Optimal Automation[‡]

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Abstract

This paper studies the joint design of regulation, robot tax, and income tax. We establish a job assignment model with a continuum of sectors micro-founded by task-based automation. The model nicely captures the complex reaction of wages to the robot inputs in the real economy. We then study optimal policies in the spirit of Mirrlees. Our findings are multifaceted. (i) We identify the endogeneity of skill premium to job assignment as a novel reason for the intervention in automation, i.e., non-optimality of production efficiency. (ii) Both robot tax and regulation are desirable. Robot tax operates through the substitution channel and regulation governs the automation channel. (iii) The regulation is concentrated in low- and middle-income sectors, where the robot tax is also in place. For other sectors, a subsidy for robots is desirable. (iv) Under appropriate regulation, robot tax (subsidy), ranging from -2% to 0.7%, is very low. (v) Lastly, with the advancement of technology, intervention phases out gradually, and income tax assumes a more significant redistributive role.

Keywords. Automation; Robot; Regulation; Job Assignment; Optimal Taxation. **JEL**. H21; E24; E64; J31.

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1 Introduction

"*Shared prosperity emerged because, and only when, the direction of technological advances and society's approach to dividing the gains were pushed away from arrangements that primarily served a narrow elite*." (Acemoglu and Johnson (2023). The question is: How to make technological advances universal? Tax and regulation are among the primary measures that humanity takes in dealing with the automation process.[1] Research on policies addressing the above question has largely focused on the robot tax (e.g., see Thuemmel (2018); Guerreiro et al. (2021); Acemoglu et al. (2020)). It is perhaps surprising that notwithstanding their wide implementation and calling for in practice, regulation of automation,[2] let alone the joint design of regulation and tax, receives little normative studies from economists.[3]

This paper formalizes a general equilibrium model capturing the rich effects of technological advances: the substitution, automation, and job assignment effects. It then studies the joint design of taxes and regulation both normatively and quantitatively. By doing so, we answer key questions underlying the centuries of policies in response to the automation process: (*i*) Is production-efficient automation socially optimal? (*ii*) If not, should governments intervene by tax or regulation? Moreover, which tax is desirable, pre-distributive indirect tax or redistributive direct tax? (*iii*) Last, given the tension between efficiency and equality, what's the optimal joint design of tax and regulation?

At the heart of the above matter lies the celebrated theorem of Diamond and Mirrlees (1971), which shows that, when the government can levy differentiated taxes on commodities, the economy should lie at the production-efficient frontier. This result has important implications for the design of tax systems. For instance, it provides an intellectual justification for opposing the taxation of intermediate goods like robots.

One key assumption of the Diamond-Mirrlees theorem is that skills are perfectly transferable across sectors (see, e.g., Gomes et al. (2018)), which is also the assumption of current studies on robot tax

[1]More than four centuries ago, Queen Elizabeth I twice denied patents for the first stocking frame knitting contraption because of concerns about its substitution for poor labor (see Acemoglu and Robinson (2013)). Today, the European Robotics Research Network has identified regulations that need to exist before robots can take off. More currently, the European Union publishes "Guidelines on Regulating Robotics" and funds the RoboLaw project. One of the key areas that these regulations typically cover is the social and economic impact including potential job displacement. As to tax, South Korea has reduced a tax break for investing in automation. The EU considered robot tax in early 2017, but it was rejected across the board. While the tax and regulation aim to help humans, the practical policies send mixed signals —— governments want to develop automatic technology without harming workers. Underlying struggles of policymakers are the questions given at the beginning.

[2]Hundreds of artificial intelligence experts and other notable figures have signed a statement to call attention to AI risk. The the statement says: "*Mitigating the risk of extinction from AI should be a global priority alongside other societal-scale risks such as pandemics and nuclear war.*" One of the concerns is about economic inequality: The benefits of AI could be unevenly distributed, leading to increased economic inequality and social tension.

[3]One excellent exception is Costinot and Werning (2023) (see also Costinot and Werning (2018)). Our paper supplements theirs in two aspects. First, Costinot and Werning (2023) study a general model to develop a sufficient-statistics approach for the policy design, while we provide a structural model micro-founded by task-based automation and job assignment. In doing so, we formalize a framework capturing many effects of technological advances. In addition, we distinguish robot tax from regulation, which is obvious in reality but ambiguous in the theory. Second, Costinot and Werning (2018) studied the regulation of technological progress. In contrast, we study the regulation of the adoption of automation technology, which is a common policy in the real economy. In our model, the regulation of automation means that the automation of certain tasks is forcibly prohibited.

(see, e.g., Guerreiro et al. (2021), Thuemmel (2018)). Without endogenous job assignment and imperfect transferability of skills, some crucial facts are overlooked in the policy design. For instance, while high-ability individuals can adapt to technological shocks by switching occupations, low-ability individuals are much harder to do so. As our results show, this realistic feature has important implications for the non-optimality of production efficiency and policy. Another gap between the reality and research is at the policy level: although taxation has been extensively studied, regulation of automation that distort production has not. The study of regulation has two levels of significance. Firstly, regulatory policies themselves are practical and quite important in the real economy. Yet it is not clear whether the Diamond-Mirrlees theorem also holds true for regulation. Secondly, when there are regulatory policies in place, the design of tax policies may differ from the situation without regulation.

Our theoritical analysis start by establishing a *task-to-talent-and-robot* job assignment model, which incorporates task-based automation (see, e.g., Acemoglu and Restrepo (2018)) into the job assignment model (see, e.g., Sattinger (1975)). In doing so, we provide a biased technology (see, e.g., Krusell et al. (2000)) micro-founded by automation and matching. Specifically, there is a continuum of sectors and talents. Within each sector, there is a contimuum of tasks, which is executed either by robots or talents assigned to the sector. In the end, the substitutability (or complementarity) between robots and talents are determined by the inter-sector substitution between sectoral goods and intra-sector substitution between robots and talents. The intra-sector substitution between robots and talents containts a traditional substitution effect in the intensive margin (substitution effect for short) and a automation effect in the extensive margin (automation effect for short) identfied by Acemoglu and Restrepo (2020). In the extensive margin, robot ruduces tasks performed by humans. In the intensive margin, it reduces labor input to each task. This technology is flexable enough to capture the realistic reactions of wages to robot inputs and differentiated automation rates across departments, which were difficult to be captured simultaneously in previous models.

We then formalize the Mirrleesian tax problem in the task-to-talent-and-robot job assignment model. Our findings are multi-fold. First, optimal automation is generally not production-efficient. Intervention in automation exhibits an indirect redistribution effect (IRE) by narrowing the wage gaps. Second, both regulation of automation and robot tax is desirable. While robot tax exerts influence via substitution effect in the intensive margin, regulation counteracts the influence of automation via job assignment in the extensive margin. Both are effective in indirect redistribution. Last, we identify the endogeneity of relative wages to job assignments (automation) as a novel reason for the intervention in production (automation or robotization).

For the last finding, remember that a key lesson from the fundamental study by Stiglitz (1982) is that IRE should be incorporated into the tax design when labor factors are imperfectly substitutable. While Naito (1999) combines Stiglitz (1982) and Diamond and Mirrlees (1971) show that indirect tax is desirable due to its IRE, Saez (2004) show that indirect tax is not necessary when occupation choice is endogenous. As a response to Saez (2004), Naito (2004) defended by pointing out that the absence of

tax's indirect redistribution effect in Saez (2004) is due to the assumption that labor factors are completely substitutable by each other. Otherwise, indirect-redistributive indirect tax is desirable even if occupation choice is endogenous. Although Naito (2004) considered endogenous job choices, under his setting, \occupation choices are not altered by policies. Our study put an end to the debate between Saez and Naito: Firstly, the endogeneity of job choices does indeed affect the effectiveness of indirect redistributive role of tax. In the sense that imperfectly substitutability between labors is no longer a sufficient condition for the IRE of taxes (contradict Naito (2004)). Secondly, even if occupation choice is endogenous, IRE is considerable when individual productivity is non-separable between the innate ability and the task he is engaged in (contradict Saez (2004)). In conclusion, whether IRE is considerable depends on whether individual productivity is separable between the innate ability and the task he is engaged in. This is because only then, will the relative wage be endogenous to the job assignment. In particular, when individual productivity is separable, the IRE in the labor income tax rule disappears and robot tax, as well as the regulation, are not desirable.

Quantitatively, we estimate parameters representing the comparative advantage of heterogenous agents by the wage distribution and reaction of wages to robot inputs. We find that regulation of automation is concentrated for low - and middle-income sectors. The robot inputs to the low-income and middle-income (high-income) sectors are subsidized (taxed). Under well-designed regulation, the robot tax is quite conservative, which ranges from -2% to 0.7%. Last, though the intervene in automation is desirable, regulation phases out gradually and the robot tax rate decreases with the progress of automatic technology. Meanwhile, redistributive income tax plays a greater role.

Literature Review. The improvement of production efficiency is one of the most important driving forces for social and economic development. There are three ways to promote production efficiency: (i) reducing misallocation; (ii) making technological progress in the intensive margin; and (iii) in the extensive margin. With intensive technological progress, more goods can be produced with given inputs.[4] With extensive technological progress, producing new products becomes possible, which is precisely the role of automation technology: Replacing manually produced products with products produced by machines or robots. The question is whether production efficient automation process is socially optimal. There is a long-standing disagreement in the academic community. As put by David Harvey: "*While Marx clearly admired capitalism's creativity he ... strongly emphasized its self-destructiveness. The Schumpeterians have all along gloried in capitalism's endless creativity while treating the destructiveness as mostly a matter of the normal costs of doing business.*"

In their seminal paper, Diamond and Mirrlees (1971) shows the optimality of the efficiency of allocation. This finding is then revised by a long list of papers (Naito (1999), Saez (2004), Naito (2004)). The literature documents endogenous wages and redistribution through the market as a crucial rea-

[4]In his book published in 1932, The Theory of Wages, J.R. Hicks classified (intensive) technological progress into three categories depending on its effect on the substitution rates of factors: capital-deepening, labor-deepening, and neutral technical progress.

son for sacrificing some production efficiency to improve the results of predistribution (see e.g., Stiglitz (2018)). Recently, studies on optimal robot tax revisited this topic (see e.g., Guerreiro et al. (2021), Thuemmel (2018), Cui et al. (2021), Costinot and Werning (2023)). Two of the most related papers to ours are Costinot and Werning (2023) and Guerreiro et al. (2021).

Costinot and Werning (2023) pioneer a sufficient-statistic welfare analysis of regulation on technological progress when the Mirrleesian tax is in the hands of government. They show that in the second best, indirect redistribution through reallocation by public policies is desirable (even it induces misallocation), while regulation of technological progress in the intensive margin is not desirable. As far as we know, it remains a problem whether maintaining the production efficiency in the extensive margin is desirable. This paper fills this gap. A combination of Costinot and Werning (2018) and the current paper leads to an interesting conclusion: while regulation of technological progress in the intensive margin is not desirable, regulation of the adoption of automatic technology is desirable.

Guerreiro et al. (2021) finds a special case where indirect tax (robot tax) is not desirable: When the level of automation technology is high enough that workers are squeezed into the same department. At this point, the factors provided by the workers are completely replaced, and the relative wages are exogenous to robot inputs. One interpretation of this crucial findings is that once a department should be fully automated, the robot investment in that department should no longer be distorted. Two questions remain. While Guerreiro et al. (2021) demonstrates that the optimal policy depends on the automation rate,[5] they have not yet answered when it's optimal for the robots to replace human beings, i.e., what's the optimal automation, which is the focus of the current paper.[6] Second, it is still unclear whether extensive choices like occupation choice invalidate the indirect redistribution of tax (see e.g., Naito (2004), Saez (2004)).[7] We answer this problem by studying optimal taxation with both imperfectly substitutable factors and endogenous job assignment. Doing so, this study jointly answers the above two questions. We find that as long as labor factors are imperfectly substituted by each other, indirect redistribution matters even if job assignment is choosen by the agents. We also find that regulation or tax is desirable in altering the job assignment. That is slowing down the substitution of robots for low-skilled workers.

There is a rising literature highlights the optimal taxation with endogenous occupation choice (see Scheuer (2014), Ales et al. (2015), Scheuer and Werning (2017), Ales et al. (2015), Thuemmel (2018)). Within this literature, our paper is mostly related to Ales et al. (2015), who studies optimal Mirrleesian income tax in a job assignment model (see also Sachs et al. (2020)). Our model is an extension of the one studied by Ales et al. (2015) in the sense that we introduce an additional factor —— robot as a type of

[5]Robot tax is needed only if laborers working in different sectors so that labor factors are imperfectly substituted by each other. When workers working in the same sector and supply perfectly substitutable factors, robot tax is not desriable.

[6]Tax policy's indirect redistribution effect through its influence on job assignment is highlighted by Ales et al. (2015). Our model is an extension of the one studied by Ales et al. (2015) in the sense that we introduce an additional factor —— robot as a type of "worker".

[7]As pointed by Naito (2004), Saez (2004) abstract the problem from substitution effects. Naito (2004) considered the effect of substitution on relative wages, but missed the effect of occupation choice on relative wages. In Naito (2004)'s two-by-two economy, equilibrium occupation choice is determined by exogenous comparative advantages. Thus, while Naito (2004) introduces occupation choice, the policy design is abstracted from its influences on the occupation choices.

"worker" and incorporate task-based automation. The manufacturer not only considers which kind of workers to employ, but also determines whether to employ workers or not. One interesting finding of this paper is that the micro-foundation of wage elasticity matters for the policy design. While Sachs et al. (2020) show that elasticity of wage with respect to the factor inputs is sufficient for the policy design no matter whether the job assignment is exogenous or endogenous. We show that micro-foundation of production matters for policy design, because of automation effects and substitution effects need to be treated differently, and the endogenity of statistics to the policy.

2 The Model Setup

2.1 Environment

Consider a static economy populated by a continuum of individuals indexed by skill $\theta \in \Theta \equiv [0,1]$, and a continuum of sectors indexed by $z \in \mathbb{Z} \equiv [0,1]$. Within each sector, there is a continuum of tasks $i \in \mathbb{I} \equiv [0,1]$. A sectoral good is an aggregator of intra-sector tasks. There is a unique final good which is an aggregator of sectoral goods. The price of final goods is normalized to be one. Skill θ distributed according to a CDF of $F_\theta(\theta)$ and PDF of $f_\theta(\theta)$ on Θ. Without loss of generality, we normalize the distribution to be uniform and govern the productivity of agents by the technology.

Preference. All individuals have the same utility $u(c,l) = c - \phi(l)$. $u : \mathbb{R}^2 \longmapsto \mathbb{R}$ is a strictly concave, twice continuously differentiable function strictly increasing in the consumption c and decreasing in labor l. Besides, $\phi(\cdot)$ satisfies the Inada conditions $\phi'(0) = 0$ and $\lim_{x\to\infty} \phi'(x) = \infty$.[8] For later use, we denote by $w(\theta), l(\theta), y(\theta) = w(\theta) l(\theta)$ and $c(\theta)$ the wage, labor supply, labor income, and consumption of a θ-skill labor.

We introduce $\vartheta : \mathbb{Z}\times\Theta \longmapsto \mathbb{R}_+$ as the conversion rate between the raw labor of the θ labor and the effective labor input to the z sector. $\vartheta(z,\theta)$ is a twice continuously differentiable nonnegative function. The consumption $c(\theta)$ and wage $w(\theta)$ subject to the following equations:

$$c(\theta) = w(\theta) l(\theta) - T(w(\theta) l(\theta)) \text{ and } w(\theta) = \int_0^1 \vartheta(z,\theta) \nu_\theta(z) w_z(z) dz, \tag{1a}$$

where $w_z(z)$ is the price of effective labor inputs. $\nu_\theta(\cdot)$ is a measurable distribution in $\mathbb{Z}$. $\nu_\theta(\cdot)$ describes the distribution of $l(\theta)$ across sectors. We set $\mathcal{D}(\theta) \equiv \left\{\nu_\theta(\cdot) \mid \int_0^1 \nu_\theta(z) dz = 1, \nu_\theta(z) \geq 0\right\}$ as the feasible set of $\nu_\theta(\cdot)$ for $\theta \in \Theta$.

Technology. Following Acemoglu and Restrepo (2018) and Acemoglu and Restrepo (2020), the final good $Q = F(\mathcal{Q})$ is a CES aggregation of a continuous of sector-specific intermediate goods $\mathcal{Q} =$

[8] This quasi-linear utility function abstracts the problem from the income effect and complementarity between consumption and leisure. This is a common assumption making the key result more transparent. Readers who are interested in these effects can refer to Atkinson and Stiglitz (1976), Mirrlees (1976), and Christiansen (1984).

$\{Q_z(z)\}_{z\in\mathbb{Z}}$. A sector-specific intermediate goods $Q_z(z)$ is a CES aggregation of a continuous of intra-sector tasks $\{Q_i(z)\}_{i\in\mathbb{I}}$. The final goods can either be consumed or used to produce robots. Without loss of generality, we assume that one unit of final goods can be used to produce one unit of robots. Formally, we consider the following technology:

$$\begin{aligned} Q_i(z) &= \kappa_i(z)\,\nu_z^B(i)\,B_z(z) + \iota_i(z)\,\nu_z^L(i)\,L_z(z); \\ Q_z(z) &= \left[\int_0^1 Q_i(z)^{\frac{\rho-1}{\rho}}\,di\right]^{\frac{\rho}{\rho-1}}, 0<\rho\leq\infty; \\ Q &= A\left[\int_0^1 \eta(z)\,Q_z(z)^{\frac{\sigma-1}{\sigma}}\,dz\right]^{\frac{\sigma}{\sigma-1}}, 0<\sigma, A\leq\infty. \end{aligned} \tag{2}$$

$B_z(z)$ and $L_z(z)$ are the total effective inputs to sector z. $\nu_z^B(i)$ and $\nu_z^L(i)$ are the share of factor inputs to task i of sector z. For later use, set $B_i(z)\equiv\nu_z^B(i)\,B_z(z)$ and $L_i(z)\equiv\nu_z^L(i)\,L_z(z)$ as robot and labor inputs to task (z,i). Productivities of robots and labor inputs, i.e., $\kappa_i:\mathbb{Z}\longmapsto\mathbb{R}_+$ and $\iota_i:\mathbb{Z}\longmapsto\mathbb{R}_+$, are continuously differentiated. For later use, set $\kappa(z)=\max_{\mathbb{I}}\kappa_i(z)$ and $\iota(z)=\max_{\mathbb{I}}\iota_i(z)$.[9] To distinguish tasks within sectors, we order i so that:

$$d\left[\frac{\iota_i(z)}{\kappa_i(z)}\right]/di>0, \forall z\in\mathbb{Z}, i\in\mathbb{I}.$$

This means that humans have a stronger comparative advantage in higher-i tasks. $\eta:\mathbb{Z}\longmapsto\mathbb{R}_+$ is a distribution parameter, which is continuously differentiated and satisfying $\int_0^1\eta(z)^\sigma\,dz=1$.[10]

Given prices, tax and regulatory policies, the representative producer chooses inputs $B_i(z)$ and $L_i(z)$ to maximize the profit:

$$Q-\int_{\mathbb{Z}}\left[1+\tau^B(z)\right]\int_{\mathbb{I}^B}B_i(z)\,didz-\int_{\mathbb{Z}}\int_0^1 w_z(z)\,L_i(z)\,didz, \tag{3}$$

where $w_z(z)$ is the price of effective labor inputs to sector z. $\tau^B(z)>-1$ is sector-specific tax on robot. $\mathbb{I}^B$ is the set of intra-sector tasks that are allowed to be automated. To prevent the collapse of equilibrium caused by excessive subsidies, we assume that:

Assumption 1 $\frac{1}{\min\{1+\tau^B(z)\}_{z\in\mathbb{Z}}}<\frac{1}{\kappa}$, *where* κ *is a comprehensive index for the level of automation technology:*[11]

$$\kappa\equiv\left[\int_0^1\eta(z)^\sigma\,\kappa(z)^{\sigma-1}\,dz\right]^{\frac{1}{\sigma-1}}.$$

[9] Notice that $\kappa_i(\cdot)$ and $\iota_i(\cdot)$ are continuously differentiated functions on the compact set, $\kappa(z)$ and $\iota(z)$ are also continuously differentiated functions in $\mathcal{Z}$.

[10] As is illustrated by Ales et al. (2015), variations in $\eta(z)$ captures the technological or preference-based variations in demand for different tasks.

[11] Note that when $\kappa(z)$ is constant $\kappa(z)=\kappa$, because we have normalized $\eta(z)$ by $\int_0^1\eta(z)^\sigma\,dz=1$.

For our baseline case with $\rho \to \infty$, we make Assumption 2 to clearify comparative advantage of humans.

Assumption 2 *Assumption A.5 When $\rho \to \infty$, the after-tax comparative advantage of robot increases in z :*

$$\frac{\partial \ln \widetilde{\kappa}(z;\theta)}{\partial \ln z} > 0, \theta \in \Theta, z \in \mathbb{Z}.$$

where $\widetilde{\kappa}(z;\theta) \equiv \frac{\iota(z)\vartheta(z,\theta)}{\kappa(z)/[1+\tau^B(z)]}$ is the after-tax comparative advantage of robot in task z.

Under Assumption 2, humans have strict comparative advantage over robots in performing high-z tasks.[12]

A Useful Technology. For specific analysis and quantative study, we specify the aggregation of production by (2) with $\rho < \infty$ and model the task-level technology by a power function, as in Chen (2019) and Guerreiro et al. (2021). This *power-function-form* firm-level technology highlights the influence of comparative advantage:

$$\kappa_i(z) = \varsigma(z)\, i^{\frac{\xi(z)-1}{\xi(z)}} \quad \text{and} \quad \iota_i(z) = \varsigma(z)\,(1-i)^{\frac{\xi(z)-1}{\xi(z)}}, i \in [0,1]. \tag{4}$$

Following the literature, we set $\varsigma(z) \equiv \left[1 + \frac{[\xi(z)-1](\rho-1)}{\xi(z)}\right]^{\frac{1}{\rho-1}}$ to normalize technology so that $\eta(z)$ alone determines the relative productivities between sector z (see equation (18)). $\xi(z)$ capture robot's comparative advantage in peroforming sector z. We assume $0 < \xi(z) < 1$ such that $\kappa_i(z)/\iota_i(z)$ strictly decreases in i. Besides, $\xi(z) > \frac{\rho-1}{\rho}$ is used to ensure the economy is well defined (see **??**).

Assumption 3 *For any $z \in \mathbb{Z}$, $0 < \xi(z) < 1$ and $\xi(z) > \frac{\rho-1}{\rho}$.*

Skill and Task. To frame the problem, we need to clearify the property of $\vartheta(\cdot,\cdot)$. To character skills, we assume that high-skill individuals have absolute advantages in performing all tasks. We then character tasks by the relationship between skills and sectoral tasks. That is high-skilled individuals have a comparative advantage in performing high-z tasks. Formally, we make the following Assumption 4:

Assumption 4 *(i) Strict absolute advantages of high-ability workers, i.e., for any $\theta \in \Theta$ and $z \in \mathbb{Z}$, $\frac{\partial \ln \vartheta(z,\theta)}{\partial \theta} > 0$; (ii) Weakly comparative advantages of high-ability worker in high-skill tasks, i.e., for any $\theta \in \Theta$ and $z \in \mathbb{Z}$:*

$$\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial z \partial \theta} \geq 0.$$

Under Assumption 4, $\vartheta(\cdot,\cdot)$ describes the absolute and comparative advantages between humans. Note that it does not necessary means that humans have higher cmparative advantages over robots

[12] Readers can refer to Acemoglu and Loebbing (2022) for a related economy with more general automation process.

in working in sectors with higher z. The inter-sector comparative advantages between different skills, together with the intra-sector comparative advantages between humans and robots, describe the comparative advantages of all humans and robots.

Policies. There are three types of policies:

(i) A direct tax on income y, i.e., $T : \mathbb{R}_+ \longmapsto \mathbb{R}$.

(ii) A sector-specific robot tax $\tau^B(z)$ on robot input $B(z)$ to sector z.

(iii) A strictly monotonic regulation is a set $\mathcal{R} = \{(z,i) \,|i \geq a_z^\star(z), \forall z \in \mathbb{Z}\}$, where $a_z^\star : \mathbb{Z} \longmapsto (0,1)$ is the cutoff of intra-sector regulation. Any task with $i < a_z^\star(z)$ in sector z is automated, while others are manual. When $a_z^\star(z) = 0$, no automation in sector z is allowed. When $a_z^\star(z) = 1$, sector z is fully automated.

Note that we consider a monotonous regulation that can be described by a cutoff function $a_z^\star : \mathbb{Z} \longmapsto (0,1)$. This consideration is rationalized by the symmetry of intra-sector tasks and the comparative advantage of labor on high i tasks. We will come back to this later.[13]

To facilitate the analysis, we assume that $\tau^B : \mathbb{Z} \longmapsto \mathbb{R}$, and $a_z^\star : \mathbb{Z} \longmapsto (0,1)$ are twice continuously differentiable. We assume that $T(\cdot)$ is twice continuously differentiable and $T'(\cdot) < 1$. As is demonstrated by Mirrlees 1971, there is no need to enforce an income tax rate higher than one. We also assume that the tax schedules are not too regressive so that first-order conditions are both necessary and sufficient to the agents' problems, which is a common assumption in the optimal tax literature (see, e.g., Jacquet et al. (2013)).[14]

For later use, we set $\mathcal{D}_L(z) \equiv \left\{ v_z^L(\cdot) \,|\, \int_0^1 v_z^L(i)\, di = 1, v_z^L(i) \geq 0 \right\}$ and $\mathcal{D}_B(z) \equiv \left\{ v_z^B(\cdot) \,|\, \int_0^{a_z^\star(z)} v_z^B(i)\, di = 1, v_z^B(i) \right.$ as the feasible set of distributions of factors. When $a_z^\star(z) = 0$, automation is forbidden in sector z, and one has $\mathcal{D}_B(z) = \emptyset$.

Baseline Case. An important baseline is $\rho \to \infty$. In this case, intra-sector tasks are perfectly substitutable by each other, and the nested-CES technology is reduced to a CES technology. The strictly monotonic regulation is a cutoff $z^\star \in [0,1)$ such that for any $z \leq z^\star$ tasks are automated, and for any

[13] Some readers may have doubts that as a form of regulation, a certain level of automation should be set so that the automation level of decentralization cannot be higher than this level. However, there is still a possibility that the automation level of decentralization may spontaneously fall below this level. This article considers a more stringent scenario (which can be referred to as strictly monotonous regulation), where all tasks below the level of regulation must be automated, while those above must be manual. This is because, in specific circumstances, any non-strictly monotonous regulated situation can be achieved through strictly monotonous regulation. For example, in the following situation: the optimal automation is monotonic, that is, there exists a certain segmentation point where tasks below this level should be automated and tasks above this level should be manually operated. Under specific technologies, this is indeed the case.

[14] Formally, denote by $l(w)$ as the labor supply of an agent whose wage rate is w. Define $\phi_*(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ by $\phi_*'(l(w)) = w$ and set $\phi_*(0) = 0$. $\phi_*(\cdot)$ is the disutility function modified by the tax. We assume that $\phi_*(\cdot)$ is strictly convex so that the FOCs are necessary and sufficient.

$z > z^\star$, tasks are manual. This case abstracts the analysis from intra-sector substitution between factors and highlights the effect of job assignment and automation. We will come back to this in section 3.3.

Objective of Government. The government chooses a feasible set of policies to maximize the social welfare $\int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta$, subject to incentive conditions, a budget constraint that is clarified below. The PDF $\widetilde{f}_\theta(\cdot)$ is a continuous schedule of Pareto weights, which helps to characterize all Pareto-constrained efficient allocations.[15] Set $\widetilde{F}(\theta) = \int_{\underline{\theta}}^{\theta} \widetilde{f}_\theta(x)\,dx$. We consider $\widetilde{F}(\theta) \geq F(\theta)$ for any $\theta \in \Theta$, which captures the preference for equality.

2.2 Equilibrium

Agent's Problem. Workers choose labor supply and distribution of labor to maximize their gross utilities subject to (1a):

$$V(\theta) = \max_{l(\theta),\{\nu_\theta(z)\}_{\mathbb{Z}}} c(\theta) - \phi(l(\theta)), \tag{5}$$

subject to $\nu_\theta(z) \in \mathcal{D}(\theta)$ and budget constraint (1a).

The representative producer's problem can be split into two parts. The final good producer maximizes profit given prizes $\{p(z)\}_{\mathbb{Z}}$:

$$\max_{\{Q(z)\}_{\mathbb{Z}}} Q - \int_0^1 p(z)\,Q_z(z)\,dz, \tag{6}$$

Intermediate goods producers choose total inputs $L_z(z) \geq 0$, $B_z(z) \geq 0$, and the distribution of factors $\left\{\nu_z^B(i), \nu_z^L(i)\right\}_{\mathbb{I}}$ to maximize the profit:

$$\max_{L_z(z),B_z(z),\{\nu_z^B(i),\nu_z^L(i)\}_{\mathbb{I}}} p(z)\,Q_z(z) - \left\{\left[1+\tau^B(z)\right]B_z(z) + w_z(z)\,L_z(z)\right\}, \tag{7}$$

subject to $\nu_z^L(i) \in \mathcal{D}_L(z)$ and $\nu_z^B(i) \in \mathcal{D}_B(z)$.

Market Clear. In equilibrium, the markets for labor, intermediate goods, and final goods are cleared. The labor market clear condition is:

$$L_z(z) = \int_0^1 \nu_\theta(z)\,\vartheta(z,\theta)\,L(\theta)\,d\theta, \tag{8}$$

where $L(\theta) = l(\theta)\,f_\theta(\theta)$ is the total raw labor supplied by θ-ability laborers.

Commodity market clearing requires that the quantity demanded in the output sector $Q_z^D(z)$ from problem (6) equals the quantity supplied $Q_z^S(z)$ from problem (7):

$$Q_z^D(z) = Q_z^S(z). \tag{9}$$

[15] See Saez and Stantcheva (2016) for generalized social welfare weights and Pareto weights.

The final goods market clear condition is:

$$\int_0^1 c(\theta) f(\theta) d\theta + \int_0^1 B_z(z) dz + G = Q, \tag{10}$$

where G is an exogenously given tax revenue.

Equilibrium. An allocation $\mathcal{A} \equiv \{B_z(\cdot), L_z(\cdot); c(\cdot), l(\cdot)\}$ is a set of measurable functions. A distribution of allocation $\mathcal{D} \equiv \left\{\{\nu_\theta(\cdot)\}_\Theta, \left\{\nu_z^B(\cdot), \nu_z^L(\cdot)\right\}_\mathbb{Z}\right\}$ is a set of measurable functions describing the distribution of factors among tasks. A price system $\{p(\cdot), w_z(\cdot)\}$ is a set of measurable functions describing the prices and wages. A policy system $\mathcal{T} \equiv \left\{\tau^B(\cdot); T(\cdot)\right\}$ and $\mathcal{R} = \{(z,i) | i \geq a_z(z), \forall z \in \mathbb{Z}\}$ is a set of measurable functions describing the tax system and a set of regulated sectoral tasks. As mentioned before, we suppose the allocation is interior. Throughout this paper, we consider the following equilibrium:

Definition 1 *Given G, a Tax-regulation Competitive Equilibrium (TRCE) is a combination of $\mathcal{A}$, $\mathcal{D}$, $\mathcal{P}$, $\mathcal{T}$, and $\mathcal{R}$ such that given $\mathcal{T}$, $\mathcal{R}$, and $\mathcal{P}$ the resulting allocation $\mathcal{A}$ and distribution of allocation $\mathcal{D}$ maximize the producers' profits (6) and (7); the labours' gross utility (5) subject to the budget constraint (1a); factors and goods markets are cleared, i.e., (8) to (10) are satisfied.*

By Walras's law, as long as endowment and output are limited, the government's budget constraint will also be satisfied under the market clear conditions and agnets' budget constraints.

3 Model Analysis

3.1 Equilibrium

Existence and Uniqueness of Equilibrium. To establish the foundation for optimal policy analysis. We need to first solve the equilibrium and analyze the existence and uniqueness of equilibrium. We demonstrate that under Assumptions 1 and 4, there exists a unique TRCE.

Proposition 1 *Under Assumptions 1 and 4, there exist a unique TRCE.*

Proof. See Appendix A.1. ■

Proposition 1 contributes to the literature on job assignment. In an economy incorporating automation, tax, and job assignment, we demonstrate when will there be a unique equilibrium.

Positive Assortative Matching. Before starting to analyze the optimal policy, we need to understand the properties of equilibrium. This helps to understand whether and how public policies can play a role. We first find that the job assignment in the TRCE is positive assortative matching.

In the baseline case:

Proposition 2 *For benchmark case, i.e., $\rho = \infty$, under Assumptions 1 to 2, there exist a job assignment $z(\theta) : \Theta \to [z^\star, 1]$ so that workers of type θ are positively matched with tasks z, i.e., $z(\theta)$ strictly monotonously increases in θ.*

Proof. See Appendix A.2 ■

In the generalized model (extension):

Proposition 3 *When $\rho < \infty$, under Assumptions 1 and 4, there exist a matching function $z(\theta) : \Theta \to \mathbb{Z}$, such that $z(\theta)$ strictly monotonously increases in θ.*

Proof. See Appendix A.3 ■

The above proposition indicates that, even with tax and regulatory interventions, job assignment will still be a positive assortative match. This finding is important because it suggests that our policies cannot change the order of wages. Besides, this discovery help us to simplify the policy analysis. That is we can focus on the positive assortative matching. Now the labor market clear condition (8) can be written as:

$$\int_0^{z(\theta)} L_z(z)\, dz = \int_0^{\theta} \vartheta(z,\theta)\, L(\theta)\, d\theta, \tag{11}$$

or equivalently,

$$\vartheta(z(\theta),\theta)\, L(\theta) = L_z(z(\theta))\, z'(\theta), \forall \theta \in \Theta. \tag{12}$$

Besides, we can set $a(\theta) = a_z(z(\theta))$ as the automation rate of the sector where θ-skilled individual working for.

Wage Gaps. We find that tax can not narrow the wage gap, i.e., $\frac{w'(\theta)}{w(\theta)}$, unless it changes job assignments. Solving the equilibrium, we have

Proposition 4 *The skill gap $\frac{w'(\theta)}{w(\theta)}$ in the equilibrium is determined by the job assignment. Specifically, it satisfies:*

$$\frac{w'(\theta)}{w(\theta)} = \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}, \forall \theta \in \Theta. \tag{13}$$

Proof. See the proof of Propositions 1 and 2.[16] ■

The above finding demonstrates that when occupation choice is free, wage gap is determined by the innate ability captured by $\vartheta(z,\theta)$ and endogenous job assignment $z(\theta)$. Therefore, progress of automation technology, i.e., κ, affect the wage inequality through its effect on job assignment. Correspondingly, policies can and can only adjust the relative wages and primary distribution through it's influence on the job assignment. One way to do this is the regulation of automation.

[16] See also the problem 7. The problem implies $w'(\theta) = \frac{\partial \vartheta(z(\theta),\theta)}{\partial \theta} p(z(\theta))$, which together with $w(\theta) = \vartheta(z(\theta),\theta)\, p(z(\theta))$ gives the above result.

Notably, the wage gap depends on the job assignment only if there is a complementarity or substitutability between the tasks and skills. In conclusion, we have the following result: Technology progress and public policies affect wage inequality only if it changes job assignment and there is a complementarity or substitutability between the tasks and skills, i.e., $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial z \partial \theta} > 0$.

Till now, without solving for the optimal policy, the tax-regulation competitive equilibrium already provides an interesting prediction. First, the job assignment is positive assortative regardless of whether there is policy intervention or not. Second, policies can adjust the skill gap through and only through its influence on the job assignment.

3.2 Decentralized Economy

This section provides the solution to the equilibrium. We consider *nested-CES* technology (2) with *power-function-form* firm-level technology (4). The detailed derivation process can be found in Appendix A.4.

Producer's Problem. By Proposition 3, we can rewrite the producer's problem as:

$$\max_{B_z(z),L_z(z)} \int_{\mathbb{Z}} p(z) Q_z^u \left(B_z(z), L_z(z)\right) dz - \int_{\mathbb{Z}} \left[1+\tau^B(z)\right] B_z(z)\, dz - \int_{\mathbb{Z}} w_z(z) L_z(z)\, dz \tag{14}$$

where

$$Q_z^u \left(B_z(z), L_z(z)\right) = \max_{a_z^u(z)\in[0,1]} Q_z \left(B_z(z), L_z(z); a_z^u(z)\right) \tag{15}$$

and

$$Q_z \left(B_z(z), L_z(z); a_z(z)\right) \equiv \max_{B_i(z),L_i(z)} \left\{ \begin{array}{c} \left[\int_0^1 Q_i(z)^{\frac{\rho-1}{\rho}} di\right]^{\frac{\rho}{\rho-1}} \mid \\ B_z(z) = \int_0^{a_z(z)} B_i(z)\, di, L_z(z) = \int_{a_z(z)}^1 L_i(z)\, di. \end{array} \right\} \tag{16}$$

One can solve the problem by backward induction. Given sectoral factor inputs $B_z(z), L_z(z)$ and the automation rate, the firm first chooses intra-sector factor inputs $B_i(z)$ and $L_i(z)$ to maxmize the output of intermediate goods (see problem (16)). It then chooses the automation rate to maximize the output given sectoral factor inputs (see problem (15)). Last, the producer chooses sectoral factor inputs $B_z(z), L_z(z)$ to maximize the profit (see problem (14)). We use the superscript "u" to indicate solution without regulation.

Aggregated Production Function. Solving the firm's problem (16), we have:[17]

$$Q_z \left(B_z(z), L_z(z); a_z(z)\right) = \left[a_z(z)^{\zeta(z)} B_z(z)^{\frac{\rho-1}{\rho}} + \left[1-a_z(z)\right]^{\zeta(z)} L_z(z)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}, \tag{17}$$

[17] See Appendix A.4.1 for the derivation of (**??**) and (**??**).

where $\zeta(z) = 1 - \frac{1}{\xi(z)}\frac{\rho-1}{\rho}$. The optimal automation $a_z^u(z)$ is a function of $(z, B_z(z), L_z(z))$:

$$a_z^u(z) = a_z^u(z, B_z(z), L_z(z)).$$

Solving problem (15), we have:

$$Q_z^u(B_z(z), L_z(z)) = \left[B_z(z)^{\xi(z)} + L_z(z)^{\xi(z)}\right]^{\frac{1}{\xi(z)}}, a_z^u(z) = \frac{1}{1 + \left[\frac{L_z(z)}{B_z(z)}\right]^{\xi(z)}}. \tag{18}$$

Given the aggregate factor inputs $B_z^u(z)$ and $L_z^u(z)$, the spontaneous automation rate $a_z^u(z)$ increases in the comparative advantage ($\xi(z)$) of robot. The elasticity of output with respect to the factor inputs can be expressed in terms of the automation rate: $\frac{\partial \ln Q_z^u(B_z(z),L_z(z))}{\partial \ln B_z(z)} = a_z^u(z)$ and $\frac{\partial \ln Q_z^u(B_z(z),L_z(z))}{\partial \ln L_z(z)} = 1 - a_z^u(z)$. Therefore, one has $a_z^u(z) = \frac{1}{1+\frac{\partial \ln Q_z^u(z)}{\partial \ln L_z(z)} / \frac{\partial \ln Q_z^u(z)}{\partial \ln B_z(z)}}$.

Price and Wage. The competitive final goods market prices satisfy the following conditions for any $(z, \theta) \in \mathbb{Z} \times \Theta$:[18]

$$p(z) = \frac{\partial Q}{\partial Q_z(z)}, 1 + \tau^B(z) = p(z)\frac{\partial Q_z(B_z(z), L_z(z); a_z(z))}{\partial B_z(z)}, \tag{19}$$

and

$$w_z(z) = \frac{\partial Q_z(B_z(z), L_z(z); a_z(z))}{\partial L_z(z)}\frac{\partial Q}{\partial Q_z(z)}, z \in \mathbb{Z}, \tag{20}$$

where $a_z(z) = a_z^u(z, B_z(z), L_z(z))$ when the automation rate is unregulated.

Wage Elasticity. Notice that the derivative of $Q_z(B_z(z), L_z(z); a_z(z))$ with respect to $a_z(z)$ is zero when $a_z(z)$ is chosen by the firm. In Appendix A.4, we formally define the wage elasticities following the literature (see e.g., Sachs et al. (2020)). Set $\gamma_{own}^{B,u}(z)$ as the wage-robot elasticity with unregulated automation. We have the following result for any $z \in \mathbb{Z}$:

$$\gamma_{own}^{B,u}(z) = \overbrace{\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)a_z^u(z)}^{Substition\ Effect} + \overbrace{\left[\frac{\rho-1}{\rho} - \xi(z)\right]a_z^u(z)}^{Automation\ Effect}, z \in \mathbb{Z}. \tag{21}$$

$\gamma_{own}^{B,u}(z)$ combines a traditional substitution part and an automation part, which is crucial for capturing the complex effect of robots on wages. If there is only a substitution effect, the wage elasticity is either positive or negative. Later in section 6, we calibrate $\xi(z)$ to fit the empirical wage-robot elasticity (see e.g., Acemoglu and Restrepo (2020)), which ranges from -13% to 2%.

[18] See equation (55) for the details of the price function.

Wage Elasticities with Regulation. The elasticity with and without regulation is quite different. When automation rate is regulated by the government, the wage-robot elasticity is

$$\gamma_{own}^{B}(z) = \frac{\overbrace{\frac{1}{\rho}-\frac{1}{\sigma}}^{Substition\ Effect}}{1+\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}\left(\frac{1-a_z(z)}{a_z(z)}\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}}, z \in \mathbb{Z}. \tag{22}$$

The sign of substitution effect is determined by $\rho - \sigma$. If the elasticity of substitution between intra-sector tasks (ρ) are larger than that between the sector-specific tasks (σ), the substition effect is negative, i.e., $\gamma_{own}^{B}(z) < 0$. Intuitively, the higher the elasticity of substitution between intra-sector tasks performed by labors and robots, the faster the sectoral wage decreases in the sectoral robot inputs. This decrease of wage rate will be weakened by the non-substitutability of sectoral goods, because as output decreases, prices will increase. As to the automation effect, notice that Assumption 3 requires $\frac{\rho-1}{\rho} - \xi(z) < 0$. Therefore, the automation effect is always negative (no matter $\rho > \sigma$ or not).

The wage-robot elasticity with and without regulation is quite different. Notice that $\gamma_{own}^{B}(z)|_{a_z(z)=a_z^u(z)} = \left(\frac{1}{\rho}-\frac{1}{\sigma}\right)a_z^u(z)$. Combination of (22) and (21) gives:

$$\gamma_{own}^{B,u}(z) = \overbrace{\gamma_{own}^{B}(z)|_{a_z(z)=a_z^u(z)}}^{Substition\ Effect} + Automation\ Effect$$

Modeling the automation is crucial for policy design. First, without modeling the automation, one can not study the optimal regulation of automation. Secondly, since the wage elasticity depends on the policy ($\gamma_{own}^{B}(z) \neq \gamma_{own}^{B,u}(z)$), one needs the micro-foundation to estimate the wage elasticity desired. For example, the economy without regulation features $\gamma_{own}^{B,u}$. However, when one designs optimal tax and regulation, γ_{own}^{B} is needed, which can not directly be derived by empirical analysis. Last, the level of wage elasticity depends on the micro foundation. When there is no endogenous automation so that only substitution effect appears in the wage elasticity, the wage elasticity is positive if $\rho < \sigma$. However, when automation is endogenous, $\gamma_{own}^{B,u}(z)$ can be negative even if $\rho < \sigma$.

3.3 Two Cases for Analysis: Baseline and Extension

In the following analysis, we will split the original problem into two cases to highlight the key to policy design. In the benchmark case, $\rho \to \infty$ so that each sector is either fully automated or manual. In the extension, $0 < \rho < \infty$ so that partially automated sectors are in place. We describe these two cases in the figure below:

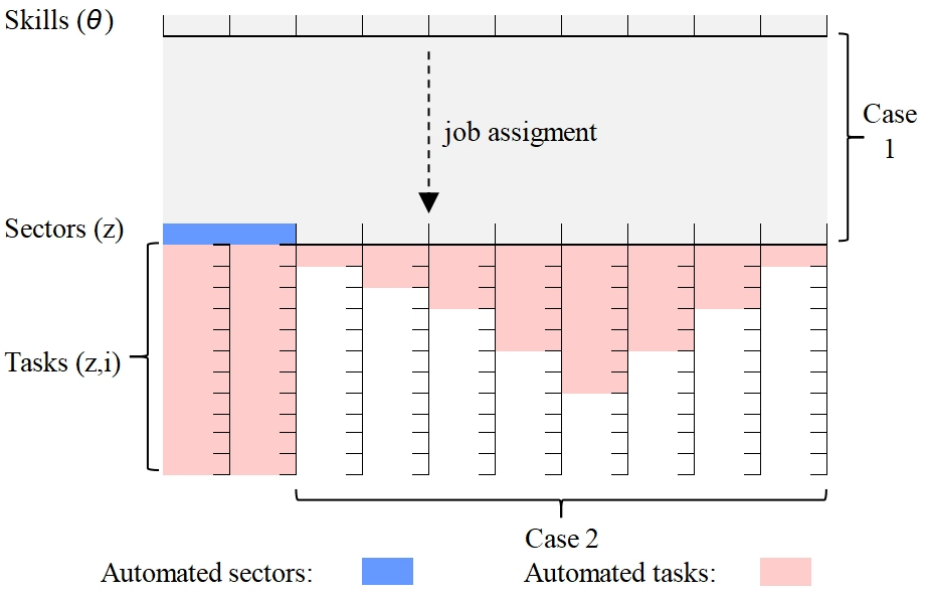


Figure 1: Two Cases

Case 1: Baseline. In the benchmark case, $\rho \to \infty$. The regulation is monotonous: there is a cutoff $z^\star$ such that for any $z \leq z^\star$ tasks are automated and for any $z > z^\star$ tasks are mannual. Assumption 2 rationalizes the monotonous inter-section regulation in the benchmark case. Under Assumption 2, human beings have strict comparative advantage over robots in performing high-z tasks.[19]

In this case, intra-sector tasks are perfectly substituted by each other. The the production function of the benchmark case can be simplified as:

$$Q = A\left[\int_0^{z^\star} \eta(z)\left[\kappa(z) B_z(z)\right]^{\frac{\sigma-1}{\sigma}} dz + \int_{z^\star}^1 \eta(z)\left[\iota(z) L_z(z)\right]^{\frac{\sigma-1}{\sigma}} dz\right]^{\frac{\sigma}{\sigma-1}},$$

because $Q_z(z) = \kappa(z) B_z(z)$ for $z \in [0, z^\star]$ and $Q_z(z) = \iota(z) L_z(z)$ for $z \in [z^\star, 1]$.

We further simplify unnecessary settings to make the key impact channels (salary matching and automation) clearer. As $\vartheta(z, \theta)$ governs the comparative advantage of labors, without loss of generality, we set $\iota(\cdot) = 1$ and let $\kappa(\cdot)$ capture the absolute advantage of robots in the benchmark. Moreover, to make the benchmark analysis transparent, we consider constant $\kappa(\cdot)$, i.e., $\kappa(\cdot) = \kappa \in \mathbb{R}_+$. In the extension, we will consider more general $\kappa(\cdot)$ and $\iota(\cdot)$.

Case 2: Extension. In the benchmark case, a sector is either fully automated or fully manual ($a_z(z)$ is either one or zero). In the extended case, where ρ is finite, and intra-sector tasks are imperfectly

[19] In the proof of Proposition 2, we demonstrate that under Assumptions 4 to 2, even the regulation is non-strict, there exists a cutoff $z^\star$ such that sectors with $z \leq z^\star$ are all automatized. Readers can refer to Acemoglu & Loebbing (2022) for a related economy where there are two cut-offs. Acemoglu & Loebbing (2022) can be treated as a generalization of our baseline economy. Our model extension can be treated as a more generalized economy.

substitutable. In this case, intra-sector automation is generally partial, where labor and robots work together to produce a sector-specific intermediate good. Under the monotonous regulation within the sector z, we have

$$Q_z(z) = \left[\int_0^{a_z(z)} \left[\kappa_i(z) B_i(z) \right]^{\frac{\rho-1}{\rho}} di + \int_{a_z(z)}^1 \left[\iota_i(z) L_i(z) \right]^{\frac{\rho-1}{\rho}} di \right]^{\frac{\rho}{\rho-1}}$$

and $Q = A \left[\int_0^1 \eta(z) Q_z(z)^{\frac{\sigma-1}{\sigma}} dz \right]^{\frac{\sigma}{\sigma-1}}$. This technology generalizes Krusell et al. (2000)'s capital-skill-complementarity technology by introducing a continuum of sectors and heterogenous labors; Acmonglu & Lobbeing (2024) by allowing intra-sector partial automation.

Regulation of automation refers to manipulating z_R in the benchmark case and $a_z(z)$ in the model extension by executive order. In both cases, regulation of automation means manipulating the job assignment directly. Alternatively, the govenment can adjust the automation rate by taxes. Note that we consider the regulation of robotic application instead of robotic technological progress (regulate the progress of $\kappa(z)$).[20]

Remarks. We consider the two cases to study both (*i*) partial and full automation of sectors; (*ii*) skill- and non-skill-neutral robots. To clarify the results, we consider full automation of the sector in the baseline and partial automation in the extension. That is, in the extension, we assume that all sectors are partially automated. Nevertheless, when $a_z^\star(z) \to 1$, partial automation case in the extension approaches the full automation case in the baseline.

Robots in baseline case is *skill-neutral* while it is *non-skill-neutral* in the extension. In the benchmark, robot inputs affect the marginal productivity of laborers as a whole. It does not affect the relative marginal productivities of labor inputs. Therefore, the benchmark case abstracts the analysis from the *substition effect* of robots and focuses on the *automation effect* or robots. Comparison between optimal policies under the benchmark and extension demonstrates the influence of substitution effect.

We focus on monotonic regulation. Notice the symmetricity of tasks across i, and the comparative advantage of humans in higher-i tasks, it's easy to see that monotonic regulation is optimal. (*i*) In the baseline, a monotonic regulation is a cut-off $z^\star \in [0,1)$, such that all sectors with $z \geq z^\star$ can not be automated. (*ii*) In the model extension, we consider monotonous intra-sector regulatory policy: there exists a cutoff $a_z^\star(z) \in [0,1]$ for any $z \in [0,1]$, such that all tasks with $i \geq a_z^\star(z)$ is manual. Note that regulation is monotonous within a sector does not mean that $a_z^\star(z)$ is a monotonic function of z. That is to say that the automation rate is not required to be monotonic in sector z, which is important in calibrating the model to the real economy.

In conclusion, the baseline is a nice starting point to get an intuition about the joint design of regulation and tax; and the extension provides a general analysis that supplements the baseline. By studying

[20] See Costinot and Werning (2018) for an excellent study on the regulation of technological process

these two cases, we provide both a transparent and comprehensive study of optimal automation.

4 Baseline

In this section, we study the policy design under the benchmark case. This model abstracts the problem from the intra-sector substitution effect between robots and labor. Moreover, automation is incorporated into the job assignment. As a result, this part highlights the influence of endogenous job assignment on optimal policies. Note that we have shown the rationale for considering assortative matching. In the following analysis, we will focus on policy design with assortative assignment.

4.1 Planning Problem

The planner's problem can be treated in several different ways. In the the heuristic argument that follows, the planner adopts a feasible direct truthtelling mechanism to implement the constrained optimal allocations that maximizes social welfare under the resource constraint and policies constraints.[21]

Under the direct truthful mechanism, an agent who reports θ should earn $y(\theta)$ units of labor income and self-selected into to $z(\theta)$ sector. After finishing the task, the labor will receive $c(\theta)$ units of consumption as a reward. In the following analysis, we clearify the machanism design problem considered here and and show the implimentability of the constrained optimal allocation in the tax-regulation equilibrium. After the machanism design problme is solved, we backout optimal policies by the tax-regulation equilibrium.

Incentive Conditions. As in the traditional Mirrleesian tax problem, labours are atomistic and treat the prices as given. Unlike the traditional case, given the prices labours in our model choose tasks to perform, which determines their wages. The labor's incentive problem can be split into two parts.

In the extensive margin, the labor chooses department to maximize its wage:[22]

$$w(\theta) = \max_{z \in [z^\star, 1]} \vartheta(z, \theta) w_z(z). \tag{23}$$

In the intensive margin, the labor chooses labor supply to maximize their utilities:

$$V(\theta) = \max_{\theta' \in \Theta} c(\theta') - \phi\left(\frac{y(\theta')}{w(\theta)}\right). \tag{24}$$

[21] Remember that following the literature, we assume that sector-specific tax on labor inputs is not enforceable (see e.g., Naito (1999); Ales et al (2014); Sachs et al (2020); JME, Rebelo et al (2020), Thummel ..), becasue it is costly to observe the real sector where labors work in. To adjust wage rates, the government has to resort to the income tax, robot taxes, as well as the regulation of automation. Nevertheless, we will discuss the case where sector-specific tax on labor factors are enforceable in Section ...

[22] In our tax equilibrium, the agent's problem is separable in the sense that no matter how much effort to be supplied, it's optimal to choose the sector that maximize the wage rate. Here, we also consider a separable truth-telling problem. We will show that the optimal allocation can be implemented in a tax equilibrium.

As is familiar from the literature, by envelope theorem, the first-order incentive condition in the intensive margin is:

$$V'(\theta) = \frac{w'(\theta)}{w(\theta)} l(\theta) \phi' \left(l \left(\theta \right) \right), \forall \theta \in \Theta. \tag{Intensive IC}$$

The first-order incentive condition in the extensive margin is:

$$\frac{d \ln \left[\vartheta \left(z \left(\theta \right), \theta \right) w_z \left(z \left(\theta \right) \right) \right]}{d\theta} = \frac{\partial \ln \vartheta \left(z \left(\theta \right), \theta \right)}{\partial \theta}, \forall \theta \in \Theta, \tag{Extensive IC}$$

or equivalently (13). The following Lemma 1 show that under common assumptions, conditions (Intensive IC) and (Extensive IC) are not only a necessary but also a sufficient condition to the agent's problem.

Lemma 1 *Under Assumption 4, the agent's problem implies the following monotonicity:*

(i) Skill gap $\frac{w'(\theta)}{w(\theta)}$ *must be non-negative;*

(ii) The second-order necessary condition of agent's intensive problem (24) requires that $y' \left(\theta \right) \geq 0$, $\forall \theta \in \Theta$. *Moreover, when for any* $\theta \in \Theta$, $y' \left(\theta \right) \geq 0$, *the local optimum of the intensive problem (24) is also global optimum.*

(iii) The second-order necessary condition of agent's extensive problem (23) requires that $z' \left(\theta \right) \geq 0$, $\forall \theta \in \Theta$. *Moreover, when for any* $\theta \in \Theta$, $z' \left(\theta \right) \geq 0$, *the local optimum of the extensive problem (23) is also global optimum.*

Proof. See Appendix A.1. ■

Part one of Lemma 1 is derived by envelop theory. It shows, due to the agent's extensive choice of sector and the high abilities' absolute advantages in producing goods, wages increase strictly in the ability. Notice that the wage rate increases in θ, traditional Mirrlees tax problem immediately implies part two of Lemma 1 (see e.g., Salanié (2003), p. 87). Analogously, we have part three of Lemma 1. It suggests that incentive compatible matching between the skills and secors are positive no matter what's the tax policies. The comparative advantage of high abilities in working in high-complexity sectors is responsible for this positive matching.

Lemma 1 suggest that the first-order approach is available under some monotonicity assumptions ($y' \left(\theta \right), z' \left(\theta \right) \geq 0$). We make the following Assumption 5 to decipline the planning problem:

Assumption 5 $y(\theta)$ *and* $z(\theta)$ *are strictly positive and strictly increasing in* θ, $\forall \theta \in \Theta$.

As is familar from the optimal tax and machanism design literature, the Spence-Mirrlees condition implies non-decreasing labor income in wage.[23] For simplicity, we assume that $y \left(\theta \right)$ strictly increases in θ. Analogously and notably, our Proposition **??** show that in any TRCE, $z(\cdot)$ is strictly monotonic under the supermodel condition. In conclusion, the supermodel condition together with Spence-Mirrlees

[23] See e.g., see **?**, p. 87.

condition guarantees the validity of the first-order approach in machanism design problem with job assignment. Assumption 5 also exclude cases with mass points. This assumption can be checked once the planning problem is solved.

Policy Constraint. A policy constraint should be enforced when sector-specific taxes on labor inputs are not free. That is

$$w(\theta) = \vartheta(z(\theta), \theta) w_z(z(\theta)), \forall \theta \in \Theta, \tag{25}$$

where

$$w_z(z) = \frac{\partial Q_z(z)}{\partial L_z(z)} p(z) \text{ and } p(z) = \frac{\partial Q}{\partial Q_z(z)}.$$

This policy constraint builds a relationship between the wage of skill and price of task with the innate ability ϑ as a wedge. This constraint together with the incentive condition in the extensive margin (13) implies that how innate ability, job assignment, as well as price allocations determining the wage gap.

Planning Problem. The planner chooses $\{V(\theta), l(\theta), z(\theta), \delta(\theta), B_z(z)\}_{\theta\in\Theta, z\in\mathbb{Z}^B}$ to maximize the social welfare $\int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta$, subject to the market clear conditions (10) to (12), incentive compatible constraints (Intensive IC) and (Extensive IC), policy constraint (25), and boundary constraint $z(\overline{\theta}) = 1$. $\delta(\theta) = z'(\theta)$ is a control variable. Note that the incentive conditions in the extensive margin (13) are not enforced because the relative wages are manipulatable by the sector-specific taxes. In Appendix A.2, we prove the implementability of the constrained optimal allocation in the tax-regulation equilibrium.

4.2 Main Results

Production Efficient Automation: A Baseline. Before studying the social optimal automation level, we investigate in the production efficient automation and robotization, which provides useful contrasts for the optimal automation and robotization. There are two ways to show the Production Efficient results. One is investigate in the Laissez-faire economy and another is studying the planning problem without concern for inequality. We here take the facility of the second way.

Definition 2 *Given labor inputs $\{L_z(z)\}_{z\in[z(\underline{\theta}),1]}$, we say the job assignment $\{z(\theta)\}_{\theta\in\Theta}$ and robot input $\{B_z(z)\}_{z\in[0,z(\underline{\theta})]}$ are production efficient if it maximizes*

$$Q - \int_{\mathbb{Z}^B} B_z(z)\, dz \tag{26}$$

subject to the boundary condition $z(\overline{\theta}) \leq 1$. Note that $z(\underline{\theta}) = z^\star$ is the automation rate.

The robot input is production efficient if the marginal production of the robot inputs equal to the social cost of robot inputs, i.e., $\frac{\partial Q}{\partial B_z(z)} = 1$. The production efficient job assignment and automation is more complicated and we save it in the Appendix A.3.

Optimal Automation. For later use, we say the planner is Utilitariam if for any $\theta \in \Theta$, $\widetilde{F}(\theta) = F(\theta)$; the planner has a weak obligation to redistribute if for any $\theta \in \Theta$, $\widetilde{F}(\theta) \geq F(\theta)$; the planner has a strict obligation to redistribute if it has a weak obligation to redistribute and there exists a positive measure $\Upsilon \in \Theta$ such that for any $\theta \in \Upsilon$, $\widetilde{F}(\theta) > F(\theta)$. To solve the planning problem we have Theorem 1:

Theorem 1 *Denote* $\{z(\theta)\}_{\theta\in\Theta}$ *and* $\{B_z(z)\}_{z\in[0,z^\star]}$ *as the optimal job assignment and robot inputs, where* $z^\star$ *is the optimal automation rate. We have the following results:*

(i) If the planner is Utilitarian, $\{z(\theta)\}_{\theta\in\Theta}$ *and* $\{B_z(z)\}_{z\in[0,z^\star]}$ *are production efficient;*

(ii) If for any $(\theta, z) \in \Theta \times \mathbb{Z}$, $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial\theta\partial z} = 0$, $\{z(\theta)\}_{\theta\in\Theta}$ *and* $\{B_z(z)\}_{z\in[0,z^\star]}$ *are production efficient, no matter whether the planner is Utilitarian or not;*

(iii) If the planner has a strict obligation to redistribute, and there exist a positive measure of set $\Delta \in \Theta \times \mathbb{Z}$ *such that* $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial\theta\partial z} > 0$, *then the production efficient job assignment is not socially optimal, while optimal robot inputs are still product-efficient in the sense that* $\frac{\partial Q}{\partial B_z(z)} = 1$.

Proof. See Appendix A.3. ■

Part one of Theorem 1 shows that regulation of automation is desirable only for redistributive reasons. Part two suggests that even if the the planner has an obligation to redistribute, the regulation of automation is not desirable when job assignments have no influence on the relative wages, which is true when there is no complementarity and substitutability between the skill and task, i.e., $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial\theta\partial z} = 0$. Last, part three shows that in common situations where the government has an obligation to redistribute and $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial\theta\partial z} \neq 0$, the regulation of automation is generally desirable.

Optimal Taxation. Proposition 5 gathers results about optimal taxes. We find that optimal robot tax is zero and optimal labor income tax in consistent of two parts: (i) the classic Mirrleesian part and (ii) the indirect redistribution effect (IRE) pioneered by Stiglitz (1982) and developed by Naito (1999), Ales et al. (2015) and Sachs et al. (2020).

Proposition 5 *We have the following results in the optimum:*

(i) Optimal robot tax is zero, i.e., $\tau^B(z) = 0$ *for any* $z \in \left[0, z^R\right]$;

(ii) For any $\theta \in \Theta$, *optimal labor tax wedge satisfies:*

$$T'(y(\theta)) = \frac{\overbrace{\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)} \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta}\left(1+\frac{1}{\varepsilon}\right)}^{Mirrleesian\ Part}}{1+\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)} \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta}\left(1+\frac{1}{\varepsilon}\right)} - \frac{\overbrace{\frac{\varphi'(\theta)}{\lambda y(\theta) f_\theta(\theta)} \gamma^L_{own}(z(\theta))}^{Indirect\ Redistribution\ Effect\ (IRE)}}{1+\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)} \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta}\left(1+\frac{1}{\varepsilon}\right)}, \tag{27}$$

where $-\varphi(\theta)$ *is the co-state of the skill premium* $\omega(\theta)$. λ *is the shadow value of government revenue.* $\gamma^L_{own}(z(\theta))$ *is the own wage-labor elasticity of wage (see equation (A10)).*

(iii) When $\vartheta(z,\theta)$ is separable, i.e., $\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} = 0$, for any $(\theta,z) \in \Theta \times \mathbb{Z}$:

$$T'(y(\theta)) = \frac{\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)} \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta}\left(1+\frac{1}{\varepsilon}\right)}{1+\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)} \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta}\left(1+\frac{1}{\varepsilon}\right)}. \tag{28}$$

(iv) Assume that $\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} > 0$ for any $(\theta,z) \in \Theta \times \mathbb{Z}$; and $\varphi(\cdot)$ is twice continuously differentiable. When $\widetilde{F}(\theta) - F(\theta) > 0$ there exist interval $(\underline{\theta},\theta_A)$ and $(\theta_B,\bar{\theta})$ such that for any $\theta \in (\underline{\theta},\theta_A)$ we have $\varphi'(\theta) > 0$, and for any $\theta \in (\theta_B,\bar{\theta})$, we have $\varphi'(\theta) < 0$, where $\underline{\theta} < \theta_A < \theta_B < \bar{\theta}$.

Proof. See Appendix A.4. ■

Part (i) suggests that robotization should be production efficient when robots are skill-neutral, i.e., when the input of robots does not affect the relative productivities of different labor inputs. While robot inputs in our benchmark case is skill-neutral, it is not under more general production technology (see our extension as well as the Translog technology considered in Sachs et al. (2020)). Therefore, the understanding of the benchmark results should be that there is no need to tax skill-neutral robot inputs, rather than that robot tax is not desirable. We will come back to this in the model extension.

Part (ii) and (iii) suggest that while robotization should be production-efficient, the job assignment is not. To see this, note that the optimal labor tax wedge is a combination of the Mirrlees (1971) tax and an indirect redistribution part as is in Stiglitz (1982). The IRE suggests that the government manipulates relative wages by adjusting job assignment $z(\theta)$, unless there is no complementarity and substitutability between skills are tasks (see the difference between equations (27) and (28)), so that altering job assignment does not change the relative wages.

To be clear, what distinguish our finding from Stiglitz (1982) is that the IRE disappear when innate ability $\vartheta(z,\theta)$ is separable between the skill and task, even if goods produced by different labors are imperfectly substituted by each other. The reason underlying this finding has been indicated by Proposition 4, which suggests that tax can only alter relative wages through its effect on job assignment and the complementarity between tasks and innate ability. IRE is effective only if there is complementarity or substitutability between innate ability and task.[24]

The comparison between equation (27) and (28) highlights the importance of considering job assignment in the tax design, without which the result can be misleading. This finding echoes the debate between Saez (2004) and Naito (2004): when/whether taxes play an indirect redistributive role? While Saez (2004) suggests that IRE is not matter when occupation choice is endogenous, Naito (2004) argues that as long as labor factor inputs to different occupations are heterogeneous, IRE matters. Our finding suggests that whether there is an IRE in the design of labor income tax depends on whether there is a complementarity (substitutability) between the innate ability θ and task z. This finding bridged the gap

[24] Another necessary condition is that labor factors of different skills are imperfectly subsitutable by each other. Otherwise the own wage elasticity $\gamma^L_{own}(z(\theta))$ is zero and the IRE disappear.

between Saez (2004) and Naito (2004), and uncovers what real matters for the IRE of tax when occupation choice is endogenous.

Part (iv) indicates that $\tau^y\left(\underline{\theta}\right) > 0$ and $\tau^y\left(\bar{\theta}\right) < 0$. As is in Stiglitz (1982), the IRE requires a higher (lower) tax rate on low-skilled (high-skilled) laborers to narrow the wage gap. This finding provides a method for testing the results of quantitative analysis.

Optimal Job Assignment. To see how preference for equality alters the optimal job assignment and automation rate, we provide Proposition **??**. In Proposition **??**, we consider a special case with separable $\vartheta\left(z,\theta\right)$, which means $\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} = 0$ for any $(\theta,z) \in \Theta \times \mathbb{Z}$.

Proposition 6 *Consider $\eta\left(\cdot\right) = f_\theta\left(\cdot\right) = 1$ to simplify the notation. We have the following results when $\vartheta\left(z,\theta\right)$ is separable, i.e., $\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial z(\theta)\partial\theta} = 0$ for any $(\theta,z) \in \Theta \times \mathbb{Z}$:*

(i) The optimal job assignment is determined by a second-order differential equation:

$$\frac{d\ln z'\left(\theta\right)}{d\theta} = \left(\varepsilon+\sigma\right)\frac{\partial \ln \vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta} - \left(\sigma-1\right)\frac{d\ln \vartheta\left(z\left(\theta\right),\theta\right)}{d\theta} - \varepsilon\frac{d\ln\left[1+\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln\theta}\frac{\widetilde{F}_\theta(\theta)-F_\theta(\theta)}{\theta f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right)\right]}{d\theta}$$

with boundary conditions $z\left(\bar{\theta}\right) = 1$ and

$$z'\left(\underline{\theta}\right) = \frac{\frac{\vartheta(z(\underline{\theta}),\underline{\theta})^{(\varepsilon+1)}}{(\kappa)^{\sigma+\varepsilon}}\left[1-(\kappa)^{\sigma-1}z\left(\underline{\theta}\right)\right]^{\frac{\sigma+\varepsilon}{\sigma-1}}}{\left\{\int_0^1 \frac{\left[\vartheta(z(\theta),\theta)^{\frac{\sigma-1}{\sigma}}z'(\theta)^{\frac{1}{\sigma}}\right]^{\frac{\sigma(\varepsilon+1)}{\sigma+\varepsilon}} f_\theta(\theta)^{\frac{\sigma-1}{\sigma+\varepsilon}}}{\left[1+\frac{d\ln\vartheta(z(\theta),\theta)}{d\ln\theta}\frac{\widetilde{F}_\theta(\theta)-F_\theta(\theta)}{\theta f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right)\right]^{\frac{(\sigma-1)\varepsilon}{\varepsilon+\sigma}}}d\theta\right\}^{\frac{\sigma+\varepsilon}{\sigma-1}}}.$$

(ii) Denote z^U as the optimal automation rate when the planner is Utilitarian. When $\vartheta\left(z,\theta\right)$ is separable, the optimal automation rate $z^\star = z^U$ and $z^\star$ increases in κ when $\sigma > 1$.

Proof. See Appendix A.5. ■

As can be seen from part one of Proposition 6 that the general result is quite involved and can not be solved by common approaches of differential equations. Part two of Proposition 6 delivers some interesting results about optimal automation.

Part two of Proposition 6 enhances part two of Theorem 1. It suggests that when $\vartheta\left(z,\theta\right)$ is separable, the optimal automation rate is not only production efficient but also independent of the planner's preference. In particular, $z^\star = z^U$. Notice that when social welfare function is utilitarian, there is no need to distort labor supplies. Thus, z^U and $z^\star$ equals the automation rate in the Laissez-faire economy, which

means in cases with skill-netural tasks the planner should not distort the automatization. In addition, we find that the optimal automation rate increases in the progress of robotic technology. This is true even if the planner has a preference for equaltiy and there is a complementarity between the ability and skill.

In more general case where $\vartheta(z,\theta)$ is not separable, i.e., there is a complementarity or substitutability between skill and tasks, the second-order differential equation becomes more complicated. The optimal automation rate is no longer the automation rate under utilitarianism. Combination of Part (i) of Proposition 5 and Proposition 6 suggests that the government should either regulate automation or embrace automation and not impose taxes on robots. This does not hold in the extended model, but the quantitative analysis (section 6) results indicate that even if a robot tax is needed, it is indeed very low.

5 Extension

In the benchmark model, we consider a stylized technology to answer the desirability of regulation and robot taxes with endogenous job assignment and automation. In that case, labors and robots within a sector are perfectly substitutable. Therefore, the production of a sector-specific goods may either be fully automated or completely manual. Under that setup, robot inputs are skill-neutral. That is the relative wages are irrlevant to the robot inputs.[25]

Another assumption of our benchmark model is that the higher the ability of worker the harder the worker to be replaced by robots. However, in the real economy, middle-skilled worker may be easier replaced by robots than those of low-skilled workers. The model extension allows partial automation, non monotonic automation rate, and non-skill-neutral robot inputs. Besides, we will study optimal taxation both with and without regulation of automation. Doing so, we highlight the role of regulation.

5.1 Model Setup

We consider technology (2) and power-function technology (4) with $\rho < \infty$, that is the technology considered in section 3.2. In addition to previous wedges, we set:

$$\tau^a\left(z\left(\theta\right)\right) = \frac{\partial Q_z\left(z\left(\theta\right)\right)}{\partial a\left(\theta\right)} \frac{\partial Q}{\partial Q_z\left(z\left(\theta\right)\right)}, \forall \theta \in \Theta,$$

as the wedge on automation. In the economy without regulation, firm's problem implies $\tau^a\left(z\left(\theta\right)\right) = 0$. When $\frac{\partial Q_z(z(\theta))}{\partial a(\theta)} > 0$, raising automation rate can increase output given the factor inputs. Therefore, positive/negative $\tau^a\left(z\left(\theta\right)\right)$ represents a regulation/promotion on automation.

[25] To be clear, remember that in the benchmark model, $w\left(\theta\right) = \eta\left(z\left(\theta\right)\right) Q^{\frac{1}{\sigma}} \left[\frac{L(\theta)}{z'(\theta)}\right]^{-\frac{1}{\sigma}} \vartheta\left(z\left(\theta\right),\theta\right)^{\frac{\sigma-1}{\sigma}}$ (see section **??**). It's easy to see that robot inputs do not alter the relative wages.

As similar to the own-wage-robot elasticity, we have the own-wage-automation elasticity:[26]

$$\gamma_{own}^{a}(z)=\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{\partial \ln Q_z(z)}{\partial \ln a_z(z)}+\left(\frac{1}{\xi(z)}\frac{\rho-1}{\rho}-1\right)\frac{a_z(z)}{1-a_z(z)},\forall z\in\mathbb{Z}. \tag{29}$$

The elasticity includs two parts. On one hand, rising automation rate may either increase or decrease $Q_z(z)$, which in turn changes the wage rate. On the other hand, increasing automation rate (decrease of tasks performed by labors) increases labor inputs to each task and decreases marginal productivities of labors. We will start with the policy design with regulation, after which we discuss policy design without regulation.

5.2 Main Results

5.2.1 With Regulation

Optimal Tax and Regulation. The planner's problem is choosing $\{B(\theta),l(\theta),V(\theta),i(\theta)\}$ to maximize the social welfare $\int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta$, subject to the market clear conditions (52) and (50), incentive compatible constraints (Intensive IC), i.e., $V'(\theta)=\frac{d\ln w(\theta)}{d\theta}l(\theta)\phi'(l(\theta))$, where wage rates satisfy $W(\theta)=w_z(z(\theta))\vartheta(z(\theta),\theta)$ and (55). We have the following results:

Proposition 7 *Optimal policies have the following properties:*

(i) Both regulation and robot tax is desirable;

(ii) The optimal robot tax and the optimal regulation satisfy:

$$\tau^B(z(\theta))=-\frac{\varphi'(\theta)}{\lambda}\frac{\gamma_{own}^{B}(z(\theta))}{B_z(z(\theta))},\forall\theta\in\Theta, \tag{30}$$

and

$$\tau^a(z(\theta))=-\frac{\varphi'(\theta)}{\lambda}\frac{\gamma_{own}^{a}(z(\theta))}{a(z(\theta))},\forall\theta\in\Theta. \tag{31}$$

In the above formulas, λ is shadow resource multiplier, and $-\varphi(\theta)$ is the optimal co-state on the skill premium $\frac{d\ln w(\theta)}{d\theta}$ (see equation (C25) for the expression of φ).

(iii) The optimal labor income tax still satisfy (27). When $\vartheta(z,\theta)$ is separable, i.e., $\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}=0$, one again has $\varphi'(\cdot)=0$ for any $(\theta,z)\in\Theta\times\mathbb{Z}$. Therefore, the optimal labor tax wedge again satisfies (28).

Proof. See Appendix C.3. ■

It is not suprise that when regulation is not enforceable, elasticities γ_{own}^{L} and γ_{own}^{B} are replaced by $\gamma_{own}^{L,u}$ and $\gamma_{own}^{B,u}$, where we use "u" to indicate the corresponding variable without regulation.

[26] See (**??**) for details.

An Illustration: Exogenous Job Assignment. The general solution off $\varphi'(\theta)$ is quite complex. We leave the solution of $\varphi'(\theta)$ to Online Appendix .To get an intuition of $\varphi'(\theta)$, we consider policy design in a model with exogenous job assignment. In this case, Proposition 9 still holds except for the expression of $\varphi'(\theta)$ is simplified. Specificly, in the exogenous assignment case, we have (See Proposition 9 in Appendix C)

$$\frac{\varphi'(\theta)}{\lambda} = -f_\theta(\theta)y(\theta)\left[\overbrace{[1-g(\theta)]\,f_y(y(\theta))y(\theta)}^{Local\ Redistribution\ Effect\ (LRE)} - \overbrace{\left[1-F_y(y(\theta))\right]\left[1-\overline{g}(\theta)\right]\frac{\varepsilon^l_w(\theta)}{\varepsilon^l_{1-\tau}(\theta)}}^{Cumulative\ Redistribution\ Effect\ (CRE)}\right]\left[1-\tau^y(\theta)\right] \quad (32)$$

where $\varepsilon^l_w(\theta)$ and $\varepsilon^l_{1-\tau}(\theta)$ are the elasticities of labor supply with respect to wage rate and net-tax income rate (see Appendix A.4.4).

$[1-g(\theta)]$ represents the welfare effect of redistributing one unit of income from a θ-skilled agent to the government. So $\gamma^L_{own}(z(\theta))[1-g(\theta)]f_y(y(\theta))y(\theta)$ caputres the total redistribution effect between θ-skilled agent and the government. To take the reactions of labor supplies into consideration and obtain the total redistribution effect, the second part should be taken into consideration. In $CRE(y)$, $\gamma^L_{own}(z(\theta))\frac{\varepsilon^l_w(\theta)}{\varepsilon^l_{1-\tau}(\theta)}$ represents the percentage decrease in $1-T'(y(\theta))$ needed to maintain the labor supply unchanged when wages increase. A decrease of $1-T'(y)$ at y increases the tax burden of all of the individuals with labor income higher than y. Thus, to derive the total effect, we should multiple $\gamma^L_{own}(z(\theta))\frac{\varepsilon^l_w(\theta)}{\varepsilon^l_{1-\tau}(\theta)}$ by $[1-\bar{g}(y)]\left[1-F_y(y)\right]$.

Optimal Robot Tax. As long as the automation (robot inputs) is not skill-neutral the regulation (robot tax) is desirable. Moreover, since the indirect redistribution effect of regulation and robot tax is sector dependent, differentiated robot tax is needed. Proposition 7 then demonstrates the optimality of production inefficiency. We take the robot tax as an illustration. Equation (30) can be transformed to be:

$$0 = \overbrace{\tau^B(z(\theta))}^{Revenue\ Effect} + \overbrace{\frac{\varphi'(\theta)}{\lambda}\frac{\gamma^B_{own}(z(\theta))}{B_z(z(\theta))}}^{Indirect\ Redistribution\ Effect}. \quad (33)$$

Following Cui et al. (2021), a heuristic derivation for (33) is as follows: Consider an extra unit of robot inputs which increases $p(z(\theta))\frac{\partial Q_z(B_z(z(\theta)),L_z(z(\theta)))}{\partial B_z(z(\theta))}$ units of output. The extra robot inputs do not change agents' gross utilities directly, while increases the net output by $p(z(\theta))\frac{\partial Q_z(B_z(z(\theta)),L_z(z(\theta)))}{\partial B_z(z(\theta))}-1$, which, according to the definition of wedges (see (19)), equals $\tau^B(z(\theta))$. This is the revenue effect of the reallocation.

The indirect redistribution effect is captured by the second term. $\frac{\gamma^B_{own}(z(\theta))}{B_z(z(\theta))}$ is the wage increase caused by one extra unit of robot input to θ sector, which multiplied by $-\frac{\varphi'(\theta)}{\lambda}$ is the welfare effect. The intuition of regulation design is just similar to the robot tax.

Optimal Automation. On the optimal automation, we have the following results:

Proposition 8 *The optimal automation satisfies*

$$\frac{1-a\left(\theta\right)}{a\left(\theta\right)}=\frac{\zeta\left(z\left(\theta\right)\right)\frac{\partial\ln Q}{\partial\ln L_{z}\left(z\left(\theta\right)\right)}Q}{1+\left(1-\frac{1}{\rho}\right)\varphi'\left(\theta\right)\gamma_{own}^{a}\left(z\left(\theta\right)\right)} \tag{34}$$

or equivalently

$$\frac{1-a\left(\theta\right)}{a\left(\theta\right)}=\frac{w\left(\theta\right)L(\theta)}{B\left(\theta\right)}+\frac{\varphi'(\theta)\left[1+\gamma_{own}^{L}\left(z\left(\theta\right)\right)\right]}{B\left(\theta\right)}. \tag{35}$$

Proof. See Appendix C.4. ■

Notice that the right side of equation (34) equals $\frac{\frac{\partial\ln Q_{z}(z(\theta))}{\partial\ln L_{z}(z(\theta))}/\frac{\partial\ln Q_{z}(z(\theta))}{\partial\ln B_{z}(z(\theta))}}{1+\left(1-\frac{1}{\rho}\right)\varphi'(\theta)\frac{\gamma_{own}^{a}(z(\theta))}{\zeta(z(\theta))\frac{\partial\ln Q}{\partial\ln B_{z}(z(\theta))}Q}}$ and $\frac{1-a_{z}^{u}(z)}{a_{z}^{u}(z)}=\frac{\partial\ln Q_{z}^{u}(z)}{\partial\ln L_{z}(z)}/\frac{\partial\ln Q_{z}^{u}(z)}{\partial\ln B_{z}(z)}$. $\left(1-\frac{1}{\rho}\right)\varphi'(\theta)\gamma_{own}^{a}\left(z\left(\theta\right)\right)$ can be treated as a wedge. Similarly, $\frac{\varphi'(\theta)\left[1+\gamma_{own}^{L}(z(\theta))\right]}{B(\theta)}$ can be treated as wedge due to the IRE.

6 Quantative Analysis

Parameterization. For simplicity, we assume that the utility is $c-\frac{l^{1+1/\varepsilon}}{1+1/\varepsilon}$ with $\varepsilon=0.33$ (Chetty (2012)) and take a production in the form of (2). The elasticity of substitution among varieties of goods is generally assumed to exceed unity. Estimates of the substitutability of labor factors in the labor and industrial organization literature typically range from 0.6 to 10 (Katz and Murphy (1992); Broda and Weinstein (2006); Hendel and Nevo (2006); Dustmann et al. (2013)).[27] We take $\sigma=3.1$ and $\rho=1.5$ as our benchmark parameters. We then consider different values of σ and ρ for robustness analysis. We choose an elasticity of substitution $\sigma=3.1$ from Heathcote, Storeslettenand Violante (2017), who structurally estimate this CES parameter for the U.S. by targeting cross-sectional moments of the joint equilibrium distribution of wages, hours, andconsumption.

We follow Rothschild and Scheuer (2013) and assume relative social welfare weights satisfy $1-\widetilde{F}_{\theta}\left(\theta\right)=\left[1-F_{\theta}\left(\theta\right)\right]^{\chi}$ where χ parametrizes the government's desire to redistribute. $\chi=1$ corresponds to utilitarian preferences. As $\chi\to\infty$, the welfare weights approach that of a Rawlsian social planner. Follow Rothschild and Scheuer (2013) and Thuemmel (2018), we set $\chi=1.3$ as the benchmark value.[28]

Remember that, following the convention of Sachs et al. (2020) and Saez and Stantcheva (2018), we have used the quantile of the wage of an agent to index the agent (θ is the quantile). In this case, the type

[27] The value $\sigma=0.6$ is from Dustmann, Frattini, and Preston (2013) who study the impact of immigration along the U.K. wage distribution and, as in our framework, group workers according to their position in the wage distribution. Katz and Murphy (1992) estimate elasticity of substitution between different inputs of skilled labor is for a CES production function. The value derived by them is 1.4.

[28] Thuemmel (2018) find that setting χ at 1.3 leads to marginal tax rates in the range of those observed empirically.

space Θ is over the interval $[0,1]$, and the distribution of type is uniform, where $f_\theta(\theta) = 1$. The productivity parameters govern the absolute and relative productivities of labor inputs. We rank occupations by wage rate. Following Ales et al. (2015), we choose the unit of tasks so that, the initial wage rate of the z-th task is exactlly the z-th wage rate, i.e., $z(\theta) = \theta$ (see Ales et al. (2015)). In the optimum, the job assignment function is endogenous.

Data. To calibrate the parameters, we first make some assumptions about the initial policies. The labor income schedule is assumed to have a constant rate of progressivity. Specifically, the original labor income tax is $T_o(y) = y - \frac{1-\varsigma}{1-\phi} y^{1-\phi}$. $\phi = 0.151$ is the rate of progressivity, and $\varsigma = -3$ governs the marginal tax rate in the lower bound (Heathcote et al. (2017)). Tax rates on the rent of robot is 37.1% (see e.g., Slavik and Yazici (2014)).[29] All tax revenue is transferred to individuals in the form of lump-sum transfers.

Follow the literature (see e.g., Mankiw et al. (2009)), we use a lognormal approximation of the U.S. labor income distribution. The mean and standard deviation of lognormal parameterization for 2019 are 10.6 and 1.02 (see e.g., Eeckhout et al. (2021)), where the unit of income is in dollars. As in Saez (2001), we infer the distribution of wages $w(\theta)$ and labor $l(\theta)$ from the labor incomes $y(\theta)$ based on individuals' first-order conditions.

Calibration. The calibration strategy for each of these parameters is described as below. First, based on the model's parameter constraints for elasticities σ and ρ, along with previous literature estimates of inter-sectoral elasticity σ, we set these two parameters to $\sigma = 3.1$ and $\rho = 1.5$, respectively. We will compare outcomes under varying values of elasticity ρ to assess robustness.

Second, the comparative advantage in household labor efficiency $\vartheta(z(\theta), \theta)$ is calibrated using wage quantile distributions from empirical data. To better capture the distributional features embedded in the data, we extend the parameterization framework for this comparative advantage originally proposed by Ales et al. (2015)), adopting the following functional form:

$$\vartheta(z(\theta), \theta) = \exp\left\{\Lambda_z \cdot z(\theta) + \Lambda_\theta \cdot \theta + \Lambda_{z\theta} \cdot [z(\theta) \cdot \theta] + \sum_{p=2}^{N_p} \Lambda_{\theta,p} \cdot \theta^p\right\},$$

and it implies the wage premium $\frac{d \log w(\theta)}{d\theta}$ follows,

$$\frac{d \log w(\theta)}{d\theta} = \Lambda_\theta + \Lambda_{z\theta} \cdot z(\theta) + \sum_{p=2}^{N_p} \Lambda_{\theta,p} p \cdot (\theta)^{p-1}$$

Given that the extended comparative advantage function form differs from Alies' original specifica-

[29] As documented by Slavik and Yazici (2014), the effective corporate tax rates on structure and equipment capital estimated by Gravelle (2011) are 32% and 26%, respectively. The interest tax rate is 15%. This implies that the overall tax on structure and equipment capital income is 1 - (1 - 0.15) ?(1 - 0.32) and 1 - (1 - 0.15) ?(1 - 0.26), respectively (i.e., 42.2% and 37.1%).

tion, it becomes necessary to additionally calibrate the polynomial parameters $\{\Lambda_{\theta,p}\}_{p=1}^{N_p}$. Specifically, we first adopt Alies' calibration approach for Λ_θ and $\Lambda_{z\theta}$, then perform a joint parameterization of $\Lambda_{\theta,p}$ to minimize the discrepancy between the wage premium distribution in our theoretical model and empirical observations.

Last, we calibrate $\xi(z)$ and $\eta(z)$ by minimize the distance between the wage-robot elasticity in the data and in our model, while guarantee $z'(\theta) = 1$. In doing so, we uses the relationship between the wage elasticity and $\xi(z)$.

$$\gamma_{own}^{B,u}(z) = \left[1 - \frac{1}{\sigma} - \xi(z)\right] \alpha_z(z).$$

Then $\eta(z)$ can be used to guarantee $z'(\theta) = 1$. Our model captures real-world features very well.

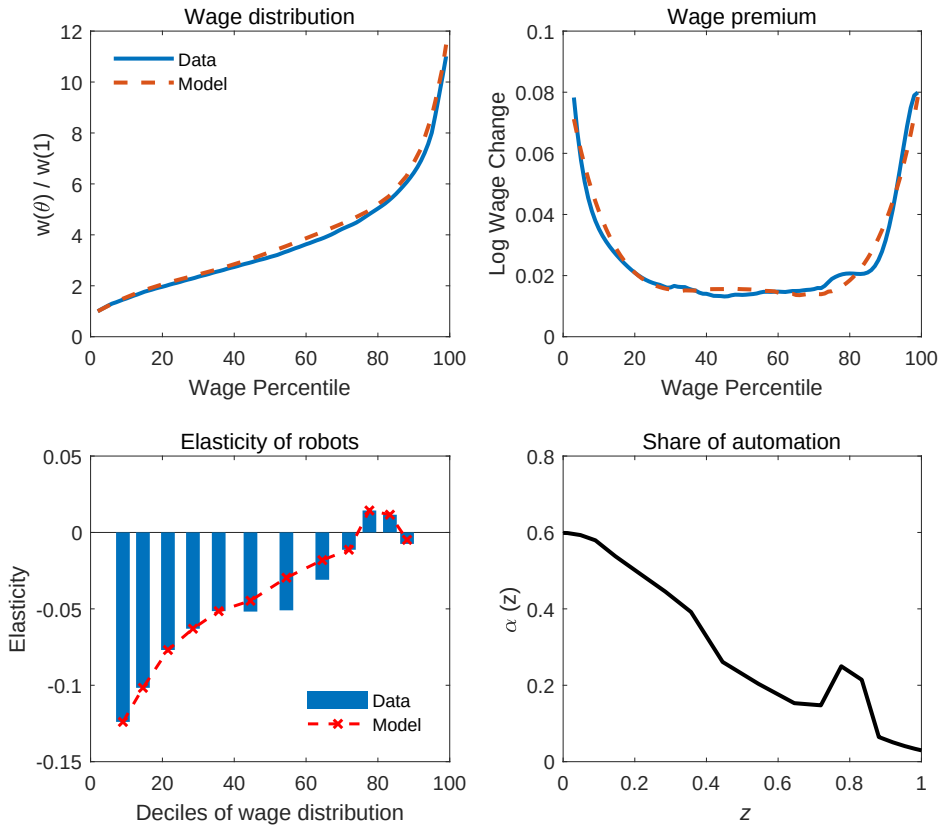


Figure 2: Calibration

Benchmark Results. The optimal policies under the benchmark parameters are presented below

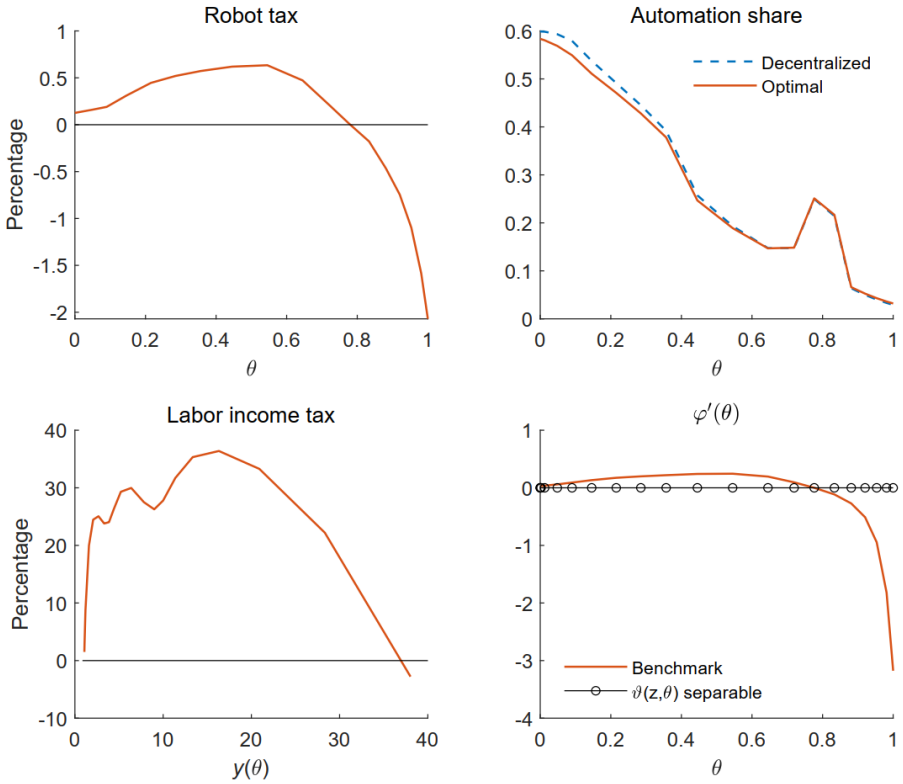


Figure 3: Optimal Policies

Counterfactual Analysis. We increases each productivity of robots $\kappa_i(z)$ by 20%. It turns out that $z(\theta)$ increases. This means that more workers are forced to engage in more difficult professions. Meanwhile, the wage gap increases.

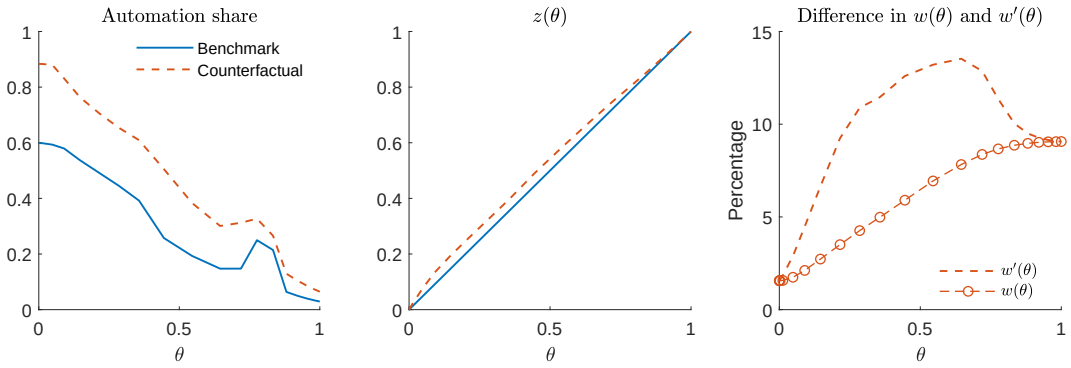


Figure 4: Counterfactual Economy

As a response to the technological advantage, intervention in production decreases while the optimal tax rate for high-income increases.

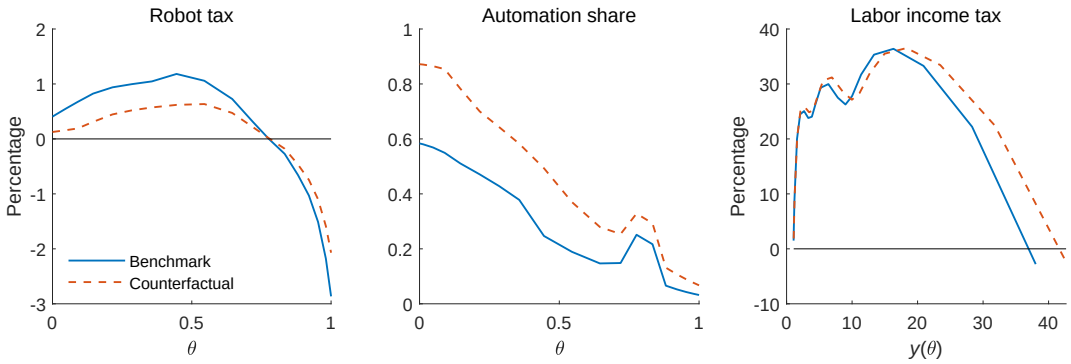


Figure 5: Counterfactual Optimal Policies

Conclusion

Is production-efficient automation socially optimal? If not, how to make automation technology universal? This paper formalizes a framework to examine the role of regulation, pre-distributive robot tax, and redistributive income tax in dealing with the impact of automation technology. We establish a job assignment model with a continuum of sectors micro-founded by task-based automation. Talents choose which sector to work for and compet with robot within each sector. Our model distinguishes automation regulation from robot tax and nicely captures the complex reaction of wages to the robot inputs in the real economy. We then study optimal public policies in the spirit of Mirrlees. Our findings are multi-faceted and suggest that the micro-foundation of technology and the package of enforcable policies are crucial for the policy design. Both robot tax and regulation are desirable and work through different channels. However, the level of robot tax an regulation is very limited and phase out with technoligical advance. Meanwhile, the role of income tax in redistribution become increasingly important.

References

ACEMOGLU, D. AND S. JOHNSON (2023): *Power and Progress: Our Thousand-Year Struggle Over Technology and Prosperity*, PublicAffairs, 1st edition ed.

ACEMOGLU, D. AND J. LOEBBING (2022): "Automation and Polarization," .

ACEMOGLU, D., A. MANERA, AND P. RESTREPO (2020): "Does the US Tax Code Favor Automation?" Tech. Rep. 27052, National Bureau of Economic Research, Inc.

ACEMOGLU, D. AND P. RESTREPO (2018): "Modeling Automation," *AEA Papers and Proceedings*, 108, 48–53.

——— (2020): "Robots and Jobs: Evidence from US Labor Markets," *Journal of Political Economy*, 128, 2188–2244.

ACEMOGLU, D. AND J. A. ROBINSON (2013): *Why Nations Fail: The Origins of Power, Prosperity, and Poverty*, New York: Currency, reprint edition ed.

ALES, L., M. KURNAZ, AND C. SLEET (2015): "Technical Change, Wage Inequality, and Taxes," *The American Economic Review*, 105, 3061–3101.

ATKINSON, A. B. AND J. E. STIGLITZ (1976): "The design of tax structure: direct versus indirect taxation," *Journal of Public Economics*, 6, 55–75.

AUTOR, D. H., L. F. KATZ, AND M. S. KEARNEY (2008): "Trends in US wage inequality: Revising the revisionists," *The Review of economics and statistics*, 90, 300–323.

BRODA, C. AND D. E. WEINSTEIN (2006): "Globalization and the Gains from Variety," *The Quarterly Journal of Economics*, 121, 541–585.

CHEN, Y. (2019): "The rise of the machines and the us labor market," Tech. rep.

CHETTY, R. (2012): "Bounds on Elasticities With Optimization Frictions: A Synthesis of Micro and Macro Evidence on Labor Supply," *Econometrica*, 80, 969–1018.

CHRISTIANSEN, V. (1984): "Which commodity taxes should supplement the income tax?" *Journal of Public Economics*, 24, 195–220.

COSTINOT, A. AND I. WERNING (2018): "Robots, Trade, and Luddism: A Sufficient Statistic Approach to Optimal Technology Regulation," Tech. Rep. w25103, National Bureau of Economic Research.

——— (2023): "Robots, Trade, and Luddism: A Sufficient Statistic Approach to Optimal Technology Regulation," *The Review of Economic Studies*, 90, 2261–2291.

CUI, X., L. GONG, AND W. LI (2021): "Supply-side optimal capital taxation with endogenous wage inequality," *Journal of Public Economics*, 198, 104421.

DIAMOND, P. A. AND J. A. MIRRLEES (1971): "Optimal Taxation and Public Production II: Tax Rules," *The American Economic Review*, 61, 261–278.

DUSTMANN, C., T. FRATTINI, AND I. P. PRESTON (2013): "The Effect of Immigration along the Distribution of Wages," *The Review of Economic Studies*, 80, 145–173.

EECKHOUT, J., C. FU, W. LI, AND X. WENG (2021): "OPTIMAL TAXATION AND MARKET POWER," .

GOMES, R., J.-M. LOZACHMEUR, AND A. PAVAN (2018): "Differential Taxation and Occupational Choice," *Review of Economic Studies*, 85, 511–557.

GUERREIRO, J., S. REBELO, AND P. TELES (2021): "Should Robots Be Taxed?" *The Review of Economic Studies*.

HEATHCOTE, J., K. STORESLETTEN, AND G. L. VIOLANTE (2017): "Optimal Tax Progressivity: An Analytical Framework*," *The Quarterly Journal of Economics*, 132, 1693–1754.

HENDEL, I. AND A. NEVO (2006): "Measuring the Implications of Sales and Consumer Inventory Behavior," *Econometrica*, 74, 1637–1673.

JACQUET, L., E. LEHMANN, AND B. VAN DER LINDEN (2013): "Optimal redistributive taxation with both extensive and intensive responses," *Journal of Economic Theory*, 148, 1770–1805.

KATZ, L. F. AND D. H. AUTOR (1999): "Chapter 26 - Changes in the Wage Structure and Earnings Inequality," in *Handbook of Labor Economics*, ed. by O. C. Ashenfelter and D. Card, Elsevier, vol. 3, 1463–1555.

KATZ, L. F. AND K. M. MURPHY (1992): "Changes in relative wages, 19631987: supply and demand factors," *The Quarterly Journal of Economics*, 107, 35–78.

KRUSELL, P., L. E. OHANIAN, J.-V. RÍOS-RULL, AND G. L. VIOLANTE (2000): "Capital-skill complementarity and inequality: A macroeconomic analysis," *Econometrica*, 68, 1029–1053.

MANKIW, N. G., M. WEINZIERL, AND D. YAGAN (2009): "Optimal Taxation in Theory and Practice," *Journal of Economic Perspectives*, 23, 147–174.

MIRRLEES, J. A. (1971): "An Exploration in the Theory of Optimum Income Taxation," *The Review of Economic Studies*, 38, 175–208.

——— (1976): "Optimal tax theory: A synthesis," *Journal of Public Economics*, 6, 327–358.

NAITO, H. (1999): "Re-examination of uniform commodity taxes under a nonlinear income tax system and its implication for production efficiency," *Journal of Public Economics*, 71, 165–188.

——— (2004): "Endogenous human capital accumulation, comparative advantage and direct vs. indirect redistribution," *Journal of Public Economics*, 88, 2685–2710.

ROTHSCHILD, C. AND F. SCHEUER (2013): "Redistributive Taxation in the Roy Model*," *The Quarterly Journal of Economics*, 128, 623–668.

SACHS, D., A. TSYVINSKI, AND N. WERQUIN (2020): "Nonlinear Tax Incidence and Optimal Taxation in General Equilibrium," *Econometrica*, 88, 469–493.

SAEZ, E. (2001): "Using Elasticities to Derive Optimal Income Tax Rates," *The Review of Economic Studies*, 68, 205–229.

——— (2004): "Direct or indirect tax instruments for redistribution: short-run versus long-run," *Journal of Public Economics*, 88, 503–518.

SAEZ, E. AND S. STANTCHEVA (2016): "Generalized Social Marginal Welfare Weights for Optimal Tax Theory," *The American Economic Review*, 106, 24–45.

——— (2018): "A simpler theory of optimal capital taxation," *Journal of Public Economics*, 162, 120–142.

SALANIÉ, B. (2003): *The economics of taxation*, The MIT Press.

SATTINGER, M. (1975): "Comparative Advantage and the Distributions of Earnings and Abilities," *Econometrica*, 43, 455–468.

SCHEUER, F. (2014): "Entrepreneurial taxation with endogenous entry," *American Economic Journal: Economic Policy*, 6, 126–63.

SCHEUER, F. AND I. WERNING (2017): "The taxation of superstars," *The Quarterly Journal of Economics*, 132, 211–270.

SLAVIK, C. AND H. YAZICI (2014): "Machines, buildings, and optimal dynamic taxes," *Journal of Monetary Economics*, 66, 47–61.

STIGLITZ, J. E. (1982): "Self-selection and Pareto efficient taxation," *Journal of Public Economics*, 17, 213–240.

——— (2018): "Pareto efficient taxation and expenditures: Pre- and re-distribution," *Journal of Public Economics*, 162, 101–119.

THUEMMEL, U. (2018): "Optimal Taxation of Robots," Tech. Rep. 7317, CESifo.

APPENDIX

A Appendix A: Equilibrium and Useful Concepts

A.1 Proof of Proposition 1

We first provide several lemmas for the proof of Proposition 1. In these lemmas, we construct consumer and producer's problems so that solutions to the original problems can be obtained by solving these constructed problems. We then show under the constructed problems, there exist a unique equilibrium. In conclusion, there exist a unique TRCE to the original problem.

Construct the producer's problem. Given positive prices $p(z)$ and wage rates $w_z(z)$, we construct the following producer's problem. The producer chooses $\left\{Q_*^D(z), Q_*^S(z), B_*(z), L_*(z), \nu_{i,*}^L(z), \nu_{i,*}^B(z)\right\}$ to maximize:

$$\pi_*^{Firm} = \max Q_* - \int_0^1 p(z)\left[Q_*^D(z) - Q_*^S(z)\right]dz - \int_0^1 \left\{B_*(z) + w_z(z) L_*(z)\right\}dz \tag{36}$$

subject to

$$Q_* = \left[\int_0^1 \eta(z) Q_*^D(z)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, Q_*^S(z) = \left[\int_0^1 Q_{i,*}(z)^{\frac{\rho-1}{\rho}} di\right]^{\frac{\rho}{\rho-1}};$$
$$Q_{i,*}(z) = \frac{\kappa_i(z)}{1+\tau^B(z)}\nu_{i,*}^B(z) B_*(z) + \iota_i(z)\nu_{i,*}^L(z) L_*(z); L_*(z) \geq 0, B_*(z) \geq 0$$

where $\nu_{i,*}^L(z) \in \mathcal{D}_L(z), \nu_{i,*}^B(z) \in \mathcal{D}_B(z)$.

Lemma A.1.1 [Consumer's problem] Solving problem (36), we have

$$B_*(z) = \left[1+\tau^B(z)\right] B(z), L_*(z) = L_z(z), Q_{i,*}(z) = Q_i(z),$$
$$Q_*^D(z) = Q_*^S(z), \nu_{i,*}^B(z) = \nu_i^B(z), \nu_{i,*}^L(z) = \nu_i^L(z),$$

where $B_z(z), L(z), Q(z), \nu_i^B(z), \nu_i^L(z)$ are solutions to the original problem **??**.

Proof: Notice that the production function of each enterprise in problem (36) is constant returns to scale, and there is no distortion. The problem can be divided into the following enterprise problem (see Chapter 5 in MWG): profit maximization problems (37) - (38) and market clear conditions for intermedi-

ate products (9).

$$\max_{Q^D(z)} Q_* - \int_0^1 p(z) Q_*^D(z) dz \tag{37}$$
$$s.t. Q_* = \left[\int_0^1 \eta(z) Q_*^D(z)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

and

$$\pi_*(z) = \max_{L_*(z),B_*(z);\nu_{i,*}^L(z),\nu_{i,*}^B(z);} p(z) Q_*^S(z) - [B_*(z) + w_z(z) L_*(z)]; \tag{38}$$
$$s.t. Q_*^S(z) = \left[\int_0^1 Q_{i,*}(z)^{\frac{\rho-1}{\rho}} di\right]^{\frac{\rho}{\rho-1}}; Q_i(z) = \frac{\kappa_i(z)}{1+\tau^B(z)} \nu_{i,*}^B(z) B_*(z) + \iota_i(z) \nu_{i.*}^L(z) L_*(z);$$
$$\int_0^1 \nu_{i,*}^L(z) di = 1; \int_0^{a_z(z)} \nu_{i,*}^B(z) di = 1; L_*(z) \geq 0, B_*(z) \geq 0; \nu_{i,*}^L(z) \in \mathcal{D}_L(z), \nu_{i,*}^B(z) \in \mathcal{D}_B(z).$$

Remember that $1+\tau^B(z) > 0$. Substituting $B_*(z)$ by $\left[1+\tau^B(z)\right] B(z)$, problem (38) can be transformed into (39)

$$\pi_*(z) = \max_{\nu_{i,*}^L(z),\nu_{i,*}^B(z);B(z);L_*(z)} p(z) Q_*^S(z) - \left\{\left[1+\tau^B(z)\right] B(z) + w_z(z) L_*(z)\right\}; \tag{39}$$
$$s.t. Q_*^S(z) = \left[\int_0^1 Q_{i,*}(z)^{\frac{\rho-1}{\rho}} di\right]^{\frac{\rho}{\rho-1}}; Q_i(z) = B(z) \kappa_i(z) \nu_{i,*}^B(z) + \iota_i(z) \nu_{i.*}^L(z) L_*(z);$$
$$L_*(z) \geq 0, \left[1+\tau^B(z)\right] B(z) \geq 0; \nu_{i,*}^L(z) \in \mathcal{D}_L(z), \nu_{i,*}^B(z) \in \mathcal{D}_B(z).$$

Solving the above problems (37) and (39), we have

$$B_*(z) = \left[1+\tau^B(z)\right] B(z), L_*(z) = L_z(z), Q_{i,*}(z) = Q_i(z),$$
$$Q_*^D(z) = Q_*^S(z), \nu_{i,*}^B(z) = \nu_i^B(z), \nu_{i,*}^L(z) = \nu_i^L(z),$$

where $B_z(z), L(z), Q(z), \nu_i^B(z), \nu_i^L(z)$ are solutions to the original problem **??** (pink words are added). In conclusion, we have **Lemma A.1.1**.□

Construct the consumer's problem. We have assumed that the first-order condition of consumer's problem is both necessary and sufficient and there is a unique solution to the problem. Set $l(w)$ as the solution to the first-order condition $w[1-T'(wl)] = \phi'(l(w))$. Since $T'(\cdot) < 1$ and $T(\cdot)$ is twice-continuously differentiable, $l(w)$ is twice-continuously differentiable. Moreover, we have assumed that $\phi(\cdot)$ satisfies the Inada conditions. Therefore, for any $w \geq 0$, $l(w)$ is finite and $l(0) = 0$.

Given the wage rates $w_z(z)$ and income tax $T(y)$, we construct the following consumer's problem:

$$V = \max_{l_*(\theta),\nu_{\theta,*}(z)\geq 0,\int_0^1 \nu_{\theta,*}(z)dz=1} \int_0^1 w_z(z) \left[\int_0^1 \vartheta(z,\theta) \nu_{\theta,*}(z) l_*(\theta) f(\theta) d\theta\right] dz - \int_0^1 \phi_*(l_*(\theta)) f(\theta) d\theta; \tag{40}$$

and

$$w_*(\theta) = \sup_{\nu_{\theta,*}(z)} \int_0^1 \vartheta(z,\theta)\, \nu_{\theta,*}(z)\, w_z(z)\, dz$$

We have the following results:

Lemma A.1.2 [Producer's problem] Solution of problem (40) satisfies $w_*(\theta) = w(\theta)$, $\nu_{\theta,*}(z) = \nu_\theta(z)$, $l_*(\theta) = l(\theta)$, $V_*(\theta) + T_*(\theta) = V(\theta)$, and $w_*(\theta) l_*(\theta) + T_*(\theta) = c(\theta)$, where $T_*(\theta) = -T(w_*(\theta) l_*(\theta))$ and $V_*(\theta) = w(\theta) l^*(\theta) - \phi_*(l^*(\theta))$. $\{w(\theta), \nu_\theta(z), l(\theta), V(\theta), c(\theta)\}$ is the solution to the original consumer's problem (**??**).

Proof: (a) Set the indirect utility function as a function of $w \geq 0$:

$$V(w) = \max_{c,l} c - \phi(l); \text{s.t. } c = wl - T(wl)$$

Remember that we have defined $\phi_*(\cdot)$ as the tax-modified disutility of labor. By definition the above problem is equivalent to the following problem■

$$V_*(w) = \max_{c^*,l^*} c^* - \phi_*(l^*); \text{s.t. } c^* = wl^*; \tag{41}$$

Since $\phi_*(\cdot)$ is strictly convex, the above problem uniquely determines $l^*(w)$ as solutions to the consumer's problem. (b) In the extensive margin, labor chooses tasks to maximize wage rate:

$$w(\theta) = \sup_{\nu_\theta(z)} \int_0^1 \vartheta(z,\theta)\, \nu_\theta(z)\, w_z(z)\, dz, s.t. \nu_\theta(z) \geq 0, \int_0^1 \nu_\theta(z)\, dz = 1 \tag{42}$$

Combination of (41) and (42) gives a unified problem for labors:

$$V_*(\theta) = \max_{c^*(\theta),l^*(\theta),w(\theta),\nu_\theta(z)} c^*(\theta) - \phi_*(l^*(\theta)); \tag{43}$$
$$s.t. c^*(\theta) = w(\theta) l^*(\theta); w(\theta) = \sup_{\nu_\theta(z)} \int_0^1 \vartheta(z,\theta)\, \nu_\theta(z)\, w_z(z)\, dz, s.t. \nu_\theta(z) \geq 0, \int_0^1 \nu_\theta(z)\, dz = 1$$

Remember that $T_*(\theta) = -T[w(\theta) l^*(\theta)]$. We have the following equations:

$$V_*(\theta) + T_*(\theta) = V(\theta); w(\theta) l_*(\theta) + T_*(\theta) = c(\theta).$$

By Chapter 12 in MWG, when residents' utility functions are all quasi-linear, the consumption demand and labor supply decisions in the market can be equivalently characterized by maximizing the utilitarian social welfare problem. Consequently, the consumer's problem (43) becomes equivalent to the following

problem:

$$V_* = \max_{c^*(\theta),l^*(\theta),w(\theta),\nu_\theta(z)} \int_0^1 c^*(\theta) f(\theta)\theta - \int_0^1 \phi_*(l^*(\theta)) f(\theta) d\theta;$$
$$\text{s.t. } c^*(\theta) = w(\theta) l^*(\theta); w(\theta) = \sup_{\nu_\theta(z)} \int_0^1 \vartheta(z,\theta) \nu_\theta(z) w_z(z) dz, s.t. \nu_\theta(z) \geq 0, \int_0^1 \nu_\theta(z) dz = 1$$

By substituting the variables in the objective function with equality constraints, the above problem can be simplified to:

$$V = \max_{l_*(\theta),\nu_{\theta,*}(z)\geq 0,\int_0^1 \nu_{\theta,*}(z)dz=1} \int_0^1 w_z(z) \left[\int_0^1 \vartheta(z,\theta) \nu_{\theta,*}(z) l_*(\theta) f(\theta) d\theta\right] dz - \int_0^1 \phi_*(l_*(\theta)) f(\theta) d\theta;$$

In conclusion, we have Lemma A.1.2.□

Lemma A.1.3 [Planner's problem] (i) In the economy, there is a one-to-one correspondence between the Walrasian equilibrium with taxes and the saddle point solution of the following problem:

$$\max_{Q_*(z),Q_{i,*}(z),B_*(z);L_*(z);\nu^L_{i,*}(z);\nu^B_{i,*}(z),l_*(\theta)} Q_* - \int_0^1 B_*(z) dz - \int_0^1 \phi_*(l_*(\theta)) f(\theta) d\theta; \tag{44}$$
$$s.t. 0 \leq Q_* \leq \left[\int_0^1 \eta(z) Q_*(z)^{1-\frac{1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, 0 \leq Q_*(z) \leq \left[\int_0^1 Q_{i,*}(z)^{\frac{\rho-1}{\rho}} di\right]^{\frac{\rho}{\rho-1}};$$
$$Q_{i,*}(z) = \frac{\kappa_i(z)}{1+\tau^B(z)} \nu^B_{i,*}(z) B_*(z) + \iota_i(z) \nu^L_{i,*}(z) L_*(z); L_*(z) = \int_0^1 \vartheta(z,\theta) \nu_{\theta,*}(z) l_*(\theta) f(\theta) d\theta$$
$$L_*(z) \geq 0, B_*(z) \geq 0; \nu_{\theta,*}(z) \geq 0, \int_0^1 \nu_{\theta,*}(z) dz = 1, \nu^L_{i,*}(z) \in \mathcal{D}_L(z), \nu^B_{i,*}(z) \in \mathcal{D}_B(z)$$

Therefore, the properties of the saddle point solution of the following problem are also the properties of the equilibrium. (ii) There exist a unique solution to the above problem under Assumptions (1) and (4).

Proof: (a) The problem (44) is the central planner's problem of maximizing the utility of the representative resident in problem (40), given the production technology in problem (36). Considering that the production function in problem (36) exhibits constant returns to scale, and the utility function in problem (40) is strictly concave, the solutions to the decentralized equilibrium in the economy is equivalent to the solution of the central planner's problem (see MWG Chapter 16). (b) The problem (44) has a strictly concave objective function and a convex choice set. Therefore, the solution to this problem exists and is necessarily unique (see Theorem 27.1 in section 27 of **??**). Thus, we only need to prove the existence of the solution to this problem. Below, we provide a step-by-step proof of existence.

Step 1. First, define the function $F(B; \{l(\theta)\}_\Theta)$ as follows, to prove that when B and $l(\theta)$ are finite

for any $\theta \in \Theta$, a solution $\left\{ Q_* (z), Q_{i,*} (z), B_* (z); L_* (z); \nu_{i,*}^L (z); \nu_{i,*}^B (z) \right\}$ exists:

$$
\begin{aligned}
F\left(B; \{l(\theta)\}_\Theta\right) \equiv & \max_{Q_*(z), Q_{i,*}(z), B_*(z); L_*(z); \nu_{i,*}^L(z); \nu_{i,*}^B(z)} \left\{ \int_0^1 \frac{\eta(z)}{1+\tau(z)} \left[\int_0^1 Q_{i,*}(z)^{\frac{\rho-1}{\rho}} di \right]^{\frac{\rho}{\rho-1}\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}} \\
s.t. Q_{i,*}(z) = & \frac{\kappa_i(z)}{1+\tau^B(z)} \nu_{i,*}^B(z) B_*(z) + \iota_i(z) \nu_{i.*}^L(z) \left[\int_0^1 \vartheta(z,\theta) \nu_{\theta,*}(z) l(\theta) f(\theta) d\theta \right]; \\
\int_0^1 B_*(z) dz = & B; \nu_{i,*}^L(z) \in \mathcal{D}_L(z), \nu_{i,*}^B(z) \in \mathcal{D}_B(z); \nu_{\theta,*}(z) \in \mathcal{D}(\theta)
\end{aligned}
$$

When $(B; l(\theta))$ is finite for any $\theta \in \Theta$, the feasible set of the above problem is known to be compact, thus ensuring the existence of a solution to the problem. Subsequently, based on Berge's Maximum Theorem, the function $F\left(B; \{l(\theta)\}_\Theta\right)$ is known to be continuous in $\left(B; \{l(\theta)\}_\Theta\right)$. Additionally, because the constraint set is on a linear space and the objective function is strictly concave, it is not difficult to see that $F\left(B; \{l(\theta)\}_\Theta\right)$ is a strictly concave function. Similarly, $F\left(B; \{l(\theta)\}_\Theta\right)$ can be found to be a homogeneous function of degree one in the inputs $\left(B; \{l(\theta)\}_\Theta\right)$. Furthermore, since the production functions are twice differentiable, the solution to the above problem is a continuously differentiable function of $\left(B; \{l(\theta)\}_\Theta\right)$.

Step 2. Given B, define the function as follows to prove that when B is finite, the solution $l_*(\theta)$ to the following problem is finite for any $\theta \in \Theta$:

$$
F(B) \equiv \max_{\{l_*(\theta)\}_\Theta} F\left(B; \{l_*(\theta)\}_\Theta\right) - \int_0^1 \phi_*(l_*(\theta)) f(\theta) d\theta
$$

Note that $\phi_*(l_*(\theta))$ satisfies $\phi_*'(0) = 0$ and $\lim_{l_*(\theta)\to\infty} \phi_*'(l_*(\theta)) = \infty$. These conditions ensure that $l_*(\theta)$ is positive and finite, so for any B, the above problem has a maximum. Since $F\left(B; \{l_*(\theta)\}_\Theta\right)$ is a strictly concave function of $\left(B; \{l_*(\theta)\}_\Theta\right)$ and $\phi_*(l_*(\theta))$ is a strictly convex function of $l_*(\theta)$, we can conclude that the value function $F(B)$ is a strictly concave function of B. Using the continuously differentiable nature of the above functions, we can determine that $F(B)$ is a continuously differentiable function of B.

Step 3. Now, prove that the following problem has a solution:

$$
\max_{B \geq 0} F(B) - B
$$

Utilizing the fact that $\frac{dF(B)}{dB}$ is monotonically decreasing, we can conclude that:

$$
\lim_{B\to+\infty} \frac{dF(B)}{dB} \leq \lim_{B\to+\infty} \frac{F(B)}{B}
$$

Notice that

$$\frac{F(B)}{B} \leq \frac{1}{\min\{1+\tau^B(z)\}} \times \max_{\int_0^1 B_*(z)dz=B} \left\{ \int_0^1 \eta(z) \left[\kappa(z) \frac{B_*(z)}{B} + \frac{\max\{1+\tau^B(z)\} \int_0^1 \iota(z)\vartheta(z,\theta)\nu_{\theta,*}(z) l(\theta) f(\theta) d\theta}{B} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}}$$

and

$$\lim_{B\to+\infty} \frac{\max\{1+\tau^B(z)\} \int_0^1 \iota(z)\vartheta(z,\theta)\nu_{\theta,*}(z) l(\theta) f(\theta) d\theta}{B} = 0.$$

We have

$$\begin{aligned}
\lim_{B\to+\infty} \frac{F(B)}{B} &\leq \lim_{B\to+\infty} \frac{1}{\min\{1+\tau^B(z)\}} \times \max_{\int_0^1 B_*(z)dz=B} \left\{ \int_0^1 \eta(z) \left[\kappa(z) \frac{B_*(z)}{B} + \frac{\max\{1+\tau^B(z)\} \int_0^1 \iota(z)\vartheta(z,\theta)\nu_{\theta,*}(z) l(\theta) f(\theta) d\theta}{B} \right]^{\frac{\sigma-1}{\sigma}} \right\} \\
&= \frac{1}{\min\{1+\tau^B(z)\}} \times \max_{\int_0^1 B_*(z)dz=B} \left\{ \int_0^1 \eta(z) \left[\kappa(z) \frac{B_*(z)}{B} \right]^{\frac{\sigma-1}{\sigma}} \right\}^{\frac{\sigma}{\sigma-1}} \\
&\leq \frac{\left\{ \int_0^1 \eta(z)^\sigma \kappa(z)^{\sigma-1} \right\}^{\frac{1}{\sigma-1}}}{\min\{1+\tau^B(z)\}}
\end{aligned}$$

Notice that Assumptions (1), we have

$$\frac{\left\{ \int_0^1 \eta(z)^\sigma \kappa(z)^{\sigma-1} \right\}^{\frac{1}{\sigma-1}}}{\min\{1+\tau^B(z)\}} < 1$$

Therefore $\lim_{B\to+\infty} \frac{F(B)}{B} < 1$ which implies a finite solution B to the above problem. When the three problems in the above three steps have solutions, the problem (44) also has a solution. □

A.2 Proof of Proposition 2

In the mainbody, we consider the strictly monotonic regulation, under which there is a cut-off for automation $z^\star \in [0,1)$ such that any tasks $z \leq z^\star$ are automated while all tasks $z \geq z^\star$ are mannual. Readers may think of another type of regulation, where all tasks below z_R are allowed to be automated, while tasks above z_R are not allowed to be automated. In the later case, the regulation may be loosed for some interval: Some tasks below z_R may not be automated if the firm finds that it is optimal to choose labor to perform the task. We call the former regulation the strictly monotonic regulation, and the later one the non-strictly monotonic regulation.

In this section, we prove Proposition 2 (see part (ii)), before which we want to prove that there is no need to consider the non-strictly monotonic regulation (see part (i)). That is to prove that the non-

strictly monotonic regulation will be reduced to be a strictly monotonic regulation. Readers who are only interested in part (ii) can skip part (i).

Part (i) Formally, we first prove that under a non-strictly monotonic regulation z_R, there exists a cutoff $z^\star$ such that any task $z \leq z^\star$ is performed by robots, while any task $z \geq z^\star$ is performed by human workers. When $z = z^\star$, the task can be performed by either a human worker or a robot. That is to say a non-strictly monotonic regulation will be reduced to be a strictly monotonic regulation and we can focus on the strictly monotonic regulation given in the main body.

Firm's problem. With the help of Lemma A.1.1 and Lemma A.1.2, for the firm in market z, we have:

(1.1) If $z > z_R$, we have

$$Q(z) = \iota(z) L(z);$$

(1.2) If $z \leq z_R$, we have

$$Q(z) = \kappa(z) B_z(z) + \iota(z) L(z)$$

In Proposition 1, we have show the exist of equilibrium. That is to say prices and wage rates are positive and finite. For any $z \in \mathbb{Z}$, one has $L(z) + B_z(z) > 0$.

If the firm employs workers, we have

$$\frac{1+\tau^B(z)}{\kappa(z)} \geq \frac{w_z(z)}{\iota(z)} = p(z) \tag{45}$$

If the firm employs robots, we have

$$\frac{w_z(z)}{\iota(z)} \geq \frac{1+\tau^B(z)}{\kappa(z)} = p(z) \tag{46}$$

In both cases, one has

$$p(z) = \eta(z) Q(z)^{-\frac{1}{\sigma}} Q^{\frac{1}{\sigma}}$$

Consumer's problem. In Proposition 1, we have show the exist of equilibrium. Therefore, $w_z(z)$ and $w(\theta)$ are positive and finite. Then by Lemma A.1.2, worker of type θ provide labor for taks z, i.e., $\nu_\theta(z) > 0$, only if

$$w(\theta) = \vartheta(z,\theta) w_z(z) \geq \vartheta(z',\theta) w_z(z'); \forall z'$$

That is to say,

$$\nu_\theta(z) > 0 \Leftrightarrow \vartheta(z,\theta) w_z(z) \geq \vartheta(z',\theta) w_z(z'), \forall z'.$$

In contrast, if task is performed by robots one has

$$\widetilde{\kappa}(z;\theta) \equiv \frac{\vartheta(z,\theta)\left[1+\tau^B(z)\right]\iota(z)}{\kappa(z)} \leq \vartheta(z,\theta) w_z(z) = w(\theta); \forall \theta \in \Theta$$

Define the set of tasks performed by robots in the economy as

$$\mathbb{Z}_B \equiv \{z|\widetilde{\kappa}(z;\theta) \leq w(\theta), \forall \theta \in \Theta\} \cap [0, z_R] \blacksquare$$

Set the set of tasks performed by workers of type θ as

$$\mathbb{Z}_B(\theta) \equiv \{z|\widetilde{\kappa}(z;\theta) \leq w(\theta)\}$$

Under Assumption 2, $\widetilde{\kappa}(z;\theta)$ is a monotonically increasing continuous function defined on the compact set $\mathbb{Z}$. Therefore, for any θ, there exists $z^\star(\theta) \leq 1$ such that $\mathbb{Z}_B(\theta) = [-1, z^\star(\theta)] \cap [0,1]$. Set $z^\star = \min\{\inf_\Theta z^\star(\theta), z_R\}$. Since $\mathbb{Z}_B = \{\cap_{\theta\in\Theta}\mathbb{Z}_B(\theta)\} \cap [0, z_R]$, we have $\mathbb{Z}_B = [-1, z^\star] \cap [0,1]$.

Similarly, set the set of tasks employing labors as:

$$\mathbb{Z}_L \equiv \{z \in \mathbb{Z}|L_z(z) > 0\}$$

Obviously, $\mathbb{Z}_L \cap \mathbb{Z}_B = \emptyset$ and $(z^\star, 1] \subset \mathbb{Z}_L$. Define $\Theta(z)$ and $\mathbf{Z}(\theta)$ as:

$$\Theta(z) \equiv \{\theta \in \Theta|\nu_\theta(z) > 0; z \in \mathbb{Z}_L\}; \mathbf{Z}(\theta) \equiv \{z \in Z|\nu_\theta(z) > 0\}.$$

$\Theta(z)$ is the set of employee types hired for the task z. $\mathbf{Z}(\theta)$ is the set of task types engaged in by employee type θ.

Part (ii). The above analysis illustrates under what circumstances tasks are performed by robots and under what circumstances they are performed by human workers. Specifically, we find that there exists a cutoff $z^\star$ such that any task $z \leq z^\star$ is performed by robots, while any task $z \geq z^\star$ is performed by human workers. When $z = z^\star$, the task can be performed by either a human worker or a robot. What remains to be proven is that within the set of tasks performed by human workers, the assignment between tasks and worker skills is positive assortative matching. This means that we need to demonstrate the existence of a continuous, monotonically increasing function from Θ to Z_L, i.e., $z(\theta) : \Theta \mapsto Z_L$, representing the job assignment.

Step 1. Prove that $\Theta(z)$ and $\mathbf{Z}(\theta)$ are not empty set.

According to the definition of $\Theta(z)$, $z \in \mathbb{Z}_L$ (i.e., it is defined only for $z \in \mathbb{Z}_L$), and at this point, $L_z(z) > 0$, which implies that $\Theta(z)$ is not an empty set. Subsequently, with the help of Proposition 1, we know that an equilibrium exists. Therefore, $w(\theta)$ must be positive at this time; otherwise, it would lead to a contradiction. This means that $\mathbf{Z}(\theta)$ cannot be an empty set.

Step 2. In this step, we prove that as correspondences, $\mathbf{Z}(\theta)$ is strictly increasing in θ (i.e., if $\theta > \theta'$, then for any $z \in \mathbf{Z}(\theta)$ and $z' \in \mathbf{Z}(\theta')$, we have $z > z'$), and $\Theta(z)$ is strictly increasing in z (i.e., if $z > z'$, then for any $\theta \in \Theta(z)$ and $\theta' \in \Theta(z')$ we have $\theta > \theta'$).

We prove the above conclusion by contradiction. If the conclusion does not hold, then there exists a

tuple $(\theta_1, \theta_2, z_1, z_2)$ such that $\theta_1 > \theta_2$ and $z_1 \leq z_2$, where $z_j \in \mathbf{Z}\left(\theta_j\right)$. This implies that:

$$\vartheta\left(z_2, \theta_2\right) w_z\left(z_2\right) \geq \vartheta\left(z_1, \theta_2\right) w_z\left(z_1\right); \vartheta\left(z_1, \theta_1\right) w_z\left(z_1\right) \geq \vartheta\left(z_2, \theta_1\right) w_z\left(z_2\right)$$

Thus, we can obtain the following results:

$$\vartheta\left(z_1, \theta_2\right) \vartheta\left(z_2, \theta_1\right) \leq \vartheta\left(z_2, \theta_2\right) \vartheta\left(z_1, \theta_1\right)$$

i.e.,

$$\frac{\vartheta\left(z_2, \theta_2\right)}{\vartheta\left(z_2, \theta_1\right)} \geq \frac{\vartheta\left(z_1, \theta_2\right)}{\vartheta\left(z_1, \theta_1\right)}$$

Since $\vartheta(z, \theta) > 0$, we can take the logarithm of both sides of the equation and divide both sides by $\theta_2 - \theta_1$:

$$\frac{\ln \vartheta\left(z_2, \theta_2\right) - \ln \vartheta\left(z_2, \theta_1\right)}{\theta_2 - \theta_1} \leq \frac{\ln \vartheta\left(z_1, \theta_2\right) - \ln \vartheta\left(z_1, \theta_1\right)}{\theta_2 - \theta_1}$$

This contradicts the fact that $\frac{\partial^2 \ln \vartheta(z, \theta)}{\partial z \partial \theta} > 0$. Similarly, it can be shown that $\Theta(z)$ is strictly increasing in z.

Step 3. In this step, we show that in the economy, there are at most countably many $\theta \in \Theta$ for which $\mathbf{Z}(\theta)$ is not a singleton set, and at most countably many $z \in \mathbb{Z}$ for which $\Theta(z)$ is not a singleton set.

First, by Step 2, we have that when $\theta_1 > \theta_2$, $\inf \mathbf{Z}\left(\theta_1\right) \geq \sup \mathbf{Z}\left(\theta_2\right)$. Set $z^{\min}(\theta) = \inf \mathbf{Z}(\theta)$, $z^{\max}(\theta) = \sup \mathbf{Z}(\theta)$; $\theta^{\min}(z) = \inf \Theta(z)$, $\theta^{\max}(z) = \sup \Theta(z)$. For $z \notin \mathbb{Z}_L$, we may define $\theta^{\min}(z) = \theta^{\max}(z) = -1$ so that for any $\theta \in \Theta(z)$, when $z \notin \mathbb{Z}_L$, it holds that $\theta \notin \Theta$. Under the above definitions, for any $\theta' \neq \theta$ and $z' \neq z$ we have■

$$\mathbf{Z}\left(\theta'\right) \cap \left(z^{\min}(\theta), z^{\max}(\theta)\right) = \varnothing; \Theta\left(z'\right) \cap \left(\theta^{\min}(z), \theta^{\max}(z)\right) = \varnothing.$$

Two different types of workers have non-overlapping sets of tasks. For two different tasks, the types of workers employed for each are non-overlapping. Because $\mathbb{Z}$ and Θ are bounded intervals on the real number line, combined with the aforementioned monotonicity, it follows that there are at most countably many $\theta \in \Theta$ for which $\mathbf{Z}(\theta)$ is not a singleton, and at most countably many $z \in \mathbb{Z}$ for which $\Theta(z)$ is not a singleton. Otherwise, $\mathbb{Z}$ and Θ would not be bounded intervals on the real number line.

Step 4. Prove that $\Theta(z)$ is a singleton set. Step 3 have proved that there are at most countably many $z \in \mathbb{Z}$ for which $\Theta(z)$ is not a singleton set. Step 4 aims to further show that for any $z \in \mathbb{Z}$, $\Theta(z)$ is a singleton set. The following proof uses contradiction. Assume $\Theta\left(z_1\right)$ is not a singleton set, where $z_1 \in \mathbb{Z}$. Since there are at most countably many $\theta \in \Theta$ for which $\mathbf{Z}(\theta)$ is not a singleton set, residents in $\left(\theta^{\min}\left(z_1\right), \theta^{\max}\left(z_1\right)\right)$ except for countably many will provide labor to firm z_1, meaning $\nu_\theta(z) = \delta\left(z - z_1\right)$ (according to the definition of the Dirac function). Considering that the sum of uncountably many non-

negative numbers cannot converge to a finite value, it follows that:

$$L_z(z_1) = \int_0^1 \nu_\theta(z)\,\vartheta(z,\theta)\,f(\theta)\,l(\theta)\,d\theta = +\infty$$

This contradict Proposition 1. In particular, it contradicts $w(\theta) > 0$. In conclusion, $\Theta(z)$ is a singleton set.

Step 5. Prove that the matching function $z(\theta)$ exists and is a strictly increasing mapping. Since $\Theta(z)$ is a singleton set, we can define a mapping $z(\theta)$, mapping the point in $\Theta(z)$ to z. Clearly, because $\Theta(z)$ is increasing with respect to z, $z(\theta)$ is an increasing function. Also, for any z, $\Theta(z)$ is non-empty, so $z(\theta)$ is a surjection. Since all $\mathbf{Z}(\theta)$ are non-empty, every individual has a corresponding task, making $z(\theta)$ a bijection from the bounded closed interval Θ to the bounded interval $\mathbb{Z}_L$. By the Intermediate Value Theorem, the function $z(\theta)$ is continuous and increasing. Since $\Theta(z)$ is strictly increasing, $z(\theta)$ is strictly increasing. Because Θ and $\mathbb{Z}_L$ are bounded intervals, and $z(\theta)$ is an increasing function, $z(\theta)$ is continuous. Since Θ is a bounded closed interval and $z(\theta)$ is continuous, $\mathbb{Z}_L$ is a bounded closed interval, meaning there exists $z^\star \in [0,1)$ such that $\mathbb{Z}_L = [z^\star, 1]$.□

A.3 Proof of Proposition 3

A non-strictly monotonic regulation is a set $\mathcal{R}^R = \{(z,i)\,|i \geq a_z^R(z), \forall z \in \mathbb{Z}\}$, where $a_z^R : \mathbb{Z} \longmapsto (0,1)$ is the cutoff of intra-sector regulation, i.e., any task with $i \geq a_z^R(z)$ in sector z is prohibit from automation. Tasks with $i < a_z^R(z)$ are not regulated, they can either be automated or not.

In contrast, a strictly monotonic regulation is a set $\mathcal{R} = \{(z,i)\,|i \geq a_z^\star(z), \forall z \in \mathbb{Z}\}$, where $a_z^\star : \mathbb{Z} \longmapsto (0,1)$ is the cutoff of intra-sector regulation. Any task with $i < a_z^\star(z)$ in sector z is automated, while others are manual. When $a_z^\star(z) = 0$, no automation in sector z is allowed. When $a_z^\star(z) = 1$, sector z is fully automated.

In the following analysis, we will first show that in the equilibrium, a non-strictly monotonic regulation will be reduced to a situtation with strictly monotonic regulation (Part (i)). We then prove Proposition 3 (Part (ii)).

Part (i). Remember that

$$\begin{aligned}
Q_i(z) &= \kappa_i(z)\,\nu_z^B(i)\,B_z(z) + \iota_i(z)\,\nu_z^L(i)\,L_z(z);\\
Q_z(z) &= \left[\int_0^1 Q_i(z)^{\frac{\rho-1}{\rho}}\,di\right]^{\frac{\rho}{\rho-1}}, 0 < \rho \leq \infty;\\
Q &= A\left[\int_0^1 \eta(z)\,Q_z(z)^{\frac{\sigma-1}{\sigma}}\,dz\right]^{\frac{\sigma}{\sigma-1}}, 0 < \sigma, A \leq \infty.
\end{aligned}$$

where

$$B_z(z) = \int_{\mathbb{I}_B} B_i(z)\,di, L_z(z) = \int_{\mathbb{I}} L_i(z)\,di$$

with $B_i(z) = \nu_z^B(i) B_z(z)$, $L_i(z) = \nu_z^L(i) L_z(z)$, $\mathbb{I}_B = \{i | i \leq a_z^R(z)\}$. By Proposition 1, $B_i(z)$ and $L_i(z)$ are non-negative, and wage rate and labor input $L_z(z)$ are positive.

Given prices, tax and regulatory policies, the representative producer chooses inputs $B_i(z)$ and $L_i(z)$ to maximize the profit:

$$Q - \int_{\mathbb{Z}} \left[1 + \tau^B(z)\right] \int_{\mathbb{I}^B} B_i(z) \, didz - \int_{\mathbb{Z}} \int_{\mathbb{I}} w_z(z) L_i(z) \, didz,$$

such that $B_i(z) \geq 0$, $L_i(z) \geq 0$. Set the following Lagrange function to solve the problem:

$$\begin{aligned}\mathcal{L} = Q &- \int_{\mathbb{Z}} \left[1 + \tau^B(z)\right] \int_0^{a_z^R(z)} B_i(z) \, didz - \int_{\mathbb{Z}} \int_{\mathbb{I}} w_z(z) L_i(z) \, didz \\ &+ \int_{\mathbb{Z}} \int_0^{a_z^R(z)} \lambda_i^B(z) B_i(z) \, didz + \int_{\mathbb{Z}} \int_{\mathbb{I}} \lambda_i^L(z) L_i(z) \, didz\end{aligned}$$

We have the following FOCs:

$$\frac{\partial \mathcal{L}}{\partial L_i(z)} = \frac{\partial \mathcal{L}}{\partial B_i(z)} = 0, \lambda_i^B(z) B_i(z) = \lambda_i^L(z) L_i(z) = 0, \lambda_i^B(z) \geq 0, \lambda_i^L(z) \geq 0.$$

which gives

$$\begin{aligned} A^{1-\frac{1}{\sigma}} \eta(z) Q^{\frac{1}{\sigma}} Q_z(z)^{\frac{1}{\rho}-\frac{1}{\sigma}} Q_i(z)^{-\frac{1}{\rho}} \kappa_i(z) &= 1 + \tau^B(z) - \lambda_i^B(z), i \leq a_z^R(z) \\ B_i(z) &= 0, i > a_z^R(z) \\ A^{1-\frac{1}{\sigma}} \eta(z) Q^{\frac{1}{\sigma}} Q_z(z)^{\frac{1}{\rho}-\frac{1}{\sigma}} Q_i(z)^{-\frac{1}{\rho}} \iota_i(z) &= w_z(z) - \lambda_i^B(z), i \in [0,1] \end{aligned}$$

By Proposition 1, $Q_i(z) > 0$. Therefore, $L_i(z)$ and $B_i(z)$ can not be zero at the same time. Then, there are two situtations:

(a) there does not exist $i^*(z) \in \left[0, a_z^R(z)\right]$ so that for $i = i^*(z)$, $\lambda_i^B(z) = \lambda_i^L(z) = 0$

(b) there exist $i^*(z) \in \left[0, a_z^R(z)\right]$ so that for $i = i^*(z)$, $\lambda_i^B(z) = \lambda_i^L(z) = 0$.

In the later case, for $i = i^*(z)$

$$\frac{\kappa_i(z)}{\iota_i(z)} = \frac{1 + \tau^B(z)}{w_z(z)}$$

Since $\frac{\kappa_i(z)}{\iota_i(z)}$ decreases in i, one has $B_i(z) > 0$ and $L_i(z) = 0$ for $i \leq i^*(z)$; $L_i(z) > 0$ and $B_i(z) = 0$ for $i > i^*(z)$. In the former case, tasks $i \in \left[0, a_z^R(z)\right]$ are all automated or manual. In both cases, there exist $a_z^\star(z) = i^*(z) \in (0,1)$, so that for any $i \leq a_z^\star(z)$, tasks are automated, while other tasks are manual.

Part (ii). We have proved that there exist $a_z^\star(z) \in (0,1)$, so that for any $i \leq a_z^\star(z)$, tasks are automated, while other tasks are manual. We can then reduce the producer's problem. We have the following solution:

$$B_i(z) = \frac{\kappa_i(z)^{\rho-1}}{\int_0^{a_z^\star(z)} \kappa_i(z)^{\rho-1} di} B_z(z) \text{ and } L_i(z) = \frac{\iota_i(z)^{\rho-1}}{\int_{a_z^\star(z)}^1 \iota_i(z)^{\rho-1} di} L_z(z).$$

Moreover, we have (see (47))

$$Q_z\left(B_z\left(z\right),L_z\left(z\right);a_z^{\star}\left(z\right)\right)=\left[\left[\left(\int_0^{a_z^{\star}(z)}\kappa_i\left(z\right)^{\rho-1}di\right)^{\frac{1}{\rho-1}}B_z\left(z\right)\right]^{\frac{\rho-1}{\rho}}+\left[\left(\int_{a_z^{\star}(z)}^1\iota_i\left(z\right)^{\rho-1}di\right)^{\frac{1}{\rho-1}}L_z\left(z\right)\right]^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}.$$

Set $\Theta\left(z\right)$ and $\mathbf{Z}\left(\theta\right)$ as

$$\Theta\left(z\right)=\left\{\theta\in\Theta|\nu_\theta\left(z\right)>0\right\};\mathbf{Z}\left(\theta\right)=\left\{z\in Z|\nu_\theta\left(z\right)>0\right\};$$

$\Theta\left(z\right)$ is the set of employee types hired for the task z. $\mathbf{Z}\left(\theta\right)$ is the set of task types engaged in by employee type θ.

Step 1. Prove that $\Theta\left(z\right)$ and $Z\left(\theta\right)$ are not empty. We provide a proof by contradiction.

The FOC with respect to $B_z\left(z\right)$ and $L\left(z\right)$ gives:

$$\left\{p\left(z\right)\frac{\partial Q_z\left(z\right)}{\partial B_z\left(z\right)}-\left[1+\tau^B\left(z\right)\right]\right\}B_z\left(z\right)=0;\left\{p\left(z\right)\frac{\partial Q_z\left(z\right)}{\partial L\left(z\right)}-w_z\left(z\right)\right\}L\left(z\right)=0$$

where

$$p\left(z\right)=\eta\left(z\right)Q_z\left(z\right)^{-\frac{1}{\sigma}}Q^{\frac{1}{\sigma}}$$

By Proposition 1, equilibrium exist and $p\left(z\right),w_z\left(z\right)$ are positve. Therefore, for any z one has $L_z\left(z\right)>0$ and $B_z\left(z\right)>0$■

If $\Theta\left(z\right)$ is empty, one has $L_z\left(z\right)=0$, which contradict the above finding. Therefore, $\Theta\left(z\right)$ is not empty.

Second, by Proposition 1, $w_z\left(z\right)$ and $w\left(\theta\right)$ are positve. Therefore $\mathbf{Z}\left(\theta\right)$ is not empty, otherwise $w\left(\theta\right)=0$.

Step 2. Prove that corresponing $\mathbf{Z}\left(\theta\right)$ strictly increases in θ, and $\Theta\left(z\right)$ strictly increases in z. Taking $\mathbf{Z}\left(\theta\right)$ as an example. It means that for any $\theta_1>\theta_2$ and $Z\left(\theta_1\right)$, $Z\left(\theta_2\right)\neq\varnothing$, one must has $\inf\left(\mathbf{Z}\left(\theta_1\right)\right)>\sup\left(\mathbf{Z}\left(\theta_2\right)\right)$. $\theta_1,\theta_2\in\Theta$. We provide a proof by contradiction.

Suppose that there are $\theta_1>\theta_2$ and $z_1\leq z_2$ such that $z_j\in\mathbf{Z}\left(\theta_j\right)$ and $\theta_j\in\Theta$, $j=1,2$. Then, one has

$$\vartheta\left(z_2,\theta_2\right)w_z\left(z_2\right)\geq\vartheta\left(z_1,\theta_2\right)w_z\left(z_1\right);\vartheta\left(z_1,\theta_1\right)w_z\left(z_1\right)\geq\vartheta\left(z_2,\theta_1\right)w_z\left(z_2\right)$$

Therefore

$$\vartheta\left(z_1,\theta_2\right)\vartheta\left(z_2,\theta_1\right)\leq\vartheta\left(z_2,\theta_2\right)\vartheta\left(z_1,\theta_1\right)\Leftrightarrow\frac{\vartheta\left(z_2,\theta_2\right)}{\vartheta\left(z_2,\theta_1\right)}\geq\frac{\vartheta\left(z_1,\theta_2\right)}{\vartheta\left(z_1,\theta_1\right)}$$

Remember that $\vartheta\left(z,\theta\right)>0$, the above inequality gives:

$$\frac{\ln\vartheta\left(z_2,\theta_2\right)-\ln\vartheta\left(z_2,\theta_1\right)}{\theta_2-\theta_1}\leq\frac{\ln\vartheta\left(z_1,\theta_2\right)-\ln\vartheta\left(z_1,\theta_1\right)}{\theta_2-\theta_1}$$

The above finding contradicts $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial z \partial \theta} > 0$.

In conclusion, when $\theta_1 > \theta_2$, one has $\inf \mathbf{Z}(\theta_1) \geq \sup \mathbf{Z}(\theta_2)$. Similarly, $\Theta(z)$ strictly increases in z. Moreover, since $\mathbb{Z}$ and Θ is a bounded real interval, there are at most countable number of $\theta \in \Theta$ such that $\mathbf{Z}(\theta)$ is not a singleton set; at most countable number of $z \in \mathbb{Z}$ such that $\Theta(z)$ is not a singleton set.

Step 3. Prove that for any z, $\Theta(z)$ is a singleton.

Suppose that there exist $z_1 \in Z$ such that $\Theta(z_1)$ is not a singleton. Notice that $Z(\theta)$ is not empty. One can see that for $\theta \in \left(\theta^{\min}(z_1), \theta^{\max}(z_1)\right)$, almost every one will provide labor to sector z_1, that is $\nu_\theta(z) = \delta(z - z_1)$. By Proposition 1, $w(\theta) > 0$. Therefore $l(\theta) > 0$. Considering that the sum of uncountably many non-negative numbers cannot converge to a finite value, it follows that:

$$L(z) = \int_0^1 \nu_\theta(z)\, \vartheta(z,\theta)\, f(\theta)\, l(\theta)\, d\theta = +\infty$$

Then, one must has $p(z) = w_z(z) = 0$ in the equilibrium, which means the wage rate of workers belong to $\Theta(z)$ is zero. This finding contradict $w(\theta) > 0$. In conclusion, $\Theta(z)$ is a singleton.

Step 4. Prove that there exist a matching function $z : \Theta \mapsto \mathbb{Z}$ and it is a monotonically increasing mapping.

Notice that $\Theta(z)$ is a singleton. We can define a mapping $z(\theta)$ from $\Theta(z)$ to z. Since $\Theta(z)$ monotonically increases in z, $z(\theta)$ must be a monotonically increasing function. On the other hand, since for any z, $\Theta(z)$ is not empty, $z(\theta)$ must be a surjection. Remember that for any θ, $\mathbf{Z}(\theta)$ is not empty. Therefore, any worker has a task they are engaged in. In conclusion, $z(\theta)$ is a bijection from the bounded closed interval Θ to the bounded closed interval $\mathbb{Z}$. According to the Intermediate Value Theorem, the function $z(\theta)$ is continuous and monotonically increasing. Since $\Theta(z)$ is a strictly monotonically increasing counterpart, it follows that $z(\theta)$ is strictly monotonically increasing.□

A.4 Decentralized Economy

A.4.1 Aggregated Production with and without Regulation

Firm-level Production with Regulation. Now we consider another technology:

$$\kappa_i(z) = \varsigma(z)\, i^{\frac{\tilde{\zeta}(z)-1}{\tilde{\zeta}(z)}} \quad \text{and} \quad \iota_i(z) = \varsigma(z)\,(1-i)^{\frac{\tilde{\zeta}(z)-1}{\tilde{\zeta}(z)}}.$$

We first solve (**??**):

$$Q_z(B_z(z), L_z(z); a_z(z)) \equiv \max_{B_i(z), L_i(z)} \left\{ \left[\int_0^1 Q_i(z)^{\frac{\rho-1}{\rho}}\, di\right]^{\frac{\rho}{\rho-1}} | B_z(z) = \int_0^{a_z(z)} B_i(z)\, di, L_z(z) = \int_{a_z(z)}^1 L_i(z)\, di. \right\}$$

Solving the above problem gives:

$$B_i(z) = \frac{\kappa_i(z)^{\rho-1}}{\int_0^i \kappa_i(z)^{\rho-1} di} B_z(z) \text{ and } L_i(z) = \frac{\iota_i(z)^{\rho-1}}{\int_i^1 \iota_i(z)^{\rho-1} di} L_z(z).$$

Then, we have:

$$\begin{aligned}
&Q_z(B_z(z), L_z(z); a_z(z)) \\
&= \left[\int_0^{a_z(z)} (\kappa_i(z) B_i)^{\frac{\rho-1}{\rho}} + \int_{a_z(z)}^1 (\iota_i(z) L_i)^{\frac{\rho-1}{\rho}} di\right]^{\frac{\rho}{\rho-1}} \\
&= \left[\int_0^{a_z(z)} \left(\kappa_i(z)^\rho \frac{B_z(z)}{\int_0^i \kappa_i(z)^{\rho-1} di}\right)^{\frac{\rho-1}{\rho}} di + \int_{a_z(z)}^1 \left(\iota_i(z)^\rho \frac{L_z(z)}{\int_i^1 \iota_i(z)^{\rho-1} di}\right)^{\frac{\rho-1}{\rho}} di\right]^{\frac{\rho}{\rho-1}} \\
&= \left[\left[\left(\int_0^{a_z(z)} \kappa_i(z)^{\rho-1} di\right)^{\frac{1}{\rho-1}} \cdot B_z(z)\right]^{\frac{\rho-1}{\rho}} + \left[\left(\int_{a_z(z)}^1 \iota_i(z)^{\rho-1} di\right)^{\frac{1}{\rho-1}} \cdot L_z(z)\right]^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}} \\
&= \left[\frac{(\xi(z)-1)(\rho-1)}{\xi(z)} + 1\right]^{\frac{-1}{\rho-1}} \left[\frac{a_z(z)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}}{\varsigma(z)^{\frac{1-\rho}{\rho}}} B_z(z)^{\frac{\rho-1}{\rho}} + \frac{(1-a_z(z))^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}}{\varsigma(z)^{\frac{1-\rho}{\rho}}} L_z(z)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}},
\end{aligned} \tag{47}$$

where $\varsigma(z) = \left[1 + \frac{[\xi(z)-1](\rho-1)}{\xi(z)}\right]^{\frac{1}{\rho-1}}$ and the last equation is derived by:

$$\left(\int_0^{a_z(z)} \kappa_i^{\rho-1} di\right)^{\frac{1}{\rho-1}} = \left(\int_0^{a_z(z)} \left[\varsigma(z) i^{\frac{\xi(z)-1}{\xi(z)}}\right]^{\rho-1} di\right)^{\frac{1}{\rho-1}} = \varsigma(z) \left(\frac{a_z(z)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}}{1 + \left(\frac{\xi(z)-1}{\xi(z)}\right)(\rho-1)}\right)^{\frac{1}{\rho-1}}$$

and

$$\left(\int_{a_z(z)}^1 \iota_i^{\rho-1} di\right)^{\frac{1}{\rho-1}} = \left(\int_{a_z(z)}^1 \left[\varsigma(z)(1-i)^{\frac{\xi(z)-1}{\xi(z)}}\right]^{\rho-1} di\right)^{\frac{1}{\rho-1}} = \varsigma(z) \left(\frac{(1-a_z(z))^{1+\frac{\xi(z)-1}{\xi(z)}(\rho-1)}}{1 + \frac{\xi(z)-1}{\xi(z)}(\rho-1)}\right)^{\frac{1}{\rho-1}}.$$

Remember that $\varsigma(z) = \left[1 + \frac{[\xi(z)-1](\rho-1)}{\xi(z)}\right]^{\frac{1}{\rho-1}}$. Equiation (47) implies (17).

$$Q_z(B_z(z), L_z(z); a_z(z)) = \left[a_z(z)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} B_z(z)^{\frac{\rho-1}{\rho}} + [1-a_z(z)]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} L_z(z)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}.$$

In addition, we have

$$Q_z^u(B_z(z), L_z(z)) = \left[B_z(z)^{\xi(z)} + L_z(z)^{\xi(z)}\right]^{\frac{1}{\xi(z)}}, a_z^u(z) = \frac{1}{1 + \left[\frac{L_z(z)}{B_z(z)}\right]^{\xi(z)}}.$$

For later use, set

$$\widetilde{\kappa}\left(z,a\left(z\right)\right)=\left(\int_{0}^{a_{z}\left(z\right)}\kappa_{i}\left(z\right)^{\rho-1}di\right)^{\frac{1}{\rho-1}}\text{ and }\widetilde{\iota}\left(z,a\left(z\right)\right)=\left(\int_{a_{z}\left(z\right)}^{1}\iota_{i}\left(z\right)^{\rho-1}di\right)^{\frac{1}{\rho-1}} \tag{48}$$

In the general case, we have

$$Q_{z}\left(B_{z}\left(z\right),L_{z}\left(z\right);a_{z}\left(z\right)\right)=\left[\left[\widetilde{\kappa}\left(z,a\left(z\right)\right)B_{z}\left(z\right)\right]^{\frac{\rho-1}{\rho}}+\left[\widetilde{\iota}\left(z,a\left(z\right)\right)L_{z}\left(z\right)\right]^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}} \tag{49}$$

In the special case, we have

$$\widetilde{\kappa}\left(z,a\left(z\right)\right)=a_{z}\left(z\right)^{\left[1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}\right]\frac{\rho}{\rho-1}}\text{ and }\widetilde{\iota}\left(z,a\left(z\right)\right)=\left[1-a_{z}\left(z\right)\right]^{\left[1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}\right]\frac{\rho}{\rho-1}}$$

Aggregated Production Function with Regulation. As in our benchmark model, the labour and capital market clear require:

$$\int_{0}^{\theta}\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)d\theta=\int_{0}^{z(\theta)}L_{z}\left(z\right)dz\quad\text{and}\quad\int_{0}^{\theta}B\left(\theta\right)d\theta=\int_{0}^{z(\theta)}B_{z}\left(z\right)d\theta,\forall\theta\in\Theta.$$

Equivalently,

$$\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)=L_{z}\left(z\left(\theta\right)\right)z'\left(\theta\right)\quad\text{and}\quad B\left(\theta\right)=B_{z}\left(z\left(\theta\right)\right)z'\left(\theta\right),\forall\theta\in\Theta. \tag{50}$$

Therefore, we have $Q_{z}\left(z\left(\theta\right)\right)=Q_{z}\left(\frac{B(\theta)}{z'(\theta)},\frac{\vartheta(z(\theta),\theta)L(\theta)}{z'(\theta)};a\left(z\left(\theta\right)\right)\right)$.

Set $a\left(\theta\right)=a_{z}\left(z\left(\theta\right)\right)$ and $\mathbb{A}\equiv\left\{a\left(\theta\right)\right\}_{\theta\in\Theta}$. Let $\mathcal{B}\equiv\left\{B\left(\theta\right)\right\}_{\theta\in\Theta}$ and $\mathcal{L}\equiv\left\{L\left(\theta\right)\right\}_{\theta\in\Theta}$. We can translate the production technology on tasks to a production technology on skills by:

$$\mathcal{F}_{\theta}\left(\mathcal{B},\mathcal{L},\mathbb{A}\right)\equiv\left[\int_{\underline{\theta}}^{\overline{\theta}}\eta\left(z\right)Q_{z}\left(\frac{B\left(\theta\right)}{z'\left(\theta\right)},\frac{\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)}{z'\left(\theta\right)};a\left(\theta\right)\right)^{\frac{\sigma-1}{\sigma}}dz\left(\theta\right)\right]^{\frac{\sigma}{\sigma-1}}, \tag{51}$$

The resource constraint is

$$\mathcal{F}_{\theta}\left(\mathcal{B},\mathcal{L},\mathbb{A}\right)-\int_{\Theta}B\left(\theta\right)d\theta-\int_{\Theta}c\left(\theta\right)f_{\theta}\left(\theta\right)d\theta\geq0. \tag{52}$$

Although our economy results in a production function of constant elasticity of substitution (CES) as is in Katz and Murphy (1992), Katz and Autor (1999), Autor et al. (2008), the wage elasticity here are quite different from the one without job assignment (see 22). As emphasized by Acemoglu et al. (2020), "though the output level in the economy can be represented by a CES aggregate of capital and labor, the implications of this setup are very different from models that assume a CES production function with factor-augmenting technologies." Moreover, our policy design suggests the crucial of distinguish the substitution effect and automation effect of robot inputs.

Technology (51) is also a generalization of the one considered by Acemoglu and Restrepo (2018) and Acemoglu and Restrepo (2020), in the sense that we introduce heterogeneous labors. Last, it extends the two-type-two-sector technology considered by Guerreiro et al. (2021) into one with a continuum of labors and job assignment.

Firm-level Production without Regulation. Now that we have derived the expression for $Q_z\left(B_z\left(z\right), L_z\left(z\right); a\right)$, i.e., (17), we can solve problem (**??**) with (17). The first-order condition of problem (**??**) implies:

$$a_z\left(z\right)^{-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}\left(B_z\left(z\right)\right)^{\frac{\rho-1}{\rho}} = \left(1-a_z\left(z\right)\right)^{-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}\left(L_z\left(z\right)\right)^{\frac{\rho-1}{\rho}}.$$

Rearranging the above equation gives:

$$\left(\frac{B_z\left(z\right)}{L_z\left(z\right)}\right) = \left(\frac{1-a_z\left(z\right)}{a_z\left(z\right)}\right)^{-\frac{1}{\xi(z)}}.$$

Therefore, we have equation (**??**):

$$a_z^u\left(z\right) = a\left(z, L_z\left(z\right), B_z\left(z\right)\right) \equiv \frac{1}{1+\left[\frac{L_z(z)}{B_z(z)}\right]^{\xi(z)}}.$$

We also have

$$a_z^u\left(z\right) = \frac{1}{1+\frac{\partial \ln Q_z^u(z)}{\partial \ln L_z(z)} \Big/ \frac{\partial \ln Q_z^u(z)}{\partial \ln B_z(z)}} \tag{53}$$

because $\frac{\partial \ln Q_z^*(z)}{\partial \ln L_z^*(z)} = \frac{L_z(z)^{\xi(z)}}{B_z(z)^{\xi(z)}+L_z(z)^{\xi(z)}}$ and $\frac{\partial \ln Q_z^*(z)}{\partial \ln B_z^*(z)} = \frac{B_z(z)^{\xi(z)}}{B_z(z)^{\xi(z)}+L_z(z)^{\xi(z)}}$.

We have

$$\begin{aligned}
\frac{\partial \ln\left[1-a_z^u\left(z\right)\right]}{\partial \ln B_z\left(z\right)} &= \frac{\xi\left(z\right)\left[\frac{L_z(z)}{B_z(z)}\right]^{\xi(z)-1}\frac{L_z(z)}{B_z(z)^2}}{\left[1+\left[\frac{L_z(z)}{B_z(z)}\right]^{\xi(z)}\right]^2}\frac{B_z\left(z\right)}{1-\frac{1}{1+\left[\frac{L_z(z)}{B_z(z)}\right]^{\xi(z)}}} \\
&= \frac{\xi\left(z\right)}{1+\left[\frac{L_z(z)}{B_z(z)}\right]^{\xi(z)}} \\
&= -\xi a^u\left(z\right).
\end{aligned}$$

Last, substitute $a_z\left(z\right)$ in $Q_z\left(B_z\left(z\right), L_z\left(z\right); a_z\left(z\right)\right)$ by $a_z\left(z\right) = a_z^u\left(z\right)$, we have

$$Q_z^u\left(B_z\left(z\right), L_z\left(z\right)\right) = \left[B_z\left(z\right)^{\xi(z)} + L_z\left(z\right)^{\xi(z)}\right]^{\frac{1}{\xi(z)}}$$

This is (**??**). In addition

$$Q=\left[\int_{\Theta}\eta\left(z\left(\theta\right)\right)Q_{z}^{u}\left(\frac{B\left(\theta\right)}{\delta\left(\theta\right)},\frac{\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)}{\delta\left(\theta\right)}\right)^{\frac{\sigma-1}{\sigma}}\delta\left(\theta\right)d\theta\right]^{\frac{\sigma}{\sigma-1}}$$

A.4.2 Wage Elasticities with and without Regulation

Wage Elasticities with Regulation. By definition of wage rate (20) and technology (2), and $w\left(\theta\right)=\vartheta\left(z\left(\theta\right),\theta\right)w_{z}\left(z\left(\theta\right)\right)$, we have

$$w\left(\theta\right)=\omega\left(\theta,z\left(\theta\right),\delta\left(\theta\right),B\left(\theta\right),L\left(\theta\right);a\left(\theta\right)\right)Q^{\frac{1}{\sigma}}, \tag{54}$$

where $\omega\left(\theta,z\left(\theta\right),\delta\left(\theta\right),B\left(\theta\right),L\left(\theta\right);a\left(z\left(\theta\right)\right)\right)\equiv\vartheta\left(z\left(\theta\right),\theta\right)\eta\left(z\left(\theta\right)\right)Q_{z}\left(\frac{B(\theta)}{\delta(\theta)},\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)};a\left(\theta\right)\right)^{\frac{-1}{\sigma}}\frac{\partial Q_{z}\left(\frac{B(\theta)}{\delta(\theta)},\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)};a(\theta)\right.}{\partial\left(\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right)}$

$$\begin{aligned}\frac{\partial Q_{z}\left(\frac{B(\theta)}{\delta(\theta)},\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)};a\left(\theta\right)\right)}{\partial\left(\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right)}&=\left[a_{z}\left(z\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}B_{z}\left(z\right)^{\frac{\rho-1}{\rho}}+\left[1-a_{z}\left(z\right)\right]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}L_{z}\left(z\right)^{\frac{\rho-1}{\rho}}\right]^{\frac{1}{\rho-1}}\left[1-a_{z}\left(z\right)\right]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}L_{z}\left(z\right)^{\frac{-1}{\rho}}\\&=Q_{z}\left(\frac{B\left(\theta\right)}{\delta\left(\theta\right)},\frac{\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)}{\delta\left(\theta\right)};a\left(\theta\right)\right)^{\frac{1}{\rho}}\left[1-a_{z}\left(z\right)\right]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}L_{z}\left(z\right)^{\frac{-1}{\rho}}\end{aligned}$$

In the equilibrium, we have:

$$\begin{aligned}&\omega\left(\theta,z\left(\theta\right),\frac{B\left(\theta\right)}{\delta\left(\theta\right)},\frac{\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)}{\delta\left(\theta\right)};a_{z}\left(z\left(\theta\right)\right)\right) \qquad (55)\\&=\vartheta\left(z\left(\theta\right),\theta\right)\eta\left(z\left(\theta\right)\right)Q_{z}\left(\frac{B\left(\theta\right)}{\delta\left(\theta\right)},\frac{\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)}{\delta\left(\theta\right)};a\left(z\left(\theta\right)\right)\right)^{\frac{1}{\rho}-\frac{1}{\sigma}}\left[1-a\left(z\left(\theta\right)\right)\right]^{1-\frac{1}{\xi(z(\theta))}\frac{\rho-1}{\rho}}\left[\frac{\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)}{\delta\left(\theta\right)}\right]^{\frac{-1}{\rho}}.\end{aligned}$$

Following Sachs et al. (2020), we define the own and cross wage elasticities by:

$$\begin{aligned}\gamma_{own}^{L}\left(z\left(\theta\right)\right)&\equiv\frac{\partial\ln\omega\left(\theta,z\left(\theta\right),\delta\left(\theta\right),B\left(\theta\right),L\left(\theta\right);a\left(z\left(\theta\right)\right)\right)}{\partial\ln L(\theta)},\gamma_{cross}^{L}\left(z,z'\right)\equiv\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln Q_{z}(z')}\frac{\partial\ln Q_{z}(z')}{\partial\ln L_{z}(z')}.\\\gamma_{own}^{B}\left(z\left(\theta\right)\right)&\equiv\frac{\partial\ln\omega\left(\theta,z\left(\theta\right),\delta\left(\theta\right),B\left(\theta\right),L\left(\theta\right);a\left(z\left(\theta\right)\right)\right)}{\partial\ln B\left(\theta\right)},\gamma_{cross}^{B}\left(z,z'\right)\equiv\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln Q_{z}(z')}\frac{\partial\ln Q_{z}(z')}{\partial\ln B_{z}(z')}\\\gamma_{own}^{a}\left(z\left(\theta\right)\right)&\equiv\frac{\partial\ln\omega\left(\theta,z\left(\theta\right),\delta\left(\theta\right),B\left(\theta\right),L\left(\theta\right);a\left(z\left(\theta\right)\right)\right)}{\partial\ln a\left(z\left(\theta\right)\right)},\gamma_{cross}^{i}\left(z,z'\right)\equiv\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln Q_{z}(z')}\frac{\partial\ln Q_{z}(z')}{\partial\ln i(z')}\end{aligned}$$

We have

$$\begin{aligned}\gamma_{own}^{B}(z) &= \left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{\partial \ln Q_z\left(B_z(z), L_z(z); a_z(z)\right)}{\partial \ln B_z(z)} \\ &= \frac{\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\left(B_z(z)\right)^{\frac{\rho-1}{\rho}} i}{\left(B_z(z)\right)^{\frac{\rho-1}{\rho}} a_z(z) + \left(L_z(z)\right)^{\frac{\rho-1}{\rho}}\left(1-a_z(z)\right)^{\frac{-1}{\xi(z)}\frac{\rho-1}{\rho}+1}} \\ &= \frac{\frac{1}{\rho}-\frac{1}{\sigma}}{1+\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}\left(\frac{1-a_z(z)}{a_z(z)}\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}}, \forall z \in \mathbb{Z}.\end{aligned}$$

and

$$\begin{aligned}\gamma_{own}^{L}(z) &= \frac{\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\left(L_z(z)\right)^{\frac{\rho-1}{\rho}}\left(1-a_z(z)\right)^{\frac{-1}{\xi(z)}\frac{\rho-1}{\rho}+1}}{\left(B_z(z)\right)^{\frac{\rho-1}{\rho}} a_z(z)^{\frac{-1}{\xi(z)}\frac{\rho-1}{\rho}+1} + \left(L_z(z)\right)^{\frac{\rho-1}{\rho}}\left(1-a_z(z)\right)^{\frac{-1}{\xi(z)}\frac{\rho-1}{\rho}+1}} - \frac{1}{\rho} \\ &= \frac{\frac{1}{\rho}-\frac{1}{\sigma}}{1+\left(\frac{B_z(z)}{L_z(z)}\right)^{\frac{\rho-1}{\rho}}\left(\frac{a_z(z)}{1-a_z(z)}\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}} - \frac{1}{\rho}, \forall z \in \mathbb{Z}.\end{aligned}$$

and

$$\gamma_{own}^{a}(z) = \left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{\partial \ln Q_z(z)}{\partial \ln a_z(z)} + \left(\frac{1}{\xi(z)}\frac{\rho-1}{\rho}-1\right)\frac{a_z(z)}{1-a_z(z)}$$

In addition, we have

$$\begin{aligned}\gamma_{own}^{z}(z(\theta)) &= \frac{\partial \ln \omega\left(\theta, z(\theta), \frac{B(\theta)}{\delta(\theta)}, \frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}; a(\theta)\right)}{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]}\frac{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]}{\partial \ln z(\theta)} + \frac{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]}{\partial \ln z(\theta)} \\ &= \left[1+\gamma_{own}^{L}(z(\theta))\right]\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln z(\theta)} \\ \gamma_{own}^{\delta}(z(\theta)) &= \frac{\partial \ln \omega\left(\theta, z(\theta), \frac{B(\theta)}{\delta(\theta)}, \frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}; a(\theta)\right)}{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]}\frac{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]}{\partial \ln \delta(\theta)} \\ &\quad + \frac{\partial \ln \omega\left(\theta, z(\theta), \frac{B(\theta)}{\delta(\theta)}, \frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}; a(\theta)\right)}{\partial \ln\left[\frac{B(\theta)}{\delta(\theta)}\right]}\frac{\partial \ln\left[\frac{B(\theta)}{\delta(\theta)}\right]}{\partial \ln \delta(\theta)} \\ &= -\gamma_{own}^{L}(z(\theta)) - \gamma_{own}^{B}(z(\theta)) \\ &= \frac{1}{\sigma}\end{aligned}$$

Wage Elasticities without Regulation. When automation rates are unregulated, we have

$$w_z^u(z) = \frac{\partial Q_z^u\left(B_z(z), L_z(z)\right)}{\partial L_z(z)}\frac{\partial Q}{\partial Q_z^u\left(B_z^*(z), L_z^*(z)\right)}$$

By definition of wage rate (**??**) and technology (2), and $w(\theta) = \vartheta(z(\theta),\theta) w_z^u(z(\theta))$, we have

$$w(\theta) = \omega^u(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta)) Q^{*\frac{1}{\sigma}}, \tag{56}$$

where

$$\begin{aligned}
&\omega^u(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta)) \\
&\equiv \vartheta(z(\theta),\theta)\eta(z(\theta)) Q_z^u \left(\frac{B(\theta)}{\delta(\theta)}, \frac{\vartheta(z(\theta),\theta) L(\theta)}{\delta(\theta)}\right)^{\frac{\sigma-1}{\sigma}} \frac{\left(\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right)^{\xi(z)-1}}{\left(\frac{B(\theta)}{\delta(\theta)}\right)^{\xi(z)} + \left(\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right)^{\xi(z)}}.
\end{aligned}$$

and $Q = A\left[\int_0^1 \eta(z) Q_z^u(z)^{\frac{\sigma-1}{\sigma}} dz\right]^{\frac{\sigma}{\sigma-1}}$.

According to (55), when the automation rate is chosen by the firm, we have:

$$\begin{aligned}
\gamma_{own}^{B,u}(z) = &\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)\left[\frac{\partial \ln Q_z(B_z(z), L_z(z); a_z^u(z))}{\partial \ln B_z(z)} + \frac{\partial \ln Q_z(B_z(z), L_z(z); a_z^u(z))}{\partial \ln a_z^u(z)} \frac{\partial \ln a_z^u(z)}{\partial \ln B_z(z)}\right] \\
&+ \left(1 - \frac{1}{\xi(z)}\frac{\rho-1}{\rho}\right) \frac{\partial \ln[1 - a_z^u(z)]}{\partial \ln B_z(z)},
\end{aligned}$$

where $\frac{\partial \ln[1-a_z^u(z)]}{\partial \ln B_z(z)} = -\xi a_z^u(z)$ and, by envelop theory, $\frac{\partial \ln Q_z(B_z(z), L_z(z); a_z(z))}{\partial \ln a_z(z)} = 0$. Therefore

$$\begin{aligned}
\gamma_{own}^{B,u}(z) &= \frac{\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)(B_z(z))^{\frac{\rho-1}{\rho}} a_z^u(z)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}}{(B_z(z))^{\frac{\rho-1}{\rho}} a_z^u(z)^{\frac{-1}{\xi(z)}\frac{\rho-1}{\rho}+1} + (L_z(z))^{\frac{\rho-1}{\rho}} (1 - a_z^u(z))^{\frac{-1}{\xi(z)}\frac{\rho-1}{\rho}+1}} + \left[\frac{\rho-1}{\rho} - \xi(z)\right] a_z^u(z) \\
&= \frac{\frac{1}{\rho} - \frac{1}{\sigma}}{1 + \left[\frac{1-a_z^u(z)}{a_z^u(z)}\right]^{\frac{\rho-1}{\rho}\frac{1}{\xi(z)}} \left(\frac{1-a_z^u(z)}{a_z^u(z)}\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}} + \left[\frac{\rho-1}{\rho} - \xi(z)\right] a_z^u(z) \\
&= \left(\frac{1}{\rho} - \frac{1}{\sigma}\right) a_z^u(z) + \left[\frac{\rho-1}{\rho} - \xi(z)\right] a_z^u(z), \forall z \in \mathbb{Z}.
\end{aligned}$$

The second equation is derived by the definition of $a_z(z)$. In the equilibrium $a_z(z) = a_z(z)$, which implies the last equation. Similarly, we have:

$$\begin{aligned}
\gamma_{own}^{L,u}(z) &= \frac{\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)(L_z(z))^{\frac{\rho-1}{\rho}} (1 - a_z^u(z))^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}}{(B_z(z))^{\frac{\rho-1}{\rho}} a_z^u(z)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} + (L_z(z))^{\frac{\rho-1}{\rho}} (1 - a_z^u(z))^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}} - \left[\frac{\rho-1}{\rho} - \xi(z)\right] a_z^u(z) - \frac{1}{\rho} \\
&= \frac{\frac{1}{\rho} - \frac{1}{\sigma}}{\left(\frac{a_z^u(z)}{1-a_z^u(z)}\right)^{\frac{\rho-1}{\rho}\frac{1}{\xi(z)}} \left(\frac{a_z^u(z)}{1-a_z^u(z)}\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} + 1} - \left[\frac{\rho-1}{\rho} - \xi(z)\right] a_z^u(z) - \frac{1}{\rho} \\
&= -\frac{1}{\sigma} - \left[1 - \frac{1}{\sigma} - \xi(z)\right] a_z^u(z), \forall z \in \mathbb{Z},
\end{aligned}$$

where we use $\frac{\partial \ln[1-a_z^u(z)]}{\partial \ln L_z(z)} = \xi a_z^u(z)$ and $\frac{\partial \ln Q_z(B_z(z),L_z(z);a_z(z))}{\partial \ln a_z(z)} = 0$.

In addition

$$\begin{aligned}
\gamma_{own}^{z,u}(z(\theta)) &= \frac{\partial \ln \omega^u(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta))}{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]} \frac{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]}{\partial \ln z(\theta)} + \frac{\partial \ln[\vartheta(z(\theta),\theta)]}{\partial \ln z(\theta)} \\
&= \left[1 + \gamma_{own}^{L,u}(z(\theta))\right] \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln z(\theta)} \\
\gamma_{own}^{\delta,u}(z(\theta)) &= \frac{\partial \ln \omega^u(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta))}{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]} \frac{\partial \ln\left[\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}\right]}{\partial \ln \delta(\theta)} + \frac{\partial \ln w_z^u(z(\theta))}{\partial \ln\left[\frac{B(\theta)}{\delta(\theta)}\right]} \frac{\partial \ln\left[\frac{B(\theta)}{\delta(\theta)}\right]}{\partial \ln \delta(\theta)} \\
&= -\gamma_{own}^{L}(z(\theta)) - \gamma_{own}^{B}(z(\theta)) \\
&= -\left(\frac{1}{\rho} - \frac{1}{\sigma}\right) a_z^u(z) - \left[\frac{\rho-1}{\rho} - \xi(z)\right] a_z^u(z) + \frac{1}{\sigma} + \left[1 - \frac{1}{\sigma} - \xi(z)\right] a_z^u(z) \\
&= \frac{1}{\sigma}
\end{aligned}$$

A.4.3 Useful Results for the Economy with Regulation

We have the following results:

$$\frac{\partial \ln Q}{\partial \ln a(\theta)} = \zeta(z(\theta)) \frac{\partial \ln Q}{\partial \ln Q_z(z(\theta))} \frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))} - \frac{a(\theta)}{1-a(\theta)} \frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))}}{1 - \frac{1}{\rho}}; \tag{57}$$

and

$$\frac{\rho-1}{\rho} \frac{\gamma_{own}^{a}(z(\theta))}{\zeta(z(\theta))} - \gamma_{own}^{B}(z(\theta)) = -\left[\gamma_{own}^{L}(z) + 1\right] \frac{a_z(z)}{1 - a_z(z)} \tag{58}$$

where $\zeta(z(\theta)) = 1 - \frac{1}{\xi(z)} \frac{\rho-1}{\rho}$, and $Q_z\left(\frac{B(\theta)}{\delta(\theta)}, \frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)}; a(\theta)\right)$; and

$$X_2(\theta) = \frac{\sigma}{\sigma-1} \frac{d}{d\theta} \left\{\beta_z(z(\theta)) z'(\theta)\right\} Q - w(\theta) f(\theta) l(\theta) \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta} \tag{59}$$

where

$$X_2(\theta) \equiv Q \left\{ \begin{array}{c} \frac{\partial \ln Q}{\partial \ln B(\theta)} \frac{B'(\theta)}{B(\theta)} + \frac{\partial \ln Q}{\partial \ln a(\theta)} \frac{a'(\theta)}{a(\theta)} + \frac{\partial \ln Q}{\partial \ln l(\theta)} \frac{l'(\theta)}{l(\theta)} \\ + \frac{\partial \ln Q}{\partial \ln \delta(\theta)} \frac{d \ln \delta(\theta)}{d\theta} + \frac{\partial \ln Q}{\partial \ln z(\theta)} \frac{d \ln z(\theta)}{d\theta} \end{array} \right\}$$

and

$$\alpha_z(z) \equiv \frac{1}{1 + \left[\frac{1-a_z(z)}{a_z(z)}\right]^{1 - \frac{1}{\xi(z)} \frac{\rho-1}{\rho}} \left[\frac{\varsigma^l(z) L_z(z)}{\varsigma^\kappa(z) B_z(z)}\right]^{\frac{\rho-1}{\rho}}}, \beta_z(z) \equiv \frac{\eta(z) [Q_z(z)]^{\frac{\sigma-1}{\sigma}}}{\int_Z \eta(z) [Q_z(z)]^{\frac{\sigma-1}{\sigma}} dz}.$$

When $\alpha_z(z) = a_z(z)$ we have $\alpha_z(z) = \frac{1}{1 + \left[\frac{\varsigma^l(z) L_z(z)}{\varsigma^\kappa(z) B_z(z)}\right]^{\xi(z)}}$ and $\left[\frac{1-\alpha_z(z)}{\alpha_z(z)}\right]^{1 - \frac{1}{\xi(z)} \frac{\rho-1}{\rho}} \left[\frac{\varsigma^l(z) L_z(z)}{\varsigma^\kappa(z) B_z(z)}\right]^{\frac{\rho-1}{\rho}} = \frac{1-\alpha_z(z)}{\alpha_z(z)}$.

(i) To derive (57), notice that

$$Q_z\left(z\left(\theta\right)\right)=\left\{a_z\left(z\left(\theta\right)\right)^{\zeta(z(\theta))}\times\left[\frac{B\left(\theta\right)}{z'\left(\theta\right)}\right]^{\frac{\rho-1}{\rho}}+\left[1-a_z\left(z\left(\theta\right)\right)\right]^{\zeta(z)}\left[\frac{\vartheta\left(z\left(\theta\right),\theta\right)l\left(\theta\right)f\left(\theta\right)}{z'\left(\theta\right)}\right]^{\frac{\rho-1}{\rho}}\right\}^{\frac{\rho}{\rho-1}}$$

$$Q=\left\{\int_Z\eta\left(z\left(\theta\right)\right)\left[\left\{\begin{array}{c}a_z\left(z\left(\theta\right)\right)^{\zeta(z(\theta))}B\left(\theta\right)^{\frac{\rho-1}{\rho}}+\\ \left[1-a_z\left(z\left(\theta\right)\right)\right]^{\zeta(z)}\left[\vartheta\left(z\left(\theta\right),\theta\right)l\left(\theta\right)f\left(\theta\right)\right]^{\frac{\rho-1}{\rho}}\end{array}\right\}^{\frac{\rho}{\rho-1}}\right]^{\frac{\sigma-1}{\sigma}}z'\left(\theta\right)^{\frac{1}{\sigma}}d\theta\right\}^{\frac{\sigma}{\sigma-1}}$$

$$=\left\{\int_Z\eta\left(z\right)\left[Q_z\left(z\right)\right]^{\frac{\sigma-1}{\sigma}}dz\right\}^{\frac{\sigma}{\sigma-1}}$$

We have

$$\frac{\partial\ln Q_z\left(z\left(\theta\right)\right)}{\partial\ln B_z\left(z\left(\theta\right)\right)}=\frac{\rho}{\rho-1}\frac{\partial\ln\left[a_z\left(z\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}B_z\left(z\right)^{\frac{\rho-1}{\rho}}+\left[1-a_z\left(z\right)\right]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}L_z\left(z\right)^{\frac{\rho-1}{\rho}}\right]}{\partial\ln B_z\left(z\left(\theta\right)\right)}$$

$$=\frac{a_z\left(z\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}B_z\left(z\right)^{\frac{\rho-1}{\rho}}}{a_z\left(z\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}\left(B_z\left(z\right)\right)^{\frac{\rho-1}{\rho}}+\left[1-a_z\left(z\right)\right]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}\left(L_z\left(z\right)\right)^{\frac{\rho-1}{\rho}}}$$

$$=\frac{1}{1+\left[\frac{1-a_z(z)}{a_z(z)}\right]^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}=\widetilde{a}_z\left(z\left(\theta\right)\right)$$

$$\frac{\partial\ln Q_z\left(z\left(\theta\right)\right)}{\partial\ln L_z\left(z\left(\theta\right)\right)}=\frac{\rho}{\rho-1}\frac{\partial\ln\left[a_z\left(z\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}B_z\left(z\right)^{\frac{\rho-1}{\rho}}+\left[1-a_z\left(z\right)\right]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}L_z\left(z\right)^{\frac{\rho-1}{\rho}}\right]}{\partial\ln L_z\left(z\left(\theta\right)\right)}$$

$$=\frac{\left[1-a_z\left(z\right)\right]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}L_z\left(z\right)^{\frac{\rho-1}{\rho}}}{a_z\left(z\right)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}\left(B_z\left(z\right)\right)^{\frac{\rho-1}{\rho}}+\left[1-a_z\left(z\right)\right]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}}\left(L_z\left(z\right)\right)^{\frac{\rho-1}{\rho}}}$$

$$\frac{\partial\ln Q_z\left(z\left(\theta\right)\right)}{\partial\ln L_z\left(z\left(\theta\right)\right)}=\frac{1}{\left[\frac{a_z(z)}{1-a_z(z)}\right]^{\zeta(z(\theta))}\left(\frac{B_z(z)}{L_z(z)}\right)^{\frac{\rho-1}{\rho}}+1}=1-\alpha_z\left(z\left(\theta\right)\right)$$

$$1=\frac{\partial\ln Q_z\left(z\left(\theta\right)\right)}{\partial\ln L_z\left(z\left(\theta\right)\right)}+\frac{\partial\ln Q_z\left(z\left(\theta\right)\right)}{\partial\ln B_z\left(z\left(\theta\right)\right)}$$

Therefore,

$$
\begin{aligned}
&\frac{\partial \ln Q_z\left(B_z(z), L_z(z); a_z(z)\right)}{\partial \ln a_z(z)} \\
&= \frac{\rho}{\rho-1} \frac{\partial \ln\left[a_z(z)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} B_z(z)^{\frac{\rho-1}{\rho}} + [1-a_z(z)]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} L_z(z)^{\frac{\rho-1}{\rho}}\right]}{\partial \ln a_z(z)} \\
&= \frac{\rho}{\rho-1} \frac{\left[1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}\right]\left[a_z(z)^{-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} B_z(z)^{\frac{\rho-1}{\rho}} - [1-a_z(z)]^{-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} (L_z(z))^{\frac{\rho-1}{\rho}}\right]}{a_z(z)^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} (B_z(z))^{\frac{\rho-1}{\rho}} + [1-a_z(z)]^{1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}} (L_z(z))^{\frac{\rho-1}{\rho}}} a_z(z) \\
&= \frac{\rho}{\rho-1} \frac{\zeta(z(\theta))\left[1-\frac{a_z(z)}{1-a_z(z)}\left[\frac{1-a_z(z)}{a_z(z)}\right]^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}\right]}{1+\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}; \zeta(z(\theta)) = 1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho} \\
&= \frac{\rho}{\rho-1}\zeta(z(\theta))\left[\frac{1}{1+\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}} - \frac{a_z(z)}{1-a_z(z)}\frac{\left[\frac{1-a_z(z)}{a_z(z)}\right]^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}{1+\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}\right] \\
&= \frac{\rho}{\rho-1}\zeta(z(\theta))\left[\frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))} - \frac{a_z(z)}{1-a_z(z)}\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))}\right] \\
&= \frac{\rho}{\rho-1}\zeta(z(\theta))\left[\alpha_z(z(\theta)) - \frac{a_z(z)}{1-a_z(z)}\left[1-\alpha_z(z(\theta))\right]\right]
\end{aligned}
$$

(ii) To derive (58), notice that, $\zeta(z(\theta)) = 1-\frac{1}{\xi(z)}\frac{\rho-1}{\rho}$,

$$
\begin{aligned}
\gamma^L_{own}(z) &= \frac{\frac{1}{\rho}-\frac{1}{\sigma}}{1+\left(\frac{B_z(z)}{L_z(z)}\right)^{\frac{\rho-1}{\rho}}\left(\frac{a_z(z)}{1-a_z(z)}\right)^{\zeta(z(\theta))}} - \frac{1}{\rho} \\
&= \left[1-\widetilde{a}_z(z(\theta))\right]\left(\frac{1}{\rho}-\frac{1}{\sigma}\right) - \frac{1}{\rho}; \\
\gamma^a_{own}(z) &= \left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{\partial \ln Q_z(z)}{\partial \ln a_z(z)} + \left(\frac{1}{\xi(z)}\frac{\rho-1}{\rho}-1\right)\frac{a_z(z)}{1-a_z(z)} \\
&= \left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{\rho}{\rho-1}\zeta(z(\theta))\frac{1-\frac{a_z(z)}{1-a_z(z)}\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}{1+\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}} - \zeta(z(\theta))\frac{a_z(z)}{1-a_z(z)} \\
&= \left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{\rho}{\rho-1}\zeta(z(\theta))\frac{1-\frac{a_z(z)}{1-a_z(z)}\frac{1-\alpha_z(z)}{\alpha_z(z)}}{1+\frac{1-\alpha_z(z)}{\alpha_z(z)}} - \zeta(z(\theta))\frac{a_z(z)}{1-a_z(z)} \\
\gamma^B_{own}(z) &= \frac{\frac{1}{\rho}-\frac{1}{\sigma}}{1+\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}} = \alpha_z(z)\left[\frac{1}{\rho}-\frac{1}{\sigma}\right]
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{\rho-1}{\rho}\frac{1}{\zeta\left(z\left(\theta\right)\right)}\gamma_{own}^{a}\left(z\left(\theta\right)\right)-\gamma_{own}^{B}\left(z\left(\theta\right)\right)\\
&=\left[\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{1-\frac{a_z(z)}{1-a_z(z)}\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}{1+\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}-\frac{\rho-1}{\rho}\frac{a_z\left(z\right)}{1-a_z\left(z\right)}\right]\\
&-\frac{\frac{1}{\rho}-\frac{1}{\sigma}}{1+\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}}\\
&=-\left[\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}{1+\left(\frac{1-a_z(z)}{a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{L_z(z)}{B_z(z)}\right)^{\frac{\rho-1}{\rho}}}+\frac{\rho-1}{\rho}\right]\frac{a_z\left(z\right)}{1-a_z\left(z\right)}\\
&=-\left[\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{1}{\left(\frac{a_z(z)}{1-a_z(z)}\right)^{\zeta(z(\theta))}\left(\frac{B_z(z)}{L_z(z)}\right)^{\frac{\rho-1}{\rho}}+1}-\frac{1}{\rho}+1\right]\frac{a_z\left(z\right)}{1-a_z\left(z\right)}\\
&=-\left[\gamma_{own}^{L}\left(z\right)+1\right]\frac{a_z\left(z\right)}{1-a_z\left(z\right)}
\end{aligned}
$$

(iii) To derive (59), note that

$$
\begin{aligned}
&\frac{\partial\ln Q}{\partial\ln B\left(\theta\right)}=\beta_z\left(z\left(\theta\right)\right)z'\left(\theta\right)\frac{\partial\ln Q_z\left(z\right)}{\partial\ln B_z\left(z\right)}|_{z(\theta)=z}=\beta_z\left(z\left(\theta\right)\right)z'\left(\theta\right)\frac{\partial\ln Q\left(\theta\right)}{\partial\ln B\left(\theta\right)};\\
&\frac{\partial\ln Q}{\partial\ln l\left(\theta\right)}=\beta_z\left(z\left(\theta\right)\right)z'\left(\theta\right)\frac{\partial\ln Q_z\left(z\right)}{\partial\ln L_z\left(z\right)}|_{z(\theta)=z}=\beta_z\left(z\left(\theta\right)\right)z'\left(\theta\right)\frac{\partial\ln Q\left(\theta\right)}{\partial\ln l\left(\theta\right)};\\
&\frac{\partial\ln Q}{\partial\ln a\left(\theta\right)}=\beta_z\left(z\left(\theta\right)\right)z'\left(\theta\right)\frac{\rho\zeta\left(z\left(\theta\right)\right)}{\rho-1}\left[\frac{\partial\ln Q_z\left(z\right)}{\partial\ln B_z\left(z\right)}-\frac{a_z\left(z\right)}{1-a_z\left(z\right)}\frac{\partial\ln Q_z\left(z\right)}{\partial\ln L_z\left(z\right)}\right]|_{z(\theta)=z}=\beta_z\left(z\left(\theta\right)\right)z'\left(\theta\right)\frac{\partial\ln Q\left(\theta\right)}{\partial\ln a\left(\theta\right)};\\
&\frac{\partial\ln Q}{\partial\ln\delta\left(\theta\right)}=\frac{1}{\sigma-1}\beta_z\left(z\left(\theta\right)\right)z'\left(\theta\right);\\
&\frac{\partial\ln Q}{\partial\ln z\left(\theta\right)}=\beta_z\left(z\left(\theta\right)\right)z'\left(\theta\right)\left\{\frac{\sigma}{\sigma-1}\frac{d\ln\eta\left(z\left(\theta\right)\right)}{d\ln z\left(\theta\right)}+\frac{\partial\ln Q\left(\theta\right)}{\partial\ln z\left(\theta\right)}\right\}
\end{aligned}
$$

We have

$$\begin{aligned}
&\frac{X_2(\theta)}{Q} = \frac{\partial \ln Q}{\partial \ln B(\theta)}\frac{B'(\theta)}{B(\theta)} + \frac{\partial \ln Q}{\partial \ln l(\theta)}\frac{l'(\theta)}{l(\theta)} + \frac{\partial \ln Q}{\partial \ln a(\theta)}\frac{a'(\theta)}{a(\theta)} \\
&+ \frac{\partial \ln Q}{\partial \ln \delta(\theta)}\frac{d\ln\delta(\theta)}{d\theta} + \frac{\partial \ln Q}{\partial \ln z(\theta)}\frac{d\ln z(\theta)}{d\theta} \\
&= \beta_z(z(\theta))z'(\theta)\left[\begin{array}{c} \frac{\partial \ln Q(\theta)}{\partial \ln B(\theta)}\frac{B'(\theta)}{B(\theta)} + \frac{\partial \ln Q(\theta)}{\partial \ln l(\theta)}\frac{l'(\theta)}{l(\theta)} + \frac{\partial \ln Q(\theta)}{\partial \ln a(\theta)}\frac{a'(\theta)}{a(\theta)} - \frac{\partial \ln Q(\theta)}{\partial \ln \delta(\theta)}\frac{d\ln\delta(\theta)}{d\theta} \\ + \frac{\partial \ln Q(\theta)}{\partial \ln z(\theta)}\frac{d\ln z(\theta)}{d\theta} + \frac{\partial \ln Q(\theta)}{\partial \ln \vartheta(z(\theta),\theta)}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} - \frac{\partial \ln Q(\theta)}{\partial \ln \vartheta(z(\theta),\theta)}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} \end{array}\right] \\
&+ \frac{\sigma}{\sigma-1}\beta_z(z(\theta))z'(\theta)\left\{\frac{d\ln\delta(\theta)}{d\theta} + \frac{d\ln\eta(z(\theta))}{d\ln z(\theta)}\frac{d\ln z(\theta)}{d\theta}\right\} \\
&= \frac{\sigma}{\sigma-1}\beta_z(z(\theta))z'(\theta)\left\{\begin{array}{c} \frac{\sigma-1}{\sigma}\frac{X_2(\theta)}{Q} + \frac{d\ln\eta(z(\theta))}{d\ln z(\theta)}\frac{d\ln z(\theta)}{d\theta} \\ + \frac{d\ln\delta(\theta)}{d\theta} - \frac{\sigma-1}{\sigma}\frac{\partial \ln Q(\theta)}{\partial \ln l(\theta)}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} \end{array}\right\}
\end{aligned}$$

By $\beta_z(z(\theta)) = \eta(z(\theta))Q(\theta)^{\frac{\sigma-1}{\sigma}}Q^{-\frac{\sigma-1}{\sigma}}$, we have:

$$\begin{aligned}
&\frac{\sigma}{\sigma-1}\beta_z(z(\theta))z'(\theta)\left\{\frac{\sigma-1}{\sigma}\frac{X_2(\theta)}{Q} + \frac{d\ln\eta(z(\theta))}{d\ln z(\theta)}\frac{d\ln z(\theta)}{d\theta} + \frac{d\ln\delta(\theta)}{d\theta}\right\} \\
&= \frac{\sigma}{\sigma-1}\beta_z(z(\theta))z'(\theta)\frac{d\ln\beta_z(z(\theta))z'(\theta)}{d\theta} = \frac{\sigma}{\sigma-1}\frac{d}{d\theta}\left\{\beta_z[z(\theta)]z'(\theta)\right\}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{X_2(\theta)}{Q} &= \frac{\sigma}{\sigma-1}\beta_z(z(\theta))z'(\theta)\left\{\frac{\sigma-1}{\sigma}\frac{X_2(\theta)}{Q} + \frac{d\ln\eta(z(\theta))}{d\ln z(\theta)}\frac{d\ln z(\theta)}{d\theta} + \frac{d\ln\delta(\theta)}{d\theta}\right\} \\
&\quad - \frac{\sigma}{\sigma-1}\beta_z(z(\theta))z'(\theta)\frac{\sigma-1}{\sigma}\frac{\partial \ln Q(\theta)}{\partial \ln l(\theta)}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} \\
&= \frac{\sigma}{\sigma-1}\frac{d}{d\theta}\left\{\beta_z(z(\theta))z'(\theta)\right\} - \beta_z(z(\theta))z'(\theta)\frac{\partial \ln Q(\theta)}{\partial \ln l(\theta)}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}
\end{aligned}$$

Notice that $Q = \left[\int_Z \eta(z)Q_z(z)^{\frac{\sigma-1}{\sigma}}dz\right]^{\frac{\sigma}{\sigma-1}}$, $\beta_z(z(\theta)) = \eta(z(\theta))Q(\theta)^{\frac{\sigma-1}{\sigma}}Q^{-\frac{\sigma-1}{\sigma}}$. We have:

$$\begin{aligned}
&\beta_z(z(\theta))z'(\theta)\frac{\partial \ln Q(\theta)}{\partial \ln l(\theta)}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} \\
&= \frac{\partial \ln Q}{\partial \ln Q_z(z(\theta))}\frac{\partial \ln Q(\theta)}{\partial \ln l(\theta)}z'(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} \\
&= \frac{\partial Q}{\partial Q_z(z(\theta))}\frac{Q_z(z(\theta))}{Q}\frac{\partial Q(\theta)}{\partial l(\theta)}\frac{l(\theta)}{Q(\theta)}z'(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} \\
&= \frac{\partial Q}{\partial l(\theta)}l(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}\frac{1}{Q} \\
&= w(\theta)f(\theta)l(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}\frac{1}{Q}
\end{aligned}$$

A combination of the above two equations gives (59).

A.4.4 Labor Elasticities

Keeping the wage constant, the elasticity of labor with respect to the net-of-tax rate along the linearized budget constraint is

$$\varepsilon \equiv \frac{\phi'(l(\theta))}{l(\theta)\phi''(l(\theta))}.$$

Keeping tax policies unchanged, denote by

$$\varepsilon_w^l(\theta) \equiv \frac{d\ln l(\theta)}{d\ln w(\theta)}$$

and

$$\varepsilon_w^y(\theta) \equiv \frac{d\ln y(\theta)}{d\ln w(\theta)} = 1 + \varepsilon_w^l(\theta) \tag{60}$$

the nonlinear elasticities of labor and labor income to wage in the equilibrium. Besides, following Sachs et al. (2020), set

$$\varepsilon_{1-\tau}^l(\theta) \equiv \frac{\varepsilon_w^y(\theta)}{1+1/\varepsilon}$$

as the nonlinear elasticity of labor with respect to the net-of-tax rate given wages.

To get an intuition of the above elasticities, take derivative of both sides of $1 - T'\left(w(\theta)l(\theta)\right) = \frac{\phi'(l(\theta))}{w(\theta)}$ with respect to θ and rearrange the equation to derive:[30]

$$\left[\frac{T''\left(y(\theta)\right)y(\theta)}{1-T'\left(y(\theta)\right)} + \frac{1}{\varepsilon}\right]\frac{d\ln l\left(\theta\right)}{d\theta} = \frac{d\ln w\left(\theta\right)}{d\theta}\left[1 - \frac{T''\left(y(\theta)\right)y(\theta)}{1-T'\left(y(\theta)\right)}\right],$$

Using $\frac{d\ln y(\theta)}{d\ln w(\theta)} = 1 + \frac{d\ln l(\theta)}{d\ln w(\theta)}$, the above equation suggests:

$$\frac{\frac{d\ln y(\theta)}{d\theta}}{\frac{d\ln w(\theta)}{d\theta}} = \frac{1+\frac{1}{\varepsilon}}{\frac{T''(y(\theta))y(\theta)}{1-T'(y(\theta))} + \frac{1}{\varepsilon}}. \tag{61}$$

One the other hand, consider a tax reform that increase the marginal tax rate of θ agent by τ: $1 - T'\left(w(\theta)l\right) - \tau = \frac{\phi'(l)}{w(\theta)}$. Total differential of the first-order condition now gives:

$$\frac{-d\tau}{1-T'\left(y(\theta)\right)} = \left[\frac{1}{\varepsilon} + \frac{T''\left(y(\theta)\right)y(\theta)}{1-T'\left(y(\theta)\right)}\right]\frac{dl(\theta)}{l(\theta)}$$

Notice that (61), we have:

$$\frac{\frac{dl(\theta)}{l(\theta)}}{\frac{-d\tau}{1-T'(y(\theta))}} = \frac{\varepsilon_w^y(\theta)}{1+\frac{1}{\varepsilon}}.$$

[30] Alternatively, one can take total differential of $1 - T'\left(wl\right) = \frac{v'(l)}{w}$ with respect to w and l to derive $\frac{d\ln l}{d\ln w} = \frac{1-\frac{T''(y)y}{1-T'(y)}}{\frac{1}{\varepsilon}+\frac{T''(y)y}{1-T'(y)}}$.

A Appendix B: Benchmark

A.1 Proof of Lemma 1

Part one. As is explained in the main body, part one of Lemma 1 is derived by the envelop theory. Taking derivative of $w(\theta)$ in (23) with respect to θ, one has (13).

Part two. Notice that the wage rate increases in θ, traditional Mirrlees tax problem immediately implies the part two of Lemma 1 (see e.g., Salanié (2003), p. 87).

In the following analysis, we focus on the part three of Lemma 1. To see when will the second-order condition satisfied, notice that the above problem implies the following conditions:

$$w\left(z'|\theta\right) \equiv \vartheta\left(z',\theta\right) p\left(z'\right) \leq \vartheta\left(z\left(\theta\right),\theta\right) p\left(z\left(\theta\right)\right) = w\left(z\left(\theta\right)|\theta\right), \forall \theta \in \Theta$$

By defintion,

$$\frac{\partial w\left(z'|\theta\right)}{\partial z'} = \frac{\partial \vartheta\left(z',\theta\right)}{\partial z'} p\left(z'\right) + \vartheta\left(z',\theta\right) p'\left(z'\right).$$

The first-order condition requires $\frac{\partial w(z'|\theta)}{\partial z'}|_{\theta=\theta(z')} = 0$:

$$0 = \frac{\partial \vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)} p\left(z\left(\theta\right)\right) + \vartheta\left(z\left(\theta\right),\theta\right) p'\left(z\left(\theta\right)\right).$$

which suggests $\frac{p'(z(\theta))}{p(z(\theta))} = -\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial z(\theta)}$ and

$$\begin{aligned} 0 = & \frac{\partial^2 \vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)^2} p\left(z\left(\theta\right)\right) z'\left(\theta\right) + \frac{\partial^2 \vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right) \partial \theta} p\left(z\left(\theta\right)\right) + \frac{\partial \vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)} p'\left(z\left(\theta\right)\right) z'\left(\theta\right) \\ & + \frac{\partial \vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)} p'\left(z\left(\theta\right)\right) z'\left(\theta\right) + \frac{\partial \vartheta\left(z\left(\theta\right),\theta\right)}{\partial \theta} p'\left(z\left(\theta\right)\right) + \vartheta\left(z\left(\theta\right),\theta\right) p''\left(z\left(\theta\right)\right) z'\left(\theta\right) \end{aligned}$$

i.e.,

$$\begin{aligned} z'\left(\theta\right) &= -\frac{\frac{\partial^2 \vartheta(z(\theta),\theta)}{\partial z(\theta) \partial \theta} p\left(z\left(\theta\right)\right) + \frac{\partial \vartheta(z(\theta),\theta)}{\partial \theta} p'\left(z\left(\theta\right)\right)}{\frac{\partial^2 \vartheta(z(\theta),\theta)}{\partial z(\theta)^2} p\left(z\left(\theta\right)\right) + \frac{\partial \vartheta(z(\theta),\theta)}{\partial z(\theta)} p'\left(z\left(\theta\right)\right) + \frac{\partial \vartheta(z(\theta),\theta)}{\partial z(\theta)} p'\left(z\left(\theta\right)\right) + \vartheta\left(z\left(\theta\right),\theta\right) p''\left(z\left(\theta\right)\right)} \\ &= -\frac{p\left(z\left(\theta\right)\right)\left[\frac{\partial^2 \vartheta(z(\theta),\theta)}{\partial z(\theta) \partial \theta} - \frac{\partial \vartheta(z(\theta),\theta)}{\partial \theta} \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial z(\theta)}\right]}{\frac{\partial^2 \vartheta(z(\theta),\theta)}{\partial z(\theta)^2} p\left(z\left(\theta\right)\right) + \frac{\partial \vartheta(z(\theta),\theta)}{\partial z(\theta)} p'\left(z\left(\theta\right)\right) + \frac{\partial \vartheta(z(\theta),\theta)}{\partial z(\theta)} p'\left(z\left(\theta\right)\right) + \vartheta\left(z\left(\theta\right),\theta\right) p''\left(z\left(\theta\right)\right)} \end{aligned}$$

The second-order condition is

$$\frac{\partial^2 w\left(z'|\theta\right)}{\partial \left(z'\right)^2}|_{z'=z(\theta)} = \left[\frac{\partial^2 \vartheta\left(z',\theta\right)}{\partial \left(z'\right)^2} p\left(z'\right) + \frac{\partial \vartheta\left(z',\theta\right)}{\partial z'} p'\left(z'\right) + \frac{\partial \vartheta\left(z',\theta\right)}{\partial z'} p'\left(z'\right) + \vartheta\left(z',\theta\right) p''\left(z'\right)\right]|_{z'=z(\theta)} \leq 0$$

Combining the above two equations give

$$z'(\theta) \geq 0.$$

To see when the first- and second-order conditions are sufficient, we set

$$w(\theta'|\theta) \equiv \vartheta(z(\theta'), \theta) p(z(\theta')), \forall (\theta, \theta') \in \Theta^2$$

Agent's problem implies

$$w(\theta'|\theta) \leq w(\theta|\theta), \forall (\theta, \theta') \in \Theta^2.$$

By defintion,

$$\frac{\partial w(\theta'|\theta)}{\partial \theta'} = \left[\frac{\partial \vartheta(z(\theta'), \theta)}{\partial z(\theta')} p(z(\theta')) + \vartheta(z(\theta'), \theta) p'(z(\theta'))\right] z'(\theta').$$

The first-order condition requires $\frac{\partial w(\theta'|\theta)}{\partial \theta'}|_{\theta=\theta'} = 0$ and $\frac{p'(z(\theta'))}{p(z(\theta'))} = -\frac{\partial \ln \vartheta(z(\theta'), \theta')}{\partial z(\theta')}$. Therefore

$$\begin{aligned}
\frac{\partial w(\theta'|\theta)}{\partial \theta'} &= \frac{\partial w(\theta'|\theta)}{\partial \theta'} - \frac{\partial w(\theta'|\theta)}{\partial \theta'}|_{\theta=\theta'} \\
&= \begin{bmatrix} \frac{\partial \vartheta(z(\theta'), \theta)}{\partial z(\theta')} p(z(\theta')) + \vartheta(z(\theta'), \theta) p'(z(\theta')) \\ -\frac{\partial \vartheta(z(\theta'), \theta')}{\partial z(\theta')} p(z(\theta')) - \vartheta(z(\theta'), \theta') p'(z(\theta')) \end{bmatrix} z'(\theta') \\
&= \frac{\partial \left[\frac{\partial \vartheta(z(\theta'), \theta^*)}{\partial z(\theta')} p(z(\theta')) + \vartheta(z(\theta'), \theta^*) p'(z(\theta'))\right]}{\partial \theta^*}|_{\theta \in [\min(\theta, \theta'), \max(\theta, \theta')]} (\theta - \theta') z'(\theta') \\
&= p(z(\theta')) \left[\frac{\partial^2 \vartheta(z(\theta'), \theta^*)}{\partial z(\theta') \partial \theta^*} + \frac{\partial \vartheta(z(\theta'), \theta^*)}{\partial \theta^*} \frac{p'(z(\theta'))}{p(z(\theta'))}\right] |_{\theta \in [\min(\theta, \theta'), \max(\theta, \theta')]} (\theta - \theta') z'(\theta') \\
&= p(z(\theta')) \left[\frac{\partial^2 \vartheta(z(\theta'), \theta^*)}{\partial z(\theta') \partial \theta^*} - \frac{\partial \vartheta(z(\theta'), \theta^*)}{\partial \theta^*} \frac{\partial \ln \vartheta(z(\theta'), \theta')}{\partial z(\theta')}\right] |_{\theta \in [\min(\theta, \theta'), \max(\theta, \theta')]} (\theta - \theta') z'(\theta') \\
&= p(z(\theta')) \vartheta(z(\theta'), \theta^*) \left[\frac{\partial^2 \ln \vartheta(z(\theta'), \theta^*)}{\partial z(\theta') \partial \theta^*}\right] |_{\theta \in [\min(\theta, \theta'), \max(\theta, \theta')]} (\theta - \theta') z'(\theta'),
\end{aligned}$$

where θ^* lies between θ' and θ. $p(z(\theta')) \vartheta(z(\theta'), \theta^*) \left[\frac{\partial \ln \vartheta(z(\theta'), \theta^*)}{\partial z(\theta') \partial \theta^*}\right] |_{\theta \in [\min(\theta, \theta'), \max(\theta, \theta')]}$ is positive under Assumption 4. When $z'(\theta') \geq 0$, we have $\frac{\partial w(\theta'|\theta)}{\partial \theta'} \geq 0$ when $\theta' < \theta$; $\frac{\partial w(\theta'|\theta)}{\partial \theta'} \leq 0$ when $\theta' > \theta$; $\frac{\partial w(\theta'|\theta)}{\partial \theta'} = 0$ when $\theta' = \theta$. Therefore, $\theta' = \theta$ is an optimal reported type. In conclusion, under Assumption 4, the first-order condition is not only necessary but also sufficient if $z'(\theta') \geq 0$ for any $\theta' \in \Theta$. □

A.2 Lemma 2: Implementability

Lemma 2 gives simpler necessary and sufficient conditions for incentive-compatibility.

Lemma 2 *Under Assumption 4 and 5:*

(i) Any tax-regulation equilibrium should satisfy the incentive constraints (Intensive IC) and (13), where the

gross utility satisfy $V(\theta) = c(\theta) - \phi(l(\theta))$*; a boundary condition* $z\left(\overline{\theta}\right)$ =1*; and the market clear conditions (10) and (12);*

(ii) In contrast, given any regulation, and allocation satisfying (10), (12), (Intensive IC), (13) and $z\left(\overline{\theta}\right) = 1$*, there exists a tax policy system and a price system together with the allocation and regulation constructs a tax-regulation equilibrium.*

Proof. Necessity. First, by definition of TRE, market clear conditions (12), (10), (**??**), policy constraints (25) and boundary condition $z\left(\overline{\theta}\right) = 1$ must be satisfied. Second, remember that we have assumed that allocations are interiors. Using the envelop theroy, (24) and (23) imply

$$V'(\theta) = \frac{w'(\theta)}{w(\theta)} l(\theta) \phi'(l(\theta))$$

and (13), where the gross utility satisfies $V(\theta) = c(\theta) - \phi(l(\theta))$. Combination of the above equation and (13) gives (Intensive IC).

Sufficiency. Remember that we have assumed that allocations are interiors. Therefore first-order conditions of producers must be satisfied. Moreover, as we consider strictly concave production functions, the first-order conditions are also sufficient. Therefore, given an allocation satisfying (12), (**??**), (10), (Intensive IC), (13), (25), $z\left(\overline{\theta}\right) = 1$ and $V(\theta) = c(\theta) - \phi(l(\theta))$, we can construct $p(z)$ by (**??**). Second, we construct wage rates $w(\theta)$. To do this, we first set an abitrary positive lower bound of wage rates. Set $w\left(\underline{\theta}\right) = \underline{w}$, where $\underline{w} > 0$. We then construct $w(\theta)$ by $w(\theta) = \underline{w} + \int_{\underline{\theta}}^{\theta} w(x)\,\omega(x)\,dx$, where $\omega(x) = \frac{\partial \ln \vartheta(z(x),x)}{\partial x}$. Then we construct $p(z)$ by $w(\theta) = \vartheta(z(\theta),\theta)\, p(z(\theta))$ and task-specific taxes $\tau(z)$ by (**??**). By constructing, the first-order conditions of producers must be satisfied. Since these conditions are both necessary and sufficient, producers' problems must be solved at these allocations under the constructed prices and taxes.

We now construct the labor income tax. By (**??**), $T'(w(\theta)l(\theta)) = 1 - \frac{\phi'(l(\theta))}{w(\theta)}$, we have the marginal tax rates at each income level appears in the equilibrium. When $T(w(\underline{\theta})l(\underline{\theta})) = \underline{T}$:

$$T(y(\theta)) = \underline{T} - \int_{y(\underline{\theta})}^{y(\theta)} T'(y)\,dy$$

To fix the income taxes we should solve for the boundary value of tax $\underline{T}$. To do this, we use agents' budget constraints. We substitute $T(y(\theta))$ in the following equation by the above equations:

$$y(\theta) - T(y(\theta)) - c(\theta) = 0. \tag{A1}$$

We want to show that there exists $\underline{T}$ such that given $\{y(\theta), V(\theta)\}_{\theta\in\Theta}$, equation (A1) is satisfied for

any $\theta \in \Theta$. To be consistent with the $\underline{\theta}$-type agent's budget constraint, $\underline{T}$ must satisfy:

$$y(\underline{\theta}) - T\left(y(\underline{\theta})\right) - c(\underline{\theta}) = 0. \tag{A2}$$

We should show that any $\underline{T}$ satisfying (A2) is also consistent with other agents' budget constraints. This is equivalent to say that:

$$y'(\theta)\left[1 - T'\left(y(\theta)\right)\right] - c'(\theta) = 0.$$

Substituting $1 - T'\left(y(\theta)\right)$ with (**??**), the above equation is equivalent to:

$$c'(\theta) - \frac{\phi'\left(l(\theta)\right)}{w(\theta)} y'(\theta) = 0.$$

The above equations hold because we have:

$$\begin{aligned} V'(\theta) &= \frac{w'(\theta)}{w(\theta)} \phi'\left(l(\theta)\right) l(\theta) \\ &= c'(\theta) - \frac{\phi'\left(l(\theta)\right)}{w(\theta)} y'(\theta) + \frac{w'(\theta)}{w(\theta)} \phi'\left(l(\theta)\right) l(\theta). \end{aligned} \tag{A3}$$

The first equation of (A3) is the incentive compatible condition, and the second equation is derived by the definition of $V(\theta)$ (i.e., $V(\theta) = c\left(\theta\right) - \phi\left(\frac{y(\theta)}{w(\theta)}\right)$). We have shown that any $\underline{T}$ satisfying the equations (A2) are consistent with all agents' budget constraints. Notice that the resource constraint is satisfied. By Walras' law, the government's budget constraint must also be satisfied.

Last, we should show that under the above constructed prices and taxes, the allocation solves workers' problems. As is suggested by Lemma 1, under Assumption 4 and 5, first-order conditions of labor's problem are both necessary and sufficient. We should show that the first-order conditions are satisfied if (Intensive IC) and (13) are satisfied. To see this notice that under the above construction $w\left(\theta\right) = \vartheta\left(z\left(\theta\right), \theta\right) p\left(z\left(\theta\right)\right)$. Takine derivatives of both sides with respect to θ, we have

$$w'\left(\theta\right) = \frac{\partial \vartheta\left(z\left(\theta\right), \theta\right)}{\partial \theta} p\left(z\left(\theta\right)\right) + \frac{\partial \vartheta\left(z\left(\theta\right), \theta\right)}{\partial z\left(\theta\right)} z'\left(\theta\right) p\left(z\left(\theta\right)\right) + \vartheta\left(z\left(\theta\right), \theta\right) p'\left(z\left(\theta\right)\right) z'\left(\theta\right).$$

Divide both sides of the above equation by $w\left(\theta\right)$ and $\vartheta\left(z\left(\theta\right), \theta\right) p\left(z\left(\theta\right)\right)$, respectively:

$$\frac{w'\left(\theta\right)}{w\left(\theta\right)} = \frac{\partial \ln \vartheta\left(z\left(\theta\right), \theta\right)}{\partial \theta} + \left[\frac{\partial \ln \vartheta\left(z\left(\theta\right), \theta\right)}{\partial z\left(\theta\right)} + \frac{p'\left(z\left(\theta\right)\right)}{p\left(z\left(\theta\right)\right)}\right] z'\left(\theta\right).$$

Notice that $z'\left(\theta\right) > 0$. Combination of the above equation and (13) gives (**??**). Next, to show (**??**) is satisfied, we use the constructed labor income tax, under which $V(\theta) = y\left(\theta\right) - T\left(y\left(\theta\right)\right) - \phi\left(\frac{y(\theta)}{w(\theta)}\right)$.

Taking derivatives of the both sides of this equation with respect to θ:

$$V'(\theta) = \left[1 - T'\left(y\left(\theta\right)\right) - \frac{\phi'\left(l\left(\theta\right)\right)}{w\left(\theta\right)}\right] y'\left(\theta\right) + \frac{w'\left(\theta\right)}{w\left(\theta\right)}\phi'\left(l\left(\theta\right)\right) l\left(\theta\right).$$

Since $y'\left(\theta\right) > 0$, the incentive condition (Intensive IC) implies (**??**).□

A.3 Proof of Theorem 1

Part (i) of Theorem 1.

Step 1. First, we solve the production efficient robot inputs and job assignment by solving a planning problem, which chooses $\{z\left(\theta\right)\}_{\theta\in\Theta}$ and $\{B_z\left(z\right)\}_{z\in[0,z(\underline{\theta})]}$ to maximize (26). The solution of this production efficient problem belongs to one of the tax-regulation equilibrium. By Proposition 1, there exist an unique solution to the planning problem. Moreover, since the problem is strictly concave, we can use the first-order conditions to derive the solution.

Notice that (12), we have

$$Q = \left[\int_0^{z(\underline{\theta})} \eta\left(z\right)\left[\kappa B_z\left(z\right)\right]^{\frac{\sigma-1}{\sigma}} dz + \int_0^1 \eta\left(z\left(\theta\right)\right)\left[L(\theta)\vartheta\left(z\left(\theta\right),\theta\right)\right]^{\frac{\sigma-1}{\sigma}} \delta\left(\theta\right)^{\frac{1}{\sigma}} d\theta\right]^{\frac{\sigma}{\sigma-1}}. \tag{A4}$$

Set the Lagrangian function:

$$\begin{aligned}\mathcal{L}\left(z,\delta,B_z;\zeta\right) = &\left[\int_0^{z(\underline{\theta})} \eta\left(z\right)\left[\kappa B_z\left(z\right)\right]^{\frac{\sigma-1}{\sigma}} dz + \int_0^1 \eta\left(z\left(\theta\right)\right)\left[l\left(\theta\right) f_\theta(\theta)\vartheta\left(z\left(\theta\right),\theta\right)\right]^{\frac{\sigma-1}{\sigma}} \delta\left(\theta\right)^{\frac{1}{\sigma}} d\theta\right]^{\frac{\sigma}{\sigma-1}} \\ &- \left[\int_0^{z(\underline{\theta})} B_z\left(z\right) dz\right] + \int_\Theta \psi(\theta)\left[z'\left(\theta\right) - \delta\left(\theta\right)\right] d\theta] + \zeta\left[1 - z\left(\overline{\theta}\right)\right]\end{aligned}$$

Under the production efficient case, the first-order conditions are:

$$B_z\left(z\right) : \frac{\partial Q}{\partial B_z\left(z\right)} - 1 = 0 \tag{A5}$$

and

$$z\left(\theta\right) : \psi'(\theta) + \frac{\partial Q}{\partial z\left(\theta\right)} = 0 \tag{A6}$$

and ($\frac{\partial Q}{\partial z(\theta)} = \frac{\partial Q}{\partial L(\theta)} L\left(\theta\right) \frac{\partial \ln\left[\eta(z(\theta))^{\frac{\sigma}{\sigma-1}}\vartheta(z(\theta),\theta)\right]}{\partial z(\theta)}$)

$$\delta\left(\theta\right) : 0 = \psi(\theta) + \frac{\partial Q}{\partial \delta\left(\theta\right)} \tag{A7}$$

and

$$z\left(\underline{\theta}\right) : \psi(\underline{\theta}) + \left[\frac{\sigma}{\sigma-1}\frac{\partial Q}{\partial B_z\left(z\left(\underline{\theta}\right)\right)} - 1\right] B_z\left(z\left(\underline{\theta}\right)\right) = 0 \tag{A8}$$

and

$$z\left(\overline{\theta}\right):\zeta=\psi(\overline{\theta}) \tag{A9}$$

The above conditions together with constraints determine the production efficient allocation. To show whether the solution to the planner's problem are production efficient, one only need to check that whether the above first-order conditions hold in the solution to the planner's problem.

Step 2. We now solve the planner's problem of maximizing welfare.

Before this, we should introduce some useful elasticities of wages. In the benchmark economy, we have $w_z(z)=\eta(z)\iota(z)\left[\iota(z)L_z(z)\right]^{\frac{-1}{\sigma}}Q^{\frac{1}{\sigma}}$ for $z\in[z_R,1]$. Notice that $L_z(z(\theta))=\frac{\vartheta(z(\theta),\theta)L(\theta)}{z'(\theta)}$, we have $w(\theta)=\omega(\theta,z(\theta),z'(\theta),L(\theta))Q^{\frac{1}{\sigma}}$, where $\omega\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)}\right)=\vartheta(z(\theta),\theta)\eta(z(\theta))\iota(z(\theta))\left[\iota(z(\theta))\vartheta(z(\theta),\theta)\frac{L(\theta)}{\delta(\theta)}\right]^{-\frac{1}{\sigma}}$, $\delta(\theta)=z'(\theta)$.

Following Sachs et al. (2020), we define the own wage-labor elasticity of wage and cross wage-labor elasticity of wage by Gâteaux derivative:

$$\gamma^L_{own}(z(\theta))\equiv\frac{\partial\ln\omega\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)}\right)}{\partial\ln L(\theta)},\gamma^L_{cross}(z,z')\equiv\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln Q_z(z')}\frac{\partial\ln Q_z(z')}{\partial\ln L_z(z')}, \tag{A10}$$

respectively, where $(\theta,\theta')\in\Theta^2$.

By definition, we have:

$$\gamma^L_{own}(z(\theta))=-\frac{1}{\sigma}\text{ and }\gamma^L_{cross}(z(\theta),z(\theta'))=\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln L(\theta')}.$$

Analogously, we define:

$$\gamma^B_{cross}(z,z')\equiv\lim_{\mu\to0}\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln Q_z(z')}\frac{\partial\ln Q_z(z')}{\partial\ln B_z(z')},z'\in[0,z^R]$$

as the cross elasticity of wage with respect to robot inputs. Under our technology, we have

$$\gamma^B_{cross}(z(\theta),z)=\frac{1}{\sigma}Q^{\frac{1-\sigma}{\sigma}}\kappa B_z(z)\left[\kappa B_z(z)\right]^{\frac{-1}{\sigma}}.$$

We define the cross wage-task elasticity of wage and own wage-task elasticity of wage as:

$$\gamma^z_{own}(z(\theta))\equiv\frac{\partial\ln\omega\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)}\right)}{\partial\ln z(\theta)},\gamma^z_{cross}(z(\theta),z(\theta'))\equiv\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln z(\theta')},\theta\neq\theta'$$

respectively, where $(\theta,\theta')\in\Theta^2$. Last, we define the cross wage-δ elasticity of wage and own wage-δ

elasticity of wage as:

$$\gamma_{cross}^{\delta}\left(z\left(\theta\right),z\left(\theta'\right)\right)\equiv\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln\delta(\theta')},\text{ and }\gamma_{own}^{\delta}\left(z\left(\theta\right)\right)\equiv\frac{\partial\ln\omega\left(\theta,z\left(\theta\right),\frac{L(\theta)}{\delta(\theta)}\right)}{\partial\ln\delta(\theta)},$$

where $(\theta,\theta')\in\Theta^2$. Since $w\left(\theta\right)=Q^{\frac{1}{\sigma}}\vartheta\left(z\left(\theta\right),\theta\right)^{\frac{\sigma-1}{\sigma}}\left[\frac{l(\theta)f_\theta(\theta)}{\delta(\theta)}\right]^{\frac{-1}{\sigma}}$, for any $(\theta,\theta')\in\Theta^2$, we have

$$\gamma_{own}^{\delta}\left(z\left(\theta\right)\right)=\frac{1}{\sigma}\text{ and }\gamma_{cross}^{\delta}\left(z\left(\theta\right),z\left(\theta'\right)\right)=\frac{1}{\sigma}\frac{\partial\ln Q}{\partial\ln\delta(\theta')}.$$

Notice that $\gamma_{cross}^{L}\left(z\left(\theta\right),z\left(\theta'\right)\right)$, $\gamma_{cross}^{z}\left(z\left(\theta\right),z\left(\theta'\right)\right)$, $\gamma_{cross}^{\delta}\left(z\left(\theta\right),z\left(\theta'\right)\right)$, and $\gamma_{cross}^{B}\left(z\left(\theta\right),z\left(\theta'\right)\right)$ are independent of θ, which means factor inputs do not affect the relative wages through the cross effects.

We now set the Lagrangian function of the planner's problem:[31]

$$\begin{aligned}
&\mathcal{L}\left(l,V,z,\delta,B_z,W;q,\psi,\varphi,\chi\right)\\
&=\int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta+\int_\Theta q(\theta)\left[\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta}l(\theta)\phi'\left(l\left(\theta\right)\right)-V'(\theta)\right]d\theta+\zeta\left[z\left(\overline{\theta}\right)-1\right]\\
&+\int_\Theta\psi(\theta)\left[\delta\left(\theta\right)-z'\left(\theta\right)\right]d\theta+\int_\Theta\varphi(\theta)\left[\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta}-W'(\theta)\right]d\theta\\
&+\int_\Theta\chi(\theta)\left[\ln w\left(\theta,z\left(\theta\right),\frac{L\left(\theta\right)}{\delta\left(\theta\right)};Q\right)-W(\theta)\right]d\theta\\
&+\lambda\left[\begin{array}{c}\left[\int_0^{z(\underline{\theta})}\eta\left(z\right)\left[\kappa B_z\left(z\right)\right]^{\frac{\sigma-1}{\sigma}}dz+\int_0^1\eta\left(z\left(\theta\right)\right)\left[L\left(\theta\right)\vartheta\left(z\left(\theta\right),\theta\right)\right]^{\frac{\sigma-1}{\sigma}}\delta\left(\theta\right)^{\frac{1}{\sigma}}d\theta\right]^{\frac{\sigma}{\sigma-1}}\\-\left[\int_0^{z(\underline{\theta})}B_z\left(z\right)dz\right]-\int_\Theta\left[V(\theta)+\phi\left(l\left(\theta\right)\right)\right]f_\theta(\theta)d\theta\end{array}\right].
\end{aligned}$$

Taking partial integrals yields:

$$-\int_\Theta q(\theta)V'(\theta)d\theta=V(\underline{\theta})q(\underline{\theta})-V(\bar{\theta})q(\bar{\theta})+\int_\Theta q'(\theta)V(\theta)d\theta,$$

[31]Note: if labor factors are perfectly substitutable by each other, we have

$$\begin{aligned}
&\mathcal{L}\left(l,V,z,\delta,B_z,W;q,\psi,\varphi,\chi\right)\\
&=\int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta+\int_\Theta q(\theta)\left[\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta}l(\theta)\phi'\left(l\left(\theta\right)\right)-V'(\theta)\right]d\theta+\zeta\left[z\left(\overline{\theta}\right)-1\right]\\
&+\int_\Theta\psi(\theta)\left[\delta\left(\theta\right)-z'\left(\theta\right)\right]d\theta\\
&+\lambda\left[\begin{array}{c}\left[\int_0^{z(\underline{\theta})}\eta\left(z\right)\left[\kappa B_z\left(z\right)\right]^{\frac{\sigma-1}{\sigma}}dz+\left[\int_0^1\eta\left(z\left(\theta\right)\right)\left[L\left(\theta\right)\vartheta\left(z\left(\theta\right),\theta\right)\right]\delta\left(\theta\right)d\theta\right]^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}\\-\left[\int_0^{z(\underline{\theta})}B_z\left(z\right)dz\right]-\int_\Theta\left[V(\theta)+\phi\left(l\left(\theta\right)\right)\right]f_\theta(\theta)d\theta\end{array}\right].
\end{aligned}$$

There is no need for regulation? So the reason for regulation is comparative advantage (endogenous occupation choice) and imperfectly substitutable labors.

and

$$-\int_{\Theta}\varphi(\theta)W'(\theta)d\theta = W(\underline{\theta})\varphi(\underline{\theta}) - W(\bar{\theta})\varphi(\bar{\theta}) + \int_{\Theta}\varphi'(\theta)W(\theta)d\theta,$$

and

$$-\int_{\Theta}\psi(\theta)z\left(\theta\right)d\theta = z(\underline{\theta})\psi(\underline{\theta}) - \psi(\bar{\theta}) + \int_{\Theta}\psi'(\theta)z(\theta)d\theta.$$

The derivatives with respect to the endpoint conditions yield the following boundary conditions:

$$q(\underline{\theta}) = q(\bar{\theta}) = 0 \text{ and } \varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0. \tag{A11}$$

Substituting the above conditions into the Lagrangian yields the following first-order conditions for $t \geq 1$ and $\theta \in \Theta$:

$$V(\theta) : \widetilde{f}_{\theta}(\theta) + q'(\theta) - \lambda f_{\theta}(\theta) = 0, \tag{A12}$$

and

$$W(\theta) : -\chi(\theta) + \varphi'(\theta) = 0, \tag{A13}$$

and

$$\begin{aligned} l(\theta) : \frac{\partial \ln \vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta} q(\theta)\phi'\left(l\left(\theta\right)\right)\left[1 + \frac{l(\theta)\phi''\left(l\left(\theta\right)\right)}{\phi'\left(l\left(\theta\right)\right)}\right] + \\ \frac{\chi(\theta)\gamma^{L}_{own}\left(z\left(\theta\right)\right) + \int_{\Theta}\chi(x)\gamma^{L}_{cross}\left(z\left(x\right),z\left(\theta\right)\right)dx}{l(\theta)} + \lambda w(\theta)\left[1 - \frac{\phi'\left(l\left(\theta\right)\right)}{w(\theta)}\right] f_{\theta}(\theta) = 0, \end{aligned} \tag{A14}$$

and

$$B_z\left(z\right) : \lambda\left[\frac{\partial Q}{\partial B_z\left(z\right)} - 1\right] + \frac{\int_{\Theta}\chi(x)\gamma^{B}_{cross}\left(x,z\right)dx}{B_z\left(z\right)} = 0, \tag{A15}$$

and

$$\begin{aligned} z\left(\theta\right) : 0 = \psi'(\theta) + \left[\begin{array}{c} q(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} l(\theta)\phi'\left(l\left(\theta\right)\right) + \\ \varphi(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} + \frac{\gamma^{z}_{own}(z(\theta))\chi(\theta) + \int_{\Theta}\chi(x)\gamma^{z}_{cross}(z(x),z(\theta))dx}{z(\theta)} \end{array}\right] \\ + \lambda\frac{\partial Q}{\partial L\left(\theta\right)} L\left(\theta\right)\frac{\partial \ln\left[\eta\left(z\left(\theta\right)\right)^{\frac{\sigma}{\sigma-1}}\vartheta\left(z\left(\theta\right),\theta\right)\right]}{\partial z\left(\theta\right)}, \end{aligned} \tag{A16}$$

and

$$\delta\left(\theta\right) : \psi(\theta) + \frac{\gamma^{\delta}_{own}\left(z\left(\theta\right)\right)\chi(\theta) + \int_{\Theta}\chi(x)\gamma^{\delta}_{cross}\left(z\left(x\right),z\left(\theta\right)\right)dx}{\delta\left(\theta\right)} + \lambda\frac{\partial Q}{\partial\delta\left(\theta\right)} = 0, \tag{A17}$$

and

$$z\left(\underline{\theta}\right) : \psi(\underline{\theta}) + \lambda\left[\frac{\sigma}{\sigma-1}\frac{\partial Q}{\partial B_z\left(z\left(\underline{\theta}\right)\right)} - 1\right] B_z\left(z\left(\underline{\theta}\right)\right) = 0, \tag{A18}$$

and

$$z\left(\bar{\theta}\right) : \psi(\bar{\theta}) + \zeta = 0. \tag{A19}$$

Using integration by parts, we have:

$$\begin{aligned}\int_{\Theta}\varphi'(x)\gamma_{cross}^{B}(x,z)\,dx &= \int_{\Theta}\gamma_{cross}^{B}(x,z)\,d\varphi(x) \\ &= -\int_{\Theta}\varphi(x)\frac{d\left[\gamma_{cross}^{B}(x,z)\right]}{dx}dx \\ &= 0.\end{aligned} \tag{A20}$$

The second equation is derived by $\varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0$ (see, e.g., (A11)). Notice that $\gamma_{cross}^{B}(x,z)$ is independent of x, we have $\int_{\Theta}\varphi(x)\frac{d\gamma_{cross}^{B}(x,z)}{dx}dx = 0$. Therefore, conditon (A15) implies and

$$\frac{\partial Q}{\partial B_z(z)} - 1 = 0, \tag{A21}$$

because $\lambda > 0$. To see $\lambda > 0$, according to (A11) and (A12), we have $\lambda = \frac{\int \widetilde{f}_{\theta}(\theta)d\theta + q(\bar{\theta}) - q(\underline{\theta})}{\int f_{\theta}(\theta)d\theta} = 1$ and

$$-q(\theta) = \widetilde{F}_{\theta}(\theta) - F_{\theta}(\theta). \tag{A22}$$

Step 3. We want to show that the policy constraint $\frac{d\ln w\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)};Q\right)}{d\theta} = \frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\theta}$ will be naturally satisfied when the welfare function is Utilitarian. Moreover, under the Utilitarian problem without policy constraint, the solution is production efficient. In conclusion, when the welfare function is Utilitarian, the solution is production efficient. To do this, we first prove that (a) the solution to the relaxed problem without policy constraint $\frac{d\ln w\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)};Q\right)}{d\theta} = \frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\theta}$ is production efficient. (b) We then show that under the Utilitarian social welfare function, the policy constraint is naturally satisfied.

(a) To show that the FOCs of the relaxed problem implies the production efficiency, set the Lagrangian function of the relaxed planning problem as

$$\begin{aligned}&\mathcal{L}(l,V,z,\delta,B_z,W;q,\psi,\varphi,\chi) \\ &= \int_{\Theta}V(\theta)\widetilde{f}_{\theta}(\theta)d\theta + \int_{\Theta}q(\theta)\left[\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\theta}l(\theta)\phi'(l(\theta)) - V'(\theta)\right]d\theta + \zeta\left[z\left(\overline{\theta}\right) - 1\right] \\ &+ \int_{\Theta}\psi(\theta)\left[\delta(\theta) - z'(\theta)\right]d\theta \\ &+ \lambda\left[\begin{array}{c}\left[\int_0^{z(\underline{\theta})}\eta(z)\left[\kappa B_z(z)\right]^{\frac{\sigma-1}{\sigma}}dz + \int_0^1\eta(z(\theta))\left[L(\theta)\vartheta(z(\theta),\theta)\right]^{\frac{\sigma-1}{\sigma}}\delta(\theta)^{\frac{1}{\sigma}}d\theta\right]^{\frac{\sigma}{\sigma-1}} \\ -\left[\int_0^{z(\underline{\theta})}B_z(z)\,dz\right] - \int_{\Theta}\left[V(\theta) + \phi(l(\theta))\right]f_{\theta}(\theta)d\theta\end{array}\right].\end{aligned}$$

The FOCs of the above relaxed problem are

$$l(\theta): \frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\theta}q(\theta)\phi'(l(\theta))\left[1 + \frac{l(\theta)\phi''(l(\theta))}{\phi'(l(\theta))}\right] + \lambda w(\theta)\left[1 - \frac{\phi'(l(\theta))}{w(\theta)}\right]f_{\theta}(\theta) = 0, \tag{A23}$$

and

$$B_z(z):\lambda\left[\frac{\partial Q}{\partial B_z(z)}-1\right]=0, \tag{A24}$$

and

$$z(\theta):\psi'(\theta)+q(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}l(\theta)\phi'(l(\theta))+\lambda\frac{\partial Q}{\partial z(\theta)}=0, \tag{A25}$$

and

$$\delta(\theta):\psi(\theta)+\lambda\frac{\partial Q}{\partial \delta(\theta)}=0. \tag{A26}$$

and (A12), (A13), (A18), and (A19). By (A11) and (A12), when the welfare function is Utilitarian, we have $q(\theta)=0$ and $\lambda=1$. Therefore (A25) can be simplified to be:

$$z(\theta):\psi'(\theta)+\lambda\frac{\partial Q}{\partial z(\theta)}=0, \tag{A27}$$

Here, conditions (A24), (A26) and (A27) are equivalent to conditions (A5) to (A7). Therefore, the solution to the above problem is production efficient.

(b) Now to show the solution of the original problem is production efficient, we only need to show that the solution to the relaxed planning problem satisfies the policy constraint $\frac{d\ln w\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)};Q\right)}{d\theta}=\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}$. Remember that under our technology,

$$\begin{aligned} w\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)};Q\right)&=\vartheta(z(\theta),\theta)^{1-\frac{1}{\sigma}}\eta(z(\theta))\left[\frac{L(\theta)}{\delta(\theta)}\right]^{-\frac{1}{\sigma}}Q^{\frac{1}{\sigma}} \\ &=\vartheta(z(\theta),\theta)^{1-\frac{1}{\sigma}}\eta(z(\theta))\left[\frac{l(\theta)f_\theta(\theta)}{\delta(\theta)}\right]^{-\frac{1}{\sigma}}Q^{\frac{1}{\sigma}} \end{aligned} \tag{A28}$$

Therefore (A26) implies:

$$\begin{aligned} \psi(\theta)&=-\frac{\partial \ln Q}{\partial \ln \delta(\theta)}\frac{Q}{\delta(\theta)}=-\frac{1}{\sigma-1}\eta(z(\theta))\left[\frac{\vartheta(z(\theta),\theta)l(\theta)f_\theta(\theta)}{\delta(\theta)}\right]^{\frac{\sigma-1}{\sigma}}Q^{\frac{1}{\sigma}} \\ &=-\frac{1}{\sigma-1}w\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)};Q\right)\frac{l(\theta)f_\theta(\theta)}{\delta(\theta)} \end{aligned} \tag{A29}$$

Similarly, by (A27), $\lambda=1$, $\frac{\partial Q}{\partial z(\theta)}=\frac{\partial Q}{\partial L(\theta)}L(\theta)\frac{\partial \ln\left[\eta(z(\theta))^{\frac{\sigma}{\sigma-1}}\vartheta(z(\theta),\theta)\right]}{\partial z(\theta)}$, and

$$\begin{aligned} \frac{\partial Q}{\partial L(\theta)}L(\theta)&=\eta(z(\theta))\left[\frac{l(\theta)f_\theta(\theta)\vartheta(z(\theta),\theta)}{\delta(\theta)}\right]^{\frac{\sigma-1}{\sigma}}\delta(\theta)Q^{\frac{1}{\sigma}} \\ &=w\left(\theta,z(\theta),\frac{L(\theta)}{\delta(\theta)};Q\right)l(\theta)f_\theta(\theta), \end{aligned}$$

we have

$$\psi'(\theta) + w\left(\theta, z(\theta), \frac{L(\theta)}{\delta(\theta)}; Q\right) l(\theta) f_\theta(\theta) \frac{\partial \ln\left[\eta(z(\theta))^{\frac{\sigma}{\sigma-1}} \vartheta(z(\theta), \theta)\right]}{\partial z(\theta)} = 0. \tag{A30}$$

The above equation combined with (A29) and (A30) delivers:

$$-\frac{1}{\sigma-1} \frac{d\left[w\left(\theta, z(\theta), \frac{L(\theta)}{\delta(\theta)}; Q\right) \frac{l(\theta) f_\theta(\theta)}{\delta(\theta)}\right]}{d\theta} + w\left(\theta, z(\theta), \frac{L(\theta)}{\delta(\theta)}; Q\right) l(\theta) f_\theta(\theta) \frac{\partial \ln\left[\eta(z(\theta))^{\frac{\sigma}{\sigma-1}} \vartheta(z(\theta), \theta)\right]}{\partial z(\theta)} = 0.$$

Rearrange the above equation:

$$\frac{d \ln\left[w\left(\theta, z(\theta), \frac{L(\theta)}{\delta(\theta)}; Q\right)^{-\frac{1}{\sigma-1}} \left[\frac{l(\theta) f_\theta(\theta)}{\delta(\theta)}\right]^{-\frac{1}{\sigma-1}} \eta(z(\theta))^{\frac{\sigma}{\sigma-1}} \vartheta(z(\theta), \theta)\right]}{d\theta} - \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta} = 0$$

Notice that $\left[\frac{l(\theta) f_\theta(\theta)}{\delta(\theta)}\right]^{-\frac{1}{\sigma-1}} \eta(z(\theta))^{\frac{\sigma}{\sigma-1}} \vartheta(z(\theta), \theta) = w\left(\theta, z(\theta), \frac{L(\theta)}{\delta(\theta)}; Q\right)^{\frac{\sigma}{\sigma-1}} Q^{-\frac{1}{\sigma-1}}$ and $\frac{d \ln Q}{d\theta} = 0$, we have:

$$\frac{d \ln w\left(\theta, z(\theta), \frac{L(\theta)}{\delta(\theta)}; Q\right)}{d\theta} = \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta}.$$

Part (ii) of Theorem 1.

We now consider the case where $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial \theta \partial z} = 0$ for any $(\theta, z) \in \Theta \times \mathbb{Z}$. Again, we solve the planning problem where the policy constraint is loosed. We then prove that the policy constraint is naturally satisfied. Since $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial \theta \partial z} = 0$ ($q(\theta) \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta \partial z(\theta)}$ is again zero), the first-order conditions of the relaxed planning problem again implies (A24), (A26) and (A27), which are equivalent to conditions (A5) to (A7). Therefore, following the **Step 3** of the proof to **Part (i)**, we have $\frac{d \ln w(\theta)}{d\theta} = \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}$.

Part (iii) of Theorem 1.

By equation (A21), we know that it is always optimal to choose robot inputs that satisfy $\frac{\partial Q}{\partial B_z(z)} = 1$. As for the job assignment, when the planner has a strict oblogation to redistribute, and there exist a positive measure of set $\Delta \in \Theta \times Z$ such that $\frac{\partial^2 \ln \vartheta(z,\theta)}{\partial \theta \partial z} > 0$, equation (A6) does not hold. Therefore, the optimal job assignment is not production efficient.□

A.4 Proof of Proposition 5

Part (i). By condition (**??**) and the definition of tax wedge, we have

$$\tau^B(z) = \frac{\partial Q}{\partial B_z(z)} - 1 = 0,$$

which is part one of Proposition 5.

Part (ii). We now prove part two of Proposition 5.

We first simplify the FOCs of the planning problem (see Step 2 of the proof of Theorem 1). Notice

that $\gamma^z_{own}(z(\theta)) = \frac{\sigma-1}{\sigma}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial z(\theta)}$ and $\gamma^z_{cross}(z(\theta), z(\theta')) = \frac{1}{\sigma}\frac{\partial \ln Q}{\partial \ln z(\theta')}$ are independent of θ:

$$\begin{aligned}\int_\Theta \chi(x)\gamma^z_{cross}(z(z), z(\theta))\,dx &= \int_\Theta \varphi'(x)\gamma^z_{cross}(z(x), z(\theta))\,dx \\ &= -\int_\Theta \varphi(x)\frac{d\left[\gamma^z_{cross}(z(x), z(\theta))\right]}{dx}dx \\ &= \varphi'(\theta)\gamma^z_{own}(z(\theta)).\end{aligned} \tag{A31}$$

The second is derived by the definitions of wage elasticities, and the last equation is derived by the symmetricity of γ^z_{cross} (see Appendix **??**). Similary to (A31), we have

$$\int_\Theta \chi(x)\gamma^\delta_{cross}(z(x), z(\theta))\,dx = 0. \tag{A32}$$

and

$$\int_\Theta \chi(x)\gamma^L_{cross}(z(x), z(\theta))\,dx = 0 \tag{A33}$$

By the above conditions, we can simplify (A15) to (A17) to (**??**) and

$$\delta(\theta): \psi(\theta) + \frac{\varphi'(\theta)}{\delta(\theta)}\gamma^\delta_{own}(\theta) + \lambda\frac{\partial Q}{\partial \delta(\theta)} = 0. \tag{A34}$$

and

$$z(\theta): \psi'(\theta) + \left[\begin{array}{c} q(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}l(\theta)\phi'(l(\theta)) + \\ \varphi(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} + \frac{\varphi'(\theta)\gamma^z_{own}(z(\theta))}{z(\theta)}\end{array}\right] + \lambda\frac{\partial Q}{\partial L(\theta)}L(\theta)\frac{\partial \ln\left[\eta(z(\theta))^{\frac{\sigma}{\sigma-1}}\vartheta(z(\theta),\theta)\right]}{\partial z(\theta)} = 0, \tag{A35}$$

and

$$l(\theta): \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}q(\theta)\phi'(l(\theta))\left[1 + \frac{l(\theta)\phi''(l(\theta))}{\phi'(l(\theta))}\right] + \frac{\varphi'(\theta)\gamma^L_{own}(\theta)}{l(\theta)} + \lambda w(\theta)\left[1 - \frac{\phi'(l(\theta))}{w(\theta)}\right]f_\theta(\theta) = 0, \tag{A36}$$

We now derive the optimal labor income tax wedge. Substitute $\int_\Theta \chi(x)\gamma^L_{cross}(z(x), z(\theta))\,dx$ in (A14) by (A33):

$$1 - \frac{\phi'(l(\theta))}{w(\theta)} = -\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}\frac{q(\theta)}{\lambda f_\theta(\theta)}\frac{\phi'(l(\theta))}{w(\theta)}\left(1 + \frac{1}{\varepsilon}\right) - \frac{\varphi'(\theta)\gamma^L_{own}(\theta)}{\lambda f_\theta(\theta)w(\theta)l(\theta)}, \tag{A37}$$

where $\varphi'(\theta)$ is determined by (A38) and $\varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0$.

$$\begin{aligned}\frac{1-\frac{\phi'(l(\theta))}{w(\theta)}}{\frac{\phi'(l(\theta))}{w(\theta)}} &= -\frac{\partial \ln \vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta}\frac{q(\theta)}{\lambda f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right) - \frac{\frac{\varphi'(\theta)\gamma^L_{own}(\theta)}{\lambda f_\theta(\theta)w(\theta)l(\theta)}}{\frac{\phi'(l(\theta))}{w(\theta)}} \\ &= -\frac{d\ln w\left(\theta\right)}{d\theta}\frac{q(\theta)}{\lambda f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right) - \frac{\varphi'(\theta)}{\lambda}\frac{\gamma^L_{own}\left(\theta\right)}{\frac{\phi'(l(\theta))}{w(\theta)}f_\theta(\theta)w(\theta)l(\theta)} \\ &= \frac{\widetilde{F}_\theta(\theta)-F_\theta(\theta)}{1-F_\theta(\theta)}\frac{1-F_w(w(\theta))}{w(\theta)f_w(w(\theta))}\left(1+\frac{1}{\varepsilon}\right) - \frac{\varphi'(\theta)}{\lambda}\frac{\gamma^L_{own}\left(\theta\right)}{\frac{\phi'(l(\theta))}{w(\theta)}f_\theta(\theta)w(\theta)l(\theta)}\end{aligned}$$

Notice that $\phi'\left(l\left(\theta\right)\right) = w\left(\theta\right)\left[1-T'\left(y\left(\theta\right)\right)\right]$, $\frac{1-F_w(w(\theta))}{w(\theta)f_w(w(\theta))}\left(1+\frac{1}{\varepsilon}\right) = \frac{1-F_y(y(\theta))}{y(\theta)f_y(y(\theta))}\frac{1}{\varepsilon^l_{1-\tau}(\theta)}$, $\frac{\widetilde{F}_\theta(\theta)-F_\theta(\theta)}{1-F_\theta(\theta)} = 1-\overline{g}\left(\theta\right)$, we have

$$\frac{\tau^y\left(\theta\right)}{1-\tau^y\left(\theta\right)} = \left[1-\overline{g}\left(\theta\right)\right]\frac{1-F_y(y(\theta))}{y(\theta)f_y(y(\theta))}\frac{1}{\varepsilon^l_{1-\tau}(\theta)} - \frac{\varphi'(\theta)}{\lambda f_\theta(\theta)}\frac{\gamma^L_{own}\left(\theta\right)}{\left[1-\tau^y(\theta)\right]y(\theta)},$$

Equivalently,

$$1-T'\left(y\left(\theta\right)\right) = \frac{1+\frac{\varphi'(\theta)}{\lambda w(\theta)l(\theta)f_\theta(\theta)}\gamma^L_{own}\left(z\left(\theta\right)\right)}{1+\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)}\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\ln\theta}\left(1+\frac{1}{\varepsilon}\right)},$$

i.e.,

$$T'\left(y\left(\theta\right)\right) = \frac{\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)}\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\ln\theta}\left(1+\frac{1}{\varepsilon}\right)}{1+\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)}\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\ln\theta}\left(1+\frac{1}{\varepsilon}\right)} - \frac{\frac{\varphi'(\theta)}{\lambda y(\theta)f_\theta(\theta)}\gamma^L_{own}\left(z\left(\theta\right)\right)}{1+\frac{\widetilde{F}(\theta)-F(\theta)}{\theta f_\theta(\theta)}\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\ln\theta}\left(1+\frac{1}{\varepsilon}\right)},$$

where $\lambda = 1$, $\gamma^L_{own}\left(\theta\right) = -\frac{1}{\sigma}$.

Part (iii). In Part (ii), we provide an equation describing the optimal tax wedge. What's implicit in the formular is $\varphi'(\theta)$ and $z\left(\theta\right)$. To solve the optimal tax rate explicitly, we substitute $\psi'(\theta)$ in (A16) by (A34):

$$\begin{aligned}0 = &-\lambda\frac{d\left[\frac{\partial Q}{\partial\delta(\theta)}+\frac{\varphi'(\theta)}{\delta(\theta)}\frac{1}{\sigma}\right]}{d\theta} + q(\theta)\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta\partial z\left(\theta\right)}l(\theta)\phi'\left(l\left(\theta\right)\right) + \varphi(\theta)\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta\partial z\left(\theta\right)} \\ &+\frac{\varphi'(\theta)\gamma^z_{own}\left(z\left(\theta\right)\right)}{z\left(\theta\right)} + \lambda L\left(\theta\right)\frac{\partial Q}{\partial L\left(\theta\right)}\frac{\partial\ln\left[\eta\left(z\left(\theta\right)\right)^{\frac{\sigma}{\sigma-1}}\vartheta\left(z\left(\theta\right),\theta\right)\right]}{\partial z\left(\theta\right)}.\end{aligned} \tag{A38}$$

Remember that $Q = \left[\int_{\mathbb{Z}^B}\eta\left(z\right)\left[\kappa B_z\left(z\right)\right]^{\frac{\sigma-1}{\sigma}}dz + \int_0^1\eta\left(z\left(\theta\right)\right)\left[\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)\right]^{\frac{\sigma-1}{\sigma}}\delta\left(\theta\right)^{\frac{1}{\sigma}}d\theta\right]^{\frac{\sigma}{\sigma-1}}$. We can substitude $\frac{\partial Q}{\partial\delta(\theta)}$ in the above equation by $\frac{\partial Q}{\partial\delta(\theta)} = \frac{1}{\sigma}\frac{\sigma}{\sigma-1}\eta\left(z\left(\theta\right)\right)\left[\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)\right]^{\frac{\sigma-1}{\sigma}}\delta\left(\theta\right)^{\frac{1-\sigma}{\sigma}}Q^{\frac{1}{\sigma}}$. What left to be done is solving $\{\delta\left(\theta\right),z\left(\theta\right),l(\theta),B_z\left(z\right)\}$ by ODEs (A35), (A36) and (A38) together with boundary conditions (A18), (A19), $\psi(\bar{\theta}) = -\frac{\varphi'(\bar{\theta})}{\delta(\bar{\theta})}\gamma^\delta_{own}\left(\bar{\theta}\right) - \lambda\frac{\partial Q}{\partial\delta(\bar{\theta})}$, $\psi(\underline{\theta}) = -\frac{\varphi'(\underline{\theta})}{\delta(\underline{\theta})}\gamma^\delta_{own}\left(\underline{\theta}\right) - \lambda\frac{\partial Q}{\partial\delta(\underline{\theta})}$ and constraints of the planning problem.

Substituting $\gamma_{own}^{z}\left(z\left(\theta\right)\right)$, $\frac{\partial Q}{\partial\delta(\theta)}$, and $\frac{\partial Q}{\partial L(\theta)}L\left(\theta\right)$ in (A38) by $\gamma_{own}^{z}\left(z\left(\theta\right)\right)=\frac{\sigma-1}{\sigma}\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial z(\theta)}$, $\frac{\partial Q}{\partial\delta(\theta)}=\frac{1}{\sigma-1}\frac{w(\theta)l(\theta)f_{\theta}(\theta)}{z'(\theta)}$,[32] and $\frac{\partial Q}{\partial L(\theta)}L\left(\theta\right)=w\left(\theta\right)l\left(\theta\right)f_{\theta}\left(\theta\right)$, we can rewrite (A38) as below:

$$\begin{aligned}0=&-\frac{d\left[\frac{1}{\sigma-1}\frac{w(\theta)l(\theta)f_{\theta}(\theta)}{z'(\theta)}+\frac{\varphi'(\theta)}{z'(\theta)}\frac{1}{\sigma}\right]}{d\theta}-\left[\widetilde{F}_{\theta}(\theta)-F_{\theta}(\theta)\right]\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta\partial z\left(\theta\right)}l(\theta)\phi'\left(l\left(\theta\right)\right)\\&+\varphi(\theta)\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta\partial z\left(\theta\right)}+\frac{\sigma-1}{\sigma}\frac{\varphi'(\theta)}{z\left(\theta\right)}\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)}\\&+w\left(\theta\right)l\left(\theta\right)f_{\theta}\left(\theta\right)\frac{\partial\ln\left[\eta\left(z\left(\theta\right)\right)^{\frac{\sigma}{\sigma-1}}\vartheta\left(z\left(\theta\right),\theta\right)\right]}{\partial z\left(\theta\right)}\end{aligned}$$

Notice that $\frac{l(\theta)f_{\theta}(\theta)}{z'(\theta)}=\frac{w(\theta)^{-\sigma}\eta(z(\theta))^{\sigma}}{\vartheta(z(\theta),\theta)^{1-\sigma}}Q^{-\frac{1}{\sigma}}$ (see equation (A28)), i.e., $\frac{w(\theta)l(\theta)f_{\theta}(\theta)}{z'(\theta)}=\frac{w(\theta)^{1-\sigma}\eta(z(\theta))^{\sigma}}{\vartheta(z(\theta),\theta)^{1-\sigma}}Q^{-\frac{1}{\sigma}}$. The above equation is equavilent to

$$\begin{aligned}0=&-\frac{1}{\sigma-1}w\left(\theta\right)l\left(\theta\right)f_{\theta}\left(\theta\right)\frac{d\ln\left[\frac{w(\theta)^{1-\sigma}\eta(z(\theta))^{\sigma}}{\vartheta(z(\theta),\theta)^{1-\sigma}}Q^{-\frac{1}{\sigma}}\right]}{d\theta}-\varphi''(\theta)\frac{1}{\sigma}+\varphi'(\theta)\frac{z''\left(\theta\right)}{z'\left(\theta\right)}\frac{1}{\sigma}\\&-z'\left(\theta\right)\left[\widetilde{F}_{\theta}(\theta)-F_{\theta}(\theta)\right]\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta\partial z\left(\theta\right)}l(\theta)\phi'\left(l\left(\theta\right)\right)+z'\left(\theta\right)\varphi(\theta)\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta\partial z\left(\theta\right)}\\&+\frac{\sigma-1}{\sigma}\frac{\varphi'(\theta)}{z\left(\theta\right)}\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)}z'\left(\theta\right)\\&+w\left(\theta\right)l\left(\theta\right)f_{\theta}\left(\theta\right)\left[\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)}+\frac{\sigma}{\sigma-1}\frac{d\ln\eta\left(z\left(\theta\right)\right)}{dz\left(\theta\right)}\right]z'\left(\theta\right)\end{aligned}\tag{A39}$$

Notice that

$$\begin{aligned}\frac{d\ln\left[\frac{w(\theta)^{1-\sigma}\eta(z(\theta))^{\sigma}}{\vartheta(z(\theta),\theta)^{1-\sigma}}Q^{-\frac{1}{\sigma}}\right]}{d\theta}&=(1-\sigma)\left[\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta}-\frac{d\ln\vartheta\left(z\left(\theta\right),\theta\right)}{d\theta}\right]+\sigma\frac{d\ln\eta\left(z\left(\theta\right)\right)}{d\theta}\\&=(\sigma-1)\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)}z'\left(\theta\right)+\sigma\frac{d\ln\eta\left(z\left(\theta\right)\right)}{d\theta}\end{aligned}$$

Equation (A39) can be rearranged as below:

$$\begin{aligned}&\varphi''(\theta)-\varphi'(\theta)\left[(\sigma-1)\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial z\left(\theta\right)}z'\left(\theta\right)+\frac{z''\left(\theta\right)}{z'\left(\theta\right)}\right]-\varphi(\theta)\sigma\frac{\partial^{2}\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta\partial z\left(\theta\right)}z'\left(\theta\right)\\&=-\sigma z'\left(\theta\right)\left[\widetilde{F}_{\theta}(\theta)-F_{\theta}(\theta)\right]\frac{\partial^{2}\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta\partial z\left(\theta\right)}l(\theta)\phi'\left(l\left(\theta\right)\right),\end{aligned}\tag{A40}$$

The above second order ODE together with the following boundary conditions can be used to determine

[32] *Remember that* $Q=\left[\int_{\mathcal{Z}^{B}}\eta\left(z\right)\left[\kappa B_{z}\left(z\right)\right]^{\frac{\sigma-1}{\sigma}}dz+\int_{0}^{1}\eta\left(z\left(\theta\right)\right)\left[\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)\right]^{\frac{\sigma-1}{\sigma}}\delta\left(\theta\right)^{\frac{1}{\sigma}}d\theta\right]^{\frac{\sigma}{\sigma-1}}$. *and* $w\left(\theta,z\left(\theta\right),\frac{L(\theta)}{\delta(\theta)};Q\right)=\vartheta\left(z\left(\theta\right),\theta\right)^{1-\frac{1}{\sigma}}\eta\left(z\left(\theta\right)\right)\left[\frac{L(\theta)}{\delta(\theta)}\right]^{-\frac{1}{\sigma}}Q^{\frac{1}{\sigma}}$. *We have* $\frac{\partial Q}{\partial\delta(\theta)}=\frac{1}{\sigma}\frac{\sigma}{\sigma-1}\eta\left(z\left(\theta\right)\right)\left[\vartheta\left(z\left(\theta\right),\theta\right)L\left(\theta\right)\right]^{\frac{\sigma-1}{\sigma}}\delta\left(\theta\right)^{\frac{1-\sigma}{\sigma}}Q^{\frac{\sigma-1}{\sigma}\left[\frac{\sigma}{\sigma-1}-1\right]}=\frac{1}{\sigma-1}\frac{w(\theta)l(\theta)f_{\theta}(\theta)}{z'(\theta)}$.

$\varphi(\theta)$ and $z(\theta)$:

$$\varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0, z\left(\bar{\theta}\right) = 1.$$

In the case with $\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} = 0$, $\ln \vartheta\left(z\left(\theta\right),\theta\right)$ can be expressed in the following form:

$$\ln \vartheta\left(z\left(\theta\right),\theta\right) = \ln \Lambda_1\left(z\left(\theta\right)\right) + \ln \Lambda_2\left(\theta\right),$$

where $\Lambda_1 : Z \to R$ and $\Lambda_2 : \Theta \to R$ are two functions satisfying $\frac{d \ln \Lambda_2(\theta)}{d \ln \theta} > 0$ (because $\frac{w'}{w} = \frac{d \ln \Lambda_2(\theta)}{d\theta} > 0$) and $\frac{d \ln \Lambda_1(z)}{d \ln z} > 0$ (under Assumptions 2, we have $\frac{d \ln \Lambda_1(z)}{d \ln z} > 0$).

Substitude $\ln \vartheta\left(z\left(\theta\right),\theta\right)$ in (A40) by $\ln \vartheta\left(z\left(\theta\right),\theta\right) = \ln \Lambda_1\left(z\left(\theta\right)\right) + \ln \Lambda_2\left(\theta\right)$:

$$\frac{d \ln \varphi'(\theta)}{d\theta} = \frac{d \ln \Lambda_1\left(z\left(\theta\right)\right)^{\sigma-1} z'\left(\theta\right)}{d\theta},$$

Solve the above ODE for $\varphi'(\theta)$:

$$\varphi'(\theta) = C_1 \sigma \Lambda_1\left(z\left(\theta\right)\right)^{\sigma-1} z'\left(\theta\right) = \frac{d}{d\theta}\left[C_1 \Lambda_1\left(z\left(\theta\right)\right)^{\sigma}\right],$$

where C_1 is a constant. Solve the above ODE:

$$\varphi(\theta) = C_1 \Lambda_1\left(z\left(\theta\right)\right)^{\sigma} + C_2$$

where C_2 is a constant. Notice that $\varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0$, $1 = z\left(\bar{\theta}\right) > z\left(\underline{\theta}\right) \geq 0$, and $\frac{d \ln \Lambda_1(z)}{d \ln z} > 0$. If $\Lambda_1\left(z\left(\bar{\theta}\right)\right) > \Lambda_1\left(z\left(\underline{\theta}\right)\right)$, $\varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0$ holds only if $C_1 = C_2 = 0$. As a result, the solution to (A40) is:

$$\varphi(\theta) = 0, \forall\theta \in \left[\underline{\theta}, \bar{\theta}\right].$$

Therefore, $\varphi'(\theta) = 0$ is true for all $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$. Then (27) can be reduced to (28).

Part (iv). Set $a\left(\theta\right) = (\sigma - 1)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial z(\theta)} z'\left(\theta\right)$ and $b\left(\theta\right) = \sigma \frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} z'\left(\theta\right)$, where $b\left(\theta\right)$ is a positive function on the interval $\left[\underline{\theta}, \bar{\theta}\right]$, because $z'\left(\theta\right) > 0$. When $\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} > 0$ and $\widetilde{F}_\theta(\theta) - F_\theta(\theta) > 0$, we have the below inequality holds for all $\theta \in \left(\underline{\theta}, \bar{\theta}\right)$ by (A40):

$$\varphi''(\theta) - \left[a\left(\theta\right) + \frac{z''\left(\theta\right)}{z'\left(\theta\right)}\right]\varphi'(\theta) - b\left(\theta\right)\varphi(\theta) < 0. \tag{A41}$$

The above ODE together with the boundary conditions implies the property of $\varphi(\theta)$ on the boundary, i.e., part (iv) of Proposition 5. Intuitively, the preference for equality ($\widetilde{F}_\theta(\theta) - F_\theta(\theta) > 0$) is crucial for the optimal tax rate on the boundary, which is captured by the property of $\varphi'(\theta)$.

Formally, set the following notation for analysis:

$$\begin{aligned} A(\theta) &= \exp\left\{\int_{\underline{\theta}}^{\theta} -\left[a(x) + \frac{z''(x)}{z'(x)}\right] dx\right\} \\ &= \exp\left\{-\left[\int_{\underline{\theta}}^{\theta} a(x)\, dx + \int_{\underline{\theta}}^{\theta} \frac{d \ln z'(x)}{dx} dx\right]\right\} \\ &= \exp\left\{-\left[\int_{\underline{\theta}}^{\theta} a(x)\, dx + \ln z'(\theta) - \ln z'(\underline{\theta})\right]\right\} \\ &= \exp\left\{\int_{\underline{\theta}}^{\theta} -a(x)\, dx\right\} \frac{z'(\underline{\theta})}{z'(\theta)} > 0. \end{aligned} \tag{A42}$$

Then, by (A41), for all $\theta \in (\underline{\theta}, \bar{\theta})$, we have below inequality:

$$\frac{d}{d\theta}\left[A(\theta)\, \varphi'(\theta)\right] < A(\theta)\, b(\theta)\, \varphi(\theta),$$

By integration, for any $\theta \in (\underline{\theta}, \bar{\theta}]$, we have

$$\left[A(\theta)\, \varphi'(\theta)\right] - A(\underline{\theta})\, \varphi'(\underline{\theta}) < \int_{\underline{\theta}}^{\theta} A(x)\, b(x)\, \varphi(x) dx. \tag{A43}$$

Combination of the above equation and (A42) gives the following result for any $\theta \in (\underline{\theta}, \bar{\theta}]$:

$$\varphi'(\theta) < \varphi'(\underline{\theta}) \exp\left\{\int_{\underline{\theta}}^{\theta} a(x)\, dx\right\} \frac{z'(\underline{\theta})}{z'(\theta)} + \frac{1}{z'(\theta)} \times \int_{\underline{\theta}}^{\theta} \exp\left\{\int_{x}^{\theta} a(s)\, ds\right\} b(x)\, z'(x)\, \varphi(x) dx, \tag{A44}$$

Now we prove the following results.

Step 1. First, there dose not exist an interval $[\underline{\theta}, \theta_1)$ or $(\theta_2, \bar{\theta}]$, such that for any $\theta \in [\underline{\theta}, \theta_1)$ or $\theta \in (\theta_2, \bar{\theta}]$, $\varphi(\theta) = 0$. We prove by contradiction. If there exist such an interval, one has $\varphi''(\theta) = \varphi'(\theta) = \varphi(\theta) = 0$ for such an interval. $\varphi'(\theta) = \varphi(\theta) = 0$ contradicts (A44), because $z'(\theta) > 0$ when $\frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} > 0$.

Since there dose not exist an interval $[\underline{\theta}, \theta_1)$ or $(\theta_2, \bar{\theta}]$, such that for any $\theta \in [\underline{\theta}, \theta_1)$ or $\theta \in (\theta_2, \bar{\theta}]$, $\varphi(\theta) = 0$, and $\varphi(\cdot)$ is twice continuously differentiable, there must exist intervals $[\underline{\theta}, \theta_1)$ and $(\theta_2, \bar{\theta}]$ such that $\varphi(\theta)$ is always positive or negative in the intervals.

Step 2. Second, we prove that there does not exist $[\underline{\theta}, \theta_1)$ such that $\varphi(\theta)$ is negative in the interval, and there doest not exist $(\theta_2, \bar{\theta}]$ such that $\varphi(\theta)$ is negative in the interval. To see this, first note that $\varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0$ and, by Step 1, $\varphi(\theta)$ is not always zero in $[\underline{\theta}, \bar{\theta}]$. Then, since $\varphi(\cdot)$ is twice continuously differentiable, there exist $\theta \in (\underline{\theta}, \bar{\theta})$ such that $\varphi'(\theta) = 0$. Set θ_1^* as the smallest θ such that $\varphi'(\theta) = 0$.

We first prove that "suppose there exist $[\underline{\theta}, \theta_1)$ such that $\varphi(\theta)$ is negative in the interval, then for any $\theta \in (\underline{\theta}, \theta_1^*)$, $\varphi(\theta) < 0$." To see this, note that if the above assertion does not hold there exist $\underline{\theta}_1$ and $\bar{\theta}_1$ such that $\underline{\theta} < \underline{\theta}_1 < \bar{\theta}_1 < \theta_1^*$ and $\varphi(\underline{\theta}_1)\varphi(\bar{\theta}_1) < 0$, which will contradict the definition of θ_1^*. To see this, consider two situations: (i) $\varphi(\underline{\theta}_1) < 0$ and $\varphi(\bar{\theta}_1) > 0$. At this situtation, since $\varphi(\underline{\theta}) = 0$, there exist $\widehat{\theta}_1^l \in (\underline{\theta}, \underline{\theta}_1)$ such that $\varphi'(\widehat{\theta}_1^l) < 0$, and exist $\widehat{\theta}_1^u \in (\underline{\theta}_1, \bar{\theta}_1)$ such that $\varphi'(\widehat{\theta}_1^u) > 0$. Then there exist θ_1^{**} such

that $\varphi'(\theta_1^{**}) = 0$ and $\theta_1^{**} < \widehat{\theta}_1^u < \overline{\theta}_1 < \theta_1^*$, which contradictd the definition of θ_1^*. (ii) $\varphi(\underline{\theta}_1) > 0$ and $\varphi(\overline{\theta}_1) < 0$. Similary, we have $\widehat{\theta}_1^l \in (\underline{\theta}, \underline{\theta}_1)$ such that $\varphi'(\widehat{\theta}_1^l) > 0$, and $\widehat{\theta}_1^u \in (\underline{\theta}_1, \overline{\theta}_1)$ such that $\varphi'(\widehat{\theta}_1^u) < 0$. Then we have $\theta_1^{**} < \widehat{\theta}_1^u < \overline{\theta}_1 < \theta_1^*$ such that $\varphi'(\theta_1^{**}) = 0$, which again contradictd the definition of θ_1^*.

What remains to be proven is that it is not true that "for any $\theta \in (\underline{\theta}, \theta_1^*)$, $\varphi(\theta) < 0$." Suppose for any $\theta \in (\underline{\theta}, \theta_1^*)$, $\varphi(\theta) < 0$. One must has $\varphi'(\underline{\theta}) < 0$. Then, by (A43),

$$A(\theta_1^*)\varphi'(\theta_1^*) - A(\underline{\theta})\varphi'(\underline{\theta}) < \int_{\underline{\theta}}^{\theta_1^*} A(x)\, b(x)\, \varphi(x) dx < 0.$$

Therefore, $A(\theta_1^*)\varphi'(\theta_1^*) < A(\underline{\theta})\varphi'(\underline{\theta}) \leq 0$, which gives $\varphi'(\theta_1^*) < 0$. $\varphi'(\theta_1^*) < 0$ contradict $\varphi(\theta_1^*) = 0$. In conclusion, we have proved that there does not exist $[\underline{\theta}, \theta_1)$ such that $\varphi(\theta)$ is negative in the interval. Similarly, we can prove that there dose not exist an interval $(\theta_2, \overline{\theta}]$, such that for any $\theta \in (\theta_2, \overline{\theta}]$, $\varphi(\theta) = 0$.

Step 3. Last, we prove that for any $\theta \in (\underline{\theta}, \overline{\theta})$, one has $\varphi(\theta) \neq 0$. We prove this by contradiction. Suppose there exist $\theta_3 \in (\underline{\theta}, \overline{\theta})$, such that $\varphi(\theta_3) = 0$. For intervals $[\underline{\theta}, \theta_3)$ and $(\theta_3, \overline{\theta}]$, by Step 1 and 2, one has the following results: There exist $\widehat{\theta}_3^l$ and $\widehat{\theta}_3^u$ satisfying $\widehat{\theta}_3^l < \theta_3 < \widehat{\theta}_3^u$ such that for $\theta \in \left(\widehat{\theta}_3^l, \theta_3\right)$, $\varphi(\theta) > 0$; for $\theta \in \left(\theta_3, \widehat{\theta}_3^u\right)$, $\varphi(\theta) > 0$. Remember that $\varphi(\theta_3) = 0$. One can see that θ_3 is local minimum point. Since $\varphi(\cdot)$ is twice continuously differentiable, one has $\varphi'(\theta_3) = \varphi(\theta_3) = 0$ and $\varphi''(\theta_3) \geq 0$, which contradict (A41).

In conclusion, we have $\varphi(\theta) > 0$ for any $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$. Since $\varphi(\cdot)$ is twice continuously differentiable and $\varphi(\underline{\theta}) = \varphi(\overline{\theta}) = 0$, there must exist interval $(\underline{\theta}, \theta_A)$ and $\left(\theta_B, \bar{\theta}\right)$ such that for any $\theta \in (\underline{\theta}, \theta_A)$ we have $\varphi'(\theta) > 0$, and for any $\theta \in \left(\theta_B, \bar{\theta}\right)$, we have $\varphi'(\theta) < 0$, where $\underline{\theta} < \theta_A < \theta_B < \bar{\theta}$.□

A.5 Proof of Proposition 6

In the case with $\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta \partial z(\theta)} = 0$, $\ln \vartheta(z(\theta), \theta)$ can be expressed in the following form:

$$\ln \vartheta(z(\theta), \theta) = \ln \Lambda_1(z(\theta)) + \ln \Lambda_2(\theta),$$

where $\Lambda_1 : Z \to R$ and $\Lambda_2 : \Theta \to R$ are two functions satisfying $\frac{d \ln \Lambda_2(\theta)}{d \ln \theta} > 0$ (because $\frac{w'}{w} = \frac{d \ln \Lambda_2(\theta)}{d\theta} > 0$) and $\frac{d \ln \Lambda_1(z)}{d \ln z} > 0$ (under Assumptions 2, we have $\frac{d \ln \Lambda_1(z)}{d \ln z} > 0$).

Part (i). By (A11) and (A12) we have $q(\theta) = F_\theta(\theta) - \widetilde{F}_\theta(\theta)$ and $\lambda = 1$, which together with (A21) gives:

$$B_z(z) : \frac{\partial Q}{\partial B_z(z)} = 1.$$

Notice that $\frac{\partial Q}{\partial B_z(z)} = \eta(z)\kappa^{\frac{\sigma-1}{\sigma}} B_z(z)^{-\frac{1}{\sigma}} Q^{\frac{1}{\sigma}}$. We have:

$$\eta(z)\kappa^{\frac{\sigma-1}{\sigma}} B_z(z)^{-\frac{1}{\sigma}} Q^{\frac{1}{\sigma}} = 1.$$

Rearrange the above equation gives:

$$B_z(z) = \eta(z)^{\sigma} \kappa^{\sigma-1} Q \tag{A45}$$

where

$$Q = \left[\frac{\int_{\underline{\theta}}^{\overline{\theta}} \eta(z(\theta)) \left[\Lambda_1(z(\theta)) \Lambda_2(\theta) l(\theta) f_\theta(\theta)\right]^{\frac{\sigma-1}{\sigma}} z'(\theta)^{\frac{1}{\sigma}} d\theta}{1 - \int_0^{z(\underline{\theta})} \eta(z)^{\sigma} \kappa^{\sigma-1} dz} \right]^{\frac{\sigma}{\sigma-1}}. \tag{A46}$$

Substitude $\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta}$ in (28) by $\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta} = \frac{d \ln \Lambda_2(\theta)}{d \ln \theta}$:

$$\phi'(l(\theta)) = w(\theta)\left[1 - \tau^y(\theta)\right] = \frac{w(\theta)}{1 + \frac{\widetilde{F}(\theta) - F(\theta)}{\theta f_\theta(\theta)} \frac{d \ln \Lambda_2(\theta)}{d \ln \theta} \left(1 + \frac{1}{\varepsilon}\right)},$$

Remeber that

$$w(\theta) = \left\{ \Lambda_1(z(\theta))^{1-\frac{1}{\sigma}} \Lambda_2(\theta)^{1-\frac{1}{\sigma}} \eta(z(\theta)) \left[\frac{l(\theta) f_\theta(\theta)}{z'(\theta)}\right]^{-\frac{1}{\sigma}} Q^{\frac{1}{\sigma}} \right\} \tag{A47}$$

Combination of the above two equations gives:

$$l(\theta) = \left[\frac{\eta(z(\theta))^{\frac{\sigma-1}{\sigma}} \Lambda_1(z(\theta))^{\frac{\sigma-1}{\sigma}} \Lambda_2(\theta)^{\frac{\sigma-1}{\sigma}} f_\theta(\theta)^{-\frac{1}{\sigma}} z'(\theta)^{\frac{1}{\sigma}}}{1 + \frac{d \ln \Lambda_2(\theta)}{d \ln \theta} \frac{\widetilde{F}_\theta(\theta) - F_\theta(\theta)}{\theta f_\theta(\theta)} \left(1 + \frac{1}{\varepsilon}\right)} \right]^{\frac{\sigma\varepsilon}{\varepsilon+\sigma}} Q^{\frac{\varepsilon}{\varepsilon+\sigma}}$$

When $\eta(\cdot) = f_\theta(\cdot) = 1$, the above equation can be simplified as

$$l(\theta) = \left[\frac{\Lambda_1(z(\theta))^{\frac{\sigma-1}{\sigma}} \Lambda_2(\theta)^{\frac{\sigma-1}{\sigma}} z'(\theta)^{\frac{1}{\sigma}}}{1 + \frac{d \ln \Lambda_2(\theta)}{d \ln \theta} \frac{\widetilde{F}_\theta(\theta) - F_\theta(\theta)}{\theta f_\theta(\theta)} \left(1 + \frac{1}{\varepsilon}\right)} \right]^{\frac{\sigma\varepsilon}{\varepsilon+\sigma}} Q^{\frac{\varepsilon}{\varepsilon+\sigma}}. \tag{A48}$$

Take the differential of both sides of equation (A47) and use $\frac{d \ln w(\theta)}{d\theta} = \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}$:

$$\begin{aligned} \frac{d \ln w(\theta)}{d\theta} &= \frac{d \ln \eta(z(\theta))}{d\theta} + \frac{\sigma-1}{\sigma} \frac{d \ln \Lambda_1(z(\theta))}{d\theta} + \frac{\sigma-1}{\sigma} \frac{d \ln \Lambda_2(\theta)}{d\theta} - \frac{1}{\sigma} \frac{d \ln l(\theta) f_\theta(\theta)}{d\theta} + \frac{1}{\sigma} \frac{d \ln z'(\theta)}{d\theta} \\ &= \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta} = \frac{d \ln \Lambda_2(\theta)}{d\theta} \end{aligned}$$

Take the differential of both sides of equation (A48) and substituting it into the above equation yields the following expression:

$$\frac{d \ln z'(\theta)}{d\theta} = (\varepsilon+1) \frac{d \ln \Lambda_2(\theta)}{d\theta} - (\sigma-1) \frac{d \ln \Lambda_1(z(\theta))}{d\theta} - \varepsilon \frac{d \ln \left[1 + \frac{d \ln \Lambda_2(\theta)}{d\theta} \frac{\widetilde{F}_\theta(\theta) - F_\theta(\theta)}{\theta f_\theta(\theta)} \left(1 + \frac{1}{\varepsilon}\right)\right]}{d\theta} \tag{A49}$$

To close the above differential equation, we need a boundary condition. In the following analysis,

we derive the boundary condition.

Substitude $l(\theta)$ in (A46) by the above equation:

$$Q^{\frac{\sigma-1}{\sigma}\left[1-\frac{\varepsilon}{\varepsilon+\sigma}\right]}=\frac{\int_{\underline{\theta}}^{\overline{\theta}}\frac{\left[\Lambda_1(z(\theta))^{\frac{\sigma-1}{\sigma}}\Lambda_2(\theta)^{\frac{\sigma-1}{\sigma}}z'(\theta)^{\frac{1}{\sigma}}\right]^{\frac{\sigma(\varepsilon+1)}{\varepsilon+\sigma}}}{\left[1+\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\frac{\widetilde{F}_\theta(\theta)-F_\theta(\theta)}{\theta f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right)\right]^{\frac{(\sigma-1)\varepsilon}{\varepsilon+\sigma}}}d\theta}{1-\int_0^{z(\underline{\theta})}\eta(z)^{\sigma}\kappa^{\sigma-1}dz}. \tag{A50}$$

Equation (A47) can be transformed to be:

$$\begin{aligned}\frac{w(\theta)l(\theta)f_\theta(\theta)}{z'(\theta)}&=\Lambda_1(z(\theta))^{1-\frac{1}{\sigma}}\Lambda_2(\theta)^{1-\frac{1}{\sigma}}\left[\frac{l(\theta)}{z'(\theta)}\right]^{1-\frac{1}{\sigma}}Q^{\frac{1}{\sigma}} \\ &=\frac{\Lambda_1(z(\theta))^{\frac{(\sigma-1)(\varepsilon+1)}{\varepsilon+\sigma}}\Lambda_2(\theta)^{\frac{(\sigma-1)(\varepsilon+1)}{\varepsilon+\sigma}}}{\left[1+\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\frac{\widetilde{F}_\theta(\theta)-F_\theta(\theta)}{\theta f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right)\right]^{\frac{(\sigma-1)\varepsilon}{\varepsilon+\sigma}}}z'(\theta)^{-\frac{\sigma-1}{\varepsilon+\sigma}}Q^{\frac{\varepsilon+1}{\varepsilon+\sigma}}\end{aligned} \tag{A51}$$

where the second equation is derived by (A48).

By (A17) and (A18), $\frac{\partial Q}{\partial\delta(\theta)}=\left[\frac{\sigma}{\sigma-1}\frac{\partial Q}{\partial B_z(z(\underline{\theta}))}-1\right]B_z(z(\underline{\theta}))-\frac{\gamma^{\delta}_{own}(z(\underline{\theta}))\varphi'(\underline{\theta})}{\delta(\underline{\theta})}$. Remember that when $\vartheta(z,\theta)$ is separable, $\varphi'(\underline{\theta})=0$. Therefore:

$$\frac{\partial Q}{\partial\delta(\underline{\theta})}=\left[\frac{\sigma}{\sigma-1}\frac{\partial Q}{\partial B_z(z(\underline{\theta}))}-1\right]B_z(z(\underline{\theta})) \tag{A52}$$

Notice that $\frac{\partial Q}{\partial\delta(\theta)}=\frac{1}{\sigma-1}\frac{w(\theta)l(\theta)f_\theta(\theta)}{z'(\theta)}$, and $B_z(z(\underline{\theta}))=\eta(z(\underline{\theta}))^{\sigma}\kappa^{\sigma-1}Q$ (see (A45)). The above equation gives:

$$\frac{w(\underline{\theta})l(\underline{\theta})f_\theta(\underline{\theta})}{z'(\underline{\theta})}=\eta(z(\underline{\theta}))^{\sigma}\kappa^{\sigma-1}Q=\kappa^{\sigma-1}Q \tag{A53}$$

Substituding $\frac{w(\underline{\theta})l(\underline{\theta})f_\theta(\underline{\theta})}{z'(\underline{\theta})}$ in (A51) by (A53) and $Q^{1-\frac{\varepsilon+1}{\varepsilon+\sigma}}$ by (A50) (note that $Q^{\frac{\sigma-1}{\sigma}\left(1-\frac{\varepsilon}{\varepsilon+\sigma}\right)}=Q^{1-\frac{\varepsilon+1}{\varepsilon+\sigma}}$), we have a boundary condition:

$$\begin{aligned}&\frac{\Lambda_1(z(\underline{\theta}))^{\frac{(\sigma-1)(\varepsilon+1)}{\varepsilon+\sigma}}\Lambda_2(\underline{\theta})^{\frac{(\sigma-1)(\varepsilon+1)}{\varepsilon+\sigma}}}{\left[1+\frac{d\ln\Lambda_2(\underline{\theta})}{d\ln\theta}\frac{\widetilde{F}_\theta(\underline{\theta})-F_\theta(\underline{\theta})}{\underline{\theta}f_\theta(\underline{\theta})}\left(1+\frac{1}{\varepsilon}\right)\right]^{\frac{(\sigma-1)\varepsilon}{\varepsilon+\sigma}}}z'(\underline{\theta})^{-\frac{\sigma-1}{\varepsilon+\sigma}} \\ &=\frac{\kappa^{\sigma-1}}{1-\kappa^{\sigma-1}\int_0^{z(\underline{\theta})}1dz}\int_{\underline{\theta}}^{\overline{\theta}}\frac{\Lambda_1(z(\theta))^{\frac{(\sigma-1)(\varepsilon+1)}{\varepsilon+\sigma}}\Lambda_2(\theta)^{\frac{(\sigma-1)(\varepsilon+1)}{\varepsilon+\sigma}}z'(\theta)^{-\frac{\sigma-1}{\varepsilon+\sigma}}}{\left[1+\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\frac{\widetilde{F}_\theta(\theta)-F_\theta(\theta)}{\theta f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right)\right]^{\frac{(\sigma-1)\varepsilon}{\varepsilon+\sigma}}}z'(\theta)d\theta\end{aligned} \tag{A54}$$

Rearrange (A54) and use $\widetilde{F}_\theta(\underline{\theta}) - F_\theta(\underline{\theta}) = 0$:

$$z'(\underline{\theta}) = \frac{\frac{\vartheta(z(\underline{\theta}),\underline{\theta})^{(\varepsilon+1)}}{(\kappa)^{\sigma+\varepsilon}}\left[1-(\kappa)^{\sigma-1}z(\underline{\theta})\right]^{\frac{\sigma+\varepsilon}{\sigma-1}}}{\left\{\int_0^1 \frac{\left[\vartheta(z(\theta),\theta)^{\frac{\sigma-1}{\sigma}}z'(\theta)^{\frac{1}{\sigma}}\right]^{\frac{\sigma(\varepsilon+1)}{\sigma+\varepsilon}} f_\theta(\theta)^{\frac{\sigma-1}{\sigma+\varepsilon}}}{\left[1+\frac{d\ln\vartheta(z(\theta),\theta)}{d\ln\theta}\frac{\widetilde{F}_\theta(\theta)-F_\theta(\theta)}{\theta f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right)\right]^{\frac{(\sigma-1)\varepsilon}{\varepsilon+\sigma}}}d\theta\right\}^{\frac{\sigma+\varepsilon}{\sigma-1}}}. \tag{A55}$$

Part (ii). Take integration of both sides of the equation (A49) gives:

$$\begin{aligned}&-\varepsilon\ln\left[1-\frac{F_\theta(\theta)-\widetilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]+(\varepsilon+1)\ln\Lambda_2(\theta)+\ln C_1\\&=(\sigma-1)\ln\Lambda_1(z(\theta))+\ln z'(\theta).\end{aligned}$$

where C_1 is a constant. Equivalently,

$$C_1\frac{\Lambda_2(\theta)^{\varepsilon+1}}{\left[1-\frac{F_\theta(\theta)-\widetilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}=\Lambda_1(z(\theta))^{\sigma-1}z'(\theta). \tag{A56}$$

Take integration of both sides of the above equation again:

$$C_1\int_\theta^{\overline{\theta}}\frac{\Lambda_2(\theta)^{\varepsilon+1}}{\left[1-\frac{F_\theta(\theta)-\widetilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}d\theta=\int_\theta^{\overline{\theta}}\Lambda_1(z(\theta))^{\sigma-1}z'(\theta)\,d\theta=\int_{z(\theta)}^1\Lambda_1(x)^{\sigma-1}dx \tag{A57}$$

Set $\Lambda(z)\equiv 1-\int_z^1\Lambda_1(x)^{\sigma-1}dx$. Notice that $\Lambda(z)$ satisfies $\Lambda\left(z\left(\overline{\theta}\right)\right)=1$. Using $\Lambda(z)$ and (A57), we have:

$$C_1\int_{\underline{\theta}}^{\overline{\theta}}\frac{\Lambda_2(\theta)^{\varepsilon+1}}{\left[1-\frac{F_\theta(\theta)-\widetilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}d\theta=1-\Lambda(z(\underline{\theta})) \tag{A58}$$

which implies

$$C_1=\frac{1-\Lambda(z(\underline{\theta}))}{\int_{\underline{\theta}}^{\overline{\theta}}\frac{\Lambda_2(\theta)^{\varepsilon+1}}{\left[1-\frac{F_\theta(\theta)-\widetilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}d\theta}.$$

Substitute C_1 in (A57) by the above equation:

$$\left[1-\Lambda(z(\underline{\theta}))\right]\frac{\int_\theta^{\overline{\theta}}\frac{\Lambda_2(\theta)^{\varepsilon+1}}{\left[1-\frac{F_\theta(\theta)-\widetilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}d\theta}{\int_{\underline{\theta}}^{\overline{\theta}}\frac{\Lambda_2(\theta)^{\varepsilon+1}}{\left[1-\frac{F_\theta(\theta)-\widetilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}d\theta}=1-\Lambda(z(\theta)). \tag{A59}$$

Substitute $z'(\theta)$ in (A55) by (A56):

$$z'(\underline{\theta}) = \frac{\frac{[\Lambda_1(z(\underline{\theta}))\Lambda_2(\underline{\theta})]^{(\varepsilon+1)}}{(\kappa)^{\sigma+\varepsilon}}}{\left\{\frac{\int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{\left[1-\frac{F_\theta(\theta)-\tilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}\Lambda_2(\theta)^{\varepsilon+1}f_\theta(\theta)d\theta}{1-(\kappa)^{\sigma-1}z(\underline{\theta})}\right\}^{\frac{\sigma+\varepsilon}{\sigma-1}}[C_1]^{\frac{(\varepsilon+1)}{\sigma-1}}}. \tag{A60}$$

Notice that (A56) implies:

$$C_1\frac{\Lambda_2(\underline{\theta})^{\varepsilon+1}}{\Lambda_1(z(\underline{\theta}))^{\sigma-1}} = z'(\underline{\theta}).$$

Substitute $z'(\underline{\theta})$ in (A60) by the above equation:

$$\frac{[\Lambda_1(z(\underline{\theta}))]^{\varepsilon+\sigma}}{(\kappa)^{\sigma+\varepsilon}} = C_1^{\frac{\varepsilon+\sigma}{\sigma-1}}\left\{\frac{\int_{\underline{\theta}}^{\overline{\theta}} \frac{\Lambda_2(\theta)^{\varepsilon+1}}{\left[1-\frac{F_\theta(\theta)-\tilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}d\theta}{1-(\kappa)^{\sigma-1}z(\underline{\theta})}\right\}^{\frac{\sigma+\varepsilon}{\sigma-1}}. \tag{A61}$$

Notice that C_1 satisfied (A58), i.e.,

$$C_1 = \frac{[1-\Lambda(\underline{z})]}{\int_{\underline{\theta}}^{\overline{\theta}} \frac{\Lambda_2(\theta)^{\varepsilon+1}}{\left[1-\frac{F_\theta(\theta)-\tilde{F}_\theta(\theta)}{\theta f_\theta(\theta)}\frac{d\ln\Lambda_2(\theta)}{d\ln\theta}\left(1+\frac{1}{\varepsilon}\right)\right]^{\varepsilon}}d\theta}.$$

Substitute C_1 in (A61) by (A58):

$$\frac{\Lambda_1(z(\underline{\theta}))}{[1-\Lambda(z(\underline{\theta}))]^{\frac{1}{\sigma-1}}} = \frac{\kappa}{\left[1-(\kappa)^{\sigma-1}z(\underline{\theta})\right]^{\frac{1}{\sigma-1}}}.$$

Equivalently,

$$\frac{\Lambda_1(z(\underline{\theta}))^{\sigma-1}z(\underline{\theta})}{\int_{z(\underline{\theta})}^1 \Lambda_1(x)^{\sigma-1}dx} = \frac{(\kappa)^{\sigma-1}z(\underline{\theta})}{1-(\kappa)^{\sigma-1}z(\underline{\theta})} \tag{A62}$$

i.e.,

$$\int_{z(\underline{\theta})}^1 \Lambda_1(x)^{\sigma-1}dx + \Lambda_1(z(\underline{\theta}))^{\sigma-1}z(\underline{\theta}) - \frac{\Lambda_1(z(\underline{\theta}))^{\sigma-1}}{(\kappa)^{\sigma-1}} = 0.$$

First, it can be seen that $z(\underline{\theta})$ is independent of social welfare weights. Moreover, $z(\underline{\theta})$ increases in κ if $\sigma > 1$, $\Lambda_1(z(\underline{\theta}))$, $\Lambda_1'(z(\underline{\theta})) > 0$. To see this, first note that the above equation gives:

$$z(\underline{\theta}) - \frac{1}{(\kappa)^{\sigma-1}} = -\frac{\int_{z(\underline{\theta})}^1 \Lambda_1(x)^{\sigma-1}dx}{\Lambda_1(z(\underline{\theta}))^{\sigma-1}} < 0.$$

Second, notice that when $\Lambda_1\left(z\left(\underline{\theta}\right)\right), \Lambda_1'\left(z\left(\underline{\theta}\right)\right) > 0$:

$$\frac{d\left[\int_{z(\underline{\theta})}^{1}\Lambda_1\left(x\right)^{\sigma-1}dx+\Lambda_1\left(z\left(\underline{\theta}\right)\right)^{\sigma-1}z\left(\underline{\theta}\right)-\frac{\Lambda_1(z(\underline{\theta}))^{\sigma-1}}{(\kappa)^{\sigma-1}}\right]}{d\kappa}>0,$$

and

$$\begin{aligned}&\frac{d\left[\int_{z(\underline{\theta})}^{1}\Lambda_1\left(x\right)^{\sigma-1}dx+\Lambda_1\left(z\left(\underline{\theta}\right)\right)^{\sigma-1}z\left(\underline{\theta}\right)-\frac{\Lambda_1(z(\underline{\theta}))^{\sigma-1}}{(\kappa)^{\sigma-1}}\right]}{dz\left(\underline{\theta}\right)}\\&=\left(\sigma-1\right)\Lambda_1\left(z\left(\underline{\theta}\right)\right)^{\sigma-2}\left[z\left(\underline{\theta}\right)-\frac{1}{\left(\kappa\right)^{\sigma-1}}\right]\Lambda_1'\left(z\left(\underline{\theta}\right)\right)<0.\end{aligned}$$

Therefore, $z\left(\underline{\theta}\right)$ increases in κ under our setup. And from (**??**) we can find that the $z\left(\underline{\theta}\right)$ is only determined by κ, and we know that $z^{\star}=z^{U}$ because it is independent on the social welfare weights. Moreover, the regulation is not desirable which is indicated by condition (A52).□

C Appendix C: Extensions

C.1 Proposition 9 Exogenous Job Assignment

Proposition 9 *Parts one to three of Proposition 9 still holds except that the expression for* $\frac{\varphi'(\theta)}{\lambda}$ *changes to take policies' effects on job assignment into consideration. The optimal automation satisfies:*

Proof. See Appendix C.2. ■

C.2 Proof of Proposition 9: Exogenous Job Assignment

We set $w\left(\theta,z\left(\theta\right),\delta\left(\theta\right),L\left(\theta\right),B\left(\theta\right),a\left(\theta\right),\mathcal{L},\mathcal{B},\mathbb{A}\right)=\vartheta\left(z\left(\theta\right),\theta\right)w_z\left(z\left(\theta\right)\right)$ as the wage function in terms of θ when there is no regulation. $Q_z\left(\theta\right)$ as the short form of $Q_z\left(\frac{B(\theta)}{\delta(\theta)},\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)};a\left(\theta\right)\right)$. Set Largangian function as:

$$\begin{aligned}\mathcal{L}=&\int_{\Theta}G\left(V(\theta)\right)\widetilde{f}_{\theta}(\theta)d\theta+\int_{\Theta}q(\theta)\left[\omega(\theta)l(\theta)\phi'\left(l\left(\theta\right)\right)-V'(\theta)\right]d\theta\\&+\int_{\Theta}\varphi(\theta)\left[\omega(\theta)-W'(\theta)\right]d\theta+\int_{\Theta}\chi(\theta)\left[\ln\left[\omega\left(\theta,z\left(\theta\right),\delta\left(\theta\right),B\left(\theta\right),L\left(\theta\right);a\left(\theta\right)\right)Q^{\frac{1}{\sigma}}\right]-W(\theta)\right]d\theta\\&+\lambda\left[\begin{array}{c}A\left[\int_{\Theta}\eta\left(z\left(\theta\right)\right)Q_z\left(\frac{B(\theta)}{\delta(\theta)},\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)};a\left(\theta\right)\right)^{\frac{\sigma-1}{\sigma}}\delta\left(\theta\right)d\theta\right]^{\frac{\sigma}{\sigma-1}}\\-\int_{\Theta}B\left(\theta\right)+\left[V(\theta)+\phi\left(l\left(\theta\right)\right)\right]f_{\theta}(\theta)d\theta\end{array}\right].\end{aligned}$$

Taking partial integrals yields:

$$-\int_{\Theta} q(\theta)V'(\theta)d\theta = V(\underline{\theta})q(\underline{\theta}) - V(\bar{\theta})q(\bar{\theta}) + \int_{\Theta} q'(\theta)V(\theta)d\theta,$$

and

$$-\int_{\Theta} \varphi(\theta)W'(\theta)d\theta = W(\underline{\theta})\varphi(\underline{\theta}) - W(\bar{\theta})\varphi(\bar{\theta}) + \int_{\Theta} \varphi'(\theta)W(\theta)d\theta.$$

Optimal Tax. The derivatives with respect to the endpoint conditions yield the following boundary conditions:

$$q(\underline{\theta}) = q(\bar{\theta}) = 0 \text{ and } \varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0. \tag{C2}$$

Substituting the above conditions into the Lagrangian yields the following first-order conditions for $t \geq 1$ and $\theta \in \Theta$:

$$V(\theta) : \widetilde{f}_{\theta}(\theta)G'\left(V(\theta)\right) + q'(\theta) - \lambda f_{\theta}(\theta) = 0, \tag{C3}$$

and

$$\omega(\theta) : \varphi(\theta) + q(\theta)l(\theta)\phi'\left(l\left(\theta\right)\right) = 0 \tag{C4}$$

and

$$B(\theta) : \lambda\left[\frac{\partial Q}{\partial B(\theta)} - 1\right] + \frac{\chi(\theta)\gamma^{B}_{own}\left(z(\theta)\right) + \int_{\Theta}\chi(x)\gamma^{B}_{cross}\left(z(x), z(\theta)\right)dx}{B(\theta)} = 0, \tag{C5}$$

and

$$W(\theta) : -\chi(\theta) + \varphi'(\theta) = 0, \tag{C6}$$

and

$$\begin{aligned} l(\theta) : \frac{\partial \ln \vartheta\left(z\left(\theta\right), \theta\right)}{\partial \theta} q(\theta)\phi'\left(l\left(\theta\right)\right)\left[1 + \frac{l(\theta)\phi''\left(l\left(\theta\right)\right)}{\phi'\left(l\left(\theta\right)\right)}\right] + \\ \frac{\chi(\theta)\gamma^{L}_{own}\left(z\left(\theta\right)\right) + \int_{\Theta}\chi(x)\gamma^{L}_{cross}\left(z\left(x\right), z\left(\theta\right)\right)dx}{l(\theta)} + \lambda w(\theta)\left[1 - \frac{\phi'\left(l\left(\theta\right)\right)}{w(\theta)}\right] f_{\theta}(\theta) = 0, \end{aligned} \tag{C7}$$

Notice that the cross elasticities are symmetric. The above conditions can be simplies as

$$B(\theta) : \left[\frac{\partial Q}{\partial B(\theta)} - 1\right] = -\frac{\chi(\theta)\gamma^{B}_{own}\left(z(\theta)\right)}{\lambda B(\theta)}, \tag{C8}$$

and

$$l(\theta) : \frac{\partial \ln \vartheta\left(z\left(\theta\right), \theta\right)}{\partial \theta} q(\theta)\phi'\left(l\left(\theta\right)\right)\left[1 + \frac{l(\theta)\phi''\left(l\left(\theta\right)\right)}{\phi'\left(l\left(\theta\right)\right)}\right] + \frac{\chi(\theta)\gamma^{L}_{own}\left(z\left(\theta\right)\right)}{l(\theta)} + \lambda w(\theta)\left[1 - \frac{\phi'\left(l\left(\theta\right)\right)}{w(\theta)}\right] f_{\theta}(\theta) = 0, \tag{C9}$$

According to (C2) and (C3), we have $\lambda = 1$ and $-q(\theta) = \widetilde{F}_{\theta}(\theta) - F_{\theta}(\theta)$. Therefore, $\frac{q'(\theta)}{\lambda} = \left[1 - g\left(y(\theta)\right)\right] f_{\theta}(\theta)$ and $-\frac{q(\theta)}{\lambda} = \left[1 - \overline{g}\left(y(\theta)\right)\right]\left[1 - F_{\theta}(\theta)\right]$ where $\overline{g}\left(y(\theta)\right) = \frac{1-\widetilde{F}_{\theta}(\theta)}{1-F_{\theta}(\theta)}$ and $g\left(y(\theta)\right) = \frac{\widetilde{f}_{\theta}(\theta)}{f_{\theta}(\theta)}$. Notice that $\gamma^{B}_{cross}\left(z(x), z(\theta)\right)$

is independent of x, we have $\int_{\Theta}\chi(x)\gamma^{B}_{cross}\left(z(x),z(\theta)\right)dx=\gamma^{B}_{cross}\left(z(x),z(\theta)\right)\int_{\Theta}\chi(x)dx=0$. Last, using $\tau^{B}\left(z(\theta)\right)=\frac{\partial Q}{\partial B_{z}(z(\theta))}-1=\frac{\partial Q}{\partial B(\theta)}-1$, (C5) implies:

$$\tau^{B}\left(z(\theta)\right)=-\frac{\frac{\chi(\theta)}{\lambda}\gamma^{B}_{own}\left(z(\theta)\right)}{B(\theta)}, \tag{C10}$$

where $\varphi'(\theta)=\chi(\theta)$ and by (C4):

$$\begin{aligned}\frac{\varphi'(\theta)}{\lambda}&=l(\theta)\phi'(l(\theta))\left[-\frac{q'(\theta)}{\lambda}-\frac{q(\theta)}{\lambda}\frac{d\ln l(\theta)}{d\theta}\left[1+\frac{l(\theta)\phi''(l(\theta))}{\phi'(l(\theta))}\right]\right] \\ &=f_{\theta}(\theta)y(\theta)\left[\begin{array}{c}-\left[1-g\left(y(\theta)\right)\right]\left[1-\tau^{y}(\theta)\right]+\\ \frac{1-F_{y}(y(\theta))}{f_{y}(y(\theta))y(\theta)}\left[1-\overline{g}\left(y(\theta)\right)\right]\left[1-\tau^{y}(\theta)\right]\frac{\varepsilon+1}{\varepsilon}\frac{d\ln l(\theta)}{d\ln w(\theta)}\frac{d\ln w(\theta)}{d\ln y(\theta)}\end{array}\right] \\ &=f_{\theta}(\theta)y(\theta)\left[\begin{array}{c}-\left[1-\tau^{y}(\theta)\right]\left[1-g\left(y(\theta)\right)\right]+\\ \frac{1-F_{y}(y(\theta))}{f_{y}(y(\theta))y(\theta)}\left[1-\tau^{y}(\theta)\right]\left[1-\overline{g}\left(y(\theta)\right)\right]\frac{\varepsilon^{l}_{w}(y(\theta))}{\varepsilon^{l}_{1-\tau}(y(\theta))}\end{array}\right].\end{aligned} \tag{C11}$$

The second equation of (**??**) is derived by substituting $\frac{1}{\varepsilon}$ for $\frac{l(\theta)\phi''(l(\theta))}{\phi'(l(\theta))}$ and substituting $\left[1-g\left(y(\theta)\right)\right]f_{\theta}(\theta)$ and $\left[1-\overline{g}\left(y(\theta)\right)\right]\left[1-F_{\theta}(\theta)\right]$ for $\frac{q'(\theta)}{\lambda}$ and $-\frac{q(\theta)}{\lambda}$, respectively. The relationship between $q(\theta)$ and social welfare weights can be derived as similar to that in the benchmark model. The third equation of (**??**) is derived by the definitions of labor elasticities:

$$\frac{\varepsilon+1}{\varepsilon}\frac{d\ln l(\theta)}{d\ln w(\theta)}\frac{d\ln w(\theta)}{d\ln y(\theta)}=\frac{\varepsilon+1}{\varepsilon}\frac{\varepsilon^{l}_{w}\left(y(\theta)\right)}{1+\varepsilon^{l}_{w}\left(y(\theta)\right)}=\frac{\varepsilon^{l}_{w}\left(y(\theta)\right)}{\varepsilon^{l}_{1-\tau}\left(y(\theta)\right)}.$$

Substituting (**??**) into (**??**) gives:

$$\frac{\tau^{B}\left(z(\theta)\right)}{1-\tau^{y}(\theta)}=\left[\left[1-g\left(y(\theta)\right)\right]-\frac{1-F_{y}(y(\theta))}{f_{y}(y(\theta))y(\theta)}\left[1-\overline{g}\left(y(\theta)\right)\right]\frac{\varepsilon^{l}_{w}\left(y(\theta)\right)}{\varepsilon^{l}_{1-\tau}\left(y(\theta)\right)}\right]\frac{\gamma^{B}_{own}\left(z(\theta)\right)y(\theta)}{B_{z}\left(z(\theta)\right)/f_{\theta}(\theta)}.$$

Similarly, using the symetricity of cross wage-labor elasticity, (C7) can be simplied as

$$\frac{1-\frac{\phi'(l(\theta))}{w(\theta)}}{\frac{\phi'(l(\theta))}{w(\theta)}}=-\frac{\partial\ln\vartheta\left(z\left(\theta\right),\theta\right)}{\partial\theta}\frac{q(\theta)}{\lambda f_{\theta}(\theta)}\left(1+\frac{1}{\varepsilon}\right)-\frac{\frac{\chi(\theta)}{\lambda}\gamma^{L}_{own}\left(z\left(\theta\right)\right)}{\frac{\phi'(l(\theta))}{w(\theta)}w(\theta)f_{\theta}(\theta)l(\theta)}.$$

Replace $\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\theta}$ by $\frac{w'(\theta)}{w(\theta)}$ and $\chi(\theta)$ by (C10):

$$\frac{1-\frac{\phi'(l(\theta))}{w(\theta)}}{\frac{\phi'(l(\theta))}{w(\theta)}}=-\frac{w'\left(\theta\right)}{w\left(\theta\right)}\frac{q(\theta)}{\lambda f_{\theta}(\theta)}\left(1+\frac{1}{\varepsilon}\right)+\tau^{B}\left(z(\theta)\right)\frac{B(\theta)}{y(\theta)f_{\theta}(\theta)}\frac{\gamma^{L}_{own}\left(z\left(\theta\right)\right)}{\gamma^{B}_{own}\left(z\left(\theta\right)\right)}\frac{1}{\frac{\phi'(l(\theta))}{w(\theta)}}.$$

Replace $\frac{\phi'(l(\theta))}{w(\theta)}$ by $1-\tau^y(\theta)$, $-\frac{w'(\theta)}{w(\theta)}\frac{q(\theta)}{\lambda f_\theta(\theta)}$ by $-\frac{w'(\theta)}{w(\theta)}/\frac{y'(\theta)}{y(\theta)}\frac{q(\theta)}{\lambda y(\theta) f_y(y(\theta))}$ and $-\frac{q(\theta)}{\lambda}$ by $[1-\overline{g}(y(\theta))][1-F_\theta(\theta)]$:

$$\frac{\tau^y(\theta)}{1-\tau^y(\theta)}=[1-\overline{g}(\theta)]\frac{1-F_y(y(\theta))}{y(\theta)f_y(y(\theta))}\frac{1}{\varepsilon^l_{1-\tau}(\theta)}+\frac{\tau^B(z(\theta))}{1-\tau^y(\theta)}\frac{B(\theta)}{y(\theta)f_\theta(\theta)}\frac{\gamma^L_{own}(z(\theta))}{\gamma^B_{own}(z(\theta))} \tag{C12}$$

where we uses the definitions of labor elasticities to replace $\frac{w'(\theta)}{w(\theta)}/\frac{y'(\theta)}{y(\theta)}\left(1+\frac{1}{\varepsilon}\right)$ by $\frac{1}{\varepsilon^l_{1-\tau}(\theta)}$.

The above equation together with (C12) gives:

$$\begin{aligned}\frac{\tau^y(\theta)}{1-\tau^y(\theta)}=&[1-\overline{g}(\theta)]\frac{1-F_y(y(\theta))}{y(\theta)f_y(y(\theta))}\frac{1}{\varepsilon^l_{1-\tau}(\theta)}\\&+\left[[1-g(\theta)]-\frac{1-F_y(y(\theta))}{f_y(y(\theta))y(\theta)}[1-\overline{g}(\theta)]\frac{\varepsilon^l_w(\theta)}{\varepsilon^l_{1-\tau}(\theta)}\right]\gamma^L_{own}(z(\theta)).\end{aligned}$$

Regulation. As to the optimal regulation, we have the following additional first-order conditions:

$$a(\theta):\lambda\frac{\partial Q_z(\theta)}{\partial a(\theta)}\frac{\partial Q}{\partial Q_z(\theta)}+\frac{\chi(\theta)}{a(\theta)}\frac{\partial\ln\omega(\theta,z(\theta),\delta(\theta),B(\theta),L(\theta);a(\theta))}{\partial\ln a(\theta)}=0, \tag{C13}$$

where we have simplified the first-order condition by the symmetricity of the cross wage elasticity.

Notice that $\frac{\partial\ln\omega(\theta,z(\theta),\delta(\theta),B(\theta),L(\theta);a(\theta))}{\partial\ln a(\theta)}=\gamma^a_{own}(z)$ and $\tau^a(z(\theta))=\frac{\partial Q_z(z(\theta))}{\partial a(\theta)}\frac{\partial Q}{\partial Q_z(z(\theta))}$, we have

$$\tau^a(\theta)=-\frac{\chi(\theta)}{\lambda}\frac{\gamma^a_{own}(z(\theta))}{a(\theta)}, \tag{C14}$$

where by equation (C11),

$$\frac{\chi(\theta)}{\lambda}=f_\theta(\theta)y(\theta)[1-\tau^y(\theta)]\left[[1-\overline{g}(y(\theta))]\frac{1-F_y(y(\theta))}{f_y(y(\theta))y(\theta)}\frac{\varepsilon^l_w(y(\theta))}{\varepsilon^l_{1-\tau}(y(\theta))}-[1-g(y(\theta))]\right],\theta\in\Theta. \tag{C15}$$

□

C.3 Proof of Proposition 7: Optimal Policies with Regulation

Set Largangian function as:

$$\begin{aligned}\mathcal{L}=&\int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta+\int_\Theta q(\theta)\left[\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\theta}l(\theta)\phi'(l(\theta))-V'(\theta)\right]d\theta\\&+\int_\Theta\psi(\theta)[\delta(\theta)-z'(\theta)]d\theta+\int_\Theta\varphi(\theta)\left[\frac{\partial\ln\vartheta(z(\theta),\theta)}{\partial\theta}-W'(\theta)\right]d\theta\\&+\int_\Theta\chi(\theta)\left[\ln\left[\omega(\theta,z(\theta),\delta(\theta),B(\theta),L(\theta);a(\theta))Q^{\frac{1}{\sigma}}\right]-W(\theta)\right]d\theta\\&+\lambda\left[\left[\int_\Theta\eta(z(\theta))Q_z\left(\frac{B(\theta)}{\delta(\theta)},\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)};a(\theta)\right)^{\frac{\sigma-1}{\sigma}}\delta(\theta)d\theta\right]^{\frac{\sigma}{\sigma-1}}-\int_\Theta B(\theta)+[V(\theta)+\phi(l(\theta))]f_\theta(\theta)d\theta\right].\end{aligned}$$

Taking partial integrals yields:

$$-\int_{\Theta} q(\theta)V'(\theta)d\theta = V(\underline{\theta})q(\underline{\theta}) - V(\bar{\theta})q(\bar{\theta}) + \int_{\Theta} q'(\theta)V(\theta)d\theta,$$

and

$$-\int_{\Theta} \varphi(\theta)W'(\theta)d\theta = W(\underline{\theta})\varphi(\underline{\theta}) - W(\bar{\theta})\varphi(\bar{\theta}) + \int_{\Theta} \varphi'(\theta)W(\theta)d\theta,$$

and

$$-\int_{\Theta} \psi(\theta)z'(\theta)\,d\theta = \psi(\underline{\theta}) - \psi(\bar{\theta}) + \int_{\Theta} \psi'(\theta)z(\theta)d\theta.$$

The derivatives with respect to the endpoint conditions yield the following boundary conditions:

$$q(\underline{\theta}) = q(\bar{\theta}) = 0 \text{ and } \varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0 \text{ and } \psi(\underline{\theta}) = \psi(\bar{\theta}) = 0$$

Substituting the above conditions into the Lagrangian yields the following first-order conditions for $t \geq 1$ and $\theta \in \Theta$:

As in the proof of Proposition 9, we again have (C3), (C4), (C8), (C9), and

$$\lambda = 1; q(\theta) = F(\theta) - \widetilde{F}(\theta); \int_{\Theta} \chi(\theta)d\theta = \varphi(\bar{\theta}) - \varphi(\underline{\theta}) = 0.$$

In addition, we have

$$a(\theta) : \lambda\frac{\partial Q}{\partial a(\theta)} + \chi(\theta)\frac{\partial \ln\left[\omega\left(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta)\right)\right]}{\partial a(\theta)} = 0 \tag{C16}$$

and

$$\delta(\theta) : \psi(\theta) + \chi(\theta)\frac{\partial \ln\omega\left(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta)\right)}{\partial \delta(\theta)} + \lambda\frac{\partial Q}{\partial \delta(\theta)} = 0, \tag{C17}$$

and

$$\begin{aligned} z(\theta) : 0 = {} & \psi'(\theta) + q(\theta)\frac{\partial^2 \ln\vartheta\left(z(\theta), \theta\right)}{\partial\theta\partial z(\theta)}l(\theta)\phi'\left(l(\theta)\right) + \varphi(\theta)\frac{\partial^2 \ln\vartheta\left(z(\theta), \theta\right)}{\partial\theta\partial z(\theta)} \\ & + \chi(\theta)\frac{\partial \ln\omega\left(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta)\right)}{\partial z(\theta)} + \lambda\frac{\partial Q}{\partial z(\theta)}, \end{aligned} \tag{C18}$$

where we have simplified the first-order conditions by the symmetricity of the cross wage elasticity.

It's easy to see that we still have (**??**) and (**??**). Notice that $\frac{\partial \ln\omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))}{\partial \ln a(\theta)} = \gamma^{a}_{own}(z)$ and $\tau^{a}\left(z(\theta)\right) = \frac{\partial Q_z(z(\theta))}{\partial a(\theta)}\frac{\partial Q}{\partial Q_z(z(\theta))}$, we have (31).

The ODE System for the Solution of Planning Problem. By the first-order conditions we have the following equations:

Multiply both sides of (C8) by $B'(\theta)$:

$$Q\frac{B'(\theta)}{B(\theta)}\frac{\partial \ln Q}{\partial \ln B(\theta)} = B'(\theta) - \varphi'(\theta)\frac{B'(\theta)}{B(\theta)}\frac{\partial \ln \omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))}{\partial \ln B(\theta)}$$

Multiply both sides of (C9) by $l'(\theta)$:

$$\begin{aligned} Q\frac{\partial \ln Q}{\partial \ln l(\theta)}\frac{l'(\theta)}{l(\theta)} = &-q(\theta)\frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta}\left[\phi'(l(\theta)) + l(\theta)\phi''(l(\theta))\right] l(\theta)\frac{l'(\theta)}{l(\theta)} + \phi'(l(\theta)) f_\theta(\theta) l(\theta)\frac{l'(\theta)}{l(\theta)} \\ &- \varphi'(\theta)\frac{\partial \ln \omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))}{\partial \ln l(\theta)}\frac{l'(\theta)}{l(\theta)}, \end{aligned}$$

Multiply both sides of (C16) by $a'(\theta)$:

$$Q\frac{\partial \ln Q}{\partial \ln a(\theta)}\frac{a'(\theta)}{a(\theta)} = -\varphi'(\theta)\frac{\partial \ln \omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))}{\partial \ln i(\theta)}\frac{i'(\theta)}{i(\theta)};$$

Multiply both sides of (C17) by $z''(\theta)$:

$$Q\frac{\partial \ln Q}{\partial \ln \delta(\theta)}\frac{d \ln \delta(\theta)}{d\theta} = -\psi(\theta)\frac{d\delta(\theta)}{d\theta} - \varphi'(\theta)\frac{\partial \ln \omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))}{\partial \ln \delta(\theta)}\frac{d \ln \delta(\theta)}{d\theta},$$

where $\frac{\partial \ln \omega(\theta, z(\theta), \delta(\theta), l(\theta), B(\theta), a(\theta))}{\partial \ln \delta(\theta)} = \frac{1}{\sigma}$.

Multiply both sides of (C18) by $z'(\theta)$:

$$\begin{aligned} &Q\frac{\partial \ln Q}{\partial \ln z(\theta)}\frac{d \ln z(\theta)}{d\theta} \\ &= -\psi'(\theta) z'(\theta) - q(\theta)\frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} l(\theta)\phi'(l(\theta)) z'(\theta) - \varphi(\theta)\frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} z'(\theta) \\ &- \varphi'(\theta)\frac{\partial \ln \omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))}{\partial \ln z(\theta)}\frac{d \ln z(\theta)}{d\theta} \end{aligned}$$

By the above equations, we have

$$\overbrace{Q\left\{\frac{B'(\theta)}{B(\theta)}\frac{\partial \ln Q}{\partial \ln B(\theta)}+\frac{\partial \ln Q}{\partial \ln l(\theta)}\frac{l'(\theta)}{l(\theta)}+\frac{\partial \ln Q}{\partial \ln a(\theta)}\frac{a'(\theta)}{a(\theta)}+\frac{\partial \ln Q}{\partial \ln \delta(\theta)}\frac{d\ln \delta(\theta)}{d\theta}+\frac{\partial \ln Q}{\partial \ln z(\theta)}\frac{d\ln z(\theta)}{d\theta}\right\}}^{X_2(\theta)} \tag{C19}$$

$$\begin{aligned}
&= B'(\theta)-\varphi'(\theta)\overbrace{\left[\begin{array}{c}\frac{B'(\theta)}{B(\theta)}\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}+\frac{\partial \ln \omega(\theta)}{\partial \ln l(\theta)}\frac{l'(\theta)}{l(\theta)}+\frac{\partial \ln \omega(\theta)}{\partial \ln z(\theta)}\frac{d\ln z(\theta)}{d\theta}\\ +\frac{\partial \ln \omega(\theta)}{\partial \ln a(\theta)}\frac{a'(\theta)}{a(\theta)}+\frac{\partial \ln \omega(\theta)}{\partial \ln \delta(\theta)}\frac{d\ln \delta(\theta)}{d\theta}\end{array}\right]}^{X_1(\theta)}\\
&-q(\theta)\left[\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}\left[\phi'(l(\theta))+l(\theta)\phi''(l(\theta))\right]l(\theta)\frac{l'(\theta)}{l(\theta)}+\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial \theta \partial z(\theta)}l(\theta)\phi'(l(\theta))z'(\theta)\right]\\
&+\phi'(l(\theta))f_\theta(\theta)l(\theta)\frac{l'(\theta)}{l(\theta)}-\varphi(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial \theta \partial z(\theta)}z'(\theta)-\psi(\theta)\frac{dz'(\theta)}{d\theta}-\psi'(\theta)z'(\theta)
\end{aligned}$$

For simplicity, we have set notations $X_1(\theta)$ and $X_2(\theta)$ in the above equation.
By the definition of $\omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))$, we have (see (48) for $\widetilde{\iota}$ and $\widetilde{\kappa}$)

$$\begin{aligned}
&\ln \omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))\\
&= \ln \vartheta(\theta, z(\theta)) + \left(1-\frac{1}{\sigma}\right)\ln A+\frac{1}{\sigma}\ln Q+\ln \widetilde{\iota}(z(\theta), a(\theta))-\frac{1}{\rho}\ln\left[\frac{\vartheta(\theta, z(\theta))\, l(\theta) f_\theta(\theta)}{\delta(\theta)}\right]\\
&+\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{\rho}{\rho-1}\ln\left\{\begin{array}{c}\left[\frac{\widetilde{\iota}(z(\theta),a(\theta))\vartheta(\theta,z(\theta))l(\theta)f_\theta(\theta)}{\delta(\theta)}\right]^{1-\frac{1}{\rho}}\\ +\left[\frac{\widetilde{\kappa}(z(\theta),a(\theta))B(\theta)}{\delta(\theta)}\right]^{1-\frac{1}{\rho}}\end{array}\right\}
\end{aligned}$$

By the definition of $X_1(\theta)$ and $\omega(\theta)$:

$$\frac{d\ln \omega(\theta)}{d\theta}=\frac{\partial \ln \omega(\theta)}{\partial \theta}+X_1(\theta)$$

Remember that $\frac{d\ln \omega(\theta)}{d\theta}=\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}$. We have

$$\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}=\frac{\partial \ln \omega(\theta)}{\partial \theta}+X_1(\theta)$$

Without loss of generality, consider $f_\theta(\theta) = 0$. We have

$$\begin{aligned}\frac{\partial \ln \omega(\theta)}{\partial \theta} &= \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta} - \frac{1}{\rho}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta} + \\ &\left(\frac{1}{\rho} - \frac{1}{\sigma}\right)\frac{\left[\frac{\widetilde{\iota}(z(\theta),a(\theta))\vartheta(\theta,z(\theta))l(\theta)f_\theta(\theta)}{\delta(\theta)}\right]^{1-\frac{1}{\rho}}}{\left[\frac{\widetilde{\iota}(z(\theta),a(\theta))\vartheta(\theta,z(\theta))l(\theta)f_\theta(\theta)}{\delta(\theta)}\right]^{1-\frac{1}{\rho}} + \left[\frac{\widetilde{\kappa}(z(\theta),a(\theta))B(\theta)}{\delta(\theta)}\right]^{1-\frac{1}{\rho}}}\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta} \\ &= \frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta} - \left[\frac{1}{\rho} - \left(\frac{1}{\rho} - \frac{1}{\sigma}\right)\left(\frac{L_z(z(\theta))\widetilde{\iota}(z(\theta),a(\theta))}{Q_z(z(\theta))}\right)^{1-\frac{1}{\rho}}\right]\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}\end{aligned}$$

Therefore

$$X_1(\theta) = \left[\frac{1}{\rho} - \left(\frac{1}{\rho} - \frac{1}{\sigma}\right)\left(\frac{L_z(z(\theta))\widetilde{\iota}(z(\theta),a(\theta))}{Q_z(z(\theta))}\right)^{1-\frac{1}{\rho}}\right]\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta} \tag{C20}$$

On the other hand, by (59)

$$X_2(\theta) = \frac{\sigma}{\sigma-1}\frac{d}{d\theta}\left\{\beta_z(z(\theta))z'(\theta)\right\}Q - w(\theta)f(\theta)l(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}$$

Then, we can rewrite (C19) as

$$\begin{aligned}0 = {} & X_1(\theta)\varphi'(\theta) + \frac{\sigma}{\sigma-1}\frac{d}{d\theta}\left\{\beta_z(z(\theta))z'(\theta)\right\}Q - w(\theta)f(\theta)l(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta} - B'(\theta) \\ & + \left\{\left(1+\frac{1}{\varepsilon}\right)q(\theta)\phi'(l(\theta))f(\theta)\right\}l'(\theta) + q(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}\phi'(l(\theta))l(\theta)z'(\theta) \\ & + \varphi(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}z'(\theta) + \frac{d}{d\theta}\left\{\psi(\theta)z'(\theta)\right\}\end{aligned} \tag{C21}$$

Remember that

$$\delta(\theta) : \psi(\theta) + \frac{1}{\sigma}\frac{\varphi'(\theta)}{z'(\theta)} + \frac{1}{\sigma-1}\beta_z(z(\theta))Q = 0,$$

because $\beta_z(z) \equiv \frac{\eta(z)[Q_z(z)]^{\frac{\sigma-1}{\sigma}}}{\int_Z \eta(z)[Q_z(z)]^{\frac{\sigma-1}{\sigma}}dz}$ and $\frac{\partial \ln \omega(\theta,z(\theta),\delta(\theta),l(\theta),B(\theta),a(\theta))}{\partial \ln \delta(\theta)} = \frac{1}{\sigma}$. We have

$$\psi(\theta)z'(\theta) = -\frac{1}{\sigma}\varphi'(\theta) - \frac{z'(\theta)}{\sigma-1}\beta_z(z(\theta))Q$$

Substitute $\psi(\theta)z'(\theta)$ in (C21) by the above equation

$$
\begin{aligned}
0 = {} & X_1(\theta)\varphi'(\theta) + \frac{\sigma}{\sigma-1}\frac{d}{d\theta}\left\{\beta_z(z(\theta))z'(\theta)\right\}Q - w(\theta)f(\theta)l(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} - B'(\theta) \\
& + \left\{\left(1+\frac{1}{\varepsilon}\right)q(\theta)\phi'(l(\theta))\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} - \phi'(l(\theta))f(\theta)\right\}l'(\theta) \\
& + q(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}\phi'(l(\theta))l(\theta)z'(\theta) \\
& + \varphi(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}z'(\theta) - \frac{d}{d\theta}\left\{\frac{1}{\sigma}\varphi'(\theta) + \frac{z'(\theta)}{\sigma-1}\beta_z(z(\theta))Q\right\}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 = {} & -\frac{1}{\sigma}\varphi''(\theta) + X_1(\theta)\varphi'(\theta) + \frac{d}{d\theta}\left\{\beta_z(z(\theta))z'(\theta)\right\}Q - w(\theta)f(\theta)l(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} \\
& - B'(\theta) + \varphi(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}z'(\theta) \\
& + \left\{\left(1+\frac{1}{\varepsilon}\right)q(\theta)\phi'(l(\theta))\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} - \phi'(l(\theta))f(\theta)\right\}l'(\theta) \\
& + q(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}\phi'(l(\theta))l(\theta)z'(\theta)
\end{aligned}
\tag{C22}
$$

Notice that $\beta_z(z(\theta))z'(\theta)Q = \frac{\partial Q}{\partial l(\theta)}l(\theta) + \frac{\partial Q}{\partial B(\theta)}B(\theta) = w(\theta)l(\theta)f(\theta) + \left[1-\varphi'(\theta)\frac{\partial \ln \omega(\theta)}{\partial B(\theta)}\right]B(\theta) = w(\theta)l(\theta)f(\theta) + B(\theta) - \varphi'(\theta)\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}$. Substitute $\beta_z(z(\theta))z'(\theta)Q$ in (C22) by $w(\theta)l(\theta)f(\theta) + B(\theta) - \varphi'(\theta)\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}$:

$$
\begin{aligned}
0 = {} & -\frac{1}{\sigma}\varphi''(\theta) + X_1(\theta)\varphi'(\theta) + \frac{d}{d\theta}\left\{w(\theta)l(\theta)f(\theta) + B(\theta) - \varphi'(\theta)\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right\} \\
& - w(\theta)f(\theta)l(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} - B'(\theta) + \varphi(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}z'(\theta) \\
& + \left\{\left(1+\frac{1}{\varepsilon}\right)q(\theta)\phi'(l(\theta))\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} - \phi'(l(\theta))f(\theta)\right\}l'(\theta) \\
& + q(\theta)\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}\phi'(l(\theta))l(\theta)z'(\theta)
\end{aligned}
$$

Rearranging the above equation gives

$$
\begin{aligned}
0 = & -\left[\frac{1}{\sigma}+\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right]\varphi''(\theta)+\left\{X_1(\theta)-\frac{d}{d\theta}\left[\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right]\right\}\varphi'(\theta) \\
& +\frac{d\left[w(\theta) l(\theta) f(\theta)\right]}{d\theta}-w(\theta) f(\theta) l(\theta) \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta} \\
& +\varphi(\theta) \frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} z'(\theta) \\
& +\left\{\left(1+\frac{1}{\varepsilon}\right) q(\theta) \phi'(l(\theta)) \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta}-\phi'(l(\theta)) f(\theta)\right\} l'(\theta) \\
& +q(\theta) \frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} \phi'(l(\theta)) l(\theta) z'(\theta)
\end{aligned}
\tag{C23}
$$

Remember that $f(\theta)=1$, we have $\frac{d[w(\theta)l(\theta)f(\theta)]}{d\theta}=\left[\frac{d\ln w(\theta)}{d\theta}+\frac{d\ln l(\theta)}{d\theta}\right]w(\theta)l(\theta)f(\theta)=\left[\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}+\frac{d\ln l(\theta)}{d\theta}\right]w(\theta)$ Substitute $\frac{d[w(\theta)l(\theta)f(\theta)]}{d\theta}$ in (C23):

$$
\begin{aligned}
0 = & -\left[\frac{1}{\sigma}+\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right]\varphi''(\theta)+\left\{X_1(\theta)-\frac{d}{d\theta}\left[\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right]\right\}\varphi'(\theta) \\
& +\varphi(\theta) \frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} z'(\theta) \\
& +\left\{\left(1+\frac{1}{\varepsilon}\right) q(\theta) \phi'(l(\theta)) \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta}-\phi'(l(\theta)) f(\theta)+w(\theta) f(\theta)\right\} l'(\theta) \\
& +q(\theta) \frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} \phi'(l(\theta)) l(\theta) z'(\theta)
\end{aligned}
\tag{C24}
$$

Notice that $q(\theta)=F(\theta)-\widetilde{F}(\theta)$, $w(\theta)f(\theta)=\frac{\partial Q}{\partial l(\theta)}$ and $l(\theta):-\phi'(l(\theta))f_\theta(\theta)\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta}\frac{F(\theta)-\widetilde{F}(\theta)}{\theta f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right)+\phi'(l(\theta))f_\theta(\theta)-w(\theta)f_\theta(\theta)=\frac{\varphi'(\theta)}{l(\theta)}\gamma^l_{own}(z(\theta))$. The above equation can be simplified as

$$
\begin{aligned}
& \left[\frac{1}{\sigma}+\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right]\varphi''(\theta)-\left\{X_1(\theta)-\frac{d}{d\theta}\left[\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right]-\frac{l'(\theta)}{l(\theta)}\gamma^l_{own}(z(\theta))\right\}\varphi'(\theta) \\
& =\varphi(\theta) \frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} z'(\theta)+q(\theta) \frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} \phi'(l(\theta)) l(\theta) z'(\theta)
\end{aligned}
$$

Notice that $\phi'(l(\theta))=w(\theta)[1-T'(y(\theta))]$ and $\frac{d\ln\phi'(l(\theta))}{d\theta}=\frac{1}{\varepsilon}\frac{d\ln l(\theta)}{d\theta}=\frac{d\ln w(\theta)}{d\theta}+\frac{d\ln[1-T'(y(\theta))]}{d\theta}$ and $\frac{d\ln w(\theta)}{d\theta}=\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}$. We have

$$
\begin{aligned}
& \left[\frac{1}{\sigma}+\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right]\varphi''(\theta)-\left\{\begin{array}{c} X_1(\theta)-\frac{d}{d\theta}\left[\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right] \\ -\varepsilon\left[\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta}+\frac{d\ln[1-T'(y(\theta))]}{d\theta}\right]\gamma^l_{own}(z(\theta)) \end{array}\right\}\varphi'(\theta) \\
& =\varphi(\theta) \frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} z'(\theta)+q(\theta) \frac{\partial^2 \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} \phi'(l(\theta)) l(\theta) z'(\theta),
\end{aligned}
\tag{C25}
$$

where $X_1(\theta)$ is given by (C20) and

$$\frac{1}{\sigma}+\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}=\frac{1}{\sigma}+\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\left[\frac{B_z(z(\theta))\widetilde{\kappa}(z(\theta),a(\theta))}{Q_z(z(\theta))}\right]^{1-\frac{1}{\rho}}>0$$

In conclusion, the solution of planning problem satisfies the following system:

First, with given $\{z(\theta), l(\theta), B(\theta), a(\theta)\}$ and Q, $\varphi(\theta)$ is determined by the following ODE system: (C25), where $1-T'(y(\theta))=\frac{v'(l(\theta))}{w(\theta)}$, and the boundary conditions $\varphi(\underline{\theta})=\varphi(\bar{\theta})=0$.

Second, with given $\varphi'(\theta)$, the job assignment function $z(\theta)$, labor inputs $l(\theta)$, robots inputs $B(\theta)$, and automation rates $a(\theta)$ are determined by:

$$\begin{aligned}
z(\theta): \frac{d\ln w(\theta)}{d\theta}&=\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}\\
l(\theta): \frac{v'(l(\theta))}{w(\theta)}&=\frac{1+\frac{\varphi'(\theta)}{w(\theta)l(\theta)f_\theta(\theta)}\gamma^l_{own}(z(\theta))}{1-\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \ln \theta}\frac{\left[1-\widetilde{F}(\theta)\right]-[1-F(\theta)]}{\theta f_\theta(\theta)}\left(1+\frac{1}{\varepsilon}\right)}\\
B(\theta): \frac{\partial Q}{\partial B(\theta)}&=1-\frac{\varphi'(\theta)}{B(\theta)}\gamma^B_{own}(z(\theta))\\
a(\theta): \frac{1-a(\theta)}{a(\theta)}&=\frac{w(\theta)L(\theta)+\varphi'(\theta)\left[1+\gamma^L_{own}(z(\theta))\right]}{B(\theta)}
\end{aligned}$$

with boundary conditons $z(\underline{\theta})=0, z(\bar{\theta})=1$. Third, the costant Q satisfies the following integration conditon:

$$Q=\left[\int_\Theta \eta(z(\theta))Q_z\left(\frac{B(\theta)}{\delta(\theta)},\frac{\vartheta(z(\theta),\theta)L(\theta)}{\delta(\theta)};a(\theta)\right)^{\frac{\sigma-1}{\sigma}}\delta(\theta)d\theta\right]^{\frac{\sigma}{\sigma-1}}.$$

The Property of $\varphi'(\theta)$. First, set the following notations:

$$\widetilde{a}(\theta)=\frac{X_1(\theta)-\frac{d}{d\theta}\left[\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}\right]-\varepsilon\left[\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial \theta}+\frac{d\ln[1-T'(y(\theta))]}{d\theta}\right]\gamma^l_{own}(z(\theta))}{\frac{1}{\sigma}+\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}}$$

and

$$\widetilde{b}(\theta)=\frac{\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}}{\frac{1}{\sigma}+\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}}$$

and

$$\widetilde{c}(\theta)=\frac{\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)}\phi'(l(\theta))l(\theta)z'(\theta)}{\frac{1}{\sigma}+\frac{\partial \ln \omega(\theta)}{\partial \ln B(\theta)}}$$

We have

$$\varphi''(\theta)-\widetilde{a}(\theta)\varphi'(\theta)-\widetilde{b}(\theta)\varphi(\theta)=\left[F(\theta)-\widetilde{F}(\theta)\right]C(\theta)$$

Note that when $\frac{\partial^2 \ln \vartheta(z(\theta),\theta)}{\partial\theta\partial z(\theta)} = 0$ holds for any $\theta \in \Theta$, one has $\widetilde{B}(\theta) = C(\theta) = 0$ and

$$\varphi''(\theta) - \widetilde{a}(\theta)\varphi'(\theta) = 0 \tag{C26}$$

Therefore

$$\varphi'(\theta) = \varphi'(\underline{\theta}) \exp\left(\int_{\underline{\theta}}^{\theta} \widetilde{a}(s)\, ds\right)$$

Set $A(\theta) = \exp\left(\int_{\underline{\theta}}^{\theta} \widetilde{a}(s)\, ds\right) > 0$. Note that $A(\theta) > 0$. One has $\varphi'(\theta) = A(\underline{\theta})\varphi'(\underline{\theta})$. Therefore $\varphi(\theta) = \varphi(\underline{\theta}) + \varphi'(\underline{\theta})\int_{\underline{\theta}}^{\theta} A(s)\, ds$. In particular, $\varphi(\overline{\theta}) = \varphi(\underline{\theta}) + \varphi'(\underline{\theta})\int_{\underline{\theta}}^{\overline{\theta}} A(s)\, ds$. Notice that $\varphi(\underline{\theta}) = \varphi(\overline{\theta}) = 0$ and $\int_{\underline{\theta}}^{\overline{\theta}} A(s)\, ds > 0$. One has $0 = 0 + \varphi'(\underline{\theta})\int_{\underline{\theta}}^{\overline{\theta}} A(s)\, ds$, which implies $\varphi'(\underline{\theta}) = 0$. Then it's easy to see that for any $\theta \in \Theta$, one has $\varphi'(\theta) = 0$.

C.4 Proof of Proposition 8: Optimal Automation

Equation (C16) implies

$$\frac{\varphi'(\theta)}{Q} \frac{\partial \ln \omega(\theta, z(\theta), \delta(\theta), B(\theta), L(\theta); a(\theta))}{\partial \ln a(\theta)} + \frac{\partial \ln Q}{\partial \ln a(\theta)} = 0, \tag{C27}$$

where $\frac{\partial \ln Q}{\partial \ln a(\theta)} = \frac{\partial \ln Q_z(\theta)}{\partial \ln a(\theta)} \frac{\partial \ln Q}{\partial \ln Q_z(\theta)}$.

Combination of (57) and (C27) gives:

$$\frac{\varphi'(\theta)}{Q\zeta(z(\theta)) \frac{\partial \ln Q}{\partial \ln Q_z(z(\theta))}} \gamma^a_{own}(z(\theta)) + \frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))} - \frac{a(\theta)}{1-a(\theta)} \frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))}}{1 - \frac{1}{\rho}} = 0, \tag{C28}$$

i.e.,

$$\frac{\varphi'(\theta)}{Q \frac{\partial \ln Q}{\partial \ln Q_z(z(\theta))}} \frac{\gamma^a_{own}(z(\theta))}{\zeta(z(\theta)) \frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}} + \frac{1 - \frac{a(\theta)}{1-a(\theta)} \left(\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))} / \frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}\right)}{1 - \frac{1}{\rho}} = 0 \tag{C29}$$

Notice that

$$\begin{aligned}
\frac{\partial \ln Q_z(z(\theta))}{\partial \ln B(\theta)} &= \frac{\rho}{\rho - 1} \frac{\partial \ln \left\{ a_z(z(\theta))^{\zeta(z(\theta))} \times \left[\frac{B(\theta)}{z'(\theta)}\right]^{\frac{\rho-1}{\rho}} + [1 - a_z(z(\theta))]^{\zeta(z)} \left[\frac{\vartheta(z(\theta),\theta) l(\theta) f(\theta)}{z'(\theta)}\right]^{\frac{\rho-1}{\rho}} \right\}}{\partial \ln B(\theta)} \\
&= \frac{\rho}{\rho - 1} \frac{a_z(z(\theta))^{\zeta(z(\theta))} \times \left[\frac{B(\theta)}{z'(\theta)}\right]^{\frac{-1}{\rho}}}{a_z(z(\theta))^{\zeta(z(\theta))} \times \left[\frac{B(\theta)}{z'(\theta)}\right]^{\frac{\rho-1}{\rho}} + [1 - a_z(z(\theta))]^{\zeta(z)} \left[\frac{\vartheta(z(\theta),\theta) l(\theta) f(\theta)}{z'(\theta)}\right]^{\frac{\rho-1}{\rho}}} \frac{B(\theta)}{z'(\theta)} \\
&= \frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}
\end{aligned}$$

We have

$$Q\frac{\partial \ln Q}{\partial \ln Q_z\left(z\left(\theta\right)\right)}\frac{\partial \ln Q_z\left(z\left(\theta\right)\right)}{\partial \ln B_z\left(z\left(\theta\right)\right)} = Q\frac{\partial \ln Q}{\partial \ln Q_z\left(z\left(\theta\right)\right)}\frac{\partial \ln Q_z\left(z\left(\theta\right)\right)}{\partial \ln B\left(\theta\right)}$$
$$= Q\frac{\partial \ln Q}{\partial \ln B\left(\theta\right)} = \frac{\partial Q}{\partial B\left(\theta\right)}B\left(\theta\right)$$

Replace $Q\frac{\partial \ln Q}{\partial \ln Q_z(z(\theta))}\frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}$ in (C29) by the above equation and rearrange the equation:

$$\frac{1-a\left(\theta\right)}{a\left(\theta\right)} = \frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))} / \frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}}{1+\left(1-\frac{1}{\rho}\right)\varphi'(\theta)\frac{\gamma^a_{own}(z(\theta))}{\zeta(z(\theta))\frac{\partial Q}{\partial B(\theta)}B(\theta)}} = \frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))}}{1+\left(1-\frac{1}{\rho}\right)\varphi'(\theta)\frac{\gamma^a_{own}(z(\theta))}{Q\zeta(z(\theta))\frac{\partial \ln Q}{\partial \ln Q_z(z(\theta))}\frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}}\frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}}$$
$$= \frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))}\frac{\partial \ln Q}{\partial \ln Q_z(z(\theta))}Q}{1+\left(1-\frac{1}{\rho}\right)\varphi'(\theta)\frac{\gamma^a_{own}(z(\theta))}{\zeta(z(\theta))}} = \frac{\zeta\left(z\left(\theta\right)\right)\frac{\partial \ln Q}{\partial \ln L_z(z(\theta))}Q}{1+\left(1-\frac{1}{\rho}\right)\varphi'(\theta)\gamma^a_{own}\left(z\left(\theta\right)\right)}$$

where $\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))} = \frac{1}{\left[\frac{a_z(z)}{1-a_z(z)}\right]^{\zeta(z(\theta))}\left(\frac{B_z(z)}{L_z(z)}\right)^{\frac{\rho-1}{\rho}}+1} = 1-\alpha_z\left(z\left(\theta\right)\right)$ and $\beta_z\left(z\right) = \frac{\partial \ln Q}{\partial \ln Q_z(z)} = \frac{\eta(z)Q_z(z)^{\frac{\sigma-1}{\sigma}}}{\int_Z \eta(z)Q_z(z)^{\frac{\sigma-1}{\sigma}}dz}$.

We have

$$a\left(\theta\right) = \frac{1}{1+\frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))} / \frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}}{1+\left(1-\frac{1}{\rho}\right)\varphi'(\theta)\frac{\gamma^a_{own}(z(\theta))}{\zeta(z(\theta))\frac{\partial Q}{\partial B(\theta)}B(\theta)}}} = \frac{1}{1+\frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))} / \frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}}{1+\frac{1-\frac{1}{\rho}}{\zeta(z(\theta))}\frac{\frac{\varphi'(\theta)}{B(\theta)}\gamma^a_{own}(z(\theta))}{1-\frac{\varphi'(\theta)}{B(\theta)}\gamma^B_{own}(z(\theta))}}}$$

$$= \frac{1+\frac{1-\frac{1}{\rho}}{\zeta(z(\theta))}\frac{\frac{\varphi'(\theta)}{B(\theta)}\gamma^a_{own}(z(\theta))}{1-\frac{\varphi'(\theta)}{B(\theta)}\gamma^B_{own}(z(\theta))}}{1+\frac{1-\frac{1}{\rho}}{\zeta(z(\theta))}\frac{\frac{\varphi'(\theta)}{B(\theta)}\gamma^a_{own}(z(\theta))}{1-\frac{\varphi'(\theta)}{B(\theta)}\gamma^B_{own}(z(\theta))} + \frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))}}{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}}}$$

where the second equation is derived by the first order contiion with respect to $B\left(\theta\right)$ (see equation (C8)):

$$\frac{\partial Q}{\partial B\left(\theta\right)} = 1-\frac{\varphi'(\theta)}{B\left(\theta\right)}\frac{\partial \ln \omega\left(\theta, z\left(\theta\right), \delta\left(\theta\right), B\left(\theta\right), L\left(\theta\right); a\left(\theta\right)\right)}{\partial \ln B\left(\theta\right)} = 1-\frac{\varphi'(\theta)}{B\left(\theta\right)}\gamma^B_{own}\left(z\left(\theta\right)\right).$$

Alternatively, we have

$$\frac{1-a\left(\theta\right)}{a\left(\theta\right)} = \frac{\frac{\partial \ln Q_z(z(\theta))}{\partial \ln L_z(z(\theta))} / \frac{\partial \ln Q_z(z(\theta))}{\partial \ln B_z(z(\theta))}}{\frac{1-\frac{1}{\rho}}{\zeta(z(\theta))}\frac{\frac{\varphi'(\theta)}{B(\theta)}\gamma^a_{own}(z(\theta))}{1-\frac{\varphi'(\theta)}{B(\theta)}\gamma^B_{own}(z(\theta))}+1}$$

Remember that in the decentralized economy without regulation (see equation (53)), $\frac{1-a^u_z(z)}{a^u_z(z)} = \frac{\partial \ln Q^u_z(z(\theta))}{\partial \ln L_z(z(\theta))} / \frac{\partial \ln Q^u_z(z($

Therefore, $\frac{1-\frac{1}{\rho}}{\zeta(z(\theta))}\frac{\frac{\varphi'(\theta)}{B(\theta)}\gamma^a_{own}(z(\theta))}{1-\frac{\varphi'(\theta)}{B(\theta)}\gamma^B_{own}(z(\theta))}$ can be treated as a automation wedge introduced by the planner.

The first-order contiion with respect to $B(\theta)$ implies (see equation (C8)):

$$\frac{1}{\frac{\partial \ln Q}{\partial \ln B(\theta)}} = \frac{Q}{B(\theta)} \frac{1}{1 - \frac{\varphi'(\theta)}{B(\theta)} \gamma_{own}^{B}(z(\theta))}$$

Notice that $\frac{\frac{\partial \ln Q_z(z)}{\partial \ln L_z(z)}}{\frac{\partial \ln Q_z(z)}{\partial \ln B_z(z)}} = \frac{\frac{\partial \ln Q}{\partial \ln L(\theta)}}{\frac{\partial \ln Q}{\partial \ln B(\theta)}} = \frac{w(\theta)L(\theta)/Q}{\frac{\partial \ln Q}{\partial \ln B(\theta)}}$, we have

$$\frac{\frac{\partial \ln Q_z(z)}{\partial \ln L_z(z)}}{\frac{\partial \ln Q_z(z)}{\partial \ln B_z(z)}} = \frac{w(\theta) L(\theta)}{Q} \frac{Q}{B(\theta)} \frac{1}{1 - \frac{\varphi'(\theta)}{B(\theta)} \gamma_{own}^{B}[z(\theta)]}$$

Therefore

$$\begin{aligned}
\frac{1 - a(\theta)}{a(\theta)} &= \frac{\frac{w(\theta)L(\theta)}{B(\theta)} \frac{1}{1 - \frac{\varphi'(\theta)}{B(\theta)} \gamma_{own}^{B}[z(\theta)]}}{\frac{\left(1 - \frac{1}{\rho}\right)}{\zeta(z(\theta))} \frac{\frac{\varphi'(\theta)}{B(\theta)} \gamma_{own}^{a}[z(\theta)]}{1 - \frac{\varphi'(\theta)}{B(\theta)} \gamma_{own}^{B}[z(\theta)]} + 1} \\
&= \frac{1}{1 + \frac{\varphi'(\theta)}{B(\theta)} \left[\frac{\left(1 - \frac{1}{\rho}\right)}{\zeta(z(\theta))} \gamma_{own}^{a}(z(\theta)) - \gamma_{own}^{B}(z(\theta)) \right]} \frac{w(\theta) L(\theta)}{B(\theta)}
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{a(\theta)}{1 - a(\theta)} \frac{w(\theta) L(\theta)}{B(\theta)} &= 1 + \frac{\varphi'(\theta)}{B(\theta)} \left[\frac{\frac{\rho - 1}{\rho}}{\zeta(z(\theta))} \gamma_{own}^{a}(z(\theta)) - \gamma_{own}^{B}(z(\theta)) \right] \\
&= 1 + \frac{\varphi'(\theta)}{B(\theta)} \left[\frac{\frac{\rho - 1}{\rho}}{\zeta(z(\theta))} \gamma_{own}^{a}(z(\theta)) - \gamma_{own}^{B}(z(\theta)) \right]
\end{aligned}$$

Use the relationship between wage elasticities $\frac{\rho - 1}{\rho} \frac{1}{\zeta(z(\theta))} \gamma_{own}^{a}(z) - \gamma_{own}^{B}(z) = -\left[\gamma_{own}^{L}(z) + 1\right] \frac{a_z(z)}{1 - a_z(z)}$ (see equation (58)):

$$\begin{aligned}
\frac{a(\theta)}{1 - a(\theta)} \frac{w(\theta) L(\theta)}{B(\theta)} &= 1 + \frac{\varphi'(\theta)}{B(\theta)} \left[\frac{\frac{\rho - 1}{\rho}}{\zeta(z(\theta))} \gamma_{own}^{a}(z(\theta)) - \gamma_{own}^{B}(z(\theta)) \right] \\
&= 1 - \frac{\varphi'(\theta)}{B(\theta)} \left[1 + \gamma_{own}^{L}(z(\theta))\right] \frac{a(\theta)}{1 - a(\theta)}
\end{aligned}$$

i.e.,

$$\frac{1 - a(\theta)}{a(\theta)} = \frac{w(\theta) L(\theta) + \varphi'(\theta) \left[1 + \gamma_{own}^{L}(z(\theta))\right]}{B(\theta)}$$

C.5 Proof of Proposition ??: Optimal Policies without Regulation

When regulation is not enforceable, the planner's problem is choosing $\{B(\theta), l(\theta), V(\theta), z(\theta), \delta(\theta)\}$ to maximize the social welfare $\int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta$, subject to the market clear conditions (52) and (50), incentive compatible constraints (Intensive IC) and (Extensive IC), where wage rates satisfy $w(\theta) = \omega^u(\theta, z(\theta), \delta(\theta), L(\theta), B(\theta)) Q^{*\frac{1}{\sigma}}$. When regulation is enforceable, one additional control variable is enabled. The planner now chooses $\{B(\theta), l(\theta), V(\theta), z(\theta), \delta(\theta)\}$ to maximize the social welfare $\int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta$ subject to the above constraints.

Set Largangian function as:

$$\begin{aligned}
\mathcal{L} = & \int_\Theta V(\theta)\widetilde{f}_\theta(\theta)d\theta + \int_\Theta q(\theta)\left[\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} l(\theta)\phi'(l(\theta)) - V'(\theta)\right]d\theta \\
& + \int_\Theta \psi(\theta)\left[\delta(\theta) - z'(\theta)\right]d\theta + \int_\Theta \varphi(\theta)\left[\frac{\partial \ln \vartheta(z(\theta),\theta)}{\partial\theta} - W'(\theta)\right]d\theta \\
& + \int_\Theta \chi(\theta)\left[\ln\left[\omega^u(\theta, z(\theta), \delta(\theta), L(\theta), B(\theta)) Q^{\frac{1}{\sigma}}\right] - W(\theta)\right]d\theta \\
& + \lambda\left[\left[\int_\Theta \eta(z(\theta)) Q_z^u \left(\frac{B(\theta)}{\delta(\theta)}, \frac{\vartheta(z(\theta),\theta) L(\theta)}{\delta(\theta)}\right)^{\frac{\sigma-1}{\sigma}} \delta(\theta)\, d\theta\right]^{\frac{\sigma}{\sigma-1}} - \int_\Theta B(\theta) + \left[V(\theta) + \phi(l(\theta))\right] f_\theta(\theta)d\theta\right].
\end{aligned}$$

Taking partial integrals yields:

$$-\int_\Theta q(\theta)V'(\theta)d\theta = V(\underline{\theta})q(\underline{\theta}) - V(\bar{\theta})q(\bar{\theta}) + \int_\Theta q'(\theta)V(\theta)d\theta,$$

and

$$-\int_\Theta \varphi(\theta)W'(\theta)d\theta = W(\underline{\theta})\varphi(\underline{\theta}) - W(\bar{\theta})\varphi(\bar{\theta}) + \int_\Theta \varphi'(\theta)W(\theta)d\theta,$$

and

$$-\int_\Theta \psi(\theta)z(\theta)\,d\theta = z(\underline{\theta})\psi(\underline{\theta}) - \psi(\bar{\theta}) + \int_\Theta \psi'(\theta)z(\theta)d\theta.$$

The derivatives with respect to the endpoint conditions yield the following boundary conditions:

$$q(\underline{\theta}) = q(\bar{\theta}) = 0 \text{ and } \varphi(\underline{\theta}) = \varphi(\bar{\theta}) = 0. \tag{C30}$$

Substituting the above conditions into the Lagrangian yields the following first-order conditions for $t \geq 1$ and $\theta \in \Theta$:

$$V(\theta) : \widetilde{f}_\theta(\theta) + q'(\theta) - \lambda f_\theta(\theta) = 0, \tag{C31}$$

and

$$B(\theta) : \lambda\left[\frac{\partial Q}{\partial B(\theta)} - 1\right] + \frac{\chi(\theta)\gamma_{own}^{B,u}(z(\theta)) + \int_\Theta \chi(x)\gamma_{own}^{B,u}(z(x), z(\theta))\,dx}{B(\theta)} = 0, \tag{C32}$$

and

$$W(\theta): -\chi(\theta) + \varphi'(\theta) = 0, \tag{C33}$$

and

$$\begin{aligned} l(\theta): & \frac{\partial \ln \vartheta\left(z\left(\theta\right), \theta\right)}{\partial \theta} q(\theta) \phi'\left(l\left(\theta\right)\right) \left[1 + \frac{l(\theta)\phi''\left(l\left(\theta\right)\right)}{\phi'\left(l\left(\theta\right)\right)}\right] \\ & + \frac{\chi(\theta)\gamma_{own}^{L,u}\left(z\left(\theta\right)\right) + \int_{\Theta} \chi(x)\gamma_{cross}^{L,u}\left(z\left(x\right), z\left(\theta\right)\right) dx}{l(\theta)} + \lambda w(\theta) \left[1 - \frac{\phi'\left(l\left(\theta\right)\right)}{w(\theta)}\right] f_{\theta}(\theta) = 0, \end{aligned} \tag{C34}$$

and

$$z\left(\theta\right): \psi'(\theta) + \left[\begin{array}{c} q(\theta) \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} l(\theta) \phi'\left(l\left(\theta\right)\right) + \\ \varphi(\theta) \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} + \frac{\int_{\Theta} \chi(x) \frac{\partial \ln \omega^{u}(x, z(x), \delta(x), L(x), B(x))}{\partial \ln z(\theta)} dx}{z(\theta)} \end{array} \right] + \lambda w\left(\theta\right) L\left(\theta\right) \left[\begin{array}{c} \frac{\sigma}{\sigma - 1} \frac{1}{z(\theta)} \\ + \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial z(\theta)} \end{array} \right] = 0, \tag{C35}$$

and

$$\delta\left(\theta\right): \psi(\theta) + \frac{\int_{\Theta} \chi(x) \frac{\partial \ln \omega^{u}(x, z(x), \delta(x), L(x), B(x))}{\partial \ln \delta(\theta)} dx}{\delta\left(\theta\right)} + \frac{1}{\sigma - 1} \lambda \frac{w\left(\theta\right) L\left(\theta\right)}{\delta\left(\theta\right)} = 0, \tag{C36}$$

and

$$z\left(\underline{\theta}\right): \psi(\underline{\theta}) = 0. \tag{C37}$$

Notice that $\int_{\Theta} \chi(x) dx = 0$ and cross elasticities are independent of θ. We have:

$$B_{z}\left(z\right): \frac{\partial Q}{\partial B\left(\theta\right)} - 1 = -\frac{\chi(\theta)\gamma_{own}^{B,u}\left(z\left(\theta\right)\right)}{\lambda B\left(\theta\right)}, \tag{C38}$$

and

$$l(\theta): \frac{\partial \ln \vartheta\left(z\left(\theta\right), \theta\right)}{\partial \theta} q(\theta) \phi'\left(l\left(\theta\right)\right) \left[1 + \frac{l(\theta)\phi''\left(l\left(\theta\right)\right)}{\phi'\left(l\left(\theta\right)\right)}\right] + \frac{\chi(\theta)\gamma_{own}^{L,u}\left(z\left(\theta\right)\right)}{l(\theta)} + \lambda w(\theta) \left[1 - \frac{\phi'\left(l\left(\theta\right)\right)}{w(\theta)}\right] f_{\theta}(\theta) = 0, \tag{C39}$$

and

$$z\left(\theta\right): \psi'(\theta) + \left[\begin{array}{c} q(\theta) \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} l(\theta) \phi'\left(l\left(\theta\right)\right) + \varphi(\theta) \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial \theta \partial z(\theta)} \\ + \frac{\chi(\theta)\gamma_{own}^{z,u}(z(\theta))}{z(\theta)} \end{array} \right] + \lambda w\left(\theta\right) L\left(\theta\right) \left[\begin{array}{c} \frac{\sigma}{\sigma - 1} \frac{1}{z(\theta)} \\ + \frac{\partial \ln \vartheta(z(\theta), \theta)}{\partial z(\theta)} \end{array} \right] = 0, \tag{C40}$$

and

$$\delta\left(\theta\right): \psi(\theta) + \frac{\chi(\theta)\gamma_{own}^{\delta,u}\left(z\left(\theta\right)\right)}{\delta\left(\theta\right)} + \frac{1}{\sigma - 1} \lambda \frac{w\left(\theta\right) L\left(\theta\right)}{\delta\left(\theta\right)} = 0, \tag{C41}$$

It's easy to see that we still have (**??**) and (**??**) except for that elasticities without regulation are replaced by elasticities with regulation.