The Financial Accelerator in Continuous Time*

Zongwu Cai and Jingxian Hu

[a]Department of Economics, University of Kansas, Lawrence, KS 66045, USA

[b]Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan, Shandong 250100, China

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1 Introduction

Since Bernanke, Gertler, and Gilchrist (1999) established the concept of the financial accelerator, it has become a widely used tool to analyze the amplification and propagation effects in the financial market and their roles in the business cycle. Specifically, the unexpected movements in the price of capital induce the movements in the net worth, and the pro-cyclical movements in the net worth induce the counter-cyclical movements in the external finance premium, thus leading the feedback between the real sector and the financial sector in the economy. The financial accelerator also helps to study the transmission mechanism of the monetary policy by embedding a credit channel into a New Keynesian framework.

Even though the financial accelerator was mainly applied for understanding the financial turbulence and the business cycles, Bernanke et al. (1999) did not assume that the capital market is in a risky dynamic system. For example, an idiosyncratic productivity shock to the return on capital is the only source of uncertainty in the capital market in Bernanke et al. (1999), yet the key financial variables, such as the capital and the capital price, were assumed to be deterministic.

With this background, it is natural for us to develop the analysis in Bernanke et al. (1999) by adding uncertainty into the dynamic system. Specifically, what is the property of the financial accelerator under uncertainty? When we add risk terms to the capital market,

*Contact information: Z. Cai (caiz@ku.edu) and J. Hu (jingxianhu@sdu.edu.cn).

will the amplification and propagation effects of the financial accelerator still exist? If so, how are these effects different from those in Bernanke et al. (1999)?

To answer the questions above, we start from a macroeconomics model that has a financial sector, and derive the optimal choices and equilibrium conditions in the capital market, which form the structural equations that we estimate in the empirical study. We build our analysis mainly based on Bernanke et al. (1999), Brunnermeier and Sannikov (2016), and Posch (2016). Following Bernanke et al. (1999), we construct the capital market with micro-foundations in a partial equilibrium framework. These micro-foundations make it possible for us to explain the source and the structure of the financial fluctuations. Due to the tractability reason, we do not study the financial accelerator in a general equilibrium framework. However, we provide an extension of our model to include the households, the price-setting of retailers, and the fiscal and monetary policy in the appendix, and leave it for our future research agenda. Following Brunnermeier and Sannikov (2016), we set up the capital market in continuous time and add the stochastic processes of financial variables to the original model of Bernanke et al. (1999). In this setting, we keep the non-linearity and the risk terms in order to study the financial accelerator effects in a stochastic dynamic system. Following the econometric methods for continuous-time models in Posch (2016), we estimate the continuous-time structural models in the capital market and examine the financial accelerator effects empirically with the real-world financial and macroeconomic data. This approach is different from calibrating the proposed model quantitatively.

There are several new features in our paper.

Compared to Bernanke et al. (1999), our paper adds more stochastic processes and uncertainty in the capital market. Firstly, we introduce an aggregate shock to the evolution of the capital K_t, in addition to the idiosyncratic productivity shock to the capital return for each entrepreneur. Secondly, Bernanke et al. (1999) assumes that the capital K_t and the capital price Q_t are deterministic, whereas we assume that they are stochastic as diffusion processes. The riskiness exists systematically in our paper. These extensions benefits greatly from the recent development of the continuous-time macro-finance literature, such as Brunnermeier and Sannikov (2016). By enriching the dynamics environment of the model, we are able to investigate more of the implications of the higher order terms, or the risks, which are particularly critical for the financial market.

We deviate from Bernanke et al. (1999) by adapting the model into continuous time. The

continuous-time framework has several advantages compared to the discrete-time framework. Firstly, we can utilize the powerful mathematical tools in the theoretical analysis, such as the stochastic calculus and the differential equations. Secondly, the computation methods have been well developed in the continuous-time models, such as the numerical methods to solve the partial differential equations, which can help to find the global solutions instead of using the local approximations. For example, in Bernanke et al. (1999), the log-linearization was applied to simplify the nonlinear system into the linear relationships in order to make the financial accelerator effect transparent and computationally attractable. However, our paper keeps the non-linearity by allowing the functional format of the idiosyncratic shock and the value functions, as well as the higher order risk terms to stay in the model. Thirdly, the continuous-time model is closely related with the high frequency methods in econometrics. With the cutting-edge estimation methods in econometrics, we are able to take the model to the macroeconomic and financial data, and find more empirical implications for the theoretical model, which builds the meaningful and promising research avenues in the future.

The empirical examinations are important for the macroeconomics models, which makes the models credible by connecting them to the real-world and provides persuasive policy suggestions. We follow Posch (2009), Posch (2016), and Posch (2020) on the econometric methods to estimate the macroeconomic structural models in continuous time, in order to examine the propagation and amplification effects of the financial accelerator. Thus, we deviate from Bernanke et al. (1999) by taking the model to the data, especially when the frequencies of the financial data and the macroeconomic data are mixed. Our empirical approach is different from the calibration approach which is a widely applied routine in the macroeconomics literature, including Bernanke et al. (1999).

Our contribution along the lines of literature Posch (2009), Posch (2016), and Posch (2020) is to add the capital market with the micro-foundations in continuous time. From this perspective, we also contribute to the continuous-time macro-finance literature as in Brunnermeier and Sannikov (2016) by adding the moral hazard problem between the entrepreneur and the bank as in Bernanke et al. (1999). Even though we can find the continuous-time contract theory literature close to our setting, our emphasis on the financial accelerator in the continuous time framework is an innovation compared to the existing literature to the best of our knowledge.

Therefore, we aim to renovate the financial accelerator in Bernanke et al. (1999) with the continuous-time analytical tools and the high frequency empirical methods. With these advances, we are able to better understand the financial market fluctuations and the transmission mechanism from the capital market to the real economy.

2 The Model

Our model is composed of the capital market, which we focus on in this paper, as well as the households, the retailers, and the government (fiscal and monetary policy). In particular, the capital market consists of the entrepreneurs, the banks, the capital producer, and the retail goods producer. Since we mainly aim to study the financial accelerator effects, we concentrate on the capital market (in the sense of partial equilibrium) and leave the investigation of the general equilibrium to our future research. The Appendix A provides such a preliminary extension.

Figure 1 below provides a full picture of the relations among different sectors in the general equilibrium model. Even though we are not actually using some sectors in this paper, we thought that understanding the relations and the mechanisms in a complete setting is helpful for viewing the role and the significance of the capital market in the whole system.

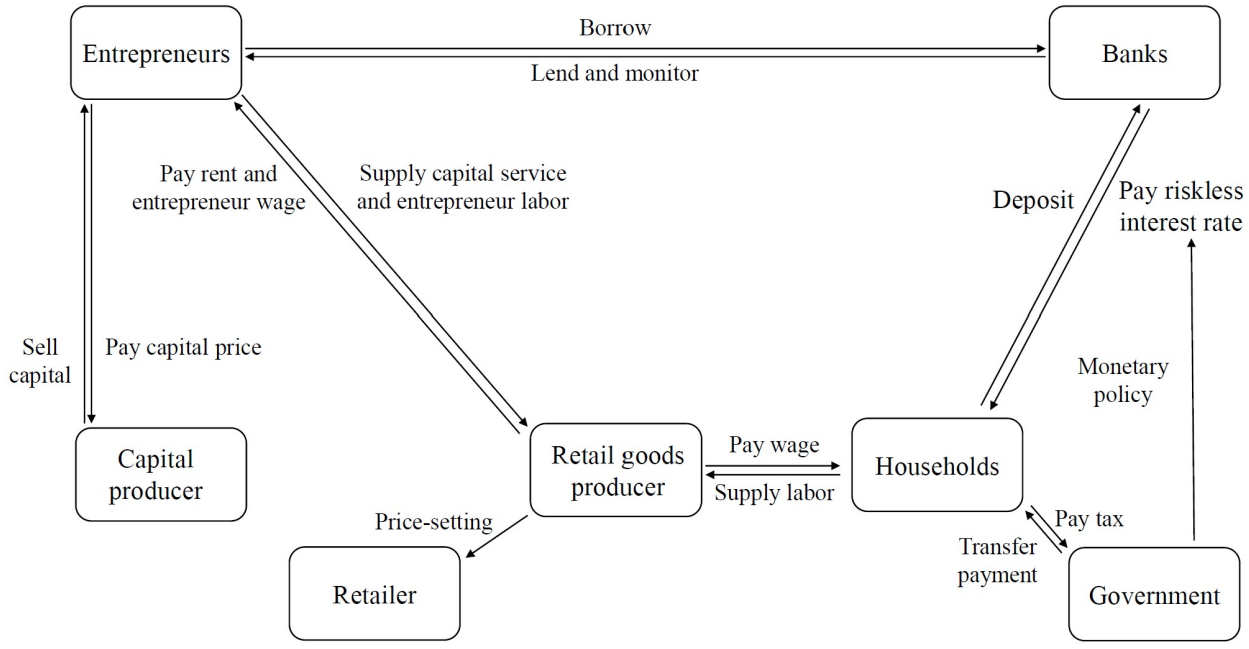


Figure 1: The structure of the model.

2.1 The Capital Market

Our paper sets up the entrepreneur's optimization problem in a capital market with micro-foundations. In particular, we assume that the capital market is composed of the entrepreneurs, the banks, the capital producers, and the retail goods producers. The entrepreneurs buy the capital from the capital producers, and provide the capital service to the retail goods producers in order to produce the output (the consumption goods). To support the purchase of the capital, the entrepreneurs need to finance in the capital market to acquire external funding in addition to their own net worth (the internal funding). Specifically, the entrepreneur borrows from the bank with a loan contract where the loan rate is correlated with an idiosyncratic productivity shock to the return on capital, as well as a monitoring cost incurred by the moral hazard problem as a friction in the financial market.

2.1.1 The Retail Goods Producer and the Capital Producer

Assume that there is a retail goods producer that uses capital in combination with labor to produce the output. The production function is assumed to be constant returns to scale in the Cobb-Douglas type:

$$Y_t(z) = A_t K_t^{\alpha} \left[H_t^{\Omega} \left(H_t^e \right)^{1-\Omega} \right]^{1-\alpha}. \tag{1}$$

where $Y_t(z)$ is the output of the retail good z, A_t denotes the technology level, K_t is the capital.

The total labor L_t is a composite of household labor H_t and entrepreneurial labor H_t^e

$$L_t = H_t^{\Omega} \left(H_t^e \right)^{1-\Omega}. \tag{2}$$

Assume that the retail goods producer aims to maximize his profit, given by the following profit function:

$$\max_{\{K_t, H_t, H_t^e\}} \Pi_t = P_t(z) Y_t(z) - W_t H_t - W_t^e H_t^e - Z_t K_t, \tag{3}$$

where Π_t denotes the profit, $P_t(z)$ is price of the retail good z, W_t is the household labor wage, W_t^e is the entrepreneurial labor wage, and Z_t stands for the rent paid to the capital service. The optimization is subject to the production function:

$$Y_t(z) = A_t K_t^{\alpha} \left[H_t^{\Omega} \left(H_t^e \right)^{1-\Omega} \right]^{1-\alpha}.$$

The first order conditions with respect to K_t, H_t, and H_t^e yield the optimal factor price, which are the rent of capital service Z_t, the worker's wage W_t, and the entrepreneur's wage

W_t^e respectively:

$$Z_t = \frac{\alpha P_t(z) Y_t(z)}{K_t}, \quad W_t = (1-\alpha)\Omega \frac{P_t(z) Y_t(z)}{H_t}, \quad W_t^e = (1-\alpha)(1-\Omega)\frac{P_t(z) Y_t(z)}{H_t^e}. \tag{4}$$

Assume that there is a capital producer maximizing the profit from producing the capital, with respect to the investment I_t:

$$\max_{\{I_t\}} \left[Q_t K_t \Phi\left(\frac{I_t}{K_t}\right) - I_t \right],$$

where Q_t denotes the capital price. The first order condition delivers the solution of the optimal investment:

$$I_t = \delta K_t + \frac{1}{c}\left(1 - \frac{1}{Q_t}\right) K_t. \tag{5}$$

In particular, the technology for producing new capital goods, $\Phi\left(\frac{I_t}{K_t}\right)$, is given by:

$$\Phi\left(\frac{I_t}{K_t}\right) = \frac{I_t}{K_t} - \frac{1}{2} c \left(\frac{I_t}{K_t} - \delta\right)^2 \tag{6}$$

which states that investing $\frac{I_t}{K_t}$ units of capital produces $\frac{I_t}{K_t} - \frac{1}{2} c \left(\frac{I_t}{K_t} - \delta\right)^2$ units of new capital, where c stands for the adjustment cost parameter and δ denotes the depreciation rate.

2.1.2 The Capital and the Asset Prices

In contrast to Bernanke et al. (1999), where $K_{t+1} = \Phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta) K_t$, a deterministic process of K_t, we assume that K_t follows the following diffusion process:

$$\frac{dK_t}{K_t} = \left[\Phi\left(\frac{I_t}{K_t}\right) - \delta\right] dt + \sigma_t^K dW_t^K, \tag{7}$$

where $\Phi\left(I_t/K_t\right) - \delta$ serves as the drift function in the above diffusion model, and $\sigma_t^K dW_t^K$ denotes an aggregate risk of the capital.

Assume that the capital price follows a geometric Brownian motion:

$$\frac{dQ_t}{Q_t} = \mu_t^Q dt + \sigma_t^Q dW_t^Q \tag{8}$$

According to Itô's lemma, we can derive the capital gain as:

$$\frac{d\left(Q_t K_t\right)}{Q_t K_t} = \left[\Phi\left(\frac{I_t}{K_t}\right) - \delta + \mu_t^Q\right] dt + \sigma_t^K dW_t^K + \sigma_t^Q dW_t^Q$$

The net rate of return on capital r_t^k is composed of three terms, the dividend yields $\frac{Y_t(z)-I_t}{Q_tK_t}dt$, the capital gain $\frac{d(Q_tK_t)}{Q_tK_t}$, and the capital service rent revenue $\frac{Z_tK_t}{Q_tK_t}dt$. Thus, the evolution of the net rate of return on capital can be written as:

$$\begin{aligned} dr_t^k =& \frac{Y_t(z)-I_t}{Q_tK_t}dt + \frac{d\left(Q_tK_t\right)}{Q_tK_t} + \frac{Z_tK_t}{Q_tK_t}dt \\ =& \frac{Y_t(z)-\delta K_t - \frac{1}{c}\left(1-\frac{1}{Q_t}\right)K_t}{Q_tK_t}dt + \left[\Phi\left(\frac{I_t}{K_t}\right) - \delta + \mu_t^Q\right]dt + \frac{\alpha P_t(z)Y_t(z)}{Q_tK_t}dt \\ &+ \sigma_t^K dW_t^K + \sigma_t^Q dW_t^Q \\ \equiv& \mu_t^{r^k}dt + \sigma_t^K dW_t^K + \sigma_t^Q dW_t^Q. \end{aligned} \tag{9}$$

where

$$\mu_t^{r^k} \equiv \frac{Y_t(z)-\delta K_t - \frac{1}{c}\left(1-\frac{1}{Q_t}\right)K_t}{Q_tK_t} + \left[\Phi\left(\frac{I_t}{K_t}\right) - \delta + \mu_t^Q\right] + \frac{\alpha P_t(z)Y_t(z)}{Q_tK_t}$$

2.1.3 The Entrepreneurs and the Banks

Entrepreneurs are assumed to be risk-neutral and can absorb any aggregate risk. An entrepreneur who manages firm j purchase capital K_t^j at the price paid per unit of capital Q_t. His ex post gross return on capital for firm j is $\omega^j R_t^k$. R_t^k denotes the gross rate of return of capital, and we have $R_t^k = 1 + r_t^k$. The ω^j is an idiosyncratic shock and is i.i.d. with c.d.f. $F(\omega)$ with $E(\omega) = 1$. The hazard rate is defined as $h(\omega) = dF(\omega)/[1 - F(\omega)]$ and $\partial(\omega\, h(\omega))/\partial\omega > 0$. To operate the firm j, the entrepreneur borrows B_t^j from the bank at the loan rate RL_t^j, which forms a loan contract. Note that B_t^j in the bank comes from the household's deposit D_t (at equilibrium $D_t = B_t$). The deposit pays a riskless gross interest rate $R_t \equiv 1 + r_t$.

The uncertainty of ω^j is mainly used to capture the possibility of the entrepreneur's default, which is contingent on the realization of the capital return R_t^k. In particular,

$$\begin{cases} \text{when } \omega^j \geq \bar{\omega}^j \quad \text{(survive)}: \text{ the entrepreneur repays the bank } \mathrm{RL}_t^j B_t^j \\ \text{when } \omega^j < \bar{\omega}^j \quad \text{(default)}: \text{ the entrepreneur defaults} \end{cases}$$

Thus, $\bar{\omega}^j$ is defined by the level of the idiosyncratic shock that makes the entrepreneur's gross return on capital enough to cover the loan repayment:

$$\bar{\omega}^j R_t^k Q_t K_t^j = RL_t^j B_t^j. \tag{10}$$

To introduce the financial friction, similar to Bernanke et al. (1999), we follow the "costly state verification" approach as in Townsend (1979). Specifically, they assumed that the bank

must pay a fixed monitoring cost, a proportion μ of the gross return on capital $\omega^j R_t^k Q_t K_t^j$, in order to observe an individual entrepreneur's realized return. It is also due to this friction that the un-collateralized external finance may be more expensive than the internal finance and there exists an external finance premium in the model by Bernanke et al. (1999).

With this borrowing environment set up, the optimization problem of the entrepreneur can be described as below. The financing balance of the entrepreneur follows:

$$d\left(Q_t K_t^j\right) = dN_t^j + dB_t^j, \tag{11}$$

This equation states that the entrepreneur's purchase of capital is financed by the external funds, which is the borrowing from the bank, B_t^j, and the internal funds, which is his own net worth, N_t^j.

Given the possibility of the entrepreneur's default, the return of the entrepreneur and the bank under each situation can be summarized as follows:

$$\begin{cases} \text{when } \omega^j \geq \bar{\omega}^j (\text{survive}) \begin{cases} \text{the entrepreneur gets } \omega^j R_t^k Q_t K_t^j - RL_t^j B_t^j \\ \text{the bank gets } RL_t^j B_t^j \end{cases} \\ \text{when } \omega^j < \bar{\omega}^j (\text{default}) \begin{cases} \text{the entrepreneur gets } 0 \\ \text{the bank gets } (1-\mu)\omega^j R_t^k Q_t K_t^j \end{cases} \end{cases}$$

Note that when the entrepreneur defaults, the bank gets the entrepreneur's residual value but need to pay the monitoring cost.

Then, the entrepreneur's optimization problem is to maximize his expected return and determine his optimal choice of capital:

$$\max_{\left\{\bar{\omega}^j, K_t^j\right\}} E\left\{\int_{\bar{\omega}^j}^{\infty} \left[\omega^j R_t^k Q_t K_t^j - \bar{\omega}^j R_t^k Q_t K_t^j\right] dF(\omega)\right\} \tag{12}$$

which can be written as:

$$\max_{\left\{\bar{\omega}^j, K_t^j\right\}} E\left\{\int_{\bar{\omega}^j}^{\infty} \omega^j R_t^k Q_t K_t^j dF(\omega) - \left[1 - F\left(\bar{\omega}^j\right)\right] \bar{\omega}^j R_t^k Q_t K_t^j\right\}$$

where $[1 - F(\bar{\omega}^j)]$ stands for the probability of no default. Note that $\int_{\bar{\omega}^j}^{\infty} dF(\omega) = 1 - F(\bar{\omega}^j)$.

The maximization problem above is subject to the bank's participation constraint:

$$\left[1 - F\left(\bar{\omega}^j\right)\right] RL_t^j B_t^j + (1-\mu) \int_0^{\bar{\omega}^j} \omega^j R_t^k Q_t K_t^j dF(\omega) \geq R_t B_t^j \tag{13}$$

which states that the bank receives an expected return on the loan contract which is no less than the opportunity cost of the loan contract. The opportunity cost here is a riskless return

$R_t B_t^j$ that the bank can earn if not lending to the entrepreneur. The cost of monitoring the entrepreneur is $\mu \int_0^{\bar{\omega}^j} \omega^j dF(\omega) R_t^k Q_t K_t^j$. Note that given the optimal choices $\bar{\omega}^*$ and K_t^*, the bank's participation constraint is binding. Following the moral hazard problem set-up in the contract theory, we use the bank's Bellman equation as constraint in the entrepreneur's maximization problem and choose the loan contract to ensure that bank's Bellman equation is satisfied.

This constraint can be alternatively written into a functional format:

$$[\Gamma(\bar{\omega}^j) - \mu G(\bar{\omega}^j)] R_t^k Q_t K_t^j = R_t B_t^j = R_t \left(Q_t K_t^j - N_t^j\right)$$

where $\Gamma\left(\bar{\omega}^j\right) \equiv \int_0^{\bar{\omega}^j} \omega dF(\omega) + \bar{\omega}^j \int_{\bar{\omega}^j}^{\infty} dF(\omega)$ and $G\left(\bar{\omega}^j\right) \equiv \int_0^{\bar{\omega}^j} \omega dF(\omega)$. Further, we have the derivatives $\Gamma'\left(\bar{\omega}^j\right) = 1 - F\left(\bar{\omega}^j\right)$ and $G'\left(\bar{\omega}^j\right) = \bar{\omega}^j F'\left(\bar{\omega}^j\right)$. From this alternative format of the constraint, we have:

$$R_t^k Q_t K_t^j = \frac{R_t \left(Q_t K_t^j - N_t^j\right)}{\Gamma(\bar{\omega}^j) - \mu G(\bar{\omega}^j)}$$

With the definition of $\Gamma(\cdot)$, the entrepreneur's objective function can be rewritten into the following format:

$$\max_{\left\{\bar{\omega}^j, K_t^j\right\}} E\left\{\int_0^{\infty} [1 - \Gamma(\bar{\omega}^j)] R_t^k Q_t K_t^j\right\}$$

which is equivalent to the following one by applying the binding participation constraint of the bank:

$$\max_{\left\{\bar{\omega}^j, K_t^j\right\}} E\left\{\int_0^{\infty} [1 - \Gamma(\bar{\omega}^j)] \frac{R_t \left(Q_t K_t^j - N_t^j\right)}{\Gamma(\bar{\omega}^j) - \mu G(\bar{\omega}^j)}\right\}$$

Following the literature in the contract theory, the entrepreneur's optimization problem is also subject to the bank's incentive compatibility constraint, such that given the loan contract, the entrepreneur's optimal K_t^* satisfies the bank's maximization problem simultaneously.

Lastly, the entrepreneur's optimization problem is subject to the entrepreneur's budget constraint, which tells how the aggregate net worth of the entrepreneurs (who are still in business) evolves:

$$dN_t = \gamma d\left(R_t^k Q_t K_t\right) - \gamma d\left(R_t B_t\right) - \gamma d\left[\mu R_t^k Q_t K_t G(\bar{\omega})\right] + d\left(W_t^e H_t^e\right) \tag{14}$$

To derive this budget constraint as above, we follow the definition of the entrepreneur's net worth from Bernanke et al. (1999). The aggregate net worth of the entrepreneurs who are

still in business is composed of their equity value V_t (the wealth accumulated from operating firms) and their entrepreneurial labor income $W_t^e H_t^e$ as follows:

$$dN_t = d\left(\gamma V_t\right) + d\left(W_t^e H_t^e\right) \tag{15}$$

where γ stands for the portion of entrepreneurs who survive. Note that different from Bernanke et al. (1999), we write the equations into the differential format in continuous time. Further, the entrepreneur equity value equals the gross earnings on holdings of capital less the repayment of borrowings:

$$dV_t = d\left(R_t^k Q_t K_t\right) - d\left[R_t B_t + \mu R_t^k Q_t K_t G(\bar{\omega})\right] \tag{16}$$

Note that the repayment of borrowing not only includes the riskless amount, but also includes the monitoring cost that appears as an external financing premium. Applying the Itô's lemma, we can derive:

$$\begin{aligned} d\left(R_t B_t\right) &= dB_t + r_t dB_t + B_t dr_t + dr_t dB_t \\ &= \left(1 + r_t\right) d\left(Q_t K_t\right) + Q_t K_t r_t dt - \left(1 + r_t\right) dN_t - N_t r_t dt \end{aligned}$$

$$\begin{aligned} d\left(r_t^k Q_t K_t\right) &= r_t^k d\left(Q_t K_t\right) + Q_t K_t dr_t^k + dr_t^k d\left(Q_t K_t\right) \\ &= r_t^k \left[\Phi\left(\frac{I_t}{K_t}\right) - \delta + \mu_t^Q\right] Q_t K_t dt + \mu_t^{r^k} Q_t K_t dt + \left(\sigma_t^K\right)^2 Q_t K_t dt + \left(\sigma_t^Q\right)^2 Q_t K_t dt \\ &\quad + \left(1 + r_t^k\right) \sigma_t^K Q_t K_t dW_t^K + \left(1 + r_t^k\right) \sigma_t^Q Q_t K_t dW_t^Q \end{aligned}$$

$$\begin{aligned} d\left(W_t^e H_t^e\right) &= d\left[(1-\alpha)(1-\Omega) P_t(z) Y_t(z)\right] \\ &= (1-\alpha)(1-\Omega) A_t H_t^{\Omega(1-\alpha)} \left(H_t^e\right)^{(1-\Omega)(1-\alpha)} d\left[P_t(z) K_t^\alpha\right] \end{aligned}$$

$$\begin{aligned} d\left[P_t(z) K_t^\alpha\right] &= P_t(z) dK_t^\alpha + K_t^\alpha dP_t(z) + dP_t(z) dK_t^\alpha \\ &= P_t(z) \left\{\alpha K_t^\alpha \left[\Phi\left(\frac{I_t}{K_t}\right) - \delta\right] + \frac{1}{2}\alpha(\alpha-1) K_t^\alpha \left(\sigma_t^K\right)^2 + \pi_t K_t^\alpha\right\} dt + P_t(z) \alpha K_t^\alpha \sigma_t^K dW_t^K \end{aligned}$$

By substituting these results into (14), we can rewrite the evolution of the net worth as:

$$dN_t = \frac{1}{\left[1 - \gamma\left(1 + r_t\right)\right]} \left(\mu_{N_t} dt + \sigma_{N_t}^K dW_t^K + \sigma_{N_t}^Q dW_t^Q\right) \tag{17}$$

which is the financial accelerator equation in Bernanke et al. (1999). In particular, $\mu_{N_t} dt$, $\sigma_{N_t}^K dW_t^K$ and $\sigma_{N_t}^Q dW_t^Q$ are given by:

$$\mu_{N_t} dt = \gamma r_t N_t dt - \gamma r_t Q_t K_t dt$$

$$
\begin{aligned}
&+\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)\right]\left[\Phi\left(\frac{I_t}{K_t}\right)-\delta+\mu_t^Q\right]Q_tK_tdt\\
&+\gamma[1-\mu G(\bar{\omega})]\left(\sigma_t^K\right)^2Q_tK_tdt+\gamma[1-\mu G(\bar{\omega})]\left(\sigma_t^Q\right)^2Q_tK_tdt\\
&+\gamma[1-\mu G(\bar{\omega})]Q_tK_t\mu_t^{r^k}dt\\
&+(1-\alpha)(1-\Omega)P_t(z)Y_t(z)\left\{\alpha\left[\Phi\left(\frac{I_t}{K_t}\right)-\delta\right]+\frac{1}{2}\alpha(\alpha-1)\left(\sigma_t^K\right)^2+\pi_t\right\}dt
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\sigma_{N_t}^KdW_t^K=&\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)\right]\sigma_t^KQ_tK_tdW_t^K\\
&+\gamma[1-\mu G(\bar{\omega})]\sigma_t^KQ_tK_tdW_t^K\\
&+\alpha(1-\alpha)(1-\Omega)P_t(z)Y_t(z)\sigma_t^KdW_t^K
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\sigma_{N_t}^QdW_t^Q=&\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)\right]\sigma_t^QQ_tK_tdW_t^Q\\
&+\gamma[1-\mu G(\bar{\omega})]\sigma_t^QQ_tK_tdW_t^Q
\end{aligned}
\tag{20}
$$

Given the objective function of the entrepreneur (with the binding participation constraint of the bank embedded) and the entrepreneur's budget constraint, the Hamilton-Jacobi-Bellman (HJB) equation can be written as:

$$
\rho V\left(N_t\right)=\max_{\{\bar{\omega},K_t\}}\left\{\begin{array}{l}
\left[1-\Gamma(\bar{\omega})\right]\frac{(1+r_t)(Q_tK_t-N_t)}{\Gamma(\bar{\omega})-\mu G(\bar{\omega})}+\frac{1}{\left[1-\gamma(1+r_t)\right]}\mu_{N_t}V'\left(N_t\right)\\
+\frac{1}{2}\frac{1}{\left[1-\gamma(1+r_t)\right]^2}\left(\sigma_{N_t}^K\right)^2V''\left(N_t\right)+\frac{1}{2}\frac{1}{\left[1-\gamma(1+r_t)\right]^2}\left(\sigma_{N_t}^Q\right)^2V''\left(N_t\right)
\end{array}\right\}
\tag{21}
$$

where $V(\cdot)$ denotes the value function and N_t serves as the state variable. Then, we can derive the first order condition with respect to $\bar{\omega}$ as:

$$
\begin{aligned}
0=&\frac{\left[F(\bar{\omega})-1\right](1+r_t)\left(Q_tK_t-N_t\right)}{\Gamma(\bar{\omega})-\mu G(\bar{\omega})}-\frac{\left[1-\Gamma(\bar{\omega})\right](1+r_t)\left(Q_tK_t-N_t\right)\left[1-F(\bar{\omega})-\mu\bar{\omega}F'(\bar{\omega})\right]}{\left[\Gamma(\bar{\omega})-\mu G(\bar{\omega})\right]^2}\\
&+\frac{V'\left(N_t\right)}{\left[1-\gamma\left(1+r_t\right)\right]}\left\{\begin{array}{l}
-\gamma\mu G'(\bar{\omega})Q_tK_t\left(1+r_t^k\right)\left[\Phi\left(\frac{I_t}{K_t}\right)-\delta+\mu_t^Q\right]\\
-\gamma\mu G'(\bar{\omega})Q_tK_t\left(\sigma_t^K\right)^2-\gamma\mu G'(\bar{\omega})Q_tK_t\left(\sigma_t^Q\right)^2\\
-\gamma\mu G'(\bar{\omega})Q_tK_t\mu_t^{r^k}
\end{array}\right\}\\
&+\frac{1}{\left[1-\gamma\left(1+r_t\right)\right]^2}\sigma_{N_t}^KV''\left(N_t\right)\left[-\gamma\mu G'(\bar{\omega})Q_tK_t\left(1+r_t^k\right)\sigma_t^K-\gamma\mu G'(\bar{\omega})Q_tK_t\sigma_t^K\right]\\
&+\frac{1}{\left[1-\gamma\left(1+r_t\right)\right]^2}\sigma_{N_t}^QV''\left(N_t\right)\left[-\gamma\mu G'(\bar{\omega})Q_tK_t\left(1+r_t^k\right)\sigma_t^Q-\gamma\mu G'(\bar{\omega})Q_tK_t\sigma_t^Q\right]
\end{aligned}
\tag{22}
$$

Similarly, the first order condition with respect to K_t is:

$$
\begin{aligned}
0 =& \frac{\left[1-\Gamma(\bar{\omega})\right](1+r_t)\,Q_t}{\Gamma(\bar{\omega})-\mu G(\bar{\omega})} \\
&+ \frac{V'(N_t)}{\left[1-\gamma(1+r_t)\right]}
\left\{
\begin{array}{l}
-\gamma r_t Q_t \\
+\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)\right]\left\{\Phi'\left(\frac{I_t}{K_t}\right)Q_t K_t+\left[\Phi\left(\frac{I_t}{K_t}\right)-\delta+\mu_t^Q\right]Q_t\right\} \\
+\gamma[1-\mu G(\bar{\omega})]\left(\sigma_t^K\right)^2 Q_t+\gamma[1-\mu G(\bar{\omega})]\left(\sigma_t^Q\right)^2 Q_t \\
+\gamma[1-\mu G(\bar{\omega})]\left\{\left[\frac{\partial Y_t(z)}{\partial K_t}-\delta-\frac{1}{c}\left(1-\frac{1}{Q_t}\right)\right]+\alpha P_t(z)\frac{\partial Y_t(z)}{\partial K_t}\right\} \\
+\gamma[1-\mu G(\bar{\omega})]\left\{\Phi'\left(\frac{I_t}{K_t}\right)Q_t K_t+\left[\Phi\left(\frac{I_t}{K_t}\right)-\delta+\mu_t^Q\right]Q_t\right\} \\
+\alpha(1-\alpha)(1-\Omega)P_t(z)Y_t(z)\Phi'\left(\frac{I_t}{K_t}\right)
\end{array}
\right\} \\
&+ \frac{1}{\left[1-\gamma(1+r_t)\right]^2}\sigma_{N_t}^K V''(N_t)
\left\{
\begin{array}{l}
\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)\right]\sigma_t^K Q_t \\
+\gamma[1-\mu G(\bar{\omega})]\sigma_t^K Q_t \\
+\alpha(1-\alpha)(1-\Omega)P_t(z)\frac{\partial Y_t(z)}{\partial K_t}\sigma_t^K
\end{array}
\right\} \\
&+ \frac{1}{\left[1-\gamma(1+r_t)\right]^2}\sigma_{N_t}^Q V''(N_t)\left\{\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)\right]\sigma_t^Q Q_t+\gamma[1-\mu G(\bar{\omega})]\sigma_t^Q Q_t\right\}
\end{aligned}
\tag{23}
$$

2.1.4 The Capital Market Equilibrium

The capital market equilibrium consists of the following key equations. We take them from the previous sections, and simplify further where appear necessary with the algebra.

(1) The production function: $Y_t(z) = A_t K_t^{\alpha}\left[H_t^{\Omega}\left(H_t^e\right)^{1-\Omega}\right]^{1-\alpha}$

(2) The optimal factor prices: $P_t(z)Y_t(z) = \frac{\mathrm{Z}_t K_t}{\alpha} = \frac{W_t H_t}{(1-\alpha)\Omega} = \frac{W_t^e H_t^e}{(1-\alpha)(1-\Omega)}$

(3) The optimal investment: $\frac{I_t}{K_t} = \delta + \frac{1}{c}\left(1-\frac{1}{Q_t}\right)$

(4) The evolution of capital: $dK_t = \left[\frac{1}{c}\left(1-\frac{1}{Q_t}\right)-\frac{1}{2c}\left(1-\frac{1}{Q_t}\right)^2\right]K_t dt + \sigma_t^K K_t dW_t^K$

(5) The evolution of the rate of return on capital:

$$
\begin{aligned}
dr_t^k =& \frac{Y_t(z)-\delta K_t-\frac{1}{c}\left(1-\frac{1}{Q_t}\right)K_t+\alpha P_t(z)Y_t(z)}{Q_t K_t}dt + \left[\frac{1}{c}\left(1-\frac{1}{Q_t}\right)-\frac{1}{2c}\left(1-\frac{1}{Q_t}\right)^2+\mu_t^Q\right]dt \\
&+ \sigma_t^K dW_t^K + \sigma_t^Q dW_t^Q
\end{aligned}
$$

(6) The evolution of the entrepreneur's net worth:

$$
dN_t = \frac{1}{\left[1-\gamma(1+r_t)\right]}\left(\mu_{N_t}dt + \sigma_{N_t}^K dW_t^K + \sigma_{N_t}^Q dW_t^Q\right)
$$

where

$$
\begin{aligned}
\mu_{N_t}dt = &\gamma r_t N_t dt - \gamma r_t Q_t K_t dt \\
&+ \left[\gamma\left(r_t^k - r_t\right) - \gamma\mu G(\bar{\omega})\left(1 + r_t^k\right) + \gamma - \gamma\mu G(\bar{\omega})\right] \times \\
&\quad \left[\frac{1}{c}\left(1 - \frac{1}{Q_t}\right) - \frac{1}{2c}\left(1 - \frac{1}{Q_t}\right)^2 + \mu_t^Q\right] Q_t K_t dt \\
&+ \gamma[1 - \mu G(\bar{\omega})]\left(\sigma_t^K\right)^2 Q_t K_t dt + \gamma[1 - \mu G(\bar{\omega})]\left(\sigma_t^Q\right)^2 Q_t K_t dt \\
&+ \gamma[1 - \mu G(\bar{\omega})]\left[Y_t(z) - \delta K_t - \frac{1}{c}\left(1 - \frac{1}{Q_t}\right) K_t + \alpha P_t(z) Y_t(z)\right] dt \\
&+ (1-\alpha)(1-\Omega) P_t(z) Y_t(z) \times \\
&\quad \left\{\alpha\left[\frac{1}{c}\left(1 - \frac{1}{Q_t}\right) - \frac{1}{2c}\left(1 - \frac{1}{Q_t}\right)^2\right] + \frac{1}{2}\alpha(\alpha - 1)\left(\sigma_t^K\right)^2 + \pi_t\right\} dt
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{N_t}^K dW_t^K = &\left[\gamma\left(r_t^k - r_t\right) - \gamma\mu G(\bar{\omega})\left(1 + r_t^k\right) + \gamma - \gamma\mu G(\bar{\omega})\right] \sigma_t^K Q_t K_t dW_t^K \\
&+ \alpha(1-\alpha)(1-\Omega) P_t(z) Y_t(z) \sigma_t^K dW_t^K
\end{aligned}
$$

$$
\sigma_{N_t}^Q dW_t^Q = \left[\gamma\left(r_t^k - r_t\right) - \gamma\mu G(\bar{\omega})\left(1 + r_t^k\right) + \gamma - \gamma\mu G(\bar{\omega})\right] \sigma_t^Q Q_t K_t dW_t^Q
$$

(7) The first order condition with respect to $\bar{\omega}$ in the entrepreneur's optimization problem:

$$
\begin{aligned}
0 = &\frac{[F(\bar{\omega}) - 1]\,(1 + r_t)\,(Q_t K_t - N_t)}{\Gamma(\bar{\omega}) - \mu G(\bar{\omega})} - \frac{[1 - \Gamma(\bar{\omega})]\,(1 + r_t)\,(Q_t K_t - N_t)\,[1 - F(\bar{\omega}) - \mu\bar{\omega}F'(\bar{\omega})]}{[\Gamma(\bar{\omega}) - \mu G(\bar{\omega})]^2} \\
&- \frac{V'(N_t)}{[1 - \gamma(1 + r_t)]}\gamma\mu G'(\bar{\omega}) Q_t K_t \left\{\begin{array}{l} \left(2 + r_t^k\right)\left[\frac{1}{c}\left(1 - \frac{1}{Q_t}\right) - \frac{1}{2c}\left(1 - \frac{1}{Q_t}\right)^2 + \mu_t^Q\right] \\ + \left(\sigma_t^K\right)^2 + \left(\sigma_t^Q\right)^2 \\ + \frac{Y_t(z) - \delta K_t - \frac{1}{c}\left(1 - \frac{1}{Q_t}\right) K_t + \alpha P_t(z) Y_t(z)}{Q_t K_t} \end{array}\right\} \\
&- \frac{1}{[1 - \gamma(1 + r_t)]^2}\sigma_{N_t}^K V''(N_t)\,\gamma\mu G'(\bar{\omega}) Q_t K_t \sigma_t^K \left(2 + r_t^k\right) \\
&- \frac{1}{[1 - \gamma(1 + r_t)]^2}\sigma_{N_t}^Q V''(N_t)\,\gamma\mu G'(\bar{\omega}) Q_t K_t \sigma_t^Q \left(2 + r_t^k\right)
\end{aligned}
$$

(8) The first order condition with respect to K_t in the entrepreneur's optimization problem:

$$
\begin{aligned}
0 =& \frac{\left[1-\Gamma(\bar{\omega})\right](1+r_t)\,Q_t}{\Gamma(\bar{\omega})-\mu G(\bar{\omega})} \\
&+ \frac{V'(N_t)}{\left[1-\gamma(1+r_t)\right]}\left\{\begin{array}{l}
-\gamma r_t Q_t \\
+\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)+\gamma-\gamma\mu G(\bar{\omega})\right] \\
\left\{-\delta-\frac{1}{c}\left(1-\frac{1}{Q_t}\right)+\left[\frac{1}{c}\left(1-\frac{1}{Q_t}\right)-\frac{1}{2c}\left(1-\frac{1}{Q_t}\right)^2+\mu_t^Q\right]Q_t\right\} \\
+\gamma[1-\mu G(\bar{\omega})]\left(\sigma_t^K\right)^2 Q_t+\gamma[1-\mu G(\bar{\omega})]\left(\sigma_t^Q\right)^2 Q_t \\
+\gamma[1-\mu G(\bar{\omega})]\left\{\left[\frac{\partial Y_t(z)}{\partial K_t}-\delta-\frac{1}{c}\left(1-\frac{1}{Q_t}\right)\right]+\alpha P_t(z)\frac{\partial Y_t(z)}{\partial K_t}\right\} \\
-\alpha(1-\alpha)(1-\Omega)P_t(z)Y_t(z)\left[\frac{\delta+\frac{1}{c}\left(1-\frac{1}{Q_t}\right)}{Q_t K_t}\right]
\end{array}\right\} \\
&+ \frac{1}{\left[1-\gamma(1+r_t)\right]^2}\sigma_{N_t}^K V''(N_t)\left\{\begin{array}{l}
\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)+\gamma-\gamma\mu G(\bar{\omega})\right]\sigma_t^K Q_t \\
+\alpha(1-\alpha)(1-\Omega)P_t(z)\frac{\partial Y_t(z)}{\partial K_t}\sigma_t^K
\end{array}\right\} \\
&+ \frac{1}{\left[1-\gamma(1+r_t)\right]^2}\sigma_{N_t}^Q V''(N_t)\left[\gamma\left(r_t^k-r_t\right)-\gamma\mu G(\bar{\omega})\left(1+r_t^k\right)+\gamma-\gamma\mu G(\bar{\omega})\right]\sigma_t^Q Q_t
\end{aligned}
$$

Appendix A An Extension to the General Equilibrium

Appendix A.1 The Households

Following Bernanke et al. (1999), we assume that each household consumes C_t, holds money M_t/P_t, works H_t with the real wage W_t, and deposits its savings in the bank D_t, which pays the riskless rate of return r_t. In addition, T_t is the lump sum taxes paid to the government and Π_t is the dividends that the household receives from owning the retail firms. The household is assumed to be risk-averse.

The household's optimization problem is to maximize its utility:

$$\max E_0 \int_0^{\infty} e^{-\rho t} \left[\ln C_t + \eta_M \ln \left(\frac{M_t}{P_t} \right) + \eta_H \ln \left(1 - H_t \right) \right] dt, \tag{24}$$

where the utility function is given by $U\left(C_t, \frac{M_t}{P_t}, H_t\right) = \ln C_t + \eta_M \ln \left(\frac{M_t}{P_t}\right) + \eta_H \ln \left(1 - H_t\right)$, subject to a wealth evolution equation:

$$dD_t = W_t H_t dt + r_t D_t dt + \Pi_t dt + TR_t dt - C_t dt - T_t dt - \left(\frac{M_t}{P_t} \right) \pi_t dt. \tag{25}$$

The wealth constraint is slightly moderated from the household's budget constraint in Bernanke et al. (1999).

The HJB equation for the households can be written as:

$$0 = \max \left\{ \begin{array}{l} \left[\ln C_t + \eta_M \ln \left(\frac{M_t}{P_t} \right) + \eta_H \ln \left(1 - H_t \right) \right] - \rho J \left(D_t \right) \\ + \left[W_t H_t + r_t D_t + \Pi_t + TR_t - C_t - T_t - \left(\frac{M_t}{P_t} \right) \pi_t \right] J' \left(D_t \right) \end{array} \right\} \tag{26}$$

where $J\left(D_t\right)$ is the value function.

The first order conditions with respect to C_t, $\frac{M_t}{P_t}$, and H_t imply that:

$$C_t = \frac{1}{J'\left(D_t\right)}, \quad \frac{M_t}{P_t} = \frac{\eta_M}{J'\left(D_t\right) \pi_t}, \quad H_t = 1 - \frac{\eta_H}{J'\left(D_t\right) W_t}. \tag{27}$$

Appendix A.2 The Price Setting

Following the classic New Keynesian literatures, the retailer sector determines the price-setting process and introduces the friction brought about by the price inertia in the nominal variables. Thus, the price stickiness is another factor that accounts for the market imperfection. Due to the inertia of the price adjustment, the market cannot clear immediately, which create the space for the monetary policy intervention to smooth out the inflation. It is

assumed that the retailers are monopolistic competitive. The retailer z sells the retail goods $Y_t(z)$ at the price $P_t(z)$. The final goods, Y_t, are the following composite of individual retail goods:

$$Y_t = \left[\int_0^1 Y_t(z)^{\frac{\varepsilon-1}{\varepsilon}} dz\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{28}$$

with $\varepsilon > 1$. Correspondingly, the final goods price is also a composite of the retail price:

$$P_t = \left[\int_0^1 P_t(z)^{1-\varepsilon} dz\right]^{\frac{1}{1-\varepsilon}}. \tag{29}$$

The demand curve facing each retailer is solved to be:

$$Y_t(z) = \left(\frac{P_t(z)}{P_t}\right)^{-\varepsilon} Y_t. \tag{30}$$

Then, the retailer chooses the sale price $P_t(z)$, taking as given the demand curve and the price of wholesale goods, P_t^w (note that $P_t^w \equiv \frac{P_t}{X_t}$ and X_t is the gross markup of the retail goods over the wholesale goods).

Following Calvo (1983), only part of the retailers is free to change the price with a price-change signal. The retailers who are able to change the price set the price at P_t^* and the demand given this price is denoted as $Y_t^*(z)$. The other part of the retailers is assumed to not receive a price change signal during the period $s-t$, where s denotes a future date ($s > t$) and $e^{-\xi(s-t)}$ denote the probability of not adjusting the price during $s-t$. The optimization problem of the retailer that can change its price is as follows. Bernanke et al. (1999) provided such an optimization problem in discrete time. In particular, the retailer chooses the P_t^* to maximize the expected discounted profits:

$$\max \sum_{k=0}^{\infty} \Gamma^k E_{t-1}\left[\Lambda_{t,k} \frac{P_t^* - P_{t+k}^w}{P_{t+k}} Y_{t+k}^*(z)\right], \tag{31}$$

where the discount rate $\Lambda_{t,k} \equiv \beta C_t / C_{t+k}$ is the household intertemporal marginal rate of substitution, which the retailer takes as given. Differentiating the expected discounted profit with respect to P_t^* implies that the optimally set price satisfies:

$$\sum_{k=0}^{\infty} \Gamma^k E_{t-1}\left\{\Lambda_{t,k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}^*(z)\left[\frac{P_t^*}{P_{t+k}} - \left(\frac{\varepsilon}{\varepsilon-1}\right)\frac{P_{t+k}^w}{P_{t+k}}\right]\right\} = 0. \tag{32}$$

In our continuous-time approach, we follow Posch (2020) and moderate the retailer's optimization problem from Bernanke et al. (1999) as below.

$$\max E_t\left\{\int_t^{\infty} e^{-\xi(s-t)} \Lambda_{t,s} \frac{P_t^* - P_s^w}{P_s^w} Y_s^*(z) ds\right\}, \tag{33}$$

where $\Lambda_{t,s} \equiv e^{-\rho(s-t)}\frac{C_t}{C_s}$, subject to the demand for the individual retail goods, $Y_s^*(z) = \left(\frac{P_t^*}{P_s}\right)^{-\varepsilon} Y_s$. Then, the first order condition implies that

$$\begin{aligned} &E_t\left\{\int_t^\infty e^{-\xi(s-t)}\Lambda_{t,s}(1-\varepsilon)\left(\frac{P_t}{P_s}\right)^{1-\varepsilon} P_t^* Y_s ds\right\} + \\ &E_t\left\{\int_t^\infty e^{-\xi(s-t)}\Lambda_{t,s}\varepsilon\left(\frac{P_s^w}{P_s}\right)\left(\frac{P_t}{P_s}\right)^{-\varepsilon} P_t Y_s ds\right\} = 0. \end{aligned} \tag{34}$$

Now, we assume that the final good price P_t can be aggregated from the retail price $P_t(z)$ along the dimension of time t, instead of along the dimension of the retailer z. Thus, P_t can be rewritten in the following integration:

$$P_t^{1-\varepsilon} = \int_0^t P_s(z)^{1-\varepsilon} de^{-(t-s)\xi}. \tag{35}$$

Thus,

$$\frac{dP_t^{1-\varepsilon}}{dt} = -\xi e^{-\xi t}\int_0^t \xi e^{\xi s} P_s(z)^{1-\varepsilon} ds + e^{-\xi t}\left[\xi e^{\xi t} P_t(z)^{1-\varepsilon} + 0\right] = \xi\left[P_t(z)^{1-\varepsilon} - P_t^{1-\varepsilon}\right]. \tag{36}$$

Also, we have $\frac{dP_t^{1-\varepsilon}}{dt} = (1-\varepsilon)P_t^{-\varepsilon}\frac{dP_t}{dt}$, which implies together with (36) that

$$\xi\left[P_t(z)^{1-\varepsilon} - P_t^{1-\varepsilon}\right] = (1-\varepsilon)P_t^{-\varepsilon}\frac{dP_t}{dt}. \tag{37}$$

Next, we define that the inflation rate π_t is $\pi_t \equiv \frac{dP_t/dt}{P_t}$. Then, from (37), we can get that

$$\pi_t \equiv \frac{dP_t/dt}{P_t} = \frac{\xi}{1-\varepsilon}\left\{\left[\frac{P_t(z)}{P_t}\right]^{1-\varepsilon} - 1\right\}. \tag{38}$$

Note that after the optimal $P_t(z)$ is chosen as P_t^*,

$$\pi_t = \frac{\xi}{1-\varepsilon}\left\{\left[\frac{P_t^*}{P_t}\right]^{1-\varepsilon} - 1\right\}. \tag{39}$$

We can rewrite the first order condition (34) as

$$\frac{P_t^*}{P_t} = \frac{\varepsilon}{\varepsilon-1}\frac{E_t\left\{\int_t^\infty e^{-\xi(s-t)}\Lambda_{t,s}\left(\frac{P_s^w}{P_s}\right)\left(\frac{P_t}{P_s}\right)^{-\varepsilon} Y_s ds\right\}}{E_t\left\{\int_t^\infty e^{-\xi(s-t)}\Lambda_{t,s}\left(\frac{P_t}{P_s}\right)^{1-\varepsilon} Y_s ds\right\}} = \frac{\varepsilon}{\varepsilon-1}\frac{x_{1,t}}{x_{2,t}}, \tag{40}$$

where we define:

$$\begin{aligned} x_{1,t} &\equiv E_t\left\{\int_t^\infty e^{-\xi(s-t)}\Lambda_{t,s}\left(\frac{P_s^w}{P_s}\right)\left(\frac{P_t}{P_s}\right)^{-\varepsilon} Y_s ds\right\} \\ x_{2,t} &\equiv E_t\left\{\int_t^\infty e^{-\xi(s-t)}\Lambda_{t,s}\left(\frac{P_t}{P_s}\right)^{1-\varepsilon} Y_s ds\right\}. \end{aligned} \tag{41}$$

Differentiating $x_{1,t}$ and $x_{2,t}$ with respect to t, respectively to obatin

$$\begin{aligned}\frac{dx_{1,t}}{dt} =& -\left(\frac{P_t^w}{P_t}\right) Y_t \\ &+ \int_t^\infty \left(\frac{P_s^w}{P_s}\right) Y_s \left[e^{-(s-t)\xi}\xi\Lambda_{t,s}\left(\frac{P_t}{P_s}\right)^{-\varepsilon} + e^{-(s-t)\xi}\Lambda_{t,s}\rho\left(\frac{P_t}{P_s}\right)^{-\varepsilon} - e^{-(s-t)\xi}\Lambda_{t,s}\varepsilon\left(\frac{P_t}{P_s}\right)^{-\varepsilon}\pi_t\right] ds,\end{aligned}$$

and some algebra leads to the following,

$$\frac{dx_{1,t}}{dt} = (\xi + \rho - \varepsilon\pi_t)\, x_{1,t} - \left(\frac{P_t^w}{P_t}\right) Y_t \tag{42}$$

and

$$\begin{aligned}\frac{dx_{2,t}}{dt} =& -Y_t \\ &+ \int_t^\infty Y_s \left[e^{-(s-t)\xi}\xi\Lambda_{t,s}\left(\frac{P_t}{P_s}\right)^{1-\varepsilon} + e^{-(s-t)\xi}\Lambda_{t,s}\rho\left(\frac{P_t}{P_s}\right)^{1-\varepsilon} + e^{-(s-t)\xi}\Lambda_{t,s}(1-\varepsilon)\left(\frac{P_t}{P_s}\right)^{1-\varepsilon}\pi_t\right] ds,\end{aligned}$$

so that

$$\frac{dx_{2,t}}{dt} = [\xi + \rho + (1-\varepsilon)\pi_t]\, x_{2,t} - Y_t. \tag{43}$$

By (39), (40) and (41), we have

$$\pi_t = \frac{\xi}{1-\varepsilon}\left[\left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{x_{1,t}}{x_{2,t}}\right)\right]^{1-\varepsilon} - \frac{\xi}{1-\varepsilon}. \tag{44}$$

Differentiating (44) with respect to $\frac{x_{1,t}}{x_{2,t}}$ yields

$$d\pi_t = \xi\left(\frac{\varepsilon}{\varepsilon-1}\right)^{1-\varepsilon}\left(\frac{x_{1,t}}{x_{2,t}}\right)^{-\varepsilon} d\left(\frac{x_{1,t}}{x_{2,t}}\right). \tag{45}$$

Further,

$$d\left(\frac{x_{1,t}}{x_{2,t}}\right) = \left[-\pi_t\left(\frac{x_{1,t}}{x_{2,t}}\right) - \frac{\left(\frac{P_t^w}{P_t}\right) Y_t}{x_{2,t}} + \frac{Y_t x_{1,t}}{(x_{2,t})^2}\right] dt,$$

so that,

$$\begin{aligned}d\pi_t &= \xi\left(\frac{\varepsilon}{\varepsilon-1}\right)^{1-\varepsilon}\left(\frac{x_{1,t}}{x_{2,t}}\right)^{1-\varepsilon}\left[-\pi_t - \frac{\left(\frac{P_t^w}{P_t}\right) Y_t}{x_{1,t}} + \frac{Y_t}{x_{2,t}}\right] dt \\ &= [(1-\varepsilon)\pi_t + \xi]\left[-\pi_t - \frac{\left(\frac{P_t^w}{P_t}\right) Y_t}{x_{1,t}} + \frac{Y_t}{x_{2,t}}\right] dt,\end{aligned} \tag{46}$$

which captures the dynamics of the inflation rate and is the nonlinear version of the New Keynesian Phillips Curve.

Appendix A.3 The Government

The government is assumed to be a monetary-fiscal authority. It adjusts the mix of financing between the money creation M_t and the lump-sum taxes T_t. The government has an expenditure G_t and its budget constraint evolves as:

$$d\left(\frac{M_t}{P_t}\right) = dG_t - dT_t + \left(\frac{M_t}{P_t}\right)(\mu_M - \pi_t)\,dt, \tag{47}$$

Additionally, we define the real money supply as $m_t \equiv \frac{M_t}{P_t}$, and

$$\begin{aligned} &\frac{dM_t}{M_t} = \mu_M dt, \quad \frac{dP_t}{P_t} = \pi_t dt, \\ &G_t = \bar{G}, \quad dT_t = T_t dt \end{aligned} \tag{48}$$

Through the open market operations (adjusting the monetary supply), the monetary policy follows the Taylor's rule given by:

$$dr_t = \left[\chi_\pi\left(\pi_t - \pi_t^*\right) + \chi_y\left(\frac{Y_t}{Y_t^*} - 1\right) - \chi_r\left(r_t - r_t^*\right)\right]dt + \sigma_t^r\sqrt{(r_t - \underline{r})\,(\bar{r} - r_t)}dW_t^r. \tag{49}$$

where the monetary policy rate is increased, when the inflation rate goes up, and is decreased when the policy rate was increased since last period. Here we assume that the monetary policy rate is equal to the riskless deposit interest rate in the bank (r_t).

Appendix A.4 The Resource Constraint

The resource constraint states the point where the supply meets the demand in the economy. In particular, the final output produced is spent on the household consumption, the entrepreneurial consumption, the investment, the government expenditure, and the aggregate monitoring costs:

$$Y_t = C_t + C_t^e + I_t + G_t + \mu\int_0^{\bar{\omega}} \omega dF(\omega) R_t^k Q_t K_t, \tag{50}$$

where C_t^e is the entrepreneurial consumption and $C_t^e = (1-\gamma)V_t$ (the entrepreneurs who fail in t consume the residual equity).

Appendix A.5 The General Equilibrium

At this moment, we do not consider the general equilibrium with the following equations in addition to the capital market equilibrium in section 2.1.4, due to the tractability. However,

we will definitely leave the general equilibrium results for our future research agenda.

$$C_t = \frac{\frac{M_t}{P_t}\pi_t}{\eta_M} = \frac{(1-H_t)\,W_t}{\eta_H}$$

$$d\pi_t = [(1-\varepsilon)\pi_t + \xi]\left[-\pi_t - \frac{\left(\frac{P_t^w}{P_t}\right)Y_t}{x_{1,t}} + \frac{Y_t}{x_{2,t}}\right]dt$$

$$dr_t = \left[\chi_\pi\left(\pi_t - \pi_t^*\right) + \chi_y\left(\frac{Y_t}{Y_t^*} - 1\right) - \chi_r\left(r_t - r_t^*\right)\right]dt + \sigma_t^r\sqrt{(r_t - \underline{r})\,(\bar{r} - r_t)}dW_t^r$$

$$Y_t = C_t + C_t^e + I_t + G_t + \mu\int_0^{\bar{\omega}} \omega dF(\omega) R_t^k Q_t K_t$$

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