Assortative Matching with Private Information

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Abstract

We study matching markets with two-sided private information where agents care about their partner's unobserved type. Competing platforms post fees and promised partner distributions, while privately-informed agents direct their search to preferred platforms. Under common ranking and supermodularity, we prove that equilibrium must be separating: each platform attracts exactly one type from each side. We characterize positive assortative matching equilibria and provide sufficient conditions for their existence. The mechanism for separation depends crucially on willingness-to-pay: when more desirable types have higher willingness-to-pay for any match, they separate through high fees; when they have lower willingness-to-pay, they separate through low matching rates.

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1 Introduction

This paper develops a model of markets with two-sided private information where heterogeneous agents care directly about their trading partner's unobserved type. Platforms compete to match agents, but cannot observe the agents' type. Instead, agents sort by selecting the platform that maximizes their expected utility. Our framework aims to provide a foundation for studying markets ranging from online dating and job search platforms to peer-to-peer exchanges where both sides have private information about attributes that matter to their potential partners.

The key innovation is a competitive search model where platforms post terms-of-trade for two sides of a market: binding fee commitments and promised distributions of trading partners. While platforms can commit to fees, promised distributions are cheap talk because agents' types are private information. Yet in equilibrium these promises must be kept. For example, if a platform promises to attract high-quality types to one side of the market but charges fees that only low-quality types find worthwhile, the promise is not credible and no one will try to trade at this terms-of-trade.

Unlike mechanism design problems where agents care only about transfers, here agents care about whom they match with. A worker cares about job quality, while the firm cares about worker skill. Dating app users care about the character of potential matches. In markets for expertise, buyers care about sellers' knowledge about the quality of what they are selling, while sellers care about buyers' ability to assess quality. This direct preference over partner types, combined with two-sided private information, fundamentally shapes how markets operate.

Our main results hold under two natural assumptions: first, that all agents agree on who is desirable (Common Ranking), and second, that more desirable agents have a stronger preference for more desirable partners (Supermodularity). Under these conditions, we prove that any terms-of-trade that maximizes platform profits must attract exactly one type from each side of the market (Corollary 1). Moreover, only downward incentive constraints bind: a platform may need to distort the terms-of-trade in order to thwart lower agent types' desire to participate, but in equilibrium higher agent types are not tempted to come to the platform (Corollary 2). These results emerge from competitive pressure between platforms. Competition determines everyone's outside option, the value of matching using a different platform.

Separation requires observationally equivalent agents to choose different terms-of-trade, depending on their private information. How separation occurs depends crucially on whether agents have increasing or decreasing willingness-to-pay for matches. With increasing will-

ingness-to-pay, where higher types value any given match more than lower types, separation works through fees. For instance, in labor markets, a high-skill worker may value matching with any given firm more than a low-skill worker does. Higher types pay more to access platforms that attract better partners, while all types match at the maximum feasible rate (Proposition 1). The intuition is straightforward: since higher types value matching more, they can be separated by charging fees that lower types find prohibitively expensive relative to their valuation.

With decreasing willingness-to-pay, where higher types value any given match less than lower types, separation works through matching probabilities. In disease transmission models, healthy individuals value any given match less than sick individuals do, because matching comes with a risk of becoming sick. Higher (healthier) types match less frequently to avoid undesirable partners, while fees are zero (Proposition 2). Here, reducing the matching rate is an effective screening device because the opportunity cost of not matching is greater for lower types who value each match more highly.

When platforms face positive advertising costs, both mechanisms operate simultaneously but in opposite directions depending on willingness-to-pay (Proposition 3). With increasing willingness-to-pay, higher types pay higher fees and match more frequently. With decreasing willingness-to-pay, higher types pay lower fees and match less frequently. These opposing patterns reflect how platforms must cover their costs: with increasing willingness-to-pay, platforms can charge high-value types more and so are able to provide high matching rates. With decreasing willingness-to-pay, high types accept low matching rates to avoid bad partners. Since this is cheaper to provide, competition between platforms drives down fees.

Our results illuminate how real-world matching markets successfully separate types despite asymmetric information. In practice, we see both fees and matching frequencies utilized. Building on early insights of Veblen (1900), research by Spence (1973), Pesendorfer (1995), and Bagwell and Bernheim (1996) showed how wasteful expenditures can signal an individual's unobserved qualities. Conversely, research by Guerrieri and Shimer (2014) shows that illiquidity can serve a similar function, with the owners of high quality assets setting a high price that attracts few buyers. This separates them from the owners of lower quality assets because the high-quality owners are more willing to accept the risk of not selling the asset.

Our theory offers a prediction about when we would expect to see wasteful expenditures (high fees paid to platforms) and when we would expect to see illiquidity (low matching probabilities) in real-world markets. The method of signaling depends on whether more desirable types are more or less eager to match with any given partner, compared to the alternative of not matching. Dating markets provide a particularly vivid illustration of these mechanisms at work.

Consider Tinder's experiments with premium tiers. For years, Tinder displayed a gold badge for premium subscribers, but recently removed this visible indicator. Our theory suggests an interpretation of this decision. If in equilibrium premium subscribers are more desirable to other users, perhaps because they are more serious about finding partners, then the gold badge serves as a valuable signal worth paying for. But if premium subscribers are less desirable on average, perhaps subscribing because they struggle to find matches using the free tier, then the badge becomes a stigma. Removing the visible badge would then increase premium subscriptions by eliminating this negative signal. Even more dramatically, Tinder briefly offered a "Select" tier costing $499 per month with minimal additional features. Such an extreme price could either signal wealth and serious intent (making the badge valuable) or desperation (making it toxic). The tier's quick discontinuation suggests the latter interpretation prevailed.

In this second scenario where premium users have a higher willingness-to-pay but are less desirable, our theory makes an additional prediction: separation should occur through more desirable users choosing not to pay for Tinder Gold and consequently receiving fewer matching opportunities, i.e. limited daily swipes and less frequent appearances in others' swipe decks. Less desirable users with higher willingness-to-pay would subscribe to premium tiers and receive more matching opportunities through unlimited swiping and boosted visibility. While more desirable users likely still achieve more actual matches despite fewer opportunities, the platform separates types by controlling access to the matching process itself. This illustrates how platforms may use multiple instruments, pricing and matching frequencies, to facilitate sorting when types are unobservable.

Related Literature

Our paper contributes to several strands of literature. Most directly, we extend the competitive search framework to environments with two-sided private information where agents care about partner types. Early competitive search models had neither heterogeneity nor private information (Moen, 1997; Shimer, 1996). Subsequent research has extended the basic model to allow for two-sided heterogeneity in order to study assortative matching (Shi, 2001; Shimer, 2005; Eeckhout and Kircher, 2010), but still assumes there is no private information.

Guerrieri, Shimer and Wright (2010) and Guerrieri and Shimer (2014) introduced private information to competitive search but maintained one-sided private information.[1] For example, workers have private information but firms do not, or buyers have private information

[1]There are other works that study environments with one-sided private information and search frictions, for example, Kim and Kircher (2015); Auster and Gottardi (2019); Albrecht et al. (2024); Auster et al. (2025).

about their valuation but sellers care only about price. The crucial difference in our environment is that both sides have private information and both sides care about their partner's type. This creates a fundamentally different screening problem. In Guerrieri, Shimer and Wright (2010), separation occurs when the uninformed party posts a contract to attract a particular type of informed party. In our model, both sides simultaneously screen and are screened, creating a fixed-point problem in beliefs about who participates in each market.

The literature on matching with private information has taken different approaches. Hoppe, Moldovanu and Sela (2009) study assortative matching when agents send costly signals, assuming matching is assortative in signals. Their single-crossing conditions ensure higher types send higher signals in equilibrium. Damiano and Li (2007) and Hoppe, Moldovanu and Ozdenoren (2011) analyze monopolistic platforms that screen privately informed agents. These papers focus on monopoly pricing and mechanism design rather than competitive markets.

Our competitive search approach differs in three key ways. First, we endogenize both fees and matching rates as screening instruments, showing that separation can work through either channel. Second, platform competition drives different predictions: monopolists extract rents while competition transfers surplus to agents. Third, we allow general payoff functions, showing that increasing versus decreasing willingness-to-pay fundamentally changes how markets operate.

Our model fills an important gap. While Myerson and Satterthwaite (1983) provides the canonical model for bilateral trade with private information, they assume that there is a single buyer and a single seller. Moreover, each agent cares only about prices and their own private information, not directly about their trading partner's characteristics. In contrast, we focus on environments with many buyers and sellers where each agent cares both about prices (fees) and about about whom they trade with.

Our technical approach builds on the mechanism design literature with type-dependent outside options. Lewis and Sappington (1989) show that countervailing incentives arise when types differ in their outside options, potentially causing both upward and downward incentive constraints to bind. Jullien (2000) characterizes optimal contracts with type-dependent participation constraints. In our setting, competition among platforms endogenously determines outside options (market utilities), and we show this competition ensures only downward constraints bind. This helps keep our model tractable, particularly for the analysis of positive assortative matching.

The paper also connects to the two-sided markets literature initiated by Rochet and Tirole (2003), Rochet and Tirole (2006) and Armstrong (2006). While that literature focuses on network effects and platform pricing with observable characteristics, we analyze how

platforms facilitate matching when characteristics are unobservable. Weyl (2010) studies monopoly platform design with heterogeneous users, but maintains the assumption that users care only about the number, not the types, of users on the other side.

Our distinction between increasing and decreasing willingness-to-pay connects to the broader screening literature. In insurance markets, Rothschild and Stiglitz (1976) show that high-risk individuals have higher willingness-to-pay for any given coverage level, leading to positive correlation between coverage and premiums. Einav and Finkelstein (2011) survey evidence of "advantageous selection," where low-risk types have higher willingness-to-pay for any given coverage level, reversing standard predictions. Our model shows how both cases arise naturally depending on whether type affects the marginal utility of matching.

Roadmap

Section 2 presents the model, introducing platforms that facilitate matching between privately informed agents. Section 3 provides motivating examples including marriage markets, labor markets, expertise markets, and disease transmission. Section 4 establishes our main theoretical results on separation and the nature of binding incentive constraints. Section 5 characterizes positive assortative matching in a symmetric environment, showing how separation works through fees versus quantities. Section 6 extends the analysis to study asymmetric environments and negative assortative matching. Section 7 concludes. Proofs are in the Appendix.

2 Model

2.1 Platforms and Agents

We consider a static model of a two-sided market with three sets of risk-neutral participants: a-side agents, b-side agents, and platforms. We let $s \in \{a, b\}$ denote one side of the market and $\bar{s}$ denote the other side, so if $s = a$, $\bar{s} = b$ and vice versa.

There exists a fixed measure I^s of s-side agents. Each agent has private information about their type $i \in \mathbb{I}^s$, a compact subset of the unit interval. The distribution of types on side s is governed by an exogenous cumulative distribution function $F^s : \mathbb{I}^s \to [0, 1]$.

When a type-i, side-s agent matches with a type-j, side-$\bar{s}$, i receives a payoff $u^s(i, j)$ before paying any platform fees. When an agent fails to match, their payoff is normalized to zero. We impose the following key assumption on preferences:

Assumption 1 (Common Ranking) *Take any s, $i \in \mathbb{I}^s$, and $j, j' \in \mathbb{I}^{\bar{s}}$. If $j > j'$,*

$u^s(i,j) > u^s(i,j')$.

This assumption requires that all agents on a given side of the market agree on the ranking of potential partners from the other side. That is, if any type-i, side-s agent prefers to match with j rather than j', then all other side-s agents share this preference. Given this commonality in preferences, we adopt the convention of labeling types such that a higher index indicates a partner who gives a higher payoff. Thus, $j > j'$ if and only if all agents on the other side of the market prefer to match with j to j'. Finally, we assume that u^s is continuous in its first argument.

There is also a fixed measure 1 of homogeneous platforms, each of which chooses its advertising effort at constant unit cost $c \geq 0$. This advertising effort facilitates matching between agents on the two sides of the market, with costs covered through endogenous fees paid by the agents. In our leading case, advertising effort is free, $c = 0$, but we also consider the possibility that advertising effort is costly, $c > 0$.

2.2 Terms-of-trade and Payoffs

A terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b}$ specifies four objects: the fees ϕ^a and ϕ^b charged to matched agents on each side and the distributions G^a and G^b of agent types participating on each side. We allow either ϕ^a or ϕ^b to be negative but assume $\phi^a + \phi^b \geq 0$. We denote by $\mathbb{T}$ the set of feasible terms-of-trade, consisting of all tuples where ϕ^a and ϕ^b are real numbers with nonnegative sum, and G^a and G^b are probability distributions with support on subsets of $\mathbb{I}^a$ and $\mathbb{I}^b$, respectively.

Agents direct their search toward particular terms-of-trade $\tau \in \mathbb{T}$. If a type-i, side-s agent succeeds in trading at terms-of-trade $\tau \in \mathbb{T}$ with probability $\lambda^s \in [0,1]$, their expected payoff is

$$U^s(i,\tau,\lambda^s) \equiv \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j) dG^{\bar{s}}(j) - \phi^s \right). \tag{1}$$

That is, with probability λ^s the agent trades, receiving an expected gross payoff equal to their expected utility $u^s(i,j)$ across potential trading partners drawn from the distribution of agents on the other side of the market, $G^{\bar{s}}$, less the platform fee ϕ^s.

Platforms facilitate these trades. For a platform advertising terms-of-trade $\tau \in \mathbb{T}$ with associated matching probabilities λ^s, the expected gross profit per unit of advertising effort is

$$V(\tau,\lambda^a,\lambda^b) \equiv m(\lambda^a,\lambda^b)(\phi^a + \phi^b) \tag{2}$$

where $m(\lambda^a,\lambda^b)$ denotes the platform's matching probability as a function of the agents' matching probability, discussed in the next subsection. Each unit of advertising effort at

terms-of-trade τ results in matches with probability $m(\lambda^a, \lambda^b)$, generating revenue $\phi^a + \phi^b$. The platform's net profit subtracts the advertising cost c from this gross profit.

2.3 Matching Function and Platform Matching Probability

To understand the platform matching probability m, we start with a more familiar object, the matching function M. We assume that in each market, the number of matches is a constant returns to scale function of the platform's advertising effort, the number of a-side agents, and the number of b side agents. Let α denote the advertising effort and $n^s \geq 0$ denote the number of s-side agents per unit of advertising effort at some terms-of-trade τ. Then the number of matches is $\alpha M(n^a, n^b)$, where $M : \mathbb{R}^2_+ \to \mathbb{R}_+$. Consistent with the constant returns to scale assumption, M is nondecreasing and strictly concave in (n^a, n^b), with $M(n^a, n^b) \leq \min\{n^a, n^b\}$. We also assume M is differentiable.

Agents' matching probability is related to the matching function via

$$\lambda^s = L^s(n^a, n^b) \equiv \frac{M(n^a, n^b)}{n^s} \leq 1. \tag{3}$$

for $s = a, b$ and $n^s > 0$. The numerator is the matching probability per unit of advertising effort, while the denominator is s-side agents per unit of advertising effort, and so the ratio is matching probability per s-side agent. This is well-defined when n^a and n^b are both strictly positive and one can handle the cases where n^a and/or n^b converge to either zero or infinity through appropriate limits.

Using this, we can express the platform matching probability via the identity

$$m(L^a(n^a, n^b), L^b(n^a, n^b)) \equiv M(n^a, n^b).$$

This is the probability that the platform matches as a function of the agents' matching probabilities, rather than as a function of the agent-advertising ratios. It is this object that we use in equation (2).

By varying n^a and n^b, we trace out a set of feasible agent matching probabilities:

$$\mathbb{\Lambda} \equiv \{(L^a(n^a, n^b), L^b(n^a, n^b)) | n^a, n^b \geq 0\} \subset [0, 1]^2.$$

The platform matching probability is a function $m : \mathbb{\Lambda} \to [0, 1]$. From the assumptions on the matching function, we get that $\mathbb{\Lambda}$ is a down set: if $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$, then $(\hat{\lambda}^a, \hat{\lambda}^b) \in \mathbb{\Lambda}$ for all

$\hat{\lambda}^a \in [0, \lambda^a]$ and $\hat{\lambda}^b \in [0, \lambda^b]$. We also define

$$\mathbb{\Lambda}^o \equiv \{(\lambda^a, \lambda^b) \in \mathbb{\Lambda} | m(\lambda^a, \lambda^b) > 0\} \subset \mathbb{\Lambda}.$$

This excludes from the set $\mathbb{\Lambda}$ the set of points (λ^a, λ^b) where $m(\lambda^a, \lambda^b) = 0$.

We can also go back from the platform matching probability to the matching dfunction by first finding the number of s-side agents per unit of advertising effort. For all $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$,

$$n^s = N^s(\lambda^a, \lambda^b) \equiv \frac{m(\lambda^a, \lambda^b)}{\lambda^s}. \tag{4}$$

We then recover the matching function via the identity

$$M(N^a(\lambda^a, \lambda^b), N^b(\lambda^a, \lambda^b)) = m(\lambda^a, \lambda^b).$$

Parametric Example. A parametric example illustrates these properties and clarifies the domain restriction in $\mathbb{\Lambda}$. Suppose M takes a CES form:

$$M(n^a, n^b) = (1 + (n^a)^{-\gamma} + (n^b)^{-\gamma})^{-\frac{1}{\gamma}}, \tag{5}$$

where $\gamma > 0$ and $1/(1+\gamma)$ is the elasticity of substitution between n^a and n^b. This matching function is continuous, increasing, strictly concave, and satisfies $M(n^a, n^b) \leq \min\{n^a, n^b\}$ for all $(n^a, n^b) \in \mathbb{R}^2_+$.

To express this in terms of matching probabilities, we first write equation (3) as

$$\lambda^s = L^s(n^a, n^b) = \left(\frac{1 + (n^a)^{-\gamma} + (n^b)^{-\gamma}}{(n^s)^{-\gamma}}\right)^{-\frac{1}{\gamma}}.$$

Since γ is positive, λ^s lies between 0 and 1. Inverting this gives us

$$n^s = N^s(\lambda^a, \lambda^b) = \left(\frac{1 - (\lambda^a)^{\gamma} - (\lambda^b)^{\gamma}}{(\lambda^s)^{\gamma}}\right)^{\frac{1}{\gamma}}.$$

Substituting back into equation (5) gives

$$m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^{\gamma} - (\lambda^b)^{\gamma})^{\frac{1}{\gamma}}. \tag{6}$$

We can equally go from the platform matching probability (6) back to the matching function (5).

By varying (n^a, n^b) in the positive real quadrant, we find that the feasible domain for

matching probabilities is

$$\mathbb{\Lambda} = \{(\lambda^a, \lambda^b) | (\lambda^a, \lambda^b) \geq 0 \text{ and } (\lambda^a)^\gamma + (\lambda^b)^\gamma \leq 1\},$$

with $\mathbb{\Lambda}^o = \{(\lambda^a, \lambda^b) | (\lambda^a, \lambda^b) \geq 0 \text{ and } (\lambda^a)^\gamma + (\lambda^b)^\gamma < 1\}$. Thus the matching probabilities have a bounded domain $\mathbb{\Lambda}$ that is down set.[2]

2.4 Equilibrium

Our equilibrium concept builds on the competitive search literature pioneered by Moen (1997) and extended to environments with private information by Guerrieri, Shimer and Wright (2010). In these models, market-makers post terms-of-trade, and agents on both sides of the market direct their search toward their preferred terms. The resulting equilibrium combines price posting with rational expectations about matching probabilities.

The key innovation of the competitive search approach, relative to random search models, is that it allows market participants to use posted terms-of-trade to solve selection and incentive problems. As in Moen (1997), platforms in our setting act as market-makers who compete to attract agents. However, our setting differs in two important ways. First, following Guerrieri, Shimer and Wright (2010), we allow for private information about agent types. Second, platforms must simultaneously attract agents from both sides of the market, making this a two-sided matching problem with private information.

We break the definition of equilibrium into two parts. First, we define a partial equilibrium where everyone takes as given the agents' *market utility* $\bar{U}^s(i)$, the utility that a type-i, side-s agent can obtain at their best possible terms-of-trade. The partial equilibrium concept requires that platforms cannot profitably deviate by offering a different terms-of-trade, taking these market utilities as given.

In the second step, we impose conditions that endogenously determine these market utilities. The market clearing conditions ensure that the total measure of each type choosing to trade equals the available supply of that type, while free entry ensures that platforms earn zero profits.

More formally, we begin with the partial equilibrium definition:

Definition 1 *A **partial equilibrium** $\{T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}\}$ is two sets $T \subseteq T^p \subseteq \mathbb{T}$ as well as functions $\Lambda^s : T^p \to \mathbb{\Lambda}$ and $\bar{U}^s : \mathbb{I}^s \to \mathbb{R}+$ such that:*

1. *(Optimal Search) For all $\tau = (\phi^s, G^s)_{s=a,b} \in T^p$ and $s \in a, b$,*

[2] We obtain points where $\lambda^s = 0 < \lambda^{\bar{s}}$ by taking the limit as $n^s \to \infty$ for fixed $n^{\bar{s}}$. Points with $\lambda^a = \lambda^b = 0$ correspond to the limit as n^a and n^b both grow without bound. We get points with $(\lambda^a)^\gamma + (\lambda^b)^\gamma = 1$ by taking the limit as n^a and n^b converge to zero with a fixed ratio.

(a) $\bar{U}^s(i) \geq U^s(i, \tau, \Lambda^s(\tau))$ *for all* $i \in \mathbb{I}^s$*;*

(b) $\int_{\mathbb{I}^s} \bar{U}^s(i) dG^s(i) = \int_{\mathbb{I}^s} U^s(i, \tau, \Lambda^s(\tau)) dG^s(i)$*;*

2. *(Impossible Terms-of-Trade) For all* $\tau = (\phi^s, G^s)_{s=a,b} \notin T^p$*, there is no* $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ *such that*

   (a) $\bar{U}^s(i) \geq U^s(i, \tau, \lambda^s)$ *for all* $s \in \{a, b\}$ *and* $i \in \mathbb{I}^s$*; and*

   (b) $\int_{\mathbb{I}^s} \bar{U}^s(i) dG^s(i) = \int_{\mathbb{I}^s} U^s(i, \tau, \lambda^s) dG^s(i)$ *for all* $s \in \{a, b\}$*;*

3. *(Profit Maximization)* $T = \arg\max_{\tau \in T^p} V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$.

Let us first understand why the conditions in part 1 (Optimal Search) capture reasonable restrictions on behavior. Consider condition 1(a), which requires $\bar{U}^s(i) \geq U^s(i, \tau, \Lambda^s(\tau))$ for all agent types. Suppose this inequality were violated for some side s, type-i agent. Then these agents would strictly prefer this terms-of-trade to their market utility $\bar{U}^s(i)$, causing them to flock to this terms-of-trade. This influx would drive down the matching probability $\Lambda^s(\tau)$ until the inequality is restored. Just as in traditional competitive search models, entry continues until agents are indifferent between this terms-of-trade and their market utility. Thus, any equilibrium must satisfy condition 1(a).

Condition 1(b) requires that agents who are supposed to come to the terms-of-trade (those in the support of G^s) obtain exactly their market utility. To see why this is necessary, suppose the expected utility were strictly less than market utility for some types in the support of G^s. Then these agents would prefer to search elsewhere, making it impossible for the terms-of-trade to yield the promised type distribution G^s. Conversely, condition 1(a) tells us that expected utility cannot exceed market utility. Thus, 1(b) is required for the promised type distribution to be consistent with agents' directed search incentives.

The second component of the definition identifies terms-of-trade that are impossible given market utilities. For instance, some terms-of-trade involve fees so high that no agent would be willing to pay them regardless of the matching probabilities. Alternatively, the promised type distribution G^s might be inconsistent with agents' incentives. For example, it may promise that only high types will participate when in fact only low types would be willing to do so. The impossible terms-of-trade condition formalizes this by requiring that for any $\tau \notin T^p$, there exist no feasible matching probabilities, $(\lambda^a, \lambda^b) \in \mathbb{\Lambda}$ that could make these terms-of-trade satisfy condition 1.

The third component determines which possible terms-of-trade at a given level of market utility. Given the matching probabilities $\Lambda^s(\tau)$ for each possible terms-of-trade, platforms choose terms-of-trade $T \subseteq T^p$ that maximize their profit per unit of advertising. This

optimization reflects platforms' optimal behavior given agents' responses encoded in the matching probabilities.

This definition extends the competitive search framework to accommodate two-sided matching with private information. The matching probabilities $\Lambda^s(\tau)$ play a role analogous to market tightness in traditional competitive search models, but now must simultaneously clear both sides of the market while respecting agents' incentives. The partition into possible and impossible terms-of-trade provides a tractable way to incorporate these incentive constraints into the equilibrium concept.

A partial equilibrium determines which terms-of-trade could emerge given fixed market utilities $\bar{U}^s(i)$. A competitive search equilibrium (CSE) endogenously determines these market utilities through free entry of platforms and market clearing.

Definition 2 *A **competitive search equilibrium** is a partial equilibrium* $\{T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}\}$ *and a measure* μ *on the set of profit-maximizing terms-of-trade* T *such that:*

1. *(Free Entry)* $c = V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$ *for all* $\tau \in T$*;*

2. *(Market Clearing)* $\forall s \in \{a, b\}$ *and* $I \subseteq \mathbb{I}^s$*,*

$$\int_I dF^s(i) I^s \geq \int_T \int_I N^s(\Lambda^a(\tau), \Lambda^b(\tau)) dG^s(i) d\mu(\tau)$$

   with equality if $\bar{U}^s(i) > 0$ *for all* $i \in I$*.*

3. *(Market Utility) For all* $s \in \{a, b\}$ *and* $i \in \mathbb{I}^s$ *with* $\bar{U}^s(i) > 0$*,* $\bar{U}^s(i) = \max_{\tau \in T} U^s(i, \tau, \Lambda^s(\tau))$*.*

Let us examine each component of this definition. First, we introduce a measure μ over the set of profit-maximizing terms-of-trade T. This measure captures both the intensive margin (how much advertising there is at each terms-of-trade) and extensive margin (which terms-of-trade are offered). This formulation allows for a rich set of equilibrium outcomes, in particular ones where different platforms attract different distributions of agents at different terms-of-trade.

The free entry condition ensures that in equilibrium, no platform can earn positive profits net of the advertising cost c. If profits were positive at some terms-of-trade $\tau \in T$, more platforms would enter, offering these terms until the profit is competed away. Conversely, if profits were negative, platforms would exit until profits rise to zero or that terms-of-trade disappears entirely.

The market clearing condition equates the supply and demand for each set of agents. For any measurable set of types I, the left side represents the total supply of these types (I^s is the total measure of s-side agents). The right side integrates across all active terms-of-trade

to find the total measure of these types being used, accounting for the matching probabilities $\Lambda^s(\tau)$ and promised type distributions G^s.

The weak inequality in market clearing allows for the possibility that some agent types earn zero market utility and do not participate in any terms-of-trade. However, if all types in set I earn strictly positive market utility, then supply must exactly equal demand. This reflects that agents cannot be excluded from terms-of-trade offering them strictly positive utility in equilibrium.

Finally, market clearing does not impose any restrictions on sets with zero measure. That is, market clearing only needs to hold for almost every type of agent with $\bar{U}^s(i) > 0$. The last condition fills this gap, imposing that every agent who has positive market utility must get that level of utility in some terms-of-trade that gives platforms zero profits, i.e. at some $\tau \in T$.

2.5 Separation and Assortative Matching

We focus on two distinct characteristics of matching patterns that can emerge in equilibrium. The first concerns whether different types mix together on the same side of a single market, while the second concerns how types match across different markets. These patterns are important for characterizing equilibrium in two-sided markets.

First, we ask whether the equilibrium involves different types participating on the same side of a market, or whether types always separate into distinct markets. This motivates our definition of separation:

Definition 3 (Separation) *A terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in \mathbb{T}$ is separating if for $s \in \{a, b\}$ there exists an i^s such that $dG^s(i^s) = 1$. A (partial or competitive search) equilibrium is separating if every profit-maximizing terms-of-trade $\tau \in T$ is separating.*

In a separating equilibrium, each active market attracts exactly one type from each side. This means that in equilibrium, an agent knows exactly who they will match with when they go to any active markets. This contrasts with pooling outcomes where multiple types might search in the same market.

Second, we examine whether there is systematic matching between agent types across markets. The classical notion of assortative matching describes whether high types match with other high types. In our setting with multiple markets, we need a definition that works across markets:

Definition 4 (Positive Assortative Matching) *Take any subset of terms-of-trade $\hat{T} \subset \mathbb{T}$. Select two elements $(\phi^s_k, G^s_k)_{s=a,b} \in \hat{T}$ for $k \in 1, 2$ and numbers i_k in the support of G^a_k*

and j_k in the support of G_k^b. If any such numbers satisfies $(i_1 - i_2)(j_1 - j_2) \geq 0$, then $\hat{T}$ has positive assortative matching (PAM). A (partial or competitive search) equilibrium has PAM if T has PAM.

To understand this definition, consider two markets and pick any type i_1 that participates on the a-side of market 1 and any type i_2 that participates on the a-side of market 2. Similarly, pick types j_1 and j_2 from the b-sides of these markets. PAM requires that if $i_1 > i_2$, then $j_1 \geq j_2$. That is, if we find a higher a-side type in market 1 than in market 2, we must also find a weakly higher b-side type there. This captures the idea that higher types match together, but extends it to environments with multiple markets and possible pooling of types.

We can analogously define Negatively Assortative Matching (NAM) by requiring $(i_1 - i_2)(j_1 - j_2) \leq 0$. Under NAM, finding a higher a-side type in market 1 than in market 2 implies we must find a weakly lower b-side type there.

Importantly, separation and assortative matching are logically distinct properties. Consider a simple example with types $\mathbb{I}^s = \{0, 1\}$ and three markets in T:

- Market 1: a-side type 0 matches exclusively with b-side type 0
- Market 2: a-side type 1 matches exclusively with b-side type 1
- Market 3: a mixture of a-side types 0 and 1 both match with b-side type 0

This equilibrium exhibits PAM: whenever we find a higher a-side type in one market than another, we never find a strictly lower b-side type there. However, it is not separating because market 3 pools different a-side types. Conversely, markets could be separating but match high types with low types, violating PAM.

3 Motivating Examples

Before characterizing the equilibrium outcomes, we introduce four motivating examples.

Marriage Market. Suppose the agents are looking for partners for marriage. Here we interpret types as the attractiveness of agents, and their roles as men and women. The utility from forming a marriage depends on the private type of both the individual and their marriage partner. As a parametric example, we assume the utility function is:

$$u^s(i, j) = \left(i^{\frac{\theta-1}{\theta}} + j^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}.$$

In this example, $\theta > 0$ is the substitution elasticity. This utility function satisfies the common ranking assumption. Additionally, it is supermodular and satisfies increasing willingness-to-pay, two additional assumptions which we introduce in Section 4.

Labor Market. We can also use the same structure to obtain a model of the labor market. Spence (1973) assumed that workers' human capital was unobservable, while their education choices were observable. Here we assume that both firm productivity and worker human capital are unobservable and only transfers (i.e. wages) and queues (unemployment and vacancy rates) can be used to signal type. More precisely, assume side-a firms hire side-b workers. Then $u^a(i,j)$ is the output produced when a firm with productivity i hires a worker with human capital j and $u^b(j,o)$ is the nonpecuniary amenity that the worker derives from the match.

Market for Expertise. Suppose side-a is the seller of an asset while side-b is the buyer. Sellers hold a single asset which may be of two qualities, S or B, with equal probability. Quality S assets are naturally retained by the seller, while quality B assets are more valuable if purchased by the buyer. For example, the asset may be a mortgage and the quality may represent the difficulty of servicing the mortgage. The seller (mortgage issuer) may be better able to handle a hard-to-service loan, while the buyer (investment bank) may have the deep pockets to hold other loans. If the seller ultimately holds a equality S asset, their payoff is 1, while it is -1 if they hold a quality B asset. The buyer's payoffs are the opposite: 1 if they purchase a quality B asset and -1 if they purchase a quality S asset.

While the seller does not know the quality of the asset, he knows how easy it is to observe the quality, a number i. And while the buyer also cannot directly observe the quality of the asset, he knows his expertise in assessing asset quality, a number j. If a type i seller matches with a type j buyer for an asset with quality S, they observe a normally distributed random variable with mean -1 and precision $\tau(i,j)$, an increasing function. If it is instead an asset with quality B, they observe a normally distributed random variable with mean 1 and the same precision $\tau(i,j)$. They then jointly decide whether to trade the asset and make any specified transfers.[3]

It is straightforward to show that there is a gain from trading the asset if and only if the signal is positive. This means that the pre-transfer expected payoff of both the buyer and the seller from this meeting is

$$u(i,j) = \frac{1}{2} - \Phi(1, \tau(i,j)),$$

[3]This example is inspired by Farboodi, Kondor and Kurlat (2025).

where $\Phi(1,\tau)$ is the probability that a normally distributed random variable with mean 1 and precision τ is negative. Since $\Phi(1,\tau)$ is decreasing in the precision τ, these payoffs are increasing in both i and j and, under appropriate restrictions on τ, may be supermodular.

To understand the payoffs, proceed as follows: If the asset is of quality S, with probability $\frac{1}{2}$, trade occurs with probability $1-\Phi(-1,\tau)=\Phi(1,\tau)$, resulting in payoff -1 for the buyer. If the asset if of quality B, also with probability $\frac{1}{2}$, trade occurs with probability $1-\Phi(1,\tau)$, resulting in payoff 1 for the buyer. If there is no trade, the buyer's payoff is zero. Adding this up gives the buyer's expected payoff. We can find the seller's expected payoff symmetrically.

Communicable Disease. Suppose individuals are looking to interact, and their types are their probability of being healthy. If a healthy individual interacts with the sick individual, they may become sick, incurring an expected cost κ, the product of the probability of getting sick and the cost of being sick. Additionally, all individuals, healthy or sick individuals gain 1 from an interaction. Sick people do not care whether they interact with sick or healthy people. All individuals know the probability that they are healthy, their type.

In summary, if a side-s agent who is healthy with probability i matches with an individual who is healthy with probability j, the value of the interaction is $u^s(i,j)=1-\kappa i(1-j)$. More specifically, the probability that the individual gets sick is proportional to the product of the probability that they are healthy i and that their partner is sick j. This payoff structure is similar to Philipson and Posner (1993) (for HIV/AIDS) and Farboodi, Jarosch and Shimer (2021) (for COVID-19). Interestingly, while the the payoff function is increasing in the partner's probability of being healthy and is supermodular, it is decreasing in the own probability of being healthy, since healthy people stand to lose more from interacting with sick people. Thus it satisfies decreasing willingness-to-pay, the flip side of increasing willingness-to-pay and another assumption we introduce in Section 4.

4 Characterization of Equilibrium

This section develops some preliminary theoretical results. We first prove that finding a partial equilibrium is equivalent to solving a particular optimization problem. We then show that market utilities must be continuous in a CSE and use that to establish that under supermodularity, all equilibria must be separating. Finally, we establish conditions under which market utilities are monotonic in agent types, allowing us to characterize equilibria through a system of differential equations.

4.1 The Platform's Problem

It is standard in the competitive search literature to reformulate equilibrium as an optimization problem. We prove here that our formulation is amenable to such a transformation, providing a tractable way to find equilibria and establish their properties. More precisely, we show that any terms-of-trade offered in equilibrium must maximize platform profits subject to constraints which ensure that the promised mix of agents come to the terms-of-trade.

Given market utilities $\bar{U}^s$, we consider the following optimization problem:

$$\max_{\substack{(\lambda^a,\lambda^b)\in\mathbb{\Lambda},\\ \{\phi^s,G^s\}_{s=a,b}\in\mathbb{T}}} \quad m(\lambda^a,\lambda^b)(\phi^a+\phi^b) \tag{7}$$

$$\text{s.t.} \quad \bar{U}^s(i) \geq \lambda^s\left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j)dG^{\bar{s}}(j) - \phi^s\right) \quad \forall i\in\mathbb{I}^s, s\in\{a,b\},$$

$$\int_{\mathbb{I}^s}\bar{U}^s(i)dG^s(i) = \int_{\mathbb{I}^s}\lambda^s\left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j)dG^{\bar{s}}(j) - \phi^s\right)dG^s(i), \; s\in\{a,b\}.$$

The objective function is $V(\tau,\lambda^a,\lambda^b)$ defined in equation (2), the platform's expected gross profit per unit of advertising effort. The first constraint says that the market utility of a type-i, side-s agent bound their utility $U^s(i,\tau,\lambda^s)$ above, where utility is defined in equation (1). The second constraint requires that agents who do participate (those in the support of G^s) receive exactly their market utility.

Our first result establishes the close link between a solution to this optimization problem and a partial equilibrium:

Lemma 1 *Given $\bar{U}^a,\bar{U}^b$, a partial equilibrium $\{T^p,T,(\Lambda^s,\bar{U}^s)_{s=a,b}\}$ can be constructed as follows:*

- *T^p is the set of all $\tau=(\phi^s,G^s)_{s=a,b}$ for which there exist $(\lambda^s)_{s=a,b}\in\mathbb{\Lambda}$ satisfying the constraints of Problem (7);*
- *For $\tau\in T^p$, $\Lambda^s(\tau)$ is the corresponding λ^s from the constraints of (7);*
- *T is the set of $\tau\in T^p$ such that $(\Lambda^s(\tau),\phi^s,G^s)_{s=a,b}$ solves Problem (7).*

Conversely, for any partial equilibrium $\{T^p,T,(\Lambda^s,\bar{U}^s)_{s=a,b}\}$ and any $\tau=(\phi^s,G^s)_{s=a,b}\in T$, the tuple $(\Lambda^s(\tau),\phi^s,G^s)_{s=a,b}$ solves problem (7).

The proof is in Appendix A.

While it is always useful to restructure a partial equilibrium as an optimization problem, problem (7) remains challenging to solve: it is a mathematical program with equilibrium constraints, nonconvex because increasing $dG^s(i)$ from zero changes inequality constraints

into equality constraints. In the remainder of the paper, we characterize the outcome of the model by imposing additional structure on preferences and by using the notion of CSE, rather than just a partial equilibrium.

4.2 Supermodularity and Separation

To start, we prove that market utility is continuous in any CSE.

Lemma 2 *In any CSE, $\bar{U}^s$ is continuous for $s \in \{a, b\}$.*

The proof is in Appendix A. Continuity follows from agents' ability to mimic the behavior of nearby types. More precisely, since the payoff function u^s is continuous in its first argument, individuals can imitate the strategy of nearby types and get almost the same level of utility. In a CSE, all types must receive their market utility at some profit-maximizing terms-of-trade, and so that level of market utility inherits the continuity of the payoff function.

Next, we prove that if preferences are supermodular, all terms-of-trade used in a CSE only attract a single type of agent on each side of the market. First, we state the supermodularity assumption:

Assumption 2 (Supermodularity) *For every $s \in \{a, b\}$, $i, i' \in \mathbb{I}^s$ and $j, j' \in \mathbb{I}^{\bar{s}}$ with $i > i'$ and $j > j'$, $u^s(i, j) + u^s(i', j') > u^s(i, j') + u^s(i', j)$.*

Supermodularity implies that higher types have a stronger preference for matching with higher partner types. Lemma 3 below establishes that under Common Ranking and Supermodularity, any terms-of-trade used in a CSE solves the following optimization problem:

$$\begin{aligned} \max_{(\lambda^a, \lambda^b) \in \mathbb{\Lambda}, \{\phi^s, k^s\}_{s=a,b}} \quad & m(\lambda^a, \lambda^b)(\phi^a + \phi^b) \\ \text{s.t.} \quad & \bar{U}^s(i) \geq \lambda^s(u^s(i, k^{\bar{s}}) - \phi^s) \ \forall i < k^s, i \in \mathbb{I}^s, s \in \{a, b\} \\ & \bar{U}^s(k^s) = \lambda^s(u^s(k^s, k^{\bar{s}}) - \phi^s) \ \forall s \in \{a, b\}, \\ & \phi^a + \phi^b \geq 0. \end{aligned} \tag{8}$$

We call the first constraint in problem (8) the *downward incentive constraint*, since it ensures that type i on side s does not want to come to the terms-of-trade intended for type $k^s > i$. The second constraint is the *participation constraint*, which ensures that type k^s earns their market utility at this terms-of-trade.

Problem (8) differs from the problem (7) in two important ways. First, it considers only separating terms-of-trade, in which exactly one type from each side participates in each market. Second, it includes only the downward incentive constraints. We ignore the upward

incentive constraints, whcih ensure that high types do not deviate to terms-of-trade intended for lower types.

Lemma 3 *Assume Common Ranking and Supermodularity. A tuple* $\{T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}\}$ *together with a measure* μ *on* T *constitutes a CSE if and only if:*

1. *For each* $\tau = (\phi^a, \phi^b, G^a, G^b) \in T$*, there exist types* $(k^a, k^b) \in \mathbb{I}^a \times \mathbb{I}^b$ *such that the associated distributions are degenerate:* $dG^a(k^a) = dG^b(k^b) = 1$*;*

2. *For each* $\tau = (\phi^a, \phi^b, G^a, G^b) \in T$*, the fees* ϕ^s *and matching probabilities* $\lambda^s = \Lambda^s(\tau)$ *solve problem* (8) *given the types* (k^a, k^b) *and market utilities* $\bar{U}^s$*, with maximized value equal to* c*;*

3. *The set of possible terms-of-trade* T^p *and matching probabilities* Λ^a, Λ^b *satisfy the optimal search and impossible terms-of-trade conditions from Definition 1;*

4. *The measure* μ *satisfies the market clearing condition and* $\bar{U}^s$ *satisfies the market utility condition from Definition 2.*

The proof is in Appendix A.

The first two parts of Lemma 3 characterize the profit-maximizing terms-of-trade in a CSE under Supermodularity. First, such terms-of-trade attract one type from each side of the market, rather than a mixture over types. Second, such terms-of-trade solve problem (8), which in particular includes the constraints for keeping out lower types but drops the constraints from keeping out higher types. The last two parts of the Lemma tell us how to find the other elements of a CSE.

This Lemma yields two important implications for equilibrium structure. First, they immediately establish our main separation result:

Corollary 1 *Assume Common Ranking and Supermodularity. Any CSE is separating.*

To understand why pooling cannot occur in equilibrium, consider a market where multiple types participate on at least one side. Common Ranking implies that all participants would prefer to match with only the highest types from the other side of the market. This suggests creating a new market charging higher fees to attract only these high types. Supermodularity guarantees that lower types would be unwilling to pay the fees that make higher types indifferent, because the higher types value the improved matching enough to pay more than lower types could afford. While such a market might attract even higher types than those in the original pooling market, potentially making it infeasible, this just creates further profit

opportunities. Even higher fees could be charged to attract those higher types. This process continues until complete separation is achieved.

An example illustrates the need for Supermodularity. Consider a market with $\mathbb{I}^s = \{0, 1\}$ and payoffs $u^s(0,0) = 1$, $u^s(0,1) = 1.5$, $u^s(1,0) = 1.1$, and $u^s(1,1) = 1.2$ for $s \in \{a, b\}$. These payoffs satisfy Common Ranking but not Supermodularity. With matching function $m(\lambda^a, \lambda^b) = 1 - \lambda^a - \lambda^b$ on the unit simplex ($\lambda^a \geq 0$, $\lambda^b \geq 0$, and $\lambda^a + \lambda^b \leq 1$), market utilities $\bar{U}^s(0) = 0.33$ and $\bar{U}^s(1) = 0.25$, and appropriate population distributions ($F^s(0) = 0.124$ and $c \approx 0.293$), we find a CSE with pooling. Specifically, a single market attracts both types on both sides, with $G^s(0) = F^s(0)$, $\lambda^s \approx 0.320$, and $\phi^s \approx 0.406$ for $s \in \{a, b\}$.

Second, the lemma establishes that only the constraint of excluding lower types can bind in equilibrium. More precisely, we say *all upward incentive constraints are slack* in a CSE if for any active market $\tau = (\phi^s, G^s)_{s=a,b} \in T$ and type i in the support of G^s, $\bar{U}^s(i') > U^s(i', \tau, \Lambda^s(\tau))$ for all $i' > i$.

Corollary 2 *Assume Common Ranking and Supermodularity. In any CSE, all upward incentive constraints are slack.*

Slack upward incentive constraints are especially useful for characterizing equilibria with positive assortative matching (PAM), as it allows us to solve for market utilities recursively starting from the lowest type.

This result contrasts with the counterveiling incentives literature (Lewis and Sappington, 1989), which emphasizes how heterogeneity in individuals' outside option may lead to incentive constraints that bind in either direction. Here the outside option is individuals' market utility. In partial equilibrium, where market utility is exogenous, upward incentive constraints may bind here as well. But in a CSE, competition between platforms determines market utility. Under Common Ranking and Supermodularity, this competition ensures that only lower types are tempted to come to the market intended for higher types.

4.3 Monotonicity of Market Utility

To further characterize equilibrium, we impose monotonicity in agents' value of matching. This additional structure ensures that market utility varies systematically with agent types, simplifying the analysis of incentive constraints.

Assumption 3 (Increasing WTP (IWTP)) $\forall\, s = a, b,\ i > i' \in \mathbb{I}^s,\ j \in \mathbb{I}^{\bar{s}},\ u^s(i,j) \geq u^s(i',j)$.

Assumption 4 (Decreasing WTP (DWTP)) $\forall\, s = a, b,\ i > i' \in \mathbb{I}^s,\ j \in \mathbb{I}^{\bar{s}},\ u^s(i,j) \leq u^s(i',j)$.

We refer to these assumptions collectively as monotone willingness-to-pay (WTP). Note that Common Ranking imposes that everyone prefers to match with higher types of partners. IWTP imposes that higher types get more from matching with anyone, while DWTP imposes that they get less.

Under monotone WTP, market utility inherits the monotonicity of underlying preferences:

Lemma 4 *Assume Common Ranking. In any CSE:*

1. *if there is IWTP, then $\bar{U}^s(i)$ is weakly increasing in i.*
2. *If there is DWTP, then $\bar{U}^s(i)$ is weakly decreasing in i.*

The proof is again in Appendix A. The proof essentially works by eliminating the fees from the constraints in problem (7).

$$\bar{U}^s(i') - \bar{U}^s(i) \geq \lambda^s \int_{\mathbb{I}^{\bar{s}}} (u^s(i', j) - u^s(i, j)) dG^{\bar{s}}(j)$$

for any terms-of-trade that attracts type-i, side-s agents and any other i'. With IWTP, the right hand side is positive for all $i' > i$, while with DWTP it is positive for all $i' < i$, proving the result. This monotonicity result provides the foundation for our subsequent analysis of equilibrium matching patterns, where we turn to continuous types.

4.4 Continuous Type Distribution and First-order Approach

We have established that all equilibrium markets must be separating under Common Ranking and Supermodularity (Lemma 3). We have also proved that market utility $\bar{U}^s$ is monotone under Monotone WTP (Lemma 4). We now develop tools for characterizing these markets when types are continuously distributed. Throughout this section, we focus on the case where the support of the type distribution is the unit interval, $\mathbb{I}^s = [0, 1]$ for $s \in \{a, b\}$.

Our first step is to establish sufficient regularity of market utilities to justify a local approach to incentive constraints:

Lemma 5 *Assume Common Ranking, Supermodularity, and Monotone Willingness-to-Pay. If $\mathbb{I}^s = [0, 1]$ for $s = a, b$, then in any CSE, market utility $\bar{U}^s(i)$ is almost everywhere differentiable. For any active market $\tau \in T$ that attracts types (k^a, k^b), if $k^s \in (0, 1)$ and $\bar{U}^s$ differentiable at k^s,*

$$\bar{U}^{s\prime}(k^s) = \Lambda^s(\tau) u_1^s(k^s, k^{\bar{s}}),$$

where u_1^s denotes the partial derivative of u^s with respect to its first argument.

The proof is in Appendix A.

For any market attracting types k^a and k^b with $\bar{U}^s$ differentiable at k^s, we can use this lemma to replace the downward incentive constraints in Problem (8) with the *local incentive constraint*:

$$\begin{aligned} \max_{(\lambda^a,\lambda^b)\in\mathbb{\Lambda},\{\phi^s,k^s\}_{s=a,b}} \quad & m(\lambda^a,\lambda^b)(\phi^a+\phi^b) \\ \text{s.t.} \quad & \bar{U}^{s\prime}(k^s) = \lambda^s u_1^s(k^s,k^{\bar{s}}) \ \forall s\in\{a,b\} \\ & \bar{U}^s(k^s) = \lambda^s(u^s(k^s,k^{\bar{s}})-\phi^s) \ \forall s\in\{a,b\}, \\ & \phi^a+\phi^b \geq 0. \end{aligned} \tag{9}$$

We can then eliminate the fees ϕ^a, ϕ^b from this problem using the participation constraint and the contact rates λ^a, λ^b using the local incentive constraint. This gives us the gross value of advertising terms-of-trade to attract (k^a, k^b):

$$\hat{V}(k^a,k^b) = m\left(\frac{\bar{U}^{a\prime}(k^a)}{u_1^a(k^a,k^b)}, \frac{\bar{U}^{b\prime}(k^b)}{u_1^b(k^b,k^a)}\right)\sum_{s=a,b}\left(u^s(k^s,k^{\bar{s}}) - \frac{\bar{U}^s(k^s)u_1^s(k^s,k^{\bar{s}})}{\bar{U}^{s\prime}(k^s)}\right). \tag{10}$$

In order for a platform advertising this terms-of-trade to obtain this value, it must satisfy three conditions. First, the combination of matching probabilities must be feasible:

$$\left(\frac{\bar{U}^{a\prime}(k^a)}{u_1^a(k^a,k^b)}, \frac{\bar{U}^{b\prime}(k^b)}{u_1^b(k^b,k^a)}\right) \in \mathbb{\Lambda}.$$

Second, the sum of the fees must be nonnegative:

$$\sum_{s=a,b}\left(u^s(k^s,k^{\bar{s}}) - \frac{\bar{U}^s(k^s)u_1^s(k^s,k^{\bar{s}})}{\bar{U}^{s\prime}(k^s)}\right) \geq 0.$$

And finally all of the downward incentive constraints must be satisfied, not just the local incentive constraint. If any of these conditions fails, no terms-of-trade matches k^a and k^b at this level of market utility.

Equation (10) captures how local incentive constraints shape the platform's profits. The matching rates are determined by the elasticity of agents' market utility $\bar{U}^s$ relative to their payoff u^s, while the total surplus must cover both the direct utility from matching and an information rent term reflecting agents' local incentive constraints. This expression reveals important features of the equilibrium construction. When $k^a, k^b > 0$ and market utilities $\bar{U}^s(i)$ are known, $\hat{V}(k^a, k^b)$ is completely determined by incentive constraints. There is no optimization problem left to solve. The local incentive constraints pin down the matching

rates, and these in turn determine the platform's revenue through the matching probability m. Thus, finding active markets reduces to identifying pairs (k^a, k^b) that yield the highest value of $\hat{V}(k^a, k^b)$, subject to market clearing.

However, some challenges remain. First, we need to determine the market utilities $\bar{U}^s(i)$ themselves. Second, we need to verify that all downward incentive constraints are satisfied, not just local ones. Third, markets that attract the lowest types ($k^a = 0$ or $k^b = 0$) require special treatment because there are no downward incentive constraints and so local incentive constraints don't bind for these types. These markets must solve the unconstrained platform problem, which then provides boundary conditions for the market utilities of higher types. Similarly, we can only compute $\hat{V}(k^a, k^b)$ at points where $\bar{U}^a$ and $\bar{U}^b$ are differentiable, which we have so far established is true almost everywhere.

This structure suggests a recursive approach to finding equilibria: solve the unconstrained problem for the lowest types, use this to initiate the market utilities, and then ensure higher-type markets satisfy both the local incentive constraints and market clearing. We pursue this approach in detail when we study PAM in the next section.

5 Positive Assortative Matching with Symmetry

In this section, we study a particular class of equilibria that combine positive assortative matching (PAM) with symmetry between the two sides of the market. This focus serves two purposes. First, such equilibria are particularly tractable as the analysis reduces to a one-dimensional problem. Second, many real-world matching markets exhibit substantial symmetry between sides (for example, dating markets or peer-to-peer trading platforms), making this a natural case to understand thoroughly.

To begin, we assume that advertising is free, $c = 0$. In this case, we characterize the equilibrium matching probabilities and fees when there is a PAM equilibrium and derive a set of sufficient conditions that guarantee the existence of such a PAM equilibrium.

We then allow for a positive advertising cost, $c > 0$. We characterize a PAM equilibrium and show numerically that a PAM equilibrium exists under reasonably general conditions.

5.1 Symmetric Environment

We impose symmetry through four key assumptions:

- equal populations on each side: $I^a = I^b = I$;
- common type distributions: $F^a = F^b = F$ with support $\mathbb{I}^a = \mathbb{I}^b = \mathbb{I}$;

- symmetric match utilities: $u^a(i,j) = u^b(j,i) = u(i,j)$ for all i, j, satisfying Common Ranking, Supermodularity, and Monotone WTP; and
- symmetric matching technology: $m(\lambda^a, \lambda^b) = m(\lambda^b, \lambda^a)$.

For simplicity, we also work in an environment in which there are no gaps in the type distribution, so without further loss of generality $I = [0, 1]$. This simplifies our exposition but is not essential for our results.

We then look for a symmetric CSE with PAM, meaning it satisfies three properties:

- type-i, side-a agents match with type-i, side-b agents for all $i \in \mathbb{I}$;
- same-type agents on different sides have the same market utility: $\bar{U}^a(i) = \bar{U}^b(i) = \bar{U}(i)$ for all $i \in \mathbb{I}$;
- markets for same-type agents have symmetric terms: In the active market τ that attracts type-i agents on both sides, $\lambda^a(\tau) = \lambda^b(\tau) \equiv \ell(i)$ and fees $\phi^a(\tau) = \phi^b(\tau) \equiv \Phi(i)$.

By definition, such an equilibrium exhibits PAM.

We show that under symmetry, it is straightforward to characterize a PAM equilibrium when one exists. First, we know immediately who matches with whom. Second, rather than working with the two dimensional feasible set of matching probabilities $\mathbb{\Lambda}$, we can work with the one-dimensional set $[0, \bar{\lambda}]$, where $m(\bar{\lambda}, \bar{\lambda}) = 0$. Thus $\bar{\lambda}$ is the highest matching probability that can be obtained by both sides of the market under symmetry. Using these simplifications, we show that $\bar{U}$, ℓ, and Φ satisfy an ordinary differential equation with known boundary conditions.

5.2 Increasing Willingness-to-Pay

To begin our analysis, we assume that there is no cost of advertising, $c = 0$. We start by characterizing an environment with Common Ranking, Supermodularity, and IWTP. A parametric case illustrates our general results. Assume that the utility function has a constant elasticity of substitution,

$$u^s(i,j) = \left(\frac{1}{2} i^{\frac{\theta-1}{\theta}} + \frac{1}{2} j^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

for $s = a, b$ and some $\theta > 0$. When $\theta = 1$, this reduces to the multiplicatively separable case, $u^s(i,j) = \sqrt{ij}$. As $\theta \to 0$, the payoff function is Leontief, $u^s(i,j) = \min\{i, j\}$, while for large values of θ, the two sides of the market are very substitutable.

We also assume the matching function has a constant elasticity of substitution,

$$m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma)^{\frac{1}{\gamma}},$$

where $\gamma > 0$ and $1/(1+\gamma)$ is the elasticity of substitution in the matching function $M(n^a, n^b)$. The support of the possible agent matching rates is $\mathbb{\Lambda} = \left\{(\lambda^a, \lambda^b) | 1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma \geq 0\right\}$.

For this parametric case, we prove that there is a PAM equilibrium if and only if $\theta \leq 1+\gamma$. This condition is the same as one would obtain from the complete information setting (Eeckhout and Kircher, 2010): PAM is sustained when the elasticity of substitution in either the matching function or production function is small.[4]

If a PAM equilibrium exists, all active markets have the same matching probability, $\ell(i) = 2^{-1/\gamma} = \bar{\lambda}$, so $m(\ell(i), \ell(i)) = 0$. That is, there is enough (free) advertising to saturate all the markets. On the other hand, higher types of agents pay higher fees, $\Phi(i) = \frac{i}{2}$, absorbing half the output from the match (note $u^s(i,i) = i$). These fees exist only because of private information and are dissipated across the platforms by the saturation of advertising, which ensures that almost all advertising does not lead to match creation. Finally, higher types of agents have higher market utility $\bar{U}(i) = 2^{-\left(1+\frac{1}{\gamma}\right)} i$.

This example is a special case of a more general class of assumptions:

Assumption 5

(a) For all $i > j > 0$, $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{\Lambda}^o$;

(b) For all $i > 0$, *either* $u_1(i,0) = 0$ *or* $u(0,i) + \tilde{\phi}(i) \leq 0$ *or* $\left(\bar{\lambda}\frac{u(0,0)}{u(0,i)+\tilde{\phi}(i)}, \tilde{\ell}(i)\right) \notin \mathbb{\Lambda}^o$, *where*

$$\tilde{\phi}(i) \equiv u(i,0) - \frac{u_1(i,0)}{u_1(i,i)}\left(u(0,0) + \int_0^i u_1(k,k)dk\right) \quad \text{and} \quad \tilde{\ell}(i) \equiv \bar{\lambda}\frac{u_1(i,i)}{u_1(i,0)}.$$

Assumption 5(a) ensures that creating a market matching i and j with $i > j > 0$ is not profitable. In our CES example, the condition reduces to

$$\frac{1}{2}i^{-\frac{\gamma}{\theta}} + \frac{1}{2}j^{-\frac{\gamma}{\theta}} > \left(\frac{1}{2}\left(i^{-\frac{\gamma}{\theta}}\right)^{\frac{\theta-1}{\gamma}} + \frac{1}{2}\left(j^{-\frac{\gamma}{\theta}}\right)^{\frac{\theta-1}{\gamma}}\right)^{\frac{\gamma}{\theta-1}}$$

Using Jensen's inequality, one can verify that this holds whenever $\theta < 1 + \gamma$.

Assumption 5(b) ensures that creating a market matching $i > 0$ with 0 is not profitable. In our CES example, $\theta \leq 1$ implies $u_1(i,0) = 0$ for all $i > 0$, so 5(b) is satisfied. For $\theta > 1$, $u_1(i,0) = 2^{\theta/(1-\theta)}$ and $u(0,i) = 2^{\theta/(1-\theta)}i$, both positive, while $\tilde{\phi}(i) = 0$. It follows that 5(b)

[4]For the exact inequality, equivalent to $\theta \leq 1 + \gamma$, see equation (15) in Eeckhout and Kircher (2010).

is satisfied if and only if $\bar{\lambda}\frac{u_1(i,i)}{u_1(i,0)} \geq 1$. For $\theta > 1$, this holds if and only if $\theta \leq 1+\gamma$, the same condition we reported earlier.

We now state our general characterization of PAM with IWTP:

Proposition 1 *Assume Common Ranking, Supermodularity, IWTP, and Assumption 5. If $c = 0$, there is a PAM CSE. In the market attracting type i agents, the matching probability is $\ell(i) = \bar{\lambda}$; the fee solves $\Phi'(i) = u_2(i,i)$ with $\Phi(0) = 0$; and market utility solves $\bar{U}'(i) = \bar{\lambda}u_1(i,i)$ with $\bar{U}(0) = \bar{\lambda}u(0,0)$.*

The proof is in Appendix B. It works as follows. First, we assume that there is PAM with the indicated market utility function and solve for the contact rates and fees in all (i,i) markets by solving a symmetric version of problem 9. Second, we verify that all the global incentive constraints are satisfied. Neither of these conditions uses Assumption 5. We then prove that there is no profitable deviation through the creation of a new terms-of-trade. From Lemma 3, we know that we can look exclusively at terms-of-trade that attract a single type on each side. Assumption 5(a) rules out the profitability of a terms-of-trade that attracts $i > 0$ on one side and $j > 0$ on the other. At such a terms-of-trade, local incentive constraints bind on both sides of the market. Assumption 5(b) rules out the profitability of a terms-of-trade that attracts $i > 0$ on side and 0 on the other side, so there is only a local incentive constraint on the first side.

Conditional on having a PAM equilibrium, the equilibrium outcomes resemble the ones from the parametric case of two CESs. All types have the same matching probability $\bar{\lambda}$, which is the maximal symmetric matching probability feasible under the matching function. Since higher types gain more from matching with IWTP, trying to exclude lower types by rationing matches is counterproductive. On the other hand, fees are increasing to ensure that lower types, who gain less from matching with a higher type, are willing to match with their own type rather than paying a higher fee to match with a higher type.

We can also examine the expertise model from Section 3. Here we assume

$$u(i,j) = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{ij/2}} e^{-t^2} dt,$$

so the variance of the signal from a type i seller and type b buyer is $1/(ij)$. This always satisfies Common Ranking and IWTP. It satisfies Supermodularity if $ij < 1$, and so we restrict the type space to the interval $[0,1]$. On the other hand, $u(i,j)$ is always log-submodular ($\log u(i,j)$ is submodular), which makes PAM less likely.

We find that Assumption 5(a) is satisfied if $i^2 \leq \frac{1+\gamma-\sqrt{1+2\gamma}}{\gamma} < 1$, where γ is the matching function parameter (elasticity $1/(1+\gamma)$). Additionally, Assumption 5(b) is always satisfied

because $u_1(i,0) = 0$. Thus there is PAM if and only if Assumption 5(a) is satisfied. In this case, the matching probability is $\bar{\lambda} = 2^{-1/\gamma}$ for everyone, fees cost half of equilibrium output, $\Phi(i) = u(i,i)/2$, and half of market utility is lost to private information, $\bar{U}(i) = \bar{\lambda}u(i,i)/2$. Conversely, when Assumption 5(a) is violated, there is no equilibrium with PAM.

5.3 Decreasing Willingness-to-Pay

We now turn to the case of DWTP. We still assume Common Ranking and Supermodularity, and also impose zero advertising costs, $c = 0$, throughout this section. We start with a parameteric example. Let $u^s(i,j)$ satisfy

$$u^s(i,j) = 1 - \kappa i(1-j)$$

for $s = a, b$, with support $\mathbb{I} = [0,1]$ and $0 < \kappa < 4$. This is our model of disease transmission. A type-i individual is healthy with probability i. They gain 1 from a match but suffer a utility loss if they get sick, which may happen whenever they match with a sick partner. κ represents the product of the cost of getting sick and the probability of disease transmission from a sick person to a healthy person. This function satisfies Common Ranking, Supermodularity, and DWTP.

We again assume the matching function has a constant elasticity of substitution, $m(\lambda^a, \lambda^b) = (1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma)^{1/\gamma}$ with $\gamma > 0$, as in our IWTP example.

For this parameteric case, we find that there is a PAM equilibrium if and only if $\kappa \leq 2(1+\gamma)$. Note that we have already assumed $\kappa < 4$, which ensures $u(i,i) > 0$ for all i. We need the assumption that $\kappa \leq 2(1+\gamma)$ to rule out the profitability of creating a $(0,i)$ match.

If a PAM equilibrium exists, all active markets charge a fee of zero, $\Phi(i) = 0$, so no resources are wasted on advertising. Instead, market utility $\bar{U}(i)$ and matching probabilities $\ell(i)$ are globally decreasing functions. In general, the functional forms are not very illuminating, but we highlight here the explicit solution when $\kappa \to 4$. In that case, $\bar{U}(i) = (1-2i)e^{-2i/(1-2i)}$ for $i \in [0, \frac{1}{2}]$ and $\bar{U}(i) = 0$ for $i > \frac{1}{2}$. The matching rate is $\ell(i) = e^{-2i/(1-2i)}/(1-2i)$ for $i \in [0, \frac{1}{2}]$ and $\ell(i) = 0$ for $i > \frac{1}{2}$, so lower types are kept out of the market by rationing meetings, while types above $\frac{1}{2}$ give up on meetings altogether.

Once again, this example is a special case of a more general class of assumptions:

Assumption 6

(a) $u(i,j) + u(j,i) \leq u(i,i)\frac{u_1(i,j)}{u_1(i,i)} + u(j,j)\frac{u_1(j,i)}{u_1(j,j)}$

(b) For all $i > 0$, either $u(0,i) + \hat{\phi}(i) \leq 0$ or $\left(\bar{\lambda}\frac{u(0,0)}{u(0,i)+\hat{\phi}(i)}, \hat{\ell}(i)\right) \notin \mathbb{A}^o$, where

$$\hat{\phi}(i) \equiv u(i,0) - \frac{u_1(i,0)}{u_1(i,i)}u(i,i) \quad \text{and} \quad \hat{\ell}(i) \equiv \bar{\lambda}\frac{u(0,0)u_1(i,i)}{u(i,i)u_1(i,0)}\exp\left(\int_0^i \frac{u_1(k,k)}{u(k,k)}dk\right).$$

Again, Assumption 6(a) ensures that creating a market matching i and j with $i > j > 0$ is not profitable. In our disease example, this condition is always satisfied. Assumption 6(b) rules out the profitability of a market matching $i > 0$ with 0. In our disease example, this assumption holds if and only if $\kappa \leq 2(1+\gamma)$.

We now state the general characterization of PAM with DWTP:

Proposition 2 *Assume Common Ranking, Supermodularity, DWTP, and Assumption 6. If $c = 0$, there is a PAM CSE. In the market attracting type i agents, the matching probability solves $\frac{\ell'(i)}{\ell(i)} = -\frac{u_2(i,i)}{u(i,i)}$, with $\ell(0) = \bar{\lambda}$ and the fee is $\Phi(i) = 0$, $\forall i$; market utility solves $\frac{\bar{U}'(i)}{\bar{U}(i)} = \frac{u_1(i,i)}{u(i,i)}$, with $\bar{U}(0) = \bar{\lambda}u(0,0)$.*

The proof is in Appendix B and follows the same structure as the proof of Proposition 1. The general characterization is similar to our example: fees are zero, but instead matching probabilities are used to exclude lower types from terms-of-trade intended for higher types. Interestingly, even though matching probabilities are used to exclude agents, relative market utility (i.e. $\bar{U}(i)/\bar{U}(0)$) and relative matching probabilities ($\ell(i)/\ell(0)$) are determined entirely by preferences u, not the matching technology m.

Why is rationing used with DWTP when fees were used with IWTP? In the DWTP case, fees are ineffectual at excluding lower types from high type market, because lower types value the match more and so are more willing to pay high fees. On the other hand, reducing the matching probability is effective, exactly because the opportunity cost of failing to match is greater for lower types.

We stress, however, that the equilibrium use of fees versus matching probabilities depends only on the derivative of utility with respect to its first argument, not on equilibrium levels of payoffs. For example, in the disease case, types with $i > \frac{1}{2}$ have a higher equilibrium payoff than lower types ($u(i,i) > u(\frac{1}{2},\frac{1}{2})$), yet still are rationed through matching probabilities rather than fees. The relevant distinction is about whether lower types get more from matching with a high type than a high type gets from the same match.

5.4 Positive Advertising Costs

We now consider the case where advertising is costly, $c > 0$. In this case, we show that both rationing and fees are used in equilibrium, but whether higher types are more or less rationed

and pay higher or lower fees depends on the direction of the monotonicity of willingness-to-pay.

The intuition for this is straightforward. For a match between two type zero agents, there is no information problem, since incentive constraints only bind downwards (Corollary 2). Still, there must be some rationing ($\lambda < \bar{\lambda}$) and some fees ($\phi > 0$) so that the platforms can cover their advertising costs. With IWTP, higher types face higher fees for the same reason as in the case without advertising costs. But if fees are higher, advertising would be more profitable, unless the platform meeting rate falls. And so $m(\lambda, \lambda)$ must be lower for higher types, or equivalently the agents' meeting rate must be higher.

With DWTP, higher types have a lower matching probability, but this means that platforms have a higher matching probability. If fees did not adjust, advertising would be more profitable. Thus competition between platforms must drive down the fees paid by the higher types.

Our formal analysis proceeds constructively. First, we note that any equilibrium with PAM must solve problem (9), specialized to the case where $k^a = k^b$. Moreover, the maximized value of the problem must be c. That is, for all $i \in (0, 1)$, we have

$$\begin{aligned} c = \max_{\lambda \in [0, \bar{\lambda}]} \; & 2m(\lambda, \lambda)\phi \\ \text{s.t.} \; & \bar{U}'(i) = \lambda u_1(i, i), \\ & \bar{U}(i) = \lambda(u(i, i) - \phi), \\ & \phi \geq 0. \end{aligned}$$

Since $c > 0$, we know that any solution to this problem must have $\phi > 0$. We can thus drop the nonnegativity constraint on ϕ and use the participation constraint to eliminate ϕ from the objective function:

$$\begin{aligned} c = \max_{\lambda \in [0, \bar{\lambda}]} \; & 2m(\lambda, \lambda) \left(u(i, i) - \frac{\bar{U}(i)}{\lambda} \right) \\ \text{s.t.} \; & \bar{U}'(i) = \lambda u_1(i, i). \end{aligned} \tag{11}$$

If we knew $\bar{U}(i)$, this problem would uniquely pin down λ. But since we do not know $\bar{U}(i)$, we instead use the problem to determine it.

First, solve the problem when $i = 0$, so the local incentive constraint is slack. In this case, we can solve the dual, $\bar{U}(0) = \max_{\lambda \in [0, \bar{\lambda}]} h(\lambda, 0)$, where

$$h(\lambda, i) \equiv \lambda \left(u(i, i) - \frac{c}{2m(\lambda, \lambda)} \right). \tag{12}$$

This pins down $\bar{U}(0)$ as well as $\ell(0) = \ell^*(0)$, where in general $\ell*(i) \equiv \arg\max_{\lambda \in [0,\bar{\lambda}]} h(\lambda, i)$ is the matching rate in an (i,i) market if there were no local incentive constraint. Note that if $c \geq 2m(0,0)u(0,0)$, $\bar{U}(0) = 0$. We instead assume in this section that $c < 2m(0,0)u(i,i)$ for all i, so an unconstrained (i,i) market could cover the advertising costs c.

Next, the objective function in problem (11) tells us that if there is a PAM equilibrium, $\bar{U}(i) = h(\ell(i), i)$, where $\ell(i)$ is the equilibrium value of λ in an (i,i) market. Differentiating this expression with respect to i and using the functional form of h in equation (12) gives us

$$\bar{U}'(i) = h_1(\ell(i), i)\ell'(i) + \ell(i)\big(u_1(i,i) + u_2(i,i)\big).$$

The local IC constraint states $\bar{U}'(i) = \ell(i)u_1(i,i)$, so we can simplify this to get

$$h_1(\ell(i), i)\frac{\ell'(i)}{\ell(i)} = -u_2(i,i). \tag{13}$$

This is an ordinary differential equation which we can solve uniquely using the boundary condition $\ell(0) = \ell^*(0)$. This provides a precise set of necessary conditions for any equilibrium with PAM in this environment.

The structure of equation (13) reveals how incentive constraints distort matching probabilities relative to the full information benchmark $\ell^*(i)$. The left-hand side captures the rate of change in matching probabilities, scaled by h_1 which measures the marginal value of increasing the meeting probability. The right-hand side involves $u_2(i,i)$, which captures how an agent's utility changes when their partner's type increases.

The following Proposition combines these insights:

Proposition 3 *Assume Common Ranking, Supermodularity, and Monotone WTP. Also assume* $2u(i,i)m(0,0) > c > 0$. *If there exists a symmetric CSE with PAM, then:*

1. *The matching probabilities* $\ell(i)$ *solve ordinary differential equation* (13) *with limiting condition* $\ell(0) = \ell^*(0)$;
2. *Market utilities satisfy* $\bar{U}(i) = h(\ell(i), i)$;
3. *Fees satisfy* $\Phi(i) = u(i,i) - \bar{U}(i)/\ell(i)$;
4. *These solutions satisfy all global incentive constraints.*

Moreover, with

- *IWTP,* $\bar{U}(i)$, $\ell(i)$, *and* $\Phi(i)$ *are increasing in* i *and* $\ell(i) \in (\ell^*(i), \bar{\lambda}]$ *for all* $i > 0$;
- *DWTP,* $\bar{U}(i)$, $\ell(i)$, *and* $\Phi(i)$ *are decreasing in* i *and* $\ell(i) \in (0, \ell^*(i))$ *for all* $i > 0$.

The proof is in Appendix B.

The discussion in the text glosses over four technical issues. First, at $i = 0$, the optimization problem is unconstrained, so the first order condition $h_1(\ell(0), 0) = 0$ holds. This in turn means that we cannot directly apply the ordinary differential equation (13) at $i = 0$. Instead, we look for a solution to the differential equation with the property that $\lim_{i\to 0} \ell(i) = \ell^*(0)$.

Second, Lemma 4 tells us that market value is monotone and so almost everywhere differentiable. This means that the differential equation (13) only holds almost everywhere, leaving open the possibility of discontinuities in ℓ function. The proof of Proposition 3 establishes that ℓ must in fact be continuous.

Third, there are two such solutions, one with ℓ increasing and one with ℓ decreasing. We show in the proof of Proposition 3 that the first solution is valid with IWTP and the second with DWTP.

Lastly, is it possible that the solution to the differential equation (13), combined with the boundary conditions, give us matching probabilities and equilibrium utilities that are non-positive? We show this can never be the case in the proof of Proposition 3. These technical considerations, while subtle, are crucial for a complete understanding of the equilibrium structure. They highlight how the matching technology's properties interact with agents' preferences to determine the appropriate solution concept.

Proposition 3 has several notable implications. First, the differential equation characterization of equilibrium has remarkable simplicity and power. While we only imposed local incentive constraints, the resulting allocation automatically satisfies global incentive constraints. This relies on the way that CSE pins down market utility, which functions as agents' outside option.

Second, the direction of Monotone Willingness-to-Pay completely determines the structure of equilibrium utilities, contact rates, and fees. With IWTP, higher types receive higher utilities and face higher contact rates and fees. With DWTP, the pattern reverses. This stands in contrast to the case with observable types, where the direction of these patterns depends on whether the equilibrium payoff $u(i, i)$ is increasing or decreasing, not on the willingness-to-pay.

Third, private information systematically distorts allocations relative to the observable types benchmark. With IWTP, equilibrium contact rates exceed the full-information optimum $\ell^*(i)$. The free entry condition then implies that fees must also be higher to cover platform costs. With DWTP, these distortions reverse: both contact rates and fees fall below their full-information levels. These distortions reflect platforms' optimal response to incentive constraints: they adjust contact rates and fees to make mimicking less attractive to lower types.

We close this section by noting that, while Proposition 3 provides a complete characterization of symmetric PAM equilibria when they exist, it does not guarantee existence. Once we find candidate equilibrium utilities $\bar{U}(i)$ by solving the ODE, we know all incentive constraints will be satisfied, both local and global. However, we still must verify that no platform would prefer to create a market matching different types. When $c = 0$, Propositions 1 and 2 provide sufficient conditions for a PAM equilibrium to exist, but we have not derived useful conditions when $c > 0$. Still, since we know market utility in this candidate equilibrium, we just need to check the value of $\hat{V}(k^a, k^b)$ in equation (10), a straightforward numerical computation.

6 Extensions to Baseline Model

The previous sections established fundamental properties of equilibrium in our two-sided matching market with private information: equilibria are separating, market utilities are continuous and monotonic, and the platform's problem can be reformulated as a constrained optimization. Moreover, with continuous types, local incentive constraints may imply global constraints. We also provide a precise characterization of PAM when the two sides of the model are symmetric. In this section, we first build on the characterization to describe PAM without symmetry. We then analyze negative assortative matching. In both cases, our characterization is based on solving an ordinary differential equation system to find the proposed equilibrium, and then checking whether market utility is consistent with the proposed equilibrium.

6.1 Positive Assortative Matching without Symmetry

In this section, we discuss an algorithm to compute a PAM CSE without imposing that the environment is symmetric. For exposition simplicity, we focus on cases where all types participate: the condition (2) in the definition of a competitive search equilibrium holds with equality for types. We discuss the other cases briefly at the end of this section.

Our characterization follows two steps. First, in a PAM equilibrium, for the type 0's market to clear, $\sigma(0) = 0$. That is, some platforms post terms-of-trade that attracts the lowest types on both sides. From Lemma 3, this terms-of-trade is undistorted. Lemma 6 summarizes the details of this terms-of-trade. Secondly, we solve other equilibrium terms-of-trade as solution of a system of odes, detailed in Proposition 4.

Lemma 6 *Assume Common Ranking and Supermodularity. In any PAM CSE,* $\sigma(0) = 0$

and $(\lambda^s(0))_{s=a,b}, \bar{U}^b(0)$ *solve the following unconstrained problem, for a fixed* $\bar{U}^a(0) = \upsilon$*:*

$$c = \max_{\lambda^a,\lambda^b} m(\lambda^a,\lambda^b)\left(u^a(0,0) + u^b(0,0) - \frac{\upsilon}{\lambda^a} - \frac{\bar{U}^b(0)}{\lambda^b}\right), \tag{14}$$

with the solution denoted as $(\lambda_0^{s*}(\upsilon))_{s=a,b}$ *and* $\bar{U}_0^{b*}(\upsilon)$*.*

Lemma 7 *Assume common ranking, supermodularity, and monotone willingness-to-pay. In any PAM CSE,* $\lambda^s(i)$ *is continuous.*

The proof is in Appendix C.

We characterize the PAM equilibrium using a system of odes in (λ^a, λ^b), summarized in Proposition 4.

Proposition 4 *Assume common ranking, supermodularity and continuous type distribution. A PAM CSE is characterized by* $(\lambda^s, \bar{U}^s)_{s=a,b}$*,* σ *that solve*

$$\nu^a(i)\frac{\ell^{a\prime}(i)}{\ell^a(i)} = -\iota(i) - u_2^b(\sigma(i), i), \qquad \nu^b(i)\frac{\ell^{b\prime}(i)}{\ell^b(i)} = \iota(i) - \sigma'(i)u_2^a(i,\sigma(i)), \tag{15}$$

$$\bar{U}^{a\prime}(i) = \ell^a(i)u_1^a(i,\sigma(i)), \qquad \bar{U}^{b\prime}(\sigma(i)) = \ell^b(i)u_1^b(\sigma(i),i), \tag{16}$$

$$\sigma'(i) = \frac{\ell^a(i)}{\ell^b(i)}\frac{I^a\bar{g}^a(i)}{I^b\bar{g}^b(\sigma(i))}, \tag{17}$$

with the boundary conditions

$$\sigma(0) = 0,\ \sigma(1) = 1,\ \lambda^s(0) = \lambda_0^{s*}(\bar{U}^a(0)), \bar{U}^b(0) = \bar{U}_0^{b*}(\bar{U}^a(0)),$$

where

$$\nu^a(i) = \left(\frac{\epsilon_1(\ell^a(i),\ell^b(i))}{m(\ell^a(i),\ell^b(i))}c + \frac{\bar{U}^a(i)}{\ell^a(i)}\right),\ \nu^b(i) = \left(\frac{\epsilon_2(\ell^a(i),\ell^b(i))}{m(\ell^a(i),\ell^b(i))}c + \frac{\bar{U}^b(\sigma(i))}{\ell^b(i)}\right), \tag{18}$$

$$\iota(i) = \sigma'(i)\nu^a(i)\frac{u_{1,2}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))} - \nu^b(i)\frac{u_{1,2}^b(\sigma(i),i)}{u_1^b(\sigma(i),i)},$$

$$\epsilon_1(\lambda^a,\lambda^b) = \frac{\lambda^a m_1(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)},\ \ \epsilon_2(\lambda^a,\lambda^b) = \frac{\lambda^b m_2(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)}.$$

The proof is in Appendix C.

Discussion. It is possible that there is no solution to the differential equation system from Proposition 4 where $\bar{U}^s(i) \geq 0$ for all $i \in \mathbb{I}^s$. In this case, the PAM equilibrium involves

some types not participating. Under increasing willingness-to-pay, there are three possible non-participating scenarios. First, it could be that the side-a low types are not participating: there is a threshold type $\underline{i}^a$ such that $\bar{U}^a(i) = 0$ for all $i \leq \underline{i}^a$. In this case, the assignment function starts from $\underline{i}^a$. The second case involves the same pattern, on side b. Lastly, it is possible that the lowest types on both sides do not participate. This happens when none of the terms-of-trade that attract side-s type-0 covers the advertising costs.

Finally, the differential equations in Proposition 4 only characterize a PAM CSE if one exists. One still has to compute the value of creating an (i, j) market, as in equation (10), and verify that this does not exceed the advertising cost for any pair (i, j).

6.2 Negative Assortative Matching

Next, we discuss an algorithm to compute a CSE with negative assortative matching (NAM), where the higher types match with lower types. First, we focus on characterization of the $(0, 1)$ market. In this market, the incentive constraint on side-a is irrelevant. Given the market utility $\bar{U}^b(1)$ and λ^b, the matching proability for side-a type-0 and her market utility are solutions to:

$$c = \max_{\lambda^a} m\left(\lambda^a, \lambda^b\right)\left(u^a(0,1) + u^b(1,0) - \frac{\bar{U}^a(0)}{\lambda^a} - \frac{\bar{U}^b(1)}{\lambda^b}\right). \tag{19}$$

For other types, we can solve an ordinary differential equation system:

Proposition 5 *Assume common ranking, supermodularity and continuous type distribution. A NAM CSE is characterized by $(\lambda^s, \bar{U}^s)_{s=a,b}$, σ that solve*

$$\nu^a(i)\frac{\ell^{a\prime}(i)}{\ell^a(i)} = -\iota(i) - u_2^b(\sigma(i), i), \qquad \nu^b(i)\frac{\ell^{b\prime}(i)}{\ell^b(i)} = \iota(i) - \sigma'(i)u_2^a(i, \sigma(i)), \tag{20}$$

$$\bar{U}^{a\prime}(i) = \ell^a(i)u_1^a(i, \sigma(i)), \qquad \bar{U}^{b\prime}(\sigma(i)) = \ell^b(i)u_1^b(\sigma(i), i), \tag{21}$$

$$\sigma'(i) = -\frac{\ell^a(i)}{\lambda^b(\sigma(i))}\frac{I^a\bar{g}^a(i)}{I^b\bar{g}^b(\sigma(i))}, \tag{22}$$

with the boundary conditions

$$\sigma(0) = 1,\ \sigma(1) = 0,\ (\lambda^s(0), \bar{U}^s(0))\ \textit{solve problem 19 given}\ (\lambda^{\bar{s}}(1), \bar{U}^{\bar{s}}(1)).$$

where

$$\nu^a(i) = \left(\frac{\epsilon_1(\ell^a(i), \ell^b(i))}{m(\ell^a(i), \ell^b(i))}c + \frac{\bar{U}^a(i)}{\ell^a(i)}\right),\ \nu^b(i) = \left(\frac{\epsilon_2(\ell^a(i), \ell^b(i))}{m(\ell^a(i), \ell^b(i))}c + \frac{\bar{U}^b(\sigma(i))}{\ell^b(i)}\right), \tag{23}$$

$$\iota(i) = \sigma'(i)\nu^a(i)\frac{u^a_{1,2}(i,\sigma(i))}{u^a_1(i,\sigma(i))} - \nu^b(i)\frac{u^b_{1,2}(\sigma(i),i)}{u^b_1(\sigma(i),i)},$$

$$\epsilon_1(\lambda^a,\lambda^b) = \frac{\lambda^a m_1(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)},\ \ \epsilon_2(\lambda^a,\lambda^b) = \frac{\lambda^b m_2(\lambda^a,\lambda^b)}{m(\lambda^a,\lambda^b)}.$$

7 Concluding Remarks

How do individuals sort in the presence of private information? We use a competitive search model to answer that question. We prove that under Common Ranking and Supermodularity, any profit maximizing terms of trade only attracts one type of agent to each side of the market. We then characterize PAM under those two assumptions, monotone Willingness-to-Pay, and symmetry. When advertising costs are zero and there is IWTP, only fees differ across active markets, and the matching rate is driven to the highest feasible value. With DWTP, only matching rates differ across markets, and fees are driven to zero. With positive advertising costs, whether we have IWTP or DWTP determines whether fees and matching rates are increasing or decreasing in the agents' type.

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Appendix

A Proofs for Section 4

Proof of Lemma 1. First, we verify that the construction yields a partial equilibrium. We check each condition of Definition 1:

1. (Optimal Search) Take any $\tau \in T^p$. By construction:
   (a) $\bar{U}^s(i) \geq U^s(i, \tau, \Lambda^s(\tau))$ for all $i \in \mathbb{I}^s$, since this is exactly the first constraints of Problem (7);
   (b) $\int_{\mathbb{I}^s} \bar{U}^s(i) dG^s(i) = \int_{\mathbb{I}^s} U^s(i, \tau, \Lambda^s(\tau)) dG^s(i)$, from the second constraints.
2. (Impossible Terms-of-Trade) For any $\tau \notin T^p$, by construction there exist no $(\lambda^s)_{s=a,b} \in \mathbb{\Lambda}$ satisfying the constraints of Problem (7). This directly implies there are no $(\lambda^s)_{s=a,b}$ satisfying conditions 2(a) and 2(b) of Definition 1
3. (Profit Maximization) Take any $\tau \in T^p$ with $\tau \notin T$. By construction, $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ does not solve Problem (7). Therefore, there exists some $(\hat{\lambda}^s, \hat{\phi}^s, \hat{G}^s)_{s=a,b}$ satisfying the constraints with strictly higher objective value. Let $\hat{\tau} = (\hat{\phi}^s, \hat{G}^s)_{s=a,b}$, so $\Lambda^s(\hat{\tau}) = \hat{\lambda}^s$. Then $\hat{\tau} \in T^p$ and $V(\hat{\tau}, \Lambda^a(\hat{\tau}), \Lambda^b(\hat{\tau})) > V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$

For the converse, take any partial equilibrium $\{T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}\}$ and any $\tau = (\phi^s, G^s)_{s=a,b} \in T$. We must show that $(\Lambda^s(\tau), \phi^s, G^s)_{s=a,b}$ solves Problem (7). From Definition 1 parts 1(a) and 1(b), this tuple satisfies the constraints of Problem (7). Suppose it is not optimal. Then there exists some $(\hat{\lambda}^s, \hat{\phi}^s, \hat{G}^s)_{s=a,b}$ satisfying the constraints with strictly higher objective value. Let $\hat{\tau} = (\hat{\phi}^s, \hat{G}^s)_{s=a,b}$. The constraints of Problem (7) imply $\hat{\tau} \in T^p$ with $\Lambda^s(\hat{\tau}) = \hat{\lambda}^s$. But then $V(\hat{\tau}, \Lambda^a(\hat{\tau}), \Lambda^b(\hat{\tau})) > V(\tau, \Lambda^a(\tau), \Lambda^b(\tau))$, so part 3 of Definition 1 implies $\tau \notin T$, a contradiction. ■

Proof of Lemma 2. Suppose not, so $\bar{U}^s$ is discontinuous at some point $i^* \in [0, 1]$. Then there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exist a point i with $|i - i^*| < \delta$ and $|\bar{U}^s(i) - \bar{U}^s(i^*)| > \epsilon$.

First assume it is possible to find such a point with $\bar{U}^s(i^*) - \bar{U}^s(i) > \epsilon$. By the third condition in the definition of competitive search equilibrium, type i^* must obtain utility $\bar{U}^s(i^*) = \Lambda^s(\tau)(\int u^s(i^*, j) dG^{\bar{s}}(j) - \phi^s)$ in some market with terms-of-trade $\tau = (\phi^s, G^s)_{s=a,b} \in T$.

Continuity of u^s in its first argument implies we can choose δ sufficiently small such that $\int u^s(i^*, j)dG^{\bar{s}}(j) - \int u^s(i, j)dG^{\bar{s}}(j) < \epsilon$. Since $0 \leq \Lambda^s(\tau) \leq 1$, this means

$$\bar{U}^s(i) < \bar{U}^s(i^*) - \epsilon = \Lambda^s(\tau)\left(\int u^s(i^*, j)dG^{\bar{s}}(j) - \phi^s\right) - \epsilon < \Lambda^s(\tau)\left(\int u^s(i, j)dG^{\bar{s}}(j) - \phi^s\right).$$

The first inequality comes from the assumed discontinuity in $\bar{U}^s$. The equation is the indifference condition of i^*. The second inequality uses the continuity of u^s. But this implies that the terms-of-trade τ do not satisfy condition 1(a) in the definition of partial equilibrium for the type i agent, a contradiction.

If instead we have $\bar{U}^s(i) - \bar{U}^s(i^*) > \epsilon$, we reverse the role of i and i^* in the proof, but the argument is otherwise unchanged. ■

Proof of Lemma 3.

Necessity. We first prove that any CSE $\left(T^p, T, (\Lambda^s, \bar{U}^s)_{s=a,b}, \mu\right)$ must satisfy these conditions. Condition 4 is a piece of the definition of CSE. Condition 3 holds in any partial equilibrium, and a CSE is a partial equilibrium. Thus Condition 3 holds as well.

Next, take any $\tau \in T$. Let $\lambda^s = \Lambda^s(\tau)$. Lemma 1 implies that $(\lambda^s, \phi^s, G^s)_{s=a,b}$ must solve problem (7) with maximal value c (from the free entry condition in the definition of CSE). We prove the solution to problem (7) must have a degenerate distribution G^s and moreover must solve problem (8).

Suppose to the contrary that either G^a or G^b is non-degenerate. Let k_1^s be the largest s-side agent with a binding incentive constraint:

$$k_1^s = \max\left\{i \in \mathbb{I}^s | \bar{U}^s(i) = \lambda^s\left(\int_{\mathbb{I}^{\bar{s}}} u^s(i, j)dG^{\bar{s}}(j) - \phi^s\right)\right\} \tag{24}$$

u^s is continuous in its first argument by assumption, and $\bar{U}^s$ is continuous in a CSE (Lemma 2). Then since $\mathbb{I}^s$ is compact, the type k_1^s exists. Moreover, k_1^s exceeds all points in the support of G^s, strictly so for a positive measure on at least one side given non-degeneracy.

Define new fees $\phi_1^s \equiv u^s(k_1^s, k_1^{\bar{s}}) - \bar{U}^s(k_1^s)/\lambda^s$. Common ranking implies $\phi_1^s \geq \phi^s$ with strict inequality on at least one side, proving $\phi_1^a + \phi_1^b > \phi^a + \phi^b$.

We next prove that for any $k' < k_1^s$, $\bar{U}^s(k') \geq \lambda^s(u^s(k', k_1^{\bar{s}}) - \phi^s)$:

$$\begin{aligned}
\lambda^s(u^s(k', k_1^{\bar{s}}) - \phi_1^s) &= \lambda^s \int_{\mathbb{I}^{\bar{s}}} (u^s(k', k_1^{\bar{s}}) - u^s(k', j)) dG^{\bar{s}}(j) \\
&\qquad + \lambda^s \int_{\mathbb{I}^{\bar{s}}} u^s(k', j) dG^{\bar{s}}(j) - \lambda^s \phi_1^s \\
&\leq \lambda^s \int_{\mathbb{I}^{\bar{s}}} (u^s(k_1^s, k_1^{\bar{s}}) - u^s(k_1^s, j)) dG^{\bar{s}}(j) \\
&\qquad + \lambda^s \int_{\mathbb{I}^{\bar{s}}} u^s(k', j) dG^{\bar{s}}(j) - \lambda^s \phi_1^s \\
&= \lambda^s \phi_1^s - \lambda^s \phi^s + \lambda^s \int_{\mathbb{I}^{\bar{s}}} u^s(k', j) dG^{\bar{s}}(j) - \lambda^s \phi_1^s \\
&\leq \bar{U}^s(k').
\end{aligned}$$

The first equality adds and subtracts $\int_{\mathbb{I}^{\bar{s}}} u^s(k', j) dG^{\bar{s}}(j)$ and reorders terms. The first inequality uses supermodularity, together with $k' < k_1^s$ and $k_1^{\bar{s}} \geq j$ for all j in the support of $G^{\bar{s}}$. The second equality uses the definition of ϕ_1^s as well as the binding constraint in the definition of k_1^s in equation (24). The final inequality collects terms and uses the incentive constraint in the original problem, $\bar{U}^s(k') \geq \lambda^s\big(\int_{\mathbb{I}^{\bar{s}}} u^s(k', j) dG^{\bar{s}}(j) - \phi^s\big)$.

We do not claim $\bar{U}^s(k') \geq \lambda^s(u^s(k', k_1^{\bar{s}}) - \phi^s)$ for $k' > k_1^s$. Instead, we construct monotone sequences (k_n^s, ϕ_n^s) by iteratively finding the highest type willing to pay more to match with previous highest types. Suppose for some $n \geq 2$, we have found non-decreasing sequences of types $k_1^s \leq \cdots \leq k_{n-1}^s$ as well as sequences of fees $\phi_1^s, \ldots, \phi_{n-1}^s$ with the following properties:

1. $\bar{U}^s(k_{n-1}^s) = \lambda^s\big(u^s(k_{n-1}^s, k_{n-1}^{\bar{s}}) - \phi_{n-1}^s\big)$, $s \in \{a, b\}$;
2. $\bar{U}^s(k') \geq \lambda^s\big(u^s(k', k_{n-1}^{\bar{s}}) - \phi_{n-1}^s\big)$ for all $k' < k_{n-1}^s$, $k' \in \mathbb{I}^s$, $s \in \{a, b\}$;
3. $\phi_{n-1}^a + \phi_{n-1}^b \geq \phi_1^a + \phi_1^b$.

We have done this when $n = 2$ and proceed by induction to extend this to any n.

Let

$$k_n^s \equiv \max\left(\arg\max_{k' \in \mathbb{I}^s}\left(u^s(k', k_{n-1}^s) - \frac{\bar{U}^s(k')}{\lambda^s}\right)\right). \tag{25}$$

That is, k_n^s is the largest element of the set of maximizers of $u^s(k', k_{n-1}^s) - \bar{U}^s(k')/\lambda^s$. The first two properties above imply that for all $k' < k_{n-1}^s$,

$$u^s(k', k_{n-1}^{\bar{s}}) - \frac{\bar{U}^s(k')}{\lambda^s} \leq \phi_{n-1}^s = u^s(k_{n-1}^s, k_{n-1}^{\bar{s}}) - \frac{\bar{U}^s(k_{n-1}^s)}{\lambda^s}.$$

Thus $k_n^s \geq k_{n-1}^s$. Also define

$$\phi_n^s \equiv u^s(k_n^s, k_n^{\bar{s}}) - \frac{\bar{U}^s(k_n^s)}{\lambda^s}$$

The same arguments as for $n = 1$ imply $\phi_n^s \geq \phi_{n-1}^s$ and $\bar{U}^s(k') \geq \lambda^s(u^s(k', k_n^{\bar{s}}) - \phi_n^s)$ for all $k' < k_n^s$.

The sequences (k_n^a, k_n^b) are nondecreasing on the compact set $\mathbb{I}^a \times \mathbb{I}^b$ and so converge to $(k^{a*}, k^{b*}) \in \mathbb{I}^a \times \mathbb{I}^b$. This means the ϕ^s converges as well, to (ϕ^{a*}, ϕ^{b*}), with $\phi^{a*} + \phi^{b*} \geq \phi_1^a + \phi_1^b > \phi^a + \phi^b$. Continuity of $\bar{U}^s$ and u^s implies

1. $\bar{U}^s(k^{s*}) = \lambda^s\big(u^s(k^{s*}, k^{\bar{s}*}) - \phi^{s*}\big)$, $s \in \{a, b\}$;
2. $\bar{U}^s(k') \geq \lambda^s\big(u^s(k', k^{\bar{s}*}) - \phi^{s*}\big)$ for all $k' < k^{s*}$, $k' \in \mathbb{I}^s$, $s \in \{a, b\}$;
3. $\phi^{a*} + \phi^{b*} \geq \phi_1^a + \phi_1^b > \phi^a + \phi^b$.

Finally, we claim the second point, the incentive constraint, extends to all $k' \in \mathbb{I}^s$, including $k' > k^{s*}$. To prove this, suppose to the contrary that there is a $k' > k^{s*}$ with $\bar{U}^s(k') < \lambda^s\big(u^s(k', k^{\bar{s}*}) - \phi^{s*}\big)$. Eliminate ϕ^{s*} using the first bullet point:

$$\bar{U}^s(k') < \lambda^s\big(u^s(k', k^{\bar{s}*}) - u^s(k^{s*}, k^{\bar{s}*})\big) + \bar{U}^s(k^{s*}).$$

But continuity of u^s and convergence of $k_n^{\bar{s}}$ to $k^{\bar{s}*}$ implies that there exists an n such that

$$\bar{U}^s(k') < \lambda^s\big(u^s(k', k_n^{\bar{s}}) - u^s(k^{s*}, k_n^{\bar{s}})\big) + \bar{U}^s(k^{s*}),$$

or equivalently

$$u^s(k', k_n^{\bar{s}}) - \frac{\bar{U}^s(k')}{\lambda^s} > u^s(k^{s*}, k_n^{\bar{s}}) - \frac{\bar{U}^s(k^{s*})}{\lambda^s},$$

From the definition of k_{n+1}^s (equation (25)), this implies $k' > k^{s*}$, which in turn contradicts monotonicity of the sequence k_n^s with convergence to k^{s*}. So we conclude that $\bar{U}^s(k') \geq \lambda^s\big(u^s(k', k^{\bar{s}*}) - \phi^{s*}\big)$ for all $k' \in \mathbb{I}^s$, $s \in \{a, b\}$.

Now let G^{s*} be the cumulative distribution function that is degenerate at k^{s*} and consider the alternative term-of-trade $(\lambda^s, \phi^{s*}, G^{s*})_{s=a,b}$. By construction this satisfies the constraints in problem (7) and it attains a higher value than the original term-of-trade because $\phi^{a*}+\phi^{b*} > \phi^a + \phi^b$ and (λ^a, λ^b) is unchanged. This proves that the original term-of-trade does not solve problem (7). That is, any solution to the problem (7) must have a degenerate type distribution on both sides of the market.

This proves the first two conditions in the statement of the Lemma, since any solution to problem (7) must have a degenerate type distribution and slack upward constraints, and

thus also solve problem (8). Following the same steps, we also find that any solution to problem (8) also solves problem (7).

Sufficiency. For sufficiency, we need to prove that a tuple satisfying the four conditions constitutes both a partial equilibrium and a CSE. Let's verify each requirement:

First, for the partial equilibrium conditions from Definition 1:

1. Optimal Search and Impossible Terms-of-Trade follow directly from condition 3;
2. For Profit Maximization, we proceed in steps. Condition 1 ensures each $\tau \in T$ is separating, and condition 2 ensures it solves problem (8). Given the equivalence between problems (7) and (8) when condition 1 is satisfied, Lemma 1 allows us to map from the solution to problem (8) back to a partial equilibrium. Thus condition 2 ensures that each $\tau \in T$ satisfies the final partial equilibrium requirement, profit maximization.

Next, for the additional CSE requirements from Definition 2:

1. Free Entry is satisfied because condition 2 ensures each $\tau \in T$ achieves value exactly equal to c;
2. Market Clearing follows directly from condition 4.

Thus, the tuple constitutes a CSE. ■

Proof of Lemma 4. This result follows directly from the incentive constraints in problem (7).

Choose any s-side type-i. If $\bar{U}^s(i) > 0$, the market for these agents clears in a CSE. That is, there is an equilibrium terms-of-trade that attracts them, with matching probability λ^s, fee ϕ^s, and partner distribution $G^{\bar{s}}(j)$. From problem (7):

$$\bar{U}^s(i) = \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i,j) dG^{\bar{s}}(j) - \phi^s \right)$$

Conversely, all other types i' weakly prefer not to come to this market

$$\bar{U}^s(i') \geq \lambda^s \left(\int_{\mathbb{I}^{\bar{s}}} u^s(i',j) dG^{\bar{s}}(j) - \phi^s \right).$$

Subtracting inequalities gives us

$$\bar{U}^s(i') - \bar{U}^s(i) \geq \lambda^s \int_{\mathbb{I}^{\bar{s}}} \big(u^s(i',j) - u^s(i,j) \big) dG^{\bar{s}}(j).$$

Now we use the monotone WTP assumptions. With IWTP, the right hand side of this last inequality is nonnegative when $i' > i$, so $\bar{U}^s$ is nondecreasing. With DWTP, the right hand side is nonnegative when $i' < i$, so $\bar{U}^s$ is nonincreasing.

Finally, if $\bar{U}^s(i) = 0$, nonnegativity of $\bar{U}^s$ implies $\bar{U}^s(i') \geq \bar{U}^s(i)$ for all i', ensuring weak monotonicity in this case as well. ■

Proof of Lemma 5. From Lemma 4, Monotone Willingness-to-Pay implies that market utilities $\bar{U}^s(i)$ are monotone. By Lebesgue's differentiation theorem, any monotone function is differentiable almost everywhere.

Now fix a $\tau \in T$, with $\lambda^s = \Lambda^s(\tau)$ and G^s degenerate at k^s for $s \in \{a, b\}$. Also assume $k^a \in (0, 1)$, so that for all small positive ϵ, $0 \leq k^a - \epsilon < k^a + \epsilon \leq 1$. The constraints from problem (7) require

$$\begin{aligned}
\bar{U}^a(k^a + \epsilon) - \bar{U}^a(k^a) &\geq \lambda^a(u^a(k^a + \epsilon, k^b) - u^a(k^a, k^b)), \\
\bar{U}^a(k^a - \epsilon) - \bar{U}^a(k^a) &\geq \lambda^a(u^a(k^a - \epsilon, k^b) - u^a(k^a, k^b)),
\end{aligned}$$

or

$$\begin{aligned}
\frac{\bar{U}^a(k^a + \epsilon) - \bar{U}^a(k^a)}{\epsilon} &\geq \lambda^a \frac{(u^a(k^a + \epsilon, k^b) - u^a(k^a, k^b))}{\epsilon}, \\
\frac{\bar{U}^a(k^a) - \bar{U}^a(k^a - \epsilon)}{\epsilon} &\leq \lambda^a \frac{(u^a(k^a, k^b) - u^a(k^a - \epsilon, k^b))}{\epsilon}.
\end{aligned}$$

Since $\bar{U}^a$ is differentiable almost everywhere, it must be $\bar{U}^{a\prime}(k^a) = \lambda^a u_1^a(k^a, k^b)$ at such points.

A symmetric argument establishes the result for side b. ■

B Proof for Section 5

Proof of Proposition 1.

Characterizing a PAM CSE. To start, we show that under the proposed equilibrium utilities, $\bar{U}(0) = \bar{\lambda} u(0, 0)$ and $\bar{U}'(i) = \bar{\lambda} u_1(i, i)$, all i solve a symmetric version of the opti-

mization problem (9):

$$\begin{aligned} \max_{\lambda \in [0,\bar{\lambda}]} \; & 2m(\lambda,\lambda)\left(u(i,i) - \frac{\bar{U}(i)}{\lambda}\right) \\ \text{s.t.} \; & \bar{U}(j) - \bar{U}(i) \geq \lambda(u(j,i) - u(i,i)) \; \forall j < i, j \in \mathbb{I}, \\ & u(i,i) - \frac{\bar{U}(i)}{\lambda} \geq 0. \end{aligned} \tag{26}$$

We also prove the maximized value is zero. Recall that $\bar{\lambda}$ is the highest matching probability that can be obtained by both sides of the market under symmetry, so $m(\bar{\lambda},\bar{\lambda}) = 0$.

The first constraint in problem (26) evaluated at j slightly smaller than i implies $\lambda \geq \bar{\lambda}$ when evaluated at the market utility:

$$\bar{U}'(i) = \lim_{j \to i} \frac{\bar{U}(j) - \bar{U}(i)}{j - i} \leq \lambda \lim_{j \to i} \frac{u(j,i) - u(i,i)}{j - i} = \lambda u_1(i,i),$$

where we reverse the inequality because we have divided both sides by $j - i < 0$. Since $\bar{U}'(i) = \bar{\lambda} u_1(i,i)$ and $u_1(i,i) > 0$, this implies $\lambda \geq \bar{\lambda}$. The requirement that $m(\lambda,\lambda) \geq 0$ implies $\lambda \leq \bar{\lambda}$. Thus given the level of market utility $\bar{U}(i)$, the only feasible matching rate is $\bar{\lambda}$ if there is to be an active (i,i) market.

Next we prove that the second constraint problem (26) is satisfied. Again use $\bar{U}(0) = \bar{\lambda} u(0,0)$ and $\bar{U}'(i) = \bar{\lambda} u_1(i,i)$. Since $u_2(i,i) > 0$, this implies $\bar{U}'(i) < \bar{\lambda}(u_1(i,i) + u_2(i,i))$. Hence for $i > 0$, $\bar{\lambda} u(i,i) > \bar{U}(i)$, so the second constraint is satisfied. This implies that the fee charged in this market is positive, $\Phi(i) = u(i,i) - \frac{\bar{U}(i)}{\bar{\lambda}} > 0$; equivalently, $\Phi(0) = 0$ and $\Phi'(i) = u_2(i,i) > 0$.

Verifying Downward ICs. So far in characterizing the equilibrium outcome, we have only incorporated the local incentive constraints. This step verifies that the equilibrium outcomes indeed satisfy all the global downward incentive constraints. More precisely:

$$\bar{U}(j) - \bar{U}(i) \geq \ell(i)(u(j,i) - u(i,i)), \quad \forall j \leq i.$$

To do so, we use the characterization of the equilibrium utility to write

$$\bar{U}(j) - \bar{U}(i) = -\bar{\lambda} \int_j^i u_1(x,x)dx \geq -\bar{\lambda} \int_j^i u_1(x,i)dx = \bar{\lambda}(u(j,i) - u(i,i)),$$

where in the first equality we use $\bar{U}'(i) = \bar{\lambda} u_1(i,i)$, in the inequality we use supermodularity of $u(i,j)$, and in the last equality we solve out the integral. This proves the global ICs also

hold on the PAM CSE.

Existence of a PAM CSE. We still need to rule out the possibility that an (i,j) market is more profitable for $i \neq j$. First, assume that i and j are both positive, so in a proposed (i,j) market, we have a binding local incentive constraint on both sides:

$$\bar{U}'(i) = \lambda^a u_1(i,j),\ \bar{U}'(j) = \lambda^b u_1(j,i).$$

Using $\bar{U}'(i) = \bar{\lambda} u_1(i,i)$ and $\bar{U}'(j) = \bar{\lambda} u_1(j,j)$, we can write this as

$$\lambda^a = \bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)},\ \ \lambda^b = \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}.$$

By assumption, however, $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{\Lambda}^o$, and any combination of such matching probabilities is infeasible.

Alternatively, if $j = 0 < i$, then there is only a binding incentive constraint on one side of the market: $\bar{U}'(i) = \lambda^a u_1(i,0)$. Assumption 5 (b) rules the profitable deviations to $(i,0)$ markets in steps. First, if $u_1(k,0) = 0$ for all k, then for some $k < i$

$$\bar{U}(k) - \bar{U}(i) = -\bar{\lambda}\int_k^i u_1(x,x)dx < -\bar{\lambda}\int_k^i u_1(x,0)dx = 0,$$

In the first equality, we use the characterization of the PAM CSE; in the first inequality, we use the supermodularity; and in the last equality, we evaluate the integral. This means for any λ^a, $\bar{U}(k) - \bar{U}(i) < 0 < \lambda^b(u(k,0) - u(i,0))$ violating the incentive constraint for type k in the $(i,0)$ market. Thus the $(i,0)$ market is infeasible.

Otherwise, we can find λ^a that satisfies the downward ICs. Inverting the participation constraints, we have the sum of fees on either side of the market must be

$$\phi^a + \phi^b = u(0,i) + u(i,0) - \frac{\bar{\lambda}u(0,0)}{\lambda^b} - \frac{u_1(i,0)}{u_1(i,i)}\left(u(0,0) + \int_0^i u_1(k,k)dk\right)\lambda^a \leq u(0,i) + \tilde{\phi}(i)$$

where in the inequality we use the definition of $\tilde{\phi}(i)$ and the fact $\frac{\bar{\lambda}u(0,0)}{\lambda^b} \geq 0$. Thus, if $u(0,i) + \tilde{\phi}(i) \leq 0$, there are no terms-of-trade that deliver a positive sum of fees to the platform, ruling out the profitable deviations to $(i,0)$ markets.

Otherwise, there is (λ^a, λ^b) that satisfies the incentive constraints and delivers a positive sum of fees. When $\lambda^b = \bar{\lambda}\frac{u(0,0)}{u(0,i)+\phi}$, the sum of fees is zero, and the sum of fees is strictly positive (negative) if $\lambda^b > (<)\bar{\lambda}\frac{u(0,0)}{u(0,i)+\phi}$. If $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,0)}, \bar{\lambda}\frac{u(0,0)}{u(0,i)+\tilde{\phi}(i)}\right) \notin \mathbb{\Lambda}^o$, then any pair of matching probability that delivers positive sum of fees is infeasible from the matching function, ruling

out the profitable devaition to $(i,0)$ market.

CES Case. Assume a CES payoff function:

$$u(i,j) = \left(\frac{1}{2}i^{\frac{\theta-1}{\theta}} + \frac{1}{2}j^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}.$$

Also assume a CES matching function

$$m(\lambda^a, \lambda^b) = \left(1 - (\lambda^a)^\gamma - (\lambda^b)^\gamma\right)^{\frac{1}{\gamma}}.$$

This implies $\mathbb{\Lambda} = \{(\lambda^a, \lambda^b) | (\lambda^a)^\gamma + (\lambda^b)^\gamma \leq 1\}$, and $\bar{\lambda} = 2^{-1/\gamma}$.

Note that in this case, $u(i,i) = i$ and $u_1(i,i) = u_2(i,i) = \frac{1}{2}$ for all i. This implies $\bar{U}(i) = \bar{\lambda} i/2$ and $\Phi(i) = i/2$ for all i. That is, if a PAM equilibrium exists, half the gains from trade are lost because of the private information problem.

For the CES case, the condition $\left(\bar{\lambda}\frac{u_1(i,i)}{u_1(i,j)}, \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)}\right) \notin \mathbb{\Lambda}$ can be expressed as

$$\frac{1}{2}i^{\frac{\gamma}{\theta}} + \frac{1}{2}j^{\frac{\gamma}{\theta}} > \left(\frac{1}{2}i^{\frac{\theta-1}{\theta}} + \frac{1}{2}j^{\frac{\theta-1}{\theta}}\right)^{\frac{\gamma}{\theta-1}}$$

Alternatively, let $x = i^{\frac{\theta-1}{\theta}}$ and $y = j^{\frac{\theta-1}{\theta}}$. Then we require

$$\frac{1}{2}x^{\frac{\gamma}{\theta-1}} + \frac{1}{2}y^{\frac{\gamma}{\theta-1}} > \left(\frac{1}{2}x + \frac{1}{2}y\right)^{\frac{\gamma}{\theta-1}}$$

Using Jensen's inequality, this holds for all $x \neq y$ if and only if $\gamma > \theta - 1$. Conversely, if $\theta > 1 + \gamma$, there is no PAM equilibrium because creating any other (i,j) market would be profitable.

Now consider a $(0,j)$ market. If $\theta \leq 1$, $u_1(j,0) = 0$ for all j and so incentive constraints require $\lambda^b = \lim_{i\to 0} \bar{\lambda}\frac{u_1(j,j)}{u_1(j,i)} = +\infty$. This means it is never feasible to create a $(0,j)$ market. For $\theta > 1$,

$$\lambda^b = \bar{\lambda}\frac{u_1(j,j)}{u_1(j,0)} = 2^{\frac{1+\gamma-\theta}{\gamma(\theta-1)}}.$$

This is larger than 1, meaning it is impossible to create such a market, whenever $\theta < 1 + \gamma$. This proves that for $\theta < 1 + \gamma$, there is a PAM CSE, while for larger values of θ such an equilibrium cannot exist. ■

Proof of Proposition 2. We follow the structure of the proof of Proposition 1.

Characterization. To start, we show that under the proposed equilibrium utilities, the equilibrium matching probabilities, fees, and utilities satisfy the proposed formulas. To start, we return to the optimization problem given equilibrium utilities:

$$\begin{aligned} \max_{\lambda \in [0,\bar{\lambda}]} \; & 2m(\lambda,\lambda)\left(u(i,i) - \frac{\bar{U}(i)}{\lambda}\right) \\ \text{s.t.} \; & \bar{U}(j) - \bar{U}(i) \geq \lambda(u(j,i) - u(i,i)) \; \forall j < i, j \in \mathbb{I}, \\ & u(i,i) - \frac{\bar{U}(i)}{\lambda} \geq 0. \end{aligned} \tag{27}$$

With DWTP, we can divide through the first set of constraints to get $\lambda \leq \frac{\bar{U}(j)-\bar{U}(i)}{u(j,i)-u(i,i)}$ for $j < i$. We want to show that $\Phi(i) = 0$ for any i. Suppose, to the contrary, the sum of fees is strictly positive. Then the platform can deviate to another terms-of-trade, charging the same set of fees while offering a lower matching probability on side a. Since the incentive constraint puts an upper bound on matching probabilities, this deviation is feasible. Because the matching probability of the platform decreases with λ^a, the proposed deviation delivers positive fees with a positive matching probability, which contradicts the optimization of the platforms. We then rule out the possibilities of $\Phi(i) < 0$. For the fees For $\Phi(i) < 0$ in the equilibrium, it must $\ell(i) = \bar{\lambda}$. Suppose this is indeed the case; type 0 would be strictly better off deviating to such terms-of-trade. More precisely, by participating to such a (i,i) market, the type 0 receives a payoff of:

$$\bar{\lambda}(u(0,i) - \Phi(i)) > \bar{\lambda}u(0,i) > \bar{\lambda}u(0,0) = \bar{U}(0),$$

where the first inequality uses the assumption $\Phi(i) < 0$, and the second inequality uses common ranking. This violates the IC. Thus, for $i > 0$, $\Phi(i) = 0$.

Next, zero fees imply that $\bar{U}(i) = \ell(i)u(i,i)$. Totally differentiating this condition, we have $\bar{U}'(i) = \ell'(i)u(i,i) + \ell(i)(\partial_1 u(i,i) + \partial_2 u(i,i))$. Imposing the IC $\bar{U}'(i) = \ell(i)u_1(i,i)$, we have $\ell'(i)u(i,i) = -\ell(i)u_2(i,i)$. This differential equation, with the initial condition $\ell(0) = \bar{\lambda}$, has a solution:

$$\ell(i) = \bar{\lambda} \exp\left(-\int_0^i \frac{u_2(\iota,\iota)}{u(\iota,\iota)} d\iota\right) > 0.$$

Thus we can divide the condition $\bar{U}'(i) = \bar{\lambda}u_1(i,i)$ by $\bar{U}(i) = \bar{\lambda}u(i,i)$ to get $\frac{\bar{U}'(i)}{\bar{U}(i)} = \frac{u_1(i,i)}{u(i,i)}$.

Verifying Downward ICs. We veify that

$$\bar{U}(j) - \bar{U}(i) \geq \ell(i)(u(j,i) - u(i,i)), \; \forall j \leq i$$

From the characterization of the equilibrium

$$\begin{aligned}
&\bar{U}(j) - \bar{U}(i) \\
=&-\int_j^i \ell(x)u_1(x,x)dx \geq -\int_j^i \ell(i)u_1(x,x)dx \geq -\ell(i)\int_j^i u_1(x,i)dx \\
=&\ell(i)(u(j,i) - u(i,i)),
\end{aligned}$$

where in the first equality we use the local ICs on the PAM CSE, in the first inequality we used $u_1(x,x) \leq 0$ and $\ell(i)$ is monotonically decreasing in i, in the second inequality we used $-\ell(i) < 0$ and the supermodularity of $u(i,j)$, and in the last equality we evaluate the integral. This shows the global downward ICs hold.

Sufficient Conditions We look for sufficient conditions for a PAM CSE. We conjecture that there is a PAM CSE and look at a market which attracts type i agents on side a and type j agents on side b. If all such markets are unprofitable or infeasible, then there is a PAM CSE. To start, we assume that both $i > 0$ and $j > 0$.

In the conjectured market, the local incentive constraints require

$$\lambda^a = \frac{\bar{U}'(i)}{u_1(i,j)},\ \lambda^b = \frac{\bar{U}'(j)}{u_1(j,i)}.$$

Inverting the participation for type i and type j, we get:

$$\begin{aligned}
\phi^a + \phi^b &= u(i,j) + u(j,i) - \frac{\bar{U}(i)}{\bar{U}'(i)}u_1(i,j) - \frac{\bar{U}(j)}{\bar{U}'(j)}u_1(j,i) \\
&= u(i,j) + u(j,i) - u(i,i)\frac{u_1(i,j)}{u_1(i,i)} - u(j,j)\frac{u_1(j,i)}{u_1(j,j)},
\end{aligned}$$

where in the second equality, we used the solution from PAM CSE. Under assumption 6(a), $\phi^a + \phi^b < 0$. Thus, there is no terms-of-trade attracting (i,j) and delivering non-negative payoffs to platforms.

Now consider a terms-of-trade attracting $(0,i)$, where $i > 0$. With supermodularity and decreasing WTP, it must be $u_1(i,0) < 0$ for any $i > 0$, otherwise $u_1(i,j) > 0$ for $j > i$.

We start with the case $0 < i < 1$. It is thus always possible to invert the local incentive constraint to get $\lambda^b = \frac{\bar{U}'(i)}{u_1(i,0)}$. From the participation constraint of type i in this proposed $(0,i)$ market, the fee charged is

$$\hat{\phi}(i) = u(i,0) - \frac{\bar{U}(i)}{\lambda^b} = u(i,0) - \frac{u_1(i,0)}{u_1(i,i)}u(i,i)$$

If $\hat{\phi}(i) + u(0,i) \leq 0$, the maximal sum of fees that can be charged by the platform is non-positive, meaning there is no profitable deviation to the $(0,i)$ market.

Now we consider the case $i = 1$. This case is special because the incentive constraint only puts an upper bound on the matching probability

$$\lambda^b \leq \frac{\bar{U}'(i)}{u_1(i,0)}$$

For such terms-of-trade to be incentive compatible, the matching probability on the a side has to satisfy

$$\bar{U}(k) - \bar{U}(i) \geq \lambda^a(u(k,0) - u(i,0)),\ \forall k < j,$$

Suppose $u_1(i,0) = 0$ for any i:

$$\bar{U}(k) - \bar{U}(i) = -\int_k^i \ell(x)u_1(x,x)dx \geq -\ell(0)\int_k^i u_1(x,x)dx > -\ell(0)\int_k^i u_1(x,0)dx = 0,$$

where in the first inequality we use $\ell(i)$ is decreasing in i, in the second inequality we use supermodularity, and in the last equality we use $u_1(i,0) = 0$.

Suppose otherwise, we can then use the local incentive constraint to find $\lambda^a = \frac{\bar{U}'(i)}{u_1(i,0)}$. The participation constraints imply that for such terms-of-trade, the sum of fees must be

$$\phi^a + \phi^b = u(0,i) + u(i,0) - u(0,0)\frac{\bar{\lambda}}{\lambda^b} - u(i,i)\frac{u_1(i,0)}{u_1(i,i)} \leq u(0,i) + u(i,0) - u(i,i)\frac{u_1(i,0)}{u_1(i,i)} = \hat{\phi}(i)$$

because $u(0,0)\frac{\bar{\lambda}}{\lambda^b} \geq 0$. If $\hat{\phi}(i) \leq 0$, there is no such terms-of-trade that delivers a positive sum of fees to the platforms. Otherwise, the fees are zero when $\lambda^b = \frac{\bar{\lambda}u(0,0)}{\hat{\phi}(i)}$, strictly positive (negative) when $\lambda^b > (<)\frac{\bar{\lambda}u(0,0)}{\hat{\phi}(i)}$. When $\left(\hat{\lambda}(i), \bar{\lambda}\frac{u(0,0)}{\hat{\phi}(i)}\right) \notin \mathbb{\Lambda}^o$, any matching probabilities that deliver a positive sum of fees is infeasbile under the matching function.

Disease Model Solving the differential equations characterizing the PAM equilibrium outcomes, we obtain the equilibrium utility

$$\bar{U}(i) = 2^{-\frac{1}{\gamma}}e^{-\zeta(i)}\sqrt{1 - \kappa i(1-i)},$$

where

$$\zeta(i) \equiv \sqrt{\frac{\kappa}{4-\kappa}}\left(\arctan\left(\sqrt{\frac{\kappa}{4-\kappa}}\right) - \arctan\left(\sqrt{\frac{\kappa}{4-\kappa}}(1-2i)\right)\right).$$

We now verify the sufficient conditions for the PAM equilibrium to exist. The sufficient

condition that rules out (i,j) markets, where $i \neq j$ and $i,j > 0$, requires

$$u(i,j) + u(j,i) - u(i,i)\frac{u_1(i,j)}{u_1(i,i)} - u(j,j)\frac{u_1(j,i)}{u_1(j,j)} = -\frac{(i-j)^2}{(1-i)(1-j)} < 0.$$

In the first equality, we imposed the solution from the equilibrium characterization, and the inequality always holds since $i,j \in [0,1]$. Thus, condition (a) of assumption 6 does not impose any additional restrictions on utility.

Now we move to ruling out $(0,i)$ markets with $i > 0$. Under a PAM equilibrium, the local IC on side b of this market requires that

$$\ell^b(i) = \frac{\bar{U}'(i)}{u_1(j,0)} = 2^{-\frac{1}{\gamma}}\frac{1-i}{\sqrt{1-\kappa i(1-i)}}e^{-\zeta(i)}.$$

Using this, we can recover the fees on side b. When the sum of fees is zero, we then get the fees on side a. Finally, the participation constraint on side a give us $\ell^a(i)$:

$$\ell^a(i) = 2^{-\frac{1}{\gamma}}\frac{1-i}{1-2i}.$$

For λ^a to be a positive number, it has to be $i \in \left[0, \frac{1}{2}\right)$. We want to find conditions such that $T(i) \equiv 1 - (\ell^a(i))^\gamma - (\ell^b(i))^\gamma \leq 0$, for $i \in \left[0, \frac{1}{2}\right)$. From the definition, $T(0) = 0$. We want to show that $T'(i) \leq 0$ for $i \in \left[0, \frac{1}{2}\right)$. Taking this derivative, we can show

$$sign\left(T'(i)\right) = sign\left(\left(1-\kappa i\left(1-i\right)\right)^{-\frac{\gamma}{2}-1}\exp\left(-\gamma\zeta\left(i\right)\right) - \left(1-2i\right)^{-\gamma-1}\right)$$

Thus $T'(i) \leq 0$ is equivalent to

$$H(i) \equiv -\gamma\zeta(i) + (1+\gamma)\log(1-2i) + \left(1+\frac{\gamma}{2}\right)\log(1-\kappa i(1-i)) \leq 0$$

We can also show that $H(0) = 0$. Taking derivative

$$H'(i) = -\gamma\zeta'(i) - (1+\gamma)\frac{2}{1-2i} + \left(1+\frac{\gamma}{2}\right)\frac{\kappa(1-2i)}{1-\kappa i(1-i)}$$

Using the definition of $\zeta(i)$ we have $\zeta'(i) = -\frac{\kappa}{2(1-\kappa i(1-i))}$. Thus:

$$H'(i) = \frac{2\kappa i^2 - \kappa(2-\gamma)i - 2(1+\gamma) + \kappa}{(1-2i)(1-\kappa i(1-i))}$$

To determine the sign of $H'(i)$, we focus on the sign of the numerator, a quadratic function.

Because $2\kappa > 0$, the maximum of the numerator is reached either at $i = 0$ or $i = \frac{1}{2}$. At $i = 0$, $H'(0) = \kappa - 2(1+\gamma)$; At $i = \frac{1}{2}$, $H'\left(\frac{1}{2}\right) = \frac{1}{2}(1+\gamma)(\kappa - 4)$. When $\kappa \leq \min\{2(1+\gamma), 4\}$, $H'(i) \leq 0$ for any $i \in \left[0, \frac{1}{2}\right)$. Since $H(0) = 0$, this shows that $H(i) \leq 0$ (and $T'(i) \leq 0$) for all $i \in \left[0, \frac{1}{2}\right)$. Since $T(0) = 0$, this also proves the condition b in assumption (6) is met when $\kappa \leq \min\{2(1+\gamma), 4\}$.

■

Proof of Proposition 3. We seek to characterize an (i, i) market in a competitive search equilibrium in the symmetric case. If such an equilibrium exists, then for all $i = k^a = k^b$, the maximized value of Problem (9) must be c:

$$\begin{aligned} c = \max_{\lambda \in [0, \bar{\lambda}]} & 2m(\lambda, \lambda)\left(u(i,i) - \frac{\bar{U}(i)}{\lambda}\right) \\ \text{s.t. } & \bar{U}(j) - \bar{U}(i) \geq \lambda(u(j,i) - u(i,i)) \text{ for all } j < i \\ & \bar{U}(i) \leq \lambda u(i,i) \end{aligned}$$

Moreover, since $c > 0$, any solution must satisfy the second constraint, which we thus drop:

$$\begin{aligned} c = \max_{\lambda \in [0, \bar{\lambda}]} & 2m(\lambda, \lambda)\left(u(i,i) - \frac{\bar{U}(i)}{\lambda}\right) \\ \text{s.t. } & \bar{U}(j) - \bar{U}(i) \geq \lambda(u(j,i) - u(i,i)) \text{ for all } j < i. \end{aligned} \tag{28}$$

Unconstrained Markets. First suppose the incentive constraints are all slack for some i. Using equation (4), we can rewrite the unconstrained problem in terms of the number of agents per posting, $n = N(\lambda, \lambda)$:

$$c = \max_{n \geq 0} 2\big(M^*(n)u(i,i) - n\bar{U}(i)\big) \tag{29}$$

where $M^*(n) \equiv M(n, n)$. Then since M is strictly concave, the objective has a unique maximum satisfying $M^{*\prime}(n)u(i,i) = \bar{U}(i)$. Plugging this into the fact that the objective function must equal c, $c = 2\big(M^*(n)u(i,i) - n\bar{U}(i)\big)$, we get

$$\frac{c}{2u(i,i)} = M^*(n) - nM^{*\prime}(n)$$

Again using concavity of the matching function M, the right hand side is increasing. It is equal to zero at $n = 0$ and, since $2u(i,i)m(0,0) > c$, it exceeds $c/(2u(i,i))$ as n grows without

bound.[5] Thus n is uniquely defined if an unconstrained (i,i) market exists in equilibrium; call this value $n^*(i)$.

Once we know $n = n^*(i)$, we can recover $\bar{U}(i) = \bar{U}^*(i)$ from the free entry condition: $c = 2\big(M^*(n^*(i))u(i,i) - n^*(i)\bar{U}^*(i)\big)$. We can also recover the matching probability $\ell^*(i) = L(n^*(i), n^*(i))$ from equation (3). Lastly, we can recover $\Phi^*(i)$ from $\bar{U}^*(i) = \ell^*(i)(u(i,i) - \Phi^*(i))$. Because $c > 0$, we must have positive matching probabilities and positive fees, $m(\ell(i), \ell(i)) > 0$ and $\Phi^*(i) > 0$.

If there is a competitive search equilibrium with PAM, this determines the matching probability and agent value when $i = 0$, since there are no incentive constraints in the $(0,0)$ market. For other markets, the constraints may bind.

Constrained Markets. Now suppose at least one of the incentive constraints in problem (28) is binding, i.e. the unconstrained solution violates an incentive constraint. Since the objective function is decreasing in $\bar{U}(i)$, if $\bar{U}(i) > \bar{U}^*(i)$ the objective evaluates to less than c for all values of λ. This is inconsistent with an active (i,i) market in a competitive search equilibrium. We thus conclude that $\bar{U}(i) \leq \bar{U}^*(i)$ in any competitive search equilibrium.

If $\bar{U}(i) = \bar{U}^*(i)$, the free entry condition $c = 2\big(M^*(n)u(i,i) - n\bar{U}(i)\big)$ pins down $n = n^*(i)$, which by assumption violates an incentive constraint. So we must have $\bar{U}(i) < \bar{U}^*(i)$. In this case, there are two values of n that satisfy the free entry condition

$$c = 2\big(M^*(n)u(i,i) - n\bar{U}(i)\big).$$

This is because (i) the right hand side is zero when $n = 0$; (ii) the right hand side is negative for a sufficiently large value of n since the slope of $M^*(n)$ is less than 1 and decreasing for all positive n; (iii) the right hand side is strictly increasing and strictly larger than c when $n = n^*(i)$ since $\bar{U}(i) < \bar{U}^*(i)$; and (iv) the right hand side is a strictly concave function of n. This implies that there are two solutions to this equation, one in the interval $[0, n^*(i))$ and the other in $(n^*(i), \infty)$.

Now since $M^*(n)$ is concave with $M^*(0) = 0$, agents' matching rate $M^*(n)/n$ is a decreasing function of n. Thus we can equivalently state that for $\bar{U}(i) < \bar{U}^*(i)$, there are two solutions $\lambda \in [0, \bar{\lambda}]$ to the equation

$$c = 2m(\lambda, \lambda)\left(u(i,i) - \frac{\bar{U}(i)}{\lambda}\right).$$

[5] It is straightforward to prove from the definitions of M^*, m, n, and λ that $\lim_{n\to\infty} M^*(n) - nM^{*\prime}(n) = m(0,0)$.

One of the solutions is in the interval $(\ell^*(i), \bar{\lambda}]$, while the other lies in the interval $(0, \ell^*(i))$. Which solution is relevant depends on whether we have IWTP or DWTP, and so we treat those cases separately.

IWTP. First, suppose there is IWTP. The incentive constraint in Problem (28), which only holds for $j < i$, implies a lower bound on λ:

$$\lambda \geq \sup_{j<i} \frac{\bar{U}(i) - \bar{U}(j)}{u(i,i) - u(j,i)}.$$

Now if the solution to problem (28) has $\ell(i)$ equal to the smaller solution, i.e. $\ell(i) \in (0, \ell^*(i))$, then $\ell^*(i)$ is feasible and, since $\bar{U}(i) < \bar{U}^*(i)$, attains a value in problem (28) that exceeds c, a contradiction. So it must be that the solution to problem (28) has $\ell(i)$ equal to the larger solution, $\ell(i) \in (\ell^*(i), \bar{\lambda}]$, whenever $\bar{U}(i) < \bar{U}^*(i)$ in a competitive search equilibrium in the IWTP case.

Next, we replace the global incentive constraints in Problem (28) with local ones:

$$\begin{aligned} c = &\max_{\lambda \in (\ell^*(i), \bar{\lambda}]} 2m(\lambda, \lambda) \left(u(i,i) - \frac{\bar{U}(i)}{\lambda} \right) \\ &\text{s.t. } \bar{U}'(i) = \lambda u_1(i,i). \end{aligned}$$

There is nothing to optimize here. Instead, these equations define the path of $\bar{U}(i)$ and $\ell(i)$:

$$c = 2m(\ell(i), \ell(i)) \left(u(i,i) - \frac{\bar{U}(i)}{\ell(i)} \right), \quad \bar{U}'(i) = \ell(i) u_1(i,i), \quad \text{and } \ell(i) \in (\ell^*(i), \bar{\lambda}]. \tag{30}$$

Since $\bar{U}(i)$ is continuous (Lemma 2), the first equation and the restriction that $\ell(i) \in (\ell^*(i), \bar{\lambda}]$ ensures $\ell(i)$ is continuous. Therefore from the second equation $\bar{U}'(i)$ is continuous.

Now solve the first equation in (30) for $\bar{U}(i)$, differentiate with respect to i, and eliminate $\bar{U}'(i)$ using the second equation. This gives us equation (13) in the text and establishes that it must hold at all i, since $\ell(i)$ is continuous and $\bar{U}(i)$ is continuously differentiable.

We next prove monotonicity. We know $\bar{U}(i)$ is increasing from the local incentive constraint $\bar{U}'(i) = \ell(i) u_1(i,i)$. Because $2u(0,0)m(0,0) > c$, $\bar{U}(0) > 0$, and monotonicity of $\bar{U}(i)$ implies that $\bar{U}(i) > 0$ for $i > 0$. To prove $\ell(i)$ is increasing, we first establish that for fixed i, h in equation (12) is single-peaked in $\lambda \in [0, \bar{\lambda}]$, achieving maximum value $\bar{U}^*(i)$ at $\ell(i) = \ell^*(i)$, the unconstrained case. To prove this, note that we have already established that the objective function in problem (28) is single-peaked: for all $\bar{U}(i) < \bar{U}^*(i)$, there exists

thresholds λ_1 and λ_2 with $\lambda_1 < \ell^*(i) \leq \lambda_2$ such that

$$c < 2m(\lambda, \lambda)\left(u(i,i) - \frac{\bar{U}(i)}{\lambda}\right) \Leftrightarrow \lambda \in (\lambda_1, \lambda_2).$$

Flipping this inequality around, we get that

$$\bar{U}(i) < \lambda\left(u(i,i) - \frac{c}{2m(\lambda,\lambda)}\right) = h(\lambda, i) \Leftrightarrow \lambda \in (\lambda_1, \lambda_2).$$

This proves that $h(\lambda, i)$ is single-peaked. Moreover, setting $\bar{U}(i) = \bar{U}^*(i)$ and using a similar logic implies that $\max_{\lambda \in [0,\bar{\lambda}]} h(\lambda, i) = h(\ell^*(i), i) = \bar{U}^*(i)$.

Since $\ell(i) \geq \ell^*(i)$ in the IWTP case, $h_1(\ell(i), i) < 0$. Then the differential equation (13) implies $\ell'(i) > 0$, proving ℓ is increasing. Finally, to show that $\Phi(i)$ is increasing, we totally differentiate the constraint $\bar{U}(i) = \ell(i)(u(i,i) - \Phi(i))$:

$$\bar{U}'(i) = \ell(i)(u_1(i,i) + u_2(i,i) - \Phi'(i)) + \ell'(i)(u(i,i) - \Phi(i)).$$

The local incentive constraint requires that $\bar{U}'(i) = \ell(i)u_1(i,i)$, so we can write:

$$\Phi'(i) = u_2(i,i) + \frac{\ell'(i)}{\ell(i)^2}\bar{U}(i)$$

The right-hand side is positive because of common ranking, $\ell'(i) \geq 0$, and $\bar{U}(i) > 0$. Thus $\Phi(i)$ must be increasing as well.

DWTP. With DWTP, the logic is reversed. The constraints in Problem (28) imply an upper bound the meeting rate:

$$\lambda \leq \sup_{j<i} \frac{\bar{U}(j) - \bar{U}(i)}{u(j,i) - u(i,i)}.$$

We can use that to rule out the upper solution, which is dominated by $\ell^*(i)$. We thus conclude that the solution to problem (28) has $\ell(i)$ equal to the smaller solution, $\ell(i) \in [0, \ell^*(i))$, whenever $\bar{U}(i) < \bar{U}^*(i)$ in a competitive search equilibrium in the DWTP case.

We again use the local incentive constraints to get a version of equation (30):

$$c = 2m(\ell(i), \ell(i))\left(u(i,i) - \frac{\bar{U}(i)}{\ell(i)}\right), \quad \bar{U}'(i) = \ell(i)u_1(i,i), \quad \text{and } \ell(i) \in [0, \ell^*(i)). \tag{31}$$

Again, we use continuity of $\bar{U}(i)$ to prove continuity of $\ell(i)$ and then use that to prove

continuity of $\bar{U}'(i)$. Finally, the same algebraic manipulations give us equation (13) in the text and establishes that it must hold at all i.

The monotonicity proof is also reversed. We know $\bar{U}(i)$ is decreasing from the local incentive constraint $\bar{U}'(i) = \ell(i)u_1(i,i)$. To prove $\ell(i)$ is decreasing, we use the same property of h in equation (12). Now private information leads to $\ell(i) < \ell(i^*)$ in the DWTP case, implying $h_1(\ell(i), i) < 0$. Then the differential equation (13) implies $\ell'(i) < 0$, proving ℓ is decreasing.

Finally, since $\ell(i)$ is decreasing, $m(\ell(i), \ell(i))$ is increasing and so the free-entry condition $c = 2m(\ell(i), \ell(i))\Phi(i)$ implies $\Phi(i)$ is decreasing. ■

C Proofs for Section 6

Proof of Lemma 7. We start by defining the objective function of the platforms in terms of (n^a, n^b): $W(n^a, n^b) \equiv M(n^a, n^b)\left(u^a(0,0) + u^b(0,0)\right) - \bar{U}^a(0)n^a - \bar{U}^b(0)n^b$. As M is strictly concave, there is a unique maximizer to $W(n^a, n^b)$ given any $\bar{U}^a(0)$ and $\bar{U}^b(0)$. Correspondingly, there is a unique combination $(\lambda^a(0), \lambda^b(0))$. In a PAM CSE, the market clearing condition requires that $\lim_{i\to 0^+}\sigma(i) = 0$. Otherwise, there is $j \in \mathbb{I}^b$ whose market does not clear.

Now we prove the continuity of $\lambda^s(i)$ by contradition. Suppose, to the contrary of the lemma, $\lambda^s(i)$ is not continuous at 0. Let $l^s = \lim_{i\to 0^+}\lambda^s(i)$. For $i > 0$,

$$c = \hat{V}(i, \sigma(i)) = m(\ell^a(i), \ell^b(i))\left(u(i, \sigma(i)) + u^b(\sigma(i), i) - \frac{\bar{U}^a(i)}{\ell^a(i)} - \frac{\bar{U}^b(\sigma(i))}{\ell^b(i)}\right)$$

Taking $i \to_+ 0$:

$$c = \lim_{i\to_+ 0}\hat{V}(i, \sigma(i)) = m(l_0^a, l_0^b)\left(u^a(0,0) + u^b(0,0) - \frac{\bar{U}^a(0)}{l_0^a} - \frac{\bar{U}^b(0)}{l_0^b}\right).$$

If $(l^a, l^b) \neq (\lambda^a(0), \lambda^b(0))$, we have found another maximiser for the platform payoff at $(0,0)$. A contradiction. ■

Proof of Proposition 4. It is convenient to use the following notations, where we write the payoff of platforms as a function of $(\lambda^s, k^s)_{s=a,b}$

$$\tilde{V}(\lambda^a, \lambda^b, k^a, k^b) \equiv m(\lambda^a, \lambda^b)\sum_{s=a,b}\left(u^s(k^s, k^{\bar{s}}) - \frac{\bar{U}^s(k^s)}{\lambda^s}\right),$$

and

$$\hat{\lambda}^s(k^s, k^{\bar{s}}) = \frac{\bar{U}^{s\prime}(k^s)}{u_1^s(k^s, k^{\bar{s}})}, \text{ for } s = a, b. \tag{32}$$

With this notation:

$$\hat{V}(i,j) = \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j).$$

Among the odes in the lemma, equation (16) comes directly from the local incentive constraints in lemma 5. We start by deriving equation (15).

For a fixed $i \in \mathbb{I}^a$, $\sigma(i)$ has to be the local maximum of $\hat{V}(i,j)$. Otherwise, the platforms can increase their payoffs by choosing $j' \neq \sigma(i)$, violating the optimality condition. Thus, the first-order condition holds $\hat{V}_2(i, \sigma(i)) = 0$. The same argument implies $\hat{V}_1(i, \sigma(i)) = 0$.

Given j, the impact of changing i on $\hat{V}(i,j)$ can be written in terms of three effects: the change in matching probabilities on two sides and the change in the utility from matches:

$$\begin{aligned}
&\frac{\partial \log \hat{V}(i,j)}{\partial i} \\
=&\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^a(i,j)} \hat{\lambda}_1^a(i,j) + \frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial \hat{\lambda}^b(i,j)} \hat{\lambda}_2^b(j,i) \qquad (33) \\
+&\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j), \hat{\lambda}^b(j,i), i, j)}{\partial i}. \qquad (34)
\end{aligned}$$

We now unpack the three effects in turns. Starting from the effect through matching probability λ^a. From the definition of $\tilde{V}$:

$$\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial \lambda^a} = \left(\frac{\epsilon_1(\lambda^a, \lambda^b)}{m(\lambda^a, \lambda^b)} \tilde{V}(\lambda^a, \lambda^b, k^a, k^b) + \frac{\bar{U}^a(k^a)}{\lambda^a} \right) \frac{m(\lambda^a, \lambda^b)}{\lambda^a \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}.$$

In the PAM CSE, $\tilde{V}(\ell^a(i), \ell^b(i), i, \sigma(i)) = c$, and so we can write

$$\frac{\partial \log \tilde{V}(\ell^a(i), \ell^b(i), i, \sigma(i))}{\partial \ell^a(i)} = \nu^a(i) \frac{m(\ell^a(i), \ell^b(i))}{c\ell^a(i)},$$

where we used the definition of $\nu^a(i)$ as in the lemma. The same logic establishes that $\frac{\partial \log \tilde{V}(\lambda^a, \lambda^b, k^a, k^b)}{\partial \lambda^b} = \nu^b(i) \frac{m(\ell^a(i), \ell^b(i))}{c\ell^b(i)}$. To derive $\hat{\lambda}_2^s(i,j)$, we totally differentiate the identity $\ell^a(i) = \hat{\lambda}^a(i, \sigma(i))$ and $\ell^b(i) = \hat{\lambda}^b(\sigma(i), i)$ to get:

$$\ell^{a\prime}(i) = \hat{\lambda}_1^a(i, \sigma(i)) + \hat{\lambda}_2^a(i, \sigma(i))\sigma'(i)$$

$$\ell^{b\prime}(i) = \hat{\lambda}_1^b(\sigma(i), i)\sigma'(i) + \hat{\lambda}_2^b(\sigma(i), i).$$

Differentiating equation (32), we have:

$$\hat{\lambda}_2^a(i,\sigma(i)) = -\ell^a(i)\frac{u_{1,2}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))},\ \hat{\lambda}_2^b(\sigma(i),i) = -\ell^b(i)\frac{u_{1,2}^b(\sigma(i),i)}{u_1^b(\sigma(i),i)}.$$

So

$$\hat{\lambda}_1^a(i,\sigma(i)) = \ell^{a\prime}(i) + \sigma'(i)\ell^a(i)\frac{u_{1,2}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))},$$

$$\hat{\lambda}_1^b(\sigma(i),i) = \frac{1}{\sigma'(i)}\left(\ell^{b\prime}(i) + \ell^b(i)\frac{u_{1,2}^b(\sigma(i),i)}{u_1^b(\sigma(i),i)}\right).$$

Putting this together, we have

$$\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j),\hat{\lambda}^b(j,i),i,j)}{\partial \hat{\lambda}^a(i,j)}\hat{\lambda}_1^a(i,j) = \nu^a(i)\frac{m(\ell^a(i),\ell^b(i))}{c}\left(\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \sigma'(i)\frac{u_{1,2}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))}\right),$$
$$\frac{\partial \log \tilde{V}(\hat{\lambda}^a(i,j),\hat{\lambda}^b(j,i),i,j)}{\partial \hat{\lambda}^b(i,j)}\hat{\lambda}_2^b(j,i) = \nu^b(i)\frac{m(\ell^a(i),\ell^b(i))}{c}\left(-\frac{u_{1,2}^b(\sigma(i),i)}{u_1^b(\sigma(i),i)}\right).$$

Finally, we directly compute the effect of changing matching partners on utility:

$$\frac{\partial \log \tilde{V}(\lambda^a,\lambda^b,k^a,k^b)}{\partial i} = \frac{m(\lambda^a,\lambda^b)}{\tilde{V}(\lambda^a,\lambda^b,k^a,k^b)}\left(u_1^a(k^a,k^b) + u_2^b(k^b,k^a) - \frac{\bar{U}^{a\prime}(k^a)}{\lambda^a}\right).$$

The incentive constraint (32) and the fact that in the PAM CSE, $\tilde{V}(\ell^a(i),\ell^b(i),i,\sigma(i)) = c$ means that we can write this as

$$\frac{\partial \log \tilde{V}(\lambda^a,\lambda^b,k^a,k^b)}{\partial i} = \frac{m(\lambda^a,\lambda^b)}{c}u_2^b(k^b,k^a).$$

Now we have all terms needed to calculate $\frac{\partial \hat{V}(i,j)}{\partial i}$. Setting $\frac{\partial \hat{V}(i,j)}{\partial i} = 0$ and multiplying both sides by $\frac{c}{m(\ell^a(i),\ell^b(i)}$, we get the following equation:

$$0 = \nu^a(i)\left(\frac{\ell^{a\prime}(i)}{\ell^a(i)} + \sigma'(i)\frac{u_{1,2}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))}\right) - \nu^b(i)\frac{u_{1,2}^b(\sigma(i),i)}{u_1^b(\sigma(i),i)} + u_2^b(\sigma(i),i).$$

Similarly, for the b side:

$$0 = -\nu^a(i)\frac{u_{1,2}^a(i,\sigma(i))}{u_1^a(i,\sigma(i))} + \nu^b(i)\frac{1}{\sigma'(i)}\left(\frac{\ell^{b\prime}(i)}{\ell^b(i)} + \frac{u_{1,2}^b(\sigma(i),i)}{u_1^b(\sigma(i),i)}\right) + u_2^a(i,\sigma(i)).$$

Using the definition of $\iota(i)$, we reach the formula as in the lemma.

Now we move on to equation (17). In our model, matches are one-to-one. Thus, $\forall i \in \mathbb{I}^a$, the accumulated numbers of matches on the a side must equal the accumulated numbers of matches on the b side:

$$\int_0^i \lambda^a(i')\bar{g}^a(i')di' = \int_0^{\sigma(i)} \lambda^b(\sigma^{-1}(j'))\bar{g}^b(j')dj'.$$

The equation above has to hold for any i, we differentiate both sides with respect to i:

$$\ell^a(i)\bar{g}^a(i) = \ell^b(i)\bar{g}^b(\sigma(i))\sigma'(i).$$

With the assumption that $\bar{g}^b(j) > 0$ for all $j \in \mathbb{I}^b$, we derive the ode for $\sigma(i)$:

$$\sigma'(i) = \frac{\ell^a(i)}{\ell^b(i)} \frac{I^a \bar{g}^a(i)}{I^b \bar{g}^b(\sigma(i))}.$$

For the market clearing condition to hold for all types, we further require that $\sigma(0) = 0$ and $\sigma(1) = 1$. This conclude the proof. ■