# Managing Financial Crises *

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20 March 2025

**Abstract**

Crisis resolution policies impact ex-ante behavior, causing moral hazard, time inconsistency, and other incentive issues. We show that adopting macroprudential policy without crisis resolution policies reduces welfare in the workhorse Bianchi-Mendoza (2018) model economy. The debt subsidy component of the optimal time-consistent debt tax policy is the main source of welfare gains in this model. Without the ex-post component, the ex-ante policy harms the economy, causing welfare losses relative to the unregulated equilibrium and worsening financial crisis dynamics. Thus, the benefits of the optimal time-consistent policy depend on post-crisis interventions. We also show that when ex-post policy implementation has no associated costs, there is no role for ex-ante macroprudential policy. The ex-post debt subsidy component supports borrowing and asset prices during crises, reflecting the classic lending of last resort policies advocated by Bagehot (1873).

**JEL Classification**: E61, F38, F44, H23
**Keywords:** Constrained efficiency, Financial crises, Macroprudential policy, Pecuniary externalities, Ramsey-optimal policy, Time consistency.

*We are grateful to Javier Bianchi, Huigang Chen, Paul Fontanier, Enrique Mendoza, and Eric Young for their helpful comments on an earlier version of this note.

# 1 Introduction

Managing financial crises has long concerned macroeconomics at least dating back to Bagehot (1873). In the aftermath of the hugely costly U.S. sub-prime mortgage crisis and the ensuing global financial crisis, the focus of the academic and policy debate shifted from crisis-resolution policies to the design of crisis-prevention policies—interventions enacted before a crisis materializes, aimed at curbing excessive borrowing and other financial excesses during periods of financial stability to reduce the likelihood and severity of future crises. As a result, a very large body of literature has been developed to motivate the adoption of these so-called macroprudential policies from a theoretical perspective and empirically investigate their effectiveness. However, while it is well known that crisis resolution policies can feedback on ex-ante behavior inducing moral hazard, time-inconsistency, and other incentive problems, the consequences of macroprudential policy adoption *in the absence of crisis resolution policies* is not well understood. Our research fills this gap and emphasizes the role of lender of last resort interventions as crucial not only during crisis times but also in enabling macroprudential policy effectiveness. Indeed, we show that absent ex-post interventions, ex-ante policies aimed at preventing the crisis alone can be just costly and can entail a welfare loss.

As a vehicle to convey this perspective, we adopt the workhorse framework proposed by Bianchi and Mendoza (2018), which has served as an important reference in advocating the adoption and implementation of macroprudential policy measures by central banks and regulatory authorities around the world. This is a model economy featuring an occasionally binding collateral constraint that generates endogenous financial crisis. An asset prices affects the value of the collateral, and macroprudential policy is implemented through a tax on debt, as in much of the related literature.

In order to emphasize the importance of ex-post policy intervention in enabling the effectiveness of ex ante policies and quantify the welfare costs of implementing macroprudential policies in isolation from bailout policies, we proceed in several steps. We first formulate the optimal time-consistent policy problem. In the model economy proposed by Bianchi and

Mendoza (2018), the scope for government intervention derives from need to address a pecuniary externality arising because private agents do not internalize the effects of their choices on the value of collateral. The constrained social planner allocation maximize the representative agent's utility subject to the competitive equilibrium implementability constraints, except the constraint describing the privately optimal choice of debt (the SP allocation). Hence, by construction, the SP allocation is obtained as if the planner could choose debt on behalf of the agents, internalizing its effect on the value of collateral, while letting the goods markets clear competitively. In this setting, the optimal time-consistent debt tax has two components. The first one is the "macroprudential debt tax," or "ex-ante component", which is active only when the collateral constraint is not binding. This component is unambiguously nonnegative. The second component, which we label "ex-post component," is active when the collateral constraint binds. In theory, the latter can be positive (i.e., a tax), negative (i.e., a subsidy), or zero. We then show that the critical element of the optimal time-consistent policy in this framework is the ex-post component active in crisis time when the collateral constraint binds.

We substantiate our perspective along different dimensions. We first show that under the BM calibration, the ex-post component of the optimal time-consistent policy is a debt subsidy, which is sizeable when the collateral constraint binds. Second, we demonstrate that the ex-post component is critical for the optimal time-consistent policy to be welfare enhancing. For that aim, we compute the restricted optimal time-consistent policy under the constraint that the debt tax must be nonnegative,. By construction, the resulting policy permits only ex-ante interventions. We show that this restricted policy yields a welfare loss relative to the decentralized competitive equilibrium (DE), suggesting that the bulk of the welfare gains from the unrestricted optimal time-consistent policy come from the ex-post component. In other words, in managing a financial crisis, relying only on ex-ante policy intervention is counterproductive. The benefit of ex-ante policy interventions arises in conjunction with ex-post policy support. Finally, we select only the macroprudential debt

tax as implied by the optimal plan and we compute the welfare gains from adopting only this component of the optimal policy plan in the decentralized allocation. This component by itself is not time-consistent but our welfare analysis highlights its marginal role as the welfare gains reduce to six times smaller than the gains from the optimal time-consistent policy.

The paper also discusses the role of ex-post-intervention costs in shaping the optimal policy. We do so by analyzing the role of the working capital constraint in the optimal policy problem. We then show that, in the absence of ex post policy intervention costs, i.e., in the special case of no working capital constraint, there is no scope for macroprudential policy, under certain conditions, the optimal time-consistent policy entails only the ex-post component. Specifically, we now show that in the limiting case of no working capital requirement, under certain conditions, the optimal time-consistent debt tax in the BM economy has a subsidy component only. Specifically, the optimal debt subsidy implements the allocation that would arise in the decentralized competitive equilibrium (DE) of the BM economy without the collateral constraint—the unconstrained equilibrium (UE).[1] This allocation is the first best, conditional on the asset market structure.

The broader point of the paper connects directly to the classic economic role of Central Banks as lenders of last resort as in Bagehot (1873). Bagehot's principle advises that, during periods of financial distress, central banks should provide liquidity to solvent institutions, supporting the stability of the financial system. While Bagehot's approach involved lending against good collateral at penalty rates, the underlying principle of intervention during crises remains relevant in our context. In our analysis, an ex-post debt subsidy serves a similar purpose: it provides targeted support to maintain asset prices and stabilize the economy when collateral constraints bind.

The rest of the paper is structured as follows: Section 2 presents the model economy and

[1] Benigno et al. (2016) find a similar result in an endowment economy when other policy instruments, such as consumption taxes, are available. Katagiri et al. (2017) show that the debt subsidy can implement the UE allocation in an endowment economy with a collateral constraint that depends on the *end-of-period* capital stock but not on the beginning-of-period capital stock, as in BM.

its competitive equilibrium. Section 3 characterizes the optimal time-consistent debt tax and discusses its two components. Section 4 illustrates the importance of the ex-post component. Section 4 quantifies the welfare gains of the ex-ante component. Section 5 discusses the role of ex-post policy implementation costs and derives the optimal policy in the special case of the model economy in which there is no working capital constraint. Section 6 concludes.

# 2 The model economy and its regulated competitive equilibrium

In this section, we define a regulated CE conditional on an arbitrary debt tax rebated lump sum. The model economy can be represented by a representative household firm that solves the following problem

$$\max_{\{b_{t+1},c_t,h_t,k_{t+1},v_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$

subject to the budget and collateral constraints

$$c_t + q_t k_{t+1} + p_v v_t + \frac{b_{t+1}}{R_t(1+\tau_t)} \leq z_t F(k_t, h_t, v_t) + q_t k_t + b_t + T_t,$$
$$\frac{b_{t+1}}{R_t} - \theta p_v v_t \geq -\kappa_t q_t k_t,$$

where $c_t$ is consumption, $h_t$ is labor, $k_t$ is capital stock, $v_t$ are intermediate inputs, $b_t$ are bond holdings, $q_t$ is the asset price, $R_t > 0$ is the exogenous gross risk-free interest rate, $z_t > 0$ is the exogenous productivity, $\kappa_t > 0$ is the exogenous leverage limit, $\beta \in (0,1)$ is the discount factor, $p_v > 0$ is the constant price of intermediate inputs, and $\theta \in [0,1]$ is the working capital constraint tightness. The utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ satisfies standard assumptions: it is strictly increasing in $c$, decreasing in $h$, concave, twice continuously differentiable, and satisfies an Inada condition $\lim_{c\to 0_+} u_c(c,h) = \infty$ for all $h > 0$.[2] The production function

[2]Our argument does not require restricting $u$ to Greenwood et al. (1988) preferences assumed in BM. In the latter case, we have $u(c,h) = U(c - g(h))$, and the relevant Inada condition is simply $\lim_{x\to 0_+} U'(x) = \infty$.

$F : \mathbb{R}^3_+ \to \mathbb{R}_+$ also satisfies standard assumptions: it is strictly increasing in all arguments, concave, twice continuously differentiable, and satisfies Inada conditions. Finally, $\tau_t$ is tax (positive) or subsidy (negative) on debt that alters the incentive to borrow—rebated lump sum with $T_t \equiv -\frac{b_{t+1}}{R_t}\frac{\tau_t}{1+\tau_t}$.

Letting $\gamma_t$ and $\mu_t$ denote the normalized Lagrange multipliers on the budget and collateral constraints, respectively, the first-order conditions (FOCs) that characterize the regulated competitive equilibrium of the economy are as follows.

$$\begin{aligned}
b_{t+1}: \quad & \gamma_t = \beta R_t(1+\tau_t)\mathbb{E}_t(\gamma_{t+1}) + (1+\tau_t)\mu_t, \\
c_t: \quad & \gamma_t = u_c(c_t, h_t), \\
h_t: \quad & 0 = u_h(c_t, h_t) + \gamma_t z_t F_h(k_t, h_t, v_t), \\
k_{t+1}: \quad & q_t\gamma_t = \beta\mathbb{E}_t[\gamma_{t+1}(q_{t+1} + z_{t+1}F_k(k_{t+1}, h_{t+1}, v_{t+1})) + \mu_{t+1}\kappa_{t+1}q_{t+1}], \\
v_t: \quad & 0 = \gamma_t(z_t F_v(k_t, h_t, v_t) - p_v) - \mu_t\theta p_v.
\end{aligned}$$

The complementary slackness conditions are

$$0 = \mu_t\left(\kappa_t q_t k_t + \frac{b_{t+1}}{R_t} - \theta p_v v_t\right), \qquad \mu_t \geq 0.$$

Capital stock is in fixed aggregate supply, therefore:

$$k_t = 1.$$

Eliminating the Lagrange multiplier on the budget constraint $\gamma_t > 0$ from the FOCs of the regulated competitive equilibrium (CE) conditional on an arbitrary debt tax rebated lump sum in Section 2, we obtain the following definition.

**Definition 2.1** (Regulated competitive equilibrium)**.** *A CE is the sequences of state-contingent taxes* $\{\tau_t\}$*, allocations* $\{b_{t+1}, c_t, h_t, v_t\}$*, asset prices* $\{q_t\}$*, and Lagrange multipliers* $\{\mu_t\}$ *that*

*satisfy*

$$c_t = z_t F(1, h_t, v_t) - p_v v_t + b_t - \frac{b_{t+1}}{R_t}, \tag{1}$$

$$0 \leq \kappa_t q_t + \frac{b_{t+1}}{R_t} - \theta p_v v_t, \tag{2}$$

$$u_c(c_t, h_t) = \beta R_t (1 + \tau_t) \mathbb{E}_t(u_c(c_{t+1}, h_{t+1})) + (1 + \tau_t)\mu_t, \tag{3}$$

$$-\frac{u_h(c_t, h_t)}{u_c(c_t, h_t)} = z_t F_h(1, h_t, v_t), \tag{4}$$

$$q_t u_c(c_t, h_t) = \beta \mathbb{E}_t[u_c(c_{t+1}, h_{t+1})(q_{t+1} + z_{t+1} F_k(1, h_{t+1}, v_{t+1})) + \mu_{t+1}\kappa_{t+1} q_{t+1}], \tag{5}$$

$$p_v = z_t F_v(1, h_t, v_t) \frac{u_c(c_t, h_t)}{u_c(c_t, h_t) + \theta \mu_t}, \tag{6}$$

$$0 = \mu_t \left( \kappa_t q_t + \frac{b_{t+1}}{R_t} - \theta p_v v_t \right), \qquad \mu_t \geq 0, \tag{7}$$

*for all* $t \geq 0$*, and a transversality condition* $\lim_{t\to\infty} \mathbb{E}_0 \left( \frac{b_t}{\prod_{j=0}^{t-1} R_j} \right) = 0$*, given an initial condition* $b_0$ *and exogenous stochastic process* $\{z_t, R_t, \kappa_t\}$*.*

# 3 Optimal time-consistent debt tax policy and its components

Proposition 1 in BM, , p. 604, characterizes the optimal time-consistent debt tax $\tau_t$ that implements the constrained-efficient social planner's allocation. In this section, we discuss its decomposition into its ex-ante and ex-post components.

Consider the planner's generalized Euler equation for next-period bond holdings, i.e., equation (14) on p. 602 in BM:

$$u'_t = \beta R_t \mathbb{E}_t(u'_{t+1} - \xi_{t+1} u''_{t+1} \mathcal{Q}_{t+1} + \xi_t \Omega_{t+1}) + \xi_t u''_t q_t + \mu^*_t. \tag{8}$$

In our analysis, we focus on a tax/subsidy on debt issuance, consistent with what BM's Section II.E, p. 604. The regulated bond Euler equation (16) on p. 604 in BM corresponds to a

tax on debt repayment. These two formulations are equivalent for the purpose of implementing the SP allocation and imply similar debt tax decompositions.[3] In both formulations, the representative household firm faces the same collateral constraint

$$\frac{b_{t+1}}{R_t} - \theta p_v v_t \geq -\kappa_t q_t k_t,$$

with the corresponding Lagrange multiplier $\mu_t \geq 0$. The agent's budget constraint is

$$c_t + q_t k_{t+1} + p_v v_t + \frac{b_{t+1}}{R_t(1+\tau_t)} \leq z_t F(k_t, h_t, v_t) + q_t k_t + b_t + T_t,$$

where $T_t \equiv -\frac{b_{t+1}}{R_t}\frac{\tau_t}{1+\tau_t}$ is the corresponding lump-sum transfer. The agent's regulated bond Euler equation is then

$$u'_t = (1+\tau_t)[\beta R_t \mathbb{E}_t(u'_{t+1}) + \mu_t]. \tag{9}$$

Equating the right-hand sides of (8) and (9), we obtain

$$\tau_t = \underbrace{-\frac{\beta R_t \mathbb{E}_t(\xi_{t+1} u''_{t+1} \mathcal{Q}_{t+1})}{\beta R_t \mathbb{E}_t(u'_{t+1}) + \mu_t}}_{\tau_t^{\text{MP}}} + \underbrace{\frac{\xi_t[\beta R_t \mathbb{E}_t(\Omega_{t+1}) + u''_t q_t] + \mu_t^* - \mu_t}{\beta R_t \mathbb{E}_t(u'_{t+1}) + \mu_t}}_{\tau_t^{\text{EP}}}. \tag{10}$$

The agent's first-order condition for $v_t$—that pins down $\mu_t$—is not affected by regulation. Therefore, $\mu_t = \frac{u'_t}{\lambda_t}\mu_t^*$ and $\xi_t = \frac{\kappa_t}{u'_t}\mu_t^*$ (see proof of Proposition II in Appendix A.1 in BM).

This tax can be decomposed into two components as

$$\tau_t = \tau_t^{\text{MP}} + \tau_t^{\text{EP}}, \tag{11}$$

where $\tau_t^{\text{MP}} \geq 0$ is the *ex-ante (macroprudential) component*—referred as the "macroprudential debt tax" in BM—and $\tau_t^{\text{EP}}$ is what we call the *ex-post component*. In "good times," when the collateral constraint is not binding, $\tau_t = \tau_t^{\text{MP}}$, while in "bad times," when the collateral

[3] We show this in Appendix A.

constraint is binding, $\tau_t = \tau_t^{\text{EP}}$.

To see this, note that, under the BM calibration assumptions that imply $\xi_{t+1} = 0$ with certainty if $\mu_t^* > 0$, we have

$$\tau_t^{\text{MP}} = -\frac{\mathbb{E}_t(\xi_{t+1} u''_{t+1} \mathcal{Q}_{t+1})}{\mathbb{E}_t(u'_{t+1})} \geq 0, \tag{12}$$

which is identical to equation (17) on p. 605 in BM.[4] The ex-post component becomes

$$\tau_t^{\text{EP}} = \xi_t \left\{ \frac{\mathbb{E}_t(\Omega_{t+1})}{\mathbb{E}_t(u'_{t+1}) + \frac{\mu_t}{\beta R_t}} + \frac{1}{\beta R_t \mathbb{E}_t(u'_{t+1}) + \mu_t} \left[ u''_t q_t + \frac{u'_t}{\kappa_t} \left( 1 - \frac{u'_t}{\lambda_t} \right) \right] \right\}.$$

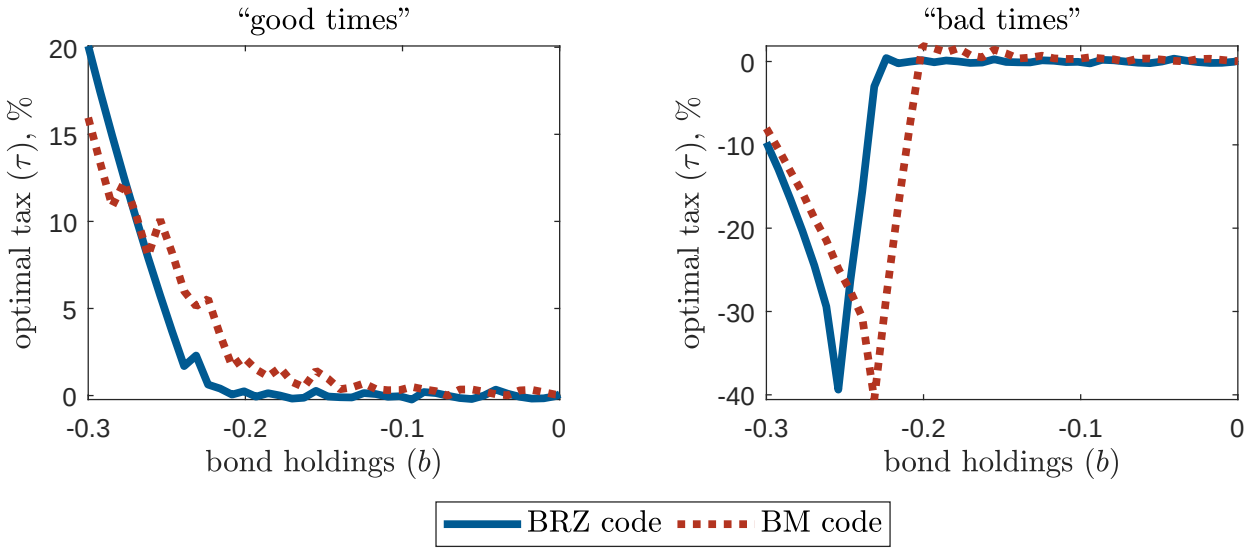
If the collateral constraint is not binding, $\tau_t^{\text{EP}} = \xi_t = \mu_t^* = 0$. If the collateral constraint is binding and $\mu_t^* > 0$, we have $\tau_t^{\text{MP}} = 0$, since $\xi_{t+1} = 0$ with certainty, showing that the optimal time-consistent debt tax in the BM economy can be decomposed into an ex-ante component—active when the collateral constraint is not binding—and an ex-post component active when the collateral constraint binds. We now discuss each of these two components, in turn.

# 4 The importance of the ex-post policy component

BM show that the sign of the ex-post component is theoretically ambiguous. In this section, we quantitatively illustrate the role and properties of the ex-post component. We show that, under the BM calibration, the optimal time-consistent policy requires both *taxing* debt in "good times" and *subsidizing* debt in "bad times," consistent with the classic theory of lending of last resort advocated by Bagehot (1873). Thus, the sign of the ex-post component is generally negative: it is a sizable debt subsidy when the collateral constraint binds, in particular during financial crisis events.

[4] BM consider a two-state Markov process for the leverage limit $\kappa_t \in \{\kappa^L, \kappa^H\}$, where $\kappa^L < \kappa^H$, $\kappa^L$ corresponds to the tight credit regime, and $\kappa^H$ corresponds to the normal credit regime. BM calibrate $\Pr(\kappa_{t+1} = \kappa^L \mid \kappa_t = \kappa^L) = 0$ (Table 1, p. 611 in BM). Numerically, the collateral constraint is binding only in the states in which $\kappa_t = \kappa^L$.

**Figure 1: Optimal time-consistent debt tax in good and bad times**




Notes: The figure plots the optimal time-consistent debt tax as a function of bond holdings conditional on the average TFP ($z$), high interest rate ($R$), and two distinct credit regimes. "Good times" corresponds to the normal credit regime ($\kappa = \kappa^H$), as in Fig. 5A in BM. "Bad times" corresponds to the tight credit regime ($\kappa = \kappa^L$), as in Fig. 2 and Fig. 3 in BM.

Figure 1 plots the optimal time-consistent debt tax as a function of bond holdings conditional on the average TFP ($z$) and high interest rate ($R$) in "good times" (normal credit regime with $\kappa = \kappa^H$) and "bad times" (tight credit regime with $\kappa = \kappa^L$). We plot the tax functions based on both our independent computation and the BM replication package. The left panel of Figure 1 corresponds to Fig. 5A in BM. In the normal credit regime, the collateral constraint is not binding independently of current bond holdings, in which case $\tau^{\text{EP}} = 0$ and $\tau = \tau^{\text{MP}}$. The macroprudential component is increasing in debt. For small enough debt levels, the collateral constraint is not binding in the next period with certainty, and $\tau = \tau^{\text{MP}} = \tau^{\text{EP}} = 0$.

Consider now the right panel of Figure 1, which is not reported in BM. In the tight credit regime, the collateral constraint is binding when current debt is sufficiently large: approximately $b \leq -0.23$ in our computation and $b \leq -0.2$ based on the BM replication package. In this region, $\tau^{\text{MP}} = 0$ and $\tau = \tau^{\text{EP}}$. The ex-post component is generally a debt

**Table 1: Optimal time-consistent policy moments and welfare gains**

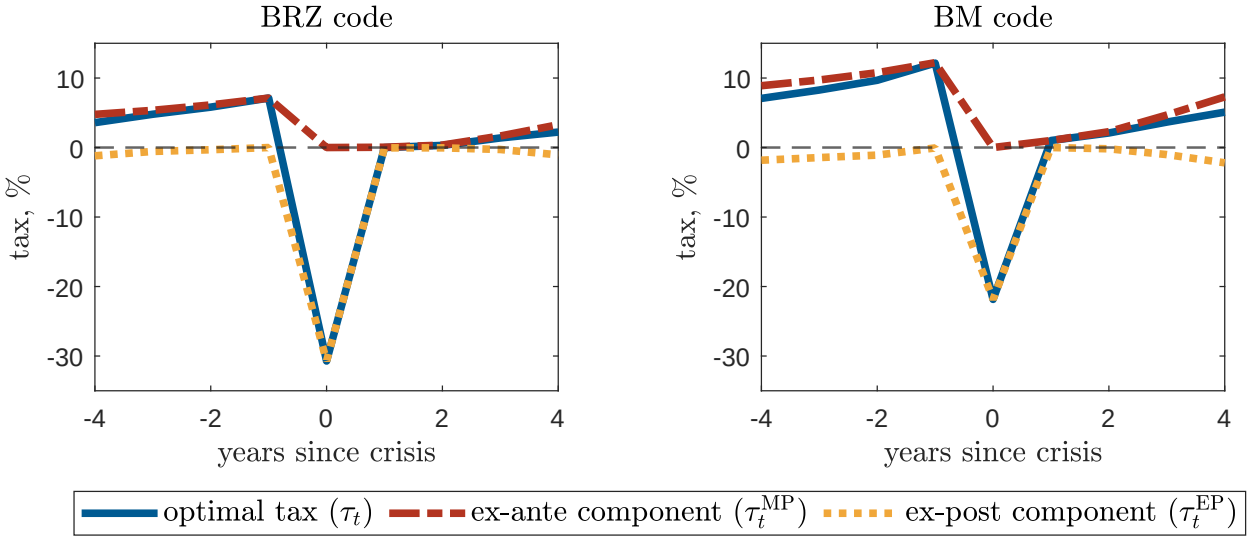
| | BRZ code | BM code |
|---|---|---|
| Optimal time-consistent debt tax moments, % | | |
| $\mathbb{E}(\tau)$ | 2.4 | 2.6 |
| $\mathbb{E}(\tau^{\text{MP}})$ | 3.9 | 3.6 |
| $\mathbb{E}(\tau^{\text{EP}})$ | -1.4 | -1.0 |
| $\min(\tau)$ | -46.4 | -48.4 |
| $\max(\tau)$ | 29.8 | 38.8 |
| $\Pr(\tau < 0)$ | 8.1 | 5.9 |
| Average welfare gain, % | | |
| Optimal time-consistent policy | 0.6 | 0.3 |

Notes: The tax moments are with respect to the SP ergodic distribution. Average welfare gains are in % of the permanent increase in net consumption $c - G(h)$, using the DE ergodic distribution weights.

subsidy, and its nonlinear shape reflects the shape of the policy function for next-period bond holdings $b'$ (Fig. 2 in BM and the top-left panel of Figure B.3 in Appendix B). The subsidy is increasing in debt when next-period debt is increasing in current debt. Similar to the left panel, when current debt is sufficiently small and the collateral constraint is not binding, it is also not binding in the next period with certainty, in which case $\tau = \tau^{\text{MP}} = \tau^{\text{EP}} = 0$.

Table 1 reports the optimal time-consistent debt tax moments and welfare gains from the optimal policy. Unlike Tables 2 and 3 in BM, which report the moments only for the macroprudential component $\tau^{\text{MP}}$, Table 1 provides the results for $\tau$, $\tau^{\text{MP}}$, and $\tau^{\text{EP}}$, confirming that the ex-post component is, on average, a debt subsidy. This subsidy exceeds 40% in certain regions of the state space and is applied with an ergodic probability of 8.1%, highlighting the importance of the ex-post component of the optimal time-consistent policy. The proper handling of nonconvexities in the BRZ code uncovers that the ex-post component is used more actively by the time-consistent planner, compared to the BM code, leading to a twofold increase in welfare gains from the optimal time-consistent policy.

**Figure 2: Optimal time-consistent policy around financial crises**




Notes: Each line is an average across all financial crisis events. The optimal tax function is evaluated at the DE states observed during the crises, as in the BM code, and decomposed into the ex-ante and ex-post components using (11).

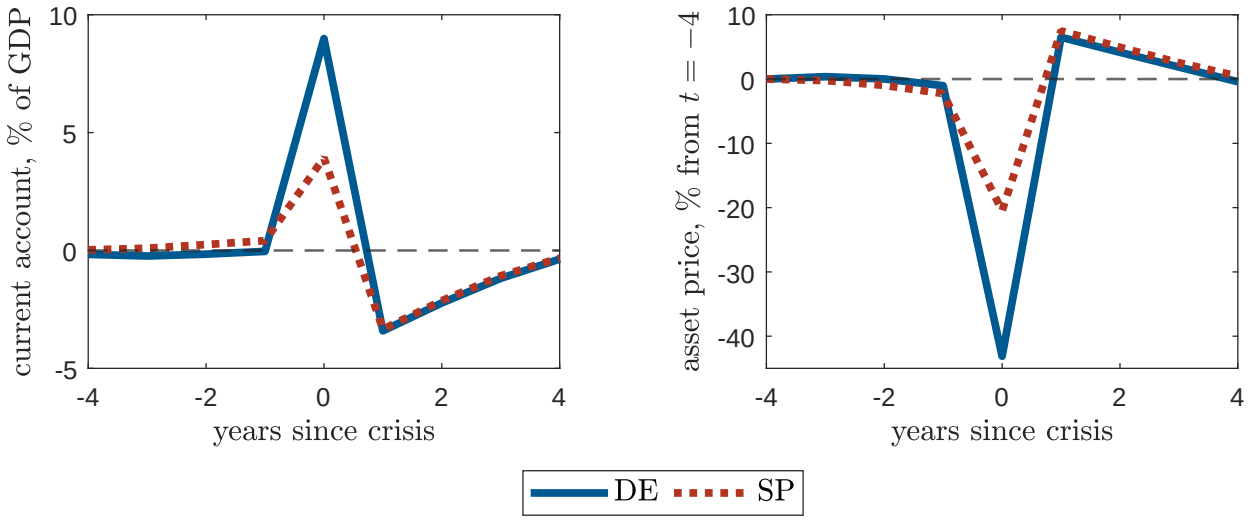
## 4.1 Financial crises dynamics

In Section III.D, BM discuss the ex-post benefits of the macroprudential debt tax $\tau^{MP}$ around financial crises. However, they do not elaborate on the role of ex-post interventions *during* financial crises.

Figure 2 illustrates the importance of the ex-post policy component in a typical financial crisis episode, both based on the BRZ code (left panel) and BM code (right panel).[5] As the economy gets closer to a financial crisis, the macroprudential component increases by 2.3 percentage points, on average, from $\tau_{-4}^{MP} = 4.8\%$ to $\tau_{-1}^{MP} = 7.1\%$ (BRZ code). When a financial crisis occurs, $\tau_0^{MP} = 0$, and the policy response is driven by the ex-post intervention that averages at $\tau_0^{EP} = -30.7\%$. In the right panel (BM code), the dynamics are similar, but the average magnitude of ex-ante (ex-post) interventions is larger (smaller) compared

[5]Following BM (p. 612–614), we identify a financial crisis as a DE event in which the current account is greater than its two standard deviations over a 100,000-period simulation (49,100-period simulation in the BM replication package). Numerically, a financial crisis occurs only if the collateral constraint is binding, although the converse is not true: the collateral constraint may be binding, but the current account is not extreme enough to be a crisis.

**Figure 3: Financial crises dynamics based on the DE and SP allocations**




Notes: Each line is an average across all financial crisis events. The policy functions are evaluated at the DE states observed during the crises, as in the BM code.

to the left panel, which is associated with two times smaller welfare gains from the optimal time-consistent policy, as shown in Table 1.

Figure 3 shows how the optimal time-consistent policy affects the financial crises dynamics of the current account in % of GDP (left panel) and the asset price (right panel). When the crisis occurs and the collateral constraint binds, there is large deleveraging in the DE, so that the current account reverts, on average, from -0.2% of GDP at $t = -4$ to 9% of GDP at $t = 0$. In contrast, in the SP allocation, the ex-post debt subsidy supports borrowing, and the current account increases only to 3.9% of GDP. In turn, a smaller current account reversal in the SP allocation is the result of a smaller asset price drop (by 20.5% from $t = -4$ to $t = 0$ compared to 43.1% in the DE).

Intuitively, the ex-post component of the optimal policy supports the value of collateral in times of crisis, similar to the lending of last resort policy suggested by Bagehot (1873). In the BM framework, the asset price support is brought about by the subsidy utilized by the social planner as opposed to a central bank in Bagehot (1873), but the rationale and the result of the intervention are the same. In the DE of the BM economy, the substantial

asset price drop arises because private agents do not take into account the impact of their individual choices on the asset price that affects the other agents' collateral values. This creates a pecuniary externality when the collateral constraint binds. By subsidizing the amount that agents can borrow, the social planner revalues the collateral. In this sense, the planner's action allows agents to borrow against *good* collateral as in Bagehot (1873). *Good* should be interpreted here as being correctly valued from the social planner's perspective.

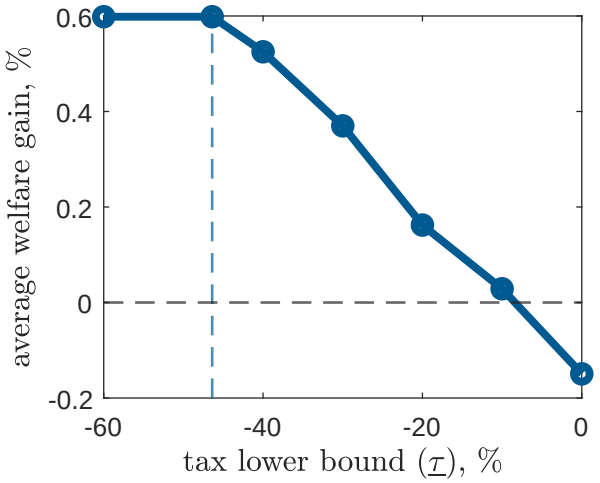
# 5 The welfare costs of the ex-ante policy component

In Section 4, we illustrated the role and importance of the ex-post policy component in rationalizing from a normative perspective the benefits of the ex-ante component.

In Appendix A.4, BM describe a time-consistent planning problem in which the tax on debt is constrained to be nonnegative. Hence, this problem generates a policy that entails only ex-ante interventions and corresponds to the standard notion of "macroprudential policy" as used in the real world to counter the build-up of systemic risk before a crisis materializes. However, BM do not provide a quantitative analysis of the policy described in Appendix A.4. Furthermore, BM report the moments of the ex-ante component $\tau_t^{\text{MP}}$ alongside the welfare gains from the SP allocation (i.e., from the optimal time-consistent policy $\tau_t$) in Tables 2 and 3 and label $\tau_t^{\text{MP}}$ as "optimal macroprudential tax" in Fig. 5. Consequently, the reader may erroneously associate the benefits of the optimal policy $\tau_t$ to its ex-ante component $\tau_t^{\text{MP}}$, since BM do not discuss the quantitative role of the ex-post component $\tau_t^{\text{EP}}$.

We proceed in two steps. We first compute the optimal time-consistent policy subject to a lower bound on the debt tax $\tau_t$. When the lower bound is zero, as in Appendix A.4 in BM, the restricted policy is associated with a welfare *loss* rather than a gain, indirectly implying that the welfare gains from the (unrestricted) optimal time-consistent policy are largely driven by its ex-post component. Second, we compute the regulated competitive

**Figure 4: Welfare gains from the restricted optimal time-consistent policy**




Notes: Average welfare gains are in % of the permanent increase in net consumption $c - G(h)$, using the DE ergodic distribution weights, and correspond to the optimal time-consistent policy constrained by (13). The dashed vertical line at -46.4% is min($\tau$) from Table 1.

equilibrium (CE) conditional on the ex-ante component $\tau_t^{\text{MP}}$ of the optimal time-consistent policy. We show that this *time-inconsistent* policy is associated with significantly smaller welfare gains compared to the optimal time-consistent policy.

## 5.1 Restricted optimal time-consistent policy

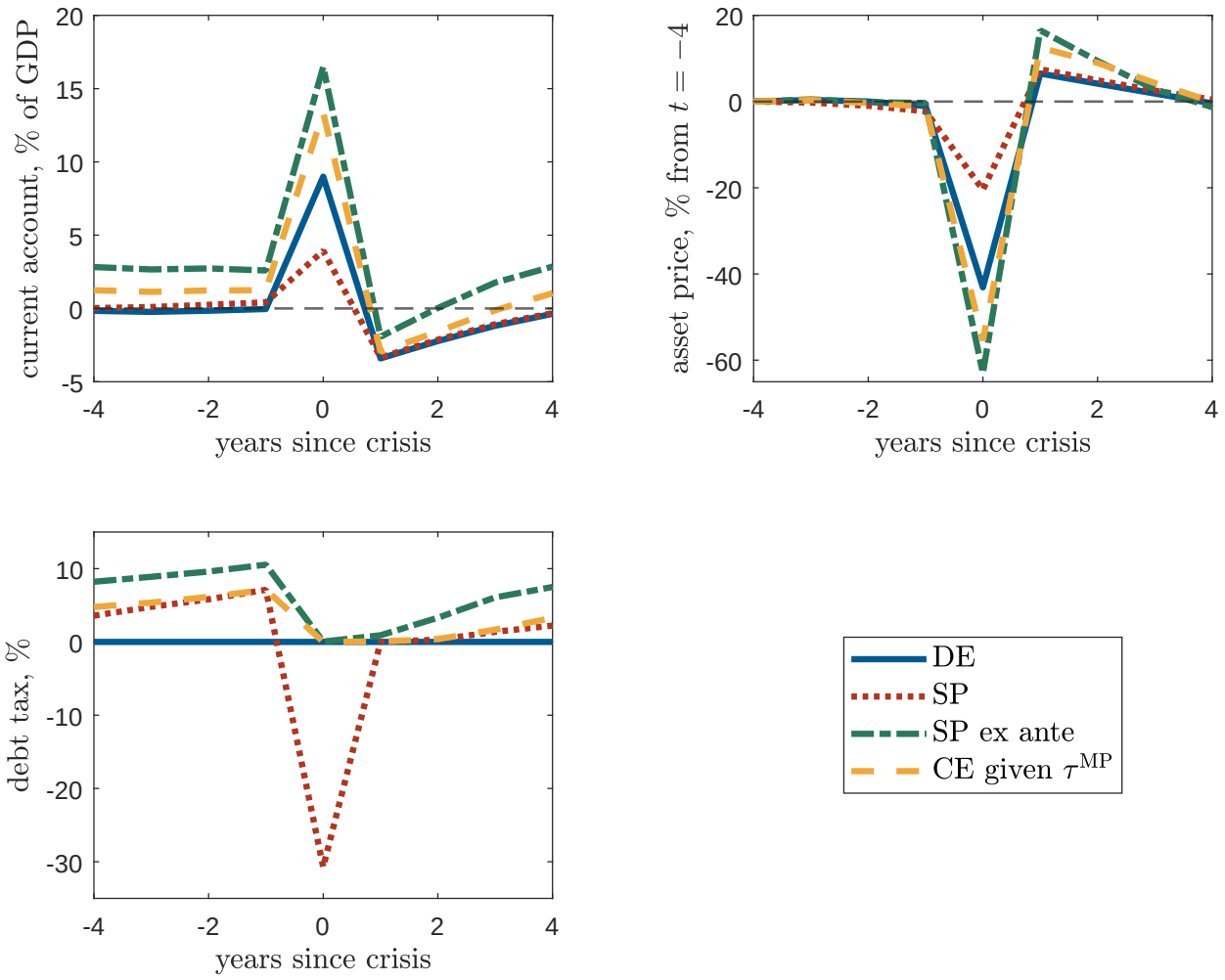
Consider the optimal time-consistent policy problem in which the tax on debt is constrained by a lower bound, i.e.,

$$\tau_t \geq \underline{\tau}, \tag{13}$$

where $\tau_t$ is given by the regulated bond Euler equation (9) in Appendix 3 and can be expressed in terms of allocations only. The case of $\underline{\tau} = 0$ corresponds to ruling out ex-post interventions, which is the problem considered theoretically in Appendix A.4 in BM.

Figure 4 reports the welfare gains from the optimal time-consistent policy constrained by (13). As we restrict the ex-post policy component to be progressively smaller, we find progressively smaller welfare gains, which turn into welfare losses when $\underline{\tau} \approx -8.3\%$. When

**Figure 5: Financial crises dynamics under alternative policies**




Notes: Each line is an average across all financial crisis events. The policy functions are evaluated at the DE states observed during the crises, as in the BM code. "SP ex ante" corresponds to the optimal time-consistent policy constrained by (13) with $\underline{\tau} = 0$. "CE given $\tau^{MP}$" corresponds to the regulated CE conditional on the ex-ante component of the unrestricted optimal time-consistent policy.

$\underline{\tau} = 0$, there is a -0.15% welfare *loss*, compared to a 0.6% welfare gain from the unrestricted optimal time-consistent policy (Table 1). Therefore, the ex-post policy component is essential for a time-consistent policy to be welfare-enhancing.

In Figure 5, we compare the implications of alternative policies around a typical financial crisis episode. The welfare loss from the time-consistent ex-ante policy (labeled "SP ex ante") is associated with its poor performance in crisis times. Under the time-consistent ex-ante

**Table 2: Financial crisis probabilities and welfare gains**

| | BRZ code | BM code |
|---|---|---|
| Financial crisis probability, % | | |
| Decentralized competitive equilibrium (DE) | 3.94 | 3.96 |
| Optimal time-consistent policy (SP) | 0.01 | 0.02 |
| Macroprudential component (CE given $\tau^{\text{MP}}$) | 2.72 | 1.94 |
| Optimal time-consistent ex-ante policy (SP ex ante) | 0.26 | – |
| Average welfare gain, % | | |
| Optimal time-consistent policy (SP) | 0.6 | 0.3 |
| Macroprudential component (CE given $\tau^{\text{MP}}$) | 0.1 | -0.11 |
| Optimal time-consistent ex-ante policy (SP ex ante) | -0.15 | – |

Notes: The financial crisis probability is the frequency of financial crises based on a 100,000-period simulation after a 1,000-period burn-in (49,100-period simulation in the BM replication package), using the DE crisis threshold, consistent with footnote 19 in BM. Average welfare gains are in % of the permanent increase in net consumption $c - G(h)$, using the DE ergodic distribution weights. The crisis probability and welfare gains from the macroprudential component $\tau^{\text{MP}}$ are based on the regulated CE conditional on $\tau^{\text{MP}}$. We use our code to compute the regulated CE conditional on $\tau^{\text{MP}}$ extracted from the BM replication package. Optimal time-consistent ex-ante policy is the optimal time-consistent policy constrained by (13) with $\underline{\tau} = 0$.

policy, the current account, on average, jumps from 2.8% of GDP at $t = -4$ to 16.5% at $t = 0$ (compared to an increase from -0.2% to 9% in the DE), while the asset price, on average, falls by 63% from $t = -4$ to $t = 0$ (compared to 43.1% in the DE). This is despite the tax rate being cut to 0% at $t = 0$, hitting the lower bound. Hence, the time-consistent ex-ante policy exacerbates the crisis dynamics because it fails to support the agents' borrowing and the value of collateral.

Table 2 shows that the financial crisis probability decreases from 3.94% in the DE to 0.26% under the time-consistent ex-ante policy, since agents, on average, borrow significantly less in the latter case. However, the crisis probability under the unrestricted optimal time-consistent policy (i.e., the SP allocation) is much lower at 0.01%. These results clearly illustrate that, in the BM economy, simply reducing borrowing ex ante does not maximize welfare, nor does it minimize the crisis probability.

### 5.2 The "macroprudential debt tax"

We now compute the regulated CE conditional on the ex-ante component $\tau^{\text{MP}}$ of the optimal time-consistent policy. The "macroprudential debt tax" $\tau^{\text{MP}}$ is the main focus of the policy discussion in BM and is a policy that is similar to those evaluated in many empirical studies in the literature and frequently advocated by policymakers. The macroprudential debt tax as a standalone policy is *time inconsistent* because a time-consistent planner would have an incentive to deviate and subsidize debt in the states in which the collateral constraint is binding, as reflected by the ex-post component $\tau^{\text{EP}}$ of the optimal time-consistent policy.

Table 2 shows that the welfare gain from $\tau^{\text{MP}}$ is only 0.1%, compared to a 0.6% welfare gain from the optimal time-consistent policy (BRZ code). On the other hand, if we construct $\tau^{\text{MP}}$ using the BM replication package, we obtain a -0.11% welfare *loss*. The macroprudential debt tax reduces the crisis probability only to 2.72% (0.01% under the optimal time-consistent policy) and performs poorly in crisis times (Figure 5). Hence, even though the (time-inconsistent) macroprudential debt tax can be welfare enhancing, it is much less effective than the optimal (time-consistent) policy.

## 6 The working capital constraint as an ex-post policy implementation cost

Although the working capital constraint assumption (i.e., $\theta > 0$ in equation (3) on p. 595 in BM) is important from a positive point of view, this constraint introduces an intermediate input distortion, which makes ex-post interventions costly in the sense of Benigno et al. (2016) and has important normative implications. Intuitively, the planner can borrow more when the collateral constraint is binding either by increasing the value of collateral or by decreasing the use of intermediate inputs, a fraction of which must be financed in advance due to the working capital constraint. The latter generates output losses that impose an endogenous cost to the use of ex-post policies that support more borrowing, making them relatively less

effective and creating more scope for the use of ex-ante macroprudential policies. The critical role of the ex-post policy component becomes even more transparent if we consider a special case of the BM economy without the working capital constraint, i.e., the limiting case of $\theta = 0$.

The limiting case of no working capital requirement is also important because it is the benchmark case in the seminal literature on collateral constraints and pecuniary externalities (Kiyotaki and Moore, 1997; Lorenzoni, 2008; Dávila and Korinek, 2018; Jeanne and Korinek, 2019). For example, in Appendix D, E.2, and K.2, BM focus on the economy without the working capital constraint when extending the model with capital adjustment costs, investigating the quantitative implications of the optimal policy under commitment, and comparing their results to those in Jeanne and Korinek (2019).

In this section, we first show that if $\theta = 0$, under certain regularity conditions, the optimal time-consistent policy $\tau$ implements the unconstrained DE allocation that would arise in the economy without the collateral constraint—the first best conditional on the asset market structure. In that case, the ex-post subsidy component is the only component of the optimal time-consistent policy, i.e., $\tau_t(s^t) \leq 0$ for all dates $t$ and histories of shocks $s^t$. We then demonstrate that, the same result holds if $\theta \neq 0$ but the policymaker can also set optimally a tax on intermediate inputs that addresses the additional distortion introduced by the working capital constraint. Intuitively, as the working capital constraint introduces a distortion in the agent's demand for intermediate inputs—in addition to a distortion in the demand for bonds—so one would need a second policy instrument to correct that distortion if the goal is to achieve the first best. If the policymaker can additionally set optimally a tax on intermediate inputs, the optimal time-consistent debt tax is identical to that in the case of no working capital requirement—it is a subsidy that implements the UE allocation.

## 6.1 The special case of no intermediate input distortion

In this section, we define the unconstrained equilibrium of the model economy (UE) and show that it is the first best, conditional on the asset market structure.

A UE is obtained from Definition 2.1 by discarding the collateral constraint and setting $\tau_t = \mu_t = 0$ for all $t \geq 0$.

**Definition 6.1** (Unconstrained equilibrium)**.** *A UE is a sequence of state-contingent allocations* $\{b_{t+1}, c_t, h_t, v_t\}$ *that satisfies* (1), (4),

$$u_c(c_t, h_t) = \beta R_t \mathbb{E}_t(u_c(c_{t+1}, h_{t+1})), \tag{14}$$

$$p_v = z_t F_v(1, h_t, v_t), \tag{15}$$

*for all* $t \geq 0$*, and a transversality condition* $\lim_{t\to\infty} \mathbb{E}_0 \left( \frac{b_t}{\prod_{j=0}^{t-1} R_j} \right) = 0$*, given an initial condition* $b_0$ *and exogenous stochastic process* $\{z_t, R_t\}$*.*

Let $\bar{q}_t$ denote the UE asset price. Note that $\{\bar{q}_t\}$ need not be part of Definition 6.1, since it can be constructed from (5) based on the UE allocation after imposing $\mu_t = 0$ for all $t \geq 0$. Specifically, $\{\bar{q}_t\}$ is the solution to the fixed-point equation

$$\bar{q}_t u_c(c_t, h_t) = \beta \mathbb{E}_t[u_c(c_{t+1}, h_{t+1})(\bar{q}_{t+1} + z_{t+1} F_k(1, h_{t+1}, v_{t+1}))]. \tag{16}$$

It follows that the UE allocation is the *first best* in the BM economy.

**Lemma 6.1.** *The UE allocation of Definition 6.1 is the first best, conditional on the asset market structure. That is, it is the solution to*

$$\max_{\{b_{t+1}, c_t, h_t, v_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$$

*subject to the resource constraint* (1)*.*

*Proof.* Follows from comparing the FOCs to (4), (14), and (15). ■

## 6.2 Optimal costless time-consistent policy

In this section, we show that in the limiting case of no working capital requirement ($\theta = 0$), under certain conditions, there exists a time-consistent policy $\{\tau_t\}$ that implements the UE allocation in the regulated CE. This policy satisfies $\tau_t \in (-1, 0]$ for all $t \geq 0$, and thus it has a subsidy component only. We also show that if $\theta \neq 0$ but the policymaker can additionally set optimally a tax on intermediate inputs $\tau_t^v$, under the same conditions, there exists a time-consistent policy $\{\tau_t, \tau_t^v\}$, with $\tau_t \in (-1, 0]$, that implements the UE allocation.

Consider the UE allocation $\{b_{t+1}, c_t, h_t, v_t\}$ of Definition 6.1 and the corresponding asset price $\{\bar{q}_t\}$ given by (16). To simplify the exposition, define

$$u_{c,t} \equiv u_c(c_t, h_t), \qquad \alpha_t \equiv \beta \mathbb{E}_t(u_{c,t+1} z_{t+1} F_k(1, h_{t+1}, v_{t+1})) > 0,$$

which implies $\bar{q}_t u_{c,t} \equiv \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t(\alpha_{t+k})$, using (16). We also define

$$\bar{y}_t \equiv u_{c,t} \left( \kappa_t \bar{q}_t + \frac{b_{t+1}}{R_t} - \theta p_v v_t \right), \tag{17}$$

which is the *excess collateral value* in marginal utility units evaluated at the UE allocation. Hence, if $\bar{y}_t \geq 0$, the UE allocation satisfies the collateral constraint, and if $\bar{y}_t < 0$, the UE allocation violates the collateral constraint, in which case it has more debt than in the baseline DE with the collateral constraint.

Let us adopt the following standard notation: $S$ is the set of possible states of nature $s_t \in S$ at any date $t \geq 1$, $s^t = (s_0, s_1, \ldots, s_t)$ is a specific history of states, and $S^t$ is the set of all possible histories at $t \geq 0$, with $S^0 = \{s_0\}$ and $s_0 \in S$. By construction, $S^t \equiv S^{t-1} \times S$ for all $t \geq 1$. Any variable $x_t$ is a Borel-measurable function $x_t : S^t \to \mathbb{R}$, and $\{x_t\}$ is a sequence $\{x_t\}_{t=0}^{\infty}$ of such functions. With a slight abuse of notation, we will write $\{x_t\} \subseteq A$, meaning that $x_t(s^t) \in A$ for all $t \geq 0$ and $s^t \in S^t$.

Consider the following definition.

**Definition 6.2.** *$\{\epsilon_t\} \subseteq \mathbb{R}_+$ is an admissible perturbation if it satisfies two conditions:*

1. *If $\bar{y}_t(s^t) \geq 0$, then $\epsilon_t(s^t) = 0$.*

2. *If $\bar{y}_t(s^t) < 0$, then $\{\{\epsilon_{t+j}(s^{t+j})\}_{s^{t+j} \in S^{t+j}|s^t}\}_{j \geq 1}$ is such that*

$$-\frac{\bar{y}_t(s^t)}{\kappa_t(s^t)} = \sum_{k=1}^{\infty} \beta^k \mathbb{E}_t \left\{ \alpha_{t+k} \left[ \prod_{j=1}^{k} (1 + \epsilon_{t+j}) - 1 \right] \right\}. \tag{18}$$

In other words, an "admissible perturbation" is a sequence of nonnegative-valued functions with two properties: (1) a function evaluates to zero whenever the UE allocation satisfies the collateral constraint; (2) if the UE allocation violates the collateral constraint, there are future histories in which the function takes positive values to satisfy (18).

The next proposition shows that if $\theta = 0$, any admissible perturbation of Definition 6.2 maps to a time-consistent policy $\{\tau_t\}$ that implements the UE allocation.

**Proposition 6.1.** *If $\theta = 0$, for any $\{\epsilon_t\}$ satisfying Definition 6.2, there exists a sequence of state-contingent taxes $\{\tau_t\} \subseteq (-1, 0]$, asset prices $\{q_t\}$, and Lagrange multipliers $\{\mu_t\}$ that together with the UE allocation $\{b_{t+1}, c_t, h_t, v_t\}$ satisfies Definition 2.1. The policy $\{\tau_t\}$ is Ramsey optimal and time consistent.*

*Proof.* By Definition 6.1, the UE allocation satisfies (1) and (4). If $\theta = 0$, (6) is equivalent to (15). Conditional on $\{\mu_t\}$, if we set

$$\tau_t = -\frac{\mu_t}{\mu_t + u_c(c_t, h_t)} \in (-1, 0], \tag{19}$$

then (3) is equivalent to (14). We are left to construct $\{q_t, \mu_t\}$ that together with the UE allocation satisfy (2), (5), and (7).

Define $\widetilde{q}_t \equiv q_t u_{c,t}$ and $\widetilde{\mu}_t \equiv 1 + \frac{\kappa_t}{u_{c,t}} \mu_t \geq 1$. Since there is a unique inverse mapping $q_t = \frac{\widetilde{q}_t}{u_{c,t}}$ and $\mu_t = \frac{u_{c,t}}{\kappa_t}(\widetilde{\mu}_t - 1)$, constructing $\{q_t, \mu_t\}$ is equivalent to constructing $\{\widetilde{q}_t, \widetilde{\mu}_t\}$.

Given these definitions, (5) can be written as

$$\widetilde{q}_t = \alpha_t + \beta \mathbb{E}_t(\widetilde{q}_{t+1}\widetilde{\mu}_{t+1}),$$

which yields a solution for $\{\widetilde{q}_t\}$ conditional on $\{\widetilde{\mu}_t\}$ and the UE allocation:

$$\widetilde{q}_t = \alpha_t + \sum_{k=1}^{\infty} \beta^k \mathbb{E}_t \left( \alpha_{t+k} \prod_{j=1}^{k} \widetilde{\mu}_{t+j} \right) + \underbrace{\lim_{k\to\infty} \beta^k \mathbb{E}_t \left( \widetilde{q}_{t+k} \prod_{j=1}^{k} \widetilde{\mu}_{t+j} \right)}_{=0 \text{ with no bubbles}}. \tag{20}$$

Writing (2) and (7) in terms of $\{\widetilde{q}_t, \widetilde{\mu}_t\}$, we get

$$\kappa_t \widetilde{q}_t + u_{c,t} \frac{b_{t+1}}{R_t} \geq 0, \qquad (\widetilde{\mu}_t - 1)\left( \kappa_t \widetilde{q}_t + u_{c,t} \frac{b_{t+1}}{R_t} \right) = 0, \qquad \widetilde{\mu}_t \geq 1.$$

Consider setting

$$\widetilde{\mu}_t = 1 + \epsilon_t. \tag{21}$$

Using (20), the associated $\{\widetilde{q}_t\}$ satisfies

$$\widetilde{q}_t = \bar{q}_t u_{c,t} + \sum_{k=1}^{\infty} \beta^k \mathbb{E}_t \left\{ \alpha_{t+k} \left[ \prod_{j=1}^{k} (1 + \epsilon_{t+j}) - 1 \right] \right\} \geq \bar{q}_t u_{c,t}.$$

Define $\widetilde{y}_t \equiv \kappa_t \widetilde{q}_t + u_{c,t} \frac{b_{t+1}}{R_t}$ and note that $\widetilde{q}_t \geq \bar{q}_t u_{c,t}$ implies $\widetilde{y}_t \geq \bar{y}_t$ for all $(t, s^t)$. Suppose $\bar{y}_t(s^t) \geq 0$. By Definition 6.2, $\epsilon_t(s^t) = 0$, so (21) implies $\widetilde{\mu}_t(s^t) = 1$. Since $\widetilde{y}_t(s^t) \geq \bar{y}_t(s^t) \geq 0$, (2) and (7) are satisfied at this $(t, s^t)$. Now suppose $\bar{y}_t(s^t) < 0$. By Definition 6.2, (18) must hold, which implies $\widetilde{q}_t(s^t) = -\frac{u_{c,t}(s^t)}{\kappa_t(s^t)} \frac{b_{t+1}(s^t)}{R_t(s^t)}$, and thus $\widetilde{y}_t(s^t) = 0$. Moreover, by Definition 6.2, $\epsilon_t(s^t) \geq 0$, so (21) implies $\widetilde{\mu}_t(s^t) \geq 1$. Hence, (2) and (7) are satisfied at this $(t, s^t)$, and thus at all $(t, s^t)$.

We have constructed $\{\tau_t\} \subseteq (-1, 0]$ and $\{q_t, \mu_t\}$ that together with the UE allocation $\{b_{t+1}, c_t, h_t, v_t\}$ satisfy Definition 2.1. To see that $\{\tau_t\}$ is Ramsey optimal and time consistent, observe that by Lemma 6.1, the UE allocation is the first best, conditional on the asset market

structure. Since any implementable allocation is weakly inferior to the UE allocation, the constructed $\{\tau_t\}$ is trivially Ramsey optimal. Furthermore, the UE allocation of Definition 6.1 can be equivalently defined as a set of policy functions generated by the solution to the Bellman equation

$$v(b,z,R) = \max_{b',h,v} \left[ u\left( zF(1,h,v) - p_v v + b - \frac{b'}{R}, h \right) + \beta \mathbb{E}(v(b',z',R') \mid z,R) \right]. \tag{22}$$

Consequently, from the perspective of any $(t,s^t)$, the optimal decision is to choose a policy $\{\{\tau_{t+j}(s^{t+j})\}_{s^{t+j}\in S^{t+j}|s^t}\}_{j\geq 0}$ consistent with the policy functions generated by (22)—that is, to continue implementing the UE allocation. This completes the proof. ■

We now identify conditions under which there exists an admissible perturbation of Definition 6.2. A *necessary* condition is that for all $(t,s^t)$ in which $\bar{y}_t(s^t) < 0$, there exists $s^{t+i} \in S^{t+i} \mid s^t$ in which $\bar{y}_{t+i}(s^{t+i}) < 0$. In other words, whenever the collateral constraint evaluated at the UE allocation is violated, it must also be violated in the future with a positive probability. Intuitively, the collateral constraint must be relevant in the UE ergodic distribution, which should trivially be the case, since the UE has more debt than in the DE with the collateral constraint.

The following assumption provides a stronger *sufficient* condition that allows constructing an admissible $\{\epsilon_t\}$ in closed form.

**Assumption 6.1.** *For all $(t,s^t)$, if $\bar{y}_t(s^t) < 0$, there exists $i \geq 1$, such that $\bar{y}_{t+j} \geq 0$ almost surely (a.s.) for all $j \in \{1,2\ldots,i-1\}$, $\bar{y}_{t+i} < 0$ a.s., and*

$$\beta^i \mathbb{E}_t \left( \frac{\bar{y}_{t+i}}{\kappa_{t+i}} \right) - \frac{\bar{y}_t(s^t)}{\kappa_t(s^t)} \geq 0. \tag{23}$$

In words, Assumption 6.1, first, requires that whenever the collateral constraint evaluated at the UE allocation is violated, it must also be violated in the future—this part is the necessary condition discussed above—and at the closest such date, it must be violated

with probability one. Intuitively, the latter condition should be satisfied at the UE ergodic distribution, as long as the distribution of bond holdings is located sufficiently far to the left from the distribution in the DE with the collateral constraint. Furthermore, Assumption 6.1 requires a stationarity condition (23) imposing that the magnitude of the violation of the collateral constraint evaluated at the UE allocation does not increase in discounted terms. In particular, the UE bond holdings distribution must be bounded below, which is ensured by the natural borrowing limit that arises due to the Inada condition on $u$ and the transversality condition. We verify Assumption 6.1 numerically in Section 6.3.

The next proposition demonstrates that Assumption 6.1 is sufficient for the existence of an admissible perturbation of Definition 6.2. Proposition 6.1 then implies the existence of a policy $\{\tau_t\}$ that implements the UE allocation.

**Proposition 6.2.** *Under Assumption 6.1, there exists an admissible perturbation $\{\epsilon_t\}$ of Definition 6.2.*

*Proof.* For all $(t, s^t)$, if $\bar{y}_t(s^t) \geq 0$, we set $\epsilon_t(s^t) = 0$, consistent with Definition 6.2. We will construct the remaining sections of $\{\epsilon_t\}$ going forward in time along the branches $s^\infty \in S^\infty$ of the contingency tree.

For an arbitrary $s^\infty \in S^\infty$, first, find $(t, s^t)$ in which $\bar{y}_t(s^t) < 0$ and either $t = 0$ or $\bar{y}_{t-j}(s^{t-j}) \geq 0$ for all $j \in \{1, 2, \ldots, t\}$ and $s^{t-j}$ in the same branch. If there is no such $(t, s^t)$, the construction for this branch is complete. Otherwise, without loss of generality, we can set $\epsilon_t(s^t) = 0$, since Definition 6.2 does not restrict $\epsilon_t(s^t)$ in this case. By Assumption 6.1, there exists $i \geq 1$, such that $\bar{y}_{t+j} \geq 0$ a.s. for all $j \in \{1, 2 \ldots, i-1\}$, and $\bar{y}_{t+i} < 0$ a.s. If $\{\{\epsilon_{t+j}(s^{t+j})\}_{s^{t+j} \in S^{t+j}|s^t}\}_{j \geq i}$ satisfies Definition 6.2, (18) must hold a.s. at $s^{t+i} \in S^{t+i} \mid s^t$:

$$-\frac{\bar{y}_{t+i}(s^{t+i})}{\kappa_{t+i}(s^{t+i})} = \sum_{k=1}^{\infty} \beta^k \mathbb{E}_{t+i} \left\{ \alpha_{t+i+k} \left[ \prod_{j=1}^{k} (1 + \epsilon_{t+i+j}) - 1 \right] \right\}. \tag{24}$$

Multiplying both sides by $\beta^i(1+\epsilon_{t+i}(s^{t+i}))$ and rearranging,

$$\epsilon_{t+i}(s^{t+i})\sum_{k=i+1}^{\infty}\beta^k\mathbb{E}_{t+i}(\alpha_{t+k})-\beta^i(1+\epsilon_{t+i}(s^{t+i}))\frac{\bar{y}_{t+i}(s^{t+i})}{\kappa_{t+i}(s^{t+i})}$$
$$=\sum_{k=i+1}^{\infty}\beta^k\mathbb{E}_{t+i}\left\{\alpha_{t+k}\left[\prod_{j=i}^{k}(1+\epsilon_{t+j})-1\right]\right\}. \quad (25)$$

Now rearranging (18) at $s^t$,

$$\begin{aligned}-\frac{\bar{y}_t(s^t)}{\kappa_t(s^t)}&=\sum_{k=1}^{i}\beta^k\mathbb{E}_t\left\{\alpha_{t+k}\left[\prod_{j=1}^{k}(1+\epsilon_{t+j})-1\right]\right\}+\sum_{k=i+1}^{\infty}\beta^k\mathbb{E}_t\left\{\alpha_{t+k}\left[\prod_{j=1}^{k}(1+\epsilon_{t+j})-1\right]\right\}\\&=\beta^i\mathbb{E}_t(\alpha_{t+i}\epsilon_{t+i})+\sum_{k=i+1}^{\infty}\beta^k\mathbb{E}_t\left\{\alpha_{t+k}\left[\prod_{j=i}^{k}(1+\epsilon_{t+j})-1\right]\right\}\\&=\mathbb{E}_t\left[\epsilon_{t+i}\sum_{k=i}^{\infty}\beta^k\alpha_{t+k}-\beta^i(1+\epsilon_{t+i})\frac{\bar{y}_{t+i}}{\kappa_{t+i}}\right],\end{aligned}$$

where the second equality follows from $\epsilon_{t+j}=0$ a.s. for all $j\in\{1,2\ldots,i-1\}$, and the third equality follows from the law of iterated expectations and (25). We thus obtain a restriction on $\{\epsilon_{t+i}(s^{t+i})\}_{s^{t+i}\in S^{t+i}|s^t}$:

$$\beta^i\mathbb{E}_t\left(\frac{\bar{y}_{t+i}}{\kappa_{t+i}}\right)-\frac{\bar{y}_t(s^t)}{\kappa_t(s^t)}=\mathbb{E}_t\left[\epsilon_{t+i}\left(\sum_{k=i}^{\infty}\beta^k\alpha_{t+k}-\beta^i\frac{\bar{y}_{t+i}}{\kappa_{t+i}}\right)\right]. \quad (26)$$

The stationarity condition (23) of Assumption 6.1 postulates that the left-hand side of (26) is nonnegative. Since $\bar{y}_{t+i}<0$ a.s. and, by definition, $\alpha_t$ is a positive-valued function for all $t$, we have $\sum_{k=i}^{\infty}\beta^k\alpha_{t+k}-\beta^i\frac{\bar{y}_{t+i}}{\kappa_{t+i}}>0$ a.s. Hence, we can find infinitely many $\{\epsilon_{t+i}(s^{t+i})\}_{s^{t+i}\in S^{t+i}|s^t}\subseteq\mathbb{R}_+$ satisfying (26). For example, we can set $\epsilon_{t+i}(s^{t+i})=\bar{\epsilon}_{t+i}(s^t)$ for

all $s^{t+i} \in S^{t+i} \mid s^t$, in which case we get an explicit solution

$$\begin{aligned}\bar{\epsilon}_{t+i}(s^t) &= \left[\mathbb{E}_t\left(\sum_{k=i}^{\infty}\beta^k\alpha_{t+k} - \beta^i\frac{\bar{y}_{t+i}}{\kappa_{t+i}}\right)\right]^{-1}\left[\beta^i\mathbb{E}_t\left(\frac{\bar{y}_{t+i}}{\kappa_{t+i}}\right) - \frac{\bar{y}_t(s^t)}{\kappa_t(s^t)}\right] \\ &= \frac{\beta^i\mathbb{E}_t\left(\frac{\bar{y}_{t+i}}{\kappa_{t+i}}\right) - \frac{\bar{y}_t(s^t)}{\kappa_t(s^t)}}{\beta^i\mathbb{E}_t\left(\bar{q}_{t+i}u_{c,t+i} - \frac{\bar{y}_{t+i}}{\kappa_{t+i}}\right)},\end{aligned} \tag{27}$$

where the second equality follows by definition of $\alpha_t$. Going forward from $s^t$ to $s^{t+i}$ in the same branch $s^\infty \in S^\infty$ and repeating the steps above, we ensure that (24) holds, validating the construction at $s^t$. Continuing in the same fashion, and doing that for all $s^\infty \in S^\infty$, we will have constructed an admissible perturbation $\{\epsilon_t\}$ of Definition 6.2. ■

We conclude this section by demonstrating that an analog of Proposition 6.1 holds when $\theta \neq 0$ but the policymaker can additionally set optimally a tax on intermediate inputs $\tau_t^v$. We also note that Proposition 6.2 remains true without change.

The agent's budget constraint is now

$$c_t + q_t k_{t+1} + (1+\tau_t^v)p_v v_t + \frac{b_{t+1}}{R_t(1+\tau_t)} \leq z_t F(k_t, h_t, v_t) + q_t k_t + b_t + T_t,$$

where $T_t \equiv \tau_t^v p_v v_t - \frac{b_{t+1}}{R_t}\frac{\tau_t}{1+\tau_t}$. Hence, (6) is replaced by

$$p_v = z_t F_v(1, h_t, v_t)\frac{u_c(c_t, h_t)}{u_c(c_t, h_t)(1+\tau_t^v) + \theta\mu_t}. \tag{28}$$

We then get the following proposition.

**Proposition 6.3.** *Suppose the policymaker can additionally set optimally a tax on intermediate inputs $\tau_t^v$. Then for any $\{\epsilon_t\}$ satisfying Definition 6.2, there exists a sequence of state-contingent taxes $\{\tau_t\} \subseteq (-1, 0]$ and $\{\tau_t^v\}$, asset prices $\{q_t\}$, and Lagrange multipliers $\{\mu_t\}$ that together with the UE allocation $\{b_{t+1}, c_t, h_t, v_t\}$ satisfies Definition 2.1 with (6) replaced by (28). The policy $\{\tau_t, \tau_t^v\}$ is Ramsey optimal and time consistent. Proposition 6.2*

*remains true without change.*

*Proof.* Conditional on the UE allocation and $\{\mu_t\}$, set

$$\tau_t^v = -\frac{\theta \mu_t}{u_c(c_t, h_t)},$$

which ensures that (28) is equivalent to (15). Given an updated definition $\widetilde{y}_t \equiv \kappa_t \widetilde{q}_t + u_{c,t}\left(\frac{b_{t+1}}{R_t} - \theta p_v v_t\right)$, the proof of Proposition 6.1 applies here. Proposition 6.2 remains true without change, since none of the arguments there depend on the value of $\theta$. ■

## 6.3 Numerical example

In this section, we apply Propositions 6.1 and 6.2 to construct numerically the optimal time-consistent debt tax $\{\tau_t\}$ that implements the UE allocation.

We assume the same preferences and technology as in BM, which implies that (4) and (15) give immediately the policy functions for labor $h(z)$ and intermediate inputs $v(z)$. Computing the UE then reduces to computing the fixed point—the policy function for next-period bond holdings $b'(b, z, R)$—of the Euler equation (14), using (1) to back out the policy function for consumption $c(b, z, R)$.

We set the same structural parameters as in BM. For simplicity, we focus on the productivity process $\{z_t\}$ as a source of uncertainty, using the 3-state Markov process calibrated in BM. We use the other Markov processes calibrated in BM to set $R_t = 1.012$, the intermediate state of the 3-state process, and $\kappa_t = 0.886$, the mean of the 2-state process with respect to its stationary distribution.

We approximate the policy function $b'(b, z)$ with a piecewise linear function for each $z$ on a grid of 1,000 points, bounding its ergodic distribution. The grid points are equally spaced in logarithmic units, having more points in original units toward the lower bound, accounting for a greater curvature of the policy function close to the natural borrowing limit. At each state $(b, z)$, given a guess for the next-period policy function, we solve the Euler equation

(14) numerically for the current choice $b'$, iterating until convergence. After converging on the UE allocation, we solve the fixed-point equation (16) for the asset price function $\bar{q}(b, z)$ and use (17) to construct the excess collateral value $\bar{y}(b, z)$.

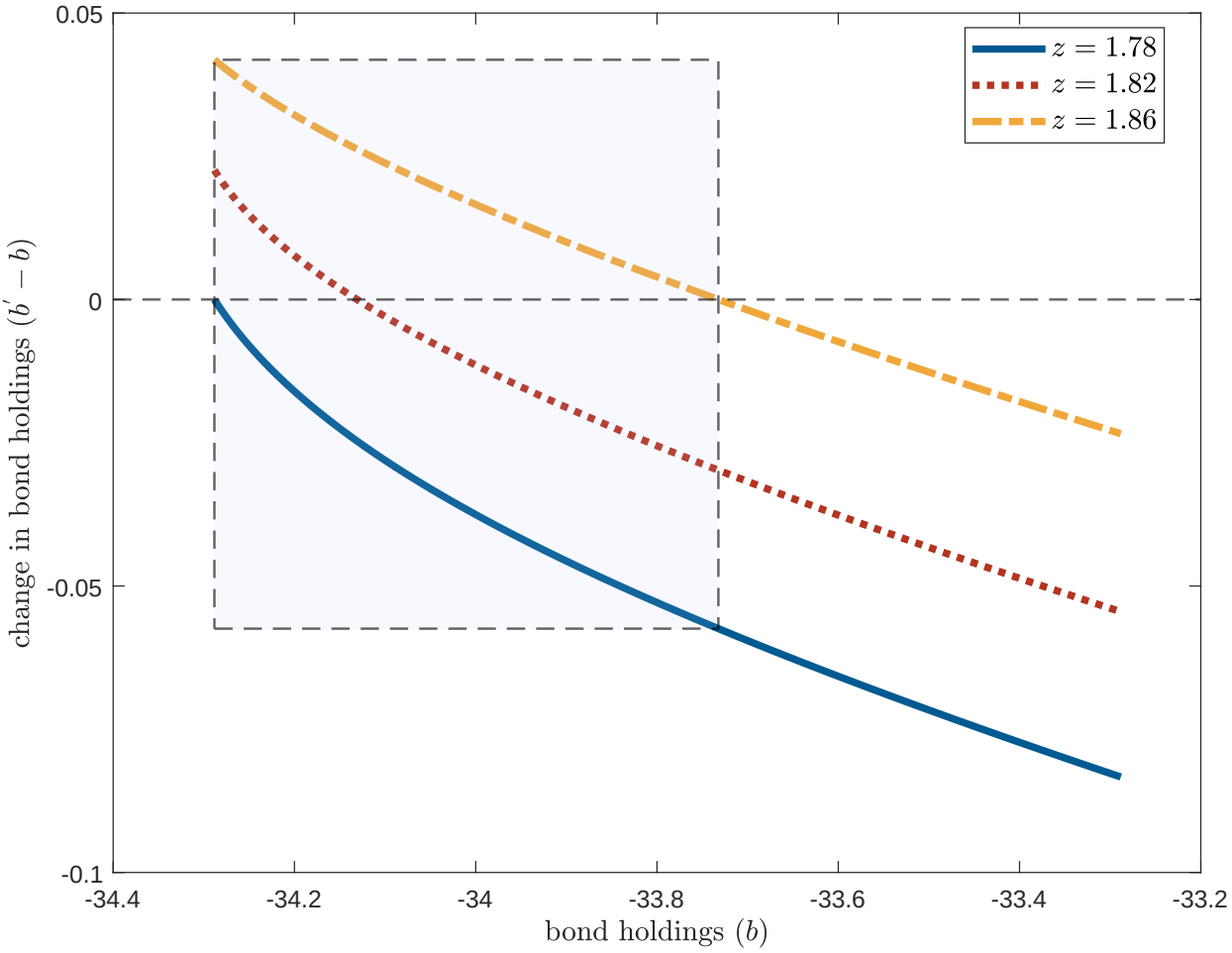
We simulate the UE for 100,000 periods, with a 1,000-period burn-in, and use (27) to construct a time series for the admissible perturbation $\{\epsilon_t\}$, which maps directly to a time series of transformed CE Lagrange multipliers $\{\widetilde{\mu}_t\}$ using (21). We then construct a time series of CE Lagrange multipliers $\{\mu_t\}$ using the inverse transformation $\mu_t = \frac{u_{c,t}}{\kappa_t}(\widetilde{\mu}_t - 1)$. Finally, we get a time series of the optimal time-consistent debt taxes $\{\tau_t\}$ using (19).

Figure 6 displays the change in bond holdings policy function $b'(b, z) - b$ for each $z$. The natural borrowing limit and the agent's relative impatience ($\beta R < 1$) imply the existence of a bond holdings ergodic distribution, which is bounded by the left and right sides of the lightly shaded rectangle in Figure 6. The left side corresponds, approximately, to the natural borrowing limit.[6] When the level of debt is not extreme, it is optimal for the impatient agent to accumulate debt and consume more today independently of $z$. In fact, if $z$ is low and the agent is relatively poor, it is always optimal to increase debt to prevent consumption from being too low. However, when $z$ is relatively high and the agent has a lot of debt, it is optimal instead to deaccumulate debt to insure against the possibility of a low $z$ and low wealth in the next period. This precautionary saving motive in the presence of the natural borrowing limit is key for the existence of an ergodic distribution of bond holdings (Aiyagari, 1994).

Figure 7 displays the empirical distributions—both histograms and kernel density function estimates—of the UE bond holdings $b$, the UE-based excess collateral value $\bar{y}/u_c$ given by (17), the optimal time-consistent debt tax $\tau$ given by (19) that implements the UE allocation in the regulated CE of Definition 2.1, and the corresponding CE Lagrange multiplier

[6] Numerically, we cannot set the lower bound of bond holdings $\underline{b}$ exactly to the natural borrowing limit $\hat{b}$, since it implies the infinite marginal utility of consumption due to the Inada condition. If $\underline{z}$ is the lowest productivity state, for any $\underline{b} > \hat{b}$, we will have $b'(\underline{b}, \underline{z}) < \underline{b}$. Hence, numerically, we must restrict the choice $b'$ by $\underline{b}$ to avoid extrapolation. This lower bound $\underline{b}$ can be chosen sufficiently close to $\hat{b}$, such that $\Pr(b_t = \underline{b}) = 0$ in any reasonably long simulation. Specifically, we set $\underline{b} = \hat{b} + 1\text{e}{-8}$.

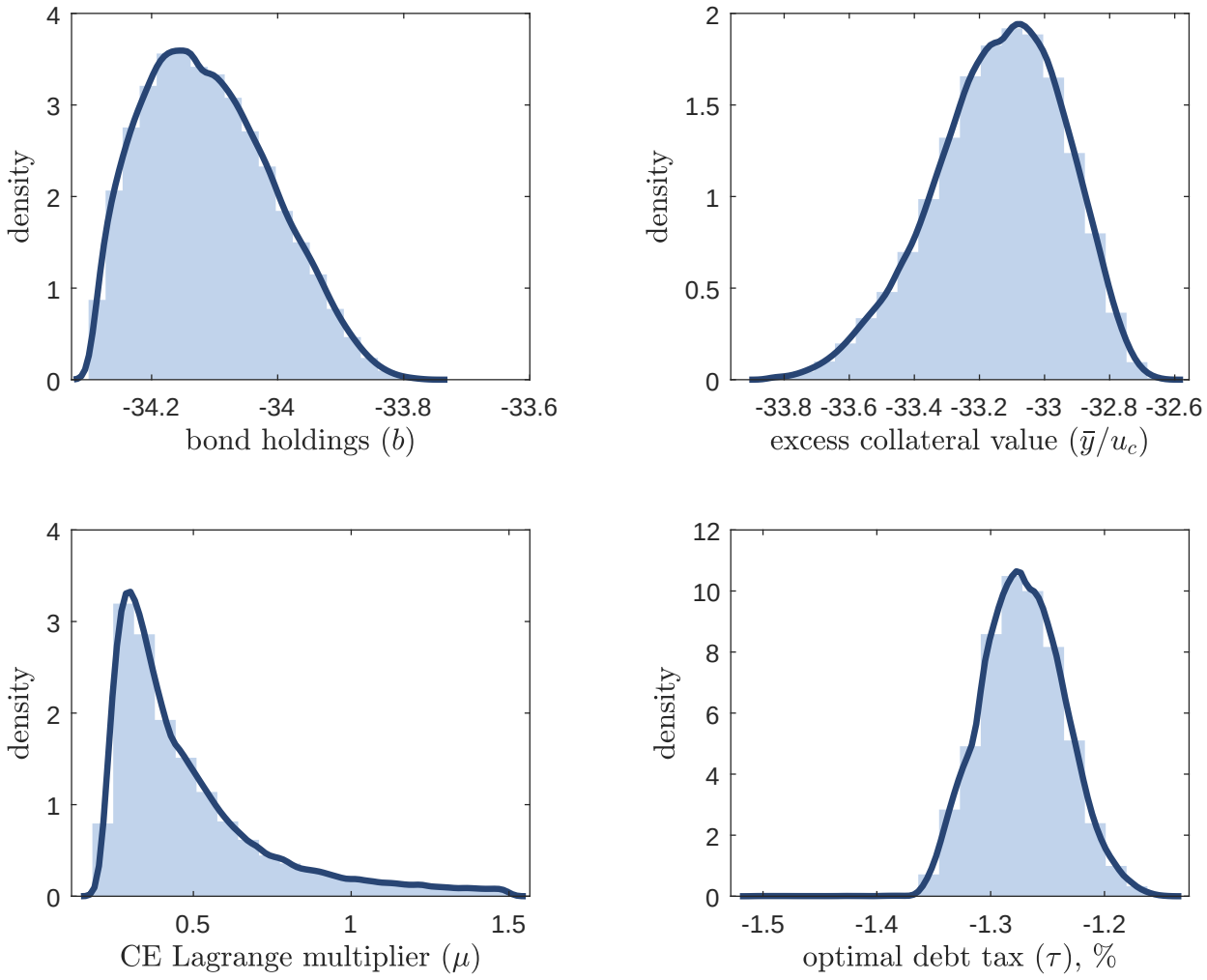
**Figure 6: UE change in bond holdings policy function**




Notes: The lightly shaded rectangle is the ergodic set in the bond holdings/change in bond holdings space.

$\mu$. The empirical distributions are based on a 100,000-period simulation with a 1,000-period burn-in. The optimal time-consistent debt tax is constructed according to Proposition 6.1, conditional on an admissible perturbation $\{\epsilon_t\}$ of Definition 6.2, constructed according to Proposition 6.2. The collateral constraint is always violated at the UE allocation in the ergodic distribution, that is, we have $\Pr(\bar{y}_t < 0) = 1$ a.s. Consequently, $i = 1$ a.s. in the sense of Assumption 6.1. Moreover, the stationarity condition (23) of Assumption 6.1 is always satisfied in the simulation, which is confirmed by $\mu_t > 0$ a.s. and $\tau_t < 0$ a.s., as clear from the figure.

**Figure 7: Empirical distributions**




Notes: The empirical distributions are based on a 100,000-period simulation with a 1,000-period burn-in. The bars represent the histograms with a density normalization on the y-axis. The solid lines are the probability density function estimates based on a normal kernel. Bond holdings $b$ and the excess collateral value $\bar{y}/u_c$ given by (17) are based on the UE allocation. The debt tax $\tau$ is the optimal time-consistent debt tax (19) that implements the UE allocation, conditional on the admissible perturbation $\{\epsilon_t\}$ based on (27). The Lagrange multiplier $\mu$ is the multiplier that arises in the regulated CE of Definition 2.1 conditional on the optimal debt tax. The Lagrange multiplier distribution is heavily right skewed, with $\max_t(\mu_t) \approx 23,393.8$, so a truncated distribution is plotted.

The empirical distribution of bond holdings is right skewed, reflecting the natural borrowing limit. The location of the distribution of the excess collateral value is determined by

the distribution of bond holdings, but its left skew comes from the prominent left skew of the UE asset price distribution. The distribution of the implied CE Lagrange multiplier is extremely right skewed, so we plot a truncated distribution, but the important property is that the Lagrange multiplier is always positive, i.e., the collateral constraint would always be strictly binding in the CE. The distribution of the optimal time-consistent debt tax is left skewed but has a relatively low variance, at least, in the ergodic distribution, with most of its mass concentrated in the $[-1.4\%, -1.1\%]$ interval. By construction, the optimal tax never takes positive values, and thus does not have a macroprudential component—it has a *subsidy component* only, reflecting *underborrowing* in the DE compared to the UE.

# 7 Conclusion

It is well known that crisis resolution policies feedback on ex-ante behavior inducing moral hazard, time inconsistency, and other incentive problems. In this paper, we show that adopting macroprudential policy *in the absence of crisis resolution policies* is welfare-reducing in the workhorse Bianch-Mendoza (2018) model economy. The debt subsidy component of the optimal time-consistent debt tax policy is the main source of welfare gains in this benchmark model economy. When ex-post interventions are not available, the time-consistent policy is detrimental to the economy, producing welfare losses relative to the unregulated competitive equilibrium and exacerbating the financial crisis dynamics. Therefore, the effectiveness and benefits of the optimal time-consistent policy are largely driven by post-crisis interventions, crucial in determining both economic outcomes and welfare gains. We also show that when there are no costs associated with ex-post policy implementation, there is no scope for macroprudential policy implementation. The debt subsidy component of the optimal policy supports borrowing and asset prices in crisis times, echoing the classic lending of last resort policies advocated by Bagehot (1873).

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# Appendices

## A Tax on debt repayment

In this appendix, we derive the optimal time-consistent tax policy with the tax placed on debt replayment. The regulated bond Euler equation (16) on p. 604 in BM is the same as in the case of a tax on debt issuance. The budget constraint is

$$c_t + q_t k_{t+1} + p_v v_t + \frac{b_{t+1}}{R_t} \leq z_t F(k_t, h_t, v_t) + q_t k_t + (1 + \tau_{t-1}) b_t + T_t,$$

where $T_t \equiv -\tau_{t-1} b_t$ is the lump-sum transfer that rebates the taxed amount back to the agent. The regulated bond Euler equation is

$$u'_t = \beta R_t (1 + \tau_t) \mathbb{E}_t(u'_{t+1}) + \mu_t. \tag{A.1}$$

Equating the right-hand sides of (8) and (A.1), we obtain

$$\tau_t = \underbrace{-\frac{\mathbb{E}_t(\xi_{t+1} u''_{t+1} \mathcal{Q}_{t+1})}{\mathbb{E}_t(u'_{t+1})}}_{\tau_t^{\text{MP}}} + \underbrace{\frac{\xi_t[\beta R_t \mathbb{E}_t(\Omega_{t+1}) + u''_t q_t] + \mu_t^* - \mu_t}{\beta R_t \mathbb{E}_t(u'_{t+1})}}_{\tau_t^{\text{EP}}}. \tag{A.2}$$

We note here that the expression for $\tau_t$, derived in the proof of Proposition 1 in Appendix A.2 in BM, misses the term corresponding to $\mu_t^* - \mu_t$ in the numerator of $\tau_t^{\text{EP}}$ in (A.2).

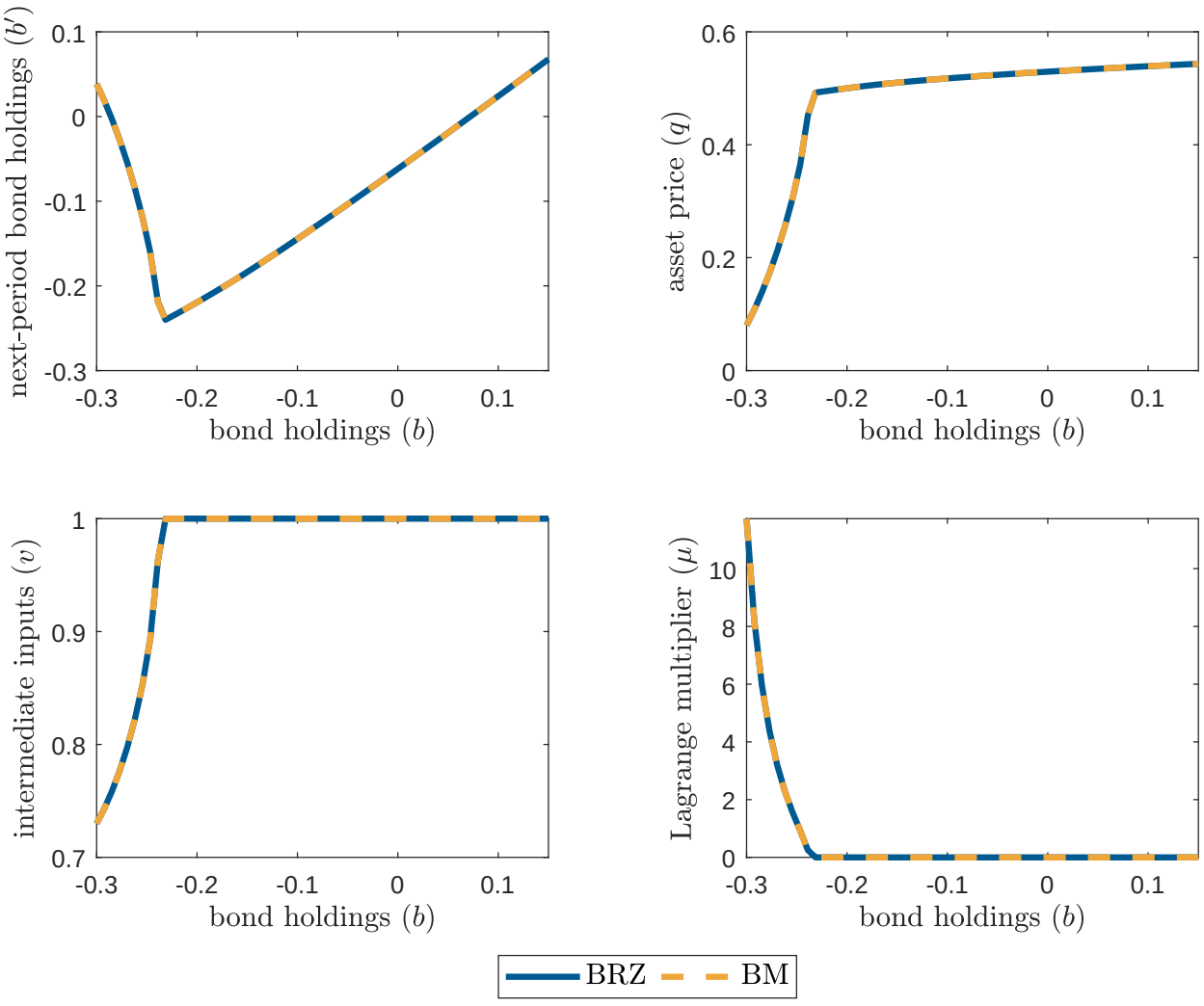
The macroprudential component in (A.2) is identical to (12) obtained placing the tax on debt issuance. Comparing the ex-post components in (10) and (A.2), we see that the only difference is the addition of $\mu_t > 0$ in the denominator of $\tau_t^{\text{EP}}$ in (10). Therefore, the ex-post component in the debt issuance formulation has the same sign but a lower magnitude than in the debt repayment formulation, i.e., $\left|\tau_t^{\text{EP, debt issuance}}\right| < \left|\tau_t^{\text{EP, debt repayment}}\right|$.

# B Numerical differences in replicating BM

The DE allocation we obtain is identical to that in the BM replication package (Figures B.1 and B.2). However, there are significant differences between the SP allocations. Specifically, we find mostly underborrowing rather than overborrowing when the collateral constraint is binding (Figure B.3), and overborrowing is quantitatively much smaller when the collateral constraint is not binding (Figure B.4), consistent with the fact that the ex-post component of the optimal policy is critical in this economy. As a result, there are twice as large welfare gains from the SP allocation in our computation compared to the BM replication package (0.6% versus 0.3%), as shown in Table 2.

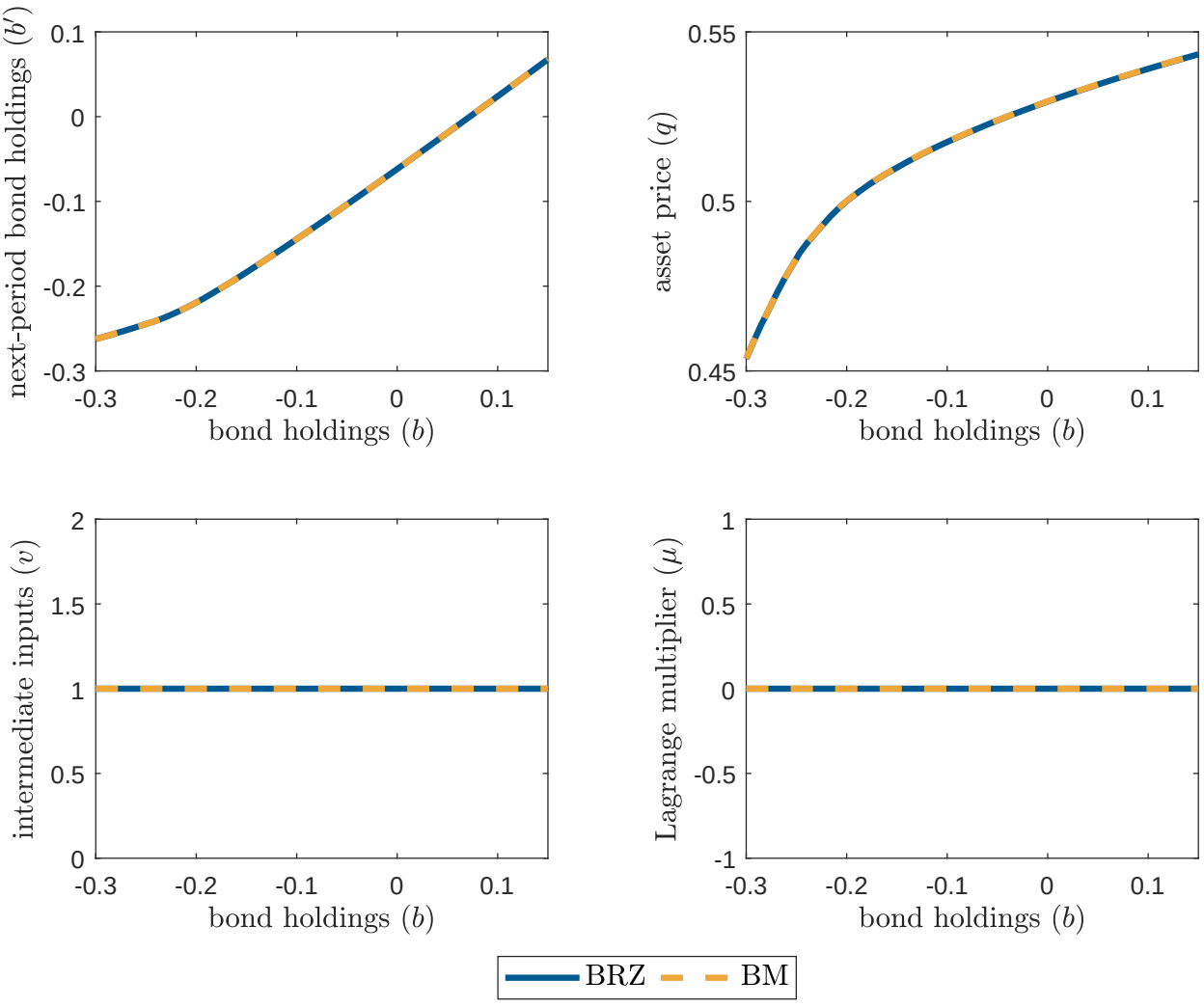
These differences can arise because the BM Fortran code does not fully account for the nonconvexity of the planner's feasible set when the collateral constraint is binding. First, as illustrated in Figure B.5, the planner's best response function may have multiple local maxima, and the Fortran routine `mnbrak` may fail to bracket the global optimum. Second, as illustrated in Figure B.6, for a given level of next-period bond holdings, there may be multiple intermediate input/asset price pairs that satisfy the binding collateral constraint and the asset pricing equation, and the Fortran routine `zbrac` may fail to bracket the welfare-maximizing root.

**Figure B.1: DE policy functions in the tight credit regime**



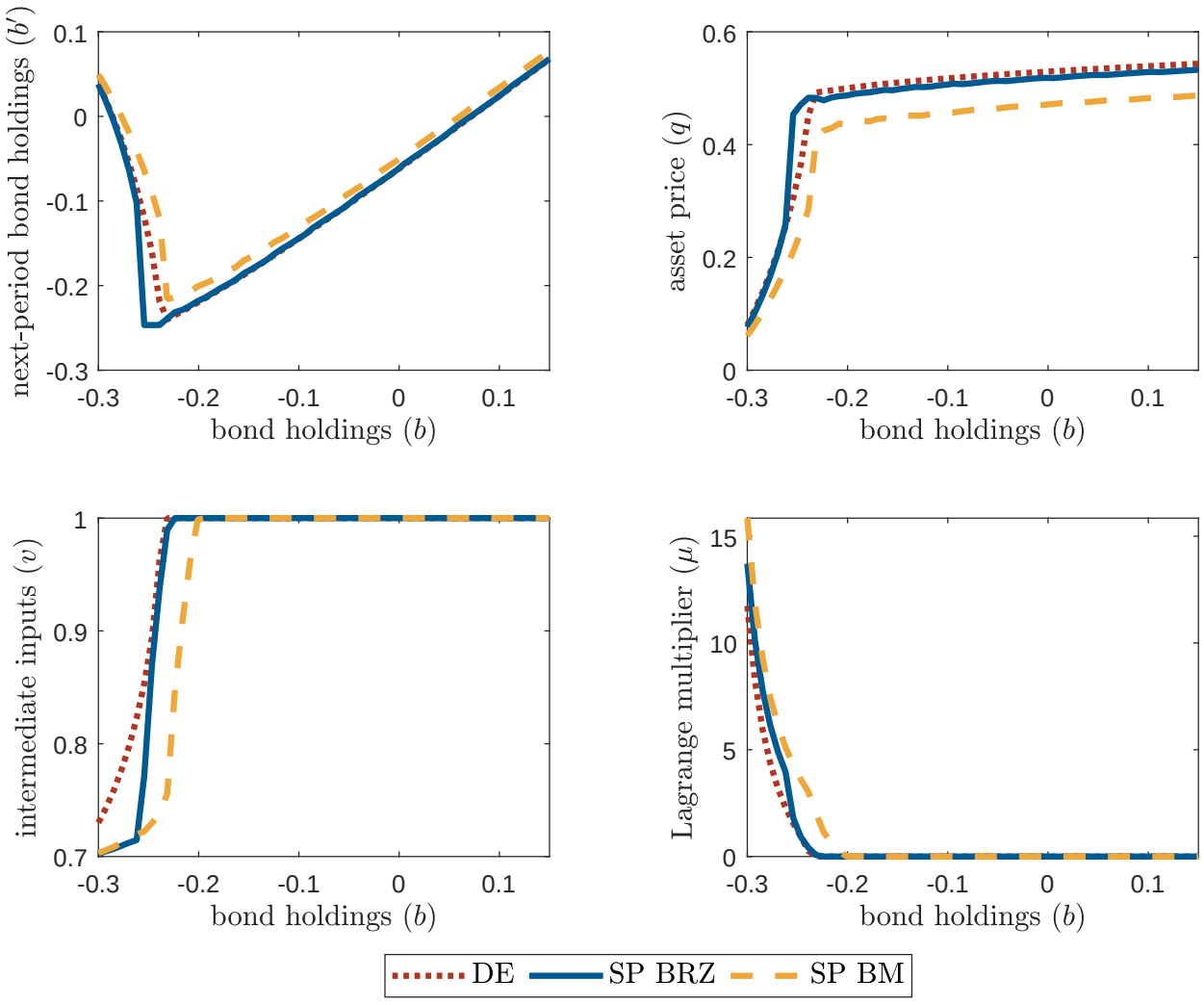

Notes: The policy functions are conditional on average $z$, high $R$, and tight credit regime ($\kappa = \kappa^L$), as in Fig. 2 and Fig. 3 in BM.

**Figure B.2: DE policy functions in the normal credit regime**



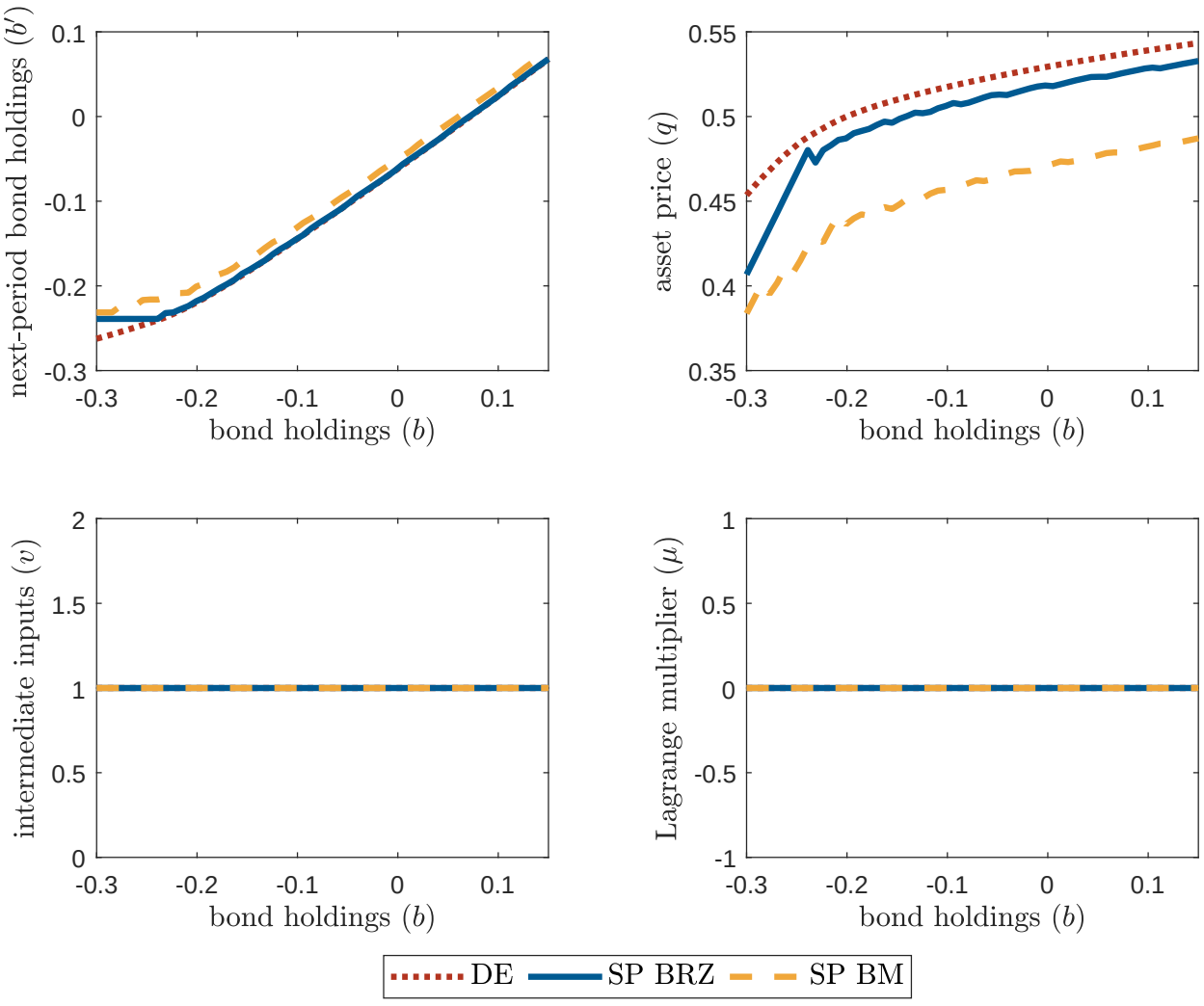

Notes: The policy functions are conditional on average $z$, high $R$, and normal credit regime ($\kappa = \kappa^H$), consistent with Fig. 5A in BM.

**Figure B.3: DE and SP policy functions in the tight credit regime**



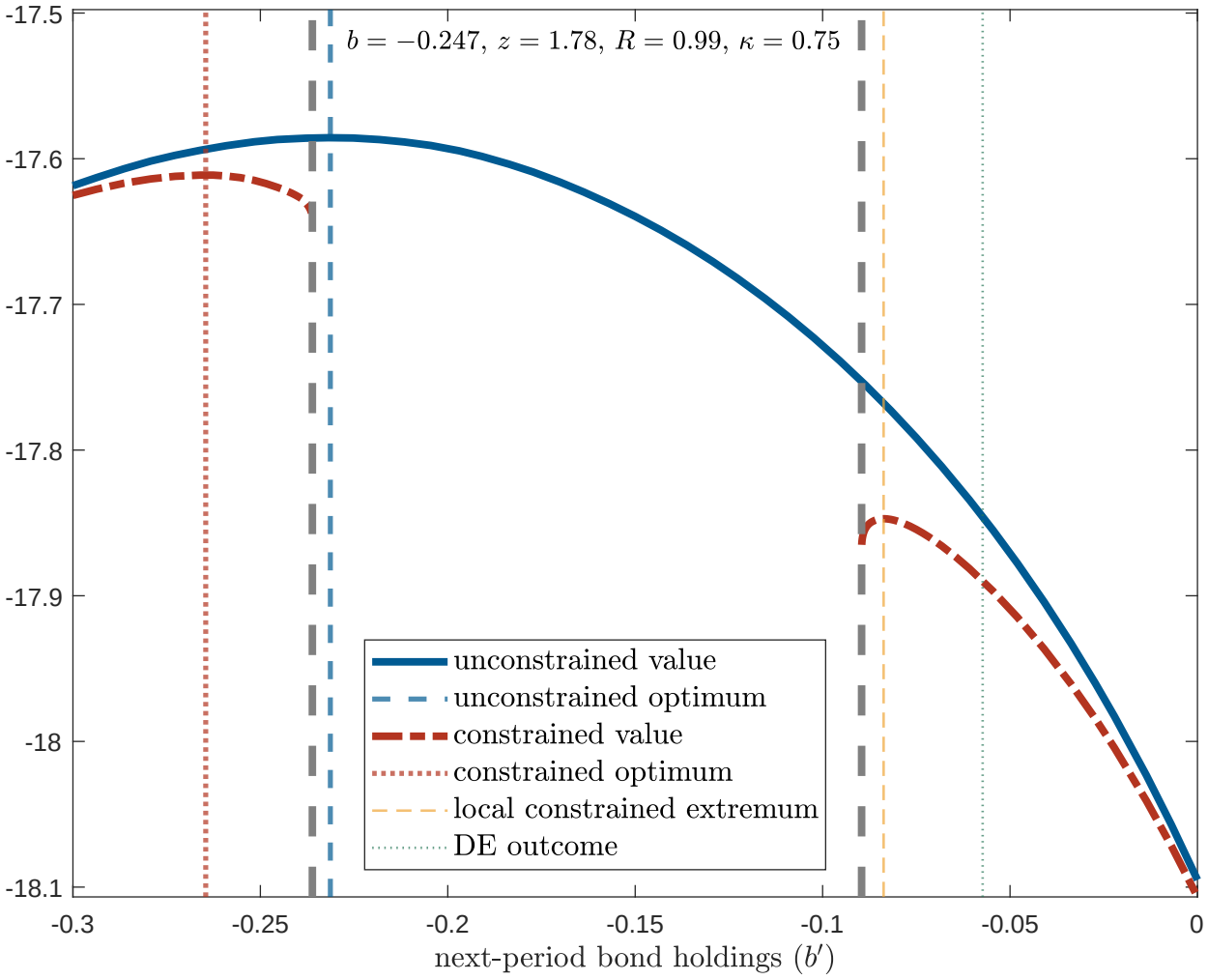

Notes: The policy functions are conditional on average $z$, high $R$, and tight credit regime ($\kappa = \kappa^L$), as in Fig. 2 and Fig. 3 in BM.

**Figure B.4: DE and SP policy functions in the normal credit regime**



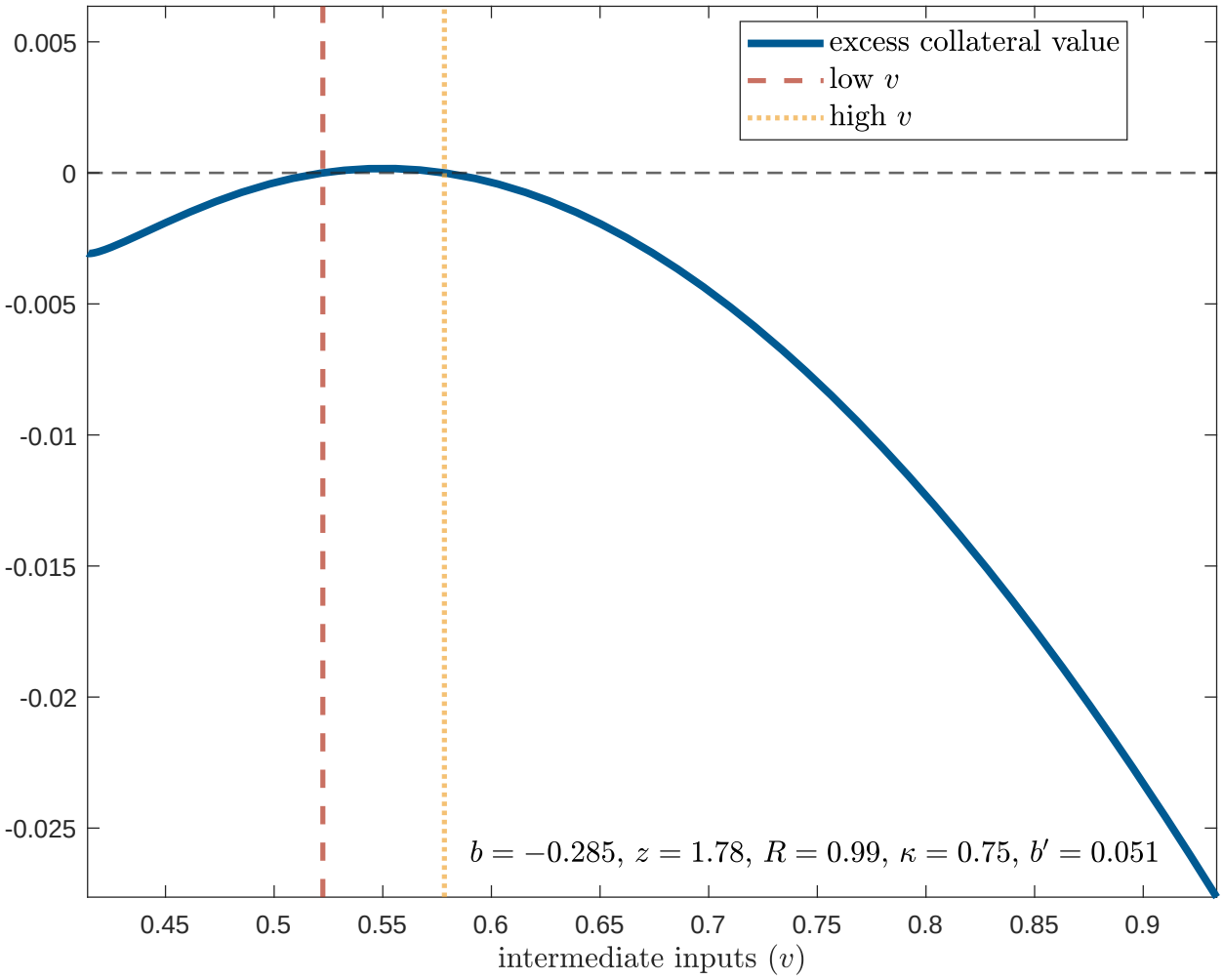

Notes: The policy functions are conditional on average $z$, high $R$, and normal credit regime ($\kappa = \kappa^H$), consistent with Fig. 5A in BM.

**Figure B.5: SP best response to DE with multiple constrained local extrema**




Notes: The planner's values are the maximized expressions in the Bellman equation represented as functions of next-period bond holdings $b'$. These values are conditional on the state $b = -0.247$, $z = 1.78$, $R = 0.99$, $\kappa = 0.75$, and next-period policy functions being the DE policy functions. The unconstrained value ignores the collateral constraint, and the unconstrained optimum violates the collateral constraint, so it is an infeasible choice for the planner. The constrained value is conditional on the binding collateral constraint. The constrained feasible set is nonconvex (disconnected): conditional on $b'$ being in the region between the two dashed gray vertical lines, there does not exist intermediate inputs ($v$) and asset price ($q$) that satisfy the binding collateral constraint and the asset Euler equation. If one were to use the bracketing routine `mnbrak`, starting from the neighborhood of the DE outcome, as in the BM code, the routine would bracket the suboptimal local constrained extremum.

**Figure B.6: Multiple constrained roots**




Notes: The excess collateral value $\kappa q + \frac{b'}{R} - \theta p_v v$ is conditional on the state $b = -0.285$, $z = 1.78$, $R = 0.99$, $\kappa = 0.75$, next-period bond holdings $b' = 0.051$, and next-period policy functions being the DE policy functions. The current asset price $q$ is given by the asset Euler equation. If one were to use the bracketing routine `zbrac`, starting from the neighborhood of the unconstrained level of intermediate inputs, as in the BM code, the routine would fail to bracket the roots. The largest root is welfare-maximizing.