Taxing the Rich?
A Theory of Income and Wealth Inequality*

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Abstract

Recently, it has been argued that a progressive wealth tax may significantly benefit the distribution of welfare in society with minimal adverse effects on real economic activity. This paper evaluates the merits of this view within a dynamic general equilibrium framework that microfounds empirically plausible income and wealth distributions as arising from a fundamental agency problem between managers and entrepreneurs, on the one hand, and capital markets, on the other hand. In this framework, wealth taxes distort the effort that managers and entrepreneurs expend to create firm value, which capital markets reward them for, by shifting their choice of projects towards less productive ventures. When wealth taxes are broad-based enough to affect not just individuals at the top of the wealth distribution, who contribute the most to firms' productivity, they help reduce inequality but depress capital accumulation and output in the economy. The model accounts well for the degree of income and wealth inequality observed in the United States, including the very different degrees of concentration of the distributions of income and wealth at the top. The model also implies a substantial output loss from wealth taxes of the magnitude currently being debated. But wealth taxes levied on the very wealth reduce inequality at the top at a modest cost in terms of output. The reduction in inequality that wealth taxes lead to would be achieved at a much lower cost by modestly increasing the progressivity of income taxes rather than by introducing progressive wealth taxes.

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1 Introduction

An intense debate on inequality in the United States has stemmed from the high degree of concentration of U.S. income and wealth. Based on data from the Survey of Consumer Finances (SCF) for 2016, in the United States households in the top 1% of the income distribution hold over 27% of pre-tax household income, whereas households in the top 1% of the wealth distribution hold over 39% of household wealth. These distributions are much more unequal than in all other OECD countries, where, for instance, the wealth share of the top 1% of households ranges between 10% and 25%.

This high degree of inequality has spurred an intense debate on both its causes and the possible remedies for it. Specifically, many have argued that a progressive wealth tax may have large beneficial effects on the distribution of welfare in a society and effectively no adverse effects on real economic activity (see, among others, Saez and Zucman [2019] and the references therein). This literature has mostly abstracted from the potentially critical role that wealth accumulation plays in aligning the private incentives of firms' managers, executives, and entrepreneurs, in short referred to as *managers*, with those of the ultimate owners of firms, which are central to the creation of wealth. We argue that this omission helps explain why even a very progressive taxation of wealth has little impact on incentives and, hence, only limited distortionary effects.

In this paper, we propose a novel framework for the study of income and wealth inequality, in which the accumulation of wealth helps alleviate agency frictions between managers and firm owners, thereby supporting output and productivity growth in the aggregate. We first show that this framework successfully reproduces the distributions of income and wealth in the United States, in particular, their characteristic thick right tails. We then use our framework to evaluate the merits of augmenting the current U.S. tax system with wealth taxes, relative to increasing its overall progressivity through general income or consumption (VAT) taxes. We find that wealth taxes impact the incentives of managers to select profitable projects and build up managerial expertise, which they do by exerting effort and acquiring human capital. In this framework, a progressive wealth tax, by affecting both of these margins, has a large distortionary impact on the economy, in sharp contrast to the presumption of the advocates of such a tax. The same decrease in inequality implied by such a tax could be achieved by a modest increase in the progressivity of the income tax in the current U.S. system at a much smaller cost in terms of aggregate output.

Our approach differs from two large strands of literature on wealth inequality: models of entrepreneurship and Aiyagari-Bewley models of incomplete financial markets.[1] The first strand of literature has focused on the role of entrepreneurs in accounting for the top percentiles of the wealth distribution, as in Quadrini [2000] and Cagetti and De Nardi [2006]. One issue with the interpretation of wealth inequality implied by this work is that in the data a sizable fraction of income earned

[1] A third branch of this literature emphasizes that the endogenous response of human capital accumulation to higher income tax rates dampens the optimal progressivity of optimal income taxes. This point was emphasized by Badel et al. [2020].

by those in the top percentiles of the wealth distribution is *labor* income, mostly attributable to executives and managers of publicly-owned firms financed by equity and corporate debt, rather than *capital* income generated from the assets of sole owners. For instance, examining the income sources of those in the top 1% of the distribution of household income (respectively, wealth) in the United States between 1989 and 2016 reveals that wages and salaries account for 50% (respectively, 38%) of it, business income from farms and sole proprietorship for 14% (respectively, 10%), capital income from mutual funds, interests, and accrued dividends for 32% (respectively, 49%), and other income forms such as inheritance, scholarships, and settlements from lawsuits for 4% (respectively, 2%).

The second strand of literature on wealth inequality has largely emphasized the role of earnings inequality in generating wealth inequality, and originally stems from models that feature consumers supplying labor, whose productivity varies stochastically over time, and facing exogenously incomplete financial markets, where consumers can invest only in uncontingent bonds. This incompleteness thus provides consumers with an incentive to accumulate savings for precautionary reasons. However, because savings incentives taper off at high levels of wealth, these models have a difficult time reproducing the observed thicker tail of the distribution of wealth than that of income, which is a hallmark of empirical wealth distributions (see Benhabib et al. [2017] and Benhabib and Bisin [2018]).[2]

The models in this strand of literature can nevertheless generate a fatter-tailed distribution of wealth than income, once extended to include investors who earn an idiosyncratic return on investments, that is, once consumers are allowed to invest in risky assets. (See, for example, Angeletos [2007], Benhabib et al. [2011], and Benhabib and Bisin [2018].) But these augmented models then lead to a well-known *no-diversification puzzle*: if agents face risk when investing, why don't they diversify their portfolios? This is in a nutshell the challenge to this literature on wealth inequality, namely, how to account for the dispersion of returns in economies in which agents have a natural desire to hedge risk.

Our approach to this puzzle builds on the simple idea that if firm owners aim at providing incentives for managers to act in their firms' best interest, they need to expose managers to the idiosyncratic risk of the firms they manage. In this sense, to discipline the agency problem between managers and firm owners, managers in our framework are induced to act like investors facing idiosyncratic investment opportunities, in the spirit of traditional incomplete financial markets models. Our resolution to the no–diversification puzzle is that managers do not diversify their portfolios because their compensation contracts are *optimally* structured to prevent them from doing so: returns on managers' savings are closely tied to the idiosyncratic risk of their firms for incentive reasons. This is the key mechanism that generates income and wealth inequality in our framework, which provides a micro-foundation

[2]One branch of the literature deals with this issue by adding an *awesome* state in which an consumer's income randomly jumps up temporarily by a large amount so that these temporarily very rich consumers save vast amounts as a precaution for the time when normal states return (see Castaneda et al. [2003] and the somewhat critical discussion in Benhabib and Bisin [2018]).

of market incompleteness that is consistent with the observed income and wealth distributions and sheds light on the primitive determinants of inequality. In particular, in contrast to Aiyagari–Bewley models, wealth spreads out not because of exogenous restrictions on portfolio holdings but rather because of optimal incentive provision in the face of moral hazard considerations. Importantly, our proposed framework manages to account for both income and wealth inequality because it gives rise to stochastic returns on assets that do not taper off as wealth increases, since the agency problem at the heart of our model if anything becomes more severe at high levels of wealth due to standard income effects.

As for our results on wealth taxation, our approach emphasizes that proponents of wealth taxes abstract from a critical margin: by controlling managers' compensation and so wealth accumulation, managerial compensation contracts help solve the conflict of interests between managers and firms. They do so by rewarding successful managers with wealth increments that move closely with a firm's fortunes. But such compensation schemes necessarily expose managers to risk. Hence, unlike what is often argued, managers' compensation is naturally both high and variable for incentive reasons. By distorting the contractual alignment of the incentives of firms and managers, wealth taxes then exacerbate agency frictions and depress output and productivity as a result. We show this would be indeed the case for the U.S. economy under popular wealth tax proposals.

Our model builds on the fundamental incentive problem stemming from the notion that it is relatively easy to monitor workers but difficult to monitor managers, despite managers' actions being critical for firms' productivity and returns. For this reason, managers are commonly provided with incentives for performance through contractual arrangements that explicit link their compensation to firm value through forms of variable pay such as bonuses, shares, stock options, and similar asset grants. Whereas this moral hazard considerations have long been known to be important determinants of managerial pay, the novelty of our approach consists in formalizing the notion that managers' compensation is structured so as to provide incentives that work by *concentrating* rather than *diversifying* their portfolios. This incentive mechanism is the crucial determinant of wealth inequality in the economy we propose. Formally, we expand on several themes in related literatures that account for the observed income and wealth distributions, namely, the important roles of managers in firms and of endogenous human capital accumulation but, critically, explicitly account for the distortions to managers' incentives arising from progressive income and wealth taxes.

We consider a general equilibrium economy that consists of a real sector, populated by consumers and firms producing an homogeneous consumption good, and a financial sector, populated by intermediaries who provide capital to producers. In this economy, consumers choose whether to be managers or workers in a firm in each period as in the span-of-control model of Lucas (1978). Consumers could be owner-entrepreneurs, in which case their income might be classified as business income, but they could also be top managers, who are not necessarily owners of their firms and whose earnings might be classified as labor income. Consumers or *agents* begin with some initial

endowments of human capital as a worker and human capital as a manager. In each period, their unobserved effort choice affects the probability distribution over their future human capital and, hence, their efficiency units of labor. Managers' efficiency units of labor are not observed and their compensation is governed by incentive-compatible Markovian labor contracts that link their current earnings and continuation wealth to the output of the firm they manage. Workers' efficiency units of labor are instead observed and they are paid a wage per unit of the observable amount of their efficiency units of labor. The motivation for these assumptions is that it is much easier to monitor the effective output of workers, who perform mostly routine and well-defined tasks, than it is to monitor that of top managers, who perform complex tasks that entail ample latitude and discretion. Instead, the best way to provide a manager with incentives to exert the desired amount of effort is to connect their compensation to the output of the group of workers that a manager leads.

Since it is optimal to tie a manager's compensation to the output of a firm, we endogenize the feature that the effective return on a manager's wealth has an idiosyncratic component based on the particular division or firm that the manager leads. In this sense, we derive the return structure that Angeletos (2007) exogenously assumed as part of the optimal compensation contract for managers.

A simple intuition for our results is as follows. In the economy described, financial intermediaries, which are akin to capital markets under diffuse firm ownership, compensate managers with wages and wealth that depend on their productivity so as to balance the goal of incentivizing managers to work and exert effort, on the one hand, against the goal of insuring them against production risk they cannot control, according to the standard risk-incentive trade-off of moral hazard models. But what is novel in our framework relative to standard dynamic moral hazard models, is that since it is optimal to tie a manager's compensation to the manager's productivity, the effective return on a manager's accumulated wealth from working features an idiosyncratic component arising from a manager's productivity process. Then, under standard CRRA preferences over consumption and leisure with parameter σ, which guarantee a homogeneity of degree $1-\sigma$ in assets and human capital for a manager's value of employment, it is possible to show that an optimal contract in general leads to an asset policy such that future assets grow approximately linearly in current assets. Hence, in equilibrium, wealth undergoes a compound growth with a stochastic per-period growth rate that depends on a manager's productivity so that if wealth and human capital approximately grow at the same rate, wealth follows a random walk process. Such a process is known to generate an arbitrarily large degree of inequality in the distribution of wealth. We derive a bounded stationary distribution, which reproduces the ones typically observed in the data, by introducing "death" for managers, which act like a reflecting barrier for the wealth process.

A very similar logic applies to the distribution of income, not just that of wealth. So, how does our model give rise to a distribution of wealth that has a thicker right tail than the distribution of income? Mechanically, such a result arises whenever future assets grow with current assets *faster* than wages increase with current assets or, equivalently, by a manager's intertemporal budget constraint,

whenever the slope of the consumption schedule in a manager's idiosyncratic productivity is not too steep. Intuitively, this feature occurs when it is difficult to infer a manager's effort from a manager's realized productivity, that is, when the moral hazard problem is hard to solve in that the so-called likelihood ratio of effort signals extracted from a manager's productivity does not increase fast enough with the signal observed. Intuitively, in this case, high managerial productivity does not convey much of a "good news" about effort (see Milgrom (1981)). In this case, inducing a steep wealth accumulation path helps support managerial incentives, since wages are not that powerful an incentive instrument. Accordingly, controlling a manager's assets in addition to a manager's wage allows financial intermediaries to fine-tune the variability of managers' pay within as well as across periods. Since, through asset grants, intermediaries can reward managers' productivity not just with greater current consumption, as wages do, but also with greater future consumption, they can provide the same incentives for performance at lower cost, due to discounting and managers' risk aversion, by reducing the variability of *current* consumption, that is, of wages. But then incentives for effort is optimally provided by increasing the variability of *future* consumption, that is, of assets and wealth.

Our preliminary simulations show that even a simple version of the model accounts well for the increase in wealth inequality in the United States over the past 30 years as well as for the concentration in observed income and wealth. Such a model implies a substantial aggregate output loss from wealth taxes of the magnitude currently debated, associated with a reduction in inequality that would be achieved at a much lower cost by a modest increase in the degree of progressivity of the taxation of income.

2 Data

Our empirical analysis is based on three main datasets. The first one is the Survey of Consumer Finances (SCF), which we use to estimate the distributions of wealth and income for the United States as a whole and for select groups of income earners—namely, managers and workers. The second dataset is Execucomp, which we use to study the compensation of managers in publicly-traded firms. We complement these datasets with balance-sheet information of publicly traded firms from Compustat and equity prices from CRSP. Finally, we obtain information on ownership on publicly traded firms from the official records of the Security and Exchange Commission (SEC). Additional details on variable definitions, data construction, and sample selection can be found in Appendix A.2.

2.1 Variable Definition

Here we discuss the definition of the main variables we use in our quantitative exercises.

Wealth and Income. We draw information on household income and wealth from the Survey of Consumer Finance (SCF), which is a triennial survey running from 1989. The SCF provides

cross-sectional information about household wealth and portfolio composition in each year of the survey, income for the prior year, and demographic characteristics, including the occupation of the household head and whether the household head, or any other member of the household, has an active management role in any business owned by the family. To provide accurate information about the concentration of income and wealth at the top of the two distributions, the SCF over-samples rich households, making it the primary source of information on income and wealth inequality in the US.

In terms of the main variables of interest, we measure households' net worth as the difference between total assets and total liabilities. Total assets are the sum of financial wealth and non-financial wealth. Financial wealth consists of safe assets—savings accounts, bonds, checking accounts, retirement accounts, and the cash value of life insurance—and risky assets—mutual funds and directly held stocks. Non-financial wealth consists of the (self-reported) value of private businesses, real estate, and vehicles.[3] We calculate total liabilities as the sum of short-term debt—credit card debt and consumer loans—and long-term debt—student debt, mortgages and real state debt, vehicle debt, and other debt. We measure income as the sum of all labor and capital income within the household excluding social security income and government transfers. In particular, total income equals the sum of wages and salaries, business and farm income, interest earned, dividends, real estate income, and realized capital gains. We define labor earnings as the sum of wages and salaries, and business and farm income.[4]

Managers and Workers. The SCF provides information about the occupation of the household head, the structure of the head's compensation, whether or not the head is an active management role in a private business, and the head's employment status (employed or self-employed). We use this information to identify *managers*, which we define as households whose head is in a managerial or professional-specialty occupation and is paid through stock grants or options; or if any member of the household has an active management role on a private business owned by the household; or if the head of the household is self-employed, excluding consultants or a partner in a law firm.[5] Note that this definition of a manager augments the common notion of an entrepreneur used in the literature, typically defined as a self-employed active business owner (Cagetti and De Nardi [2006]), with the notion of a manager consistent with the framework we propose—namely, workers assigned projects that are critical to a firm's productivity (capital) whose compensation exposes them to the risk of the firm they manage for incentive reasons (agency frictions of the moral-hazard type). The rest of

[3]Although this information in SCF is only self-reported, the degree of wealth concentration is broadly comparable with information available from administrative data such as IRS. See Smith et al. [2023].

[4]Two potentially important sources of income are not considered in SCF. The first is the value of delayed compensation in the form of stock options or grants. The SCF asks whether the head received any of these payments but does not record the value. Second, SCF does not record any information that allows the estimation of unrealized capital gains, which have been shown to be an important source of income for households at the top of the wealth distribution (see Ozkan et al. [2023] and Fagereng et al. [2019]).

[5]We exclude consultants and law firm partners from our definition since we want to capture individuals who are self-employed but typically manage other workers.

the households are classified as *workers*. According to these definitions, 20% of all households in the 2019 SCF are classified as managers.

Managers of Publicly Traded Companies. Although the SCF covers a representative sample of the U.S. population, it only provides cross-sectional information about managers' and workers' characteristics and compensation. An important aspect of our analysis concerns the role of managers in determining the success of the firms they manage. In order to estimate managers' contribution to firms' productivity and the degree to which their own productivity may increase over time, we exploit panel information on manager compensation and firm outcomes from a sample of executives of large publicly-traded U.S. corporations drawn from the S&P's Compustat Execucomp. This dataset contains annual information on the demographic characteristics, labor market experience, and compensation of top executives—namely, chief executive officer, chief financial officers, and the additional (if any) three highest-paid executives—of publicly traded U.S. firms from 1992 to 2022.[6] Most of the firm-level information starts in 1994, although for a smaller sample of firms—basically those in the S&P500—the data can be extended a few years prior.

Crucially for our study, Execucomp provides longitudinal information about executive wage and non-wage compensation. Wage compensation consists of salary, bonuses, and non-equity-based pay. Non-wage or equity-based compensation consists of the value of awarded stock and options and other forms of deferred compensation. The value of awarded stocks and options, which is calculated by Execucomp, is intended to capture the fair value of the stocks and options on the day of the award. For a subsample of executives, Execucomp also provides information about an executive's tenure within the firm, tenure as CEO, age, and other socio-demographic information.

For each executive in each year, we merge annual data on the executive's compensation with annual balance sheet information and the annual returns of the executive's firm from Compustat and CRSP, respectively. Annual returns are calculated by compounding monthly returns—including dividends—within a calendar year t for a particular firm as $1 - r(t) = \Pi_{k=1}^{12}[1 + r(k)]$. The merged sample contains 53,040 unique executives and 3,852 unique firms, which account for 22% of all firms and 50% of all sales in Compustat in 2019. We use this information to estimate managers' productivity and human capital accumulation function as we discuss in the following sections. Additional details on variables definition and sample construction can be found in Appendix A.2.2.

2.2 Distributions of Income and Wealth

We start our analysis by characterizing the distribution of income and wealth for managers and workers. Table 1 shows population shares and cross-sectional moments of the distribution of income

[6]The database covers over 3,462 unique companies, both active and inactive, that are part of the S&P 1500 plus companies that were originally part of this group and are still publicly traded. Inactive firms are firms that were public at some point and when sampled, are private.

and wealth for the overall U.S. population and separately for managers and workers in 2019 using data from SCF. Not surprisingly, managers are typically richer than workers in terms of both income and wealth. Indeed, as the table shows, although only 20% of households are classified as managers, and they hold 58% of wealth and 44% of the total income in the economy.[7] Their average wealth (income) is about $2.2 millions ($191 thousands) relative to $393 thousand ($60 thousands) for workers. Most of the difference in wealth and income across managers and workers is due to differences in the right tails of the distributions across the two groups. Namely, managers at the top 0.1% of their within-group wealth (income) distribution own about $100 million ($6.2 million), whereas workers at the top 0.1% of their within-group wealth (income) distribution own about $22 million ($1.8 million).

Table 2 shows concentration measures of wealth and income at different quantiles of the wealth distribution in 2019. As expected, wealth is highly concentrated at the top, and most of it is held by managers. In the population, the share of wealth held by the top 0.1% of households is 14.2%. Among these households, those who are managers hold 12.2% of total wealth, whereas workers who are at the top 0.1% of the wealth distribution, hold only 1.3% of total wealth. Wealthy households also hold a significant fraction of the total income in the economy, about 5.2%, most of which is held by managers (4.6% by managers and 0.6% by workers). Similarly, although labor income is less concentrated than wealth and total income, it is the managers who hold the larger fraction of labor income among wealthy households. [8]

On average, managers and workers hold similar portfolios of assets in terms of their composition, albeit for workers it is tilted toward real estate (47% of their portfolio) whereas for managers it is tilted toward housing and private businesses (30% and 35% of their portfolio, respectively). Larger differences, however, emerge at the top of the wealth distribution—shown in the right panels of Table 3. At the top 0.1% of the wealth distribution, managers typically hold a large fraction of their portfolio in private businesses, whereas wealthy workers hold most of their portfolio in risky financial assets (stocks and mutual funds). Correspondingly, rich managers draw their income mostly from business and capital income (86%). Instead, for workers at the top of the wealth distribution, their main income sources and wages and capital income.

Table 4 further characterizes the wealth distribution of managers in terms of their wealth share across different types of businesses, whether they started the business they own, and the size distribution of these businesses. In general, wealthy managers who own a business tend to own a limited liability company (LLC) or a corporation (panel A), they have founded the company they manage (panel B), run companies with more than 500 employees, and generate more than 10 million dollars in sales per year (panels C and D respectively).

[7] Figure 7 in the Appendix shows that the share of managers in the population, wealth, and income has remained stable since 1998 with a small increase in the wealth share in 2019.

[8] Table A.1 displays within-group measures of concentration of income and wealth. These show that wealth and income are more concentrated at the top for managers than for workers.

Managers and Entrepreneurs. Our notion of manager expands the notion of entrepreneurs typically considered in the literature (see Cagetti and De Nardi [2006]). To draw a more direct comparison between our definition and that of the rest of the literature, Table 5 shows the population share of managers, workers, and entrepreneurs in the U.S. and within percentiles of the wealth and income distribution. The first two columns show our baseline definition of manager and worker. The third column (Occ.) shows those households whose head has a manager or professional occupation. For entrepreneurs, we use three different definitions i) households where a member has an active management role in the private business they own (ABO or Active Business Owner); ii) households whose head is self-employed (SE); iii) and the intersection between the previous two definitions, that is, Self-Employed Active Business Owners.

Similar to other papers in the literature, we find that entrepreneurs are overrepresented at the top of the wealth and income distributions. For instance, Active Business Owners (ABO) account for 12.2% of the population but 70.4% (64.0%) of the households in the top 0.1% of the wealth (income) distribution. Importantly, only a small fraction of these top entrepreneurs do not have a managerial occupation, as shown in the last three columns of Table 5. In particular, only 11.9% of the 70.4% of the active business owners are not managers. Hence we conclude that understanding the decisions of managers, rather than simple entrepreneurs, is crucial to understanding the behavior of households at the top of the wealth and income distributions.

2.3 Managerial Compensation and Firm Returns

To better understand the incentives of managers at the top of the income distribution, we study the distribution of compensation and firm returns for managers of publicly traded firms. We start with a brief comparison with respect to our main sample of managers in SCF. Panel A of Table 6 shows cross-sectional moments of the distribution of manager's compensation and age in 2019 from Execucomp. On average, managers of publicly traded firms earn $3.5 million a year, most of which is accounted for equity compensation ($2.8 million).[9] Panel B of Table 6 reports a few descriptive statistics about managers in our SCF sample. Relative to these managers, Execucomp executives tend to be older and earn salaries in the top 10% of the overall earnings distribution.

Importantly, in the model we propose in the next section, managers' compensation is tightly linked to firms' returns. We find a similar pattern in the data. Figure 2 shows the relationship between firm annual returns—calculated as the average return between year t and $t-3$—and manager's annual compensation. In all plots, we control for executive age, firm size, year-fixed effects, and executive fixed effects. Panel A shows that total compensation is positively correlated with firm

[9]Managers compensation has increased significantly over time, as shown in Figure 1. Most of this increase is accounted for by an increase in incentive-based compensation, which is defined as the sum of bonuses, the value of stock awards, the value of option awards, and other non-equity incentive payments. In contrast, non-incentive-based compensation, which is the sum of salaries, pension contributions, and other compensation, has remained relatively constant over the same period.

returns. As expected, incentive-based compensation is more positively correlated with returns than non-incentive-based compensation, as shown in panels B and C. Panels D to F of Figure 2 show that these patterns have even stronger results for CEO compensation.

Furthermore, our model predicts that richer managers will invest a larger fraction of their wealth in the company they manage and should finance a larger fraction of the capital in the firm. Although Execucomp does not report the amount of wealth of managers, we can use information from the Security and Exchange Commission (SEC) to obtain information on the share of equity owned by a manager of a publicly traded firm they work for. To this end, we retrieve information from the official filings for the SEC, that requires that individuals and companies that own more than 5% of a traded company report this to the authorities. SEC files contain about 58 thousand year-file observations from 1995 to 2022. About 29 thousand observations are individuals rather than firms (e.g. mutual funds). We merge this information to Execucomp and identify 984 unique executives accounting for about 10% of all the year-files in the sample.

From this data, we obtain two key variables, i) the percentage share of the company that the individual owns (from SEC) and ii) the market value of the firm they manage (from merging Execucomp with CRSP), which allows us to compute the equity wealth of a manager. The results are shown in Figure 3. Panel A displays the share of ownership across all individuals in the data (including those who are not managers). We find an average ownership of about 16%. Panel B shows the correlation between managers' equity wealth and ownership share. This figure shows that, i) managers of large corporations are typically the top 1 to top 0.1% of the wealth distribution (marked with vertical lines) and that ownership increases with the level of wealth: above the top 1%, the average ownership is about 21%. Our model is consistent with these observations.

3 A Model of Income and Wealth Inequality

Consider an infinite horizon economy with two types of households, managers and workers. The mass of managers is 1 and the mass of workers is N. A given dynasty of each type of household survives with probability ρ. If a dynasty ends it is immediately replaced with a new one.

The period preferences of managers are $u(c_t, e_t)$ where (c_t, e_t) consumption and e_t effort respectively. Workers have the same preferences $u(c_{nt}, e_{nt})$ where (c_{nt}, e_{nt}) are their consumption and effort. Both types of households discount the future at rate β.

The economy has a single good per period which is produced by a continuum of firms. Each firm is managed by a single manager who uses capital and workers' labor to produce a single good. The production technology of a specific firm is given by the constant returns to scale function

$$F(zh, k, n), \tag{1}$$

where h is the human capital of the manager, k is capital input and n is the workers' labor input

of that firm and z is a manager-specific productivity shock. The density of this shock depends on manager's effort and is given by $f(z|e)$. The effort level of the manager is the private information of the manager, whereas the realization of the shock z is public information.

The human capital of the manager is specific to the manager and can be used at any firm. The capital stock should be thought of as including both tangible and intangible capital. We describe the details of the aggregation that leads to a single capital stock below. Each manager's human capital evolves according to

$$h_{t+1} = \Psi(h_t, i_t) + (1 - \delta_h)h_t, \tag{2}$$

for a function Ψ specified below, where i_t the amount of goods devoted to human capital accumulation and δ_h is the depreciation rate of human capital. Each worker's human capital evolves according to

$$h_{nt+1} = z_{nt+1}h_{nt}^{\gamma_n}. \tag{3}$$

where the idiosyncratic exogenous shock z_n has density $f_n(z_n)$.

We turn now to developing a recursive notion of competitive equilibrium. We assume that in each period, financial intermediaries compete to attract managers by offering contracts that insure the managers against the idiosyncratic risk while providing the managers' incentives to exert effort.

The timing for the contracting problem is that at the end of any period, financial intermediaries and managers enter into contracts. At this point each manager has wealth W and human capital h, so that the state variable for a manager is (W, h). In terms of the compensation of the manager, we consider contracts which have a flat component w' which is independent of the shock z' and a risky component that depends on the returns generated by the firm. To understand the risky component, consider a firm that has k' units of capital allocated to it. The gross return of this firm, given the human capital of the manager, the amount of capital allocated to the firm, and the flat component of the manager's compensation is given by $R(zh, k, w)$ which is defined by

$$R(zh, k, w)k = \max_n [F(zh, k, n) - w_n n - w + (1 - \delta)k] \tag{4}$$

which implies an optimal level worker employment, $\bar{n}(h, k, w, z)$ where we suppress dependence on the market wage of labor w_n. Let R_s denote the market return on a diversified portfolio. We assume that the risky component of the manager's compensation is given is linear in the excess return of the firm and is given by $\phi\left[R(zh, k, w) - R_s\right]k$ where ϕ is the *loading factor* on the excess return. Thus, the manager of a given firm receives a higher compensation if that firm outperforms the market and receives a lower compensation if that firm underperforms relative to the market. It will turn out that the expected value of the risky component of a manager's compensation is zero.

We now formalize the contracting problem. Consider a manager who enters into a contract with wealth W and human capital h. A *contract* $x = (c, e, i, A', w', \phi, k', W'(z'))$ for this manager consists

of consumption c, investment in human capital i, and savings A' that satisfy the budget constraint

$$c + i + A' \leq W, \tag{5}$$

a flat component of compensation w', the loading factor ϕ, an allocation of capital k' to the firm, a recommended level of effort e, and next period's wealth and human capital $(W'(z'), h')$. Next period's wealth satisfies the transition law

$$W' = w' + \phi \left[R(z'h', k', w') - R_s\right] k' + R_s A' \tag{6}$$

and human capital satisfies (2).

Next, we show that competition among financial intermediaries implies zero profits in that the expected gross return of a firm must equal the market return so that

$$\int R(z'h', k', w') f(z'|e) dz = R_s. \tag{7}$$

To see this result, note that the return to intermediary in each state z' after paying the workers $w_n n'$ and the managers the flat component w' is $R(z'h', k', w')k'$. Hence, after also paying the risky component $\phi \left[R(z'h', k', w') - R_s\right] k'$ the return is

$$R(z'h', k', w')k' - \phi \left[R(z'h', k', w') - R_s\right] k'.$$

Since the intermediary can always obtain $R_s k'$ from investing k' in the market, the expected return from investing in the firm must satisfy

$$\int \left\{R(z'h', k', w')k' - \phi \left[R(z'h', k', w') - R_s\right] k'\right\} f(z|e) dz \geq R_s k'$$

which after some rearrangement implies that the left side of (7) must be at least as high as the right side. If this inequality is strict, then an intermediary can offer a slightly higher flat compensation, keeping the other elements of the contract fixed and attract all managers. Thus, competition among financial intermediaries implies (7).

Finally, we derive the incentive constraint of the manager. To do so, let $V(W', h')$ denote the continuation value of manager. The incentive compatibility constraint is

$$e \in \arg\max_e u(c(W, h), e) + \beta\rho \int_z V\left(W'(W, h, z), h'(W, h)\right) f(z|e) dz \tag{8}$$

where $c(W, h), W'(W, h, z)$ and $h'(W, h)$ are specified in the contract.

Then, the *contracting problem* for this manager with the intermediary is

$$V(W, h) = \max_x \; u(c, e) + \beta\rho \int_z V\left(W', h'\right) f(z'|e) dz' \tag{9}$$

subject to the manager's budget constraint (5), the transition laws for wealth (6) and human capital (2), the zero profit condition for intermediaries (7), and the incentive compatibility for a manager's

effort (8). The solution to this contracting problem generates optimal contracting functions

$$\{c(W,h), i(W,h), A'(W,h), w'(W,h), \phi(W,h), k'(W,h), W'(W,h,z), h'(W,h), e(W,h)\}. \quad (10)$$

Given these contracting functions we can construct $R(zh,k,w)k$ and then use (4) to construct the worker demand function $n(W,h,z) = \bar{n}(h,k(W,h),w(W,h),z)$.

We turn now to workers who solve standard consumption-savings problems with incomplete markets as in Aiyagari-Bewley models. We adopt the same timing assumptions for workers as for managers. At the end of any period workers have wealth W_n and human capital h_n, so that the state variable for a worker is (W_n, h_n). Workers choose how much to consume and save subject to budget constraint

$$c_n + A'_n \leq W_n. \quad (11)$$

The also choose how much effort to exert. The transition laws for wealth and human capital are given by

$$W'_n = w_n h'_n e_n + R_s A'_n \text{ and } h'_n = z'_n h_n^{\gamma} \quad (12)$$

Workers solve the following recursive problem

$$V_n(W_n,h_n) = \max_{c_n,e_n,W'_n} u_n(c_n,e_n) + \beta\rho \int V_n(W'_n,h'_n) f_n(z_n) dz_n \quad (13)$$

subject to (11), and the transition laws (12). The solution for the worker generate policy functions $c_n(W_n,h_n), e_n(W_n,h_n), W'_n(W_n,h_n,z_n)$ and the new human capital $h'_n = z'_n h_n^{\gamma}$.

Next, we describe the timing of birth and deaths of a dynasty and how wealth is transfered from dying dynasties. In terms of timing we assume that after production has taken place, payments are made to suppliers of worker labor, managerial labor, and capital. After these payments are made, but before consumption occurs, a fracton $1-\rho$ of the dynasties of workers and managers die. They are replaced by newborn dynasties. We assume that our economy has an equilibrium with a stationary distribution $\mu(W,h)$ for managers and $\mu_n(W_n,h_n)$ for workers. The wealth and human capital of a newborn manager is given by a convex combination of a random draw of $\mu(W,h)$ and the mean of that distribution, with weights a and $1-a$ respectively. Likewise, wealth and human capital of a newborn workers given in the same way. Clearly, this way of redistributing wealth and human capital preserves the stationarity of the distributions.

A stationary equilibrium consists of a measure of managers $\mu(W,h)$ over their states, a measure of workers, $\mu_n(W_n,h_n)$ over their states, contracting policy functions (10), and a worker demand function $n(W,h,z)$, worker policy functions, and value functions for managers and workers $V(W,h)$ and $V_n(W_n,h_n)$, a worker wage rate w_n and a market-wide rate of return R_s, which *i)* solve the contracting problem for managers (9), *ii)* solve the worker's problem (13), *iii)* satisfy labor market

clearing,

$$\int [n(W,h,z)f(z|e(W,h))dz]\, d\mu(W,h) = N \int [e_n(W_n,h_n)z_n h_n^{\gamma_n} f_n(z_n)dz_n]\, d\mu_n(W_n,h_n)$$

iv) satisfy goods market clearing,

$$\int [c(W,h)+i(W,h)]\, d\mu(W,h) + N\int c_n(W_n,h_n)d\mu_n(W_n,h_n) + \delta \int k(W,h)d\mu(W,h)$$
$$= \int [F(zh,k(W,h),n(W,h,z))f(z|e(W,h))dz]\, d\mu(W,h)$$

v) satisfy capital market clearing,

$$\int A'(W,h)d\mu(W,h) + N\int A'_n(W_n,h_n)d\mu_n(W_n,h_n) = \int k(W,h)d\mu(W,h)$$

vi) for managers, the measure of wealth and human capital (W,h) replicates itself in that for any Borel subset B of (W,h)

$$\mu(B) = \int \left[I_{\{[W'(W,h,z),h'(W,h)]\in B\}} f(z|e(W,h))dz\right] d\mu(W,h)$$

where I denotes the indicator variable,

vii) for workers, the measure of wealth and human capital $\mu_n(W_n,h_n)$ replicates itself in that for any Borel subset B of (W_n,h_n)

$$\mu_n(B) = \int \left[I_{\{[W'(W_n,h_n),h'_n(h_n,z_n)]\in B\}} f_n(z_n)dz_n\right] d\mu_n(W_n,h_n)$$

where $h'_n(h_n,z_n) = z_n h_n^{\gamma_n}$.

3.1 How Our Model Generates Wealth Inequality

In order to understand how our model generates wealth inequality, it is useful to begin with a stripped-down version of our model. Here let the human capital of managers fully depreciate each period, $\delta_h = 1$, and let dynasties never die, $\rho = 1$, and let the utility function be

$$u(c,e) = \log c + v(e). \tag{14}$$

Since $\delta_h = 1$, human capital of managers is no longer a state variable. The following result holds.

Proposition 1. *Under these assumptions the value function has the form $V(W) = A + \log W$. Effort $e = \bar{e}$ and the loading factor $\phi = \bar{\phi}$ are constant, $c' = \bar{c}W$, $k' = \bar{k}W, i = \bar{\imath}W, w' = \bar{w}W, A' = \bar{A}W$, $W'(W,z') = \bar{R}(z)W$ are linear in wealth, where $\tilde{R}(z) \equiv \left\{\bar{w}\bar{h} + [\phi R(z') + (1-\phi)R_s]\,\bar{A}\right\}$. Furthermore, for this problem to have a stationary distribution, it must be true that $E(\tilde{R}(z)) = 1$.*

Using the form of the utility function (14), the contracting problem is to choose $x = (c, i, A', w', \phi, k', e, W'(z'))$ to solve

$$V(W) = \max_x \left\{(1-\beta\rho)\log c + v(e) + \beta\rho \int_z V\left[W'(z)\right] f(z'|e)dz'\right\} \tag{15}$$

subject to the budget constraint, the transition law for wealth, the zero profit constraint, and the incentive constraint, given by

$$c + i + A' \leq W \tag{16}$$

$$W'(z) = w' + \phi \left[R(z'Bi, k', w') - R_s \right] k' + R_s A' \tag{17}$$

$$\int R(z'h', k', w')k' f(z'|e)dz = R_s k' \tag{18}$$

$$e \in \arg\max_e \log c + v(e) + \beta \int_z V\left(W'(W, z)\right) f(z|e)dz. \tag{19}$$

In the appendix, we prove this result. This proposition implies that, conditional on survival, wealth follows a geometric random walk. Specifically, taking logs of the transition law for wealth gives

$$\log W' = \log W + \log \bar{R}(z). \tag{20}$$

Hence, absent death ($\rho = 1$), starting from any initial distribution of wealth, over time the cross section distribution of wealth increases without bound. Next, we introduce a birth and death process of the type considered above the distribution can be made stationary.

In our discrete time model it is difficult to solve for the stationary distribution in closed form. We consider a version of the transition equation for wealth when the length of time of each period is Δ and show that as Δ converges to zero, the distribution of wealth converges to a two-sided Pareto. Here calendar age of an agents evolves via $t_n = t_{n-1} + \Delta$ where n is the number of periods that agent is alive. Let the probability of death in any period be $\rho\Delta$. Here conditional on survival, (20) becomes

$$\log W_{t_{n+1}} = \log W_{t_n} + \log R_{t_{n+1}}$$

We suppose that when a dynasty dies, it is replaced by a newborn dynasty with wealth $\log W_0$ drawn from a normal distribution with mean $\log \bar{W}_0$ and variance $\sigma^2\Delta$. We suppose that $\log R_t$ is normal with mean 0 and variance $\sigma^2\Delta$.

3.2 Workings of the Model

To shed light on how our model can reproduce not just the observed degree of concentration in income and wealth but also the entire distribution of each, we consider now a simple version of our model.

4 Quantification

Here, we first discuss how we discipline the parameters of our model, starting with the key technologies of output production, of managers' human capital acquisition, and of managers' productivity conditional on their effort. We maintain that both managers and workers discount the future by the

factor β, which we set to 0.994, and their probability of survival ρ is 0.9917 so that a dynasty is three-generation lived and $1/(\beta\rho) = 1.4$, which corresponds to an annual risk-free interest rate is 4%.

As described in our parameterization of the production function, we will normalize the level of labor-augmenting total factor productivity Φ. This fact implies that we will be free to normalize the expected value of z—or rather, in practice, one of the three values of $z \in \{z_1, z_2, z_3\}$—and of n—or, more precisely, $\bar{h}_n$ is normalized to 1, while the size of the worker population, N, is recovered as the ratio of workers in the total U.S. population.

4.1 Model Parameterization

We provide here details about the parameterization of the model.

Production Function. The production function is Cobb-Douglas of the form

$$y_t = F(k_t, n_t, z_t h_{t-1}) = (z_t h_{t-1})^{1-\alpha-\gamma} k_t^{\alpha} (\Phi n_t)^{\gamma},$$

where the constant Φ is labor-augmenting total factor productivity. In pinning down its parameters, an important accounting issue arises. In SCF, the ratio of wealth to income net of depreciation is around 8, whereas in, say, NIPA the ratio of physical capital to income net of depreciation is about 5. Hence, as is well known, wealth in the United States simply does not simply map into capital. To reconcile these two statistics, we follow the approach in Hall [2001] and interpret the difference between the two statistics as due to the existence of intangible capital, based on the idea that the aggregate corporate market-book ratio reflects the value of this latter type of capital.

Formally, consider the augmented production function

$$y_t = F(k_{1t}, k_{2t}, n_t, z_t h_t \ell) = \Gamma^{\gamma} k_1^{\alpha_1} k_2^{\alpha_2} n^{\gamma} (zh\ell)^{1-\alpha-\gamma},$$

with $\alpha = \alpha_1 + \alpha_2$, where k_1 is measured (tangible) capital and k_{2t} is unmeasured (intangible) capital with laws of motion $k_{1t+1} = (1-\delta)k_{1t} + x_{1t}$ and $k_{2t+1} = (1-\delta)k_{2t} + x_{2t}$. Given intertemporal prices p_t, production choices are now solution to the problem

$$\max_{n_t, x_{1t}, x_{2t}} \sum_{t=0}^{\infty} p_t \left[y_t - w_{nt} n_t - w_t z_t h_t \ell - x_{1t} - x_{2t} \right]$$

subject to the laws of motion of the two types of capital. Since the optimal choice of capital $i = 1, 2$ satisfies $p_t = p_{t+1}(F_{it+1} + 1 - \delta)$, firms equate the marginal product of capital of the two types of capital ($F_{1t+1} = F_{2t+1}$), which, using the Cobb-Douglas form of the production function, implies $\alpha_1 y_t / k_{1t} = \alpha_2 y_t / k_{2t}$ or $k_{1t} = \alpha_1 k_{2t} / \alpha_2$. Letting let $k_t = k_{1t} + k_{2t}$ and using that $\alpha = \alpha_1 + \alpha_2$, it follows that $k_{1t} = \alpha_1 k_t / \alpha$ and $k_{2t} = \alpha_2 k_t / \alpha$. Thus, the production function can be rewritten as

$$y_t = \Gamma^{\gamma} \left(\frac{\alpha_1}{\alpha} k_t \right)^{\alpha_1} \left(\frac{\alpha_2}{\alpha} k_t \right)^{\alpha_2} n_t^{\gamma} (z_t h_{t-1} \ell)^{1-\alpha-\gamma} = A \Gamma^{\gamma} k_t^{\alpha} n_t^{\gamma} (z_t h_{t-1} \ell)^{1-\alpha-\gamma} \tag{21}$$

where $A = (\alpha_1/\alpha)^{\alpha_1} (\alpha_2/\alpha)^{\alpha_2}$. Hence, it is immediate that an economy with tangible and intangible

capital and the production function in (21) is equivalent (in terms of aggregates) to an economy with the single capital stock $k_t = k_{1t} + k_{2t}$ with law of motion $k_{t+1} = (1-\delta)k_t + x_t$ and production function $y_t = \Gamma^\gamma A k_t^\alpha n_t^\gamma (z_t h_{t-1}\ell)^{1-\alpha-\gamma}$.

In light of this equivalence result, we can reconcile SCF and FoF as follows. Aggregate investment in intangible capital, X_{2t}, is (mistakenly) treated as an intermediate good in the national income accounts rather than as an investment good, as it should be—upper–case letters denote aggregate variables. So *measured output* in these accounts is $Y_{mt} = Y_t - X_{2t}$. In a steady state, $X_2 = \delta K_2$ and $K_2 = \alpha_2 K/\alpha$ so that $X_2 = \delta\alpha_2 K/\alpha$ so that measured aggregate output is $Y_m = Y - \delta\alpha_2 K/\alpha$ and "true" output is $Y = Y_m + \delta\alpha_2 K/\alpha$. We measure the wealth-to-net-income ratio ω in SCF as

$$\omega \equiv \frac{K}{Y_m - \delta K_1} = \frac{K}{Y - \delta K} \tag{22}$$

in steady state, where net income $Y_m - \delta K_1$ is the sum of labor and capital income and equals measured income minus depreciated tangible capital $(Y_m - \delta K_1)$. The second equality in (22) follows from the definition of measured output as $Y_{mt} = Y_t - X_{2t}$ and that $X_2 = \delta K_2$ in steady state so that $Y_m - \delta K_1 = Y - X_2 - \delta K_1 = Y - \delta K_1 - \delta K_2 = Y - \delta K$, where $K = K_1 + K_2$. By (22),

$$\frac{Y}{K} = \frac{1}{\omega} + \delta. \tag{23}$$

We measure $\theta = K_1/Y_m$ from the Flow of Funds as the ratio of gross fixed capital to measured GDP. Next, by using that $K = K_1 + K_2$ and $K_2/K_1 = \alpha_2/\alpha_1$ in steady state as implied by $k_{1t} = \alpha_1 k_t/\alpha$ derived above, we obtain that

$$\omega = \frac{K}{Y_m - \delta K_1} = \left(\frac{K_1 + K_2}{Y_m}\right)\frac{Y_m}{Y_m - \delta K_1} = \frac{K_1}{Y_m}\left(1 + \frac{K_2}{K_1}\right)\frac{Y_m}{Y_m - \delta K_1} = \theta\left(1 + \frac{\alpha_2}{\alpha_1}\right)\frac{Y_m}{Y_m - \delta K_1}.$$

Simple manipulations based on $Y_m = Y - X_2$, $X_2 = \delta K_2$, $\alpha = \alpha_1 + \alpha_2$, and (23) then lead to

$$\omega = \theta\left(\frac{\alpha}{\alpha_1}\right)\frac{Y - \delta\left(\frac{\alpha_2}{\alpha}\right)K}{Y - \delta K} = \theta\left(\frac{\alpha}{\alpha_1}\right)\frac{\frac{Y}{K} - \delta\left(\frac{\alpha-\alpha_1}{\alpha}\right)}{\frac{Y}{K} - \delta} = \theta\left(\frac{\alpha}{\alpha_1}\right)\left[\frac{\frac{1}{\omega} + \delta - \delta\left(1 - \frac{\alpha_1}{\alpha}\right)}{\frac{1}{\omega} + \delta - \delta}\right] = \theta\left(\frac{\alpha}{\alpha_1} + \delta\omega\right),$$

where the next-to-last equality uses (23). Rearranging terms gives that

$$Q \equiv \frac{K}{K_1} = \frac{\alpha}{\alpha_1} = (1 - \delta\theta)\frac{\omega}{\theta}, \tag{24}$$

where the first equality follows from $K = K_1 + K_2$ and $K_2/K_1 = \alpha_2/\alpha_1$ in steady state, where Q denotes implied Tobin's Q defined as the ratio of the market value of capital to its replacement value. Hence, we can use Tobin's Q as measured from the Flow of Funds and (24) to back out δ as

$$\delta = \frac{1}{\theta} - \frac{Q}{\omega}. \tag{25}$$

Lastly, in SCF we can also measure the ratio of aggregate labor income to aggregate net income as

$$\ell \equiv \frac{\bar{w} + w_n N \bar{h}_n}{Y - \delta K} = \frac{(1-\alpha)Y}{Y - \delta K} = \frac{1-\alpha}{1 - \frac{\delta}{\frac{1}{\omega} + \delta}} \quad \text{or} \quad \alpha = 1 - \ell\left(1 - \frac{\delta}{\frac{1}{\omega} + \delta}\right) \tag{26}$$

where the last equality in the expression for ℓ uses (23). Instead, (24) implies that

$$\alpha_1 = \frac{\alpha\theta}{(1-\delta\theta)\omega} \tag{27}$$

For example, if $\delta = 0.08$ as is standard, $\omega = 8.65$ measured from SCF as aggregate wealth over aggregate income in 2019, $\ell = 0.89$, and $\theta = 3.7$ measured from the Flow of Funds as the sum of non-financial assets of B.101, B.103, and B.104 divided by GDP in 2019, we obtain that

$$\alpha = 1 - 0.89\left(1 - \frac{0.08}{\frac{1}{8.65} + 0.08}\right) \approx 0.47 \quad \text{and} \quad \alpha_1 = \frac{3.7\alpha}{(1 - 3.7 \cdot 0.08)8.65} \approx 0.29 \tag{28}$$

Note that these values imply a value of $Q = \alpha/\alpha_1 \approx 1.65$, which is very similar to the average value of Q in the data, measured from the Flow of Funds for 2019 as equities/net worth = 1.70, which therefore validates the choice for δ. To instead match exactly the measure of Q using (25), we would alternatively obtain that

$$\delta = \frac{1}{3.7} - \frac{1.70}{8.65} \approx 0.074, \alpha = 1 - 0.89\left(1 - \frac{0.074}{\frac{1}{8.65} + 0.074}\right) \approx 0.46, \text{ and } \alpha_1 = \frac{3.7\alpha}{(1 - 3.7 \cdot 0.074)8.65} \approx 0.27,$$

which are values quite similar to those recovered in (28). Finally, we back out γ from the ratio of labor income of workers to the labor income of workers and managers as measured in SCF.

Distribution of Manager Productivity Conditional on Effort. Our approach relies on two important features of our model. Recall that

$$R_{t+1}(z) = R_s + \frac{1-\gamma}{\alpha}(R_s - 1 + \delta)\left[\frac{z_{t+1}^{\frac{1-\alpha-\gamma}{1-\gamma}}}{\int_z z^{\frac{1-\alpha-\gamma}{1-\gamma}} f(z|e_t;\theta)dz} - 1\right] = R_s + \frac{1-\gamma}{\alpha}(R_s - 1 + \delta)x_{t+1}(z), \tag{29}$$

which can easily be measured in the data from observed firm returns, once α, γ, and δ are known as argued. If we define the term in square brackets as $x_{t+1}(z)$, then

$$x_{t+1}(z) = \frac{R_{t+1}(z) - R_s}{\frac{1-\gamma}{\alpha}(R_s - 1 + \delta)}.$$

Letting $\widetilde{z}_{t+1} \equiv z_{t+1}^{\frac{1-\alpha-\gamma}{1-\gamma}}$ and $\lambda_t \equiv \int_z z^{\frac{1-\alpha-\gamma}{1-\gamma}} f(z|e_t;\theta)dz$, it follows that

$$1 + x_{t+1}(z) = \frac{z_{t+1}^{\frac{1-\alpha-\gamma}{1-\gamma}}}{\int_z z^{\frac{1-\alpha-\gamma}{1-\gamma}} f(z|e_t;\theta)dz} = \frac{\widetilde{z}_{t+1}}{\lambda_t} \tag{30}$$

For any possible choice of e_t, multiple z_{t+1} are possible, but importantly the set of potential realizations is *constant* across effort levels.

For simplicity and to be able to ensure that standard regularity conditions (such as the monotone likelihood ratio property) hold, we assume that manager productivity z can take one of three values, $z \in \{z_1, z_2, z_3\}$ with $z_1 < z_2 < z_3$, and specify its probability mass as a function of a manager's effort as $\Pr(z_3|e) = 1 - \exp(-\xi e)$ and $\Pr(z_1|e)/\Pr(z_2|e) = (\xi e)^{-\nu}$. Notice that this density function is defined by the three support points $\{z_1, z_2, z_3\}$ and the two parameters (ξ, v), which we recover as

follows. Suppose that we can identify managers who choose the same level of effort e_t—first important assumption—by their wage, which is a measure of their human capital, or wealth. For instance, imagine that managers with the highest (respectively, lowest) level of wealth exert the same, although possibly different, level of effort.[10] Then λ_t is constant across them, which implies that by (30), the distribution of their realized $1+x_{t+1}(z)$ identifies $(\widetilde{z}_1/\lambda_t, \widetilde{z}_2/\lambda_t, \widetilde{z}_3/\lambda_t)$, with $z_1 < z_2 < z_3$, at any point in time. If one support point of the distribution of z is known, say, z_1—second important assumption—then the ratio of $\widetilde{z}_1$ to $\widetilde{z}_1/\lambda_t$ identifies λ_t and from $\widetilde{z}_2/\lambda_t$ and $\widetilde{z}_3/\lambda_t$, we can identify $\{\widetilde{z}_2, \widetilde{z}_3\}$. Hence, (λ_t, z_2, z_3) are easily recoverable. In turn, the probability masses of $(\widetilde{z}_1/\lambda_t, \widetilde{z}_2/\lambda_t, \widetilde{z}_3/\lambda_t)$ identify $\Pr(z_{t+1} = z_1|e_t)$, $\Pr(z_{t+1} = z_2|e_t)$, and $\Pr(z_{t+1} = z_3|e_t)$. If these three probabilities are completely determined by two parameters alone so that θ consists only of these parameters, we can identify θ from just the probabilities $\{\Pr(z_{t+1} = z_1|e_t), \Pr(z_{t+1} = z_2|e_t), \Pr(z_{t+1} = z_3|e_t)\}$, provided that two of these probabilities are invertible functions of θ.

Once the parameters of $\{\Pr(z_{t+1} = z_1|e_t), \Pr(z_{t+1} = z_2|e_t), \Pr(z_{t+1} = z_3|e_t)\}$ are recovered, by this argument, we can infer the effort of the remaining groups of managers. Specifically, if, say, e_t is monotone with wealth, then we can repeat the steps just described for managers at percentiles p-th and $p+1$-th of the wealth distribution who exert the corresponding level of effort e'_t and from the identified $\{\Pr(z_{t+1} = z_1|e'_t), \Pr(z_{t+1} = z_2|e'_t), \Pr(z_{t+1} = z_3|e'_t)\}$, back out e'_t, once the parameters that determine the map between managers' effort level and these probability masses are known.

In practice, we proceed as follows. We start with the sample of managers obtained from ExecuComp as described in Section 2. We collapse this database to one observation per firm averaging managers compensation within a firm. This reduces the sample to 55,029 firm-year observations with an average of about 1,900 firms per year. We merge this data with information on firm-level returns from CRSP. Since some firms-year have missing price information, the sample is reduced to 53,097 observations between 1992 and 2020.

We calculate monthly returns including dividends from CRSP, which we then aggregate at the annual level. We obtain annual average returns as the weighed average return $r_{st} = \sum_i \omega_{it} r_{it}(z) / \sum_i \omega_{it}$, where ω_{it} is the market capitalization of the firm at the end of the year.[11] We include observations on firms whose managers' average salary is above the 25-th percentile of the within-year salary distribution to ensure that average executives' wages—namely, wages, salaries, and bonuses—in the resulting Execucomp sample of firms is in the top 5% of the distribution of managers' salaries in SCF, which we identify as managers exerting the *highest possible* level of effort.[12] This restriction reduces the sample to 45,117 observations. For this sample, we calculate $1 + x_{it+1}(z)$ as in (30), which connects firm (observable) returns to managers' productivity (for managers who exert the same

[10] For an effort policy monotone in cash-on-hand and human capital, these levels of effort would be, respectively, the maximum and the minimum.

[11] We obtain information on market capitalization from CRSP data as the (absolute) value of price at the end of the year (PRC) times the number of outstanding shares (1,000*SHROUT).

[12] As we show in Table A.3 in the appendix, our estimation results are robust changing our sample to only focusing on CEO rather than all the managers available in Execucomp or if we exclude the Great Recession.

level of effort). Importantly, we restrict our sample to observations with $1 + x_{t+1}(z) \geq 0$, since z_{it+1} is strictly positive in our model. We are left with a sample 37,746 observations since 1993. Figure 9 in the appendix shows the distribution of returns used in our analysis.

To recover the distribution of $\{z_t\}$ in practice, we proceed as follows. We first recover the distribution of $\widetilde{z}_{t+1}/\lambda_t$ from the distribution of actual firm returns, pooled across years, based on Equation (30). We then discretize the distribution of $\widetilde{z}_{t+1}/\lambda_t$. To ensure that this discretization accurately reproduces the right tail of the distribution of $\{z_t\}$, which is inherently linked to the distribution of income and wealth according to our model, we choose $\widetilde{z}_3/\lambda_t$ to correspond to the 99th percentile of the distribution of $\widetilde{z}_{t+1}/\lambda_t$. We then divide the remaining probability mass (99%) into two equal-probability clusters, and pin down $\widetilde{z}_2/\lambda_t$ and $\widetilde{z}_3/\lambda_t$ as the medians of these two clusters. After normalizing z_2 to 0.1 and using the fact that at this stage α and γ are known, we back out λ_t from $\widetilde{z}_2/\lambda_t$ so that from λ_t, $\widetilde{z}_1/\lambda_t$, and $\widetilde{z}_3/\lambda_t$, we can determine the values of $\widetilde{z}_1$ and $\widetilde{z}_3$. Finally, we obtain z_1 and z_3 from $\widetilde{z}_1$ and $\widetilde{z}_3$, respectively, using that $\widetilde{z}_{t+1} \equiv z_{t+1}^{\frac{1-\alpha-\gamma}{1-\gamma}}$ and the values of α and γ. Accordingly, we estimate that $\{z_1, z_2, z_3\} = \{0.0214; 0.1; 16, 382.1110\}$. We also recover $\{\Pr(z_{t+1} = z_1|e'_t), \Pr(z_{t+1} = z_2|e'_t), \Pr(z_{t+1} = z_3|e'_t)\}$ for executives at each decile of the distribution of wages all executives in our Execucomp sample, which, as discussed above, pin down the effort choices of managers earning those wages.

Human Capital Process. Once manager productivity is known, we use the information on manager compensation to recover the parameters of the human capital process. Recall that according to our model, a manager's wage in period t is

$$w_{t+1} = \frac{1-\alpha-\gamma}{1-\gamma}\left(\int_z z^{\frac{1-\alpha-\gamma}{1-\gamma}} f(z|e_t;\theta)dz\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} h_t = \left(\frac{1-\alpha-\gamma}{1-\gamma}\right)\lambda_t^{\frac{1-\gamma}{1-\alpha-\gamma}} h_t. \tag{31}$$

Hence, with λ_t identified as detailed above, we can recover a manager's human capital in given year from the manager's wages and salaries in the year as

$$h_t = w_t\left(\frac{1-\gamma}{1-\alpha-\gamma}\right)\left(\frac{1}{\lambda_t}\right)^{\frac{1-\gamma}{1-\alpha-\gamma}}.$$

We measure w_{it} as the sum of salaries and bonuses for each executive in our sample after trimming the bottom 1% of this distribution—typically managers with zero salaries. We further restrict our sample to information available after 2007 and to executives of firms with information on total asset value. These selection criteria reduce the sample to 148,556 executive-years observations. Recall that the human capital function is given by $h_{t+1} = (1-\delta_h)h_t + \Psi(h_t, \iota_t)$, where ι_t is the time investment in acquiring human capital in period t. Given the estimated $\{h_t\}$, the investment function $\Psi(\cdot)$ is non-nonparametrically identified up to ι under standard assumptions, if the depreciation rate of human capital δ_h is known. We set δ_h at 0.1. Also, we parameterize the investment function $\Psi(h_t, \iota_t)$

as

$$\Psi(h,\iota) = B_h\iota - \Phi\left(\frac{\iota}{h}\right)h = B_h\iota - \left(\frac{\phi_h}{1-\xi_h}\right)\left[\left(\frac{\iota}{h}\right)^{1-\xi_h} - 1\right]h, \tag{32}$$

with parameters $\phi_h > 0$ and $\xi_h \geq 0$ such that $\xi_h \neq 1$ and the properties that $\Phi(1) = 0$, $\Phi'(x) = \phi_h(x)^{-\xi_h}$, and $-\Phi''(x)x/\Phi'(x) = \xi_h$.

Utility Function. The utility function of managers and workers is

$$u(c,e) = \frac{c^{1-\sigma_c} - 1}{1-\sigma_c} + \chi\frac{(1-e)^{1-\sigma_e} - 1}{1-\sigma_e}.$$

Intuitively, we pin down these parameters by targeting the effort profile recovered as discussed as well as the ratio ϕ of an executive equity in the executive's firm over firm market capitalization. Formally, it is apparent from the first-order conditions for consumption and effort from a manager's contracting problem that once all other parameters are backed out, those conditions evaluated for managers with different levels of human capital pin down σ_c, χ, and σ_e. In our baseline exercises, we take $\sigma_c \to 1$.

4.2 Model Fit

Expand:

1) Is it correct that our Execucomp sample corresponds to the top 5% of I distribution (correct) and the top 0.1% of W distribution in SCF (we do not know if this is for ALL Executives, only those we can match to SEC files, which are about 900 individuals)?

2) Is it correct that the Execucomp subsample for the figure you attached is selected further because it imposes the qualifying condition for an SEC filing for the executive (the executives needs to own at least 5% of the firm shares)

3) To what top percentiles of the I and W distribution the Execucomp sample in the picture corresponds to? For wealth, do we not really know. But we do know that those in the picture have enough equity wealth to be in the top 1% of the wealth distribution. As for income, we can check. I can do that later.

Untargeted Moments. Besides the moments directly targeted in the estimation step, our model is also consistent with key empirical observations on the distribution of ownership over the wealth and income distributions. In particular, our model predicts that as managers get richer, their share in the company they manage should increase as the increase their exposure to idiosyncratic risk. We can very this observation in the SCF data. We start by sorting managers by their net worth and then calculating the share of ownership they have on the company they manage (which corresponds to ϕ in the model

5 Impact of Taxes

Through the lens of our model, we evaluate here the impact of alternative taxation regimes, including consumption taxes, wealth taxes, and income taxes.

5.1 Progressive Wealth Taxes

Consider a progressive tax as currently implemented in Norway,

$$\tau(A) = \begin{cases} 0 & A < A_\tau \\ \tau_A(A - A_\tau) & A \geq A_\tau \end{cases}$$

The tax applies to wealth in excess of the cutoff A_τ with rate τ_A (0.85%). For example, in Norway A_τ is set to 60th percentile of distribution of wealth. We measure the progressivity of the tax by the percentile where A_τ cuts. Namely, higher A_τ implies higher progressivity, and if $A_\tau = 0$ then taxes are linear. In the presence of wealth taxes, the only change to the managerial contracting problem is due to their effect on the budget constraint, which becomes

$$c + i + A' + \tau(A') \leq W.$$

In our wealth tax experiments, we set the tax rate τ_A for different progressivity levels A_τ to raise a fixed amount of revenues—namely, 0.4% of baseline GDP, which corresponds to the amount of tax revenues raised in Norway through wealth taxes. Specifically, we consider four levels of progressivity: 1) linear tax with $\tau_A = 0.08\%$; 2) wealth tax above the 60th percentile of the baseline wealth distribution with a marginal tax rate above the threshold of $\tau_A = 0.09\%$; 3) wealth tax above 90th percentile of the baseline wealth distribution with a marginal tax rate above the threshold of $\tau_A = 0.16\%$; and 4) wealth tax above 99th percentile of the baseline wealth distribution with a marginal tax rate above the threshold of $\tau_A = 0.44\%$. At these specific tax rates τ_A, the four wealth tax regimes raise the same tax revenue in a dynamic scoring sense, i.e., at the new steady-state equilibrium of the resulting model.

Figure 6 depicts our wealth tax regimes for the third and fourth tax regimes we consider, with thresholds at the 90th and 99th percentiles of the baseline wealth distribution. As the wealth tax becomes more progressive, it affects more and more only wealth rich agents and, for a given tax revenue raised, the marginal rate for such agents increases.

Table 8 reports the results of these four wealth tax regimes in our model. We note a large fall in GDP per dollar of revenue raised for thresholds below the 90th percentile, which produce a loss of around 3 dollars of GDP per dollar of tax revenue raised; but for higher threshold, such as at the 99th percentile, the economy only loses 1.3 dollars of GDP per dollars of tax revenue raised.

Moreover, we note a modest decrease in wealth inequality but only at the top.

5.1.1 Intuition for the Nonmonotonic Effect of Wealth Taxes

Figure 4 shows the percentage change in savings relative to the no-tax baseline for our four wealth tax regimes. We note the non-monotonic effect of taxes on the lower end of the wealth distribution, where higher progressivity of taxes mitigates the drop in savings, while the effect of higher progressivity on the highest end of the wealth distribution is a larger drop in savings. Still, such an effect is evident only for extremely large values of wealth, as the top 0.1% migrates down fast. Intuitively, highly progressive wealth taxes punish luck in our model, and hence produce smaller distortions on GDP.

To understand the intuition behind the non-monotonic effect on GDP of wealth taxes as the level of progressivity increases, note that expectations of future taxes are central to saving decisions. Namely, even if current taxes are zero, agents care about crossing the boundary to the positive tax region; and, conversely, even if current taxes are high, agents care about crossing the boundary to the zero tax region. Under a linear tax, such a *crossing the boundary effect* is absent, as agents face τ_A at all wealth levels. But with boundary $A_\tau > 0$, for agents with wealth level near A_τ, this effect becomes important. Conversely, far from A_τ, the effect becomes again unimportant.

Also, when A_τ is large, the tax rate must be larger to raise a given amount of revenues, so, if the tax applies, it follows that the distortion is larger. But the set of agents to which the tax applies is smaller, and so the effect on GDP can be smaller.

Thus, when the wealth tax is particularly progressive, the managers that cross the boundary into the taxed region do so to a large extent because of luck—the realization of the low-probability high result z_h. The human capital and effort levels of these managers is not much higher than other managers that happen to be close to the boundary, which results in smaller distortions to effort and human capital choices, and hence smaller impacts on GDP.

Accordingly, Figure 5 shows the percentage change in the effort choice relative to the no-tax baseline for our four wealth tax regimes. The non-monotonicity is evident also in the effort choice, whereby the negative effect of a moderately progressive wealth tax is larger than the effect of a highly progressive wealth tax below very high values of wealth, well beyond the tax threshold.

5.2 Progressive Consumption Taxes

Next, we consider the effect of several consumption tax regimes, that have the same parametric shape as the wealth taxes, or

$$\tau(c) = \begin{cases} 0 & c < c_\tau \\ \tau_c(c - c_\tau) & c \geq c_\tau \end{cases}$$

and differ in progressivity. In the presence of consumption taxes, the only change to the managerial contracting problem is due to their effect on the budget constraint, which becomes

$$c + \tau(c) + i + A' \leq W.$$

Table 9 shows the effect of four consumption tax regimes that differ in progressivity in analogous fashion as our wealth tax experiments, with thresholds set at the 60th, 90th and 99th percentiles of the baseline income distribution. The loss in GDP for each dollar of tax revenue raise are substantially smaller under a consumption tax than under the wealth tax regimes we consider—or than under the income tax regimes we will consider below, for that matter. The impacts on GDP are 4-5 times lower for thresholds below the 90th percentile, and around two times lower in the highly progressive cases, as the marked non-monotonicity documented for wealth taxes is nearly absent for consumption taxes. Indeed, the dollar losses in GDP per dollar raised and the inequality reductions are similar across the progressive regimes considered.

The consumption tax regimes are therefore substantially less distortionary than the wealth tax regimes, but also imply very small reductions in the inequality measures. In fact, the linear consumption tax case implies exactly no distortions to both aggregate output and inequality measures—a consequence of the assumption of a unitary elasticity of intertemporal substitution, which implies that income and substitution effects of consumption taxes offset each other.

5.3 Progressive Income Taxes

Finally, defining income as $y(z) \equiv [\phi(R(z)-1)+(1-\phi)(R_s-1)]A' + w'$, or the sum of capital and labor incomes, we consider the effect of several income tax regimes. The income taxes we consider have the same parametric shape as the wealth taxes, or

$$\tau(y(z)) = \begin{cases} 0 & y(z) < y_\tau \\ \tau_y(y(z) - y_\tau) & y(z) \geq y_\tau \end{cases}$$

and differ in progressivity.

In the presence of income taxes, the only change to the managerial contracting problem is in the evolution of cash on hand, which becomes

$$W'(z) = y(z) - \tau(y(z)) + A'$$

. Table 9 shows the effect of four income tax regimes that differ in progressivity in analogous fashion as our wealth tax experiments, with thresholds set at the 60th, 90th and 99th percentiles of the baseline income distribution. Contrary to the case of wealth taxes, their negative effect on GDP raises monotonically as the degree of progressivity increases. Intuitively, while the highly progressive wealth tax tended to affect the luckiest managers mostly, the most progressive income tax regime tends to affect the most productive managers, as income taxes penalize earnings, and hence human capital accumulation, much more directly.

6 Conclusion

We have proposed a novel framework to study income and wealth inequality that resolves what we have called the Angeletos-Benhabib-Bisin puzzle. By endogenizing returns on wealth based on the optimal contractual exposure of managers and entrepreneurs to individual firm risk, such a mechanism is capable of generating thick right-tailed distributions of income and wealth, as observed in U.S. data. We view this framework as a promising one to account for observed inequality and to be used as a laboratory to investigate the impact of alternative tax regimes, such as consumption, wealth, and income taxes. Our findings suggest that wealth taxes can be highly distortionary and tend to raise little revenues. Still, they can reduce top inequality if applied to only the very wealthy.

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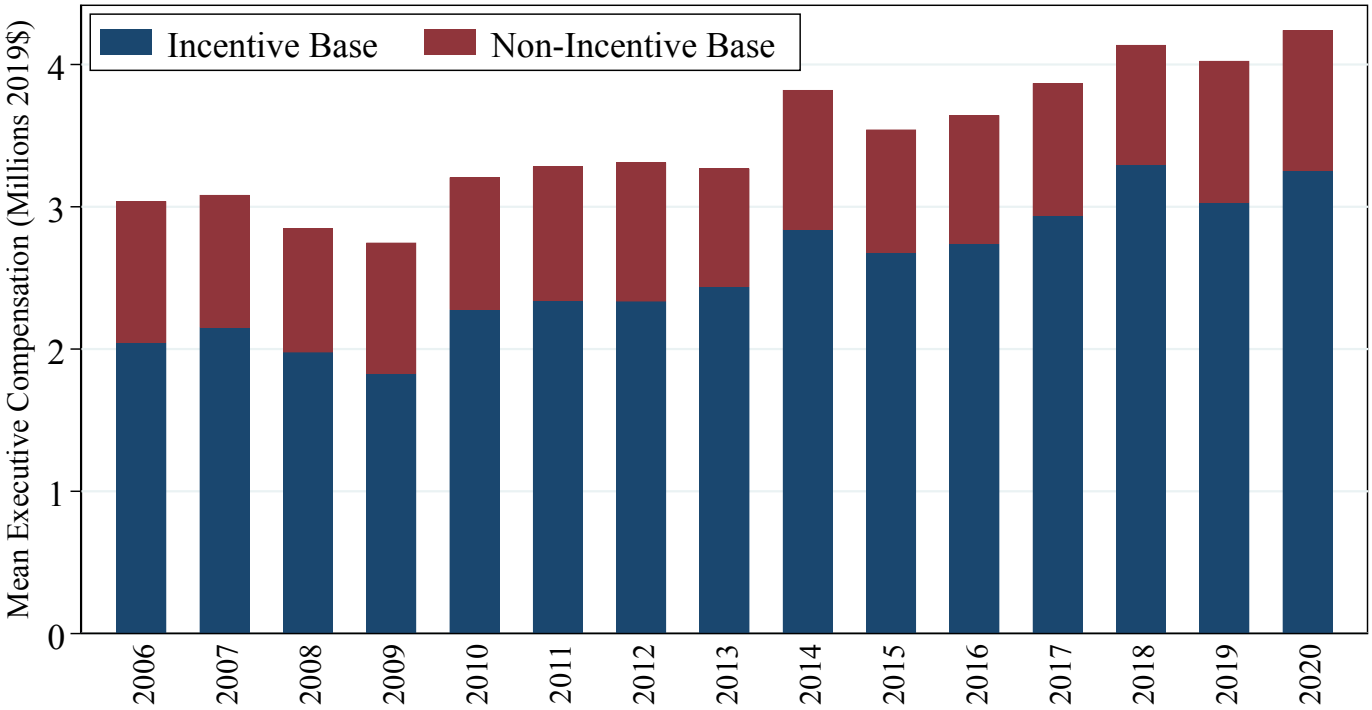
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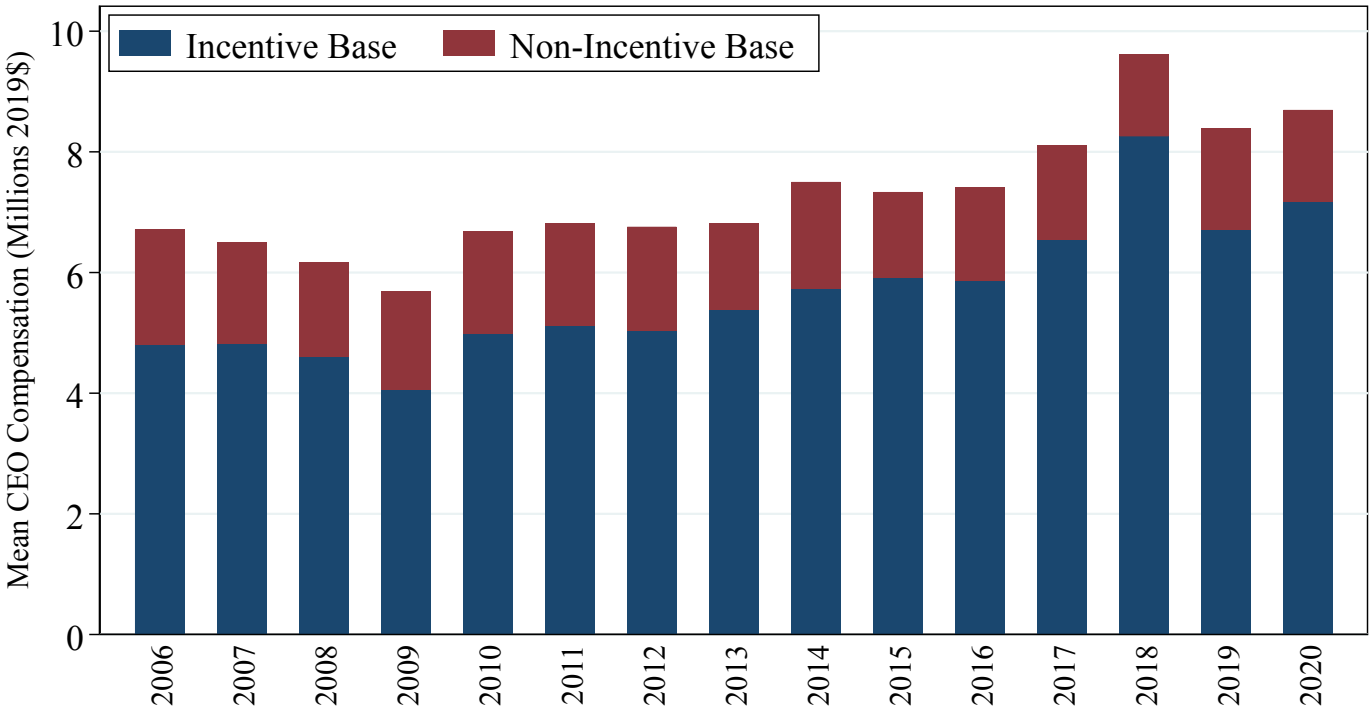
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Figure 1: Distribution of Compensation: Incentive versus Non-Incentive



(a) All Executives



(b) CEOs only

Note: Figure 1 shows the distribution of executives' compensation and the share of incentive and non-incentive base payments. Incentive-based compensation is the sum of bonuses, the value of stock awards, the value of option awards, and other non-equity incentive payments. Non-incentive-based compensation is the sum of salaries, pension contributions, and other compensation. The total number of observations is 160,940.

Table 1: Population Shares and Cross-Sectional Moments

A: Population Shares						
		Population	Wealth	Total Income	Labor Income	
Managers		0.20	0.58	0.44	0.42	
Workers		0.80	0.42	0.56	0.58	
B: Wealth Distribution (000s of $2018)						
	Mean	P10	P50	P90	P99	P99.9
Population	746.3	-0.5	121.5	1219.9	11106	43248.5
Manager	2153.2	10.2	372.8	4705.8	27819.8	107280
Worker	393.7	-1.7	87.2	804.2	5427.8	22321.7
C: Total Income Distribution (000s of $2018)						
	Mean	P10	P50	P90	P99	P99.9
Population	86.3	0	42.6	171.1	763	2970
Manager	191.2	12.8	88.5	371.3	1823.8	6258.1
Worker	60	0	35.4	134.8	365.5	1766.7
D: Labor Income Distribution (000s of $2018)						
	Mean	P10	P50	P90	P99	P99.9
Population	77	0	40.1	165.2	632.1	2261.6
Manager	163.2	11.8	84.1	336.4	1441.9	4164.5
Worker	55.4	0	33.6	129.5	342.2	1186.5

Notes: Table 1 shows cross-sectional moments of the wealth and income distribution of for managers and workers using data from the 2019 SCF.

Table 2: Concentration Measures of Wealth and Income in the Population

	Wealth			Income			Labor		
Wealth Rank	Population	Manager	Worker	Population	Manager	Worker	Population	Manager	Worker
<50	1.5	0.2	1.3	22.6	3.4	19.1	29.4	3.4	26.0
Top 10	76.5	51.4	25.0	43.1	30.0	13.1	29.7	17.7	12.0
Top 5	64.9	46.8	18.2	34.2	24.9	9.2	20.2	13.1	7.2
Top 1	37.3	29.2	8.1	16.6	12.6	4.0	6.4	3.8	2.6
Top 0.5	28.1	23.2	4.9	12.3	9.6	2.6	3.8	2.5	1.3
Top 0.1	14.2	12.4	1.8	5.2	4.6	0.7	0.7	0.6	0.1

Notes: Table 2 shows concentration measures of wealth, total income, and labor earnings using 2019 data from SCF.

Table 3: Portfolio and Income Composition

	Population			**Top 1%**			**Top 0.1%**		
	Pop.	Mngr.	Wrks.	Pop.	Mngr.	Wrks.	Pop.	Mngr.	Wrks.
A: Portfolio Composition									
Safe Financial	22.3	17.5	28.4	3.6	11.9	21.3	1.5	8.9	26.8
Risky Financial	16.1	14.8	17.6	6.8	20.3	51.7	3.2	20.1	49.1
Real State	37.4	30	47.1	4.3	16.1	17.3	1.6	11.9	10.4
Private Buss.	20.7	35.3	1.8	11.3	50.4	8.7	6.9	57.4	12.3
Vehicle Others	3.5	2.3	5	0.3	1.3	1	0.2	1.6	1.3
Leverage	13.4	9.2	18.8	0.5	2.1	1.9	0.2	1.3	0.9
B: Income Composition									
Wages	72.6	51.8	89.1	37.8	32.7	54.2	19.6	12.8	39.9
Business	16.6	33.5	3.1	30.7	38.3	6.8	31.6	40.4	5.0
Capital	10.8	14.6	7.7	31.5	29.1	39.0	48.8	46.7	55.1

Notes: Panel A of Table 3 shows the portfolio composition for the population and within managers and workers as a share of gross assets. Panel B shows the composition of income as a share of total household income. Data from SCF 2019.

Table 4: Wealth Shares of Managers and Their Firms

A: Business Type				B: How Acquired Business			
	All	Top 1%	Top 0.1%		All	Top 1%	Top 0.1%
Partner/Sole	8.2	7.7	6.8	Bought	23.4	23.7	24.1
S- & C-corp	32.0	32.6	32.9	Founded	50.2	50.6	51.1
LLC	50.5	51.4	52.6	Inherited	5.3	5.4	5.5
Not BO	9.2	8.3	7.7	Partner	8.0	7.9	7.1
				Other	3.9	4.1	4.5
				No BO	9.2	8.3	7.7

C: Firm Employment				D: Firm Sales			
	All	Top 1%	Top 0.1%		All	Top 1%	Top 0.1%
10	21.2	20.4	19.3	250K	2.9	2.3	1.9
20	3.9	3.7	3.0	1M	2.7	2.1	1.5
100	14.5	14.4	13.6	5M	7.4	6.7	5.5
500	27.0	27.9	29.3	10M	5.3	5.3	4.4
500	33.3	33.5	34.9	10M	70.3	73.2	77.1
				No BO	11.4	10.3	9.6

Notes: Panel A of Table 4 the wealth shares for different business types distribution for managers who own and actively manage a business. Managers who do not own a business are classified as Not Business Owners (Not BO). Panel B shows wealth shares for managers considering how they acquired the business they own. Panel C shows the wealth share of managers across the distribution of employment of the firms where they work. Panel D shows the wealth shares of managers across the distribution of sales of the firms they own.

Table 5: Shares of Managers, Entrepreneurs, and non-Managers Entrepreneurs

	Manager	Worker	Occ.	ABO	SE	ABO-SE	ABO	SE	ABO-SE
							Who are not Manager		
Population	20.0	80.0	29.5	12.2	14.3	9.2	5.6	7.1	4.0
Top 1% Wealth	71.4	28.6	67.0	62.3	59.9	54.2	16.1	13.8	12.2
Top 0.1% Wealth	82.0	18.0	65.6	70.4	73.2	63.1	11.9	16.2	9.2
Top 1% Income	76.7	23.3	79.2	62.4	56.2	51.5	11.5	6.5	6.2
Top 0.1% Income	69.7	30.3	83.9	64.0	58.7	53.8	6.4	6.9	5.5

Table 6: Distribution of Income and Age Among Managers

	Mean	P5	P10	P50	P90	P95	P99	P99.9
A: ExecuComp Managers in 2019 in 000's								
Wages	587	204	278	500	992	1,170	1,700	4,982
Wages-Bonus	712	230	300	542	1,151	1,500	3,750	9,650
Equity	2,836	56	209	1,480	6,621	10,190	19,502	44,807
Total	3,547	455	640	2,085	7,721	11,519	21,455	48,561
Age	55	42	45	55	63	66	74	88
B: Managers in the SCF in 2019								
Wages	108	0	0	70	222	324	737	1,964
Wages-Business	144	8	20	85	272	416	1,183	3,552
Age	47	25	29	47	67	72	83	90

Notes: Table 6 shows the compensation and age distribution for managers in Execucomp (Panel A) and SCF (Panel B). Compensation in thousands of dollars.

Table 7: Estimates of Human Capital Parameters

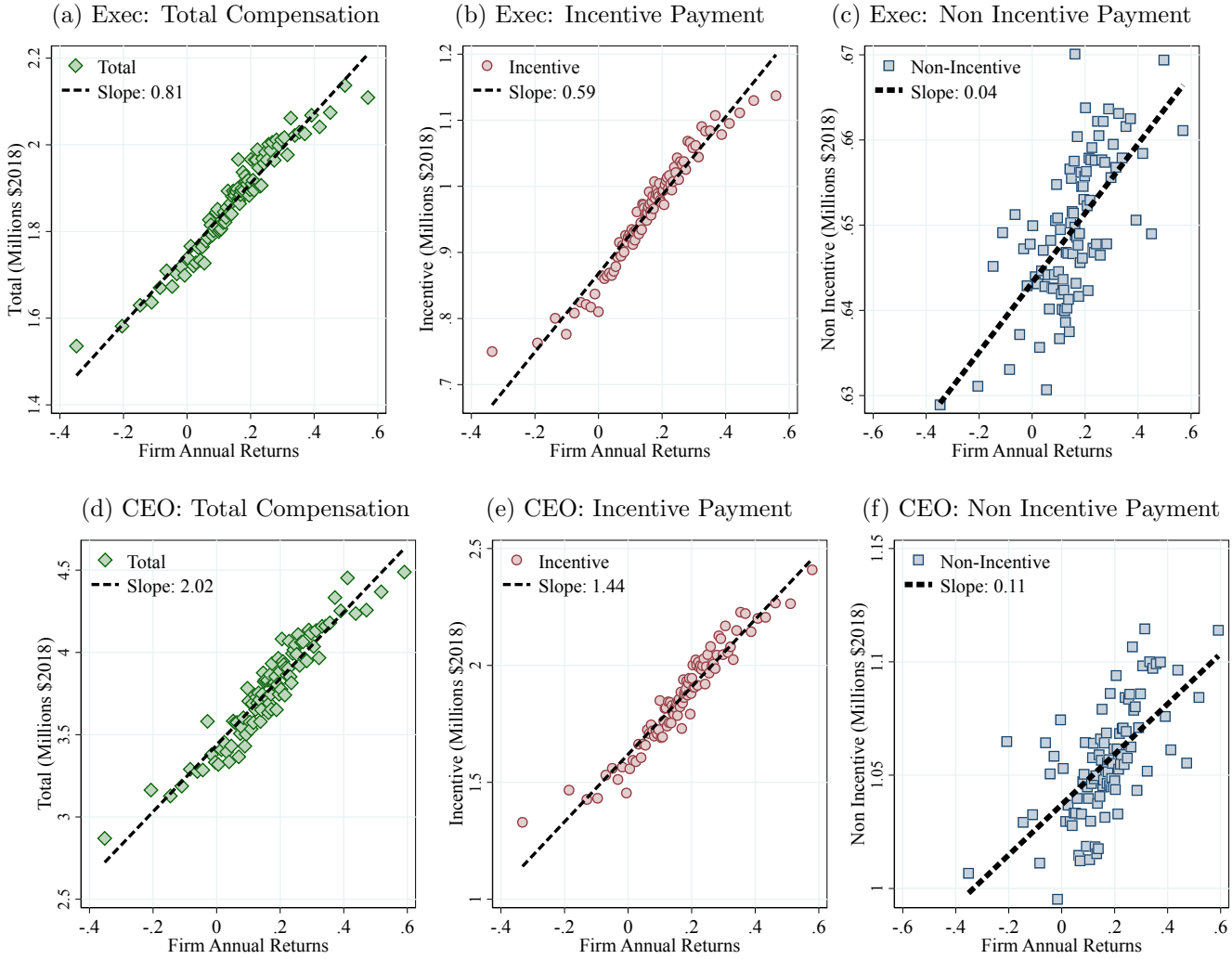
	(1)	(2)	(3)
γ_z	0.088***	0.119***	0.107***
	(0.018)	(0.036)	(0.034)
γ_h	0.934***	0.993***	0.942***
	(0.041)	(0.049)	(0.051)
Tenure		Y	Y
Age			Y
SIC			Y
R	0.44	0.51	0.55
N	56,267	9,803	9,790

Standard errors in parentheses, clustered at the firm-executive level
* $p < 0.1$, ** $p < 0.05$,*** $p < 0.01$

Table 8: Effects of different wealth tax regimes

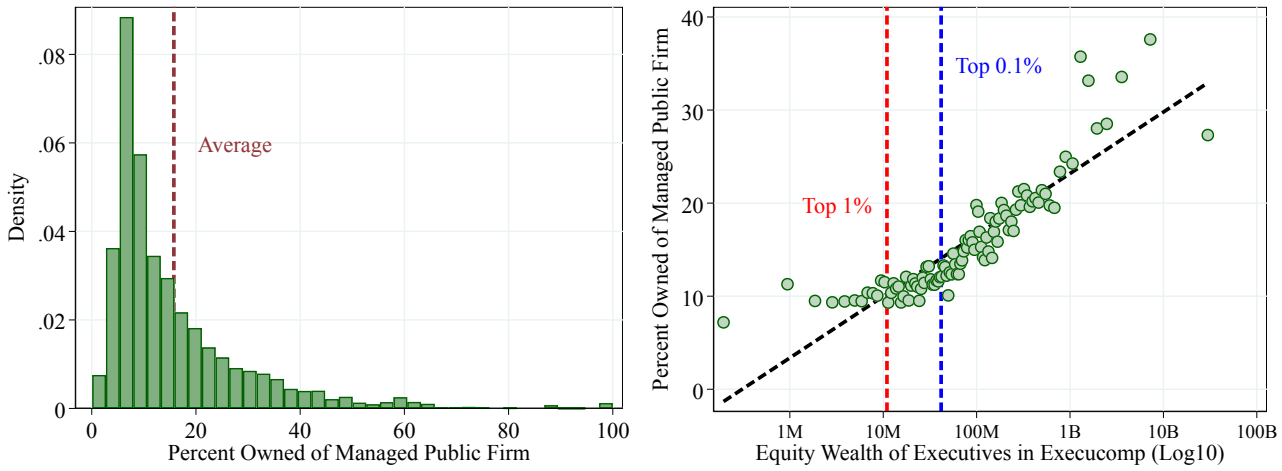
	Baseline	Wealth Tax		
Threshold (Percentile of Baseline)		Linear	60th	90th	99th
Top marginal tax, τ (%)		0.08	0.09	0.16	0.44
Δ GDP p.c. (%)		-1.39	-1.49	-1.56	-0.51
Δ GDP/Δ Tax revenue		-3.48	-3.73	-3.90	-1.28
Wealth share bottom 50%	1.13	1.16	1.19	1.23	1.16
Wealth share top 10%	74.2	74.0	73.9	72.7	73.3
Wealth share top 1%	31.0	30.7	30.4	28.6	29.1
Wealth share top 0.1%	12.1	11.8	11.6	10.1	10.1

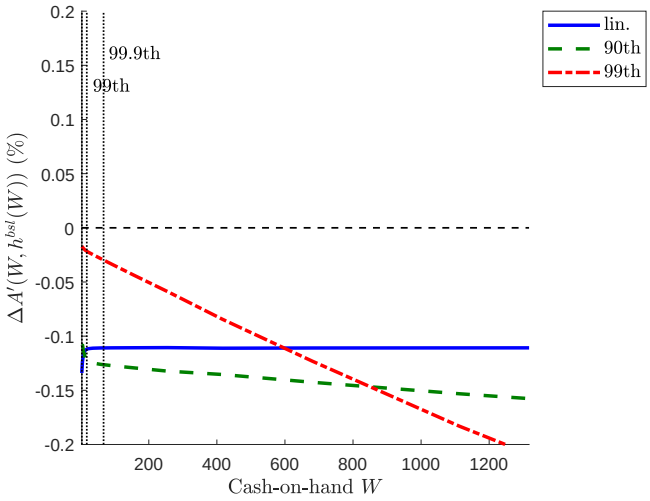
Figure 2: Firm Returns and Executives Compensation



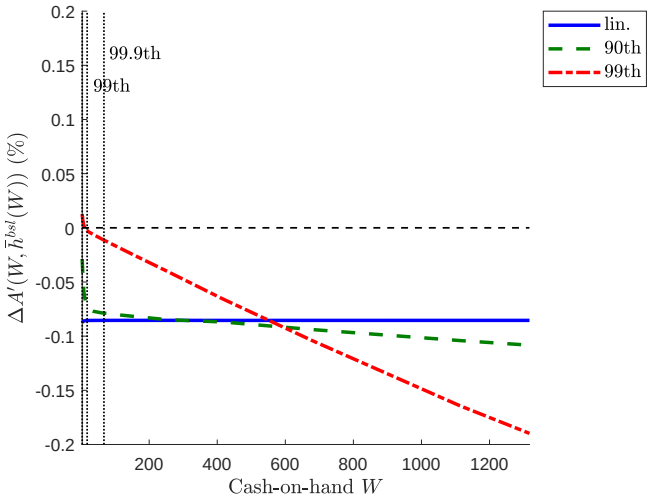
Notes: Figure 2 shows binscatter plots of firm returns and different measures of executive compensation. Total compensation in the sum of incentive and non-incentive compensation. Incentive-based compensation is the sum of bonuses, the value of stock awards, the value of option awards, and other non-equity incentive payments. Non-incentive-based compensation is the sum of salaries, pension contributions, and other compensation. Firm returns are calculated as the average annual return (including dividends) of the firm where the executive works between periods t and $t-3$ (four years average). Binscatters calculated controlling for year-fixed effect, tenure-fixed effects, firm size (employment), age fixed effects, and executive fixed effects.

Figure 3: Ownership Distribution and Wealth



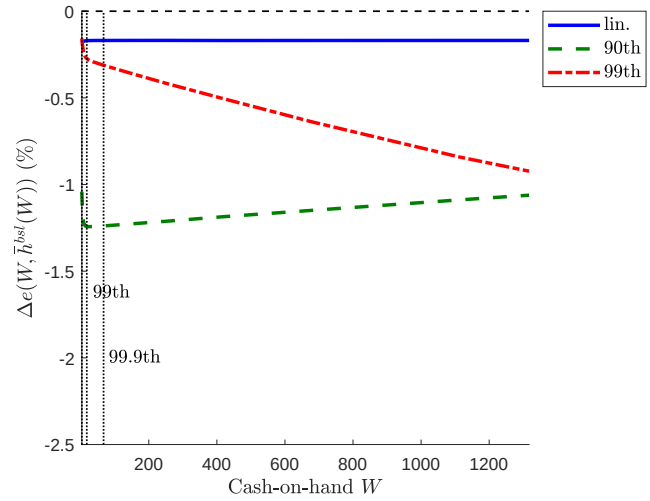


(a) Partial equilibrium: hold (R_s, w_n) fixed at baseline values

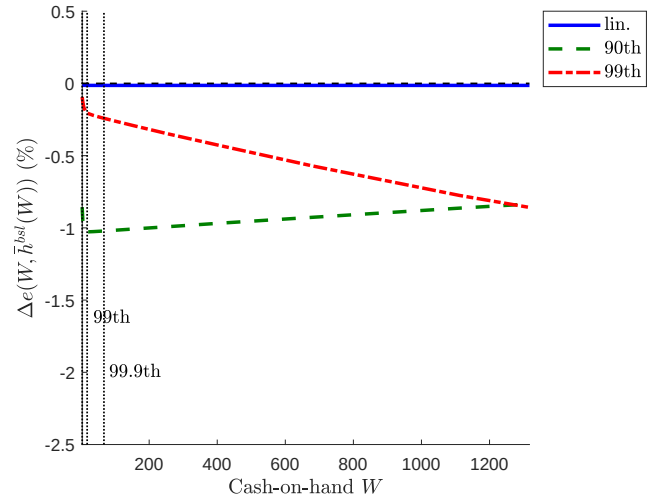


(b) General equilibrium: prices (R_s, w_n) clear markets after taxes

Figure 4: Percentage change in managerial savings relative to the no-tax baseline under wealth taxes. The x-axis covers 90th to 99.999th percentiles of the distribution of W; the vertical lines indicate the values of W associated with 90th and 99th percentiles of A'.



(a) Partial equilibrium: hold (R_s, w_n) fixed at baseline values



(b) General equilibrium: prices (R_s, w_n) clear markets after taxes

Figure 5: Percentage change in managerial effort choice relative to the no-tax baseline under wealth taxes. The x-axis covers 90th to 99.999th percentiles of the distribution of W; the vertical lines indicate the values of W associated with 90th and 99th percentiles of A'.

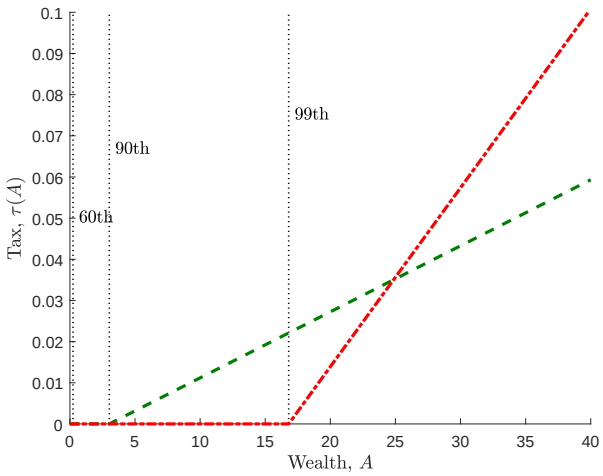


Figure 6: Wealth Tax at 90th and 99th Wealth Percentiles

Table 9: Effects of different consumption, wealth, and income tax regimes

	Consumption Tax				Wealth Tax				Income Tax			
Threshold (Pctl of Bsl Distr.)	lin.	60th	90th	99th	lin.	60th	90th	99th	lin.	60th	90th	99th
Risk-free rate, $R_s - 1$ (%)	1.44	1.44	1.44	1.44	1.52	1.52	1.51	1.46	1.45	1.47	1.48	1.47
Workers' wage rate, w_n	0.442	0.442	0.442	0.442	0.436	0.436	0.436	0.440	0.439	0.436	0.437	0.437
Top marginal tax, τ (%)	0.87	1.81	3.84	11.1	0.08	0.09	0.16	0.44	0.67	1.20	2.06	4.78
Δ GDP p.c. (%)	0.00	-0.23	-0.30	-0.26	-1.39	-1.49	-1.56	-0.51	-0.71	-1.20	-1.25	-1.27
Δ capital (%)	0.00	-0.24	-0.33	-0.27	-2.23	-2.35	-2.33	-0.72	-0.82	-1.51	-1.66	-1.54
Δ workers' labor (%)	0.00	-0.18	-0.23	-0.22	-0.10	-0.13	-0.11	-0.09	-0.05	0.17	-0.04	-0.09
Δ managers' labor (%)	0.00	-0.29	-0.37	-0.30	-1.35	-1.51	-1.91	-0.68	-1.42	-2.42	-2.01	-2.31
Aggr. wealth/income	8.36	8.36	8.35	8.36	8.24	8.24	8.25	8.33	8.34	8.31	8.30	8.32
Δ GDP/Δ Tax revenue	0.00	-0.58	-0.75	-0.65	-3.48	-3.73	-3.90	-1.28	-1.78	-3.00	-3.13	-3.18
Wealth share bot. 50%	1.13	1.14	1.13	1.13	1.16	1.19	1.23	1.16	1.15	1.17	1.20	1.21
Wealth share top 10%	74.2	74.2	74.1	74.1	74.0	73.9	72.7	73.3	74.0	73.5	72.5	71.2
Wealth share top 1%	31.0	31.0	30.8	30.9	30.7	30.4	28.6	29.1	30.7	30.0	28.0	26.0
Wealth share top 0.1%	12.1	12.1	12.0	12.0	11.8	11.6	10.1	10.2	11.9	11.3	9.62	8.61
Income share bot. 50%	5.26	5.27	5.27	5.27	5.29	5.34	5.37	5.28	5.27	5.27	5.31	5.28
Income share top 10%	55.1	55.0	55.0	55.0	54.8	54.7	53.9	54.7	54.9	54.6	54.0	54.5
Income share top 1%	22.9	22.9	22.8	22.9	22.6	22.5	21.7	22.2	22.7	22.3	21.5	23.1
Income share top 0.1%	8.64	8.62	8.59	8.62	8.46	8.34	7.57	7.51	8.51	8.23	7.66	10.2
Earnings share bot. 50%	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Earnings share top 10%	62.3	62.3	62.2	62.2	62.0	62.0	61.7	62.1	62.1	61.8	61.8	62.1
Earnings share top 1%	24.5	24.5	24.5	24.5	24.3	24.1	23.0	22.9	24.4	23.9	22.8	25.6
Earnings share top 0.1%	7.31	7.29	7.27	7.28	7.16	7.03	5.30	6.02	7.17	7.00	6.57	10.3

A Data Appendix

A.1 Additional Tables and Figures

Table A.1: Within Groups Concentration

	Wealth			Income			Labor		
	Population	Manager	Worker	Population	Manager	Worker	Population	Manager	Worker
<50	1.5	0.3	3.1	22.6	7.8	34.4	29.4	10.9	38.0
Top 10	76.5	89.0	59.3	43.1	67.5	23.6	29.7	55.8	17.5
Top 5	64.9	80.9	43.0	34.2	56.2	16.6	20.2	41.2	10.5
Top 1	37.3	50.4	19.3	16.6	28.4	7.1	6.4	12.0	3.8
Top 0.5	28.1	40.0	11.7	12.3	21.7	4.8	3.8	7.9	1.9
Top 0.1	14.2	21.5	4.3	5.2	10.3	1.2	0.7	1.9	0.1

Table A.2: Wealth-to-Income Ratio

	Population	Managers	Workers
Bottom 50	0.57	0.97	0.40
Top 10	15.34	15.43	21.85
Top 5	16.44	16.84	24.17
Top 1	19.47	18.91	23.07
Top 0.5	19.81	20.28	24.10
Top 0.1	23.46	26.83	26.98
Total	8.65	8.98	7.98

Table A.3: Robustness of Z's

Sample	**z1**	**z2**	**z3**	**ntl1**	**ntl2**	**ntl3**
Baseline	0.021	1	16382.110	0.495	0.496	0.009
CEO Only	0.022	1	16635.660	0.497	0.493	0.010
Before 2007	0.016	1	20878.760	0.496	0.494	0.010
Drop Great Recession	0.022	1	16194.620	0.495	0.496	0.009

Figure 7: Share of Managers over Time in SCF

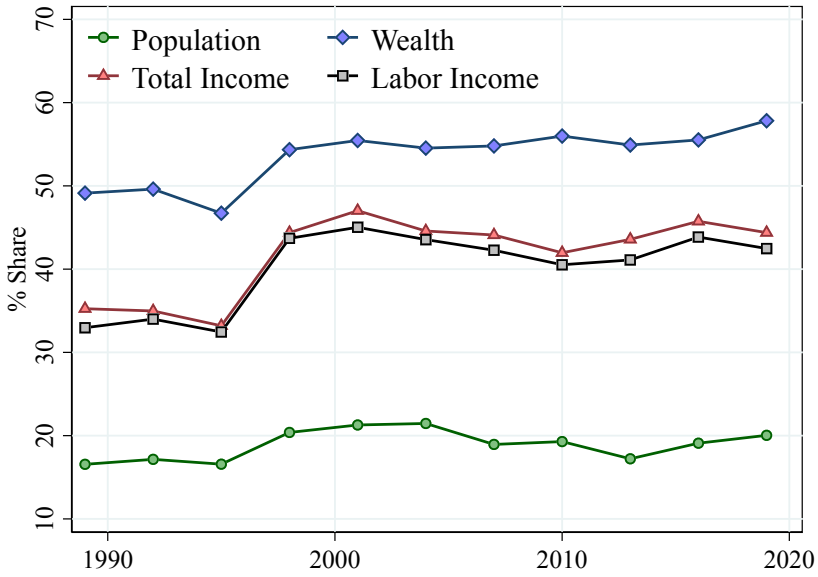
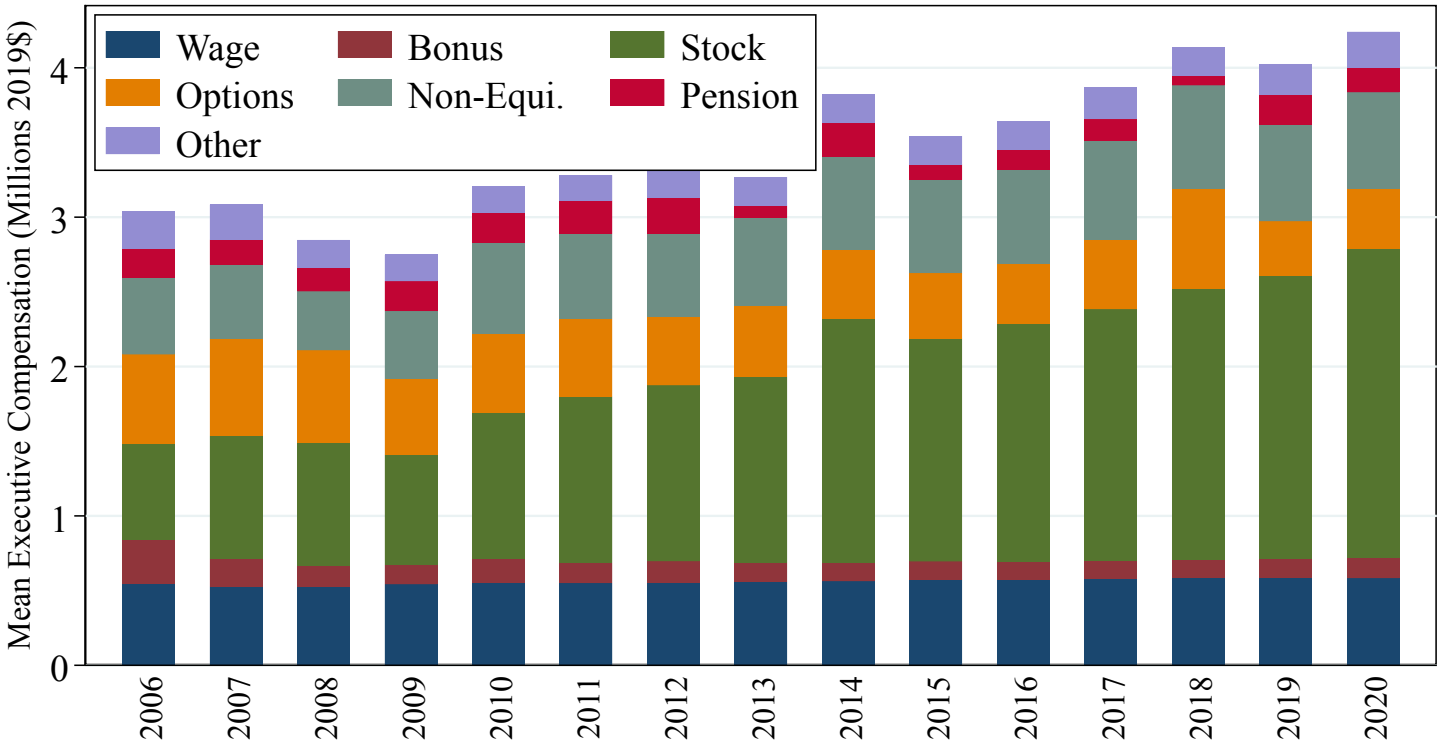
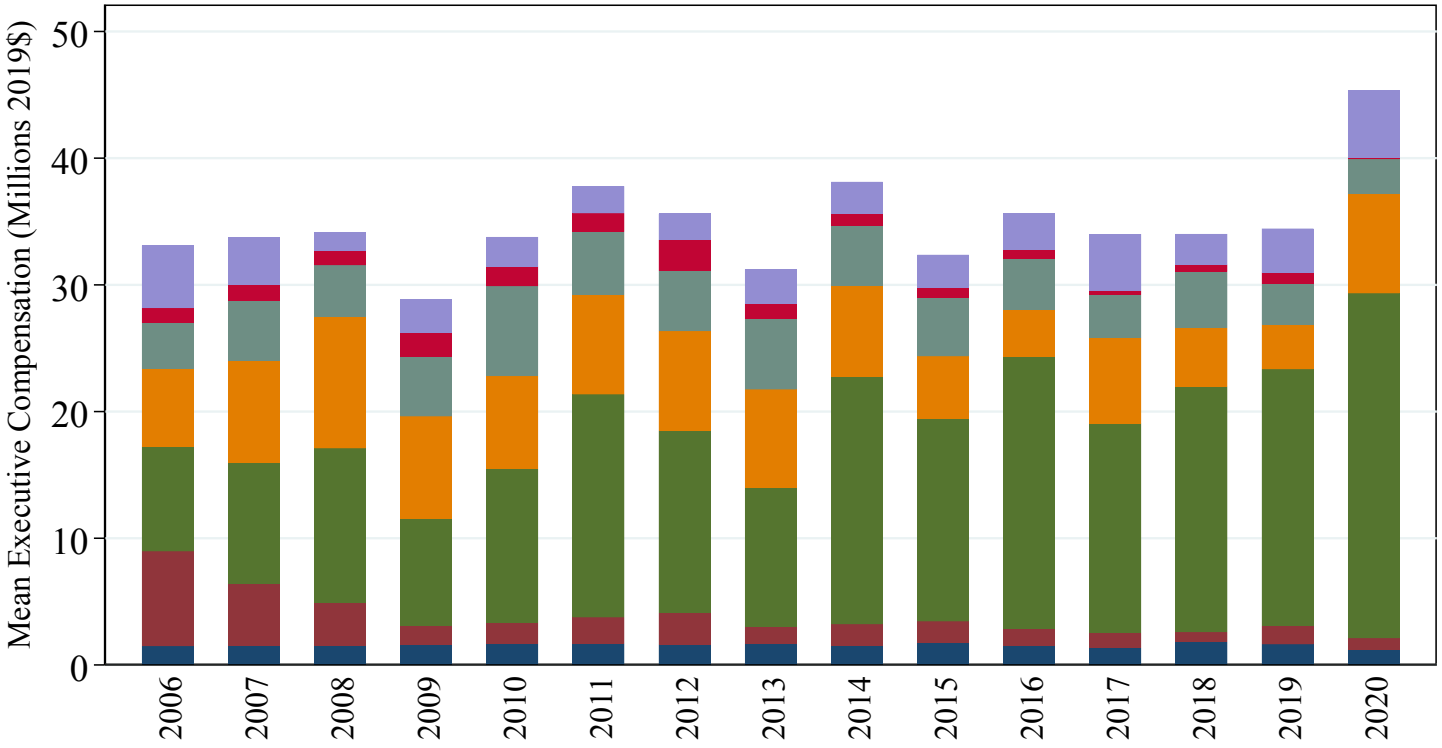


Figure 8: Distribution of Executive Compensation



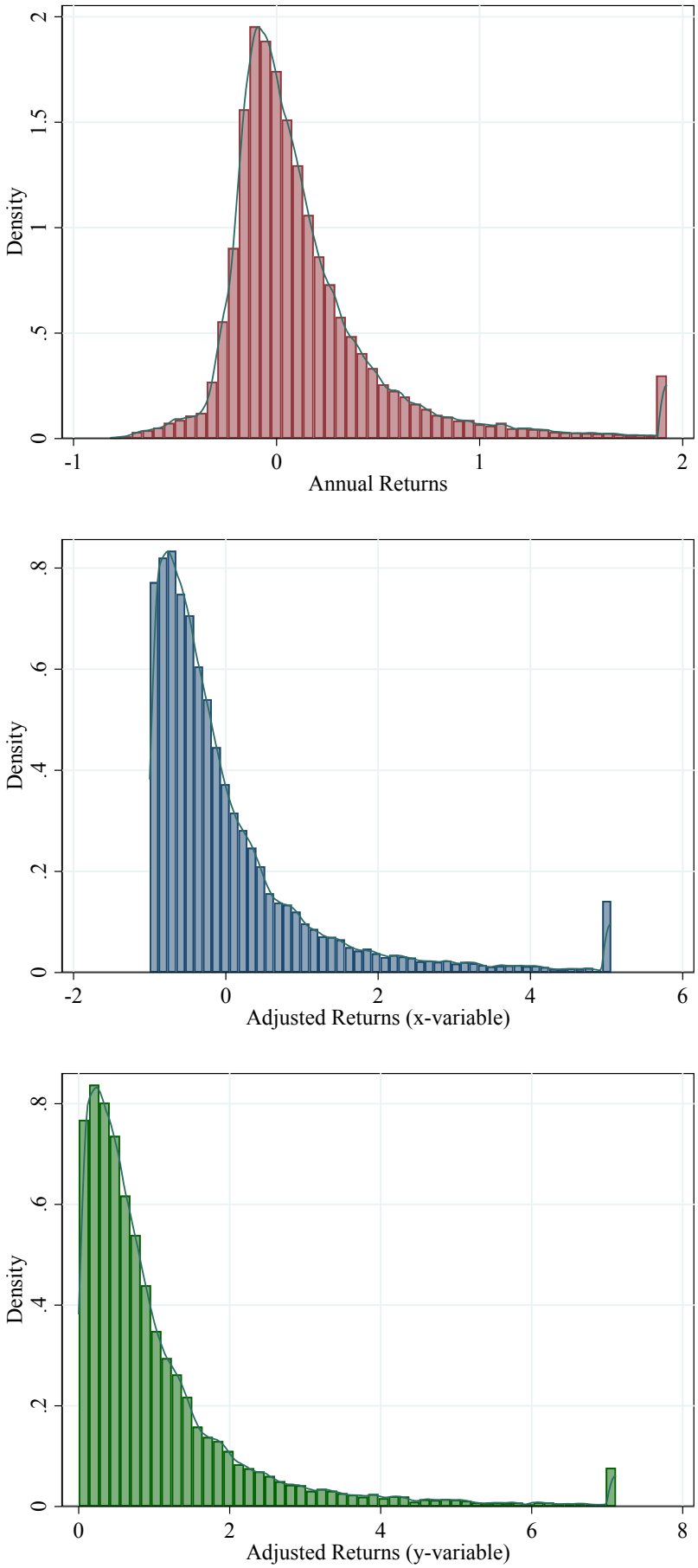
(a) All Executives



(b) Executives at Top 1% of Compensation Distribution

Note: Figure 8 shows the average compensation of executives and the distribution of different forms of payments each year. The top panel shows the average for all executives in the sample; the bottom panel shows the average for managers at the top 1% of the total compensation distribution. The total number of observations is 160,940.

Figure 9: Distribution of Returns in our data



A.2 Data Description

We describe here the data we use and the construction of the main variables.

A.2.1 Survey of Consumer Finances (SCF)

The Survey of Consumer Finances (SCF) is a triennial cross-sectional survey conducted by the Federal Reserve Board in partnership with the Department of the Treasury. It provides detailed information about U.S. household finances, including wealth, assets, and liabilities. It collects in-depth data on income, net worth, asset ownership (e.g., real estate, stocks, and retirement accounts), debt (e.g., mortgages, student loans, and credit card debt), and savings.

The SCF uses a dual-frame sampling method. The first sample is a national probability sample representative of the general U.S. population, while the second is an oversample of high-wealth households. This dual approach captures a more accurate picture of wealth distribution and the financial behaviors of both typical and high-net-worth households.

We use SCF to construct measures of net worth, total income, labor income, assets portfolio shares, and so on. To build these variables, we follow the data construction of the public dataset provided by the Federal Reserve Board. The variables used to define who is a manager, and size of the firm they manager (sales and employment) are crucial for our analysis and hence we describe them in more detail. Recall that we define as managers those households (i) whose head is in a managerial or professional-specialty occupation and is paid through stock grants or options; (ii) or if any member of the household has an active management role on a private business owned by the household; (iii) or if the head of the household is self-employed, excluding consultants or a partner in a law firm.

Starting with (i) questions X7401/X7411 ask to the head of the household "What is the official title of your (husband/wife/partner/spouse)'s job? (The title that (your/his/her/his or her) employer uses?).[13] Although the answer of this question is quite detailed—at the four-digit occupational codes—the public version of the SCF provides an aggregated version of this question, and puts managerial occupations (e.g. Chief executives, Public relations managers) together with professional occupations (e.g. Astronomers and physicists, Economists) in one category. This is too broad of a definition of manager. Hence, to better identify those managers that match our model notion, we identify those who are paid through stock or options. For that, we use the question X6797/X6798 "Some employers give their employees financial options that can be used to purchase company stock at a later time. During the past year, has your (husband/wife/partner/spouse)'s current employer given (you/him/her/him or her) any of these, either as a regular part of (your/his/her/his or her) compensation or as a bonus?".

To this set of households, we add (ii) Active Business Owners as per question X3104 which asks

[13] We report two different questions, the first for the head of the households and the second for the spouse.

"Do you (or anyone in your family living here) have an active management role in any of these businesses?" If the answer is yes, then we consider the household an Active Business Owner.

Finally, (iii) we consider as managers households whose head is self employed excluding those who are consultant or are in a partnership. In practice, we consider households categorized as 2 "Self-employed; other closely held business owned by PEU" in questions X4106/X4706.

Table **??** shows population, wealth, and income shares. Since there is a significant degree of overlap between these definitions (many active business owners are managers and are also self-employed), the sum of the shares is larger than the share of the managers in our sample (20% in Table **??**).

Table A.4: Distribution of Managers under Different Definitions

	Manager	Worker	Occupation	Active Buss. Owner	Self-Employed
Population	20.041	79.959	3.209	12.246	14.315
Top 1% W	76.442	23.558	11.56	61.978	55.818
Top 0.1% W	69.451	30.549	5.168	63.609	58.299

A.2.2 Execucomp

Execucomp defines total compensation as

$$\begin{aligned} \text{1992 format:} \quad TDC_1 &= Salary + Bonus + OtherAnnual \\ &+ TotalValueRestrictedStockgranted \\ &+ TotalValueStockOptionsgranted \\ &+ LongTermIncentivePayouts + AllOtherTotal \end{aligned}$$

$$\begin{aligned} \text{2006 format:} \quad TDC_1 &= Salary + Bonus \\ &+ NonEquityIncentivePlanCompensation \\ &+ GrantDateFairValueofStockAwards \\ &+ GrantDateFairValueofOptionAwards \\ &+ DeferredCompensation \\ &+ OtherComp. \end{aligned}$$

Execucomp claims that this preserves continuity in their measure of total compensation.

A.2.3 Ownership in Public Firms

In this Appendix, we describe the construction of the sample used to calculate results on ownership of publicly traded firms by managers. We combine three datasets. The first one comes from the official filing report of the Security and Exchanges Commission from where we obtain information on

ownership information of publicly traded firms. The second is balance sheet and equity price data for publicly traded firms from Compustat/CRSP. The last one comes from Execucomp and contains information on managers of publicly traded firms. We describe each dataset in detail below.

SEC files. We draw information from the SEC filings from forms 13D-A and 13G-A which we obtained from the WRDS website. File 13D-A (A is for addendum) is filed by individuals or entities (e.g. firms or mutual funds) that acquire more than 5% of a company's equity securities with the intent to influence or control management. A 13D-A must be filed within 10 days of crossing the 5% ownership threshold. It includes information such as the identity of the investor, the number and percentage of shares owned, the source of funds used to make the purchase, and the purpose behind the acquisition (e.g., whether the investor intends to push for changes in the company's strategy or management). The form 13G-A is a simplified version of the 13D, and it is used by investors who acquire more than 5% of a company's stock but do so with passive intent, meaning they have no intention of influencing or controlling management. This is typically filed by institutional investors (e.g., mutual funds, pension funds) who qualify as passive investors. Individuals and firms must also file with the SEC when their ownership goes below 5%. There is no information on ownership an individual or firm controls less than 5% of the share of a company. From these files, we obtain information on the number of shares owned, the percent of the firm owned, filer name and Central Index Key (CIK), firm name and the corresponding CIK. Filers can be individuals and/or firms and several filers can be part of one document although this is not very common. From this dataset, we keep only the last filing per person/year/firm with a non-missing value for the percent of shares owned.

The next step is to merge the file information to firm-level information from Compustat. For that, WRDS provides a gvkey-CIK mapping, with gvkey being a unique identifier in Compustat. This is a many-to-many matching, that is, there are multiple possible gvkey/cik pairs, mostly based on date (e.g. if a firm is bought by another entity) and individuals (e.g., several individuals owning the same firm). Keys are matched to filing dates for best gvkey/cik pair per filing. About 100 gvkey/cik pairs out of about 58,000 are not temporally separated; for these, the pair sourced from the Compustat data table is selected. After this selection, we have one row per filer/firm/year for each firm that a filer reports in any year. The extra rows are filed with NaNs.

We then merge this information to Compustat which we join using the gvkey. Some firms have multiple entries in a given year (same gvkey per year) with different or missing variables, so the values provided for total assets, market valuation, and price are averaged and spread throughout all entries for that year. Duplicate firm/year entries are then dropped, keeping the entry with the least NA values. The resulting table is merged with the expanded SEC data table by gvkey-year pairs. At this point, for each file, we have information on the percentage of ownership, the name of the firm, and market valuation of the corresponding firm for each year.

We compute the equity wealth of an individual as percent share owned times the market valuation if available, otherwise we compute equity wealth as the percent times total firm assets. Wealth for years without an explicit filing is imputed by forward-filling the percent owned through NaN values. That is, we assume that an individual has not filled, it is because his/her ownership has not changed in a given year. Finally, since an individual can own wealth in several firms, we sum across all firms within a year-individual.

Filers who are people (as opposed to firms) are filtered for by taking filers who either i) do not no report an industry code (SIC) or ii) or do not have in their registered name terms such as "LLC," "Associates," or "Treasury" that indicate that the filer is not an individual.

We then merge the data from Execucomp. Importantly, Execucomp identifies the firm that a manager works by its gvkey, which we use to merge to our dataset. If this information is not available, a token set fuzzy match using the filer and executive names is used to create an EXECID-CIK key table. All occurrences of an EXECID associated with a filer CIK are kept, but specific salary information is matched to the firm and year reported. This data is "executives." (See below for more details).

Price and wealth are CPI-adjusted to 1994 dollars. Returns are calculated as the percent change in a firm's stock price between years, and merged to the filings data on firm and year

Execucomp Merge. In this section, we describe how we merge the data from SEC to Execucomp. The main complication of this matching is that there is no direct connection between the SEC filing and the executives in Execucomp. Hence, we use the connection between firms, which we complement with a string search that directly looks at the names of the managers. We proceed as follows.

- Merge SEC data with Execucomp on gvkey and year. This creates a row for every filing/executive pair for executives associated with the subject company in that year.
- Compare filer names and executive names using a token set match. This is a fuzzy match that ignores word order (so [firstname, lastname] matches with [lastname, firstname]) and searches for the shortest substring within the tokens ([firstname, middle, lastname] matches [first initial, lastname]).
- Group by filer CIK and select the best name match (regardless of year and gvkey. Possible pairings are already restricted). Execucomp name-EXECID pairs are 1:1 so matching the name matches the EXECID. Keep only CIK-EXECID pairs with the highest score so long as that score is above 75.
- Drop duplicates. Most CIK-EXECID pairs are already 1:1, but some have multiple possibilities that cannot be distinguished based on name (e.g. Robert Parker Jr. and Robert Parker Sr. both match with Robert Parker). Multiple CIK are allowed to map to the same EXECID but each CIK is restricted to one EXECID.

- The CIK-EXECID key is inner-merged to the SEC filings data so that only filings from a CIK associated with an EXECID are kept. The full Execucomp data is then merged in with a left merge to associate salary/etc. data with the correct EXECID/gvkey/year row. This keeps all filings from people who appear in Execucomp in any year, even if they are not represented in the year that the file with the SEC (e.g. after retirement).

A.3 Sample Statistics and Results

The results of this matching process are shown in this section. Figure 10 shows that the vast majority of filers own only one firm in each period. The middle and left panels show the total number of unique filers and the total number of filers in a given year.

Figure 11 shows that the average returns is more or less the same for all filers in the data (including firms and individuals) for all individuals, and for those who we can identify as managers through Execucomp. The right panel, show the dispersion of these returns, which is lower for managers.

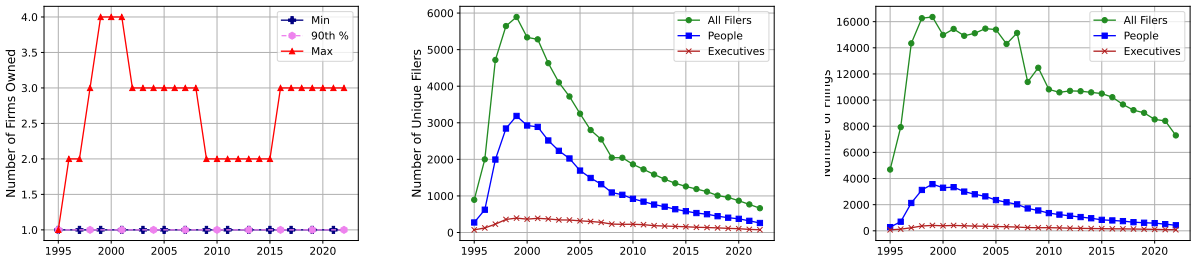


Figure 10: Number of unique filers (left) and filings (right) per year.

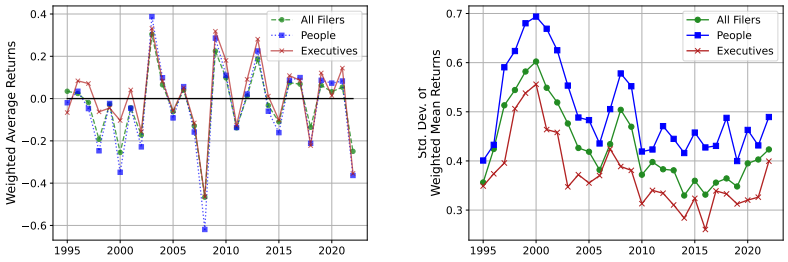


Figure 11: Mean and Standard Deviation of Returns, weighted by wealth.

B Model Appendix

B.1 Homogeneity in Simple Model with General Human Capital

Consider the more general human capital function

$$h' = G(h, n_h + \alpha_h e_m) = \Gamma_h(n_h + \alpha_h e_m),$$

with $n_h + n_p = n$ so that

$$R^G(k, w; z) \equiv \max_n \; k^\alpha(\Gamma n_p)^\gamma(z\Gamma_h(n_h + \alpha_h e_m)))^{1-\alpha-\gamma} - w_n(n_p + n_h) - w + (1-\delta)k.$$

The following result holds.

Lemma 1. *R^G is homogeneous of degree 1 in (k, w, e_m) for each z.*

Proof: Note that

$$R^G(\lambda k, \lambda w) \equiv \max_{n_p, n_h} \; \lambda k^\alpha(\Gamma \lambda n_p)^\gamma(z\Gamma_h\lambda(n_h + \alpha_h e_m)))^{1-\alpha-\gamma} - w_n\lambda(n_p + n_h) - \lambda w + (1-\delta)\lambda k \quad (33)$$

For the scaled economy with λ let $n_p(1)$, $n_h(1)$, and $e_m(1)$ be the choice with $\lambda = 1$. Clearly, if we set $n_p(\lambda) = \lambda n_p(1)$, $n_h(\lambda) = \lambda n_n(1)$, and $e_m(\lambda) = \lambda e_m(1)$, then this choice will produce $R^G(\lambda) = \lambda R^G(1)$. So if these choices for $\lambda \neq 1$ are not profit maximizing, take the choice for the λ economy and scale those for the $\lambda = 1$ economy, then this implies the original choices for the $\lambda = 1$ economy are not profit maximizing. □

Now, a contract is $x = (c, n_h, n_p, A', w', \phi, k', e, e_m, W'(z'))$, with

$$h' = \Gamma_h(n_h + \alpha_h e_m) \quad \text{and} \quad k^\alpha(\Gamma_n n_p)^\gamma(z\Gamma_h(n_h + \alpha_h e_m)))^{1-\alpha-\gamma}.$$

We can prove the following result.

Proposition 2. *Under these assumptions $V(W) = E + F\log W$. Effort $e = \bar{e}$, $e_m = 0$, and the loading factor $\phi = \bar{\phi}$ are constant, $c' = \bar{c}W$, $k' = \bar{k}W, n_h = \bar{n}_h W, n_p = \bar{n}_p W, w' = \bar{w}W, A' = \bar{A}W$ and*

$$W'(W, z') = \tilde{R}(z)W \quad (34)$$

where $\tilde{R}(z) \equiv \left\{\bar{w}\bar{h} + \left[\phi\frac{R^G(z)}{k'} + (1-\phi)R_s\right]\bar{A}\right\}$ and furthermore, the $E(\tilde{R}(z)) = 1$.

This proposition implies that wealth follows a geometric random walk. Specifically, taking logs of (34) gives

$$\log W' = \log W + \log R(z).$$

Hence, starting from initial distribution of wealth, over time the cross section distribution of wealth increases without bound. If we introduce a birth and death process of the type considered above the distribution can be made stationary.

Proof: Under the utility function (14), the contracting problem is

$$V(W) = \max_{x} \ \log c + v(e + e_m) + \beta \int_z V\left(W'\right) f(z'|e) dz' \tag{35}$$

$$c + A' \leq W, \tag{36}$$

$$W' = w' + \phi \left[\frac{R^G(k', w'; z)}{k'} - R_s \right] k' + R_s A' \tag{37}$$

$$\int R^G(k', w'; z) f(z'|e) dz = R_s k' \tag{38}$$

$$(n_p, n_h) \in \arg\max_{n_p, n_h} \ k^\alpha (\Gamma n_p)^\gamma (z \Gamma_h (n_h + \alpha_h e_m)))^{1-\alpha-\gamma} - w_n (n_p + n_h) - w + (1-\delta)k \tag{39}$$

$$(e, e_m) \in \arg\max_{e, e_m} \log c + v(e + e_m) + \beta \int V\left(W'(W, z)\right) f(z|e) dz \tag{40}$$

We begin by guessing that the continuation value function has the form $V(W) = E + F \log W$. Under this guess the constraint set of the contracting problem is homogeneous of degree 1 in the vector of choice variables excluding effort e and the loading factor ϕ together with the state W. That is, if a contract $x = (c, n_h, n_p, A', w', \phi, k', e, e_m, W'(z'))$ is in the constraint set for a particular value of the state W, then for any constant $\lambda > 0$ and an associated state λW, the contract

$$\hat{x} = (\lambda c, \lambda n_h, \lambda n_p, \lambda A', \lambda w', \phi, \lambda k', e, \lambda e_m, \lambda W'(z')$$

is also in the constraint set for that λW.

Homogeneity for (36) because these constraints are linear in the controls. For (37) and (38), note that $R^G(k, w; zh)k$ is defined by (33). Since F is constant returns to scale,

$$\max_n F(z\lambda h, \lambda k, \lambda n) - w_n \lambda n - \lambda w + (1-\delta)\lambda k = \lambda \max_n F(zh, k, n) - w_n n - w + (1-\delta)k$$

and hence, $R(z\lambda h, \lambda k, \lambda w)\lambda k = \lambda R(zh, k, w)k$. This observation together the linearity in other control variables that enter (37) and (38) establishes the relevant homogeneity. Since F is constant returns to scale and the control variables are homogeneous, (39) is also homogeneous. Finally, for (40), imposing the guess gives

$$(e, 0) \in \arg\max_{e, e_m} \ \log\, c + v(e + e_m) + \beta \int \left(E + F \log W'(z)\right) f(z|e) dz. \tag{41}$$

Our homogeneity property requires that if a given $(e, 0)$ solves (41) for some c and $W'(z)$, then the same $(e, 0)$ solves the analog of (41) when we replace c and $W'(z)$ with λc and $\lambda W'(z)$ since

$$(e, 0) \in \arg\max_{e, e_m} \log \lambda + \log\ c + v(e + e_m) + \beta F \log \lambda + \beta \int \left(E + F \log W'(z)\right) f(z|e) dz. \tag{42}$$

Hence, the constraint set is homogenous of degree 1 in our sense.

We next show that under our guess for the continuation value functions, the optimal choices for

all choice variables besides e and ϕ are linear in wealth. To do so let $\lambda = 1/W$. Using the form of our guess for the continuation value function in the objective function as well and in the incentive constraint,

$$V(W) = \max_{x} \ \log c + v(e + e_m) + \beta \int_z (E + F \log W'(z)) \, f(z'|e) dz' \tag{43}$$

We claim that if $x = (c, i, A', w', \phi, k', e, e_m = 0, W'(z'), h')$ solves the contracting problem for given wealth W, then $\hat{x} = (\lambda c, \lambda i, \lambda A', \lambda w', \phi, \lambda k', e, \lambda e_m = 0, \lambda W'(z'), \lambda h')$ for wealth λW. To prove this claim, note that under our guess the objective function is at λW is

$$\max_{x} \ \log c + \log \lambda + v(e) + \beta F \log \lambda + \beta \int_z (E + F \log W'(z)) \, f(z'|e) dz'. \tag{44}$$

Clearly, the scaled allocations $\hat{x}$ are feasible for the economy with wealth λW. Suppose by way of contradiction, the optimal allocations for the λW economy do not coincide with $\hat{x}$ and instead are $x^* \neq \hat{x}$. But then by unscaling these x^* allocations by dividing them by λ we obtain an optimal allocation for economy with wealth W that yields higher welfare then the supposed optimal allocations x.

We now show that in (34) letting $\tilde{R}(z') \equiv \left\{ \bar{w}\bar{h} + [\phi R(z') + (1 - \phi) R_s] \, \bar{A} \right\}$ and

$$E\left[\tilde{R}(z')\right] = \beta \rho R_s.$$

Hence, if wealth follows $W'(z) = \tilde{R}(z')W$, then for there to a stationary solution for wealth we need $E\left[W(z)\right] = E\left[\tilde{R}(z')W\right] = \beta \rho R_s W = W$, which by the last equality implies that $\beta \rho R_s = 1$. Substituting out the budget constraint, the human capital accumulation law, that $e_m = 0$, and writing the first order condition version of the incentive constraint, we can express the contracting problem as

$$V(W) = \max_{(A', w', \phi, k', e, i)} \ u(W - i - A') + v(e) + \beta \rho \int V(W'(z')) f(z'|e) dz', \tag{45}$$

as desired. □

B.2 Derivation of Estimating Equation

The firm hires labor w_n to solve the static profit maximization problem: where $R(zh, k, w)k$ is returns to capital

$$R(zh, k, w)k = \max_{n} \ k^{\alpha} (\Gamma n)^{\gamma} (zh)^{1-\alpha-\gamma} - w_n n - w + (1 - \delta) k \tag{46}$$

Also bring down the FOCs from the recursive contracting problem:

$$k_i': \quad \frac{V_W(W,h)}{1-\kappa_W(i,h,W)} = \beta\rho \int V_W'(z) R(zh',k',w') \left[1+\varphi_e \frac{f_e(z|e)}{f(z|e)}\right] f(z|e)dz \tag{47}$$

$$b': \quad \frac{V_W(W,h)}{1-\kappa_W(i,h,W)} = \beta\rho \int V_W'(z) R_s \left[1+\varphi_e \frac{f_e(z|e)}{f(z|e)}\right] f(z|e)dz \tag{48}$$

$$w': \quad 0 = \beta\rho \int V_W'(z)[R_w(zh',k',w')k_i'+1] \left[1+\varphi_e \frac{f_e(z|e)}{f(z|e)}\right] f(z|e)dz + \varphi_p \int R_w(zh',k',w')f(z|e)dz \tag{49}$$

$$k': \quad 0 = \beta\rho \int V_W'(z) R_k(zh',k',w')k_i' \left[1+\varphi_e \frac{f_e(z|e)}{f(z|e)}\right] f(z|e)dz + \varphi_p \int R_k(zh',k',w')f(z|e)dz \tag{50}$$

Make FOC for k' equation (50) more succinct by combining the two integrals

$$0 = \int \left(k_i' \beta\rho V_W'(z) \left[1+\varphi_e \frac{f_e(z|e)}{f(z|e)}\right] + \varphi_p \right) R_k(zh',k',w')f(z|e)dz \tag{51}$$

Step 1. The solution to the static profit maximization problem

$$\gamma \frac{\Gamma^\gamma k^\alpha n^\gamma (zh)^{1-\alpha-\gamma}}{n} = w_n \implies \gamma\Gamma^\gamma k^\alpha (zh)^{1-\alpha-\gamma} = w_n n^{1-\gamma}$$

gives the labor demand function $n(zh,k;w_n)$

$$n(zh,k;w_n) \equiv \left(\frac{\gamma\Gamma^\gamma k^\alpha (zh)^{1-\alpha-\gamma}}{w_n} \right)^{\frac{1}{1-\gamma}} \tag{52}$$

Step 2. Substitute the labor demand function into static profits (46) and divide by k to get the rate of return on capital R as function of (zh,k',w')

$$R(zh,k,w) = (1-\gamma) \left(\frac{zh}{k}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \left(\frac{\gamma\Gamma}{w_n}\right)^{\frac{\gamma}{1-\gamma}} - \frac{w}{k} + 1 - \delta \tag{53}$$

The revenue-yield of capital is

$$r(zh,k,w) \equiv R(zh,k,w) - (1-\delta) = (1-\gamma) \left(\frac{zh}{k}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \left(\frac{\gamma\Gamma}{w_n}\right)^{\frac{\gamma}{1-\gamma}} - \frac{w}{k} \tag{54}$$

Step 3. Use zero profit condition

$$R_s = \int R(zh',k',w')f(z|e(W,h))dz = (1-\gamma) \left(\frac{zh'}{k'}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \left(\frac{\gamma\Gamma}{w_n}\right)^{\frac{\gamma}{1-\gamma}} - \frac{w'}{k'} + 1 - \delta \tag{55}$$

Step 4. Take derivatives of $R(zh,k,w)$ in equation (53)

$$R_w(zh',k',w') = -\frac{1}{k'} \tag{56}$$

$$R_k(zh',k',w') = -\frac{1-\alpha-\gamma}{1-\gamma} [R(zh',k',w') - 1 + \delta] \frac{1}{k'} + \frac{\alpha}{1-\gamma} \frac{w'}{(k')^2} \tag{57}$$

$$R_{zh}(zh',k',w') = (1-\alpha-\gamma) \left(\frac{zh'}{k'}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \left(\frac{\gamma\Gamma}{w_n}\right)^{\frac{\gamma}{1-\gamma}} \frac{1}{zh'} \tag{58}$$

Step 5. Substitute the expression for R_w equation (56) into the FOC for w' equation (49) and multiply through by k', to obtain the condition that the marginal benefit of raising the wage (LHS) equals the marginal cost via the participation constraint on investors (RHS)

$$w': \quad (k'-k_i')\beta\rho\int V_W'(z)\left[1+\varphi_e\frac{f_e(z|e)}{f(z|e)}\right]f(z|e)dz=\varphi_p$$

Step 6. Substituting the FOC for b' (48) gives an expression for the Lagrange multiplier on the participation constraint φ_p as a function of outside capital invested $k'-k_i'$

$$(k'-k_i')\frac{V_W(W,h)}{R_s[1-\kappa_W(i,h,W)]}=\varphi_p \tag{59}$$

Step 7. Substitute the expression for R_k (57) into the FOC for k' (51) obtains

$$\begin{aligned}0=&-\frac{1-\alpha-\gamma}{1-\gamma}\int\left(k_i'\beta\rho V_W'(z)\left[1+\varphi_e\frac{f_e(z|e)}{f(z|e)}\right]+\varphi_p\right)R(zh',k',w')f(z|e)dz\\&+\left(\frac{1-\alpha-\gamma}{1-\gamma}(1-\delta)+\frac{\alpha}{1-\gamma}\frac{w'}{k'}\right)\int\left(k_i'\beta\rho V_W'(z)\left[1+\varphi_e\frac{f_e(z|e)}{f(z|e)}\right]+\varphi_p\right)f(z|e)dz\end{aligned}$$

Step 8. Substitute the FOC for k_i' equation (47) and the zero profit condition equation (55) into the first line; substitute the FOC for b' equation (48) into the second line to get

$$\begin{aligned}0=&-\frac{1-\alpha-\gamma}{1-\gamma}\left[k_i'\frac{V_W(W,h)}{R_s[1-\kappa_W(i,h,W)]}+\varphi_p\right]R_s\\&+\left[\frac{1-\alpha-\gamma}{1-\gamma}(1-\delta)+\frac{\alpha}{1-\gamma}\frac{w'}{k'}\right]\left[k_i'\frac{V_W(W,h)}{R_s[1-\kappa_W(i,h,W)]}+\varphi_p\right]\end{aligned}$$

and factorize out the bracketed term:

$$0=\left(-\frac{1-\alpha-\gamma}{1-\gamma}R_s+\frac{1-\alpha-\gamma}{1-\gamma}(1-\delta)+\frac{\alpha}{1-\gamma}\frac{w'}{k'}\right)\left[k_i'\frac{V_W(W,h)}{R_s[1-\kappa_W(i,h,W)]}+\varphi_p\right]$$

Step 9. In the last equation, the product of two terms is zero, with the second necessarily positive. Hence, the first term must be zero. Setting it to zero yields an expression that expresses the manager's wage w' as a share of revenue $r_s\equiv R_s-(1-\delta)$:

$$w'=\frac{1-\alpha-\gamma}{\alpha}[R_s-(1-\delta)]k'=\frac{1-\alpha-\gamma}{\alpha}r_sk'$$

The ratio of manager's wages to capital managed is proportional to outside interest rate r_s:

$$\frac{w'}{k'}=\frac{1-\alpha-\gamma}{\alpha}r_s$$

Step 10. Substitute the w'/k' ratio into revenue yield equation (54) to write revenue yield of capital as

$$r(zh',k',w')=(1-\gamma)\left(\frac{zh'}{k'}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}\left(\frac{\gamma\Gamma}{w_n}\right)^{\frac{\gamma}{1-\gamma}}-\frac{1-\alpha-\gamma}{\alpha}r_s \tag{60}$$

Step 11. Combining the revenue $r(zh',k',w')$ with the intermediary's zero-profit condition from equation

(55), $\int r(zh,k')f(z|e(W,h))dz = r_s$,

$$r_s = \int \left[(1-\gamma) \left(\frac{zh'}{k'} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \left(\frac{\gamma\Gamma}{w_n} \right)^{\frac{\gamma}{1-\gamma}} - \frac{1-\alpha-\gamma}{\alpha} r_s \right] f(z|e(W,h))dz$$

or, rearranging,

$$\frac{1-\gamma}{\alpha} r_s = \int (1-\gamma) \left(\frac{zh'}{k'} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \left(\frac{\gamma\Gamma}{w_n} \right)^{\frac{\gamma}{1-\gamma}} f(z|e(W,h))dz$$

or, multiplying by $\alpha/(1-\gamma)$,

$$r_s = \alpha \int \left(\frac{zh'}{k'} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} \left(\frac{\gamma\Gamma}{w_n} \right)^{\frac{\gamma}{1-\gamma}} f(z|e(W,h))dz$$

Step 12. Rearranging shows that physical-capital-to-human-capital ratio k'/h' depends on (W,h) only through the effort choice

$$k' = \left[\frac{\alpha}{r_s} \left(\frac{\gamma\Gamma}{w_n} \right)^{\frac{\gamma}{1-\gamma}} \int (zh'^{\frac{1-\alpha-\gamma}{1-\gamma}} f(z|e(W,h))dz \right]^{\frac{1-\gamma}{1-\alpha-\gamma}}$$

Step 13. Substitute this expression for physical capital k' back into the revenue yield of capital (60) to get

$$r(zh'(W,h),k'(W,h)) = (1-\gamma) \frac{(zh')^{\frac{1-\alpha-\gamma}{1-\gamma}} (\frac{\gamma\Gamma}{w_n})^{\frac{\gamma}{1-\gamma}}}{\frac{\alpha}{r_s} (\frac{\gamma\Gamma}{w_n})^{\frac{\gamma}{1-\gamma}} \int (zh'^{\frac{1-\alpha-\gamma}{1-\gamma}} f(z|e(W,h))dz} - \frac{1-\alpha-\gamma}{\alpha} r_s \tag{61}$$

and rearrange as

$$r(zh'(W,h),k'(W,h)) = r_s + \frac{1-\gamma}{\alpha} r_s \left(\frac{z^{\frac{1-\alpha-\gamma}{1-\gamma}}}{\frac{\alpha}{r_s} \int z^{\frac{1-\alpha-\gamma}{1-\gamma}} f(z|e(W,h))dz} - 1 \right) \tag{62}$$

This obtains the desired equation.

C Existence and Form of Stationary Distribution

C.1 Existence of a stationary distribution

This will establish the existence of the stationary distribution. Here, if an agent survives, wealth accumulates subject to the return shock, as in the simple model. Upon death, the agent is reborn with new wealth drawn from the normal distribution described next. Thus, we consider a law of motion which in log form is

$$\log W_{t+1} = \begin{cases} \log R_{t+1} + \log W_t & \text{with probability } 1-\rho \\ \log W_0 & \text{with probability } \rho \end{cases}$$

where $\{\log R_t\}_t \sim_{iid} f$, and $\log W_0 \sim N(\log W_e, \sigma_0^2)$.

It is convenient to write this transition law in logs. Let $w = \log W_t$ and $w' = \log W_{t+1}$ and $r = \log R_t$. Let $\phi_N(w)$ denote the density of the normal distribution Define the transition kernel $p(w, w')$ as the probability of moving to future wealth w' from existing wealth w. We need to show that this equation has a unique ergodic distribution when $\mathbb{E}[R_t] \in (0, 1]$, and $\mathbb{P}(R_t > 1) > 0$. Consider an agent with current wealth w. That agent survives with rate ρ. For that agent to transition from w to w' the agent must draw $r = w' - w$ on an agent with current state w surviving, which occurs with probability $f(w' - w)$, where f is the density of $\log R$. If that agent dies, which occurs with rate $1 - \rho$, the probability of drawing w' is $\log W_0 \sim N(\log W_e, \sigma_0^2)$.

Then the transition kernel p is given by

$$\begin{aligned} p(w, w') &= \\ &= (1 - \rho) f(w' - w) + \rho \frac{1}{\sqrt{2\pi}} \exp\left\{ - \left(\frac{w' - w_e}{2\sigma_0^2} \right) \right\} \end{aligned}$$

where f is the density of $\log R$. Henceforth, let M denote the *Markov operator* induced by p. It is clear that $p(w, \cdot)$ is a continuous and hence measurable density. That is, define M such that for any density φ over w,

$$\varphi'(w') = \varphi M(w') = \int_S p(w, w') \varphi(w) dw.$$

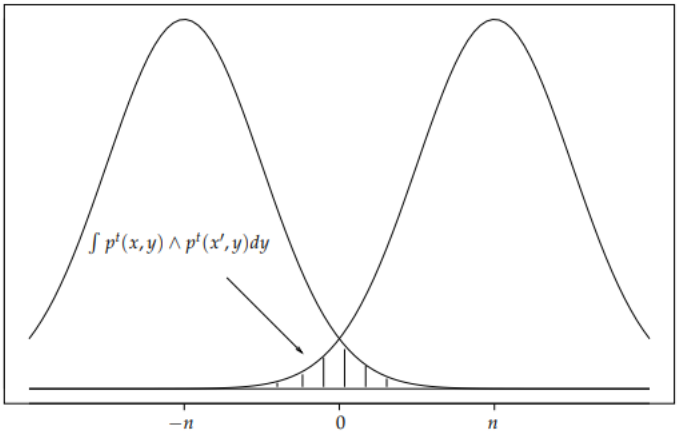
In the discrete case where w and w' lie on a bounded grid, it is classic to show that an attracting stationary distribution exists so long as the Markov chain is positive recurrent. This allows us to easily show that the Markov operator is a *uniform contraction* on the space of distributions under the L_1 norm. This is the classic stuff done in Stokey and Lucas Jr [1989], and even though the proof is not easy, an economic audience should be familiar with the idea.

However, in the continuous/mixed state space case, M may admit an attracting stationary distribution even if it is not a uniform contraction. We will use ideas from Stachurski [2009] to see how this is the case.

With the proper defintion of equivalence classes, we note that our kernel $p(x, \cdot)$ is an element of a closed subset of $L_1(\mathbb{R})$ as long as we assume that f is a measurable density. This is because $p(x, \cdot)$ is trivially integrable (integrates to one) as long as it is also measurable. Thus $p(x, \cdot)$ is an element of the closed subset of $L_1(\mathbb{R})$ of functions that integrate to one, and the set of $\{p : x \in \mathbb{R}_+\}$ is itself a proper closed subset of $L_1(\mathbb{R})$. Then, using the fact that $L_1(\mathbb{R})$ is a complete metric space we know that $\{p : x \in \mathbb{R}_+\}$ is a closed subset of a complete metric space. This will allow us to apply some theorems that will yield a stationary distribution for M.

We will prove the following theorem, and show that it applies to p and M in $D(\mathbb{R})$ the space of probability distributions on $\mathbb{R}_+$.

Theorem 8.2.10 of Stachurski [2009]. *For a transition kernel p and Markov operator M, the following three conditions imply that the dynamical system $(D(S), M)$ is globally stable. That is, it*



has an ergodic distribution.

1) There exists $t \in \mathbb{N}$ such that

$$\int \min\{p^t(x,y), p^t(x',y)\}dy > 0$$

for all $(x, x') \in S \times S$.

2) The kernel p satisfies geometric drift to the centre. i.e

$$\int q(y)p(x,y)dy \leq \alpha q(x) + \beta$$

for all x and for some q function that has precompact sublevel sets and for $\alpha < 1$.

3) There exists a continuous $m : S \to \mathbb{R}$ such that $p(x,y) \leq m(y)$ for all x, y.

Condition 1) is the notion of *contractiveness* (equivalently *mixing*).[14] It essentially says that there is a set of points of positive measure that can be reached from any starting point. We don't need it to hold integrandwise because we do not need this to in general hold over the whole space - rather it just needs to hold over some set of positive measure. So we just need the integral > 0.

Consider the following heuristic image. Imagine we are in one of the distributions - full support means we can get to any point from either distribution.

To build some more intuition note that if $\int \min\{p^t(x,y), p^t(x',y)\}dy = 0$, then $p(x', \cdot)$ and $p(x, \cdot)$ have disjoint support. If on the other hand $\int \min\{p^t(x,y), p^t(x',y)\}dy = 1$, then $p(x', \cdot) = p(x, \cdot)$,

Condition 2) gives us what's called *tightness*. Formally, tightness is the condition that:

$$\forall \varepsilon > 0,\ \exists K \subset \mathbb{R} \ \text{ where } K \text{ is compact, such that } \sup_{\psi \in \mathcal{M}} \int_{K^c} \psi(x)dx \leq \varepsilon.$$

It is a lemma that condition 2) is an easy to check definition that is equivalent to tightness.

[14] This is not to be confused with the notion of a *uniform contraction*!

Intuitively, tightness rules out problematic sequences of the form

$$\phi_n = \mathbb{1}_{[n,n+1]}.$$

In our case, tightness holds as we will see.

Condition 3) is equivalent to *uniform integrability.* Uniform integrability is the condition that:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \lambda(A) < \delta \implies \sup_{\psi \in \mathcal{M}} \int_A \psi(x)dx \leq \varepsilon.$$

It is also a lemma that this definition is equivalent to condition 3). Intuitively, uniform integrability will rule out pathological sequences of the form

$$\phi_n = n \cdot \mathbb{1}_{(0,\frac{1}{n})}.$$

This is also why we inject births subject to a continuous distribution, as a point mass birth would violate uniform integrability.

Lemma 1. *The tightness and uniform integrability conditions 2) and 3) imply that the trajectories* $\{\psi M^t : t \in \mathbb{N}\}$ *are precompact.*

I will not prove this lemma - I will leave it as the only thing to be taken for granted.

Lemma 2. *If the trajectories* $\{\psi M^t\}_{t\geq 0}$ *are precompact and* M^t *is contracting for some* $t \in \mathbb{N}$*, then* $(D(S), M)$ *is globally stable.*

Proof. Fix $\psi \in D(S)$ and define $\Gamma(\psi)$ to be the closure of $\{M^n(\psi) : n \in \mathbb{N}\}$. The set $\Gamma(\psi)$ is a compact subset of $D(S)$, because it is closed and the contraction keeps it totally bounded. Moreover, M sends $\Gamma(\psi)$ into itself since M is continuous (contracting) and $M(\Gamma(\psi)) \subset M(\Gamma(\psi))$. Hence $(\Gamma(\psi), M)$ is a dynamical system where M is contracting and $\Gamma(\psi)$ is compact. This implies the existence of a unique fixed point $\psi^\star \in \Gamma(\psi)$ where $M^n(\psi) \to \psi^\star$. Uniqueness in $D(S)$ follows because M has at most one fixed point in $D(S)$ by contraction over U.[15]

Uniqueness also follows by same argument in the proof of the contraction mapping theorem. □

Proposition 3. *The law of motion*

$$W_{t+1} = \begin{cases} R_{t+1}W_t & \text{with probability } 1-\rho \\ W_0 & \text{with probability } \rho \end{cases}$$

admits the stochastic transition kernel in logs

$$p(w, w') = (1-\rho) f\left(w' - w\right) + \rho \frac{1}{\sqrt{2\pi}} \exp\left\{-\left(\frac{w' - w_e}{2}\right)\right\}$$

[15]Suppose x^* and x^{**} were two distinct fixed points. Then $||T(x^*) - T(x^{**})|| = ||x^* - x^{**}||$. This contradicts the contraction definition.

The dynamical system $(D(\mathbb{R}), M)$ defined by p is globally stable. That is, W_{t+1} has an attracting stationary distribution $W^\star$.

Note that it is necessary to work in logs for the transition kernel since we do not have geometric drift to the centre in absolute terms. This is because even if $\log W_t$ satisfies geometric drift to the center, it does not guarantee that W_t also satisfies geometric drift to the center. However, if $\log W_t$ has a stationary distribution, then W_t must also have a stationary distribution.

Proof. Let us first check condition 1). If f is a density with full support, then

$$\int \min\{p(w, w'),\ p(\tilde{w}, w')\}dw'$$

is obviously going to be strictly positive for any choice of w, $\tilde{w}$. Moreover, if f only has positive support the argument will still hold. Thus for pretty general versions of our specification, this condition will hold.

Now we check condition 2). Let $q(w) = w$. This obviously has precompact sublevel sets on $\mathbb{R}_+$. Then:

$$\begin{aligned}
\int w' p(w, w')dw' &= (1-\rho)\int_0^\infty f(w'-w)w'dw' + \underbrace{\rho\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(\frac{w'-w_e}{2}\right)dw'}_{\rho\mathbb{E}w_0=\beta_1} \\
&= (1-\rho)\int_0^\infty f(w'-w)w'dw' + \beta_1 \\
&= (1-\rho)\int_0^\infty f(r)(r+w)d(r+w) + \beta_1 \\
&= (1-\rho)w\underbrace{\int_0^\infty f(r)dr}_{=1} + (1-\rho)\underbrace{\int_0^\infty f(r)rdr}_{=\beta_2} + \beta_1 \\
&= (1-\rho)w + \beta \\
&\leq \alpha w + \beta
\end{aligned}$$

where $\alpha \in [1-\rho, 1)$ and $\beta < \infty$ are constants, yielding the result.

Finally, condition 3) holds pretty generally. For instance, so long as $p(w, w')$ is a continuous density on $[0, \infty)$, then it will be bounded by a constant, say J, on that set for any reasonable specification. □

C.2 Stationary Distribution is a two-sided Pareto

Let the size of the time indices be denoted by Δ such that $t_n - t_{n-1} = \Delta$. So $t_n = t_{n-1} + \Delta$. Probability of death is $\rho\Delta$. Conditional on survival we have

$$W_{t_{n+1}} = R_{t_{n+1}} W_{t_n}$$

So

$$\log W_{t_{n+1}} = \log R_{t_{n+1}} + \log W_{t_n}$$

Moreover, suppose that agents are injected after death with wealth distributed as $\log W_0 \sim N(\log W_e, \Delta)$.

Let $r = \log R$. Let's suppose that $\log(R_t) \sim_{iid} N(0, \sigma^2\Delta)$. Note

$$\begin{aligned}\mathbb{E}_0[\log W_a] &= a\mathbb{E}r + \mathbb{E}\log W_e \\ &= \mathbb{E}\log W_e\end{aligned}$$

Similarly,

$$\begin{aligned}Var_0[\log W_a] &= aVar(r) + Var(\log W_0) \\ &= aVar(r) + Var(\log W_0) \\ &= a\sigma^2\Delta + \Delta^2.\end{aligned}$$

Then since $\{r_t\}$ is *i.i.d*, $\log W_a$ is the sum of a independent normal random variables with mean 1 and variance $\sigma^2\Delta$, plus a normal distribution with mean $\log W_e$ and variance Δ^2. Then, inclusive of the randomness in W_e in the injection point, $\log W_a \sim N(\log W_e, a\sigma^2\Delta + \Delta)$.

If an agent is born with wealth W_e, then with the birth point of W_e we have the wealth at age a, $\log W_a - \log W_e \sim N(0, a\sigma^2\Delta + \Delta)$. So, similarly $\log W_a \sim N(\log W_e, a\sigma^2\Delta + \Delta)$.

Let a denote the number of periods of length Δ that a person has been alive. Call a the *age count*. In the steady state, the mass of agents with age count a is given by a geometric distribution (discrete time). That is, the mass of agents at age count a is $(1-\rho\Delta)^a\rho\Delta$.

Then the stationary density of $w = \log W_a - \log W_e$ will be given by the exponential distribution characterizing the mass of ages multiplied by the normal distributions characterizing wealth at each age. That is, the following:

$$f(w) = \sum_{a=0}^{\infty}(1-\rho\Delta)^a\rho\Delta\frac{1}{\sqrt{2\pi(\sigma^2\Delta a + \Delta)}}\exp\left\{-\frac{w^2}{2(\sigma^2\Delta a + \Delta)}\right\} \tag{63}$$

$$= \sum_{a=0}^{\infty}(1-\rho\Delta)^a\rho\Delta\frac{1}{\sqrt{2\pi(\sigma^2\Delta a + \Delta)}}\exp\left\{-\frac{w^2}{2(\sigma^2\Delta a + \Delta)}\right\}da \tag{64}$$

$$= \int_0^{\infty}(1-\rho\Delta)^{\lfloor a\rfloor}\rho\Delta\frac{1}{\sqrt{2\pi(\sigma^2\Delta\lfloor a\rfloor + \Delta)}}\exp\left\{-\frac{w^2}{2(\sigma^2\Delta\lfloor a\rfloor + \Delta)}\right\}da \tag{65}$$

where equation (3) uses the floor function $\lfloor\rfloor$ to switch from summations to integrals. The floor function is is the function that takes as input a real number x, and gives as output the greatest integer less than or equal to x.

Consider the arithmetic sequence

$$\{n\}_{n=N}^{\infty} = N, N+1, N+2...$$

we use this sequence to define the associate sequence of $\{\Delta_n\}$ where $\Delta_n = \frac{1}{n}$. The sequence of Δ_n is the harmonic sequence associated with the arithmetic sequence $\{n\}_{n=N}^{\infty}$.

We will now show that for small Δ_n (large n) that this expression is approximated by the integral in Amol's notes. In particular, as $n \to \infty$ (i.e $\Delta_n \to 0$ harmonically) we show that the Amol integral holds exactly. At a high level, we are simply exploiting the property that the geometric distribution approximates an exponential distribution.

When $\Delta_n = 1$, then a is both the number of periods an agent has been alive (a counter) as well as the absolute amount of time that an agent has been alive. I want to let t_a be the absolute amount of time that a person has been alive (from 0 to t_a) and let a be the number of periods it takes for that agent to be alive for length of time t_a.

The equation $t_a = t_0 + a\Delta_n$ gives the relationship between number of periods of length Δ_n and the absolute amount of time that the person has lived. So $(t_a - t_0)/\Delta_n = a$. Letting $t_0 = 0$, we have $a = t_a/\Delta_n$. So the change of variables is from periods of life to length of life in absolute time. Define $w = \log W_t - \log W_e$.

$$\begin{aligned}
f_n(w) &= \int_0^{\infty} (1-\rho\Delta_n)^{\lfloor a \rfloor} \rho\Delta_n \frac{\exp\{-w^2/[2(\sigma^2\Delta_n\lfloor a\rfloor + \Delta_n)]\}}{\sqrt{2\pi(\sigma^2\Delta_n\lfloor a\rfloor + \Delta_n)}} da \\
&= \int_0^{\infty} (1-\rho\Delta_n)^{\lfloor t_a/\Delta_n \rfloor} \rho\Delta_n \frac{\exp\{-w^2/[2(\sigma^2\Delta_n\lfloor t_a/\Delta_n\rfloor + \Delta_n)]\}}{\sqrt{2\pi(\sigma^2\Delta_n\lfloor t_a/\Delta_n\rfloor + \Delta_n)}} d\frac{t_a}{\Delta_n} \\
&= \int_0^{\infty} (1-\rho\Delta_n)^{\lfloor t_a/\Delta_n \rfloor} \rho \frac{\exp\{-w^2/[2(\sigma^2\Delta_n\lfloor t_a/\Delta_n\rfloor + \Delta_n)]\}}{\sqrt{2\pi(\sigma^2\Delta_n\lfloor t_a/\Delta_n\rfloor + \Delta_n)}} dt_a \\
\to f_{\infty}(w) &= \int_0^{\infty} \rho e^{-\rho t_a} \frac{1}{\sqrt{2\pi\sigma^2 t_a}} \exp\left\{-\frac{w^2}{2\sigma^2 t_a}\right\} dt_a
\end{aligned}$$

The last line in this expression holds because of two limit results.

The first result is that $\lim_{n\to\infty}(1-\rho\Delta_n)^{t_a/\Delta_n} = e^{-\rho t_a}$.

To establish that $\lim_{n\to\infty}(1-\rho\Delta_n)^{t_a/\Delta_n} = e^{-\rho t_a}$ we use the well known formula that $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. Letting $x = -\rho, \Delta_n = \frac{1}{n}$ we see that $\lim_{n\to\infty}(1-\rho\Delta_n)^{1/\Delta_n} = e^{-\rho}$. Raising both sides to the power of t_a gives the result that $\lim_{n\to\infty}(1-\rho\Delta_n)^{t_a/\Delta_n} = e^{-\rho t_a}$.

Secondly, the function $\Delta_n\lfloor t_a/\Delta_n\rfloor$ converges to the t_a as Δ_n becomes small. Finally, note that since

the integrands are both continuous bounded densities, dominated convergence holds so evaluating the limit internally is a valid operation.

Notice that this expression is

$$f_\infty(w) = \int_0^\infty \rho \exp\{-\rho t_a\} \frac{1}{\sqrt{2\pi\sigma^2 t_a}} \exp\left\{-\frac{w^2}{2\sigma^2 t_a}\right\} dt_a \tag{66}$$

$$= \int_0^\infty \rho \frac{1}{\sqrt{2\pi\sigma^2 t_a}} \exp\left\{-\frac{w^2/\sigma^2}{2t_a} - \frac{2\rho t_a^2}{2t_a}\right\} dt_a \tag{67}$$

$$= \rho \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2 t_a}} \exp\left\{-\frac{p + q t_a^2}{2t_a}\right\} dt_a}_{\text{Integral similar to Amol}} \tag{68}$$

$$\tag{69}$$

where $p = w^2/\sigma^2$ and $q = 2\rho$.

This brings us back to a variation of Amol Amol's case when $\sigma = 1$. So I now include an edited version of his working here and assume that $\sigma = 1$.

Let us define

$$h(T) = \int_0^T \frac{1}{\sqrt{2\pi t_a}} \exp\left\{-\frac{p + q t_a^2}{2t_a}\right\} dt_a$$

From Amol's notes, take as given that for any $T > 0$,

$$h(T) = \frac{1}{\sqrt{q}} \left(e^{-\sqrt{pq}} \mathbf{\Phi}\left(\frac{-\sqrt{p} + T\sqrt{q}}{\sqrt{T}}\right) - e^{\sqrt{pq}} \mathbf{\Phi}\left(\frac{-\sqrt{p} - T\sqrt{q}}{\sqrt{T}}\right)\right)$$

Here, $\mathbf{\Phi}(.)$ is the cumulative density function of a standard normal distribution. The fact that it is indeed the integral is proven in the next subsection. Then, $f_\infty(w)/\rho = \lim_{T\to\infty} h(T) = \frac{e^{-\sqrt{pq}}}{\sqrt{q}}$ when $\sigma = 1$. So, when $\sigma = 1$

$$f(w) = \rho \frac{e^{-\sqrt{pq}}}{\sqrt{q}}.$$

For $w \leq 0$, $\sqrt{p} = -w$. For $w > 0$, $\sqrt{p} = w$. Thus, for $w \leq 0$,

$$f(w) = \rho \frac{e^{w\sqrt{q}}}{\sqrt{q}} = \rho \frac{e^{w\sqrt{2\rho}}}{\sqrt{2\rho}}$$

For $w > 0$,

$$f(w) = \rho \frac{e^{-w\sqrt{2\rho}}}{\sqrt{2\rho}}$$

Combining the two cases of $w > 0$ and $w < 0$, we obtain

$$f(w) = \rho \frac{\min\{e^{-w\sqrt{2\rho}}, e^{w\sqrt{2\rho}}\}}{\sqrt{2\rho}} = \rho \frac{e^{-|w|\sqrt{2\rho}}}{\sqrt{2\rho}}$$

This is thus the density of $\log(W_t/W_e)$. We now investigate the density of W_t/W_e. Define the cumulative density of $w = \log(W_{j,e,t}/W_{e,t})$ as $F(w)$. Then, for $w \leq 0$,

$$F(w) = \rho \int_{-\infty}^{w} \frac{e^{-|m|\sqrt{2\rho}}}{\sqrt{2\rho}} dm = \rho \int_{-\infty}^{w} \frac{e^{m\sqrt{2\rho}}}{\sqrt{2\rho}} dm = \rho \frac{e^{-|m|\sqrt{2\rho}}}{2\rho}\Big|_{-\infty}^{w} = \rho \frac{e^{-|w|\sqrt{2\rho}}}{2\rho} = \frac{e^{w\sqrt{2\rho}}}{2}. \tag{70}$$

This implies that at $w = 0$ we have $F(0) = 1/2$. Then, for $w > 0$ we have

$$F(w) = \frac{1}{2} + \rho \int_{0}^{w} \frac{e^{-m\sqrt{2\rho}}}{\sqrt{2\rho}} dm = \frac{1}{2} - \rho \frac{e^{-m\sqrt{2\rho}}}{2\rho}\Big|_{0}^{w} = 1 - \frac{e^{-w\sqrt{2\rho}}}{2}.$$

Now, define $\tilde{W} = W_t/W_e$. Then define $G(\tilde{w}) = F(\log w)$.

This means that we have

$$\begin{aligned} G(\tilde{w}) &= \begin{cases} \frac{e^{\log w\sqrt{2\rho}}}{2}, & \text{if } \tilde{w} < 1 \\ \frac{1}{2} - \frac{\tilde{w}^{-\sqrt{2\rho}}}{2}, & \text{if } \tilde{w} \geq 1 \end{cases} \\ &= \begin{cases} \frac{\tilde{w}^{\sqrt{2\rho}}}{2}, & \text{if } \tilde{w} < 1 \\ \frac{1}{2} - \frac{\tilde{w}^{-\sqrt{2\rho}}}{2}, & \text{if } \tilde{w} \geq 1 \end{cases} \end{aligned}$$

Differentiating for the density yield:

$$g(\tilde{w}) = \begin{cases} \frac{\sqrt{\rho}\tilde{w}^{\sqrt{2\rho}-1}}{\sqrt{2}}, & \text{if } \tilde{w} < 1 \\ \frac{\sqrt{\rho}\tilde{w}^{-\sqrt{2\rho}-1}}{\sqrt{2}}, & \text{if } \tilde{w} \geq 1 \end{cases} \tag{71}$$

This is a similar to expression to Amol's notes. I think it is a sort of two sided ParetoLet the size of the time indices be denoted by Δ such that $t_n - t_{n-1} = \Delta$. So $t_n = t_{n-1} + \Delta$. Probability of death is $\rho\Delta$. Conditional on survival we have

$$W_{t_{n+1}} = R_{t_{n+1}} W_{t_n}$$

So

$$\log W_{t_{n+1}} = \log R_{t_{n+1}} + \log W_{t_n}$$

Moreover, suppose that agents are injected after death with wealth distributed as $\log W_0 \sim N(\log W_e, \Delta)$.

Let $r = \log R$. Let's suppose that $\log(R_t) \sim_{iid} N(0, \sigma^2\Delta)$. Note

$$\begin{aligned} \mathbb{E}_0[\log W_a] &= a\mathbb{E}r + \mathbb{E}\log W_e \\ &= \mathbb{E}\log W_e \end{aligned}$$

Similarly,

$$Var_0[\log W_a] = aVar(r) + Var(\log W_0) = a\sigma^2\Delta + \Delta^2.$$

Then since $\{r_t\}$ is *i.i.d*, $\log W_a$ is the sum of a independent normal random variables with mean 1 and variance $\sigma^2\Delta$, plus a normal distribution with mean $\log W_e$ and variance Δ^2. Then, inclusive of the randomness in W_e in the injection point, $\log W_a \sim N(\log W_e, a\sigma^2\Delta + \Delta)$.

If an agent is born with wealth W_e, then with the birth point of W_e we have the wealth at age a, $\log W_a - \log W_e \sim N(0, a\sigma^2\Delta + \Delta)$. So, similarly $\log W_a \sim N(\log W_e, a\sigma^2\Delta + \Delta)$.Let a denote the number of periods of length Δ that a person has been alive. Call a the *age count*. In the steady state, the mass of agents with age count a is given by a geometric distribution (discrete time). That is, the mass of agents at age count a is $(1-\rho\Delta)^a\rho\Delta$. Then, the stationary density of $w = \log W_a - \log W_e$ will be given by the exponential distribution characterizing the mass of ages multiplied by the normal distributions characterizing wealth at each age. That is,

$$f(w) = \sum_{a=0}^{\infty}(1-\rho\Delta)^a\rho\Delta\frac{1}{\sqrt{2\pi(\sigma^2\Delta a + \Delta)}}\exp\left\{-\frac{w^2}{2(\sigma^2\Delta a + \Delta)}\right\} \tag{72}$$

$$= \sum_{a=0}^{\infty}(1-\rho\Delta)^a\rho\Delta\frac{1}{\sqrt{2\pi(\sigma^2\Delta a + \Delta)}}\exp\left\{-\frac{w^2}{2(\sigma^2\Delta a + \Delta)}\right\}da \tag{73}$$

$$= \int_0^{\infty}(1-\rho\Delta)^{\lfloor a\rfloor}\rho\Delta\frac{1}{\sqrt{2\pi(\sigma^2\Delta\lfloor a\rfloor + \Delta)}}\exp\left\{-\frac{w^2}{2(\sigma^2\Delta\lfloor a\rfloor + \Delta)}\right\}da, \tag{74}$$

where equation (3) uses the floor function $\lfloor\rfloor$ to switch from summations to integrals. The floor function is is the function that takes as input a real number x, and gives as output the greatest integer less than or equal to x.

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$$F(w) = \rho \int_{-\infty}^{w} \frac{e^{-|m|\sqrt{2\rho}}}{\sqrt{2\rho}} dm = \rho \int_{-\infty}^{w} \frac{e^{m\sqrt{2\rho}}}{\sqrt{2\rho}} dm \tag{79}$$

$$= \rho \frac{e^{-|m|\sqrt{2\rho}}}{2\rho} \Big|_{-\infty}^{w} = \rho \frac{e^{-|w|\sqrt{2\rho}}}{2\rho} = \frac{e^{w\sqrt{2\rho}}}{2}. \tag{80}$$

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Now, define $\tilde{W} = W_t/W_e$. Then define $G(\tilde{w}) = F(\log w)$. Hence, we have

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Differentiating for the density yield:

$$g(\tilde{w}) = \begin{cases} \frac{\sqrt{\rho}\tilde{w}^{\sqrt{2\rho}-1}}{\sqrt{2}}, & \text{if } \tilde{w} < 1 \\ \frac{\sqrt{\rho}\tilde{w}^{-\sqrt{2\rho}-1}}{\sqrt{2}}, & \text{if } \tilde{w} \geq 1 \end{cases} \tag{81}$$

This is the familiar two-sided Pareto distribution.