Government Intervention in the Financial Market

Jiang Wang*

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Abstract

Government intervention through its own trading in the financial market fundamentally changes the market's structure, function, behavior and outcome. We develop a parsimonious general equilibrium framework to study the impact of government trading in the financial market on market outcome and investor welfare. In the presence of market incompleteness and information asymmetry, the market's equilibrium allocation is in general not Pareto optimal, leaving room for government intervention. Any government trading when it is public will have no net effect on the equilibrium outcome as investors adjust their trades to fully offset its influence. However, when government trading is private, it can improve welfare even without any additional information. With additional information, the welfare impact of government intervention is sensitive to its design as well as details of the market's underlying structure. In addition, we show that performance measures based on market prices, such as payoff or Sharpe ratio, can be misleading in assessing government intervention policies as they also affect equilibrium prices and allocations.

JEL Codes: G11, G12, G18, D53, D61, H11

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1 Introduction

In addition to its conventional regulatory role, government also intervenes directly in the financial market as a dominant participant.[1] The Federal Reserve's massive holdings of mortgage-backed securities, the European Central Bank's large positions in corporate bonds, and the Bank of Japan's growing stock ETF purchase program are prominent examples of such interventions, as shown in Figure 1. Government's direct intervention through its own trading fundamentally changes the market's structure, function, behavior and outcome. Yet, in our conventional analysis of the financial market and asset pricing, the role of the government as a major market player has mostly been left out.

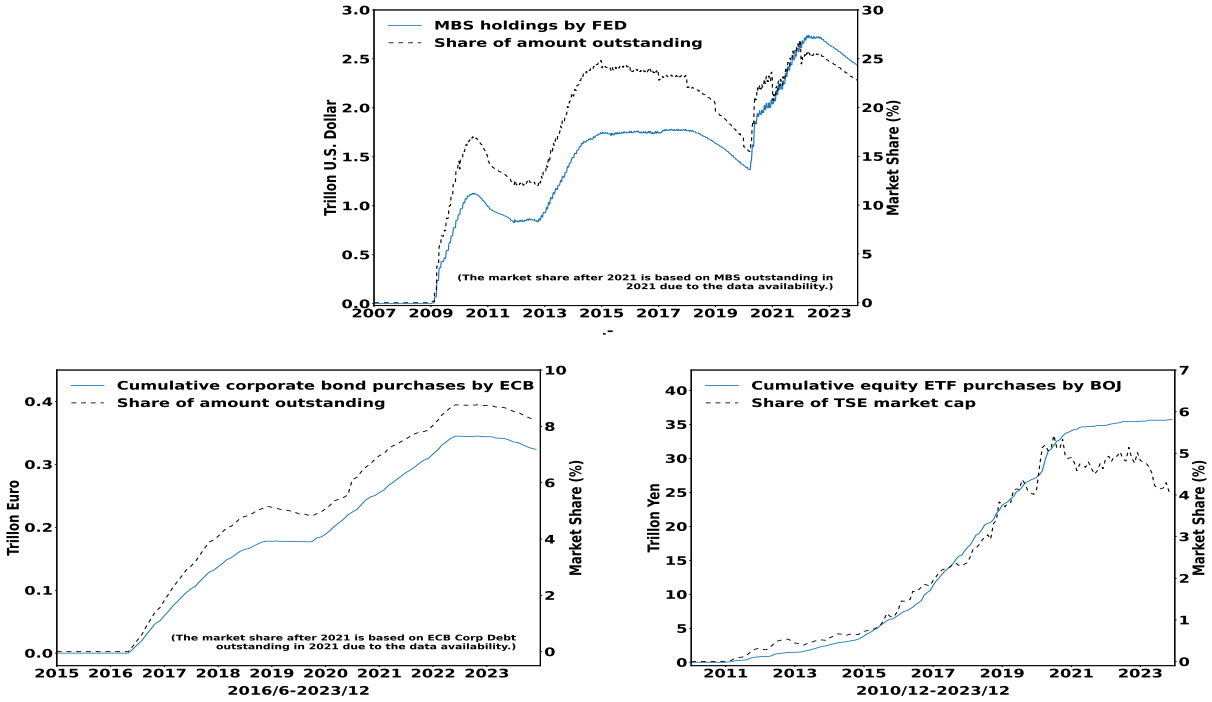


Figure 1: Holdings of non-governmental securities by the Federal Reserve, the European Central Bank, and the Bank of Japan. By 2023 year end, the Fed's MBS holdings exceeded 2.4 trillion dollars, over 22% of the overall market, ECB's corporate bond holdings were over 0.3 trillion euros, over 8% of the market, and BOJ's stock holdings reached 35.7 trillion yen, more than 4% of the market.

In this paper, we develop a parsimonious general equilibrium framework which includes the government as a unique monopolistic participant in the market. We then examine the

[1]The market here is limited to non-governmental or private securities such as stocks, corporate bonds and asset-backed securities. For governmental securities like government bonds and currencies, the government is an essential participant, hence will not be the focus of this paper. We also treat its different branches, including the central bank, as integral parts of the government.

role the government can play to improve investor welfare. We show that even without any additional information, the government can improve investors' welfare by trading in the market. When the government has additional information on the underlying state of the economy, its impact on market equilibrium and investor welfare are quite sensitive to the structural details of the economy as well as the design of the intervention policy. We further demonstrate that conventional measures to assess the performance of government interventions using market prices are incomplete and sometimes misleading.

We consider a financial market in which investors face heterogeneous income shocks. They can trade in the market to share their income risks. In addition, investors also have private information on their income risk as well as on the payoff of traded assets, which gives rise to their desire to speculate in the market. We assume that the government can directly trade in the market. Its gain/loss from trading is fully passed on to the investors and its objective is to maximize investor welfare. Hence, the economy is fully self-contained. Taking the government's trading policy as given, the investors choose their optimal trading strategies, and the market equilibrium is achieved through market clearing. Given that the market is incomplete with respect to the investors' idiosyncratic income risks, the equilibrium allocation is in general not Pareto optimal, which leaves room for government intervention. The government then chooses its intervention policy to improve investor welfare.

Government's intervention policy depends on the information it has. We consider three situations. The first situation is when the government only has public information such as asset prices. The second situation is when the government has accurate information about the underlying aggregate state variables such as aggregate income risk exposure and asset payoff. The third situation is when the government has noisy information about the underlying aggregate state variables.

When the government only has public information, it can adopt a trading policy which depends on that information, such as market prices of traded assets. We show that in a rational expectations equilibrium, the government's trades in this case will be known to the market. As a result, investors can adjust their trades to fully offset the impact of the government's trades on their net positions. Consequently, the government's trading, despite its monopolistic position, will have no impact on the market equilibrium and investor welfare. We believe this "no-impact theorem" holds in general as long as the government's trades are known to the investors, who

also bear the resulting gain/loss. Such a result is reminiscent of the notion of Ricardo equivalence in the current context.

This, however, does not mean that the government cannot do better, given that the market equilibrium does not yield optimal risk sharing. In fact, we show that the government can improve risk sharing by acting as a noise trader. In particular, it can trade in the market on pure noise, unobservable to the investors. By doing so, the government is providing extra returns to investors in the market and effectively reducing the risk premium they pay when trading to share their idiosyncratic risks. Such a price impact, within certain range, can improve investor welfare as it reduces the price cost for their risk sharing.

When the government has accurate information about the underlying state of the economy, its intervention policy can depend on these state variables. We show that for small interventions either in the direction of taking on aggregate income risk or revealing more information about asset payoff, the investors' welfare is always improved. The intuition behind these results is as follows. When the government is taking on the aggregate risk, the investors will bear less aggregate risk. In equilibrium, this will reduce the risk premium on the traded asset investors use to share risk and hence improves risk sharing and welfare. When the government is trading positively on its signal about future asset payoff, its trade will further reveal this information through the equilibrium price. This then reduces the risk concerning asset payoff and the information asymmetry among investors, and improves their risk sharing and welfare.

However, we find that when the government's intervention in the market becomes substantial, its overall impact on the market equilibrium and investor welfare becomes quite sensitive to the underlying structure of the economy and the design of the intervention policy. In particular, for the same intervention policy, the impact can be quite different for different values of the underlying parameters. Also, given the structure of the economy, small changes in the magnitude of the intervention can lead to different impacts on market outcome and investor welfare, both qualitatively and quantitatively.

When the government has noisy information about the underlying state of the economy, the situation is like a combination of the previous two cases. When it intervenes, its trades bring both information and noise into the market. The overall result then depends on how the two factors interact and which one dominates. We leave the details of this case to later discussions.

Given the government's direct intervention/trading in the market, during crisis as well as

normal times, an important question is how to measure the performance of these interventions. Some heuristic measures have been used, such as realized payoffs, without or with risk adjustments. These measures rely on market prices of the assets involved. But, as our analysis suggests, using market prices can be misleading when they are also influenced by the interventions. Our framework, however, provides a proper way to assess the intervention policies by looking at their welfare implications. Moreover, we can also use it to examine the other heuristic measures, such as expected payoff or Sharpe ratio.

In particular, we consider two intervention policies as examples. One policy is "buy low and sell high." This corresponds to government interventions during financial crisis. The government may directly or indirectly buy assets when their prices drop drastically and sell them back to the market when their prices recover. This strategy only relies on market prices and requires no additional information. It yields a positive expected payoff and a positive Sharpe ratio. But as discussed above, such a strategy has no impact on market equilibrium and investor welfare. This is simply because such a strategy can also be used by the investors. In fact, it will be, but in the opposite direction to offset the government trades. Another policy is "trade on noise," as discussed earlier. Given that the government's trades will move the prices against itself, such a strategy will yield a negative expected payoff and a negative Sharpe ratio. However, as we show in the paper, such a strategy, within a proper range, can actually improve investor welfare. These results show that extra caution is needed when using market prices to assess government intervention policies.

Related literature

As noted earlier, this paper focuses on government intervention through its trading in the financial securities market, a narrow notion of the financial market we use in this paper. Thus, we will limit our discussion on the literature that is closely related in this specific sense. In a more general sense, government intervention may also include other forms ranging from regulation and supervision/guidance, to guarantees/bailouts and taxes/subsidies, and the financial market can also include other parts of the financial system, in particular various types of financial institutions and intermediaries. There is an extensive literature on different aspects of these types of interventions, which are beyond the scope of this paper.

It is well recognized that the financial market or the securities market, when it is incomplete

and contains information asymmetry, need not lead to efficient allocations (see, for example, **?**). **?** consider potential government interventions through taxes and transfers to improve welfare. They show that full re-distribution with taxes and transfers can improve allocation efficiency. Their analysis, however, does not examine the allocation and welfare impact of government intervention through trading in the financial market in a general equilibrium setting.

In a study of Chinese government's active intervention in the stock market, **?** consider a noisy rational expectations model in the spirit of **?** in which the market is influenced by retail investors who act as noise traders. They show that government trading to partially offset the retail investors' impact can influence market outcomes such as price volatility and informational efficiency. Using a model similar to **?**, **?** consider the same problem in which the government trades strategically to stabilize price, using private information. They examine the intervention's impact on price informativeness and market liquidity.

Central banks also trade directly and substantially in the government bond market and currency market. The goals of these interventions are multi-dimensional, ranging from macro-economic targets to financial objectives, complex and less quantifiable. Building a coherent analytical framework to study these interventions is particularly challenging. Several papers have examined the specific impact of these interventions from a market-microstructure perspective using a Kyle-style setting. For example, assuming particular policy targets, **?** and **?** analyze how government intervention in the currency market influences its informational efficiency and liquidity, and **?** consider Federal Reserve's trading in the U.S. Treasuries market.

Our paper considers the government's market intervention in general and focuses on its impact on investor welfare rather than particular market outcomes such as price volatility, market liquidity or other metrics. Using a full equilibrium framework in which investors trade rationally for risk-sharing and speculation and the government's objective is the same as the investors' with a closed budget, we can examine the welfare implications of its intervention policies in a coherent and self-contained manner. Hopefully, it will provide a more complete basis to consolidate and extend the existing analysis.

There is an extensive literature on government interventions in the financial system especially during crises, focusing primarily on assistance or bailout of financial institutions.[2] The very existence of financial institutions and their potential need for government assistance rely on the

[2]See, for example, **?**, **?**, **?**, **?**, and **?**.

presence of frictions in addition to those considered in this paper. From this perspective, our framework here, which is at a more basic level, can provide a theoretical basis as well as useful insights for the analysis of more complex situations.

Our model builds on the classic work of **?**, with key extensions and modifications. In particular, we explicitly model investors' risk-sharing trading in the market instead of introducing noise traders, whose behavior is exogenously specified and welfare is not fully accounted. Also, the bond market is included in the market equilibrium, without assuming an exogenous interest rate.[3] These extensions make our model a fully general equilibrium one and allow a proper welfare analysis, which is essential for our purpose.

The remainder of the paper is organized as follows. In Section 2, we develop the basic model for our analysis. Section 3 presents the solution to the model, including the market equilibrium and the government's optimal intervention policy. Section 4 illustrates the basic properties of model when there is no government intervention. In Section 5, we show the impact of government intervention on market equilibrium and investor welfare under different information assumptions. Section 6 discusses different performance measures for government's intervention policies. Section 7 concludes. The appendix contains the proofs, technical details and additional results.

2 Model

The economy has two dates, 0 and 1, denoted by $t = 0, 1$.

2.1 Financial Market

There is a financial market consisting of two traded assets, a risk-free bond and a risky stock. The bond yields a sure payoff of 1 at date 1. The stock pays a dividend D at date 1, given by:

$$D = F + u, \tag{2.1}$$

where F and u are two independent random variables. Using the bond as the numeraire, we denote the price of the stock at date 0 by P.

[3]See **?** and **?** for variants of the static Grossman-Stiglitz noisy rational expectations model and **?**, among others, for its intertemporal extensions. **?**, **?**, and **?** introduce investors' risk-sharing trading to fully endogenize the "noise" in the market.

2.2 Investors

There is a continuum of investors, denote by $i \in [0, 1]$.

Endowment. All investors are endowed with 0 unit of the bond and 0 share of the stock. Thus, they have zero initial financial wealth, which is the market value of all their endowed asset holdings. In addition, each investor i receives a non-traded income N_i at date 1, given by:

$$N_i = Y_i\, u = (Y + y_i)\, u, \quad i \in [0, 1], \tag{2.2}$$

where Y and y_i are two independent random variables. Thus, u gives the income risk, while Y and y_i capture investors' common and idiosyncratic exposures to the income risk, respectively. Here, income risk is assumed to be the same as the risk of stock payoff, allowing the investors to use the stock to share their idiosyncratic income risks.[4]

Information. At date 0, investor i observes Y_i, which is private information. In addition, she receives a private signal S_i about the stock's future payoff F:

$$S_i = F + \varepsilon_i, \quad i \in [0, 1], \tag{2.3}$$

where ε_i represents the noise in the signal. Let I_i denote the information set of investor i at date 0. We then have $I_i = \{P, Y_i, S_i\}$.

Trading. Investors trade competitively in the market at date 0. Let θ_i denote investor i's stock holding at date 0. In general, it depends on her information at date 0. In absence of other income, the budget constraint for investor i is given by:

$$W_i = \theta_i(F + u - P) + Y_i\, u, \quad i \in [0, 1]. \tag{2.4}$$

Preferences. The investors' preferences are given by an expected utility function over their terminal wealth. For simplicity, we assume that they all have constant absolute risk aversion with risk aversion parameter $\alpha > 0$. In particular, investor i has the following utility function:

$$u_i(W_i) = -e^{-\alpha W_i}, \quad i \in [0, 1], \tag{2.5}$$

where W_i is her terminal wealth at date 1.

It is important to note that there is perfect symmetry between the investors as defined above.

[4]All we need is that the income risk is correlated with the stock payoff risk. The correlation is set to 1 in the paper merely for simplicity. The only exception is when the correlation is zero, in which case the financial market becomes useless for investors' risk sharing and yields an autarky equilibrium.

This helps to simplify our analysis without losing the essence of the model. The heterogeneity among investors in both endowment and information gives rise to their need for trading. The difference in endowment leads to the trading need for risk sharing and the difference in private information leads to the trading need for speculation.

2.3 Government

The government as a central planner can also participate in the market. We assume that the government has no additional endowment or income of its own.

Let I_G denote the information set of the government at date 0. We assume that other than P, the stock price, the government may also receive signals S_{GF} and S_{GY} about F and Y, the two aggregate state variables, given by:

$$S_{GF} = \beta_{GF} F + \varepsilon_{GF}, \tag{2.6a}$$

$$S_{GY} = \beta_{GY} Y + \varepsilon_{GY}, \tag{2.6b}$$

where ε_{GF} and ε_{GY} represent the noises in the signals.

The government can trade in the stock market. Let θ_G denote the government's stock holding at date 0 on per capita basis. It can depend on the information the government has. For tractability, we limit the government's trading policy to be linear in its signals:

$$\theta_G = h_{GP} P + h_{GF} S_{GF} + h_{GY} S_{GY} = h_G \begin{bmatrix} P & S_{GF} & S_{GY} \end{bmatrix}^\top, \quad h_G \equiv \begin{bmatrix} h_{GP} & h_{GF} & h_{GY} \end{bmatrix}, \tag{2.7}$$

where $\begin{bmatrix} \cdot & \cdots & \cdot \end{bmatrix}$ denotes a row vector/matrix and $[\cdot]^\top$ denotes its transpose. Hence, the government's trading policy is given by the three coefficients, h_{GP}, h_{GF} and h_{GY}, or h_G. For brevity, we will use θ_G to denote both the government's trading policy and its actual stock position in the market without further clarification when its meaning is clear in the context. In addition, the former is given by h_G in the case of linear policy, and is independent of the state of the economy. From (2.6), we can further rewrite the government's trading policy as:

$$\theta_G = h_{GP} P + h_{GF} \beta_{GF} F + h_{GY} \beta_{GY} Y + \varepsilon_G, \quad \varepsilon_G \equiv h_{GF} \varepsilon_{GF} + h_{GY} \varepsilon_{GY}, \tag{2.8}$$

where ε_G denotes the noise introduced into the market by the government's trading. In general, the government's trading is not directly observable to the investors, although they can form their expectations.

Let W_G denote the the government's wealth at date 1, on per capital basis. Its budget

constraint is then given by:

$$W_G = \theta_G \, (F + u - P). \tag{2.9}$$

We assume that the government's wealth at date 1 will be fully transferred to the investors. Negative transfers correspond to taxes. Given the symmetry between all the investors, we assume that government transfers will be equal among them. Thus, in the presence of government intervention, investors' terminal wealth will be:

$$W_i = \theta_i(F + u - P) + Y_i \, u + W_G = (\theta_i + \theta_G)(F + u - P) + Y_i \, u, \quad i \in [0, 1]. \tag{2.10}$$

The government's objective is assumed to be maximizing the investors' expected utility ex ante, which is:

$$\int_i \mathrm{E}\left[u_i(W_i)\right] = \int_i \mathrm{E}\left[-e^{-\alpha W_i}\right]. \tag{2.11}$$

2.4 Distributional Assumptions and Parameter Constraints

For tractability, we assume that all shocks to the economy, F, u, Y, y_i, ε_i, for all $i \in [0, 1]$, and ε_{FG}, ε_{GY} are independent and normally distributed with zero mean and variances σ_F^2, σ_u^2, σ_Y^2, σ_y^2, σ_ε^2, σ_{GF}^2, and σ_{GY}^2, respectively. Unless otherwise stated, all variances are assumed to be strictly positive to avoid degeneracies. For convenience, we also define $\sigma_D^2 \equiv \sigma_F^2 + \sigma_u^2$ and $\sigma_S^2 \equiv \sigma_F^2 + \sigma_\varepsilon^2$.

Under normality, the investors' non-traded income has unbounded values, which can yield extreme values for the utility function. In order to guarantee the expected utility to be well defined, we impose the following parameter condition:

$$\alpha^2 \sigma_u^2 \left(\sigma_Y^2 + \sigma_y^2\right) < 1. \tag{2.12}$$

In fact, under (2.12), investors' expected utility without trading is given by:

$$\mathrm{E}\left[u_i(W_i)\right] = -\frac{1}{\sqrt{1 - \alpha^2 \sigma_u^2 (\sigma_Y^2 + \sigma_y^2)}} \equiv V_{00}. \tag{2.13}$$

Thus, V_{00} is the level of their welfare under autarky. It is worth noting that V_{00} is independent of σ_F^2, the risk of stock payoff F. This is because that investors' net holding of the stock is zero. However, as we will see later, when investors can trade in the market, their welfare can become dependent on σ_F^2. This is because the risk in stock payoff brings additional risk for investors' risk sharing and speculation trades.

2.5 Common Knowledge and Rational Expectations

The structure of the economy is assumed to be common knowledge and both the investors and the government have rational expectations.

2.6 Market Equilibrium and Optimal Intervention

The notion of market equilibrium we use is the standard one of rational expectations defined as follows: (1) taking the government's trading policy and the stock price as given, investors choose their trading policy to maximize their expected utility; (2) the stock market clears, given the government and the investors' trading policies, which determines the equilibrium price and investors' stock holdings and expected utility. Finally, we consider the government's choice of trading policy to improve investor welfare.

Given the government's trading policy θ_G, investor i chooses θ_i to maximize her expected utility:

$$J(P, I_i, \theta_G) \equiv \max_{\theta_i} \ \mathrm{E}\left[u_i(W_i)|I_i, \theta_G\right] = \mathrm{E}\left[-e^{-\alpha W_i}\middle| I_i, \theta_G\right], \quad i \in [0,1], \tag{2.14}$$

where $J(P, I_i, \theta_G)$ gives the value function of investor i, conditional on her information and the government's intervention policy.

The market equilibrium is then given by the following market-clearing condition:

$$\int_i \theta_i + \theta_G = 0. \tag{2.15}$$

The government chooses its trading policy θ_G (or h_G) to maximize the overall expected utility for all investors:

$$\max_{\theta_G} \ \mathrm{E}\left[\int_i J(P, I_i, \theta_G)\right] \ \equiv \ \max_{h_G} \ V(h_G). \tag{2.16}$$

The last equality simply states that given the linear form of the government's trading policy, the choice of θ_G reduces to the choice of h_G.

2.7 Ex Ante First Best Allocation

In the economy defined above, investors have heterogeneous income risks, which give rise to their desire to trade to share risk. In addition, when they have private information, they also want to trade to speculate. In the absence of trading for risk-sharing, there will be no trading for speculation either (see, for example, **?**). Thus, investors' primary trading need is risk sharing. In particular, they want to trade to share the risk from their idiosyncratic income shocks, given

by y_i. If such a risk sharing is complete and costless, they will only be bearing their fair share of the aggregate income risk, given by Y. In this case, their ex ante expected utility will be:

$$V^* \equiv -\frac{1}{\sqrt{1-\alpha^2\sigma_u^2\sigma_Y^2}}, \tag{2.17}$$

which is independent of σ_y^2. V^* gives the investors' welfare level under the first best allocation. However, when investors have to rely on their trading in the market to share idiosyncratic income risk, which is correlated with market risk, they have to pay the corresponding market risk premium. Consequently, they may not be able to achieve the first best welfare level. This then leaves room for the government to intervene in the market to improve investor welfare.

2.8 Remarks on Model Specification

For tractability, clarity, and completeness, we have made simplifying assumptions in the model defined above. Several remarks are in order regarding these assumptions, their economic implications and possible relaxations.

Investor heterogeneity is essential to the importance of financial market, which facilitates more efficient resource/risk allocations. In our model, we introduced two forms of heterogeneity, heterogeneity in endowment and information. The heterogeneity in endowment gives rise to investors' desire to trade in the market to share risk and the heterogeneity in information gives rise to their desire to trade to speculate. We have avoided heterogeneity in preferences, although our setting can be easily extended to allow differences in investors' (absolute) risk aversion.

For the financial market, we allow only the riskless bond and the stock to be traded. Their payoffs only depend on aggregate shocks, such as the aggregate income ($Y\,u$), which contributes to aggregate consumption. Given that there exist idiosyncratic income risks, captured by y_i, the financial market is incomplete. This is the basic imperfection in our setting, which prevents investors from achieving the Pareto optimal allocation and leaves room for government intervention. It also allows the the market to be informationally inefficient in the sense that market prices in general won't fully reveal all the relevant information to investors.

For simplicity, we have assumed that the total stock and bond holdings of all investors are zero. This can be relaxed without affecting the main results of the paper.

The joint assumption of normality for the underlying shocks and constant absolute risk aversion allows us to solve the model analytically.

We have also assumed that all investors are identical ex ante. This is mainly for simplicity.

We can allow differences in investors' initial endowment of stock and bond, their exposure to aggregate income shock, the amount of idiosyncratic income shock, and the precision of private information. These extensions will of course lead to richer structures of the model as well as its implications. They will also make the analysis more complex. In order to focus on the main issues of this paper, we consider the simplest structure.

Regarding the role of the government, we assume that its objective is the same as the investors' and its budget is closed. These two assumptions are critical to our analysis on the impact of government interventions. In principle, we can allow the government to have different objectives. But this will complicate the welfare implications of its actions. Given the ex ante symmetry among the investors, we only consider the government's ex post transfer to be also symmetric. When the investors are heterogeneous ex ante, the government's intervention and transfer policies also need to be adapted accordingly.

In the model, we only allow the government to have information about aggregate state variables. However, in our setting, we can also allow the government to have information on individual investors, which will lead to richer intervention policies.

For tractability, we only consider the government's intervention (trading) policy within the linear class. Thus, any notion of optimality is only within this class. The desire for tractability forces us to consider linear equilibrium under linear policies for the government. Since the main goal of the paper is about conditions under which government intervention is warranted, we are less concerned about if a welfare-improving intervention is globally optimal or even optimal within the linear class.

We will provide more discussions on the possible extensions of the basic model in Section 7.

3 Solution to Market Equilibrium and Optimal Intervention

We now consider the solution to the market equilibrium, taken as given the government's trading policy in the form of (2.7).

In order to proceed, we need to know the form of the stock price as a function of the underlying state variables. We start with the conjecture that the stock price takes the following linear functional form in the aggregate state variables:

$$P = b\,F - c\,Y + d_F\,\varepsilon_{GF} + d_Y\,\varepsilon_{GY} = b\,F - c\,Y + \varepsilon_P, \quad \varepsilon_P \equiv d_F\,\varepsilon_{GF} + d_Y\,\varepsilon_{GY}, \tag{3.1}$$

where b, c, d_F and d_Y are constants. Our task going forward in this section is to show that the stock price of this form can indeed clear the market. In general, the price function should also include a constant term. However, as we later confirm this constant is zero in equilibrium, given our current setting. Thus, we omit it here to simplify exposition. As noted before, P also provides a public signal about the underlying state of the economy, common to all market participants including the government.

3.1 Conditional Distributions of State Variables

Given the linear price function (3.1), we can derive the distributions of the underlying state variables conditional on investors' signals, public and private. For convenience, we will express the relevant quantities in matrix form. For brevity, we only present the formal results here. Detailed derivations and expressions of all the relevant quantities are given in the appendix.

For a normal random variable x of any dimension, we denote it by $x \sim N(\bar{x}, \Sigma_{xx})$, where $\bar{x}$ denotes its mean vector and Σ_{xx} denotes its covariance matrix.

Denote the vector of signals for investor i by $X_i = [\, P \;\; Y_i \;\; S_i \,]^\top$. Thus, $I_i = X_i$ and we have:

$$X_i = [\, P \;\; Y_i \;\; S_i \,]^\top \sim N\,(0, A)\,, \tag{3.2}$$

where the mean of zero in the normal distribution represents a (3×1) vector of zeros and $A \equiv \Sigma_{XX}$ is the covariance matrix of the three signals, assumed to be positive definite. An explicit expression of A is given by (A.3a) in the appendix.

Next, we derive the distribution of F, Y, u, ε_{GF} and ε_{GY}, the five aggregate shocks to the market, conditional on investor i's information X_i. Given the stock price function in (3.1), these shocks are no longer independent conditional on P, which is in X_i. In particular, we have:

$$\varepsilon_{GY} = d_Y^{-1}[P - (b\,F - c\,Y + d_F\,\varepsilon_{GF})], \tag{3.3}$$

when $d_Y \neq 0$. (When $d_Y = 0$, we will simply let $\varepsilon_{GY} = 0$.) Hence, we only need to include the four independent shocks going forward: F, Y, u and ε_{GF}. In particular, let:

$$Z = [\, F \;\; Y \;\; u \;\; \varepsilon_{GF} \,]^\top \sim N\,(0, \Sigma_{ZZ})\,, \quad \Sigma_{ZZ} = \text{Diag}\left[\, \sigma_F^2,\, \sigma_Y^2,\, \sigma_u^2,\, \sigma_{GF}^2 \,\right], \tag{3.4}$$

where Σ_{ZZ} is the covariance matrix of Z and Diag$[\cdots]$ denotes a diagonal matrix of its elements. Given our distributional assumptions, Σ_{ZZ} reduces to a diagonal matrix with variances of F, Y,

u and ε_{GF}, respectively. The joint distribution of Z and X_i, $i \in [0,1]$, is then given by:

$$\begin{bmatrix} Z \\ X_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{ZZ} & \Sigma_{ZX} \\ \Sigma_{XZ} & A \end{bmatrix} \right), \tag{3.5}$$

where Σ_{ZX} is the covariance matrix of Z and X_i, given by (A.4) in the appendix.

Joint normality of Z and X_i implies that conditional on X_i, Z remains normal, as given in the lemma below:

Lemma 1. *Conditional on X_i, Z has the following normal distribution:*

$$Z|X_i \sim N\left(\bar{Z}_i, \Sigma_{ZZ|X}\right), \tag{3.6}$$

where

$$\bar{Z}_i \equiv \mathrm{E}\left[Z|X_i\right] = \Sigma_{ZX}A^{-1}X_i \equiv \beta_{ZX}X_i, \quad \Sigma_{ZZ|X} = \Sigma_{ZZ} - \Sigma_{ZX}A^{-1}\Sigma_{XZ}, \tag{3.7}$$

with β_{ZX} and Σ_{ZX} given by (A.7) and (A.8) in the appendix.

Here, $\Sigma_{\bullet|X}$ denotes the covariance matrix conditional on X and $\beta_{\bullet X} = [\,\beta^{(1)}_{\bullet X} \;\; \beta^{(2)}_{\bullet X} \;\; \beta^{(3)}_{\bullet X}\,]$ is a (1×3) row matrix, representing the projection coefficients of an element of Z on X_i. For convenience, we further define:

$$\tilde{Z}_i \equiv Z - \bar{Z}_i \sim N\left(0, \Sigma_{ZZ|X}\right), \quad i \in [0,1], \tag{3.8}$$

where $\tilde{Z}_i$ is a demeaned 4-dimensional normal random variable with covariance matrix $\Sigma_{ZZ|X}$.[5]

Next, we consider the government's trading in the market, which is not observable to the investors. But they can form expectations. In particular, from (2.8) and (3.3), we have:

$$\theta_G = h'_{GP}P + h'_{GZ}Z, \quad h'_{GZ} = \left[\, h'_{GF} \;\; h'_{GY} \;\; 0 \;\; h'_{G\varepsilon_{GF}} \,\right], \tag{3.9}$$

where

$$\begin{aligned} &h'_{GP} = h_{GP} + h_{GY}/d_Y, \\ &h'_{GF} = h_{GF}\beta_{GF} - h_{GY}b/d_Y, \quad h'_{GY} = h_{GY}\beta_{GY} + h_{GY}c/d_Y, \quad h'_{G\varepsilon_{GF}} = h_{GF} - h_{GY}d_F/d_Y. \end{aligned} \tag{3.10}$$

The following lemma gives the distribution of θ_G conditional on investors' information:

[5]To avoid degeneracy, we assume that A, Σ_{ZZ} and $\Sigma_{ZZ|X}$ are all strictly positive definite, which means that the elements in X_i, Z and $Z|X_i$ are not degenerate (such as having a zero variance) or linearly dependent. When that is not true, we will simply remove the degenerate or linearly dependent elements from the corresponding vectors to maintain the positive definitiveness of their covariance matrices.

Lemma 2. *From (3.9), we have:*

$$\theta_G = \bar{\theta}_{Gi} + \tilde{\theta}_{Gi}, \quad i \in [0,1], \tag{3.11}$$

where

$$\bar{\theta}_{Gi} \equiv \mathrm{E}\left[\theta_G | X_i\right] = h'_{GP}P + h'_{GZ}\bar{Z}_i = h'_{GX}X_i, \quad h'_{GX} \equiv h'_{GP}\left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right] + h'_{GZ}\beta_{ZX}, \tag{3.12a}$$

$$\tilde{\theta}_{Gi} \equiv \theta_G - \bar{\theta}_{Gi} = h'_{GZ}\tilde{Z}_i, \tag{3.12b}$$

since $\bar{Z}_i = \beta_{ZX}X_i$.

Combine (3.11) and (3.12), we have:

$$\theta_G = h'_{GX}X_i + h'_{GZ}\tilde{Z}_i. \tag{3.13}$$

The first term in (3.13) gives investor i's expected value of θ_G based on her signal X_i, and the second term gives the unexpected part.

3.2 Investors' Optimal Trading Policy and Value Function

We now consider the investors' optimal trading policy, taken as given the government's intervention/trading policy and the price function. We can rewrite investor i's wealth, $i \in [0,1]$, as follows:

$$\begin{aligned} W_i &= (\theta_i + \bar{\theta}_{Gi} + \tilde{\theta}_{Gi})[(\bar{F}_i - P) + (\tilde{F}_i + u)] + Y_i\, u \\ &= (\theta_i + \bar{\theta}_{Gi})(\bar{F}_i - P) + (\theta_i + \bar{\theta}_{Gi})\tilde{D}_i + (\bar{F}_i - P)h'_{GZ}\tilde{Z}_i + Y_i\, u + \tfrac{1}{2}\tilde{Z}_i^{\top}H_G\tilde{Z}_i, \end{aligned} \tag{3.14}$$

where

$$\tilde{D}_i \equiv \tilde{F}_i + u = h'_{DZ}\tilde{Z}_i, \quad h'_{DZ} \equiv \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \end{array}\right], \tag{3.15a}$$

$$H_G \equiv h_{DZ}^{\prime\top}h'_{GZ} + h_{GZ}^{\prime\top}h'_{DZ}. \tag{3.15b}$$

All the terms in (3.14) are linear in $\tilde{Z}_i$ except the last term which is quadratic in $\tilde{Z}_i$.

For the investors' optimization problem to be well defined, $\Sigma_{ZZ|X}^{-1} + \alpha H_G$ need to be positive definite, which will be confirmed later. We then define $\Sigma_{ZZ|XG}$ as follows:

$$\Sigma_{ZZ|XG}^{-1} \equiv \Sigma_{ZZ|X}^{-1} + \alpha H_G, \tag{3.16}$$

which is also positive definite (and symmetric). It can be viewed as a modified covariance matrix from $\Sigma_{ZZ|X}$ with αH_G. Going forward, we will also use $\sigma^2_{D|XG}$ to denote the variance of D under $\Sigma_{ZZ|XG}$, $\sigma_{DZ|XG}$ to denote the covariance between D and Z under $\Sigma_{ZZ|XG}$, etc. We then have

the following proposition:

Proposition 1. *For positive definite $\Sigma_{ZZ|XG}$, investor i's optimal trading policy is:*

$$\theta_i = -\bar{\theta}_{Gi} + h_s(\bar{F}_i - P) - h_h Y_i, \quad i \in [0,\, 1], \tag{3.17}$$

where

$$h_s \equiv \frac{1}{\alpha\sigma^2_{D|XG}}(1 - \alpha h'_{GZ}\sigma_{DZ|XG}), \quad h_h \equiv \frac{\sigma_{Du|XG}}{\sigma^2_{D|XG}}, \tag{3.18}$$

and her value function is:

$$J(W^i_t, X_i, \theta_G) = -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\, e^{-\alpha\hat{W}_i}, \tag{3.19}$$

where

$$\begin{aligned}\hat{W}_i = \tfrac{1}{2}\,\alpha\Big\{&(\theta_i + \bar{\theta}_{Gi})^2\sigma^2_{D|XG} \\ &- \left[Y_i^2\sigma^2_{u|XG} + 2\,(\bar{F}_i - P)Y_i\, h'_{GZ}\sigma_{uZ|XG} + (\bar{F}_i - P)^2 h'_{GZ}\Sigma_{ZZ|XG}h'^{\top}_{GZ}\right]\Big\}.\end{aligned} \tag{3.20}$$

Investors' optimal stock holdings given in (3.17) have a simple interpretation. The first term reflects the investor i's desire to offset the expected distortion to her terminal wealth by the government's intervention. The second term reflects her speculative trade based on the difference between her expectation of the stock's final payoff and its current price. The third term reflects her hedging trade to share the risk from her non-traded income.

For later use, investors' optimal stock holdings in (3.17) can also be rewritten as:

$$\theta_i = -h_P P - h_Y Y_i + h_S S_i = -h_P P - h_Y(Y + y_i) + h_S(F + \varepsilon_i), \quad i \in [0, 1], \tag{3.21}$$

where

$$h_P \equiv (1 - \beta^{(1)}_{FX})h_s + h'^{(1)}_{GX}, \quad h_Y \equiv h_h - \beta^{(2)}_{FX}h_s + h'^{(2)}_{GX}, \quad h_S \equiv \beta^{(3)}_{FX}h_s - h'^{(3)}_{GX}. \tag{3.22}$$

3.3 Price, Stock Holdings and Welfare in Equilibrium

Given the investors' optimal stock holdings in (3.21), the market-clearing condition (2.15) becomes:

$$0 = \int_i \theta_i + \theta_G = (-h_P P - h_Y Y + h_S F) + (h_{GP}P + h_{GF}S_{GF} + h_{GY}S_{GY}). \tag{3.23}$$

It leads to the following conditions on b, c, d_F and d_Y:

$$0 = (-h_P + h_{GP})\, b + (h_S + h_{GF}\beta_{GF}), \quad 0 = (-h_P + h_{GP})\, c + (h_Y - h_{GY}\beta_{GY}),$$
$$0 = (-h_P + h_{GP})\, d_F + h_{GF}, \quad 0 = (-h_P + h_{GP})\, d_Y + h_{GY}. \tag{3.24}$$

We then have the following proposition for the solution of the equilibrium stock price:

Proposition 2. *In the market equilibrium with the linear stock price function in the form of (3.1), the price coefficients are given by:*

$$b = \frac{1}{h_P - h_{GP}}(h_S + h_{GF}\beta_{GF}), \quad c = \frac{1}{h_P - h_{GP}}(h_Y - h_{GY}\beta_{GY}),$$
$$d_F = \frac{1}{h_P - h_{GP}} h_{GF}, \quad d_Y = \frac{1}{h_P - h_{GP}} h_{GY}. \tag{3.25}$$

The above solution for the price coefficients confirms that the linear price function (3.1) can indeed yield a market equilibrium, with the coefficients given in (3.25) and no constant term.[6]

In equilibrium, the investors' stock holdings are:

$$\theta_i = -\theta_G - h_Y y_i + h_S \varepsilon_i, \quad i \in [0, 1]. \tag{3.26}$$

It is important to note that (3.21) expresses θ_i in terms of investor i's signals (information) while (3.26) expresses it in terms of the underlying aggregate and idiosyncratic shocks. Given the symmetry among investors, their equilibrium stock holdings do not depend on the aggregate shocks F and Y except through θ_G, the government's intervention in the market. It also shows that investors adjust their stock positions θ_i to offset the distortion caused by the government's intervention θ_G.

Next, we consider the investors' welfare, given the government's trading policy. For convenience, define:

$$z_i \equiv [\, F \;\; Y \;\; y_i \;\; \varepsilon_i \;\; \varepsilon_{GF} \;\; \varepsilon_{GY} \,]^\top \sim N(0, \Sigma_{zz}), \quad i \in [0, 1], \tag{3.27a}$$
$$\Sigma_{zz} \equiv \text{Diag}\left[\, \sigma_F^2,\ \sigma_Y^2,\ \sigma_y^2,\ \sigma_\varepsilon^2,\ \sigma_{GF}^2,\ \sigma_{GY}^2 \,\right]. \tag{3.27b}$$

Note that while Z contains only aggregate shocks, z_i also includes the idiosyncratic shocks to the investors. We then express the stock price, the government's trading policy, investor i's income risk, expectations and stock holdings in equilibrium as follows:

$$P = B_P z_i, \;\; \theta_G = B_G z_i, \;\; Y_i = B_Y z_i, \;\; \bar{F}_i = B_F z_i, \;\; \theta_i = -\bar{\theta}_{Gi} + B_\theta z_i = B_G z_i + B_\theta z_i, \tag{3.28}$$

[6]The constant term can become relevant if the market's unconditional exposure to the payoff risk is not zero. This occurs for example when the initial endowment of the stock shares or the mean of Y is not zero.

where B_P, B_G, B_Y, B_F, B_θ are given in (A.23). With the matrix representation in (3.28), (3.20) becomes:

$$\hat{W}_i = \tfrac{1}{2}\,\alpha\, z_i^\top B z_i, \tag{3.29}$$

where B is given in (A.24).

From the market equilibrium given above, investors' welfare is given in following proposition:

Proposition 3. *Given the government's intervention policy (2.7), the investors' ex ante expected utility is given by:*

$$V(\theta_G) = V(h_G) = -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\frac{|\Sigma_{zz}^{-1}|^{1/2}}{|\Sigma_{zzB}^{-1}|^{1/2}}, \quad i \in [0,1], \tag{3.30}$$

where

$$\Sigma_{zzB}^{-1} = \Sigma_{zz}^{-1} + \alpha^2 B. \tag{3.31}$$

Note that V is the same for all investors, hence it does not carry the index i.[7]

3.4 Optimal Intervention Policy

Under the linear form of intervention policy, the government's optimal choice is given by:

$$h_G = \arg\max\, V(h_G). \tag{3.32}$$

The first order condition is given by:

$$\nabla_{h_G} V(h_G) \;=\; 0, \quad \nabla_{h_G} \equiv \left[\begin{array}{ccc} \dfrac{\partial}{\partial h_{GP}} & \dfrac{\partial}{\partial h_{GF}} & \dfrac{\partial}{\partial h_{GY}} \end{array}\right]^\top. \tag{3.33}$$

Optimality is ensured by the second order condition, which requires the second order derivatives of $V(h_G)$ to be negative definite:

$$\nabla_{h_G}^2 V(h_G) < 0. \tag{3.34}$$

3.5 Additional Comments

We have shown that under the government's linear intervention policy (2.7) and the linear price function (3.1), the market equilibrium is fully characterized by Propositions 1 and 2. Its existence and uniqueness will depend on the parameter values, including h_G. In what follows, we will consider situations of interest and establish the parameter conditions guaranteeing the existence of an equilibrium and its uniqueness.

[7](3.31) requires $\Sigma_{zz}^{-1} + \alpha^2 B$ to be positive definite, which can be guaranteed by proper parameter restrictions imposed in our future analysis.

4 No Government Intervention

In order to develop more understanding about the basic properties of the model, we first consider the situation when there is no government intervention. In this case, $\theta_G = 0$ or $h_G = 0$. Consequently, $H_G = 0$ and $\Sigma_{ZZ|XG} = \Sigma_{ZZ|X}$, which is positive definite.

The solution to the market equilibrium in this case follows directly from the general solution in Section 3 by simply setting $\theta_G = 0$ (or $h_G = 0$) and adjusting the appropriate state variables. However, in order to better demonstrate the economics behind the model, we sketch a few key steps when needed to illustrate the underlying mechanics.

4.1 Price Function and Conditional Distributions of State Variables

In absence of government intervention, the linear price function in (3.1) simply takes the form:

$$P = b\,F - c\,Y, \tag{4.1}$$

with b and c being constants.

Under the linear price function (4.1), F and Y become perfectly correlated conditional on P, which is part of X_i. In particular, $Y = (bF - P)/c$ when $c \neq 0$. In this case, we only need to maintain F and u as the two independent aggregate shocks in our analysis. Thus, we can simply drop Y as well as ε_{GF} and ε_{GY} from Z in (3.5) and have $Z = [F \quad u]^\top$. As shown in the appendix, conditional on X_i, we have:

$$\Sigma_{ZZ|X} = \begin{bmatrix} \sigma^2_{F|X} & 0 \\ 0 & \sigma^2_{u|X} \end{bmatrix}, \quad \sigma^2_{F|X} = \frac{1}{|A|} c^2 \sigma^2_F \sigma^2_Y \sigma^2_y \sigma^2_\varepsilon, \quad \sigma^2_{u|X} = \sigma^2_u, \tag{4.2}$$

and

$$\begin{bmatrix} \beta_{FX} \\ \beta_{uX} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} b\,\sigma^2_F(\sigma^2_Y+\sigma^2_y)\sigma^2_\varepsilon & b\,c\,\sigma^2_F\sigma^2_Y\sigma^2_\varepsilon & c^2\sigma^2_F\sigma^2_Y\sigma^2_y \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.3}$$

where $|A| = b^2\sigma^2_F(\sigma^2_Y+\sigma^2_y)\sigma^2_\varepsilon + c^2\sigma^2_Y\sigma^2_y(\sigma^2_F+\sigma^2_\varepsilon)$. In addition, $\tilde{Z}_i$ as defined in (3.8) now becomes 2-dimensional with zero mean and covariance matrix $\Sigma_{ZZ|X}$, after removing $\tilde{Y}$.

4.2 Market Equilibrium

Optimal trading policy. With $\theta_G = 0$, investor i's terminal wealth is:

$$W_i = \theta_i\,(F + u - P) + Y_i\,u, \quad i \in [0,1], \tag{4.4}$$

which is normal conditional on investor i's signal X_i. The optimal trading policy is then:

$$\theta_i = \frac{1}{\alpha\sigma^2_{D|X}}\left[(\bar{F}_i - P) - \alpha\sigma_u^2 Y_i\right] = h_s(\bar{F}_i - P) - h_h Y_i, \quad i \in [0,1], \tag{4.5}$$

where

$$h_s = \frac{1}{\alpha\sigma^2_{D|X}}, \quad h_h = \frac{\sigma_u^2}{\sigma^2_{D|X}}. \tag{4.6}$$

Since $\bar{F}_i = \beta_{FX} X_i$, where β_{FX} is given in (4.3), we can rewrite (4.5) as:

$$\theta_i = h_P P - h_Y Y_i + h_S S_i, \quad h_P = h_s\left(\beta^{(1)}_{FX} - 1\right), \quad h_Y = h_h - h_s\beta^{(2)}_{FX}, \quad h_S = h_s\beta^{(3)}_{FX}. \tag{4.7}$$

Value function. Given the investors' optimal trading policy, their value function becomes:

$$J(P, X_i) = -e^{-\alpha\hat{W}_i}, \quad i \in [0,1], \tag{4.8}$$

where

$$\hat{W}_i = \tfrac{1}{2}\,\alpha\left(\sigma^2_{D|X}\theta_i^2 - \sigma_u^2 Y_i^2\right) = \tfrac{1}{2}\,\alpha\left\{\sigma^2_{D|X}\left[h_s(\bar{F}_i - P)^2 - h_h Y_i\right]^2 - \sigma_u^2 Y_i^2\right\}. \tag{4.9}$$

Equilibrium stock price. In the absence of government intervention, market clearing leads to the following solution for the two coefficients in the price function (4.1):

$$b = -\frac{h_S}{h_P} = \frac{\beta^{(3)}_{FX}}{1 - \beta^{(1)}_{FX}}, \quad c = -\frac{h_Y}{h_P} = \frac{\alpha\sigma_u^2 - \beta^{(2)}_{FX}}{1 - \beta^{(1)}_{FX}}, \tag{4.10}$$

where $\beta^{(1)}_{FX}$, $\beta^{(2)}_{FX}$ and $\beta^{(3)}_{FX}$ are given in (4.3). Assuming $c \neq 0$, we can combine the two equations in (4.10) as follows. Let:

$$k \equiv b/c = \frac{\beta^{(3)}_{FX}}{\alpha\sigma_u^2 - \beta^{(2)}_{FX}}. \tag{4.11}$$

Substituting in $\beta^{(2)}_{FX}$ and $\beta^{(3)}_{FX}$ from (4.3), we obtain the following equation for k:

$$\alpha\,\sigma_u^2(\sigma_Y^2 + \sigma_y^2)\sigma_\varepsilon^2\,k^3 - \sigma_Y^2\sigma_\varepsilon^2\,k^2 + \alpha\,\sigma_u^2\sigma_Y^2\sigma_y^2(\sigma_S^2/\sigma_F^2)\,k - \sigma_Y^2\sigma_y^2 \;=\; 0. \tag{4.12}$$

(4.12) is a cubic equation, and its solution is given by (A.33) in the appendix. Moreover, over certain domain in the parameter space, the solution is unique, which is what we consider. Substituting the solution to k into (4.10) yields the solution to b and c, which then gives the equilibrium price.

Equilibrium stock holdings. Given the equilibrium price, we obtain investors' stock holdings

in equilibrium:

$$\theta_i = -h_Y y_i + h_S \varepsilon_i, \quad i \in [0,1], \tag{4.13}$$

where h_Y and h_S are given in (4.7).

Welfare. Next, we compute investors' welfare. With no government intervention, shocks ε_{GF} and ε_{GY} drop out of the analysis. From the general solution, (3.27) now reduces to:

$$z_i = [\, F \;\; Y \;\; y_i \;\; \varepsilon_i \,]^\top \sim N(0, \Sigma_{zz}), \quad \Sigma_{zz} = \text{Diag}\left[\, \sigma_F^2,\ \sigma_Y^2,\ \sigma_y^2,\ \sigma_\varepsilon^2 \,\right], \quad i \in [0,1], \tag{4.14}$$

and (3.28) becomes:

$$P = B_P z_i, \quad Y_i = B_Y z_i, \quad \bar{F}_i = B_F z_i, \quad \theta_i = B_\theta z_i, \tag{4.15}$$

where B_P, B_Y, B_F, B_θ, as a special case of (A.23), are given in (A.35). From (3.29), we have:

$$\hat{W}_i = \tfrac{1}{2}\, \alpha\, z_i^\top B z_i, \quad B = (\sigma_{F|X}^2 + \sigma_u^2) B_\theta^\top B_\theta - \sigma_u^2 B_Y^\top B_Y. \tag{4.16}$$

Obviously, B is significantly simplified now. From (3.30), investors' welfare now becomes:

$$V_0 \equiv V(0) = -\frac{|\Sigma_{zzB}|^{1/2}}{|\Sigma_{zz}|^{1/2}} = -\frac{1}{\sqrt{1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)}} \frac{1}{\sqrt{1+\alpha^2(\sigma_{F|X}^2+\sigma_u^2) B_\theta \Sigma_{zzY} B_\theta^\top}}, \tag{4.17}$$

where Σ_{zzB} is given in (3.31) and $\Sigma_{zzY}^{-1} = \Sigma_{zz}^{-1} + \alpha^2\sigma_u^2 B_Y^\top B_Y$.

It is worth noting that the first term in (4.17) depends only on the parameters related to investors' income shocks and risk aversion. In fact, including the minus sign, it is the welfare level in autarky V_{00}. It does not depend on the risk in stock payoff (σ_F) and the degree of information asymmetry in the market. The second term, however, does depend on the risk of stock payoff and the degree of information asymmetry through its dependence on $\sigma_{F|X}^2$ and B_θ. Moreover, $V_0 \geq V_{00}$ since the second term is no greater than 1. Obviously, trading in the market allows investors to share their income risk and hence to improve welfare.

4.3 Special Cases

In order to better understand the basic properties of the model, we consider a few special cases. Since the general solution for b, c and V_0 obtained above requires Σ_{zz} to be non-singular, which requires σ_F, σ_Y, σ_y and σ_ε to be strictly positive, it may not directly yield the corresponding solution when any of these parameters becomes zero. In this case, we need to go through the solution procedure separately to arrive at the proper results. For brevity, we only present the key results here. More details of the solution for these special cases are provided in the appendix.

Special Case 1: $\sigma_Y^2 = 0$**.** In this case, there is no aggregate income risk. The stock price becomes:

$$P = F, \tag{4.18}$$

which fully reveals F, and we have a fully-revealing equilibrium. From (3.18), the investors' optimal trading policy is given by:

$$h_s = 1/(\alpha\sigma_u^2), \quad h_h = 1. \tag{4.19}$$

In equilibrium, their stock holdings will be:

$$\theta_i = -y_i, \quad i \in [0, 1]. \tag{4.20}$$

From (4.17), their welfare is:

$$V_0 = -1. \tag{4.21}$$

With no aggregate income risk, investors can fully insure their idiosyncratic income risks through their trading in the stock market at no cost. Their welfare is then independent of their idiosyncratic income risks, which is captured by σ_y^2. Clearly, with the opportunity to trade in the market to hedge the idiosyncratic risks, the investors achieve a higher level of welfare than in autarky. In fact, they achieve the first-best welfare level in this case with zero hedging cost.

Special Case 2: $\sigma_y^2 = 0$**.** When $\sigma_y^2 = 0$, $Y_i = Y$ for all i and there is no heterogeneity in investors' income risks. Moreover, all investors observe Y. This implies that they can also infer F from the stock price P, in the form of (4.1). Thus, we have a fully-revealing equilibrium, in which the stock price reveals F to all investors and they hold the same share of the aggregate risk, given by Y.

In particular, $\sigma_{F|X}^2 = 0$ and $\sigma_{D|X}^2 = \sigma_u^2$. Investors' optimal trading policy is the same as given in (4.19). In equilibrium, the price is given by:

$$P = F - \alpha\sigma_u^2 Y, \tag{4.22}$$

and the investors' stock holdings are:

$$\theta_i = 0, \quad i \in [0, 1]. \tag{4.23}$$

In this case, investors are identical and there is no trading among them. Their welfare is simply:

$$V_0 = -\frac{1}{\sqrt{1 - \alpha^2\sigma_u^2\sigma_Y^2}}. \tag{4.24}$$

This is the first-best level, which decreases with $\sigma_u^2\sigma_Y^2$, the aggregate income risk.

Special Case 3: $\sigma_\varepsilon^2 \to 0$**.** In this case, $\varepsilon_i \to 0,\ \forall\ i \in [0,1]$, all investors observe F, and there is no information asymmetry about the stock's payoff. Given the form of the stock price, it will also fully reveal Y. Thus, we again have a fully-revealing equilibrium.

In this case, we have $\sigma_{F|X}^2 = 0$, $\sigma_{D|X}^2 = \sigma_u^2$, and investors' optimal trading policy given in (4.19). In equilibrium, the stock price is:

$$P = F - \alpha\sigma_u^2\, Y, \tag{4.25}$$

and the investors' stock holdings are:

$$\theta_i = -y_i, \quad i \in [0,1]. \tag{4.26}$$

Investors' welfare is given by:

$$V_0 = -\frac{1}{\sqrt{1-\alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)}}. \tag{4.27}$$

Parameter condition (2.12) guarantees that what is in the square root is positive.

From investors' welfare function, we observe that in the presence of aggregate risk (when $\sigma_Y > 0$), sharing idiosyncratic risks in the market is no longer cost free. Despite the fact that the investors fully share their idiosyncratic risks through trading, as (4.26) indicates, their welfare still depends on the magnitude of idiosyncratic risks as measured by σ_y^2. In particular, V_0 decreases with σ_y^2 and this dependence further depends on σ_Y^2 and σ_u^2, the total amount of aggregate risk. This is because investors need to pay the market risk premium to unload the idiosyncratic risk, which is (perfectly) correlated with the aggregate risk as captured by u. Moreover, while on average investors pay zero premium since their idiosyncratic risk exposure is zero, the risk of paying the premium reduces their welfare. The fact that $V_{00} \le V_0 \le V^*$ implies that the equilibrium allocation is in general not Pareto optimal.

Obviously, Special Case 1 and 2 are merely special cases of Special Case 3.

Special Case 4: Investors receive no private information on F**.** In this case, we drop S_i from X_i, which becomes $X_i = [\,P \ \ Y_i\,]^\top$. Since investors receive no information on F, the stock price won't depend on F but only depend on Y, which then fully reveals Y.

In this case, $\sigma_{F|X}^2 = \sigma_F^2$ and $\sigma_{D|X}^2 = \sigma_F^2 + \sigma_u^2 = \sigma_D^2$. Investors' optimal trading policy becomes:

$$h_s = 1/(\alpha\sigma_D^2), \quad h_h = \sigma_u^2/\sigma_D^2. \tag{4.28}$$

The equilibrium stock price is given by:

$$P = -\alpha\sigma_u^2\, Y, \tag{4.29}$$

and the investors' equilibrium stock holdings are:

$$\theta_i = -\frac{\sigma_u^2}{\sigma_D^2}\, y_i, \quad i \in [0,1]. \tag{4.30}$$

Their welfare is given by:

$$V_0 = -\frac{1}{\sqrt{1-\alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)-\alpha^2\sigma_u^2(1-\alpha^2\sigma_u^2\sigma_Y^2)\sigma_y^2(\sigma_F^2/\sigma_D^2)}}. \tag{4.31}$$

Clearly, in this case the investors' welfare level is below the level in Special Case 3 when they have full information on F. This is because uncertainty in F increases the risk premium for the stock, the cost for investors to use the stock as a hedging instrument.

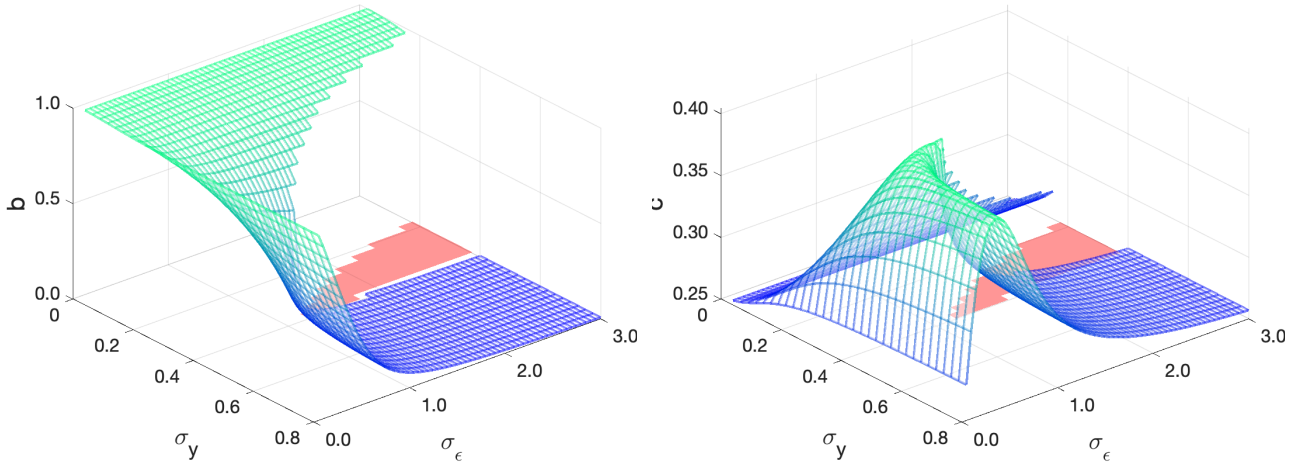
Additional comments. The analysis above reveals important properties of our model when there is no government intervention. In particular, the market equilibrium is in general not Pareto optimal. This is best illustrated by Special Cases 3 and 4 when the equilibrium is fully revealing. The main reason behind this outcome is that the market is incomplete with respect to investors' idiosyncratic income risks given by σ_y^2.

4.4 General Case

We now consider the general case. Several comments are in order before we present the results. First, we limit the parameter values allowed by the parameter constraint (2.12). Second, since the equilibrium price coefficients b and c are given by a cubic equation, they may have multiple (real) solutions. To avoid multiple roots, we limit ourselves to the parameter domain with only a unique solution. As shown in the appendix, this domain can be fully specified. Third, we only present the results with a limited set of parameter values. These results help to illustrate the key properties of the model. They are not intended to be extensive in exploring the model's full behavior. In particular, we will hold fixed the values of parameters about the aggregate market such as α, σ_F, σ_u and σ_Y and focus on parameters characterizing investor heterogeneity such as σ_y and σ_ε. While σ_y characterizes investors' heterogeneous income risk, it also reflects the idiosyncratic noise in the private signal about Y, and σ_ε reflects the idiosyncratic noise in private signals about F. Even though how the aggregate market characteristics affect the market behavior as well as interact with investor heterogeneity are also interesting aspects of

the model, for brevity, in the paper we examine primarily the heterogeneity effects, taking the aggregate environment as given.

To fix ideas, throughout the paper unless noted otherwise, we set the following values for the aggregate parameters: $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$ and $\sigma_Y = 0.5$.



(a) Price coefficient b

(b) Price coefficient c

Figure 2: Price coefficients b and c for different values of σ_y and σ_ε with no government intervention. The other parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$ and $\sigma_Y = 0.5$.

In Figure 2, we plot the equilibrium price coefficients, b and c, for different values of σ_y and σ_ε, holding constant the aggregate parameters α, σ_F, σ_u and σ_Y. Panel (a) of Figure 2 shows how b varies with σ_y, the x-axis, and σ_ε, the y-axis. There is a set of σ_y and σ_ε values which lead to multiple solutions for b (and c). This set is given by the red domain on the $\sigma_y - \sigma_\varepsilon$ plane in the figure. For convenience, we will call the red domain non-admissible and remaining area in the $\sigma_y - \sigma_\varepsilon$ plane the admissible domain. The b values we plot are for values of σ_y and σ_ε in the admissible domain, which forms a surface with an opening above the red non-admissible domain. Hence, they represent the unique solution to the equilibrium over the admissible domain.

It is worth noting that in order to better illustrate how b varies with σ_y and σ_ε, respectively, with the grid lines, we use a relatively sparse grid in the figure. Consequently, the interior edge of the b-surface simply matches the grid, and so does the edge of the red domain on the $\sigma_y - \sigma_\varepsilon$ plane. Thus, they don't look continuous. Of course, when the grid becomes dense, both boundaries will become continuous. This feature is true for the remaining figures in the paper.

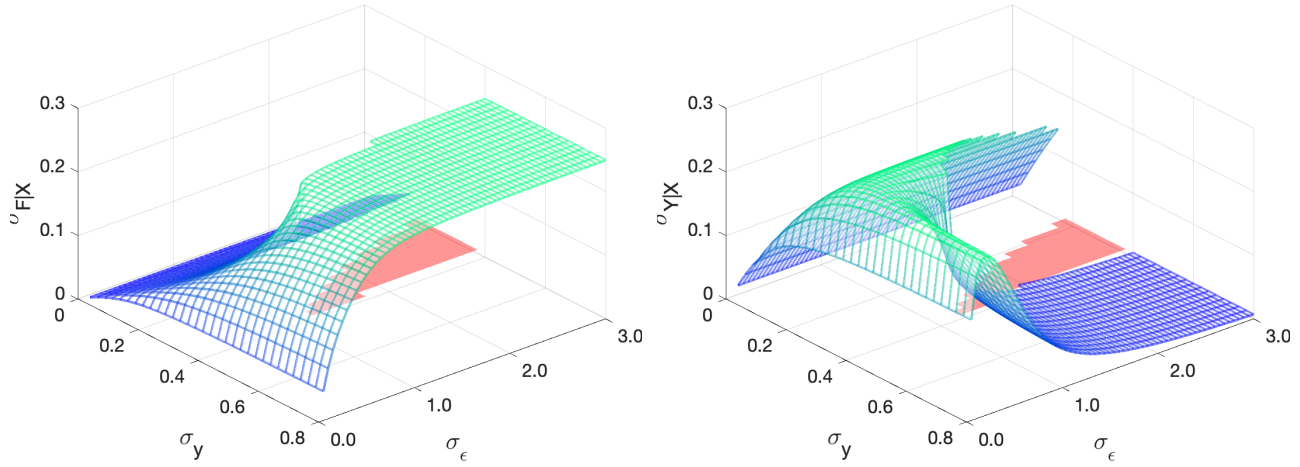
For $\sigma_y = 0$, we have Special Case 2 discussed earlier. Y is known to the investors and the stock price fully reveals F, independent of the value of σ_ε. Thus, $b = 1$, as shown by the upper

left edge of the b surface, along the y-axis. For $\sigma_\varepsilon = 0$, we have Special Case 3. In this case, investors' private signals fully reveal F and $b = 1$, independent of σ_y. This is shown by the lower left edge of the surface, along the x-axis. For small values of σ_y, the equilibrium is similar and b is close to 1. As σ_y becomes larger, reaching the range between 0.2 and 0.3, b becomes non-monotonic in σ_ε. It decreases with σ_ε when σ_ε is small and then increases with it. As the value of σ_y becomes significantly large, b becomes monotonically decreasing with σ_ε. This is intuitive. When the investors' signal about F becomes noisier, the price becomes less dependent on F.

Panel (b) of Figure 2 plots the value of c for different values of σ_y and σ_ε, within the admissible domain. Again, $\sigma_y = 0$ and $\sigma_\varepsilon = 0$ correspond to Special Case 2 and 3, respectively, for which $c = \alpha\sigma_u^2 = 0.25$. For non-trivial values of σ_y, c varies with σ_ε non-monotonically, first increasing and then decreasing with σ_ε. Since c captures the risk premium of the aggregate risk, its behavior is of critical importance to the market. The aggregate risk has two components, one associated with F and the other associated with u and Y. For $\sigma_\varepsilon = 0$, the risk concerning F is zero, the $c = \alpha\sigma_u^2$. As σ_ε increases, the risk concerning F increases, which tends to increase c. However, there is also an offsetting effect. As σ_ε increases, the price becomes less dependent on F and more dependent on Y. As a result, it becomes more informative about Y. This helps to reduce the investors' uncertainty about Y and the corresponding risk premium.

Given the equilibrium price, we can further examine other aspects of the equilibrium outcome. One important aspect is the informational efficiency of the market, which can be measured by the conditional volatility of F and Y in equilibrium, $\sigma_{F|X}$ and $\sigma_{Y|X}$, respectively. Figure 3(a) plots $\sigma_{F|X}$ for different values of σ_y and σ_ε. Again, the red region in the $\sigma_\varepsilon - \sigma_y$ plane defines the domain with multiple solutions, which we exclude from the plot. As before, for $\sigma_y = 0$ and $\sigma_\varepsilon = 0$, i.e., the σ_ε and σ_y axes, which correspond to Special Case 2 and 3, respectively, $\sigma_{F|X} = 0$ and F is fully revealed, either by the stock price or investors' private signals on F. For non-trivial values of σ_y, the price is no longer fully revealing about F, $\sigma_{F|X}$ monotonically increases with σ_y, the amount of noise in investors' private signals. The plot also shows that for a fixed (and positive) value of σ_ε, $\sigma_{F|X}$ increases with σ_ε. This is also intuitive. Holding other parameters constant, increasing σ_y also increases the noise in investors' signal Y_i about Y. This tends to reduce the information she can extract from the price, about both F and Y.

Panel (b) of Figure 3 plots $\sigma_{Y|X}$ against σ_y and σ_ε, and we see a richer behavior. In particular,



(a) Price coefficient b

(b) Price coefficient c

Figure 3: Conditional volatility of F and Y for different values of σ_y and σ_ε with no government intervention. The other parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$ and $\sigma_Y = 0.5$.

for relatively small values of σ_y, e.g., between 0 and 0.2, $\sigma_{Y|X}$ tends to increase with σ_ε. This is intuitive. Holding other parameters constant, increasing σ_ε increases the uncertainty about F, which also reduces the information investors can infer from the price about Y. However, when σ_y becomes sufficiently large, e.g., above 0.4, as σ_ε becomes large enough, e.g., exceeding 0.4, $\sigma_{Y|X}$ starts to decrease with σ_ε. This is because, for this range of σ_ε, b is decreasing as σ_ε increases and quickly approaching 0. This makes the price very informative about Y, and eventually approach Special Case 4.

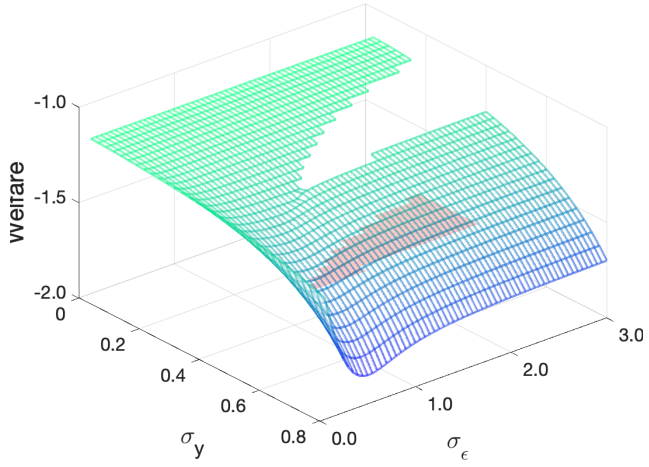


Figure 4: Investor welfare for different values of σ_y and σ_ε with no government intervention.The other parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$ and $\sigma_Y = 0.5$.

Finally, we consider the investors' welfare, which is plotted in Figure 4 for different values

of σ_y and σ_ε. In general, investors' welfare decreases with σ_y. This is true even when $\sigma_\varepsilon = 0$, which is Special case 3 and the welfare level is given by (4.31). The reason behind this result is straightforward. Even though the investors can perfectly hedge their idiosyncratic risks through trading in the financial market, the hedging is not costless. Given that the idiosyncratic income risk is perfectly correlated the aggregate income risk, hedging it entails a risk premium, which is captured by $c\,Y$. More hedging involves a higher risk premium, as explained before. This effect persists even when σ_ε is strictly positive. The behavior of welfare becomes more complex for large values of σ_y. In particular, for σ_y close to 0.8, for example, the welfare level initially decreases with σ_ε for small values of σ_ε, but then start to increase with σ_ε when it exceeds 0.5. This behavior is related to the fact as σ_ε becomes sufficiently large, the risk premium, which is proportional to c starts to decrease with σ_ε. A decreasing hedging cost helps to increase the investors' welfare.

5 Government Intervention

We now consider government intervention in the market. We will start with several special cases regarding government's information and then consider the general case.

5.1 Intervention with Public Information

The first special case we consider is when the government has no private information. In this case, the only information the government has is the stock price P. Thus, the government's trading in the market can be a function of P, that is, $\theta_G = \theta_G(P)$.[8] In this case, we do not need to assume θ_G to be linear in P.

Price function. We conjecture that the price function takes the following form:

$$P = b\,F - c\,Y + d\,\theta_G(P) \equiv P' + d\,\theta(P'), \quad P' \equiv b\,F - c\,Y, \tag{5.1}$$

where b, c, d are constants. Given that θ_G is only a function of P or equivalently P', which is public information, it will be known to all investors. Thus, their information set is unaffected by the government's intervention in this case. In particular, $X_i = [\,P' \;\; Y_i \;\; S_i\,]^\top$.

[8] In principle, the government can also intervene with pure noise with no information content. We consider this case in the next section.

Investors' optimal trading policy and value function. Investors' terminal wealth is now:

$$W_i = (\theta_i + \theta_G)(F + u - P) + Y_i\, u, \quad i \in [0,1], \tag{5.2}$$

same as (2.10). Given that θ_G is known to all investors, W_i is normal conditional on investor i's signal X_i. This corresponds to the situation in the general case when $\bar{\theta}_{Gi} = \theta_G$ and $\tilde{\theta}_{Gi} = 0$.

Investor i's optimization problem is formally similar to the maximization problem with no government intervention, which yields the following solution:

$$\theta_i = -\theta_G + \frac{1}{\alpha\sigma^2_{D|X}}\left[(\bar{F}_i - P) - \alpha\sigma_u^2 Y_i\right] = -\theta_G + h_s(\bar{F}_i - P) - h_h Y_i, \quad i \in [0,1], \tag{5.3}$$

where h_s and h_h are given in (4.6). From (5.3) we can see that investor i's optimal trading policy is to achieve her net optimal portfolio by fully offsetting any intervention from the government. This is achievable if the government's trading/position is public information.

We can further compute each investor's value function under her optimal trading policy. In particular, we have:

$$J(W_i, X_i, \theta_G) = -e^{-\alpha\hat{W}_i}, \quad i \in [0,1], \tag{5.4}$$

where

$$\hat{W}_i = \tfrac{1}{2}\,\alpha\left[\sigma^2_{D|X}(\theta_i + \theta_G)^2 - \sigma_u^2 Y_i^2\right] = \tfrac{1}{2}\,\alpha\left\{\sigma^2_{D|X}\left[h_s(\bar{F}_i - P) - h_h Y_i\right]^2 - \sigma_u^2 Y_i^2\right\}. \tag{5.5}$$

This is identical to what investors achieve in the absence of government intervention.

Market equilibrium and optimal intervention policy. From the analysis above, we conclude that when the government has no private information but trades on public information, any of its intervention will be offset by the investors through their own trading to achieve the net optimal portfolio. Thus, the market equilibrium and investor welfare will remain the same, independent of the government's intervention. Consequently, there is no merit for the government to intervene in the market using only public information. We summarize these results in the following proposition:

Proposition 4. *(No Impact Theorem) When the government intervenes with only public information, its impact will be fully offset by investors' own trading, leaving no net effect on the equilibrium price and welfare.*

This result is a form of Ricardo equivalence in the current context under the given conditions. Thus, in what follows we will simply let $h_{GP} = 0$ unless stated otherwise.

5.2 Intervention with Pure Noise

The next special case we examine is when the government has no private information but chooses to intervene nonetheless with pure noise. We assume that the government's trading/position is not public information (otherwise, we return to the previous case). In particular, we have $\beta_{GF} = \beta_{GY} = 0$ and:

$$\theta_G = h\,\varepsilon_G \tag{5.6}$$

where ε_G denotes a pure noise with a volatility of σ_G. According to the formal specification, σ_G can be either σ_{GF} or σ_{GY}. With no loss of generality, we let $\sigma_G \equiv \sigma_{GY}$ with variance $\sigma_G^2 = 1$.

In this case, the government acts as a noise trader, adding random shocks to the market. There are, however, two key differences between the government here and the noise traders in the noisy-rational-expectations models.[9] First, the government's gain/loss from noise trading is ultimately born by the investors, not left outside the model. Second, the government's objective function is the investors' welfare, which it acts strategically to improve. Thus, our model remains closed and general equilibrium in nature even when the government trades with noise.

Given the government's noise trading, the stock price now takes the following form:

$$P = b\,F - c\,Y + d\,\varepsilon_G. \tag{5.7}$$

It is worth noting that since the government knows ε_G, it can also infer $P' \equiv b\,F - c\,Y$ from P. This has two implications. First, the government can extract better information from the price about the aggregate state variables F, Y and ε_G than the investors. Second, the government can implement an intervention policy which also depends on P'. Our purpose here is to examine the impact of the government's pure noise trading, we will assume that it ignores the information P' reveals here to simplify the analysis. In the next two subsections, we will consider in more detail the situation when the government has information that investors do not have.

To further simplify our analysis, we also let $F = 0$ in this subsection. It is worth noting that when $h = 0$, this case reduces to Special Case 3 with no government intervention in Section 4. In this case, $D = u$, the state variables for the economy reduce to Y, u and ε_G. The price function now becomes:

$$P = -c\,Y + d\,\varepsilon_G. \tag{5.8}$$

[9] See **?** and also **?**, **?**, **?**, among others.

Given the intervention policy (5.6) and the price function (5.8), the solution to the market equilibrium can be easily obtained from the general solution laid out in Section 3, which is outlined below.

Market equilibrium

From the price function (5.8), we now have $Z = [Y \quad u]^\top$ and $X = [P \quad Y_i]$. Applying the general solution in Section 3, as shown in the appendix, we arrive at the following solution to market equilibrium. In particular, the investors optimal trading strategy is given by:

$$h_s = \frac{1}{\alpha\sigma_u^2}, \quad h_h = 1, \tag{5.9}$$

with (3.17) and $\bar{F}_i = 0$.

In equilibrium, the price coefficients c and d are given by the following equations:

$$0 = (\sigma_Y^2 + \sigma_y^2)k^3 - h\sigma_y^2 k^2 + \sigma_Y^2\sigma_y^2 k - h\sigma_Y^2\sigma_y^2, \quad k \equiv d/c, \tag{5.10a}$$

$$c = (\alpha\sigma_u^2)\left[1 - \frac{\sigma_y^2 k\, h}{\sigma_Y^2\sigma_y^2 + (\sigma_Y^2+\sigma_y^2)k^2}\right]. \tag{5.10b}$$

(5.10a) is a cubic equation whose general solution is given in the appendix. The investors' stock holdings in equilibrium are:

$$\theta_i = -h\,\varepsilon_G - \left(1 + c^2\sigma_Y^2 kh/|A|\right) y_i, \quad |A| = c^2\sigma_Y^2\sigma_y^2 + d^2(\sigma_Y^2+\sigma_y^2), \quad i \in [0,1]. \tag{5.11}$$

Given the solution to c and d, we can easily compute the investors' welfare from (3.30), which is given in the appendix.

Small interventions

What we consider next is the situation when the government only intervenes in the market marginally, that is, when h is small. In this case, we can obtain solutions to the equilibrium price coefficients and investor welfare level to the lowest (non-trivial) order of h. The results are summarized in the following proposition:

Proposition 5. *For h small, the equilibrium price coefficients are given by:*

$$c = \alpha\sigma_u^2[1 - (1/\sigma_Y^2)h^2] + o(h^2), \quad d = \alpha\sigma_u^2\, h + o(h^2), \tag{5.12}$$

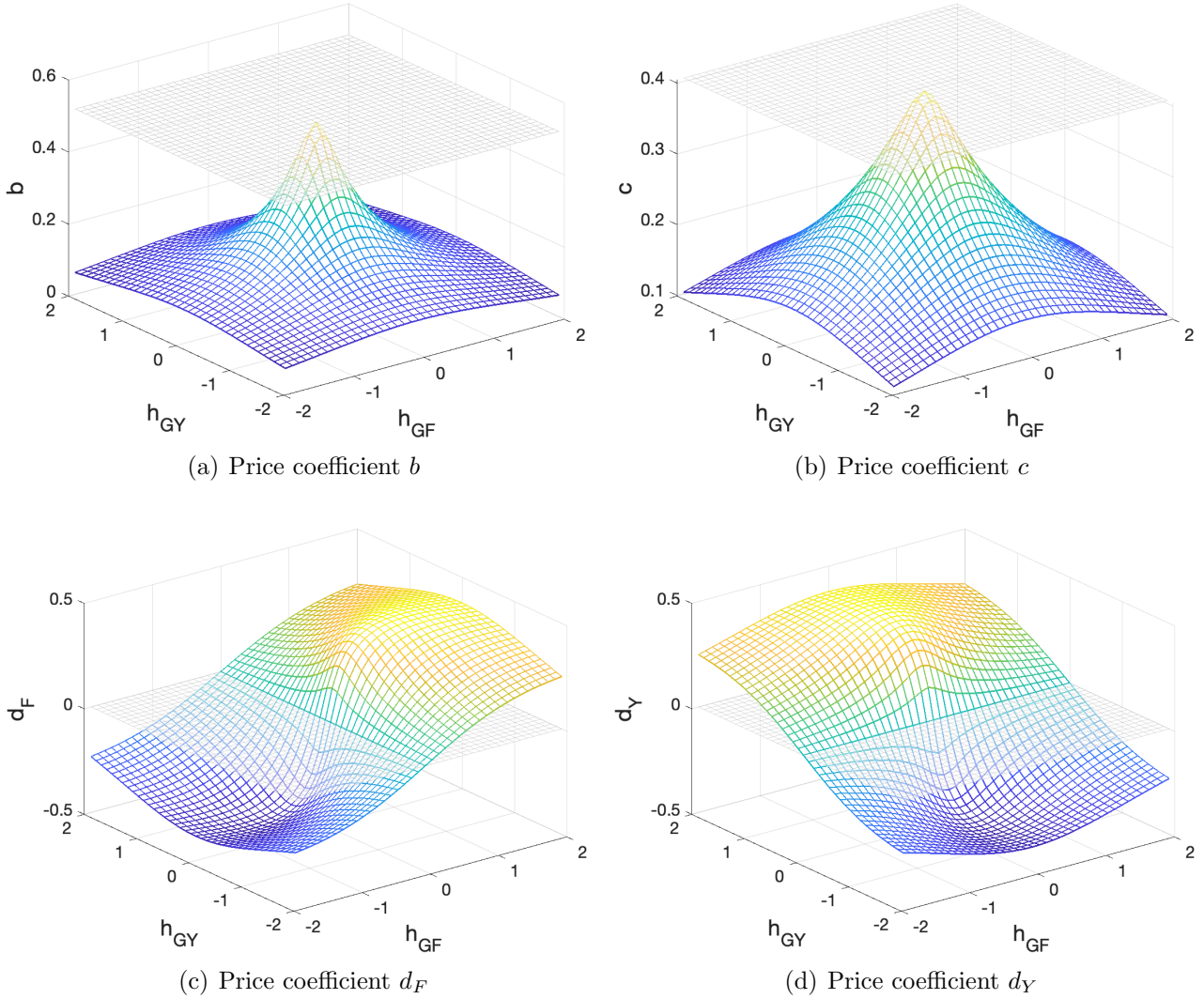
where $o(h^2)$ denotes terms of higher order of h^2. The investors' welfare is given by:

$$V(h) = -\frac{1}{\sqrt{1 - \alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)}}\left[1 - \frac{1}{2}\frac{(\alpha^2\sigma_u^2)^2\sigma_y^2(1+\alpha^2\sigma_u^2\sigma_Y^2)}{1 - \alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)}h^2\right] + o(h^2). \tag{5.13}$$

Proposition 5 shows that for small values of h, the investors' welfare increases with h^2.

Optimal intervention

We now explore the optimal intervention policy when the government trades on noise. From (3.25), we can solve for the equilibrium price coefficients b, c, d_F and d_Y. Keep in mind that in this case, there is perfect symmetry between d_F and d_Y. In addition to the aggregate parameters, we also fix the values of σ_y and σ_ε both at 0.5. Figure 5 plots the four price coefficients against different intervention policies given by h_{GF} and h_{GY}.



(a) Price coefficient b

(b) Price coefficient c

(c) Price coefficient d_F

(d) Price coefficient d_Y

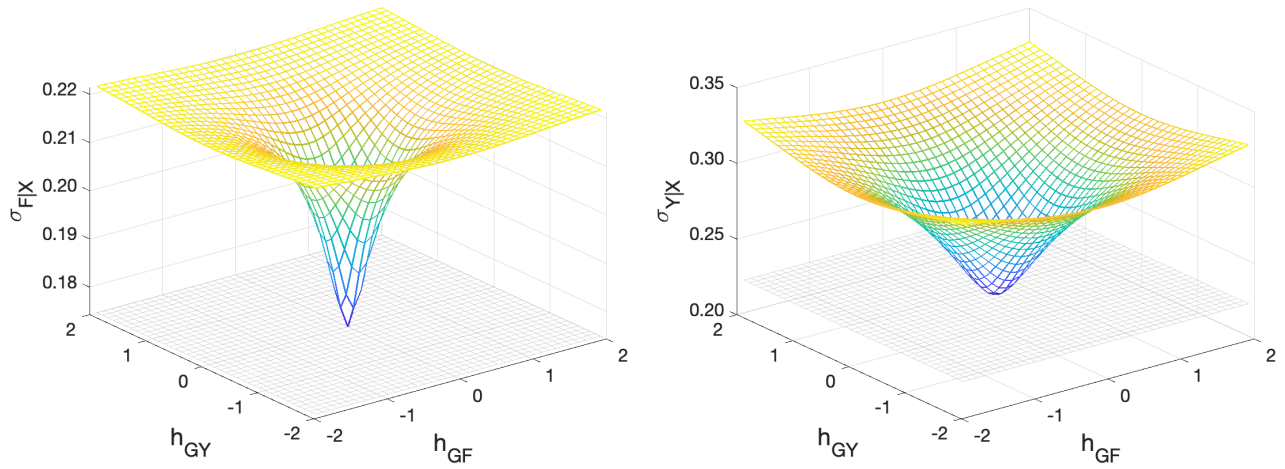
Figure 5: Price coefficients b, c, d_F and d_Y for different values of hGF and h_{GY} under government intervention with pure noise. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 0$, and $\sigma_G = 1$.

In the plots, the light gray plane shows the benchmark value when there is no intervention,

i.e., when $h_{GF} = h_{GY} = 0$. Panel (a) of Figure 5 shows that the price coefficient b decreases monotonically when $|h_{GF}|$ or $|h_{GY}|$ increases. This is not surprising. As the government trades more with noise, it introduces more noise into the price. Consequently, it reduces the information content about the fundamentals F and Y, and the price becomes less dependent on F.

Figure 5(b) shows the same pattern for price coefficient c, which decreases with the magnitude of government trading. As discussed above, the noise trading from the government has the effect of reducing the risk premium associated with aggregate income risk. This can lead to welfare gains for the investors by reducing their cost to hedge idiosyncratic income risks.

Figure 5(c) and (d) plot the price coefficients d_F and d_Y, respectively, for different values of h_{GF} and h_{GY}. Given the symmetry between h_{GF} and h_{GY} when intervention is based only on noise, we only need to consider the case of, say, changing h_{GY} while holding $h_{GF} = 0$. In this case, $d_F = 0$. From Panel (d), we see that d_Y is proportional to h_{GY} for small values of h_{GY}, reflecting the impact of the government's noise trading on the price. But the dependence weakens as h_{GY} becomes large and may eventually diminish. This is because for large values of h_{GY}, the government's trading has a large impact on the stock price, making it more transparent to the investors. As we show in Section 5.1, any known intervention by the government will be countered by the investors' offsetting trading and thus has little impact on the price.



(a) Conditional standard error of F

(b) Conditional standard error of Y

Figure 6: Conditional standard error of F and Y for different values of hGF and h_{GY} under government intervention with pure noise. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 0$, and $\sigma_G = 1$.

Next, we consider the informational efficiency of the market when government intervenes

with noise. Obviously, as the government introduces more noise into the market through its noise trading, the market's informational efficiency worsens. Figure 6 plots the conditional volatility of F and Y for different values of h_{GF} and h_{GY}. Apparently, both $\sigma_{F|X}$ and $\sigma_{Y|X}$ increase with the absolute value of h_{GF} and h_{GY}.

Finally, we consider the investors' welfare. As we have shown earlier, small amount of noise trading by the government can improve investor welfare at the margin. Figure 7 plots the investors welfare for a range of values for h_{GF} and h_{GY}. Again, the gray plane marks the welfare level without intervention, which is at the value of -1.3630. With intervention, the welfare initially increases with $|h_{GF}|$ or $|h_{GY}|$ quadratically, levels off at some optimal magnitude, and then starts to decrease. For large values of $|h_{GF}|$ and $|h_{GY}|$, government's noise trading makes investors worse off than with no intervention. Here, two factors are at work. On the one hand, government's noise trading lowers c as shown in Figure 5(b), which lowers the risk premium investors pay for hedging their income risk and improves welfare. On the other hand, government's noise trading introduces additional risk in investors' terminal wealth/consumption, which lowers welfare. Figure 7 shows that the former dominates for small amount of noise trading, and the latter dominates for large amount of noise trading.

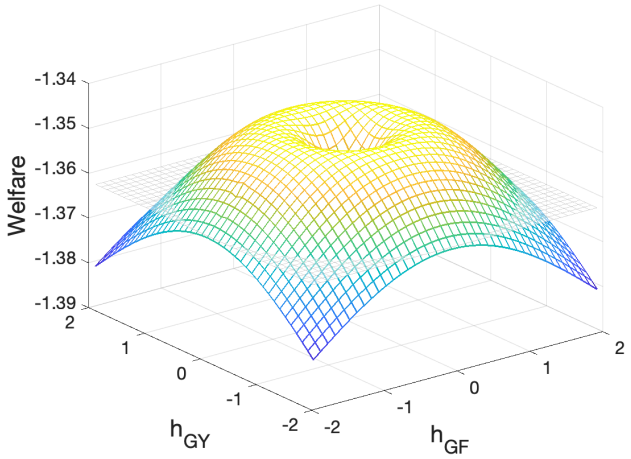


Figure 7: Investor welfare for different values of h_{GF} and h_{GY} under government intervention with pure noise. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 0$, and $\sigma_G = 1$.

For the parameter values given in Figure 7, the maximum welfare is achieved at $|h_G| = \sqrt{h_{GF}^2 + h_{GY}^2} = 0.76$ at the level of -1.3474. This is higher than the welfare level of -1.3630 with no intervention.

5.3 Intervention with Full Information

The next case we consider is when the government has perfect information on the aggregate state of the economy. In particular, we assume that $S_{GF} = F$ and $S_{GY} = Y$, that is, the government observes F and Y and $I_G = \{F, Y\}$. The solution for this case can be obtained from the general solution in Section 3 by letting $\beta_{GF} = \beta_{GY} = 1$ and $\varepsilon_{GF} = \varepsilon_{GY} = 0$, which we do in the appendix. Below we summarize the main results.

Intervention Policy, Price Function, and Conditional Distributions

Given the government's information and the linear class of its trading policy, we have:

$$\theta_G = h_{GF} F + h_{GY} Y. \tag{5.14}$$

With (5.19), the price function takes the following form:

$$P = b\,F - c\,Y, \tag{5.15}$$

where b and c are constants. Although the price function maintains the same functional form as in the case with no government intervention, the coefficients b and c will be dependent on the government's choice of h_{GF} and h_{GY}.

Given the form of the price function, F and Y become linearly dependent conditional on P. Thus, we can remove Y from the state vector Z, which now reduces to $Z = [\,F \;\; u\,]^\top$. The distribution of Z conditional on investor i's signal X_i can be obtained from (3.6) and (3.7). Since Z and X_i in this case take exactly the same form as in the case of no government intervention, as given in (A.26), Σ_{ZZ}, Σ_{ZX}, A, $\Sigma_{ZZ|X}$ and β_{ZX} will take the same form as given in (A.27), (A.28) and (4.3).

Not observing F and Y, investors can decompose the government's trading into the expected part and the unexpected part, based on their signals. In particular,

$$\theta_G = \bar{\theta}_{Gi} + \tilde{\theta}_{Gi} = (h_{GF}\bar{F}_i + h_{GY}\bar{Y}_i) + (h_{GF}\tilde{F}_i + h_{GY}\tilde{Y}_i). \tag{5.16}$$

From (5.15), $P = b\,\bar{F}_i - c\,\bar{Y}_i = b\,F - c\,Y$, which leads to $\bar{Y}_i = (1/c)(b\,\bar{F}_i - P)$ and $\tilde{Y}_i = (b/c)\,\tilde{F}_i$. Hence,

$$\theta_G = \Big[\hat{h}_{GF}\bar{F}_i - h_{GY}(b/c)P\Big] + \hat{h}_{GF}\tilde{F}_i, \quad \hat{h}_{GF} \equiv h_{GF} + (b/c)\,h_{GY}. \tag{5.17}$$

Investors' Optimal Trading Policy, Value Function, and Welfare

Following the general solution in Section 3, we now have:

$$\Sigma_{ZZ|XG} = \left(\Sigma_{ZZ|X}^{-1} + \alpha H_G\right)^{-1}, \quad H_G = \begin{bmatrix} 2\hat{h}_{GF} & \hat{h}_{GF} \\ \hat{h}_{GF} & 0 \end{bmatrix}. \tag{5.18}$$

The investors' optimal trading policy is given by:

$$\theta_i = -\left[\hat{h}_{GF}\bar{F}_i - h_{GY}(b/c)P\right] + h_s(\bar{F}_i - P) - h_h Y_i, \quad i \in [0,1], \tag{5.19}$$

where

$$h_s = \frac{1 - \alpha\hat{h}_{GF}\sigma_{DF|XG}}{\alpha\sigma_{D|XG}^2}, \quad h_h = \frac{\sigma_{Du|XG}}{\sigma_{D|XG}^2},$$

and their value function is given by:

$$J(W_i, X_i, \theta_G) = -e^{-\alpha\hat{W}_i}, \quad \hat{W}_i = \tfrac{1}{2}\,\alpha\, z_i^\top B z_i, \tag{5.20}$$

where B is given in (A.35).

In equilibrium, the stock price coefficients b and c are given by:

$$b = \frac{\beta_{FX}^{(3)} + \left(1 - \beta_{FX}^{(3)}\right)\left(\hat{h}_{GF}/h_s\right)}{1 - \beta_{FX}^{(1)} + \beta_{FX}^{(1)}\left(\hat{h}_{GF}/h_s\right)}, \quad c = \frac{(h_h/h_s) - \beta_{FX}^{(2)} + \beta_{FX}^{(2)}\left(\hat{h}_{GF}/h_s\right)}{1 - \beta_{FX}^{(1)} + \beta_{FX}^{(1)}\left(\hat{h}_{GF}/h_s\right)}, \tag{5.21}$$

and the investors' stock holdings are:

$$\theta_i = -h_{GF}F - h_{GY}Y - h_Y y_i + h_S\varepsilon_i, \quad i \in [0,1]. \tag{5.22}$$

From (3.30), the investors' welfare is given by:

$$V(h_G) = -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\frac{|\Sigma_{zzB}|^{1/2}}{|\Sigma_{zz}|^{1/2}}, \quad \Sigma_{zzB} = (\Sigma_{zz}^{-1} + \alpha^2 B)^{-1}, \tag{5.23}$$

where Σ_{zzB} is given in (3.31) as before. Clearly, investor welfare now depends on the government's intervention policy given by $h_G = [h_{GF} \;\; h_{GY}]$.

Optimal Intervention Policy

With the linear form of the government's intervention policy, the optimal choice is given by (3.33) with the second order condition (3.34).

Instead of pursuing an analytical solution for (3.33), we proceed as follows. First, we show analytically that there are welfare gains from a specific intervention policy, at least for some parameter values. Next, we show that there are in general potential welfare gains for small interventions. Finally, we examine the welfare gain from intervention numerically.

A Specific intervention policy. We now consider a special intervention policy for which the government only trades on Y: $\theta_G = h\,Y$. Furthermore, we look for the choice of h such that in equilibrium, $c = 0$. In this case, the stock price fully reveals F. Without loss of generality, we set $F = 0$. As a result, we have:

$$P = 0, \tag{5.24}$$

$\bar{F}_i = 0$, $\tilde{F}_i = 0$, and $\bar{F}_i - P = 0$. The state vector now becomes $Z = [Y \;\; u]^\top$. In this case, price reveals no information and the only private signal investor i has is $Y_i = Y + y_i$. It is worth noting that in the absence of government intervention, this case reduces to Special Case 3 in Section 4. The solution for this case is provided in the appendix, and we outline the key results here.

From Proposition 1, we have $h_h = 1$. From the equilibrium solution, we have:

$$h^* = \frac{\sigma_Y^2 + \sigma_y^2}{\sigma_y^2} \tag{5.25}$$

in order for the market to clear. In other words, $\theta_G = h^* Y$ leads to the equilibrium for which $P = 0$. The investors' stock holding in equilibrium is:

$$\theta_i = -\left(\frac{\sigma_Y^2}{\sigma_y^2} + 1\right) Y_i, \tag{5.26}$$

and her welfare is:

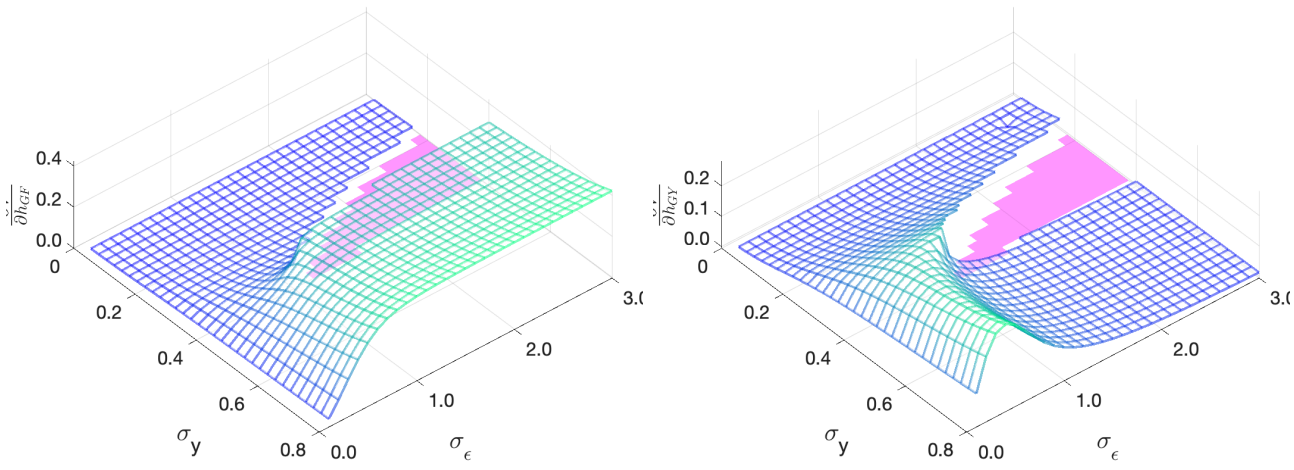
$$V(h^*) = -\frac{1}{\sqrt{1 - \alpha^2 \sigma_u^2 \sigma_Y^2 (\sigma_Y^2 + \sigma_y^2)/\sigma_y^2}}. \tag{5.27}$$

Comparing with the welfare level with no intervention, which is given in (4.27), we immediately conclude that investors' welfare is higher under intervention policy h^* when $\sigma_Y^2 < \alpha^2 \sigma_u^2 \sigma_y^4$.

Small interventions. Next, we consider the case of small interventions, that is, when h_{GF} and h_{GY} are small. In particular, we consider how investor welfare changes when h_{GF} and h_{GY} increases slightly from 0. This is captured by the two partial derivatives of V with respect to h_{GF} and h_{GY} at $h_{GF} = h_{GY} = 0$.

Figure 8 plots $\frac{\partial V}{\partial h_{GF}}(0)$ and $\frac{\partial V}{\partial h_{GY}}(0)$ in panel (a) and (b), respectively, for different values of σ_y and σ_ε. As before, the area marked red in the $\sigma_y - \sigma_\varepsilon$ plane represents the domain where the equilibrium solution is not unique, which is excluded from the plotted surface.

From Panel (a) of Figure 8 we see that $\frac{\partial V}{\partial h_{GF}}(0)$ is in general positive. This means that a small increase of h_{GF} from zero tends to increase welfare. It also means that a small negative



(a) $\frac{\partial V}{\partial h_{GF}}(0)$ (b) $\frac{\partial V}{\partial h_{GY}}(0)$

Figure 8: Derivative of investor welfare with respect to h_{GF} and h_{GY}, respectively, at $h_{GF} = h_{GY} = 0$, for different values of σ_y and σ_ε, excluding the domain marked red where there are multiple solutions to the equilibrium. The other model parameters values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, and $\sigma_Y = 0.5$.

value of h_{GF} tends to decrease welfare. In addition, for small values of σ_y, the welfare gain from a small positive h_{GF} is quite small, independent of the value of σ_ε. However, for large values of σ_y, the corresponding welfare gain can be quite substantial, and more so when the value of σ_ε becomes large. We note that for significant values of σ_y, larger σ_ε implies more information asymmetry about stock payoff F. This is also shown by $\sigma_{F|X}$ plotted in Figure 3. In this case, a positive h_{GF} tends to make the price more dependent on F and thus more informative about it. It will then helps to reduce uncertainty about F and improves welfare.

Panel (b) of Figure 8 shows that $\frac{\partial V}{\partial h_{GY}}(0)$ is also positive in general, implying a positive welfare gain for a small increase of h_{GY} from zero. However, the size of the gain behaves quite differently from $\frac{\partial V}{\partial h_{GF}}(0)$ for different values of σ_y and σ_ε. In particular, for small and medium values of σ_y, $\frac{\partial V}{\partial h_{GY}}(0)$ tends to increase with σ_y. But for large values of σ_y, e.g., exceeding 0.5, it becomes less sensitive to σ_y. In addition, for large values of σ_y, $\frac{\partial V}{\partial h_{GY}}(0)$ increases with σ_ε for small values of σ_ε and then starts to decrease with it. Again, comparing Panel (b) of Figure 8 with Panel (b) of Figure 3 showing $\sigma_{Y|X}$, we find that the two have similar patterns. This suggests when there is more information asymmetry about Y, a positive h_{GY} tends to increase the price's dependence on Y, which helps to reduce the uncertainty about it and increase welfare.

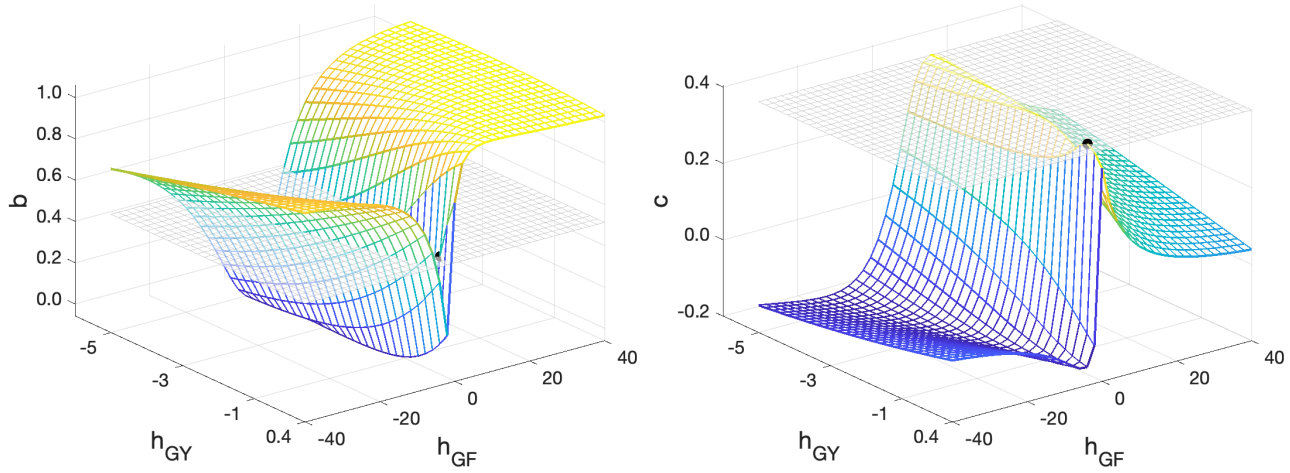
While Figure 8 shows that for small and positive h_{GF} and h_{GY}, investor welfare always improves, the magnitude of the welfare gain is highly sensitive to the values of underlying parameters, in particular, σ_y and σ_ε. This is best illustrated by 8(b) for $\frac{\partial V}{\partial h_{GY}}(0)$. Its value can change substantially with σ_ε for σ_y in the range of 0.4 to 0.8. This suggests that in assessing the welfare impact of an intervention policy, the government needs to be cautious about the details of the market's underlying structure, a point we will return to next.

General intervention policies. Finally, we consider the general intervention policy in the form of (5.14) when the government has full information on the aggregate state of the economy. As pointed out earlier, the existence and uniqueness of the market equilibrium imposes restrictions on the values of h_{GF} and h_{GY}. In particular, $\Sigma^{-1}_{ZZ|XG}$ as defined in (3.16) needs to stay positive definite for the market equilibrium to exist. Hence, we will limit the range of $h_G = [h_{GF} \;\; h_{GY}]$ such that this condition is satisfied and also the equilibrium is unique.[10] In particular, for the parameter values we choose, a sufficient condition is $h_{GY} \leq 0.4$ (for a wide rang of h_{GF}). Hence, we will limit ourselves to this range of h_G.

We first examine how government intervention impacts the equilibrium price. Figure 9 plots the price coefficients b and c, respectively, for different values of h_{GF} and h_{GY}. The black dot in the figure marks the point with $h_{GF} = h_{GY} = 0$, i.e., the value of b and c, respectively, with no government intervention. From Panel (a), we see that when h_{GF} is positive, it has a positive impact on b. This is intuitive. If the government trades positively with F, its action will make the price more dependent on F. When h_{GF} becomes very large, b approaches 1. When h_{GF} is negative, it has a negative impact on b, at least initially, for the same reason as in the case of positive h_{GF}. However, when h_{GF} becomes very negative, it will start to have a positive impact on b. This is because in this case, the government's trading starts to dominate the price and make it more informative about F. The investors rationally incorporates this information into their expectations. From Panel (a), we also see that h_{GY} in general has a positive impact on b. This is also intuitive. Since Y has a negative impact on price, trading positively on Y by the government, i.e., when $h_{GY} > 0$, reduces the impact of Y on the price. This tends to make the price more informative about F, which has the effect of increasing b. Trading negatively on Y,

[10]It is worth noting that for the parameter values we choose, the equilibrium is unique with no government intervention. However, when the government intervenes, multiple equilibrium may appear for certain ranges of h_{GF} and h_{GY}. In this case, we will limit their values to the range with a unique equilibrium. For the results shown in the paper, this situation does not occur.

i.e., $h_{GY} < 0$, has the opposite effect.



(a) Price coefficient b

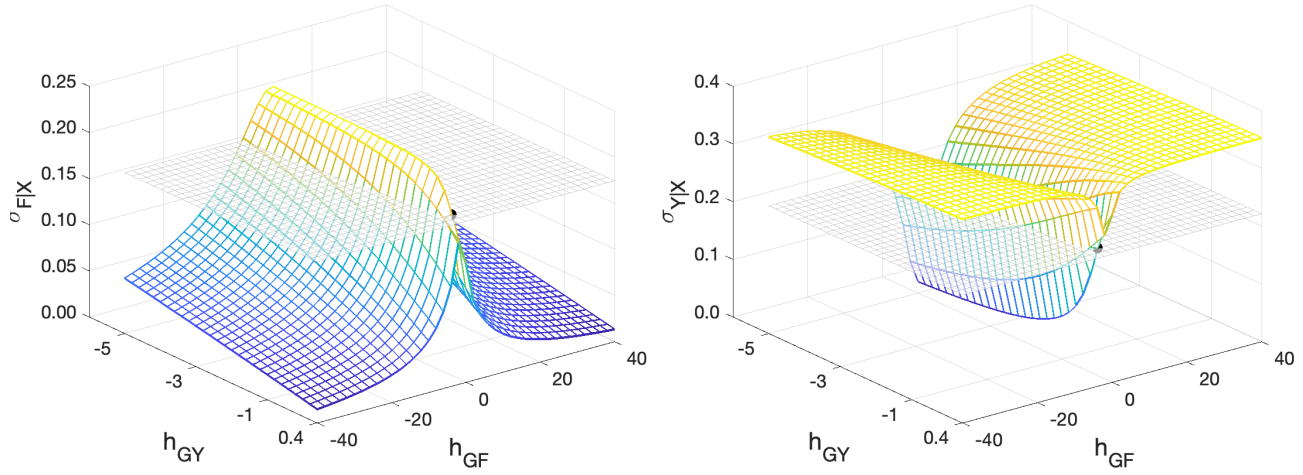
(b) Price coefficient c

Figure 9: Price coefficients b and c for different values of hGF and h_{GY} under government intervention with full information. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 1$, and $\sigma_{GF} = \sigma_{GY} = 0$.

Panel (b) of Figure 9 plots price coefficient c for different values of h_{GF} and h_{GY}. First, we observe that government intervention, independent of its direction, tends to reduce the value of c, relative to its value with no intervention, marked by the black dot. As we have seen in Special Case 3 with no government intervention, c mostly captures the risk premium for the aggregate income risk as given by Y. Thus, government intervention in general has the effect of reducing c, at least for a wide range of h_G around 0.

Figure 10 further illustrates how h_{GF} and h_{GY} influence the conditional volatility of F and Y for the investors in equilibrium. Again, the black dot marks the value with no intervention. For simplicity, we focus on relatively small interventions, that is, when h_{GF} and h_{GY} take on moderate values.

We first consider $\sigma_{F|X}$, which is shown in Figure 10(a). Holding h_{GY} constant, we see that $\sigma_{F|X}$ tends to decrease when h_{GF} becomes positive. This result is intuitive. When h_{GF} becomes positive, b increases, as Figure 9(a) shows, which makes the price more informative about F and $\sigma_{F|X}$ decrease. When h_{GF} becomes negative. We should see the opposite effect: as h_{GF} becomes negative, b decreases and the stock price becomes less dependent on F, which makes $\sigma_{F|X}$ increase. Yet, we see the opposite: $\sigma_{F|X}$ actually decreases. This occurs because when h_{GF} becomes negative, c also decreases and very steeply, as Figure 9(b) shows. This makes the price



(a) Conditional volatility $\sigma_{F|X}$

(b) Contitional volatility $\sigma_{Y|X}$

Figure 10: Conditional volatility of F and Y for different values of h_{GF} and h_{GY} under government intervention with full information. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 1$, and $\sigma_{GF} = \sigma_{GY} = 0$.

less dependent on Y, which makes it more informative about F. When this effect dominates, $\sigma_{F|X}$ also decreases when h_{GF} takes on larger negative values. We thus have the result that holding h_{GY} constant, $\sigma_{F|X}$ decreases with $|h_{GF}|$.

Holding h_{GF} constant, we see that in general $\sigma_{F|X}$ decreases with h_{GY}. When h_{GY} becomes positive, it tends to decrease c, the dependence of price on Y. This makes the price more informative about F and reduces $\sigma_{F|X}$. When h_{GY} becomes negative, we have just the opposite effect.

Interestingly, for $\sigma_{Y|X}$, we see almost the exact opposite impact of h_{GF} and h_{GY}. This is because the stock price, which is a linear function of F and Y, is observable to the investors. Hence, conditional on P, F and Y are linearly dependent. Consequently, $\sigma_{F|X}$ and $\sigma_{Y|X}$ are inversely related to each other. We can rephrase the discussion above about $\sigma_{F|X}$ to explain the behavior of $\sigma_{Y|X}$, which is omitted for brevity.

Finally, we examine the impact of government intervention on investor welfare. Figure 11 plots investors' welfare level for different values of h_{GF} and h_{GY} within the admissible range. The gray plane represents the welfare level with no intervention, i.e., when $h_{GF} = h_{GY} = 0$, which is marked by the black dot in the plot. As shown in Figure 11, for small values of h_{GF} and h_{GY}, welfare is linear in them with a positive slope. That is, for small positive (negative) values of h_{GF} and h_{GY}, the welfare increases (decreases) with the intervention intensity. The plot also

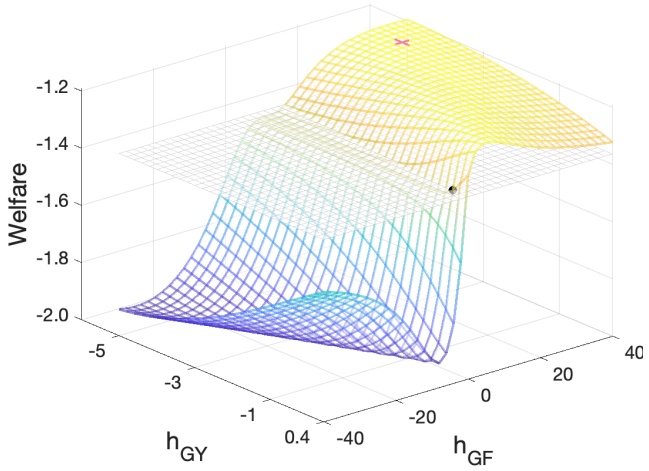


Figure 11: Investor welfare for different values of h_{GF} and h_{GY} under government intervention with full information. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 1$, $\sigma_{GF} = \sigma_{GY} = 0$.

shows that for this set of parameter values, welfare is much more sensitive to h_{GF} than h_{GY}.

Welfare is maximized at $h_{GF} = 30.0$ and $h_{GY} = -4.2$, which is marked by the red cross on the welfare surface. The corresponding welfare value is -1.1926, which is significantly higher than the welfare level of -1.3630 with no intervention. As discussed earlier, the maximum welfare achieved here is only local in the sense that it is within the linear policy class and the range of admissible h_G. What we also observe from Figure 11 is that the sensitivity of welfare to the intervention policy can vary substantially. For example, around the local maximum, welfare is quite insensitive to substantial changes in h_G. On the other hand, for h_{GY} around -0.35 and h_{GF} in the range of 0 to -10, the welfare can change substantially with small changes in h_{GF}. This is consistent with the result we show previously that the welfare impact of government intervention can be highly sensitive to the policy choice.

5.4 Intervention with Noisy Information

Given the general solution provided in Section 3, we can now explore the impact of government intervention on market equilibrium and investor welfare when it has noisy information about the underlying state of the economy. Again, we will limit ourselves to the admissible domain of θ_G allowing a unique market equilibrium.

For brevity, we will only present the impact of government intervention on welfare here. Its impact on price coefficients and conditional volatility of aggregate state variables are similar to that in the case of full information, and are reported in the appendix for completeness.

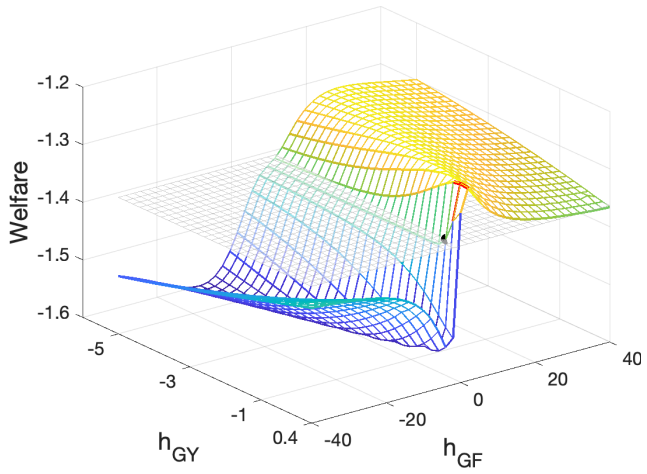


Figure 12: Investor welfare for different values of h_{GF} and h_{GY} under government intervention with noisy information. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 1$, $\sigma_{GF} = \sigma_{GY} = 0.25$.

Compared with the situation when government has full information, we see a similar pattern for the welfare impact of its interventions with moderate values of h_{GF} and h_{GY}. In particular, for small to modest interventions, welfare tends to increase with h_{GF} and h_{GY}. However, when h_{GF} becomes significantly positive, welfare starts to decrease quickly. This is related to the fact that with noisy information, strong interventions also bring more noise into the price and investors' terminal wealth, which can have undesirable welfare implications. The (local) maximum welfare is achieved at $h_{GF} = 2.0$ and $h_{GY} = 0.2$, at the value of -1.2645. Not surprisingly, this welfare level is lower than the maximum welfare level achieved with full information.

6 Evaluating Government Intervention

How to assess the government's intervention in the market through its direct trading has always been a challenging question. It requires a proper cost-benefit analysis with the right objective function. In this subsection, we explore some of the conventional assessment measures. We also develop a measure using our general equilibrium framework.

To demonstrate the key points, we consider two simple trading strategies for the government. The first is a contrarian strategy, buy low and sell high. Formally, it is given by:

$$\theta_G^{Contrarian} = -\,h\,P, \quad h > 0. \tag{6.1}$$

This strategy is reminiscent of government's interventions during market downturns. The second is a noise trading strategy given in (5.6). Neither strategy requires any information advantage for the government. We further assume that $F = 0$ to simplify the analysis. The computational details are provided in the appendix. Below we only present the results.

Payoff. The first measure is the expected payoff on the government's trading strategy. Given the government's trading strategy θ_G, its terminal payoff is:

$$W_G = \theta_G(u - P). \tag{6.2}$$

For the contrarian strategy, its expected payoff is:

$$\mathrm{E}\left[W_G^{Contrarian}\right] = \mathrm{E}\left[\theta_G^{Contrarian}(u - P)\right] = \mathrm{E}\left[-\,h\,P\,(u - P)\right] = h\,c^2\sigma_Y^2 > 0. \tag{6.3}$$

Clearly, in this case, the average payoff on the contrarian strategy is also positive.

Next, we consider the noise-trading strategy. Its expected payoff is given by:

$$\mathrm{E}\left[W_G^{Noise}\right] = \mathrm{E}\left[\theta_G^{Noise}(u - P)\right] = \mathrm{E}\left[h\,\varepsilon_G\,(u - P)\right] = -\,h\,d\,\sigma_G^2 < 0, \tag{6.4}$$

which is always negative, at least for small h.

The difference in the expected payoffs from these two strategies reflects several important aspects of the government's trading in the market. In particular, in the case of contrarian strategy, the government trades only on public information. Thus, the investors can use their own trades to fully offset the government's trades. As a result, the government's trading has no impact on the price. As a contrarian, the government earns a positive expected payoff. However, this payoff is the same for any investor in the market who follows the same strategy. In other words, the government does not earn any extra payoff than what the market offers to all investors.

In the case of noise-trading strategy, the situation is very different. The government trades on noise, which is not public information. Thus, government's noise trading will move the market price, which cannot be fully offset by the investors' offsetting trades, despite their efforts in doing so. Consequently, the government's price impact, which is in the opposite direction of its trades, will lead to negative expected payoffs.

The results above clearly demonstrate that we need to be cautious in assessing the government's trading in the market by simply using the observed market prices. These prices can be influenced by the government's trades. However, whether or not they are depends on how

observable the government's trades are. More importantly, when the government's trading does move prices, using market prices to assess its merit can be misleading. As shown in the case of noise trading by the government, it always loses money but can actually improve welfare, as we show in Section 5.2.

Sharpe ratio. In addition to expected payoff of the government's trading strategy, we can also include its risk in the assessment. A natural candidate is the Sharpe ratio (SR), given by:

$$SR = \frac{\mathrm{E}\,[W_G]}{\sigma[W_G]}, \tag{6.5}$$

where $\sigma[W_G]$ denotes the volatility of W_G. As shown in the appendix, simple computations yield the following:

$$SR^{Contrarian} = \frac{c\,\sigma_Y}{\sqrt{3\,c^2\sigma_Y^2 + \sigma_u^2}}, \qquad SR^{Noise} = -\frac{d\,\sigma_G}{\sqrt{c^2\sigma_Y^2 + 2\,d^2\sigma_G^2 + \sigma_u^2}}. \tag{6.6}$$

For reasons explained above, the Sharpe ratio for the government's contrarian strategy does not depend on its trading and is the same for all market participants. Keep in mind that the equilibrium price coefficients b and c here are not influenced by the government's trading strategy, as shown in Section 5.1. In the case of government's noise trading, however, the situation is different. Here, the government's noise trading, as characterized by h, will influence the price coefficients c and d. As shown in Section 5.2, for small values of h, d will be proportional to h. Thus, not only that the Sharpe ratio is negative, it also increases with the magnitude of the government's trading, at least when it is small.

Certainty equivalence of welfare gain. The ultimate measure of the government's intervention is the welfare gain or loss it generates for the investors. In particular, we define the certainty equivalence, CE, in terms of investor wealth as a measure of their welfare gain or loss. Specifically, we have:

$$V(W + CE(h_G), h_G = 0) = V(W, h_G), \tag{6.7}$$

where $V(W, h_G)$ is the investors' welfare level with wealth W and government trading policy h_G.

From the welfare results obtained in Section 5.1 and 5.2, we can easily compute CE for the two of the government's trading strategies under consideration. For the contrarian strategy, it yields no welfare gain. Hence, its CE will be zero. For the noise-trading strategy, however, there

is a positive welfare gain, yielding a positive CE. Mathematically, we have:

$$CE^{Contrarian} = 0, \quad \text{for all } h; \tag{6.8a}$$

$$CE^{Noise} = \frac{1}{2}\alpha^3\sigma_u^4\sigma_y^2\frac{1+\alpha^2\sigma_u^2\sigma_Y^2}{1-\alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_Y^2)}h^2 + o(h^2), \quad \text{for small } h, \tag{6.8b}$$

where the second equation comes from (5.13). It is interesting to note that despite its positive expected payoff and Sharpe ratio, the contrarian strategy adds no value by the certainty equivalence measure. In contrast, the noise-trading strategy yields a positive certainty equivalence measure despite its negative expected payoff and Sharpe ratio, which is proportional to the square of the trading magnitude h for small values of h.

The analysis above shows that the government's trading in the market cannot be simply evaluated using observed market prices. This is due to the fact that government trading will in general shift market prices, which influences investor welfare. When the government's trades are public information, they can be fully offset by investors' own trading and thus won't impact market prices. In this case, the government's trading can be evaluated using equilibrium market prices. However, in this case the intervention loses its value as it has no effect on investor welfare. When the government's trades are not public information, they will have non-trivial welfare implications. In this case, market prices will be influenced by the government's trading strategy and thus lose their validity in assessing its merits. A proper evaluation measure is based on the strategy's impact on investor welfare, which can be constructed in the context of a general equilibrium framework like ours.

7 Conclusion

In this paper, we have developed a general equilibrium model to examine the impact of government's direct trading in the market on market outcomes. We show that when the market is incomplete in allowing perfect risk sharing among investors, the government's intervention can improve welfare even when it has no additional information. We also show that any intervention based on public information does not alter market outcomes since investors can trade to offset the distortion caused by the government's trades. When the government has additional information that investors do not have, the impact of its interventions on market outcomes and investor welfare is quite sensitive to the details of the economy and the design of the intervention policy. We also show that when the government plays a monopolistic role in the market, using market

prices to assess the its intervention policies can be incomplete and misleading.

As stated in the introduction, the main objective here is to construct a parsimonious but comprehensive framework that allows us to illustrate the potential role for government intervention and its market implications. Thus, for tractability and clarity, we have made simplifying assumptions regarding the structure of the economy and the market, the objective of the government and its information and intervention policies. As discussed in Section 2.8, many of these assumptions can be relaxed if the focus turns to specific imperfections of the market and the corresponding government intervention policies. We will leave these explorations for future research.

Appendix

A.1 Parameter constraint (2.12)

Under normality, the non-traded income given in (2.2) is unbounded, which can yield extreme values for the utility function. In particular, we have:

$$W_i = Y_i\, u \tag{A.1}$$

and

$$\mathrm{E}\left[u_i(W_i)|Y_i\right] = -e^{\frac{1}{2}\alpha^2 \mathrm{V}\left[Y_i u|Y_i\right]} = -e^{\frac{1}{2}\alpha^2\sigma_u^2 Y_i^2}. \tag{A.2}$$

The expected value of the utility function above (over Y_i) is bounded only if:

$$\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2) < 1,$$

which is (2.12). Under this condition, (2.13) follows.

A.2 Solution to Market Equilibrium with Government Intervention

Conditional Distributions.

Given the price function (3.1), X_i in (3.2) and Z in (3.4), (3.5) immediately follows with:

$$A = \begin{bmatrix} \sigma_P^2 & -c\,\sigma_Y^2 & b\,\sigma_F^2 \\ -c\,\sigma_Y^2 & \sigma_Y^2+\sigma_y^2 & 0 \\ b\,\sigma_F^2 & 0 & \sigma_F^2+\sigma_\varepsilon^2 \end{bmatrix}, \quad \sigma_P^2 \equiv b^2\sigma_F^2 + c^2\sigma_Y^2 + d_F^2\sigma_{\varepsilon_{GF}}^2 + d_Y^2\sigma_{\varepsilon_{GY}}^2, \tag{A.3a}$$

$$|A| = b^2\sigma_F^2(\sigma_Y^2+\sigma_y^2)\sigma_\varepsilon^2 + c^2\sigma_Y^2\sigma_y^2(\sigma_F^2+\sigma_\varepsilon^2) + \sigma_{\varepsilon_P}^2(\sigma_Y^2+\sigma_y^2)(\sigma_F^2+\sigma_\varepsilon^2), \tag{A.3b}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} (\sigma_Y^2+\sigma_y^2)(\sigma_F^2+\sigma_\varepsilon^2) & c\,\sigma_Y^2(\sigma_F^2+\sigma_\varepsilon^2) & -b\,\sigma_F^2(\sigma_Y^2+\sigma_y^2) \\ c\,\sigma_Y^2(\sigma_F^2+\sigma_\varepsilon^2) & \sigma_{P\setminus F}^2(\sigma_F^2+\sigma_\varepsilon^2)+b^2\sigma_F^2\sigma_\varepsilon^2 & -b\,c\,\sigma_F^2\sigma_Y^2 \\ -b\,\sigma_F^2(\sigma_Y^2+\sigma_y^2) & -b\,c\,\sigma_F^2\sigma_Y^2 & \sigma_{P\setminus Y}^2(\sigma_Y^2+\sigma_y^2)+c^2\sigma_Y^2\sigma_y^2 \end{bmatrix}, \tag{A.3c}$$

where $\sigma_{P\setminus F}^2 \equiv \sigma_P^2 - b^2\sigma_F^2$, $\sigma_{P\setminus Y}^2 \equiv \sigma_P^2 - c^2\sigma_Y^2$, and $\sigma_{\varepsilon_P}^2 \equiv d_F^2\sigma_{\varepsilon_{GF}}^2 + d_Y^2\sigma_{\varepsilon_{GY}}^2$, and

$$\Sigma_{ZX} = \begin{bmatrix} b\,\sigma_F^2 & 0 & \sigma_F^2 \\ -c\,\sigma_Y^2 & \sigma_Y^2 & 0 \\ 0 & 0 & 0 \\ d_F\sigma_{\varepsilon_{GF}}^2 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \sigma_{FX} \\ \sigma_{YX} \\ \sigma_{uX} \\ \sigma_{\varepsilon_{GF}X} \end{bmatrix}. \tag{A.4}$$

The following lemma, which is a standard result about joint normal distributions, will be

repeatedly used in our computations:

Lemma A.1. *Let x be an N-dimensional normal random variable:*

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right). \tag{A.5}$$

Then, conditional on x_2, x_1 remains normal with:

$$\mathrm{E}\left[x_1|x_2\right] = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2), \quad \mathrm{C}\left[x_1, x_1|x_2\right] = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \equiv \Sigma_{11|2}, \tag{A.6}$$

where $\mathrm{C}\left[x_1, x_1|x_2\right]$ denotes the covariance matrix of x_1 conditional on x_2.

Given the joint normality of Z and X_i, $i \in [0,1]$, Lemma 1 follows from Lemma A.1 immediately. For convenience, we further define:

$$\beta_{ZX} = \Sigma_{ZZ}A^{-1} == \begin{bmatrix} \sigma_{FX}A^{-1} \\ \sigma_{YX}A^{-1} \\ \sigma_{uX}A^{-1} \\ \sigma_{\varepsilon_{GF}X}A^{-1} \end{bmatrix} \equiv \begin{bmatrix} \beta_{FX} \\ \beta_{YX} \\ \beta_{uX} \\ \beta_{\varepsilon_{GF}X} \end{bmatrix} \equiv \begin{bmatrix} \beta_{FX}^{(1)} & \beta_{FX}^{(2)} & \beta_{FX}^{(3)} \\ \beta_{YX}^{(1)} & \beta_{YX}^{(2)} & \beta_{YX}^{(3)} \\ \beta_{uX}^{(1)} & \beta_{uX}^{(2)} & \beta_{uX}^{(3)} \\ \beta_{\varepsilon_{GF}X}^{(1)} & \beta_{\varepsilon_{GF}X}^{(2)} & \beta_{\varepsilon_{GF}X}^{(3)} \end{bmatrix}. \tag{A.7}$$

Since $\sigma_{uX} = 0$, we also have $\beta_{uX} = 0$. In addition, for (3.7), we have:

$$\Sigma_{ZZ|X} = \Sigma_{ZZ} - \Sigma_{ZX}A^{-1}\Sigma_{XZ} \equiv \begin{bmatrix} \sigma_{F|X}^2 & \sigma_{FY|X} & 0 & \sigma_{F\varepsilon_{GF}|X} \\ \sigma_{FY|X} & \sigma_{Y|X}^2 & 0 & \sigma_{Y\varepsilon_{GF}|X} \\ 0 & 0 & \sigma_{u|X}^2 & 0 \\ \sigma_{F\varepsilon_{GF}|X} & \sigma_{Y\varepsilon_{GF}|X} & 0 & \sigma_{\varepsilon_{GF}|X}^2 \end{bmatrix}, \tag{A.8}$$

where the conditional variance and covariance of F, Y, u and ε_{GF} given X_i are given by:

$$\sigma_{F|X}^2 = \sigma_F^2 - \sigma_{FX}A^{-1}\sigma_{XF}, \quad \sigma_{Y|X}^2 = \sigma_Y^2 - \sigma_{YX}A^{-1}\sigma_{XY}, \quad \sigma_{u|X}^2 = \sigma_u^2, \tag{A.9a}$$

$$\sigma_{\varepsilon_{GF}|X}^2 = \sigma_{\varepsilon_{GF}}^2 - \sigma_{\varepsilon_{GF}X}A^{-1}\sigma_{X\varepsilon_{GF}}, \tag{A.9b}$$

$$\sigma_{FY|X} = -\sigma_{FX}A^{-1}\sigma_{XY}, \quad \sigma_{Fu|X} = 0, \quad \sigma_{F\varepsilon_{GF}|X} = -\sigma_{FX}A^{-1}\sigma_{X\varepsilon_{GF}}, \tag{A.9c}$$

$$\sigma_{Yu|X} = 0, \quad \sigma_{Y\varepsilon_{GF}|X} = -\sigma_{YX}A^{-1}\sigma_{X\varepsilon_{GF}}, \quad \sigma_{u\,\varepsilon_{GF}|X} = 0. \tag{A.9d}$$

Given $\tilde{Z}_i$ defined in (3.8), which is a 4-dimensional normal random variable with zero mean and covariance matrix $\Sigma_{ZZ|X}$, we have:

$$\mathrm{E}\left[F|X_i\right] = \bar{F}_i = \beta_{FX}X_i, \quad \mathrm{E}\left[Y|X_i\right] = \bar{Y}_i = \beta_{YX}X_i, \tag{A.10a}$$

$$\mathrm{E}\left[u|X_i\right] = \bar{u}_i = 0, \ \mathrm{E}\left[\varepsilon_{GF}|X_i\right] = \bar{\varepsilon}_{GF} = \beta_{\varepsilon_{GF}X}X_i, \tag{A.10b}$$

$$\mathrm{V}\left[D|X_i\right] = \mathrm{V}\left[F+u|X_i\right] \equiv \sigma_{D|X}^2 = \sigma_F^2 - \sigma_{FX}A^{-1}\sigma_{XF} + \sigma_u^2, \quad i \in [0,1], \tag{A.10c}$$

where $\sigma^2_{D|X}$ denotes the variance of $D = F + u$ conditional on X_i, which is independent of i given the symmetry between investors, and

$$\beta_{FX} = \sigma_F^2[b\,(\sigma_Y^2+\sigma_y^2)\sigma_\varepsilon^2 \quad b\,c\,\sigma_Y^2\sigma_\varepsilon^2 \quad \sigma^2_{P\setminus F}\sigma_y^2+\sigma^2_{\varepsilon_P}\sigma_Y^2], \tag{A.11a}$$

$$\beta_{YX} = \sigma_Y^2[-c\,\sigma_y^2(\sigma_F^2+\sigma_\varepsilon^2) \quad \sigma^2_{P\setminus Y}\sigma_\varepsilon^2+\sigma^2_{\varepsilon_P}\sigma_F^2 \quad b\,c\,\sigma_F^2\sigma_y^2], \tag{A.11b}$$

$$\beta_{\varepsilon_{GF}X} = d_F\sigma^2_{\varepsilon_{GF}}[(\sigma_Y^2+\sigma_y^2)(\sigma_F^2+\sigma_\varepsilon^2) \quad c\,\sigma_Y^2(\sigma_F^2+\sigma_\varepsilon^2) \quad -b\,(\sigma_Y^2+\sigma_y^2)\sigma_F^2]. \tag{A.11c}$$

Investors' Optimal Trading Policy.

From (2.7) and (3.3), we have:

$$\begin{aligned}\theta_G &= h_{GP}P + h_{GF}\beta_{GF}F + h_{GY}\beta_{GY}Y + h_{GF}\varepsilon_{GF} + h_{GY}d_Y^{-1}[P-(b\,F-c\,Y+d_F\varepsilon_{GF})]\\ &= h'_{GP}P + h'_{GF}F + h'_{GY}Y + h'_{G\varepsilon_{GF}}\varepsilon_{GF},\end{aligned} \tag{A.12}$$

which gives (3.9).

Lemma 2 follows directly from Lemma A.1.

Given investor i's wealth, $i \in [0,1]$, in (3.14), for (3.15a) we further have:

$$H_G \equiv h'^{\top}_{DZ}h'_{GZ} + h'^{\top}_{GZ}h'_{DZ} = \begin{bmatrix} 2h'_{GF} & h'_{GY} & h'_{GF} & h'_{G\varepsilon_{GF}} \\ h'_{GY} & 0 & h'_{GY} & 0 \\ h'_{GF} & h'_{GY} & 0 & h'_{G\varepsilon_{GF}} \\ h'_{G\varepsilon_{GF}} & 0 & h'_{G\varepsilon_{GF}} & 0 \end{bmatrix}. \tag{A.13}$$

We can rewrite (3.14) as:

$$W_i = W'_i + \tfrac{1}{2}\tilde{Z}_i^{\top}H_G\tilde{Z}_i = \bar{W}'_i + \tilde{W}'_i + \tfrac{1}{2}\tilde{Z}_i^{\top}H_G\tilde{Z}_i, \tag{A.14}$$

where

$$W'_i \equiv (\theta_i + \bar{\theta}_{Gi})(\bar{F}_i - P) + (\theta_i + \bar{\theta}_{Gi})\tilde{D}_i + (\bar{F}_i - P)h'_{GZ}\tilde{Z}_i + Y_i\,u, \tag{A.15a}$$

$$\bar{W}'_i \equiv \mathrm{E}\,[W'_i|X_i] = (\theta_i + \bar{\theta}_{Gi})(\bar{F}_i - P), \tag{A.15b}$$

$$\tilde{W}'_i \equiv W'_i - \bar{W}'_i = (\theta_i + \bar{\theta}_{Gi})\tilde{D}_i + (\bar{F}_i - P)h'_{GZ}\tilde{Z}_i + Y_i\,u. \tag{A.15c}$$

Note that conditional on X_i, $\tilde{W}'_i$ is normal with mean zero.

We now solve for the investors' optimal trading policy. For $\Sigma^{-1}_{ZZ|X} + \alpha H_G$ positive definite, which will be confirmed with parameter restrictions, we define $\Sigma_{ZZ|XG}$ as in (3.16), which is also positive definite (and symmetric). It can be interpreted as a modified covariance matrix from $\Sigma_{ZZ|X}$ with αH_G.

Let $|\cdot|$ denote the determinant of a matrix. The normality of $\tilde{W}_i'$ yields the following lemma:

Lemma A.2. *Given investor i's trading policy θ_i, her expected utility is:*

$$\begin{aligned}\mathrm{E}\left[-e^{-\alpha W_i}\middle|X_i\right] &= -\mathrm{E}\left[e^{-\alpha W_i' - \frac{1}{2}\alpha \tilde{Z}_i^\top H_G \tilde{Z}_i}\middle| X_i; \Sigma_{ZZ|X}\right] \\ &= -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\ \mathrm{E}\left[e^{-\alpha \bar{W}_i' - \alpha \tilde{W}_i'}\middle| X_i; \Sigma_{ZZ|XG}\right] \equiv -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}}\ e^{-\alpha \hat{W}_i},\end{aligned} \tag{A.16}$$

where $\mathrm{E}[\cdot|X_i;\Sigma]$ denotes the expectation conditional on X_i with covariance matrix Σ, and

$$\hat{W}_i \equiv \bar{W}_i' - \tfrac{1}{2}\,\alpha \mathrm{V}\left[\tilde{W}_i'|X_i;\Sigma_{ZZ|XG}\right]. \tag{A.17}$$

From Lemma A.2, the investor's optimization problem reduces to:

$$\begin{aligned}\max_{\theta_i}\ \hat{W}_i = (\theta_i + \bar{\theta}_{Gi})(\bar{F}_i - P) - \tfrac{1}{2}\,\alpha\,\big[&(\theta_i+\bar{\theta}_{Gi})^2\sigma^2_{D|XG} + 2\,(\theta_i+\bar{\theta}_{Gi})(\bar{F}_i-P)h_{GZ}'\sigma_{DZ|XG} \\ &+ 2\,(\theta_i+\bar{\theta}_{Gi})Y_i\,\sigma_{Du|XG} + (\bar{F}_i-P)^2 h_{GZ}'\Sigma_{ZZ|XG}h_{GZ}^\top + Y_i^2\sigma^2_{u|XG} \\ &+ 2\,(\bar{F}_i-P)Y_i\,h_{GZ}'\sigma_{uZ|XG}\big],\end{aligned} \tag{A.18}$$

where

$$\sigma_{DZ|XG} \equiv \left[\sigma_{DF|XG}\ \ \sigma_{DY|XG}\ \ \sigma_{Du|XG}\ \ \sigma_{D\varepsilon_{GF}|XG}\right]^\top, \tag{A.19a}$$

$$\sigma_{uZ|XG} \equiv \left[\sigma_{Fu|XG}\ \ \sigma_{Yu|XG}\ \ \sigma^2_{u|XG}\ \ \sigma_{u\varepsilon_{GF}|XG}\right]^\top. \tag{A.19b}$$

The first order condition gives the following solution for the optimal trading policy:

$$\theta_i = -\bar{\theta}_{Gi} + h_s(\bar{F}_i - P) - h_h Y_i, \tag{A.20}$$

where

$$h_s \equiv \frac{1}{\alpha\sigma^2_{D|XG}}(1 - \alpha h_{GZ}'\sigma_{DZ|XG}), \quad h_h \equiv \frac{\sigma_{Du|XG}}{\sigma^2_{D|XG}}, \tag{A.21}$$

which are reproduced in (3.17) and (3.18) in Proposition 1.

Price, Stock Holdings and Welfare in Equilibrium.

From the market-clearing condition (3.23), we have:

$$\begin{aligned}0 = &\left[(-h_P+h_{GP})\,b + (h_S+h_{GF}\beta_{GF})\right]F - \left[(-h_P+h_{GP})\,c + (h_Y - h_{GY}\beta_{GY})\right]Y \\ &+ \left[(-h_P+h_{GP})\,d_F + h_{GF}\right]\varepsilon_{GF} + \left[(-h_P+h_{GP})\,d_Y + h_{GY}\right]\varepsilon_{GY},\end{aligned} \tag{A.22}$$

which leads to (3.24). Proposition 2 immediately follows.

With the state variables included in z_i for investor i, we can write the solution to the equilibrium in the form of (3.28) and (3.29) with:

$$B_P \equiv [\, b \;\; -c \;\; 0 \;\; 0 \;\; d_F \;\; d_Y \,], \tag{A.23a}$$

$$B_G \equiv h_{GP} B_P + [\, h_{GF}\beta_{GF} \;\; h_{GY}\beta_{GY} \;\; 0 \;\; 0 \;\; h_{GF} \;\; h_{GY} \,], \tag{A.23b}$$

$$B_Y \equiv [\, 0 \;\; 1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \,], \tag{A.23c}$$

$$B_F \equiv \Big[\, \beta_{FX}^{(1)} b + \beta_{FX}^{(3)} \;\; -\beta_{FX}^{(1)} c + \beta_{FX}^{(2)} \;\; \beta_{FX}^{(2)} \;\; \beta_{FX}^{(3)} \;\; \beta_{FX}^{(1)} d_F \;\; \beta_{FX}^{(1)} d_Y \,\Big], \tag{A.23d}$$

$$B_\theta \equiv h_s (B_F - B_P) - h_h B_Y, \tag{A.23e}$$

and

$$\begin{aligned} B \equiv\; & \sigma_{D|XG}^2 B_\theta^\top B_\theta - \sigma_{u|XG}^2 B_Y^\top B_Y - (h'_{GZ} \Sigma_{ZZ|XG} h_{GZ}'^\top)(B_F - B_P)^\top (B_F - B_P) \\ & - (h'_{GZ} \sigma_{uZ|XG}) \left[(B_F - B_P)^\top B_Y + (B_F - B_P)^\top B_Y \right] \\ =\; & \left(\sigma_{D|XG}^2 h_s^2 - h'_{GZ} \Sigma_{ZZ|XG} h_{GZ}'^\top \right) (B_F - B_P)^\top (B_F - B_P) - \left(\sigma_{u|XG}^2 - \sigma_{D|XG}^2 h_h^2 \right) B_Y^\top B_Y \\ & - \left(\sigma_{D|XG}^2 h_s h_h + h'_{GZ} \sigma_{uZ|XG} \right) \left[(B_F - B_P)^\top B_Y + (B_F - B_P)^\top B_Y \right]. \end{aligned} \tag{A.24}$$

Finally, we compute the investors' ex ante expected utility under government's intervention policy (2.7). In particular, we have:

$$\begin{aligned} V(\theta_G) = V(h_G) &= - \frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}} \mathrm{E}\left[e^{-\alpha \hat{W}_i} \right] = - \frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}} \mathrm{E}\left[e^{-\frac{1}{2}\alpha^2 z_i^\top B z_i} \right] \\ &= - \frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}} \frac{|\Sigma_{zzB}|^{1/2}}{|\Sigma_{zz}|^{1/2}}, \quad i \in [0,1], \end{aligned} \tag{A.25}$$

where Σ_{zzB} is given in (3.31).

A.3 Solution to Market Equilibrium with No Government Intervention

With no government intervention, the price function is given by (4.1) and we have:

$$Z = \begin{bmatrix} F \\ u \end{bmatrix}, \quad \begin{bmatrix} Z \\ X_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{ZZ} & \Sigma_{ZX} \\ \Sigma_{XZ} & A \end{bmatrix} \right), \tag{A.26}$$

where X_i is given in (3.2) and

$$\Sigma_{ZZ} = \begin{bmatrix} \sigma_F^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}, \quad \Sigma_{ZX} = \begin{bmatrix} b\,\sigma_F^2 & 0 & \sigma_F^2 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} b^2\sigma_F^2 + c^2\sigma_Y^2 & -c\,\sigma_Y^2 & b\,\sigma_F^2 \\ -c\,\sigma_Y^2 & \sigma_Y^2 + \sigma_y^2 & 0 \\ b\,\sigma_F^2 & 0 & \sigma_F^2 + \sigma_\varepsilon^2 \end{bmatrix}. \tag{A.27}$$

(3.6) and (3.7) now become:

$$Z|X_i \sim N\left(\bar{Z}_i, \Sigma_{ZZ|X}\right), \quad \Sigma_{ZZ|X} = \begin{bmatrix} \sigma_{F|X}^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}, \tag{A.28}$$

and (4.2), (4.3) follow immediately. From (A.27), we further have:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} (\sigma_Y^2+\sigma_y^2)(\sigma_F^2+\sigma_\varepsilon^2) & c\,\sigma_Y^2(\sigma_F^2+\sigma_\varepsilon^2) & -b\,\sigma_F^2(\sigma_Y^2+\sigma_y^2) \\ c\,\sigma_Y^2(\sigma_F^2+\sigma_\varepsilon^2) & b^2\sigma_F^2\sigma_\varepsilon^2+c^2\sigma_Y^2(\sigma_F^2+\sigma_\varepsilon^2) & -b\,c\,\sigma_F^2\sigma_Y^2 \\ -b\,\sigma_F^2(\sigma_Y^2+\sigma_y^2) & -b\,c\,\sigma_F^2\sigma_Y^2 & b^2\sigma_F^2(\sigma_Y^2+\sigma_y^2)+c^2\sigma_Y^2\sigma_y^2 \end{bmatrix}, \tag{A.29}$$

Investors' optimal trading policy. Investors' optimization problem in (A.18) becomes:

$$\max_{\theta_i} \ \hat{W}_i \ = \ \theta_i(\bar{F}_i - P) - \tfrac{1}{2}\alpha\,\left(\theta_i^2\sigma_{D|X}^2 + 2\,\theta_i Y_i\,\sigma_u^2 + Y_i^2\sigma_u^2\right), \tag{A.30}$$

where we have substituted in $\sigma_{Du|X} = \sigma_u^2$. Its solution, given in (4.6), follows immediately. With $\sigma_{D|X}^2 = \sigma_{F|X}^2 + \sigma_u^2 \geq \sigma_u^2$ and $\sigma_u > 0$, we further have:

$$0 < h_s \leq \frac{1}{\alpha\sigma_u^2}, \quad 0 < h_h \leq 1.$$

Equilibrium price coefficients. Equation (4.11) for k is a cubic equation. For its solution, we introduce the following lemma for the roots of a cubic equation.

Lemma A.3. *For a cubic equation of the following canonical form:*

$$a_3x^2 + a_2x^2 + a_1x + a_0 = 0, \tag{A.31}$$

define:

$$p \equiv \frac{a_1}{3a_3} - \frac{a_2^2}{9a_3^3}, \quad q \equiv \frac{a_2a_1}{6a_3^2} - \frac{a_2^2}{27a_3^3} - \frac{a_0}{2a_3}, \quad r \equiv -\frac{a_2}{3a_3}, \tag{A.32a}$$

$$\Delta \equiv q^2 + p^3, \quad \omega \equiv \frac{-1+\sqrt{3}i}{2}, \tag{A.32b}$$

where Δ is the discriminant of the cubic equation and ω is a complex cubic root of -1. (A.31) then has three roots given by:

$$x_1 = \sqrt[3]{-q+\sqrt{q^2+p^3}} + \sqrt[3]{-q-\sqrt{q^2+p^3}} + r\,, \tag{A.33a}$$

$$x_2 = \omega\sqrt[3]{-q+\sqrt{q^2+p^3}} + \omega^2\sqrt[3]{-q-\sqrt{q^2+p^3}} + r\,, \tag{A.33b}$$

$$x_3 = \omega^2\sqrt[3]{-q+\sqrt{q^2+p^3}} + \omega\sqrt[3]{-q-\sqrt{q^2+p^3}} + r\,. \tag{A.33c}$$

Moreover, when $\Delta > 0$, there is only one real root, when $\Delta < 0$, there are three distinct real roots, and when $\Delta = 0$, there are multiple real roots.

Applying Lemma A.3 to (4.11), we immediately obtain the three roots for k with:

$$a_3 = \alpha\sigma_u^2(\sigma_Y^2+\sigma_y^2)\sigma_\varepsilon^2, \quad a_2 = -\sigma_Y^2\sigma_\varepsilon^2, \quad a_1 = \alpha\sigma_u^2\sigma_Y^2\sigma_y^2(\sigma_S^2/\sigma_F^2), \quad a_0 = -\sigma_Y^2\sigma_y^2. \tag{A.34}$$

Investor welfare. From the general solution to the market equilibrium given by (3.28)–(3.29), with no government intervention, we immediately have (4.15) with:

$$\begin{aligned} &B_P = [\,b \;\; -c \;\; 0 \;\; 0\,], \quad B_Y = [\,0 \;\; 1 \;\; 1 \;\; 0\,], \\ &B_F = \left[\,\beta_{FX}^{(1)}b+\beta_{FX}^{(3)} \;\; -\beta_{FX}^{(1)}c+\beta_{FX}^{(2)} \;\; \beta_{FX}^{(2)} \;\; \beta_{FX}^{(3)}\,\right], \quad B_\theta = h_s(B_F-B_P)-h_hB_Y. \end{aligned} \tag{A.35}$$

Given (A.35) and B in (4.16), we can take more steps in computing V_0. First, note that $|\Sigma_{zzB}| = |\Sigma_{zzB}^{-1}|^{-1}$. Next,

$$\Sigma_{zzB}^{-1} = \Sigma_{zz}^{-1}+\alpha^2 B = \Sigma_{zzY}^{-1}+\alpha^2\sigma_{D|X}^2B_\theta^\top B_\theta, \quad \Sigma_{zzY}^{-1} \equiv \Sigma_{zz}^{-1}-\alpha^2\sigma_u^2B_Y^\top B_Y. \tag{A.36}$$

In order to compute $|\Sigma_{zzB}|$, we introduce the following lemma:

Lemma A.4. *Let M be an $(n\times n)$ invertible matrix and u, v be $(n\times 1)$ column vectors. Then,*

$$|M+u\,v^\top| = |M|(1+v^\top M^{-1}u).$$

Proof of Lemma A.4. We first consider the special case when $M=I$, where I is the identity matrix. Consider the following identity:

$$\begin{bmatrix} I & 0 \\ v^\top & 1 \end{bmatrix}\begin{bmatrix} I+uv^\top & u \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I & 0 \\ -v^\top & 1 \end{bmatrix} = \begin{bmatrix} I & u \\ 0 & 1+v^\top u \end{bmatrix}.$$

The determinant of the left hand side is the product of the determinants of the three matrices on the the right hand side. Since all the matrices are triangular matrices, we immediately have:

$$|I+uv^\top| = 1+v^\top u,$$

which is what the lemma says in this case. For the general case, we have:

$$|M+uv^\top| = |M||I+(M^{-1}u)v^\top| = |M|(1+v^\top M^{-1}u). \qquad \square$$

We now apply Lemma A.4 to our welfare computations in this case. First, we have:

$$\Sigma_{zzY}^{-1} = \Sigma_{zz}^{-1}+\alpha^2\sigma_u^2B_Y^\top B_Y = \begin{bmatrix} \sigma_F^{-2} & 0 & 0 & 0 \\ 0 & \sigma_Y^{-2}-\alpha^2\sigma_u^2 & -\alpha^2\sigma_u^2 & 0 \\ 0 & -\alpha^2\sigma_u^2 & \sigma_y^{-2}-\alpha^2\sigma_u^2 & 0 \\ 0 & 0 & 0 & \sigma_\varepsilon^{-2} \end{bmatrix}. \tag{A.37}$$

The determinant of Σ_{zzY}^{-1} can be easily computed, given by:

$$|\Sigma_{zzY}^{-1}| = |\Sigma_{zz}^{-1}| \left[1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)\right] = N_Y^{-1}|\Sigma_{zz}^{-1}|, \quad N_Y^{-1} \equiv 1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2), \tag{A.38}$$

which is strictly positive under parameter condition (2.12). Hence, Σ_{zzY}^{-1} is positive definite. We can also compute $|\Sigma_{zzB}|^{-1}$ and Σ_{zzY}, which are given by:

$$\begin{aligned} |\Sigma_{zzB}^{-1}| &= |\Sigma_{zzY}^{-1} + \alpha^2\sigma_{D|X}^2 B_\theta^\top B_\theta| = |\Sigma_{zzY}^{-1}|(1 + \alpha^2\sigma_{D|X}^2 B_\theta \Sigma_{zzY} B_\theta^\top) \\ &= |\Sigma_{zz}^{-1}| \left[1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)\right] (1+\alpha^2\sigma_{D|X}^2 B_\theta \Sigma_{zzY} B_\theta^\top), \end{aligned} \tag{A.39}$$

and

$$\Sigma_{zzY} = \begin{bmatrix} \sigma_F^2 & 0 & 0 & 0 \\ 0 & N_Y(1-\alpha^2\sigma_u^2\sigma_y^2)\sigma_Y^2 & N_Y\alpha^2\sigma_u^2\sigma_Y^2\sigma_y^2 & 0 \\ 0 & N_Y\alpha^2\sigma_u^2\sigma_Y^2\sigma_y^2 & N_Y(1-\alpha^2\sigma_u^2\sigma_Y^2)\sigma_y^2 & 0 \\ 0 & 0 & 0 & \sigma_\varepsilon^2 \end{bmatrix}. \tag{A.40}$$

Investors' welfare can then be rewritten as in (4.17).

A.4 Special Cases with No Government Intervention

In this subsection, we provide more details on the results for the special cases with no intervention.

Special Case 1: $\sigma_Y^2 = 0$**.** In this case, $Z = [F \;\; u]^\top$ and $X_i = [y_i \;\; F + \varepsilon_i]$. The stock price only depends on F only, hence fully reveals F. Thus, $\sigma_{D|X}^2 = \sigma_u^2$. (A.21) now gives:

$$h_s = 1/(\alpha\sigma_u^2), \quad h_h = 1.$$

From the market-clearing condition (4.10), we have $b = 1$ and

$$|A| = \sigma_F^2\sigma_y^2\sigma_\varepsilon^2, \quad \beta_{FX} = [1 \;\; 0 \;\; 0],$$

and

$$B_P = [1 \;\; 0 \;\; 0], \quad B_Y = [0 \;\; 1 \;\; 0], \quad B_F =]1 \;\; 0 \;\; 0], \quad B_\theta = -[0 \;\; 1 \;\; 0].$$

Special Case 2: $\sigma_y^2 = 0$**.** First, the parameter condition (2.12) now becomes $\alpha^2\sigma_u^2\sigma_Y^2 < 1$. Next, investors' signals now become:

$$X_i \sim N(0, A), \quad A = \begin{bmatrix} b^2\sigma_F^2+c^2\sigma_Y^2 & -c\,\sigma_Y^2 & b\,\sigma_F^2 \\ -c\,\sigma_Y^2 & \sigma_Y^2 & 0 \\ b\,\sigma_F^2 & 0 & \sigma_F^2+\sigma_\varepsilon^2 \end{bmatrix}. \tag{A.41}$$

In addition, $|A| = b^2\sigma_F^2\sigma_Y^2\sigma_\varepsilon^2$, and

$$\beta_{FX} = \left[\beta_{FX}^{(1)} \;\; \beta_{FX}^{(2)} \;\; \beta_{FX}^{(3)}\right] = [1/b \;\; c/b \;\; 0]\,. \tag{A.42}$$

(4.10) then leads to the following solution to the equilibrium price coefficients:

$$b = 1, \quad c = \alpha\sigma_u^2. \tag{A.43}$$

Special Case 3: $\sigma_\varepsilon^2 \to 0$**.** In this case, we have:

$$X_i \sim N(0, A), \quad A = \begin{bmatrix} b^2\sigma_F^2 + c^2k^2\sigma_Y^2 & -c\,\sigma_Y^2 & b\,\sigma_F^2 \\ -c\,\sigma_Y^2 & \sigma_Y^2 + \sigma_y^2 & 0 \\ b\,\sigma_F^2 & 0 & \sigma_F^2 \end{bmatrix}, \tag{A.44}$$

and

$$|A| = k^2\sigma_F^2\sigma_Y^2\sigma_y^2, \;\; \beta_{FX} = \left[\beta_{FX}^{(1)} \;\; \beta_{FX}^{(2)} \;\; \beta_{FX}^{(3)}\right] = [0 \;\; 0 \;\; 1], \;\; \sigma_{D|X}^2 = \sigma_u^2. \tag{A.45}$$

(4.10) then leads to the solution to the same equilibrium price coefficients as given in (A.43). Furthermore,

$$B = -\sigma_u^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1/(\alpha\sigma_u^2) \\ 0 & 0 & 1/(\alpha\sigma_u^2) & -1/(\alpha^2\sigma_u^4) \end{bmatrix}. \tag{A.46}$$

Special Case 4: No Private Signals on F**.** In this case, $b = 0$. Going through the solution for market equilibrium, described in Section 4.2 with $z_i = [\,F \;\; Y \;\; y_i\,]^\top$, we have:

$$\sigma_{F|X}^2 = \sigma_F^2, \;\; \beta_{FX} = 0, \;\; B_F = 0, \;\; h_s = 1/(\alpha\sigma_D^2), \;\; h_h = \sigma_u^2/\sigma_D^2, \;\; \sigma_D^2 = \sigma_F^2 + \sigma_u^2.$$

In equilibrium, we have:

$$c = \alpha\sigma_u^2, \quad \theta_i = -\frac{\sigma_u^2}{\sigma_D^2}\,y_i, \quad B = -\sigma_u^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & \sigma_F^2/\sigma_D^2 \end{bmatrix}. \tag{A.47}$$

A.5 Government Intervention with Public Information

In this case, the investor i's optimization problem is similar to the maximization problem (A.30) with no government intervention, which reduces to:

$$\max_{\theta_i} \;\; \hat{W}_i = (\theta_i + \theta_G)(\bar{F}_i - P) - \tfrac{1}{2}\alpha\,\left[(\theta_i + \theta_G)^2\sigma_{D|X}^2 + 2(\theta_i + \theta_G)Y_i\,\sigma_{Du|X} + Y_i^2\sigma_u^2\right]. \tag{A.48}$$

The the first order condition for θ_i yields:

$$\bar{F}_i - P - \alpha\, Y_i \sigma_u^2 = \alpha\,(\theta_i + \theta_G)\sigma_{D|X}^2. \tag{A.49}$$

It then yields the solution to θ_i in (5.3). The rest of the results for this case follow easily.

A.6 Government Intervention with Pure Noise

Given the price function (5.8), we now have:

$$Z = \begin{bmatrix} Y \\ u \end{bmatrix}, \quad \Sigma_{ZZ} = \begin{bmatrix} \sigma_Y^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}, \tag{A.50a}$$

$$X_i = \begin{bmatrix} P \\ Y_i \end{bmatrix} = \begin{bmatrix} -c\,Y + d\,\varepsilon_G \\ Y + y_i \end{bmatrix}, \quad A = \begin{bmatrix} c^2\sigma_Y^2 + d^2 & -c\,\sigma_Y^2 \\ -c\,\sigma_Y^2 & \sigma_Y^2 + \sigma_y^2 \end{bmatrix}, \tag{A.50b}$$

and

$$\beta_{ZX} = \frac{1}{|A|}\begin{bmatrix} -c\,\sigma_Y^2\sigma_y^2 & d^2\sigma_Y^2 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_{ZZ|X} = \begin{bmatrix} d^2\sigma_Y^2\sigma_y^2/|A| & 0 \\ 0 & \sigma_u^2 \end{bmatrix}, \tag{A.51}$$

with

$$|A| = c^2\sigma_Y^2\sigma_y^2 + d^2(\sigma_Y^2 + \sigma_y^2), \quad A^{-1} = |A|^{-1}\begin{bmatrix} \sigma_Y^2 + \sigma_y^2 & c\,\sigma_Y^2 \\ c\,\sigma_Y^2 & c^2\sigma_Y^2 + d^2 \end{bmatrix}. \tag{A.52}$$

In addition,

$$|\Sigma_{ZZ|X}| = d^2 D_z/|A|, \quad \Sigma_{ZZ|X}^{-1} = \begin{bmatrix} |A|\sigma_u^2/(d^2 D_z) & 0 \\ 0 & 1/(\sigma_u^2) \end{bmatrix}, \quad D_z \equiv \sigma_Y^2\sigma_y^2\sigma_u^2. \tag{A.53}$$

For (3.9-3.10) and (3.13), we now have:

$$\begin{aligned} &h'_{GP} = h/d, \quad h'_{GY} = c\,(h/d), \\ &h'_{GZ} = [c\,(h/d) \;\; 0], \quad h'_{GX} = \frac{(h/d)d^2}{|A|}[\sigma_Y^2 + \sigma_y^2 \;\; c\,\sigma_Y^2]. \end{aligned} \tag{A.54}$$

Moreover,

$$H_G = \begin{bmatrix} 0 & c\,(h/d) \\ c\,(h/d) & 0 \end{bmatrix} \tag{A.55}$$

and

$$\Sigma_{ZZ|XG}^{-1} = \Sigma_{ZZ|X}^{-1} + \alpha H_G = \begin{bmatrix} |A|\sigma_u^2/(d^2 D_z) & \alpha c\,(h/d) \\ \alpha c\,(h/d) & 1/\sigma_u^2 \end{bmatrix}, \tag{A.56a}$$

$$\Sigma_{ZZ|XG} = |\Sigma_{ZZ|XG}^{-1}| \begin{bmatrix} 1/\sigma_u^2 & -\alpha c\,(h/d) \\ -\alpha c\,(h/d) & |A|\sigma_u^2/(d^2 D_z) \end{bmatrix}, \tag{A.56b}$$

where

$$|\Sigma_{ZZ|XG}^{-1}| = \frac{|A|}{d^2 D_z} - (\alpha c)^2 (h/d)^2 = |\Sigma_{ZZ|XG}|^{-1}. \tag{A.57}$$

With $D = u$, we then have:

$$\sigma_{D|XG}^2 = \sigma_{u|XG}^2 = \sigma_{Du|XG}^2 = |\Sigma_{ZZ|XG}|(|A|\sigma_u^2)/(d^2 D_z), \tag{A.58a}$$

$$\sigma_{DX|XG} = |\Sigma_{ZZ|XG}| \begin{bmatrix} -\alpha c\,(h/d) \\ |A|\sigma_u^2/(d^2 D_z) \end{bmatrix}. \tag{A.58b}$$

From (3.18), we now derive the investors' trading policy, given by h_s and h_h. Since

$$1 - \alpha h'_{GZ}\sigma_{DZ|XG} = 1 + |\Sigma_{ZZ|GX}|(h/d)^2(\alpha c)^2,$$

we have:

$$\begin{aligned} h_s &= \frac{1}{\alpha\sigma_{D|XG}^2}\left(1 - \alpha h'_{GZ}\sigma_{DZ|XG}\right) = \frac{d^2 D_z}{|A|\alpha\sigma_u^2}|\Sigma_{ZZ|XG}|^{-1}\left[1 + |\Sigma_{ZZ|XG}|(h/d)^2(\alpha c)^2\right] \\ &= \frac{d^2 D_z}{|A|\alpha\sigma_u^2}\left[|\Sigma_{ZZ|XG}|^{-1} + (h/d)^2(\alpha c)^2\right] = \frac{1}{\alpha\sigma_u^2}, \end{aligned} \tag{A.59}$$

and

$$h_h = \frac{\sigma_{Du|XG}}{\sigma_{D|XG}^2} = 1, \tag{A.60}$$

which turns out to be the same as in Special Case 3 with no government intervention. Given h_s and h_h, we can derive the equilibrium price coefficients, c and d. From (3.22), with $\beta_{FX} = 0$, we have:

$$h_P = h_s + h_{GX}^{\prime(1)} = \frac{1}{\alpha\sigma_u^2} + \frac{(h/d)d^2}{|A|}(\sigma_Y^2 + \sigma_y^2), \quad h_Y = h_h + h_{GX}^{\prime(2)} = 1 + \frac{(h/d)c\,d^2}{|A|}\sigma_Y^2. \tag{A.61}$$

From (3.25), we now have:

$$c = h_Y/h_P, \quad d = h/h_P, \quad d/c \equiv k = h/h_Y. \tag{A.62}$$

We can rewrite the last equation as:

$$\left[1 + |A|^{-1}(h/d)d^2 c\,\sigma_Y^2\right](d/c) = h, \tag{A.63}$$

which leads to the following equation for k after some algebra:

$$(\sigma_Y^2+\sigma_y^2)k^3 - h\sigma_y^2k^2 + \sigma_Y^2\sigma_y^2k - h\sigma_Y^2\sigma_y^2 = 0. \tag{A.64}$$

This is a cubic equation, which has a unique real root for small h. In particular, $k=0$ when $h=0$. Substituting the solution to k in (A.62), we have the solution to c and d:

$$c = (\alpha\sigma_u^2)\left[1 - \frac{\sigma_y^2kh}{\sigma_Y^2\sigma_y^2 + (\sigma_Y^2+\sigma_y^2)k^2}\right], \quad d = k\,c. \tag{A.65}$$

The investors' stock holdings in equilibrium is then given by:

$$\theta_i = -h_P P - h_Y Y_i = -h\,\varepsilon_G - \left(1 + c^2\sigma_Y^2kh/|A|\right) y_i. \tag{A.66}$$

Next, we consider the investors' welfare. In this case, we have $z_i = [Y \quad y_i \quad \varepsilon_G]^\top$ and $\Sigma_{zz} = \mathrm{Diag}[\sigma_Y^2, \sigma_y^2, 1]$. Consequently, (A.23) becomes:

$$B_P = [-c \;\; 0 \;\; d], \quad B_G = [0 \;\; 0 \;\; h], \quad B_Y = [1 \;\; 1 \;\; 0], \tag{A.67}$$

(and $B_F = 0$). Now (A.24) reduces to:

$$B = \frac{1}{\alpha}\left[\frac{1}{\alpha\sigma_u^2}B_P^\top B_P + (B_P^\top B_Y + B_Y^\top B_P)\right]. \tag{A.68}$$

Hence,

$$B_P^\top B_P = c^2\begin{bmatrix} 1 & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & k^2 \end{bmatrix}, \quad B_P^\top B_Y + B_Y^\top B_P = -c\begin{bmatrix} 2 & 1 & -k \\ 1 & 0 & -k \\ -k & -k & 0 \end{bmatrix},$$

and

$$\alpha^2 B = \begin{bmatrix} c^2/\sigma_u^2 - 2\alpha c & -\alpha c & -(c^2/\sigma_u^2 - \alpha c)k \\ -\alpha c & 0 & \alpha ck \\ -(c^2/\sigma_u^2 - \alpha c)k & \alpha ck & (c^2/\sigma_u^2)k^2 \end{bmatrix}.$$

From $\Sigma_{zzB}^{-1} = \Sigma_{zz}^{-1} + \alpha^2 B$, we obtain:

$$|\Sigma_{zzB}^{-1}| = \left[\frac{1}{\sigma_y^2}\left(\frac{1}{\sigma_Y^2} + \frac{c^2}{\sigma_u^2} - 2\alpha c\right) - (\alpha c)^2\right] + \frac{c^2}{\sigma_Y^2\sigma_y^2\sigma_u^2}\left[1 - \alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)\right]k^2. \tag{A.69}$$

From (A.57) and (A.69) with $|\Sigma_{zz}| = \sigma_Y^2\sigma_y^2$, we can compute the investors' welfare from (3.30).

We now consider the situation when h is small. This entails to obtaining approximate solutions to the lowest (non-trivial) order of h. From (A.63), we immediately have:

$$k = h + o(h^2)$$

where $o(h^2)$ denotes terms of higher orders of h^2. From the first equation of (A.62), we have:

$$c = (\alpha\sigma_u^2)\left[1-(1/\sigma_Y^2)h^2\right] + o(h^2), \quad d = (\alpha\sigma_u^2)\, h + o(h^2),$$

where $c_0 = \alpha\sigma_u^2$ is the solution for c when $h = 0$. As a result, we have:

$$c^2 = (\alpha\sigma_u^2)^2\left[1-(2/\sigma_Y^2)h^2\right] + o(h^2).$$

From (A.57), we have:

$$\begin{aligned}\frac{|\Sigma_{ZZ|XG}|}{|\Sigma_{ZZ|X}|} &= \left[1-(\alpha c)^2\frac{d^2 D_z}{|A|}(h/d)^2\right]^{-1} = \left[1-\frac{\alpha^2\sigma_Y^2\sigma_y^2\sigma_u^2}{\sigma_Y^2\sigma_y^2+(\sigma_Y^2+\sigma_y^2)k^2}h^2\right]^{-1} \\ &= 1+\alpha^2\sigma_u^2h^2+o(h^2).\end{aligned}$$

From (A.69), we have:

$$\frac{|\Sigma_{zzB}|}{|\Sigma_{zz}|} = \left\{\left[1+\sigma_Y^2\left(\frac{c^2}{\sigma_u^2}-2\alpha c\right)-\sigma_Y^2\sigma_y^2(\alpha c)^2\right]+\left(\frac{c^2}{\sigma_u^2}\right)\left[1-\alpha^2\sigma_u^2(\sigma_Y^2+\sigma_y^2)\right]k^2\right\}^{-1}. \quad \text{(A.70)}$$

Note that when $h = 0$, we have:

$$\frac{|\Sigma_{zzB}|_0}{|\Sigma_{zz}|_0} = \left[1-\alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)\right]^{-1} = Q^2, \quad Q \equiv 1/\sqrt{1-\alpha^2\sigma_u^2\sigma_Y^2(1+\alpha^2\sigma_u^2\sigma_y^2)}.$$

Thus,

$$\frac{|\Sigma_{zzB}|}{|\Sigma_{zz}|} = Q^2\left\{1-Q^2\alpha^2\sigma_u^2\left[1-\alpha^2\sigma_u^2(\sigma_Y^2-\sigma_y^2)\right]h^2\right\}+o(h^2).$$

We then have:

$$\begin{aligned}\frac{|\Sigma_{ZZ|XG}|}{|\Sigma_{ZZ|X}|}\frac{|\Sigma_{zzB}|}{|\Sigma_{zz}|} &= Q^2\left\{1+\alpha^2\sigma_u^2h^2\right\}\left\{1-Q^2\alpha^2\sigma_u^2\left[1-\alpha^2\sigma_u^2(\sigma_Y^2-\sigma_y^2)\right]h^2\right\}+o(h^2) \\ &= Q^2\left\{1+\alpha^2\sigma_u^2h^2-Q^2\alpha^2\sigma_u^2\left[1-\alpha^2\sigma_u^2(\sigma_Y^2-\sigma_y^2)\right]h^2\right\}+o(h^2) \\ &= Q^2\left\{1+Q^2(\alpha^2\sigma_u^2)\left[1/Q^2-1+(\alpha^2\sigma_u^2)(\sigma_Y^2-\sigma_y^2)\right]h^2\right\}+o(h^2) \\ &= Q^2\left[1-Q^2(\alpha^2\sigma_u^2)^2\sigma_y^2(1+\alpha^2\sigma_u^2\sigma_Y^2)\,h^2\right]+o(h^2)\end{aligned}$$

It then follows that:

$$V(h) = -\left\{\frac{|\Sigma_{ZZ|XG}|}{|\Sigma_{ZZ|X}|}\frac{|\Sigma_{zzB}|}{|\Sigma_{zz}|}\right\}^{1/2} = -\,Q\left[1-\frac{1}{2}Q^2(\alpha^2\sigma_u^2)^2\sigma_y^2(1+\alpha^2\sigma_u^2\sigma_Y^2)h^2\right]+o(h^2). \quad \text{(A.71)}$$

Obviously, $V(0) = -Q$, $V'(0) = 0$, and $V''(0) = \frac{1}{2}Q^3(\alpha^2\sigma_u^2)^2\sigma_y^2(1+\alpha^2\sigma_u^2\sigma_Y^2) > 0$.

A.7 Government Intervention with Full Information

General intervention policy

Given the intervention policy (5.14), price function (5.15) and the conditional distributions given in Section 5.3, we first obtain the investors' optimal trading policy and then the market equilibrium, following the general solution in Section 3. When applying the general solution, we remove all the redundant elements in the state-vector and the corresponding coefficients.

First, from

$$\theta_G = \hat{h}_{GF} F - (1/c)\, h_{GY} P, \quad \hat{h}_{GF} = h_{GF} + (b/c)\, h_{GY}$$

for (3.9) and (3.12) we have:

$$h'_{GP} = -(1/c)\, h_{GY}, \quad h'_{GZ} = [\hat{h}_{GF} \;\; 0],$$

and

$$\bar{\theta}_{Gi} = h'_{GX} X_i, \quad \tilde{\theta}_{Gi} = \hat{h}_{GF} \tilde{F}_i, \quad h'_{GX} = h'_{GP}[1 \;\; 0 \;\; 0] + h'_{GZ} \beta_{ZX}.$$

We also have:

$$\tilde{D}_i = \tilde{F}_i + u = h'_{DZ} \tilde{Z}_i, \quad h'_{DZ} = [1 \;\; 1].$$

Given the intervention policy (5.19) and the above, investor i's terminal wealth is given by:

$$W_i = (\theta_i + h_{GF} F + h_{GY} Y)(F + u - P) + Y_i\, u, \quad i \in [0, 1]. \tag{A.72}$$

We can rewrite (A.72) as:

$$W_i = W'_i + \hat{h}_{GF} \tilde{F}_i (\tilde{F}_i + u) = \bar{W}'_i + \tilde{W}'_i + \tfrac{1}{2} \tilde{Z}_i^\top H_G \tilde{Z}_i, \tag{A.73}$$

where

$$\begin{aligned}
W'_i &= (\theta_i + \bar{\theta}_{Gi})(\bar{F}_i - P) + \hat{h}_{GF}(\bar{F}_i - P)\tilde{F}_i + (\theta_i + \bar{\theta}_{Gi})(\tilde{F}_i + u) + Y_i\, u, \\
\bar{W}'_i &= \mathrm{E}\,[W'_i | X_i] = (\theta_i + \bar{\theta}_{Gi})(\bar{F}_i - P), \\
\tilde{W}'_i &= (\theta_i + \bar{\theta}_{Gi})(\tilde{F}_i + u) + (\bar{F}_i - P)\hat{h}_{GF} \tilde{F}_i + Y_i\, u,
\end{aligned}$$

and

$$\tilde{Z}_i = \begin{bmatrix} \tilde{F}_i \\ u \end{bmatrix}, \quad H_G = {h'_{DZ}}^\top h'_{GZ} + {h'_{GZ}}^\top h'_{DZ} = \begin{bmatrix} 2\hat{h}_{GF} & \hat{h}_{GF} \\ \hat{h}_{GF} & 0 \end{bmatrix}, \quad \bar{\theta}_{Gi} = \hat{h}_{GF} \bar{F}_i - h_{GY}(b/c) P,$$

which also gives $\Sigma_{ZZ|XG}$ in (5.18). Her expected utility is:

$$\mathrm{E}\left[-e^{-\alpha W_i}\middle|X_i\right] = -\frac{\left|\Sigma_{ZZ|XG}\right|^{1/2}}{\left|\Sigma_{ZZ|X}\right|^{1/2}}\, e^{-\alpha \hat{W}_i}, \quad \hat{W}_i = \bar{W}_i' - \tfrac{1}{2}\,\alpha \mathrm{V}\left[\tilde{W}_i'|X_i; \Sigma_{ZZ|XG}\right]. \tag{A.74}$$

Following the solution in Section 3, maximizing $\hat{W}$ yields the optimal trading policy given in (5.22). For the value function, from (3.20) and (3.29), we easily obtain (5.20) with the B's taking the same form in (A.35) with B given by:

$$\begin{aligned} B \equiv\ & \sigma^2_{D|XG} B_\theta^\top B_\theta - \sigma^2_{u|XG} B_Y^\top B_Y \\ & - 2\,\hat{h}_{GF}\,\sigma_{Fu|XG}(B_F - B_P)^\top B_Y - \hat{h}^2_{GF}\sigma^2_{F|XG}(B_F - B_P)^\top (B_F - B_P), \end{aligned} \tag{A.75}$$

and B_P, B_F, B_Y and B_θ are given in (A.35).[11]

We can also rewrite the investors' optimal trading policy as:

$$\theta_i = h_P P - h_Y Y_i + h_S S_i = (h_P b + h_S)\, F - (h_P c + h_Y)\, Y - h_Y y_i + h_S \varepsilon_i, \tag{A.76}$$

where

$$h_P = (h_s - \hat{h}_{GF})\beta^{(1)}_{FX} - h_s, \quad h_Y = h_h - (h_s - \hat{h}_{GF})\beta^{(2)}_{FX}, \quad h_S = (h_s - \hat{h}_{GF})\beta^{(3)}_{FX}. \tag{A.77}$$

In equilibrium, the market-clearing condition now becomes:

$$\int_i \theta_i + h_{GF} F + h_{GY} Y = (h_P b + h_S + h_{GF})\, F - (h_P c + h_Y + h_{GY})\, Y = 0, \tag{A.78}$$

which requires:

$$h_P b + h_S + h_{GF} = 0, \quad h_P c + h_Y + h_{GY} = 0. \tag{A.79}$$

The solution to (A.78) gives the solution for b and c in the equilibrium stock price, which is given in (5.21).

Investors' equilibrium stock holdings, value function and welfare follow directly from the general solution with the proper substitutions, which are given in Section 5.3.

A special intervention policy

For the special case considered here, we have $F = 0$, $\theta_G = hY$, and $P = 0$ under the particular value of h we seek to determine. For this case, the state vector becomes $Z = [Y \quad u]^\top$ and

[11] As noted above, even though there is government intervention here, the price takes the same form as in the case with no intervention. Consequently, the B's take the same form.

$\Sigma_{ZZ} = \text{Diag}[\sigma_Y^2, \sigma_u^2]$. In addition, $X_i = Y + y_i$, $A = \Sigma_{XX} = \sigma_Y^2 + \sigma_y^2$, and

$$\Sigma_{ZX} = \begin{bmatrix} \sigma_Y^2 \\ 0 \end{bmatrix}, \quad \beta_{ZX} = \begin{bmatrix} \beta_{YX} \\ 0 \end{bmatrix}, \quad \Sigma_{ZZ|X} = \begin{bmatrix} \beta_{YX}\sigma_y^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}, \quad \beta_{YX} \equiv \frac{\sigma_Y^2}{\sigma_Y^2 + \sigma_y^2}.$$

Given that $\theta_G = h\,Y$, we have:

$$\theta_G = \bar{\theta}_{Gi} + \tilde{\theta}_{Gi}, \quad \bar{\theta}_{Gi} = h\beta_{YX}Y_i, \quad \tilde{\theta}_{Gi} = h\tilde{Y}_i, \quad \tilde{Y}_i \equiv Y - \beta_{YX}Y_i.$$

We also have:

$$W_i = (\theta_i + \bar{\theta}_{Gi} + Y_i)\,u + h\,\tilde{Y}_i u = (\theta_i + \bar{\theta}_{Gi} + Y_i)\,u + \tfrac{1}{2}\tilde{Z}_i^\top H_G \tilde{Z}_i$$

and

$$H_G = \begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix}, \quad \tilde{Z}_i = \begin{bmatrix} \tilde{Y}_i \\ u \end{bmatrix}.$$

In addition,

$$\Sigma_{ZZ|X}^{-1} = \begin{bmatrix} (\beta_{YX}\sigma_y^2)^{-1} & 0 \\ 0 & (\sigma_u^2)^{-1} \end{bmatrix}, \quad \Sigma_{ZZ|XG}^{-1} = \begin{bmatrix} (\beta_{YX}\sigma_y^2)^{-1} & \alpha\,h \\ \alpha\,h & (\sigma_u^2)^{-1} \end{bmatrix}.$$

From Proposition 1, the investors' optimal trading policy is:

$$\theta_i = -\bar{\theta}_{Gi} - h_h Y_i, \quad h_h = 1, \quad i \in [0,1].$$

The market clearing condition with $P = 0$ immediately requires:

$$h^* = \frac{\sigma_Y^2 + \sigma_y^2}{\sigma_y^2} = \frac{1}{1 - \beta_{YX}}.$$

Thus, the investors' equilibrium stock holding is:

$$\theta_i = -\left(\frac{\sigma_Y^2}{\sigma_y^2} + 1\right)(Y + y_i).$$

For welfare, we have $z_i = [Y \;\; y_i]^\top$ and $\Sigma_{zz} = \text{Diag}[\sigma_Y^2, \sigma_y^2]$. From (A.23) and (A.24), we have $B_P = B_F = 0$, $B = 0$, $\Sigma_{zzB} = \Sigma_{zz}$, and

$$|\Sigma_{ZZ|X}| = \beta_{YX}\sigma_u^2\sigma_y^2, \quad |\Sigma_{ZZ|XG}| = |\Sigma_{ZZ|X}|\frac{1}{1 - \alpha^2\sigma_u^2(\sigma_Y^2 + \sigma_y^2)(\sigma_Y^2/\sigma_y^2)}.$$

Thus, Proposition 3 gives the investors' welfare under the specific intervention policy:

$$V(h^*) = -\frac{|\Sigma_{ZZ|XG}|^{1/2}}{|\Sigma_{ZZ|X}|^{1/2}} = -\frac{1}{\sqrt{1 - \alpha^2\sigma_u^2(\sigma_Y^2 + \sigma_y^2)(\sigma_Y^2/\sigma_y^2)}},$$

which is (5.27).

A.8 Evaluation of Government Intervention Policy

Here, we compute the evaluation measures for the government's intervention policies: the simple contrarian trading strategy linear in stock price and the noise trading strategy linear in noise.

Expected Payoff. For the linear contrarian strategy given in (6.1), it has no impact on the equilibrium stock price, which takes the form in (4.1). For convenience, we define the return/net payoff on one share of the stock by:

$$R \equiv F + u - P.$$

Then, the net payoff on the contrarian strategy is given by:

$$W_G = -h\,P(F + u - P) = -h\,B_P Z Z^\top B_R,$$

where the last equation is in matrix format with $Z = [F \quad u \quad Y]^\top$, $B_P = [b \quad 0 \quad -c]$ and $B_R = [(1-b) \quad 1 \quad c]$. Immediately, we have:

$$\mathrm{E}\,[W_G] = -hB_P\Sigma_{ZZ}B_R^\top = -h\,[b(1-b)\sigma_F^2 - c^2\sigma_Y^2].$$

For $F = 0$, we have (6.3).

For the noise-trading strategy with $F = 0$, we have:

$$W_G = \theta_G\,(u - P) = h\,\varepsilon_G\,(u + c\,Y - d\,\varepsilon_G).$$

Let $Z = [u \quad Y \quad \varepsilon_G]^\top$, $B_N = [0 \quad 0 \quad 1]$, $B_P = [0 \quad -c \quad d]$, and $B_R = [1 \quad c \quad -d]$. We have:

$$W_G = hB_N Z Z^\top B_R^\top.$$

It then follows that:

$$\mathrm{E}\,[W_G] = hB_N\mathrm{E}\,[ZZ^\top]B_R^\top = hB_N\Sigma_{ZZ}B_R^\top = -h\,d\,\sigma_{\varepsilon_G}^2,$$

which is negative, at least for h small.

Sharpe ratio. In order to compute the Sharpe ratio of the two strategies, we need to compute the variance of their payoffs, which are the products of two normal variables. For this purpose, we need the following lemma:

Lemma A.5. *Let z_1 and z_2 be standard normal with correlation ρ. Then,*

$$\mathrm{V}\,[z_1 z_2] = 1 + 2\,\rho^2. \tag{A.80}$$

Using the lemma, we can now compute the payoff variance for the strategies under consideration.

In general, we can write their payoffs as:

$$W_G = \pm\, h\,(B_I Z)(B_R Z).$$

Let σ_I and σ_R denote the volatility of $B_I Z$ and $B_R Z$, respectively, and ρ_{IR} their correlation. Then, we have:

$$\sigma_I^2 = B_I \Sigma_{ZZ} B_R^\top, \quad \sigma_R^2 = B_R \Sigma_{ZZ} B_R^\top, \quad \mathrm{C}\,[B_I Z, B_R Z] = B_I \Sigma_{ZZ} B_R^\top, \quad \rho_{IR} = \frac{B_I \Sigma_{ZZ} B_R^\top}{\sigma_I \sigma_R}.$$

For the contrarian trading strategy, $Z = [u \;\; Y]^\top$, $B_I = B_P = [1 \quad -c]$ and $\sigma_I = \sigma_P$. We have:

$$\mathrm{V}\,[PR] = c^2 \sigma_Y^2 (3c^2 \sigma_Y^2 + \sigma_u^2).$$

For the noise trading strategy, $Z = [u \;\; Y \;\; \varepsilon_G]^\top$, $B_I = [0 \;\; 0 \;\; d]$, and we have:

$$\mathrm{V}\,[(B_I Z)(B_R Z)] = h^2 \sigma_G^2 (c^2 \sigma_Y^2 + 2d^2 \sigma_G^2 + \sigma_u^2).$$

Combine the results above, we obtain the Sharpe ratios for the two strategies given in (6.5).

Certainty equivalence. Given the form of the value function (3.19), welfare function (3.30) and the definition of certainty equivalence (6.7), we have:

$$V(W, h_G = 0) e^{-\alpha CE(h_G)} = V(W, h_G).$$

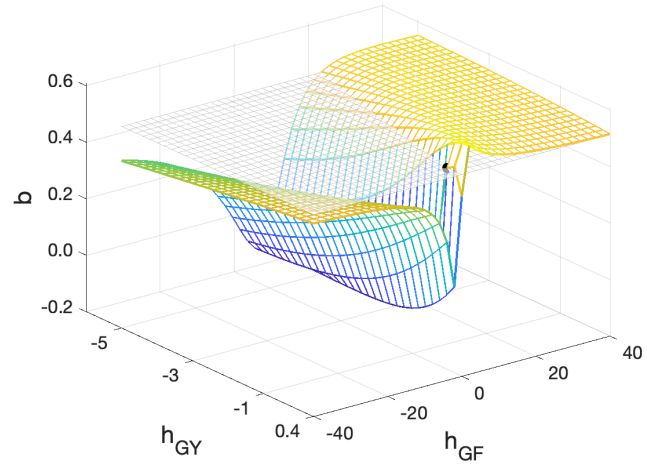
In the case of contrarian strategy, $V(W, h_G) = V(W, h_G = 0)$. Hence, $CE^{\,\text{Contrarian}} = 0$. In the case of noise trading strategy, $V(W, h_G)$ is given in (5.13) for h small. To the lowest order of h, we immediately have the second result in (6.5).

A.9 Government Intervention with Noisy Information

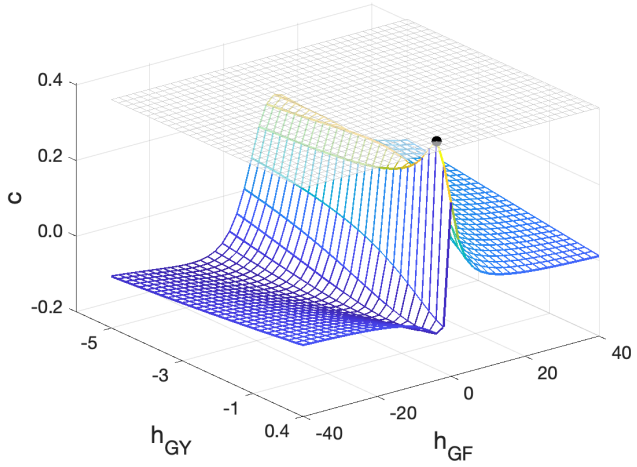
In this subsection, we illustrate the impact of government intervention on market equilibrium with noisy information. Figure A.1 plots the price coefficients b, c, d_F and d_Y, respectively, for different values of h_{GF} and h_{GY}.

For b and c, shown in Panel (a) and (b), we see similar behavior as in the case of full information, for reasons explained there. For d_F and d_Y, the results are new. For h_{GF} ranging from small negative values to large positive values, d_F is positively related to h_{GF}, as shown in Panel (c). For large negative values of h_{GF}, however, d_F becomes negatively related to it. This is expected. As h_{GF} becomes substantially large, the government's trading offsets the impact of F on the price and brings more noise from its signal on F into the price.

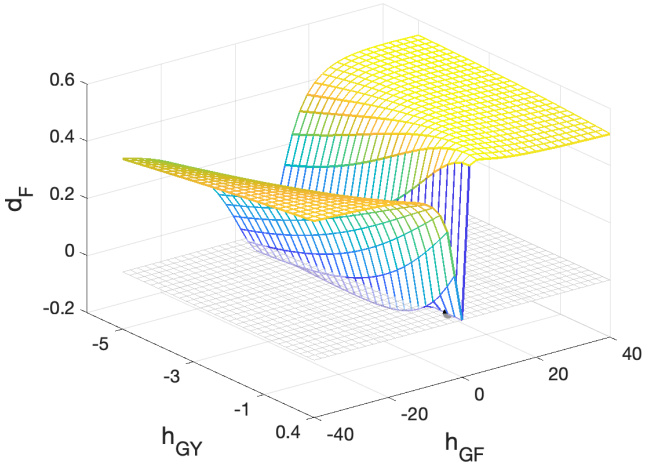
In Figure A.2, we plot the conditional volatility of F and Y for different values of h_{GF}



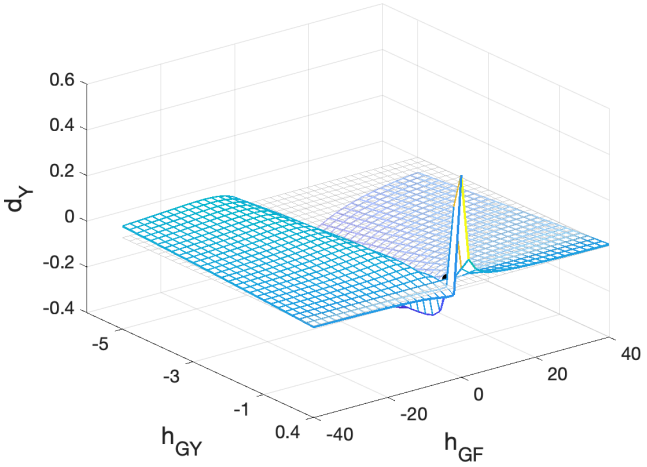
(a) Price coefficient b



(b) Price coefficient c



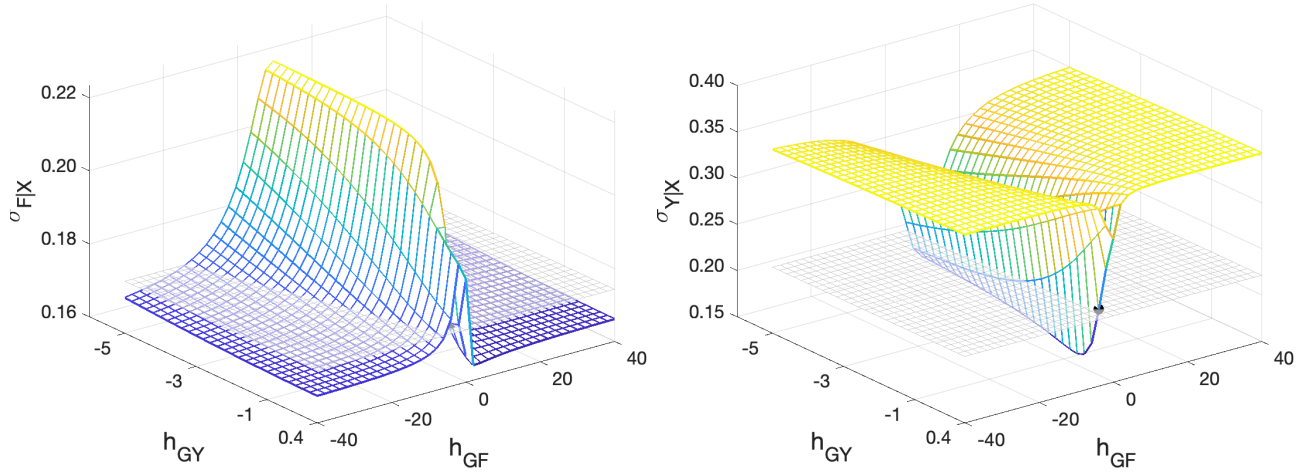
(c) Price coefficient d_F



(d) Price coefficient d_Y

Figure A.1: Price coefficients b, c, d_F and d_Y for different values of h_{GF} and h_{GY} under government intervention with noisy information. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 1$, and $\sigma_{GF} = \sigma_{GY} = 0.25$.

and h_{GY}. For both $\sigma_{F|X}$ and $\sigma_{Y|X}$, the overall patterns are similar to those in the case of full information. However, the levels of both $\sigma_{F|X}$ and $\sigma_{Y|X}$ are now significantly higher. This can be seen by the positions of the surfaces for $\sigma_{F|X}$ and $\sigma_{Y|X}$ relative to the gray planes in both panels, which represent their levels with no intervention. Such a result is not surprising given that in this case government's trading also brings noise into the price.



(a) Conditional volatility $\sigma_{F|X}$

(b) Contitional volatility $\sigma_{Y|X}$

Figure A.2: Conditional volatility of F and Y for different values of h_{GF} and h_{GY} under government intervention with noisy information. The parameter values are set at $\alpha = 4$, $\sigma_F = 0.25$, $\sigma_u = 0.25$, $\sigma_Y = 0.5$, $\sigma_y = 0.5$, $\sigma_\varepsilon = 0.5$, $\beta_{GF} = \beta_{GY} = 1$, and $\sigma_{GF} = \sigma_{GY} = 0.25$.