, h_G) = V(W, h_G = 0)$. Hence, $CE^{\,\text{Contrarian}} = 0$. In the case of noise trading strategy, $V(W, h_G)$ is given in (5.9) for h_G small. To the lowest order of h_G, we immediately have the second result in (6.5).