Sustainable Social Security*

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We study the optimal design of a sustainable social security system in an economy with demographic uncertainty. A benevolent social planner, aiming to maximize the expected utility of all generations, must balance efficient intergenerational risk sharing, maintaining contributors' participation incentives, and honoring commitments to retirees. This trade-off requires a system that blends defined-benefit and defined-contribution features with longevity adjustments: benefits are unrelated to past contributions when contribution rates are low but become linked to contributions when rates exceed a threshold to sustain participation. The theory also predicts fiscal inertia, where demographic shocks prompt gradual adjustments in contribution and benefit rates, with their magnitude and direction depending on the shocks' characteristics and their impact on participation constraints. Public savings, such as trust funds, can buffer demographic shocks, reducing reliance on higher contributions and loosening the link between benefits and contributions.

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JEL Codes: C61; D64; H55; J11.

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I Introduction

Global trends of declining population growth and rising life expectancy are exerting substantial fiscal pressure on social security systems (McGrattan and Prescott, 2017). While retrenchment reforms are widely acknowledged as essential for ensuring long-term fiscal sustainability, strong public support for these programs often constrains their implementation, making them politically challenging even when economically necessary. As a result, welfare programs exhibit significant fiscal inertia, with retrenchment typically unfolding gradually over time. Figure 1 illustrates this dynamic, depicting the total number of pension policy changes per decade across a sample of OECD countries. Positive values represent expansionary reforms, while negative values indicate contractionary measures. The data reveals a wave of rapid pension program expansions from the 1960s through the 1980s, followed by more gradual retrenchment efforts beginning in the 1990s.

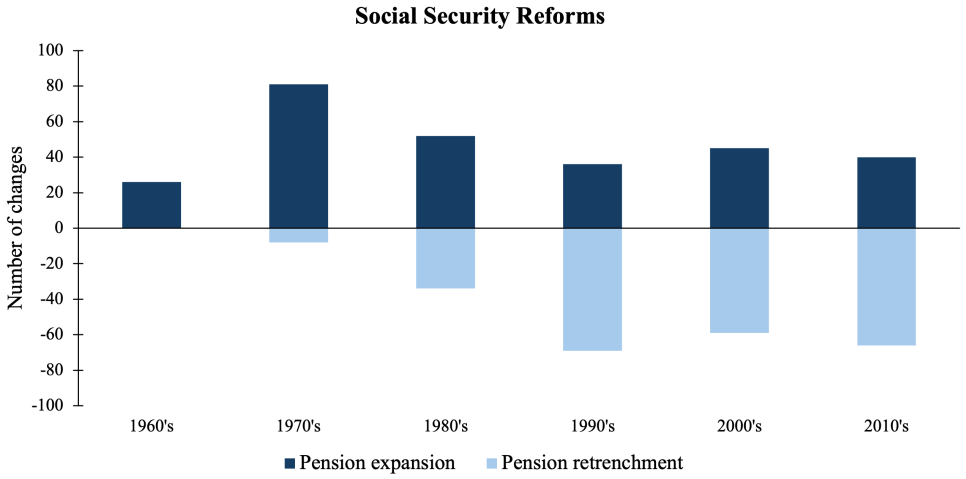


Figure 1: Changes in pension policies in ten OECD countries: Australia, Belgium, Denmark, Finland, France, Italy, Japan, New Zealand, Spain, and the United Kingdom, from 1962 to 2017. Data taken from Bi and Zubairy (2023).

Despite calls to transition to funded systems, most countries continue to rely on unfunded PAYGO social security schemes, where current workers' contributions directly finance retirees' benefits. In these systems, the contribution rate equals the product of the replacement ratio (benefits as a share of wages) and the dependency ratio (retirees relative to workers). As the dependency ratio increases, fiscal sustainability necessitates either higher contribution rates or lower replacement ratios (Hosseini and Shourideh, 2019). However, democratic governments face two key enforcement frictions when implementing such adjustments.

First, policy changes must satisfy the participation constraint, requiring the consent of currently living generations. In practice, this constraint appears to hold, as support for

social security in the US as well as in many advanced economies remains strong across demographics, including the working and retired populations, high- and low-income groups (Arnold, 2022). Second, reforms must uphold the entitlements of current retirees while phasing in future adjustments to ensure financial viability. McHale (2001) highlights that reforms often prioritize preserving current retirees' benefits while gradually addressing demographic pressures through phased adjustments.

Under limited enforcement frictions and demographic uncertainty, some key questions naturally arise: How should an optimal and sustainable social security be designed? How do the replacement ratios and contribution rates adjust to demographic shocks? What is the right balance of defined benefit and defined contribution elements for a PAYGO system?

This paper develops a standard overlapping generations (OLG) model where agents live for two periods: a working-age period, during which they contribute to the system, and a retirement period, during which they receive a pension. The model is intentionally simple, focusing on redistribution across and between generations. Young agents work at a fixed wage, ensuring that changes in the contribution rate do not distort the labor market. There is no savings technology, so the social security system does not affect the capital market. Additionally, agents are not myopic, avoiding any preference-related distortions.

The model incorporates three key features. First, demographic uncertainty affects the dependency ratio. Cocco and Gomes (2012), using the Lee-Carter model (Lee and Carter, 1992), show that post-1970 increases in life expectancy were unexpectedly large and underestimated by actuaries in both the UK and the US. For instance, the UK Pensions Commission (2005) emphasized the wide uncertainty in life expectancy projections and the need to design pension systems that account for both increasing longevity and uncertainty (quoted in Barr and Diamond, 2006). To model longevity risk, we assume that the survival probability of retirees follows a Markov process, as in Bohn (2001). While the main analysis assumes constant population growth, we also examine the impact of unexpected shocks to growth rates.

Second, there is an infinite-horizon social planner who maximizes a geometrically weighted sum of the utilities of current and future generations to share risk across generations. This approach provides a straightforward framework to analyze Pareto efficiency while maintaining a recursive structure. While much of the literature has focused on social security in deterministic settings, the significance of longevity and other risks in shaping and reforming social security systems has been widely recognized (see, e.g., De Nardi

et al., 1999, Diamond and Orszag, 2005, Kitao, 2014, Sheshinski and Caliendo, 2021, and others). In this paper, the benevolent planner seeks to share risk both across and within generations, necessitating adjustments to the replacement ratio and contribution rate in response to the dependency ratio.

Third, risk-sharing is subject to two constraints: a promise-keeping constraint, ensuring that retirees receive their promised pensions, and a participation constraint, ensuring that both the young and the old prefer the planner's allocation over a scenario without social security. For the young to pay a high contribution rate, they must expect a sufficiently high replacement ratio in retirement. Participation constraints of this type have been used successfully in models of risk sharing with infinitely-lived agents (see, for example, Thomas and Worrall, 1988, Kehoe and Levine, 1993, Alvarez and Jermann, 2000, among others) and more recently in an OLG model with endowment risk by Lancia et al. (2024). When the participation constraint binds, it transmits to the next period's promise-keeping constraint, creating an endogenous link between contributions in working age and pensions in retirement.

Our first result concerns the existence of a sustainable social security system in the presence of demographic risk. A social security scheme can be sustained when the autarky is dynamically inefficient. In this context, a necessary and sufficient condition for dynamic inefficiency is that the population growth is greater than the implied interest rate (the inverse of the marginal rate of substitution) at the autarky allocation. This condition is equivalent to the dynamic inefficiency condition in deterministic models with investment and savings that there is dynamic inefficiency when the population growth rate is larger than the equilibrium interest rate. In this case, it is well-known that introducing a PAYGO social security system is Pareto-improving (see, for example, Samuelson, 1975).[1] In both cases the reasoning is similar. Introducing a PAYGO social security system allows a transfer between the generations at higher rate, determined by the dependency ratio, than the current interest rate.[2] Therefore, we will assume that the autarky allocation is dynamically inefficient (Assumption 1).

Although we are primarily interested in longevity risk, the limited enforcement constraints impact the sustainability of social security even when longevity is deterministic. Consider a case in which there is a first best solution with a positive pension and suppose that for some historical reasons, the planner has inherited a promise to provide a larger

[1]The canonical model is a textbook standard (see, for example, Acemoglu, 2009, chapter 9).

[2]The converse of the standard result is that when there is dynamic efficiency, a funded social security scheme is preferable to an unfunded scheme because the rate of return on capital is greater than the population growth rate (see, for example, Aaron, 1966, Feldstein, 1974). When there is uncertainty and the interest rate is risky, this argument no longer applies (see, for example, Dutta et al., 2000).

pension to the current old. For the current young to accept the taxes necessary to fund this pension, they must be offered a sufficiently high pension for their old age. However, the planner can gradually reduce the pension provision over time while satisfying the constraint that young agents are prepared to fund the current pension. Within finite time, the social security system adjusts towards the optimal allocation where enforcement constraints do not bind. Although the limited enforcement constraints impact the adjustment phase, they have no impact in the long run.

The situation is different with stochastic survival probability. Depending on past promises of pension entitlements to retirees and the longevity realization, the participation constraint of taxpayers (the young generation) may either bind or be slack. If the constraint is slack, the social planner can achieve the first-best level of intergenerational redistribution and promised future entitlements that maximize the weighted utility of current and future generations. However, if the participation constraint of the young binds, the social planner needs to provide incentives and sacrifice risk-sharing. The young generation with a binding participation constraint pays contributions that are smaller than those maximizing risk-sharing possibilities and is entitled to higher future promised utility, with the burden shifted forward to future generations.

We show that the optimal and sustainable social security rule can be easily characterized. The replacement ratio in old age is positively correlated with longevity risk. There is a non-linear relation between the replacement ratio and the contribution made during the working age. When past contributions are below a certain threshold, a minimum replacement ratio is guaranteed. In contrast, when past contributions exceed this threshold, the replacement ratio increases with past contributions.

As a result, the optimal pension scheme combines automatic benefit adjustments to longevity risk with features of both defined-benefit and defined-contribution systems—characteristics commonly observed in national pension systems. For instance, in recent years, countries like Sweden, Italy, Portugal, and Latvia have transitioned to notional defined contribution systems. These systems maintain a stable contribution rate, closely link benefits to past contributions, and include mechanisms to adjust benefits in response to demographic shocks, such as changes in longevity risk.

We extend the model to include public saving through the establishment of a trust fund, where payroll taxes exceed current benefits to generate surpluses invested in assets. Many countries, such as the U.S., Canada, Japan, and Norway, have implemented trust funds to pre-fund future retirement benefits in response to demographic changes. This framework enables the analysis of optimal trust fund management to ensure the sustainability of

social security systems under limited enforcement frictions. By redeeming assets, trust funds can pay benefits without raising contribution rates, easing participation constraints and weakening the direct link between contributions and benefits.

Literature There is a large literature studying social security reform and design. Many papers assume there are parametric restrictions on the types of policy and reform that are examined. The rationale is both realism and simplicity. The literature can be divided into normative (see, for example, Bohn, 2001, Conesa and Garriga, 2008, Golosov et al., 2013) and positive approaches (see, for example, Huggett and Ventura, 1999, Conesa and Krueger, 1999, De Nardi et al., 1999, Nishiyama and Smetters, 2007, Imrohoroğlu and Kitao, 2012, Kitao, 2014, McGrattan and Prescott, 2017). In this paper, we do *not* impose parametric restrictions on policy instruments. While the resulting optimal policies are non-linear, they retain some simplistic features that are found in typical of realistic social security systems, such as a mix of defined benefit and defined contribution components.

In the design of optimal social security systems, two main approaches are used in the literature. The first approach posits a specific social welfare function to identify welfare-maximising policies (see, for example, Huggett and Parra, 2010, Golosov et al., 2016, Shourideh and Troshkin, 2017, Ndiaye, 2021). The second approach focuses on identifying Pareto-improving reforms (see, for example, Hosseini and Shourideh, 2019). Both approaches assume that policy instruments are mandatory and primarily address information frictions arising from unobservable idiosyncratic risks. In contrast, our approach focuses on aggregate risk and assumes that contributions are subject to a *voluntary* participation constraint.

In the absence of enforcement frictions, the role of aggregate risk in shaping Pareto-improving intergenerational transfers is studied by Demange and Laroque (1999), among others, who introduce the concept of *interim optimality*. This concept is akin to standard Pareto optimality once the state of the world into which agents are born is known.[3] The optimal sustainable social security system we derive is conditionally Pareto optimal, with generations receiving the same level of promises and being in the longevity states being

[3]In stochastic overlapping generations models, two forms of Pareto optimality are commonly discussed: *ex ante* and *interim* optimality. In an ex ante Pareto optimum, no generation's expected lifetime utility can be increased without reducing the expected utility of another generation. In the interim case, generations are distinguished by both the state and the date of birth. Here, a Pareto optimum is achieved when it is impossible to improve the lifetime utility of any generation without reducing that of another generation either at a different date or in a different state. These concepts are often referred to as *equal-treatment* and *conditional* Pareto optimality, respectively (see, for example, Aiyagari and Peled, 1991, Demange, 2002, Labadie, 2004).

treated equally. However, unlike Demange and Laroque (1999), the Pareto weights in our framework are *endogenously* determined. They evolve dynamically with the history of shocks, reflecting the underlying wedges generated by the frictions of limited participation.

The paper also relates to the literature on the political limits of social security where agents vote on tax and benefit rates for intergenerational transfers (see, for example, Cooley and Soares, 1999, Boldrin and Rustichini, 2000, Bassetto, 2008, Gonzalez-Eiras and Niepelt, 2008). In this literature, a social security scheme corresponds to the subgame (or Markov) perfect equilibria of a repeated (or dynamic) game. Typically, it is assumed that voters in the working-age group are pivotal. In equilibrium, a social security scheme is supported by current workers in the expectation that future workers will do the same. However, these equilibria are not necessarily Pareto optimal. In contrast, the approach presented here identifies constrained Pareto optimal intergenerational transfers that each generation unanimously agrees to respect.

II The Model

Time is discrete and indexed by $t = 0, 1, 2, \ldots, \infty$. The model consists of a pure exchange economy with an overlapping generations demographic structure. At each time t, a new generation is born and lives for two periods. Within each generation, the agents are homogeneous. Generation t enters as young in period t and survives as old in period $t+1$ with probability $\phi_{t+1} \in (0, 1]$. Denote the number of young agents by N_t^y and the number of old individuals by $N_t^o = \phi_t N_{t-1}^y$ at any time t. The gross growth rate of the young generation is $n = N_t^y / N_{t-1}^y$. Therefore, the dependency ratio, defined as the ratio of the number of old individuals to the number of young individuals, is (ϕ_t / n). The dependency ratio increases when the gross growth rate decreases or when the survival probability to old age increases.

Endowments At each time t, young agents supply labor inelastically and earn an exogenous wage of $w > 0$, while the old have a constant endowment of $\alpha > 0$. In the baseline model, we also assume that there is no storage technology to transfer endowments across periods. We will relax these assumptions subsequently.

Uncertainty In the baseline model, the only source of risk is related to longevity. The survival probability $\phi_t \in \Phi$ is stochastic, for example, due to health shocks, and can take $K \geq 1$ values denoted by $\phi(1), \phi(2), ..., \phi(K)$ ordered as $\phi(i) < \phi(i+1)$ for any

$i = 1, ..., K-1$, so that a higher state corresponds to a higher life expectancy for the old.[4] We assume that ϕ_t follows a time-homogeneous, aperiodic, finite-state Markov process with $K \times K$ transition matrix Π. The probability of transiting from state i with $\phi = \phi(i)$ to state j with $\phi = \phi(j)$ next period given by π_{ij}. The stationary distribution of Π is the probability distribution ϖ satisfying $\varpi\Pi = \varpi$.[5] Denote the expected survival probability conditional on the current ϕ by $\bar{\phi}(\phi) = \sum_j \pi_{ij}\phi(j)$ and let the history of shocks up to and including time t be $\phi^t := (\phi_0, ..., \phi_t) \in \Phi^t$.

Balanced Budget Total taxes paid by young workers are equal to the pensions paid to the old. At each time t, the social security pension received by an old agent $p_t \geq 0$ is financed by a payroll tax paid by young agents at a rate of $\tau_t \in [0, 1]$. We rule out borrowing and lending possibilities. Therefore, the budget constraint is:

$$\tau_t \cdot w = \frac{\phi_t}{n} \cdot p_t. \tag{1}$$

The replacement rate is given by p_t/w, which, by (1), is also equal to $\tau_t(n/\phi_t)$. The (gross) internal rate of return of the social security program is defined as the pension received relative to the contribution paid:

$$\vartheta_{t+1} := \frac{p_{t+1}}{\tau_t w} = \frac{n}{\phi_{t+1}} \cdot \frac{\tau_{t+1}}{\tau_t}$$

Social Security Scheme A social security scheme is a sequence of history-contingent pensions. Let $\{p\} = \{p(\phi^t) : t \geq 0, \phi^t \in \Phi^t\}$ denote this sequence where ϕ^t is the history of realizations of the mortality risk up to date t.

Households Denote the per-period consumption of the young by $c_t^y(\phi^{t-1}, \phi_t)$ and the per-period consumption of the old by $c_t^o(\phi^{t-1}, \phi_t)$. Since there is no storage, the per-period budget constraints for the young and for the old are $c_t^y(\phi^{t-1}, \phi_t) = w(1 - \tau(\phi^t))$ and $c_t^o(\phi^{t-1}, \phi_t) = \alpha + p(\phi^t)$. Note that the consumption of the young and old depends on the history (ϕ^{t-1}, ϕ_t) because the social security benefits can, in principle, depend on the history of states. In autarky, agents consume only their own endowments, that is, $c_t^y(\phi^{t-1}, \phi_t) = w$ and $c_t^o(\phi^{t-1}, \phi_t) = \alpha$ for all t and (ϕ^{t-1}, ϕ_t).

Each generation is born after the uncertainty of that period has resolved. Therefore, after birth, a generation only faces uncertainty in old age. Moreover, there is no insurance

[4] There is a substantial body of literature in actuarial finance demonstrating that changes in longevity are largely unexpected, and the design of security instruments for hedging this source of risk is relevant (see, for example, Blake and Burrows, 2001, Cocco and Gomes, 2012).

[5] That is, ϖ is the left eigenvector of the matrix Π corresponding to a unit eigenvalue.

market in which the young can insure against their longevity risk. Therefore, the only means of hedging against this source of risk is through transfers between the young and the old. There are no altruistic or bequest motives. The lifetime utility over autarky for a generation born after the history ϕ^t is

$$U(\{p\};\phi^t) := u(w - p(\phi^t)\phi_t/n)) - u(w) + \beta\mathbb{E}_t\left[\phi_{t+1}\left(u(\alpha + p(\phi^t,\phi_{t+1})) - u(\alpha)\right)\right],$$

where $u(\cdot)$ is the per-period utility function, common to both the young and the old, $\beta \in (0,1]$ is the generational discount factor, and $\mathbb{E}_t$ is the expectation over future histories conditional on ϕ^t. We assume the per-period utility function is of CRRA form, $u(c) = (c^{1-\gamma}-1)/(1-\gamma)$ with $u(c) = \log(c)$ in the limit as $\gamma \to 1$.

Benevolent Social Planner The benevolent social planner chooses a social security scheme $\{p\}$ to maximize a discounted sum of the expected utilities of all generations. Let the social planner value function (net of autarky payoffs) be:

$$V\left(\{p\};\phi^t\right) := \frac{\beta}{\delta}\frac{\phi_t}{n}\left(u(\alpha + p(\phi^t)) - u(\alpha)\right) + \mathbb{E}_t\left[\sum\nolimits_{j=t}^{\infty}(\delta n)^{j-t}U\left(\{p\};\phi^j\right)\right]$$

where $\delta \in (0,1)$ is the discount factor of the planner and $\beta/(\delta n)$ is the weight on the utility of the initial old and ϕ_t/n is the dependency ratio.[6]

In seeking to maximize the discounted sum of expected lifetime utilities, the social planner chooses a social security rule that respects the constraint that the working generation voluntarily pay contributions to cover social security benefits to the old. That is, the social planner must respect the constraint that neither the old nor the young would be better off in autarky than adhering to the specified social security rule for any history of shocks. For the old, this means they will not make a positive transfer to the young because there is no future benefit to offset such a transfer. Hence,

$$p(\phi^t) \geq 0 \quad \text{for all } t \geq 0 \text{ and } \phi^t \in \Phi^t. \tag{2}$$

The analogous participation constraint for the young requires that the conditional transfers promised for their old age sufficiently compensate for the transfer made when young so that they are no worse off than reneging on the transfer today and receiving the corresponding autarkic lifetime utility. That is,

[6]The assumption of geometric discounting for the planner is common (see, for example, Farhi and Werning, 2007).

$$U(\{p\};\phi^t) \geq 0 \quad \text{for all } t \geq 0 \text{ and } \phi^t \in \Phi^t. \tag{3}$$

Hence, at $t = 0$, the social planner chooses the history-contingent stream of social security benefits $\{p\} = \{p(\phi^t) : t \geq 0, \phi^t \in \Phi^t\}$ that maximizes:

$$V(\{p\};\phi_0), \tag{4}$$

subject to the constraint set $\Lambda := \{\{p\} \mid (2) \text{ and } (3)\}$.[7]

Definition 1. *A social security rule is sustainable if the history-dependent sequence* $\{p\} \in \Lambda$ *and optimal if it maximizes the objective in equation* (4) *subject to the promise keeping constraint that the initial old receive a benefit larger than or equal to a level* $\bar{p}_0$*:*

$$p(\phi_0) \geq \bar{p}_0. \tag{5}$$

It is well known (see, for example, Aiyagari and Peled, 1991) that a necessary and sufficient condition for the existence of a non-trivial system of transfers is that the Perron root of the state-price matrix, evaluated at autarky, is larger than one. Let $\hat{m} = \beta u'(\alpha)/u'(w)$ denote an agent's marginal rate of substitution between consumption when old and consumption when young when the survival probability were one and let $\hat{r} = 1/\hat{m}$ denote the implied interest rate for transferring consumption from young to old age. Let $\hat{Q}$ be the state-price matrix at autarky with a typical element $\hat{q}_{ij} = n\pi_{ij}\hat{m}\phi(j)/\phi(i)$. It can be seen that $\phi = (\phi(1), \ldots, \phi(K))$ is an eigenvector of $\hat{Q}$:

$$\hat{Q} \cdot \phi = n \cdot \hat{m} \cdot \phi.$$

That is $n\hat{m}$ is the corresponding eigenvalue. By the Perron-Frobenius theorem, this is the Perron root (largest eigenvalue) of $\hat{Q}$. This eigenvalue is larger than one when $n > \hat{r}$. We assume this to be the case.

Assumption 1. $n > \hat{r}$.

This is the same as the standard Samuelson dynamic inefficiency condition in the deterministic case. Aggregate consumption can be increased by transferring resources from young to old. Assumption 1 requires that the autarkic allocation is dynamically inefficient.

[7]Since utility is strictly concave and the constraints in (3) are linear in utility, the planner's objective in equation (4) is concave and the constraint set Λ is convex and compact.

Proposition 1. *Under Assumption 1, there exists a non-trivial and stationary sustainable social security rule that improves upon autarky.*

III Recursive Problem

In this section, we characterize the optimal social security rule chosen by the social planner under the limited enforcement constraints considering the following recursive formulation of the maximization problem, where $\phi = \phi(i)$ and p_j is the promised social security benefit for next period when the realised survival probability is $\phi(j)$:

$$V_i(p) = \max_{\tau,(p_j)_{j=1}^K} \frac{\beta}{\delta}\frac{\phi}{n}\left(u\left(\alpha + \tau w\frac{n}{\phi}\right) - u(\alpha)\right) + (u(w(1-\tau)) - u(w)) + (\delta n)\sum_{j=1}^K \pi_{ij}V_j(p_j)$$

subject to:

$$u\left(\alpha + \tau w\frac{n}{\phi}\right) \geq u(\alpha + p) \quad \left(\frac{\beta}{\delta}\frac{\phi}{n}\lambda\right), \tag{6}$$

$$u(w(1-\tau)) - u(w) + \beta\sum_{j=1}^K \pi_{ij}\phi(j)(u(\alpha + p_j) - u(\alpha)) \geq 0 \quad (\mu), \tag{7}$$

where (6) represent the promise-keeping constraint, which ensure that past promises are maintained, while (7) represent the young participation constraints. The policies are also subject to boundary constraints, $0 \leq \tau \leq \tau_{\max}(i)$ and $0 \leq p_j \leq p_{\max}(j)$, where $\tau_{\max}(i)$ is the largest solution in which constraint (7) holds with equality and $p_{\max}(j) = \tau_{\max}(j)wn/\phi(j)$.[8] Let $\mathcal{T}(\phi) = [0, \tau_{\max}(\phi)]$ and $\mathcal{P}(\phi) = [0, p_{\max}(\phi)]$. Let $\mathcal{P} = \cup_{\phi\in\Phi}\mathcal{P}(\phi)$ and $\mathcal{X} = \mathcal{P}\times\Phi$.

Let (p, ϕ) be the state vector, $f(p, \phi)$ be the stochastic policy function for the contribution rate, and $g_j(p, \phi))$ be the stochastic policy function for the state-contingent promised pension with $j = 1, 2, ..., K$ where $f\colon \mathcal{X} \to \mathcal{T}(\phi)$ and $g_j\colon \mathcal{X} \to \mathcal{P}(\phi(j))$. The functions $\lambda(p, \phi)$ and $\mu(p, \phi)$ are the multipliers associated with constraints (6) and (7) where $\lambda, \mu\colon \mathcal{X} \to \Re_+$.[9] Ignoring the upper and lower bound constraints, the first-order conditions are:[10]

[8] If the states $\phi(i)$ are i.i.d., then $\tau_{\max}$ is independent of i and $p_{\max}(j) = p_{\max}(i)\phi(i)/\phi(j)$.

[9] The replacement ratio is $g_j(p, \phi)/w$. The implicit rate of return on the contribution is $\vartheta_j(p, \phi) = g_j(x)/(wf(p, \phi))$.

[10] We can ignore the lower bound condition on τ because $\tau > 0$ unless $p = 0$. In the latter case, $\lambda = \mu = 0$ and the first-order condition for τ is $\hat{m} \leq \delta$ where $\hat{m} = \beta u'(\alpha)/u'(w)$. But by Assumption 1 $n > \hat{r} = 1/\hat{m}$ and so $\delta/\hat{n} < \delta n < 1$, a contradiction.

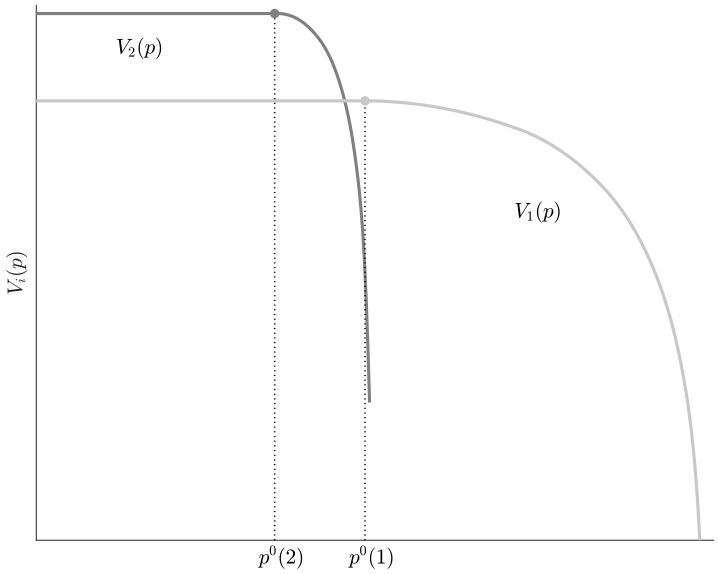


Figure 2: Value Functions. The figure plots a case with two states: $\phi \in \{\phi(1), \phi(2)\}$. The dark grey line is the value function $V_2(p)$ and the light grey line is the value function $V_1(p)$.

$$\frac{u'(w(1-f(p,\phi)))}{u'(\alpha+p)} = \frac{\beta}{\delta}\left(\frac{1+\lambda(p,\phi)}{1+\mu(p,\phi)}\right), \tag{8}$$

$$V_j'(g_j(p,\phi)) = -\mu(p,\phi)\frac{\beta}{\delta}\frac{\phi(j)}{n}u'(\alpha+g_j(p,\phi)) \quad \text{for each } j=1,2,\ldots,K. \tag{9}$$

There is also an envelope condition:

$$V_i'(p) = -\lambda\frac{\beta}{\delta}\frac{\phi}{n}u'(\alpha+p), \tag{10}$$

For the strict concavity of the utility function and the convexity of the constraint set, the value function $V_i(p)$ is a concave and decreasing function in p with $p^0(i) = \sup\{p|V_i'(p) = 0\}$. The value functions $V_i(p)$ for a case with $K = 2$ are plotted in Figure 2. The light gray line represents V_1, while the dark gray line represents V_2. The value functions are flat when the social security benefits are less than $p^0(i)$, and are monotonically decreasing when $p \in (p^0(i), p_{\max}(i)]$. For a given current entitlement to the old p and a realized survival probability ϕ, the social planner would ideally like to promise current workers benefits of $p_j = p^0(j)$. However, these promised benefits may violate their participation constraints (7), rendering the social security rule unsustainable. The presence of limited enforcement frictions therefore introduces a form of gradualism in the adjustment of policies over time. This property is better understood by first

considering two benchmark cases: i) the full enforcement case; and ii) the deterministic survival probability case.

III.1 Full Enforcement

The full enforcement (or first-best) outcome is obtained when the participation constraints specified in (7) are not imposed. Let $V_i^*(p)$ denote the value function in this case. The domain of V_i^* is $[0, wn/\phi(i)]$.

If constraint (6) is slack, then for any ϕ, the first-best pension $p^*(\phi)$ satisfies a standard Arrow-Borch risk-sharing property

$$\frac{u'\left(w - \frac{\phi}{n}p^*(\phi)\right)}{u'\left(\alpha + p^*(\phi)\right)} = \frac{\beta}{\delta}. \tag{11}$$

In the first-best, and for any ϕ, the planner chooses the pension p that maximises the weighted sum of utilities of the young and old: $\delta n u(w - (\phi/n)p) + \beta\phi u(\alpha + p)$ where δn is the weight on the young and $\beta\phi$ is the weight on the old. Maximising with respect to p gives the first-order condition in (11).[11] It follows from (11) and the concavity of the utility function that $p^*(\phi)$ is decreasing in ϕ and the first-best contribution rate $\tau^*(\phi) = (\phi/n)(p^*(\phi)/w)$ is increasing in ϕ. That is, risk is shared between generations: an increase in ϕ (a higher dependency ratio) leads to an increased contribution rate and a lower pension. The history of past shocks is irrelevant in this first-best allocation. If the discount factor of the planner is the same as the discount factor of the agent $\delta = \beta$, then the equation (11) shows that consumption is equalised between young and old.

From Definition 1, constraint (6) does not bind in the initial period if $\bar{p}_0 \leq p^*(\phi_0)$, where $\bar{p}_0$ is the pension promised to the initial old and ϕ_0 is the corresponding survival probability. In this case, the first-best contribution rate at $t = 0$ is $\tau^*(\phi_0)$, determined by condition (11) just as in every other time $t > 0$. On the other hand, if the pension promised to the initial old is large enough such that $\bar{p}_0 > p^*(\phi_0)$, then the constraint (6) binds and the initial contribution rate is correspondingly higher. Since the pension is high and the contribution rate is high, the marginal utility of the young is low, and the marginal utility of the young is high. Thus, the ratio of marginal utilities given by the condition (11) is larger than β/δ. We can write the difference in terms of a *wedge* λ_0 that is the Lagrangian multiplier on the constraint (6):

$$\frac{u'\left(w - \frac{\phi_0}{n}\bar{p}_0\right)}{u'\left(\alpha + \bar{p}_0\right)} = \frac{\beta}{\delta}\left(1 + \lambda_0\right). \tag{12}$$

[11] As explained previously, it follows from Assumption 1 and $\delta n < 1$ that $p^*(\phi) > 0$.

Note that this only occurs at date $t = 0$ if $\bar{p}_0 > p^*(\phi_0)$. In each subsequent period, the pension is set to $p^*(\phi)$ as determined by the optimality condition in (11). The pension is adjusted immediately to the first best at dates $t \geq 1$.

III.2 Deterministic Case

As an alternative benchmark, consider the case where the survival probability is deterministic and constant, but the planner respects the participation constraints of the young. Let p_t denote the pension at date t. Then, for any $t \geq 0$, constraint (7) becomes:

$$u(w - (\phi/n)p_t) - u(w) + \beta\phi\left(u(\alpha + p_{t+1}) - u(\alpha)\right) \geq 0. \tag{13}$$

When (13) holds as equality, it defines a difference equation, $p_{t+1} = y(p_t)$ where y is an increasing function with $y(p) < p$.[12] Assumption (1), together with the strict concavity of the utility function, implies that there are two stationary solutions that satisfy (13) with equality: $p = 0$ and a strictly positive solution $p = p_{\max} < wn/\phi$. The first-best pension is $p^*(\phi)$ given by equation (11) and there is a unique value $p^c > p^*(\phi)$ such that $y(p^c) = p^*(\phi)$. Hence, the optimal social security rule is determined by:

$$p_{t+1} = \begin{cases} p^*(\phi), & \text{for } p \in [0, p^c], \\ y(p), & \text{for } p \in (p^c, p_{\max}). \end{cases} \tag{14}$$

Or more simply, $p_{t+1} = g(p) = \max\{p^*(\phi), y(p)\}$. If $\bar{p}_0 > p^c$, then the pension decreases in the initial periods along $y(p)$ to $p = p^*(\phi)$. The function $g(p)$ is illustrated in Figure 3. Unlike the first-best, the convergence takes several periods, although $p^*(\phi)$ will be reached in a finite time. As with the first-best, if $\bar{p}_0 > p^*(\phi)$ the ratio of marginal utilities in equation (11) is above β/δ but declines over time rather than in one period. Intuitively, if the initial promised entitlement to the old is large, then the contribution rate on the current taxpayers is also large. The planner would like to reduce p to p^* as quickly as possible to improve welfare. But if future social security benefits are reduced too much, it will violate the participation constraint of current taxpayers. The presence of limited enforcement means that social security benefits and, in turn, contribution rates are gradually reduced, a form of *fiscal inertia*. However, in this deterministic case, the simultaneous increase in pension benefits and contribution rates that can be see in the data cannot occur. In the following section, we demonstrate that the interplay between limited enforcement and uncertainty produces more realistic dynamics in the evolution of pensions and contribution rates.

[12]The function $y(p) = u^{-1}(u(\alpha) - (\beta\phi)^{-1}(u(w - (\phi/n)p) - u(w))) - \alpha$.

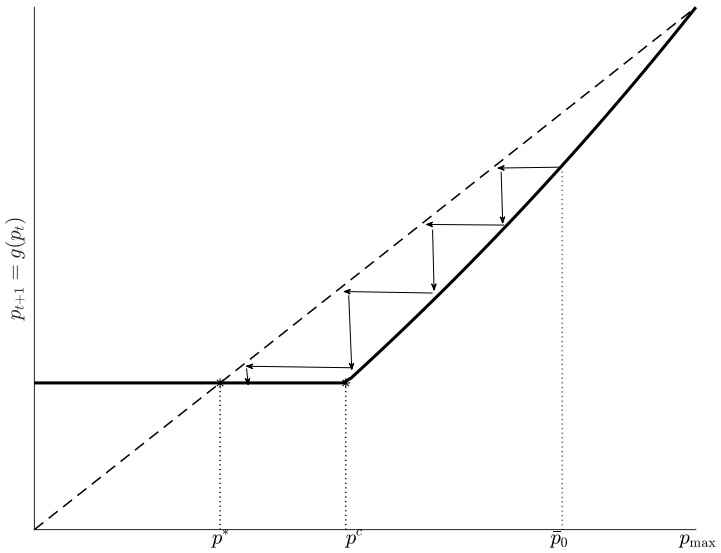


Figure 3: Social Security Rule in the Deterministic Case. For any initial $p \in [0, p_{\max}]$, the deterministic policy function is $p_{t+1} = g(p_t) = \max\{p^*(\phi), y(p_t)\}$, illustrated by the solid line.

III.3 Limited Enforcement

In this section, we consider the case where the survival probability is stochastic and the planner chooses a social security rule that satisfies the participation constraints of the young. We are interested in the properties of the policy functions $f(p, \phi)$ and $g_j(p, \phi)$ that determine the current contribution rate and the future pension promise conditional on the current pension promise p and the current survival probability ϕ.

We assume that the first-best policy is not sustainable, that is, there exists at least one state $\phi(i)$ in which the participation constraint of the young is violated at the first-best pension. That is, we assume:

Assumption 2. *There is some survival probability $\phi(i)$ such that:*

$$u\left(w - \frac{\phi(i)}{n} p^*(\phi(i))\right) - u(w) + \beta \sum_{j=1}^{K} \pi_{ij} \phi(j) \left(u(\alpha + p^*(\phi(j))) - u(\alpha)\right) < 0.$$

Assumption 2 may or may not hold, depending on the parameter values, but if it does not hold, then the long-run distribution converges to the first-best allocation described by equation (12) with the optimal pension taking on K values in the long run with probabilities ϖ, the stationary distribution of the transition matrix Π. We are however, interested in deriving the long-run properties of the solution when Assumption 2 holds. In particular, we show that when Assumption 2 holds, there is history dependence of the

optimal social security rule even in the long run.

First, consider equation (8). If the promise-keeping constraint (6) binds ($\lambda > 0$) but the participation constraint (7) does not ($\mu = 0$), then $f(p, \phi) = p\phi/(wn)$ and equation (8) becomes:

$$\frac{u'\left(w - \frac{\phi}{n}p\right)}{u'\left(\alpha + p\right)} = \frac{\beta}{\delta}\left(1 + \lambda(p, \phi)\right).$$

The promise keeping constraint (6) binds when the current promised pension p is high. Thus, the marginal utility of the old is relatively low, and because the contribution rate of the young is high, the marginal utility of the young is relatively low. Thus, exactly as in the initial period in the full enforcement outcome, if the current pension promise is high, the ratio of the marginal utilities is greater than β/δ, with the wedge between the ratio of the marginal utilities and $\beta.\delta$ measured by the Lagrangian multiplier $\lambda(p, \phi)$. On the other hand, if the promise-keeping constraint (6) does not bind ($\lambda = 0$) but the participation constraint (7) does ($\mu > 0$), then it follows from the envelope condition (10) that $V_i'(p) = 0$ and the contribution rate can be raised to $f(p, \phi) = p^0(\phi)\phi/(wn)$ without reducing the planner's payoff. In this case, equation (8) becomes:

$$\frac{u'(w(1 - f(p, \phi)))}{u'\left(\alpha + p\right)} = \frac{\beta}{\delta}\left(\frac{1}{1 + \mu(p, \phi)}\right).$$

Since the participation constraint is binding, the contribution rate cannot be raised too high, and in particular cannot be raised to the first-best level, so $p^0(\phi) < p^*(\phi)$. With the promised pension low and the current contribution restricted by the participation constraint, the ratio of marginal utilities is less than β/δ, with the wedge between the ratio of the marginal utilities and β/δ measured by the Lagrangian multiplier $\mu(p, \phi)$. The first order condition (8) shows that whether the pension is above or below the full enforcement outcome depends on the balance of the commitment and participation constraints.

The first-order condition (9) provides the link between the promise-keeping and participation constraints in adjacent periods. In particular, using the envelope condition (10) and iterating equation (9) one period forward, gives the following updating rule:

$$\lambda(g_j(p, \phi), \phi(j)) = \mu(p, \phi). \tag{15}$$

Equation (15) provides an updating rule showing that the multiplier of the participation constraint for the young in a given state (p, ϕ) equals the multiplier of the promise keeping constraint in the subsequent state $\lambda(g_j(p, \phi), \phi(j))$. Specifically, if the current

participation constraint (7) is slack ($\mu(p,\phi)=0$) for a pension promise p and a survival probability ϕ, then the promise-keeping constraint (6) in the next period will also be slack ($\lambda(g_j(p,\phi),\phi(j))=0$). In that case, the social planner chooses the highest future pension promise to maximise $V_j(p)$, that is, $g_j(p,\phi)=p^0(j)$. Since the future pension promise is chosen optimally, $f(p,\phi)=(\phi/(wn))\max\{p^0(\phi),p\}$.

It also follows from (9), together with the strict concavity of u and V (and that V is downward sloping), that the state-contingent social security benefit p_j is a strictly increasing function of the multiplier μ, say $p_j=\psi_j(\mu)$. Hence, for all $t\geq 1$, when constraint (7) binds, there is a unique value of μ, say, $\mu=\varphi_i(w-(\phi/n)p)$ that satisfies this constraint where φ_i is decreasing in $w-(\phi/n)p$.[13] Consequently, $g_j(p,\phi)=\psi_j(\varphi_i(w-(\phi/n)p))$ increases in both the current pension p and the realized survival probability ϕ.

Figure 4 provides an illustration of the optimal stochastic functions $f(p,\phi)$ and $g_j(p,\phi)$ for a case with $K=2$, $\phi(1)<\phi(2)$, and i.i.d. transition probabilities. Assume that the participation constraint does not bind for $p=p^*(\phi(1))$. The upper row of Figure 4 shows the contribution rate $f(p,\phi)$ and the lower row plots the promised future pension $g_j(p,\phi)$, $j=1,2$. The left column has $\phi=\phi(1)$ and the right column has $\phi=\phi(2)$.

The upper row shows that $f(p,\phi)$ is flat up to $p=p^0(\phi)$ and then increases with slope $\phi/(wn)$. The full enforcement outcome has $p^*(\phi(2))<p^*(\phi(2)$ and by Assumption 2, $p^0(\phi(2))<p^*(\phi(2))$. Hence, $p^0(\phi(2))<p^*(\phi(2))<p^*(\phi(1))=p^0(\phi(1))$.

The bottom row illustrates the policy functions $g_j(p,\phi(i))$. The policy functions are flat up to the point where the participation constraint starts to bind. With i.i.d. transition probabilities, there are critical values $p^c(\phi(1))$ and $p^c(\phi(2))$ such that $g_j(p,\phi(1))$, $j=1,2$, increases in p for $p>p^c(\phi(1))$ and $g_j(p,\phi(2))$, $j=1,2$, increase for $p>p^c(\phi(2))$.[14] As in the full enforcement outcome $g_1(p,\phi)>g_2(p,\phi)$. That is, risk is shared.

To understand how these policy functions generate a dynamic sequence of pension promises, consider starting with $p=p^*(\phi(1))$ with survival probability $\phi(1)$. The initial contribution rate is $\tau^*(\phi(1))$ and future pension promises contingent on the future survival probability are $p^0(\phi(j))$, $j=1,2$. Suppose that next period, the survival probability is $\phi(2)$ (see the left panels of Figure 4). To satisfy the participation constraint, the contribution rate is lower than $\tau^*(\phi(2))$ but because the policy functions are increasing in p, the promised pensions are also increased. If the high survival probability state

[13] When the transition probabilities are i.i.d., the function φ_i is independent of i. Otherwise, φ_i depends on the current survival probability through the properties of the transition matrix and how ψ_j depends on j.

[14] Where $p^c(\phi(1))\phi(1)=p^c(\phi(2))\phi(2)$.

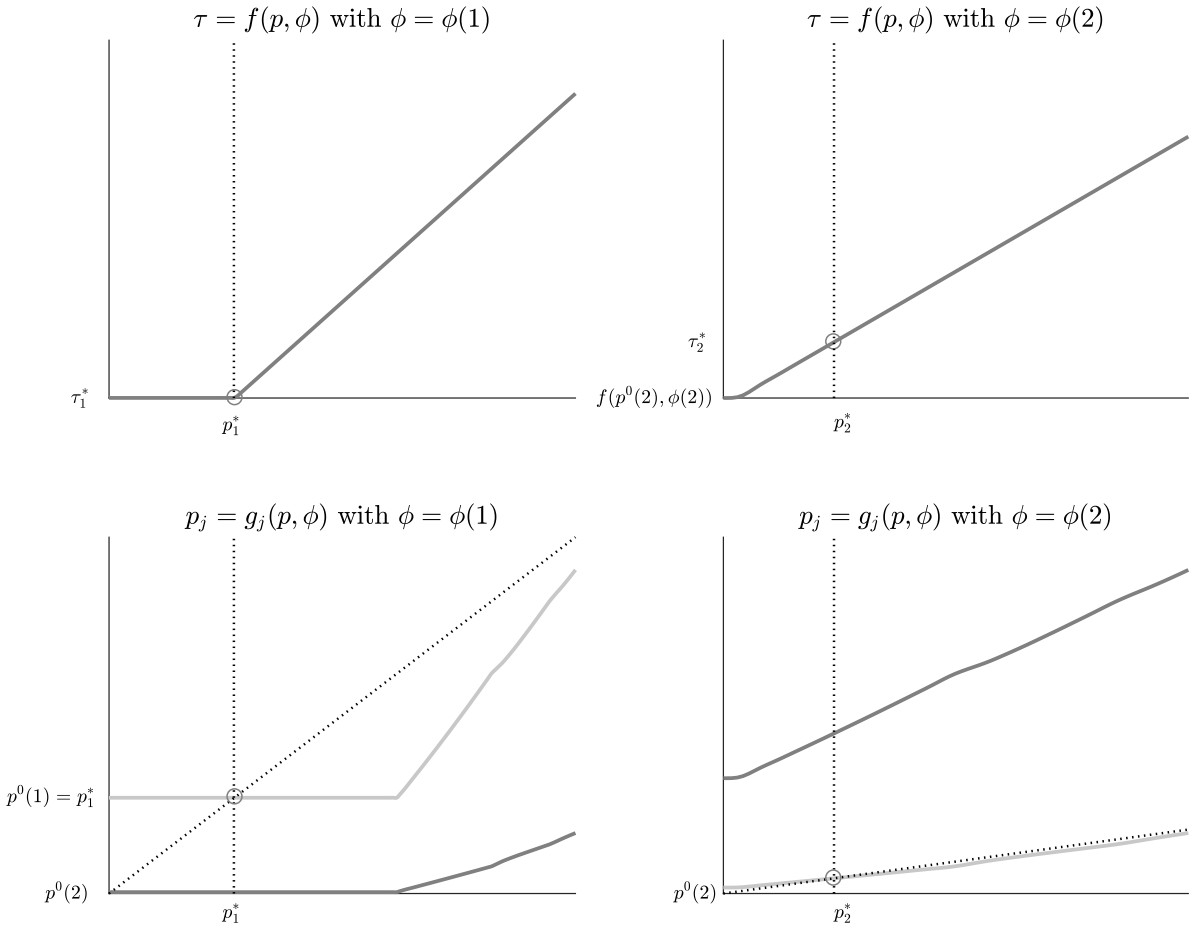


Figure 4: Social Security Rule in the Stochastic Case. The figure plots a case with two states: $\phi \in \{\phi(1), \phi(2)\}$. In the first row, the panels plot the policy $\tau = f(p, \phi)$ when $\phi = \phi(1)$ (left panel) and $\phi = \phi(2)$ (right panel). In the second row, the panels plot the policy $p_j = g_j(p, \phi)$ when $\phi = \phi(1)$ (left panel) and $\phi = \phi(2)$ (right panel). In the left panel, the light gray line is $g_1(p, \phi(1))$ and the dark gray line is $g_2(p, \phi(1))$. In the right panel, the light gray line is $g_2(p, \phi(2))$ and the dark gray line is $g_1(p, \phi(2)$. Note the diagrams are not drawn on the same scale. The 45° lines are indicated by dotted lines.

persists in subsequent periods, contribution rates increase to fulfill past promises, and future social security entitlements continue to rise, converging toward the fixed point p_2^* after an infinite sequence of realisations of $\phi(2)$. Hence a simultaneous rise of contribution rates and replacement rates can endogenously be generated thanks to the interplay of limited enforcement and uncertainty. Conversely, if a low survival probability state occurs again, the high promised entitlements to the old is maintained, but with a contribution from the next-period taxpayer that is smaller than in the previous period, as fewer individuals have survived to old age. In the illustrated example, $f(p, \phi(2))$ is such that $\mu(f(p^*(\phi(2)), \phi(1))) = 0$. Therefore, future pension promises are again $p^0(\phi(j))$, $j = 1, 2$. The history of past shocks is forgotten, allowing a new cycle to arise.

Figure 5 displays the realisations of the replacement ratio (panel A), contribution rates (panel B), and implied internal rate of return of the pension for a particular sample path of the realised survival probabilities, indicated by the blue dots. It compares the dynamics of the full enforcement outcome with those of the limited enforcement solution. Figure 5 reveals that the variability of the replacement ratio is greater in the limited enforcement outcome than in the full enforcement outcome, while the variability of contribution rates is smaller. Furthermore, the internal rate of return oscillates between the extreme values $1/\phi(2)$ and $1/\phi(1)$ and is sometimes higher and sometimes lower in the limited enforcement outcome than in the full enforcement outcome.

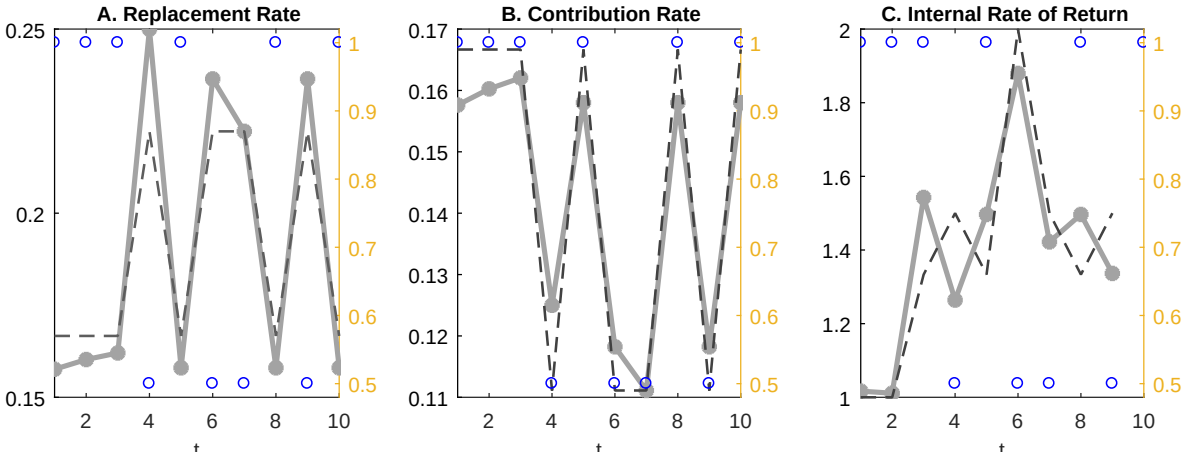


Figure 5: Simulated Paths. The panels plot a simulated path of replacement ratios (Panel A), contribution rates (Panel B), and internal rate of return of pensions (Panel C) for a case with two states: $\phi \in \{\phi(1), \phi(2)\}$, with parameters: $\phi(1) = 0.5$, $\phi(2) = 1$, $\pi_{ij} = 1/2$ for each $\phi(i)$ and $\phi(j)$, $w = 1.5$, $\alpha = 1$, $\beta = \delta = 0.9$, and $\gamma = 1.2$. The blue dots are the realization of survival probabilities ϕ whose value is reported on the right-hand-side vertical axis.

III.4 Long Run Distribution

This section considers the long-run distribution of the pair $x = (p, \phi)$. It uses the approach of Lancia et al. (2024) to show that there is a unique and countable ergodic set E with cardinality $|E| > K$ and strong convergence to the corresponding invariant distribution. For simplicity, we assume that there is at least one state, say k with survival probability $\phi(k)$, such that the first-best is sustainable in that state.

Assumption 3. *There is a survival probability $\phi(k) \in \Phi$ such that:*

$$u(w(1-\tau^*(\phi(k)))) - u(w) + \beta \sum_{j=1}^{K} \pi_{kj}\phi(j)\left(u(\alpha + g_j(p^*(\phi(k)), \phi(k))) - u(\alpha)\right) > 0.$$

Using Assumption 3, it can be shown that there is a critical value $p^c(k) > p^*(\phi(k))$

such that $g_k(p, \phi(k)) = p^*(\phi(k))$ for $p < p^c(k)$.[15]

Let $x = (p, \phi)$ denote the state that encompasses the current pension promise and current survival probability. The future evolution of x is a Markov chain defined by the transition function $P(x, A \times B) := \Pr\{x_{t+1} \in A \times B \mid x_t = (p, \phi(i))\} = \sum_{\phi(j) \in B} \pi_{ij} \mathbb{1}_A g_j(p, \phi(i))$ where $A \subseteq \mathcal{A}$, $B \subseteq \Phi$ and $\mathbb{1}_A g_j(p, \phi(i)) = 1$ if $g_j(p, \phi(i)) \in A$ and zero otherwise. The chain starts from $x_0 = (p_0, \phi_0)$ for some initial promise p_0.

The monotonicity of $g_j(p, \phi)$ in p and Assumption 3 imply that starting from any x_t, it is possible to construct a finite sequence of survival probabilities, say of length ℓ, ending with the survival probability $\phi(k)$ such that $x_{t+\ell} = (p^*(\phi(k)), \phi(k))$. This sequence is constructed so that the pension decreases at each date.[16] In this case, we say that x is *reset* to $(p^*(\phi(k)), \phi(k))$ at time $t+\ell$. Since the transition probabilities $\pi_{ij} > 0$ for each i, j, the probability of this finite sequence is also strictly positive. An immediate consequence is that Condition **M** of Stokey et al. (1989, page 348) is satisfied, and therefore there is strong convergence in the uniform metric to a unique invariant probability measure $\varphi(X)$ for $X \in \mathcal{X}$ (see, Lancia et al., 2024).

Since there is a strictly positive probability that x is reset to $(p^*(\phi(k)), \phi(k))$ in finite time, the Markov chain for x is *regenerative* and $(p^*(\phi(k)), \phi(k))$ is a regeneration point (see, for example, Foss et al., 2018).[17] For simplicity, suppose that the Markov chain starts at $x_0 = (p^*(\phi(k)), \phi(k))$. The state $x = (p, \phi)$ is said to be *accessible* from x_0 if there is a positive probability path, starting with x_0, that reaches x in finite time. Since x_0 has a positive probability mass and the set of endowment states Φ is finite and time is discrete, the associated set of all states accessible from x_0 is countable. Furthermore, the set E is an equivalence class because every $x \in E$ is accessible from x_0, and a sequence can be constructed such that there is a path from every accessible x back to x_0. Therefore, E is an absorbing set, that is, $P(x, E) = 1$ for all $x \in E$, and since no proper subset of E has this property, it is *ergodic* (see, for example, Stokey et al., 1989, chapter 11). It follows that the *expected return time* to x is finite and thus each $x \in E$ is positive recurrent. Since the ergodic set E is countable, standard results on the convergence of positive recurrent Markov chains apply.[18] To state these results, let P denote the transition matrix with

[15] Assumption 3 is a sufficient and not a necessary condition for strong convergence. We can construct examples where Assumption 3 does not hold but there is strong convergence because at least one of the policy functions is flat in the relevant range.

[16] This is analogous to the deterministic case we considered in Subsection III.2.

[17] The chain can be divided into blocks. A regeneration time is the time at which the next regeneration point is reached. The blocks are determined by these regeneration times. At each regeneration point, past shocks are forgotten and the future evolution of x is probabilistically identical. The regeneration times are also i.i.d./ and so standard results on the convergence of i.i.d./ processes can be applied.

[18] It is not necessary to assume $x_0 = (p^*(\phi(k)), \phi(k))$. The same arguments apply for any $x_0 \in E$,

elements $P(x, x') = \pi_{ij}\mathbb{1}_{p'}g_j(x)$ where $x = (p, \phi(i))$ and $x' = (p', \phi(j))$. Similarly, let $P^k(x, x')$ be the elements of the corresponding k-period transition matrix.

Proposition 2. *(i) There is pointwise convergence to a unique and non-degenerate invariant distribution $\varphi = \varphi P$ where for each $x \in E$, $\varphi(x) = \lim_{k\to\infty} P^k(\cdot, x)$. (ii) The invariant distribution is the limit of the iteration $\varphi_{t+1}(x') = \sum_{x\in E} P(x, x')\varphi_t(x)$ for any given $\phi_0(x)$. (iii) The cardinality $|E| > K$.*

Parts (i) and (ii) of Proposition 2 are standard and show convergence to a unique invariant distribution φ. Moreover, for each $x \in E$, $\varphi(x)$ is the inverse of the expected return time. The invariant distribution can be computed iteratively, given knowledge of the policy functions. In particular, for $\phi_0 = \phi(k)$, the invariant distribution can be computed starting from an initial distribution $\varphi_0(x) = 1$ if $x = (p^*(\phi(k)), \phi(k))$ and $\varphi_0(x) = 0$ otherwise..[19] Part (iii) shows that the cardinality of the ergodic set is greater than K and follows directly from Assumption 2 that the full enforcement outcome is not sustainable. This result stands in contrast to the full enforcement and deterministic benchmarks where the cardinality of the ergodic set equal to the cardinality of the set of endowment states.

III.5 Impulse Response Functions

In this subsection, we consider a one-period, unexpected change in population size, an MIT shock. We use the convergence result of the previous subsection to compute averages at the long-run invariant distribution of pensions and the corresponding contributions. Using these long-run averages, we analyze the impulse response function generated from the policy functions for the contribution rate and the future pension benefits. Our findings show that, compared to the full enforcement case, the impact of this demographic shock is amplified, prolonged, and non-linear, and the impact is asymmetric depending on the direction of the shock.

The first row of Figure 6 shows the changes in the contribution rate (left panel) and pension benefits (right panel) after a temporary increase in the population growth rate n at date $t = 0$. The second row presents the same impulse response functions for a temporary decrease in the population growth rate.

In the case of a positive demographic shock, the temporary increase in the ratio of young (tax-payers) to old (pensioners) allows the social planner, if not constrained by

since any state in E is accessible from any other. Convergence also applies for a starting point $x_0 \notin E$ since after a finite time some $x \in E$ is reached and the argument can be applied from this point.

[19] This iteration results holds more generally for any initial distribution $\phi_0(A)$.

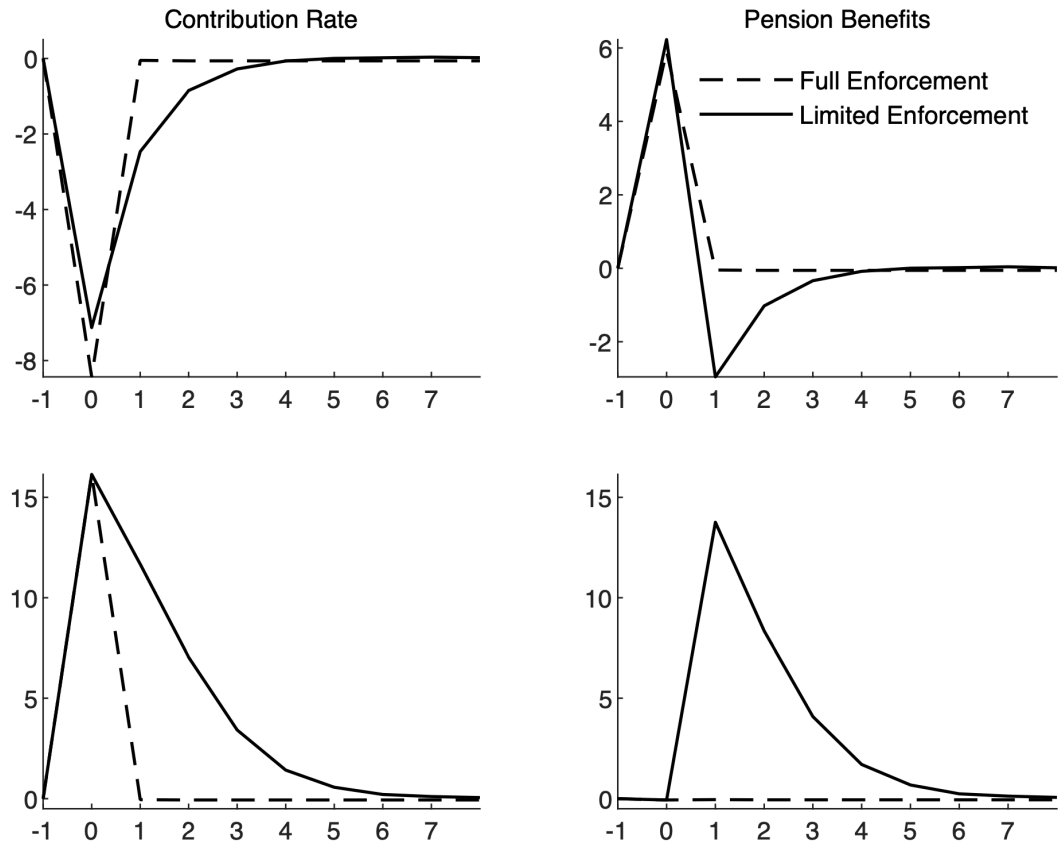


Figure 6: Impulse Response Functions for a Demographic Shock. The panels illustrate the average contribution rate (first column) and pension benefits (second column) in response to a demographic shock occuring at time $t = 0$, characterized by a temporary, unexpected 15% increase in the gross growth rate n (first row) and a 15% decrease (second row). These results are based on optimal policies derived from a numerical simulation using parameter values as in Figure 5.

limited enforcement, to reduce the burden on each contributor and raise transfers to each beneficiary. This results in a lower contribution rate and higher pension benefits for pensioners. Since the shock is temporary, the ratio of young to old returns to its pre-shock level in the following period, as does the redistribution of resources across generations. Therefore, the effect of the shock is short-lived.

However, if the social planner is constrained by limited enforcement, the positive demographic shock also relaxes the participation constraint by easing the tax burden on contributors. While pension benefits initially increase due to the relaxation of the budget constraint, they may be sharply reduced in the following period as the retiring generation experienced a slackened participation constraint when young. Contribution rates gradually return to their pre-shock levels, and pension benefits follow suit.

The impact of a negative shock, such as a temporary contraction in the population growth rate n, looks very different. With fewer contributors but unchanged pension promises, pension benefits remain stable, and the entire burden of the shock is absorbed by an increase in the contribution rate required from the young. Under full enforcement, where the young generation's participation restriction is not a concern, the social planner can quickly revert to the pre-shock conditions after just one period.

In contrast, under limited enforcement, the heavy tax burden imposed on the young due to the negative demographic shock must be compensated by significantly increasing pension promises in the period following the shock. Although the ratio of young to old returns to its pre-shock level after one period, the tightening of the participation constraint for one generation carries over, affecting future generations. As a result, reverting to pre-shock policies occurs gradually, and the effects take a long time to fully dissipate

IV Sustainable Social Security Rule

We show that in the case when the transition probabilities are i.i.d., there exists a straightforward rule that links the social security benefits received in old age with the contribution rate paid when young. As we have seen, the promise-keeping constraint (6) binds, except possibly in the initial period. Thus, the policy function $f(p,\phi)$ can be inverted to give $p = f^{-1}(\tau,\phi)$ where f^{-1} is the partial inverse. We have shown that the policy function $g_j(p,\phi)$ can be written as $g_j(p,\phi) = \psi_j(\varphi_i(w-(\phi/n)p))$ where ψ_j is an increasing function and φ_i is a decreasing function. Moreover, φ_i is independent of i when the transition probabilities are i.i.d.: $\pi_{ij} = \pi_j$. Since $f^{-1}(\tau,\phi) = \tau wn/\phi$, it follows that, when the transition probabilities are i.i.d., $g_j(p,\phi) = h_j(\tau) = \psi_j(\varphi(w(1-\tau)))$ is independent of ϕ. Furthermore, since $\mu = \varphi(w(1-\tau))$, there is a critical value of τ, say τ^{c}, such that the participation constraint binds for $\tau > \tau^{\mathrm{c}}$ and does not bind otherwise. That is, $h_j(\tau)$ is constant for $\tau < \tau^{\mathrm{c}}$ and increasing for $\tau > \tau^{\mathrm{c}}$ for each $j = 1, 2, \ldots, K$. This is formalised in the following proposition.

Proposition 3. *Suppose the transition probabilities are i.i.d., $\pi_{ij} = \pi_j$. The social security rule $h_j(\tau)$ is continuous in τ and there is a unique threshold $\tau^{\mathrm{c}} := 1 - w^{-1}u^{-1}(u(w) - \beta\sum_j \pi_j\phi(j)(u(\alpha+p^0(j)) - u(\alpha))$ such that: (i) for $\tau \le \tau^{\mathrm{c}}$, $h_j(\tau) = p^0(j)$; (ii) for $\tau > \tau^{\mathrm{c}}$, $h_j(\tau)$ is strictly increasing in τ; and (iii) for any τ, $h_{j'}(\tau) \ge h_j(\tau)$ if $\phi(j') < \phi(j)$.*

Proposition 3 demonstrates that the optimal rule is nonlinear, as illustrated in Figure 7. There exists a critical contribution rate τ^c, defined as the maximum rate below which the participation becomes nonbinding, considering as state-contingent promised pensions the

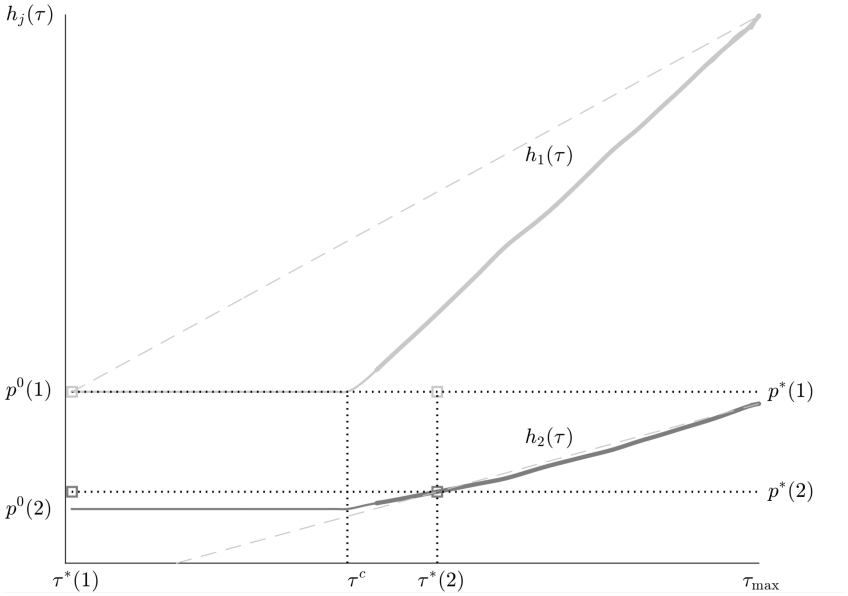


Figure 7: Sustainable Social Security Rule. The figure plots a case with two i.i.d. states: $\phi \in \{\phi(1), \phi(2)\}$. The light grey line is the policy rule $h_1(\tau)$ and the dark grey line is the policy rule $h_2(\tau)$. Square dots indicate the levels of contribution and pension benefits in the first-best case when $\phi(j) = \phi(1)$ and diamond dots indicate the levels of contribution and pension benefits in the first-best case when $\phi(j) = \phi(2)$.

resetting levels $p^0(j)$.[20] The properties of the rule depend on whether the contributions paid are above or below this endogenous threshold. The rule incorporates three key features: (i) a *guaranteed minimum level* of benefits $p^0(j)$ if contributions during working age are less than or equal to the threshold τ^c; (ii) a *contributory-related component*, with social security benefits positively linked to past contributions if those contributions exceed τ^c; and (iii) a negative indexation of social security benefits to the survival probability realized during retirement, enhancing the financial sustainability of the system.

Therefore, the optimal social security scheme requires automatic adjustments of benefits to longevity risk and a combination of flat-rate and contributory-related elements. These features resemble some properties commonly observed in national pension systems. For example, in recent years, several countries including Sweden, Italy, Portugal, and Latvia have reformed their pension systems from defined benefit to notional defined contribution ones, wherein the contribution rate is promised to remain more stable, benefits are closely tied to contributions, and rules are in place to allow for endogenous adjustment of benefits in response to demographic shocks such as the increase in survival probability.

[20]For each $\phi(i)$, we can also show that there is a τ_i^{f} such that $\tau_i^{\mathrm{f}} = \min\{\tau_i^*, \tau_{\max}\}$ and $h_j(\tau_i^{\mathrm{f}}) = \min\{p_j^*, p_{\max}(j)\}$. For some state $\phi(i)$ sufficiently high, $\tau_i^{\mathrm{f}} > \tau^{\mathrm{c}}$.

In particular, in the absence of enforcement constraints, the ideal scheme simplifies to include only flat-rate benefits linked to the survival probability. The contributory-related element arises because the enforcement of transfers is limited.

IV.1 Comparison with Alternative Social Security Rules

As discussed in the preceding section, the optimal sustainable social security rule has pension benefits that are a non-linear function of the contribution rate paid during the working age and are linked to the longevity risk faced in old age. This rule mirrors key aspects of observed pay-as-you-go state pension systems, incorporating elements of indexing to life-expectancy alongside a mixture of flat-rate and contributory-related components. However, implementing the optimal social security scheme may prove challenging if the contributory-related element are highly non-linear. Thus, in this section, we explore simple piecewise linear alternatives, both with and without a threshold, and assess their welfare and risk implications considering a two state example and i.i.d. transition probabilities.[21]

We consider three alternative social security rules. The first is an *Approximated Rule*, which approximates the optimal function $h_j(\tau)$ described in Proposition 3, by assuming that the contributory-related element for $\tau > \tau^{\mathrm{c}}$ is increasing linearly. This approximated rule is represented by the following ramp function:

$$h_j^{AR}(\tau) = \begin{cases} p^0(j) & \text{if } \tau \leq \tau^c, \\ p^0(j) + \rho^{AR}(j)(\tau - \tau^c) & \text{if } \tau > \tau^c, \end{cases}$$

where τ^c is defined in Proposition 3. If the contribution rate paid when young is below the threshold, a minimum pension benefit $p^0(j)$ is guaranteed. Above the threshold, the benefits are linearly increasing in the contribution rate. The parameter $\rho^{AR}(j)$ quantifies the responsiveness of pension benefits to changes in the contribution rate.

The second alternative rule is a *Defined-Benefit Rule*, without any contributory-related element. It simply specifies a pension benefit $\rho^{DB}(j)$ that depends only on the current state. Finally, the third rule is a *Defined-Contribution Rule*, where the pension benefit is a linear function of the past contribution rate: $h_j^{DC}(\tau) = \rho_0^{DC}(j) + \rho_1^{DC}(j) \cdot \tau$. All three alternatives exhibit links to the longevity risk. Table 1 reports the parameter values of the various policy rules for a specific numerical case. The first row also reports the value

[21] Farhi and Werning (2013) provide a similar comparison of income tax policies. They compare a non-optimized age-dependent linear tax, which they show is close to optimal, with a simpler and numerically tractable age-independent tax for which the optimum can be computed.

of the parameter for a first-best rule, given by $p^*(j)$ for all j.

Table 1: Social Security Alternatives

Social Security Rules	Parameter Values
First Best	$p^*(1) = 0.2222 \quad p^*(2) = 0.1667$
Approximated Rule	$p^0(1) = 0.2218 \quad \rho^{AR}(1) = 3.3379$ $p^0(2) = 0.1572 \quad \rho^{AR}(2) = 0.6921$
Defined-Benefit Rule	$\rho^{DB}(1) = 0.2357 \quad \rho^{DB}(2) = 0.1555$
Defined-Contribution Rule	$\rho_0^{DC}(1) = 0.1352 \quad \rho_1^{DC}(1) = 0.7836$ $\rho_0^{DC}(2) = 0.1852 \quad \rho_1^{DC}(2) = -0.1111$

Note: The table reports the parameters' value of the various policy rules for the numerical case described in Figure 5. Both the Optimum and the Approximated Rule have the same threshold, $\tau^c = 0.1534$.

The policy coefficients for the Approximated Rule are obtained by minimizing a constrained sum of squared errors, weighted by the frequency of occurrence of state (p, ϕ) in the long-run distribution of the constrained efficient allocation, between the approximated rule and the optimal function $h_j(\tau)$ for the given optimal threshold τ^c. This approach circumvents the numerical complexity associated with optimizing over a non-differentiable function. It is worth noting that while the Approximated Rule may potentially violate participation constraints, any such violations are numerically negligible, at least in this example. In contrast, both the Defined-Benefit Rule and the Defined-Contribution Rule are constructed to always maintain the participation constraints.

The numerical example underscores several key observations. The minimum guaranteed replacement ratio implied by the Approximated Rule closely aligns with what a first-best rule would recommend for each survival probability state. In comparison to a Defined-Contribution Rule, which lacks a minimum pension entitlement level, the Approximated Rule exhibits a significantly greater elasticity to changes in the contribution rate. This elasticity is notably more pronounced within the Approximated Rule in states with lower survival probability than those with high survival probability, that is, $\rho^{AR}(1) > \rho^{AR}(2)$. Compared to a first-best rule, the Defined-Benefit Rule exhibits greater variability in the pension benefits.

IV.2 Valuation of Pensions

We now study some of the implications of the model for asset-pricing. With abuse of notation let $x = (\tau, \phi(i))$ represent the current state and $x' = (h_j(\tau)\phi(j)/(nw), \phi(j))$ be the successor state in the following period. In an overlapping generation model, the stochastic discount factor is given by:

$$m(x, x') = \beta n \left(\frac{\phi(j)}{\phi(i)}\right) \left(\frac{u'\left(\alpha + h_j(\tau)\right)}{u'\left(w\left(1-\tau\right)\right)}\right).$$

The corresponding state price $q_{ij} := \pi_j m(x, x')$ represents the amount of today's pension that must be given up to secure an additional unit of pension tomorrow. It is straightforward to show that the stochastic discount factor decreases in τ. Consequently, the state price also decreases with τ. This relationship arises from the participation constraint of the young. A higher pension payment lowers the current consumption of the young, thereby requiring future consumption claims to be compensated with the promise of higher pensions. The resulting decrease in the intertemporal marginal rate of substitution is reflected in a lower stochastic discount factor. Consequently, the price of state-contingent claims falls, making state-contingent pensions more affordable.

To simplify the discussion, we restrict our attention to states in the ergodic set. In this case, the Ross Recovery Theorem applies (Ross, 2015) and we can write $m(x, x') = \varrho\psi(x)/\psi(x')$, where ϱ is the Perron root of the state-price matrix and ψ is the corresponding eigenvector. Using the first-order condition (8) and the updating rule (15):

$$\varrho = \delta n \quad \text{and} \quad \psi(x) = \frac{1}{\phi(i)u'(w(1-\tau))(1+\mu(x))};$$

The long-run interest rate is $-\log(\delta n)$. The variability of the stochastic discount factor is determined by three key factors: the variability of survival probabilities, the variability of the marginal utility of consumption for the younger generation, and the variability in the tightness of the participation constraint of the young. In a scenario with full enforcement where $\mu(x) = 0$ and $\tau = \tau^*(i)$, the stochastic discount factor varies with $\phi(i)$ but this variability is independent of past shocks. However, under limited enforcement, a connection between the stochastic discount factor and the history of shocks arises. This is because limited enforcement ties the variability of the stochastic discount factor to changes in the contribution rate τ, introducing a dynamic response to past shocks.

The stochastic discount factor can be employed to quantify the insurance premium, defined as the difference between rates of return on pension securities and a one-period bond that delivers a certain unit of the consumption good in the next period. Let

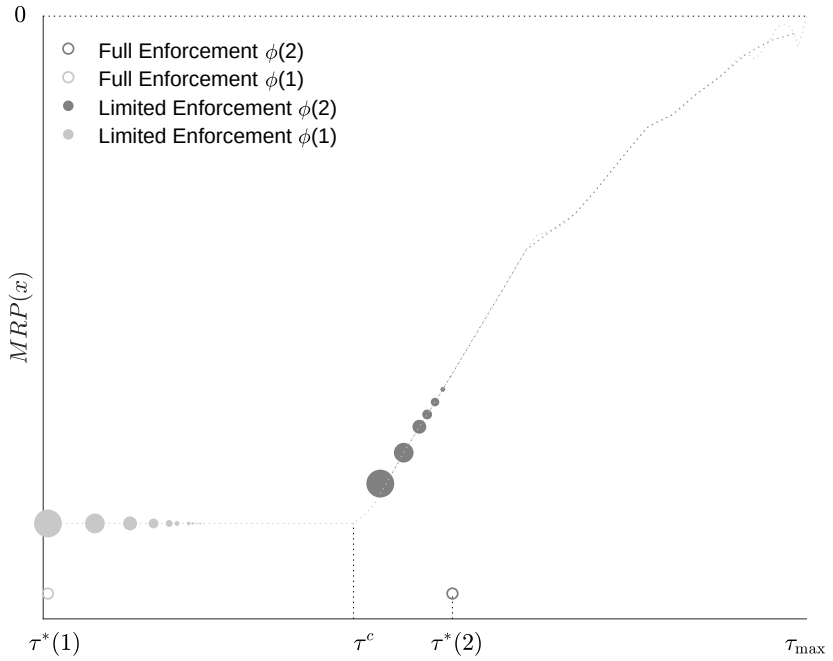


Figure 8: Multiplicative Risk Premium. The figure shows the multiplicative risk premium, $\text{MRP}(x)$, for the value of τ in the ergodic set. The illustration corresponds to the two-state case depicted in Figure 7. The light gray dots represent $\phi = \phi(1)$, while the dark gray dots represent $\phi = \phi(2)$. The filled dots indicate the limited-enforcement case, and the empty dots indicate the full-enforcement case. The size of the dot indicates the frequency of occurrence

$D^p(x) := \mathbb{E}_x[m(x,x')h_j(\tau)\phi(j)]$ denote the market value of pensions, representing the present discounted value of future pension claims conditional on the current state x. Next, define the conditional pension return as:

$$R^p(x,x') := \frac{h_j(\tau)\phi(j)}{D^p(x)}$$

The associated expected return is given by $\bar{R}^p(x) := \mathbb{E}_x[R^p(x,x')]$. The multiplicative risk premium (hereafter, MRP) is equal to the negative of the covariance between the stochastic discount factor and the conditional pension return. Formally, it can be expressed as:

$$\text{MRP}(x) := \frac{\bar{R}^p(x) - R^f(x)}{R^f(x)} = -\operatorname{cov}_x\Big[m(x,x')R^p(x,x')\Big]$$

where $R^f(x) := 1/\mathbb{E}_x[m(x,x')]$ is the risk-free rate of return in state x.

Figure 8 illustrates the MRP in both full and limited enforcement scenarios for the two-state parametric example presented in Figure 7. In both cases, the positive correlation

between the stochastic discount factor and pension returns results in a negative risk premium, indicating a positive insurance value for the social security system. Under full enforcement, the MRP is state independent, meaning that the insurance value remains constant regardless of changes in contribution rates or survival probabilities. However, under limited enforcement, the insurance value is highest when both the contribution rate and survival probability are low. As contribution rates and survival probabilities increase, the insurance value decreases, since less insurance is provided to sustain incentives for participation. Within the ergodic set and the specific example presented, the MRP reaches its maximum at $\tau = \tau^*(2)$, corresponding to the fixed point of the policy rule for the highest survival probability.

V Public Saving

We now introduce a public storage technology that allows the planner to transfer resources across periods while earning a net return on storage. In the context of social security, this implies that the budget needs not be balanced in each period. Instead, the social planner can invest in assets and use both the principal and accumulated returns to help finance pension entitlements for the elderly, supplementing contributions from the working-age population. Many governments, including those of the U.S. and Norway, accumulate trust funds to pre-fund social security benefits and ensure long-term fiscal sustainability. Incorporating this feature into the model offers new insights into the optimal management of trust funds for sustaining social security programs.

The social planner can accumulate a fund by buying one-period non-state contingent assets $a_+ \geq 0$ at a price $q > 0$, which yield a payoff of one unit of consumption in the next period. The public budget constraint (1) (with variables defined in per-adult terms) is expanded as follows:

$$S \equiv \tau w - \frac{\phi}{n} p = nqa_+ - a \quad \text{with} \quad a, a_+ \geq 0.$$

If $S > 0$, then there is a surplus which is used to accumulate the fund. In contrast, when $S < 0$ pension benefits are partly covered by the accumulated assets. The recursive formulation of the maximization problem reported in Section III can then be extended over a bi-dimensional state space as follows:

$$V_i(p,a) = \max_{a_+,\tau,(p_j)_{j=1}^K} \frac{\beta}{\delta}\frac{\phi}{n}\left(u\left(\alpha + \frac{n}{\phi}\Big(a - nqa_+ + \tau w\Big)\right) - u(\alpha)\right) + (u(w(1-\tau)) - u(w)) + (\delta n)\sum_{j=1}^{K}\pi_{ij}V_j(p_j, a_+),$$

subject to:

$$u\left(\alpha + \frac{n}{\phi}\Big(a - nqa_+ + \tau w\Big)\right) \geq u(\alpha + p) \qquad \left(\frac{\beta}{\delta}\frac{\phi}{n}\lambda\right),$$

$$u(w(1-\tau)) - u(w) + \beta\sum_{j=1}^{K}\pi_{ij}\phi(j)(u(\alpha + p_j) - u(\alpha)) \geq 0 \qquad (\mu),$$

$$a_+ \geq 0 \qquad (n\delta\chi),$$

where $p_j \in [0, p_{\max}(j, a_+)]$. Define the state vector as $x := (p, a, \phi)$, with x_+ representing its next-period value. The stochastic policy functions are: $\tau = f(x)$ for the contribution rate, $p_j = g_j(x)$ for the state-contingent promised pension, and $a_+ = z(x)$ for the non-state-contingent asset. The first-order conditions, excluding the boundary conditions of p_j, are:

$$\tau : \frac{u'(w(1-f(x)))}{u'(c^o(x))} = \frac{\beta}{\delta}\Big(\frac{1+\lambda(x)}{1+\mu(x)}\Big),$$

$$p_j : V'_{j,1}(g_j(x), z(x)) = -\mu(x)\frac{\beta}{\delta}\frac{\phi(j)}{n}u'(\alpha + g_j(x)), \text{ for each } j = 1, 2, ..., K,$$

$$a_+ : \sum_{j=1}^{K}\pi_{ij}V'_{j,2}(g_j(x), z(x)) = (1+\lambda(x))\frac{q}{\delta}\frac{\beta}{\delta}u'(c^o(x)) - \chi(x),$$

where $c^o(x) \equiv \alpha + \frac{n}{\phi}\Big(a - nqz(x) + f(x)w\Big)$. The envelope conditions are:

$$p : V'_{i,1}(p,a) = -\lambda(x)\frac{\beta}{\delta}\frac{\phi}{n}u'(\alpha + p),$$

$$a : V'_{i,2}(p,a) = (1+\lambda(x))\frac{\beta}{\delta}u'(c^o(x)).$$

Using the envelope condition with respect to p and the first-order condition with respect to p_j, we obtain the updating condition, as in the model without public saving, $\lambda(x_+) = \mu(x)$, which implies that any time the participation constraint is slack, then also the next-period promise keeping constraint is slack and the implemented pension benefit is $p^0(i,a) = \sup\{p | V'_{i,1}(p,a)\}$. Furthermore, taking the envelope with respect to a and

moving one period forward we get:

$$V'_{j,2}(p_j, a_+) = (1 + \lambda(x_+))\frac{\beta}{\delta}u'(\alpha + g_j(x)).$$

Replacing this equation into the first-order condition with respect to a_+ we get:

$$(1 + \lambda(x))\frac{q}{\delta}u'(c^o(x)) - \frac{\delta}{\beta}\chi(x) = \sum_{j=1}^{K}\pi_{ij}(1 + \lambda(x_+))u'(\alpha + g_j(x)).$$

Using the updating condition:

$$(1 + \lambda(x))u'(c^o(x)) = (>)(1 + \mu(x))\frac{\delta}{q}\sum_{j=1}^{K}\pi_{ij}u'(\alpha + g_j(x)) \text{ if } a_+ > (=)0.$$

Using the first order condition with respect to τ, we get:

$$u'(w(1 - f(x))) = (>)\frac{\beta}{q}\sum_{j=1}^{K}\pi_{ij}u'(\alpha + g_j(x)) \text{ if } a_+ > (=)0.$$

V.1 Deterministic Case

We first consider the case without uncertainty and characterize the first-best scenario, where we set $\mu_t = 0$ and $\lambda_t = 0$ for all $t \geq 0$ under the following simplifying assumption

Assumption 4. $\delta = \beta = q$.

Under this assumption, when $a_0 \geq 0$, the optimal allocation is characterized by $u'(\alpha + p_{t+1}) = u'(w(1-\tau_t))$ and $u'(c_t^o) = u'(w(1-\tau_t))$. Concavity of the utility function implies that this allocation can be achieved by keeping constant the accumulated asset $a_t = a_0$ for all t and setting pension benefits at

$$p_t = \frac{n}{\phi + n}((w - \alpha) + (1 - \beta n)a_0) \equiv p^*(a_0) \text{ for all } t \geq 0 \text{ and } a_0 \geq 0.$$

Consider an initial pension entitlement $p_0 > p^*(a_0)$, implying that the promise-keeping constraint binds, i.e., $c_0^o = \alpha + p_0$, and distorts the intra-temporal first-order condition such that $u'(w(1-\tau_0)) > u'(\alpha + p_0)$. Under full enforcement, the updating rule yields $\lambda_1 = \mu_0 = 0$, leading to next-period pension benefits $p_1 = p^*(a_1)$. However, the asset level a_1 may deviate from a_0 due to asset returns and the individual's discount factor. Since $q = \beta$, the inter-temporal condition gives $u'(w(1-\tau_0)) = u'(\alpha + p^*(a_1))$. Substituting the public budget constraint and replacing $p^*(a_1)$ in the inter-temporal first-order condition,

we derive the policy rule:

$$a_1 = z(p_0, a_0) = \frac{\phi + n}{n(1 + \beta\phi)}\Big(\frac{\phi}{\phi + n}(w - \alpha) + a_0 - \frac{\phi}{n}p_0\Big) < a_0. \tag{16}$$

From this rule, we can determine a threshold level of pension benefit $\hat{p}(a_0) := \{p_0 | z(p_0, a_0) = 0\} = \frac{n}{\phi+n}(w-\alpha) + \frac{n}{\phi}a_0$ such that if $p_0 > \hat{p}(a_0)$, then $a_1 = 0$ and $p_1 = p^*(0) = \frac{n}{n+\phi}(w-\alpha)$, while if $p^*(a_0) < p_0 \leq \hat{p}(a_0)$, then a_1 is given by Equation (16) and $p_1 = p^*(a_1)$. In all cases, the dynamics result in a one-period decline in both surpluses and pension benefits.

Now, consider the case where enforcement may be constrained by binding participation constraints. We assume that the participation constraint holds under the first-best allocation and for any asset level, ensuring that all allocations $(p^*(a_0), a_0)$ remain sustainable. The following condition suffices to satisfy this assumption.

Assumption 5. $u\Big(\frac{nw+\phi\alpha}{\phi+n}\Big) > \frac{u(w)+\beta\phi u(\alpha)}{1+\beta\phi}$.

It is straightforward to see that, for any initial condition (p_0, a_0) such that $p^*(a_0) < p_0 \leq \hat{p}(a_0)$, Assumption (5) ensures that the participation constraint is never binding. Indeed, in this case, $a_1 > 0$, and since $q = \beta$, the allocation $(p^*(a_1), a_1)$ can be reached in one period without violating the participation constraint. When $p_0 > \hat{p}(a_0)$, there exists a threshold level of pension benefits, $p^c(a_0)$, such that if $p_0 \leq p^c(a_0)$, then promising $p_1 = p^*(0)$ in the next period with $a_1 = 0$ sustainable. In other words, $p^c(a_0)$ is the critical level of p_0 that satisfies the following equation:

$$u\Big(w - \frac{\phi}{n}p_0 + a_0\Big) + \beta\phi u(\alpha + p^*(0)) = u(w) + \beta\phi u(\alpha),$$

given by:

$$p^c(a_0) = \frac{n}{\phi}\Big[w + a_0 - u^{-1}[u(w) + \beta\phi(u(\alpha) - u(\alpha + p^*(0)))]\Big].$$

If $\hat{p}(a_0) < p_0 \leq p^c(a_0)$, then there is a one-period transition to $a_1 = 0$ and $p_1 = p^*(0)$. Finally, when $p_0 > p^c(a_0)$, the participation constraint is binding and the policy rule of pension benefit $p_1 = g(p_0, a_0)$ is determined from the equation:

$$u\Big(w - \frac{\phi}{n}p_0 + a_0\Big) + \beta\phi u(\alpha + p_1) = u(w) + \beta\phi u(\alpha).$$

As in the case without public saving, the dynamics of p_t when $p_0 > p^c(a_0)$ converge gradually toward the steady state $p^*(0)$. Figure 9 depicts the various thresholds in the (a, p) state space, along with the corresponding dynamics for given initial conditions

(a_0, p_0). The dynamics feature a one-period transition to the long-run allocation with zero assets whenever the initial condition (a_0, p_0) lies below $p^c(a)$. For initial conditions above $p^c(a)$, the economy undergoes an initial sharp asset reduction followed by a gradual convergence to the long-run allocation.

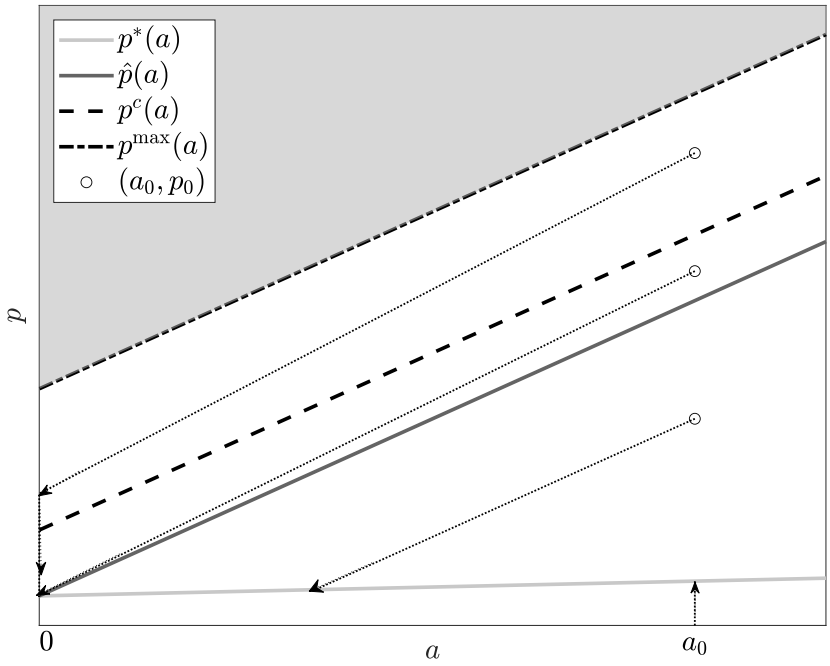


Figure 9: Phase Diagram in the Deterministic Case. The arrows illustrate how pensions and assets evolve over time, starting from an initial condition.

The two panels in Figure 10 illustrate the optimal policy rules: $p_{t+1} = g(p_t, a_t)$ for given a_t, and $a_{t+1} = z(p_t, a_t)$ for given p_t. Since $z(a, p)$ lies below the 45° line, any initial condition (a_0, p_0) that satisfies participation constraints implies asset depletion. The speed of depletion increases with the initial pension entitlement. When initial assets are positive, the next-period pension promise is a non-monotonic function of current entitlement. However, when $a_0 = 0$, it mirrors the case without public savings.

V.2 Stochastic Case

Consider now the case with uncertainty. As a benchmark, we first characterize the first-best allocation under full enforcement to isolate the effects of limited enforcement on asset accumulation and social security dynamics. Under the simplifying assumption $\beta = \delta = q$, the intra-temporal first-order condition is $u'(w(1 - f(x))) = u'(c^o(x))$ for all x. This condition leads to $c^y(x) = c^o(x) \equiv c(x)$ for all x. Given the public budget constraint, the consumption allocation then pins down the asset accumulation function $a_+ = z(x) = (1/qn)[w + (\phi/n)\alpha + a - ((\phi + n)/n)c(x)]$. In a scenario where neither

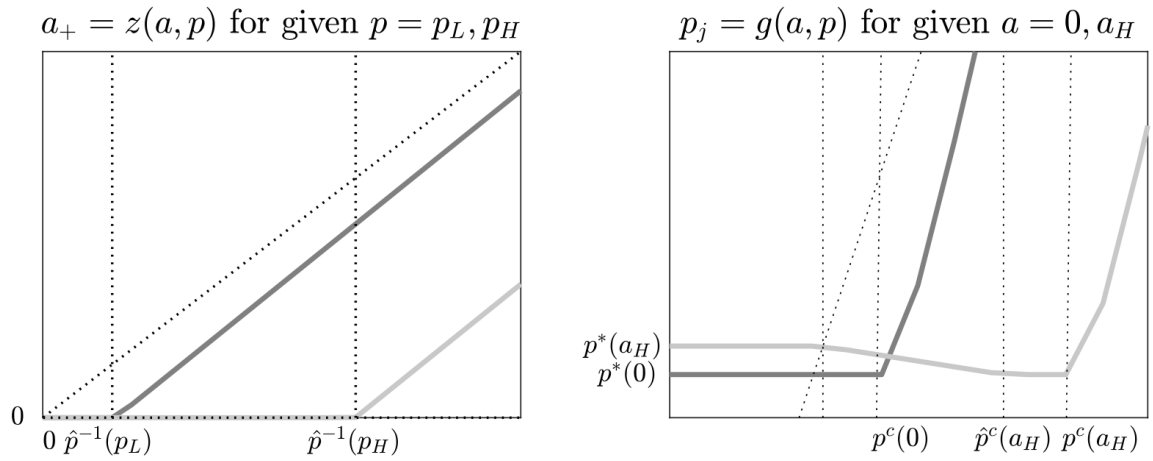


Figure 10: Asset Accumulation and Social Security Rule in the Deterministic Case. The left-hand panel plot the policy $a_+ = z(a,p)$ for given values of p equal to $p = p_L$ (dark gray function) and $p = p_H$ (light gray function) with $p_L < p_H$. The right-hand panel plot the policy $p_j = g(a,p)$ for given values of a equal to $a = 0$ (dark gray function) and $a = a_H$ (light gray function) with $a_H > 0$. The 45° lines are indicated by dotted lines.

the promise-keeping nor the participation constraints are binding, we can focus on the relevant state $x = (a, \phi)$. The intertemporal first-order condition is then given by:

$$u'(w(1 - f(a,\phi))) = (>) \sum_{j=1}^{K} \pi_{ij} u'(w(1 - f(z(a,\phi),\phi_j))) \text{ if } a_+ = z(a,\phi) > (=) 0.$$

The degree of convexity in marginal utility—or, equivalently, the behavior of the third derivative (u''')—influences the strength of precautionary saving motives. Note that even in the absence of prudence, the first-best allocation features precautionary saving due to the non-negativity constraint on assets, $a_+ \geq 0$. Specifically, there exist thresholds $\hat{a}(\phi)$ such that when $0 \leq a < \hat{a}(\phi)$, the non-negativity constraint is binding, inducing precautionary motives and implying $c(a,\phi) < \sum_{j=1}^{K} \pi_{ij} c(z(a,\phi),\phi_j)$, as shown by Carroll et al. (2021). For $a \geq \hat{a}(\phi)$, the non-negativity constraint is slack, and consumption follows a random walk when $u''' = 0$: $c(a,\phi) = \sum_{j=1}^{K} \pi_{ij} c(z(a,\phi),\phi_j)$.

The solid lines in Figure 11 depict the optimal asset dynamics and implemented pension levels in the first-best benchmark for a two-state example with $\phi(1) < \phi(2)$. The left panel shows the optimal asset path when the promise-keeping constraint is slack, i.e., $a_+ = z^*(a,\phi)$ (solid lines), which holds whenever $p_0 \leq p^*(a_0,\phi_0)$. The threshold pension level, defined as $p^*(a_0,\phi_0) := \frac{n}{n+\phi}\Big(w - \alpha - qnz^*(a_0,\phi_0) + a_0\Big)$, is plotted in the right panel. When $\phi = \phi(1)$, the low dependency ratio allows for partial savings, leading to $z^*(a,\phi) > a$. Conversely, when $\phi = \phi(2)$, the high dependency ratio forces asset drawdowns, resulting in $0 \leq z^*(a,\phi) < a$. If $a_0 = 0$, the planner has an incentive

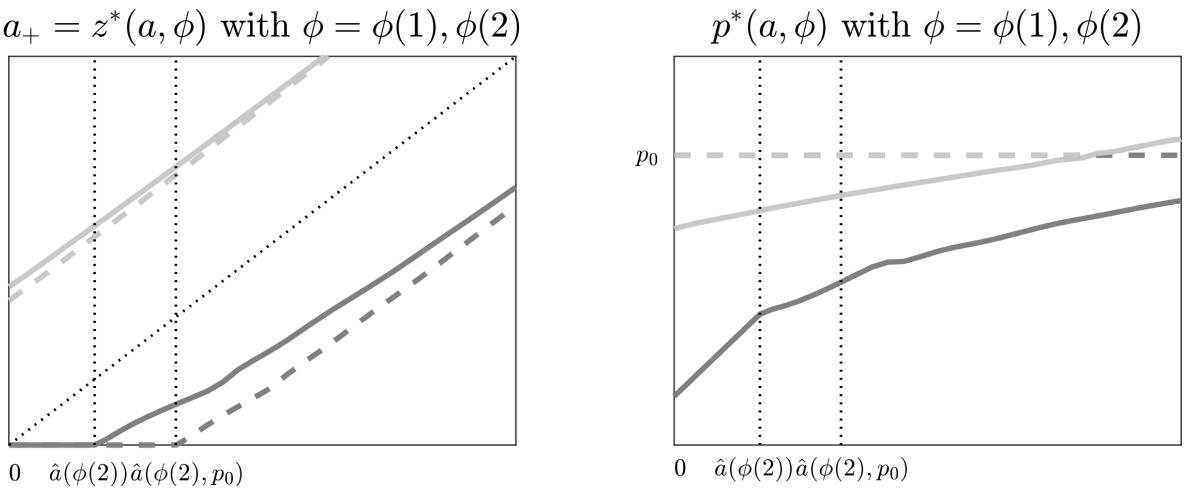


Figure 11: Asset Accumulation and Consumption in the Stochastic First-Best Case. The left panel displays the policy function $a_+ = z^*(a, \phi)$ for two values of ϕ, with $\phi(1) < \phi(2)$: the light gray curve corresponds to $\phi = \phi(1)$, and the dark gray curve to $\phi = \phi(2)$. Solid lines represent the case where $p_0 < p^*(a_0, \phi_0)$, while dashed lines indicate $p_0 \geq p^*(a_0, \phi_0)$. The 45° line is shown as a dotted line. The right panel plots the first-best level of social security, $p^*(a, \phi)$, for the corresponding values of ϕ.

to accumulate assets to relax future borrowing constraints. Consequently, the optimal consumption function, $c^*(x) = \alpha + p^*(x)$, is concave.

When $p_0 > p^*(a_0, \phi_0)$, the promise-keeping constraint in state ϕ_0 binds in the initial period. However, the updating property ensures that in the next period, $p_1 = p^*(a_1, \phi_1)$, where the asset level evolves as $a_1 = z(a_0, p_0, \phi_0)$, now explicitly depending on the initial pension level p_0. The left panel of Figure 11 illustrates an example of $z(a_0, p_0, \phi_0)$ (dashed lines) when the initial pension promise exceeds $p^*(a_0, \phi_0)$, as indicated in the right panel. A higher p_0 tightens the promise-keeping constraint, temporarily shifting the asset accumulation policy downward and accelerating asset depletion to ease the constraint in the following period. Consequently, the next-period pension is given by $p_1 = p^*(z(a_0, p_0, \phi_0), \phi_1)$ from that point onward.

We maintain Assumptions (2) and (3). Under these conditions, there always exists a state where the participation constraints bind and a state where they remain slack. The possibility of binding participation constraints persistently distorts asset accumulation, as they effectively act as an additional borrowing constraint. The social planner seeks to relax these constraints as quickly as possible while ensuring that the participation conditions of future generations are satisfied.

In the presence of public saving technology, the optimal asset management is influenced not only by the standard precautionary saving motive—where assets are accumulated

in states with a lower dependency ratio to prevent the non-negativity constraint from binding—but also by an additional social security sustainability motive. Specifically, in the presence of enforcement frictions, in states with a low dependency ratio, where participation constraints are more likely to be slack, the planner has an incentive to accumulate even more assets than under full enforcement. This accumulation allows the planner to use surpluses to cover increased pension expenditures when the dependency ratio rises, without resorting to incentive-providing future promises. At the same time, it weakens the direct link between contributions and benefits, making a purely defined-benefit pension system sustainable even under high levels of pension entitlements.

VI Conclusions

This paper presents a simple theory for analyzing the optimal social security rule in the presence of stochastic survival probability and contracting frictions due to limited enforcement. We demonstrate that under enforcement frictions, the optimal rule balances incentives for participation with the efficacy of risk sharing across generations. Remarkably, the optimal rule bears some similarities to existing social security systems in advanced economies, characterized by three main features: (i) a guaranteed minimum replacement ratio, activated if contributions during the working age fall below a specified threshold; (ii) a contributory-related component where the replacement ratio correlates positively with past contributions exceeding the threshold; and (iii) a negative correlation between social security entitlements and survival probability, whose realization remains uncertain until retirement age, thereby enhancing the social security scheme sustainability.

A key aspect of the optimal social security scheme, under limited enforcement of transfers, is the presence of a contribution threshold, beyond which benefits are positively related to the contributions paid in working age. Although it may be tempting to interpret this as implying higher contributions in youth yield larger benefits when old, the model lacks intragenerational heterogeneity. Thus, the correct interpretation pertains to contributions across different generations of the young. Hence, a relevant avenue for future research involves enriching the demographic structure of the model. This could involve incorporating more than two overlapping generations or allowing for intragenerational heterogeneity. Such extensions would facilitate addressing the interplay between intergenerational and intragenerational insurance dynamics effectively.

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