Testing for Weak Factors in Asset Pricing

Soohun Kim* Valentina Raponi† Paolo Zaffaroni‡

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Abstract

We provide a framework for testing the strength of observed weak factors in a conditional asset pricing setting. Our methodology allows for factors' strength to vary across time, an empirically sound feature. We resolve the complete absence of statistical power against the case of weakness of the risk factors that characterize existing methodologies, namely when their risk exposures are at most square summable and standard PCA is unable to estimate the corresponding risk factors. Monte Carlo evidence supports our theoretical findings with an empirically relevant sample size. We demonstrate the economic significance of detecting weak factors. Applying our method to recently proposed climate-related factors, we find that the importance of climate-related factors varies substantially over time.

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*College of Business, KAIST

†IESE Business School

‡Imperial College Business School

1 Introduction

A major challenge in empirical asset pricing is to explain the cross-section of asset returns. Factor asset-pricing models have been the dominant paradigm, narrowing the challenge to finding the complete set of priced risk factors. However, the task of identifying such complete set of "relevant" risk factors has proved to be extremely challenging. This difficulty is highlighted by the continuously increasing number of proposed candidate factors, often referred to as the "factor zoo" (Cochrane (2011)). Harvey, Liu, and Zhu (2016) even document the existence of more than four hundred risk factors proposed in the empirical asset pricing literature.

Given the considerable size of the factor zoo, it is highly likely that the candidate factor asset-pricing model could include *weak* factors, i.e. factors to which the assets being tested have minimal or no exposure. This is extremely problematic, because weak factors not only undermine the accuracy of their estimated risk premia, but also cast doubts on the validity of the entire model's pricing ability, even if all the other factors are *strong* (i.e., they exhibit considerable risk exposure across most test assets). For example, as a consequence of the inclusion of a weak factor, we may encounter cases in which a strong factor, while contributing to the explanation of the cross-section of asset returns, might be excluded from the model due to an incorrectly perceived lack of significance in its estimated risk premium. Conversely, an extremely weak factor, could paradoxically display a significant risk premium estimate, despite not being priced.

These challenges can become even more problematic when considering the potential variability of the factor's strength across different time periods. For instance, it is plausible that specific risk factors exhibit time-varying degrees of strength, being strong during financial crises while weak during periods of market stability.

Priced risk factors exhibit weaknesses for a variety of reasons, which renders their inclusion in candidate asset-pricing models particularly compelling. Weak factors may emerge when constructing factors based on anomalies or when considering macroeconomic factors, as they are typically exposed to a smaller subset of the test assets under examination. Additionally, market incompleteness can contribute to the presence of weak factors, as not all sources of risk are adequately spanned across all assets. Similarly, weak factors can come into play when constructing a portfolio that is neutral to systematic risk but instead only exposed to unsystematic risk (see Raponi, Uppal,

and Zaffaroni (2023) and DelloPreite et al. (2023)).[1] Theoretically, even the most conventional asset pricing models like the CAPM or the Consumption-CAPM do not completely rule out the possibility of weak factors.

This paper proposes a statistical methodology for assessing and making inference on the *strength* of a risk factor, or a group of them. Our innovative testing procedure offers several notable features, representing a substantial improvement over existing methodologies. First, our methodology possesses statistical power to discern when factors are considered "weak", in the sense that its contribution on the cross-section of stock returns (quantified by the magnitude of the eigenvalues in the covariance matrix of test asset's returns) is akin to the contribution of the idiosyncratic error term (which is typically associated with the boundedness of the eigenvalues in the corresponding covariance matrix). The issue of weak factors has been already investigated in the asset pricing literature (see Lettau and Pelger (2020a,b), among others), yet no current methodologies are able to effectively test for it. Our asymptotic theory, not only provides a rigorous testing procedure to identify weak factors, but also remains valid even in cases of extreme weakness, that is when the factor and the test assets are perfectly uncorrelated, thereby implying zero risk exposure. This situation is commonly referred to in the literature as the presence of *spurious* or *useless* factors (see Jagannathan and Wang (1998) and Kan and Zhang (1999)).

Second, our methodology allows for the factor's strength to vary across time, effectively identifying factors that may appear *locally* weak during certain time periods but show strength in others. In other words, we are able to test for the level of weakness (or strength) of a given risk factor even within the framework of *conditional* asset pricing models, in which the time-varying parameters remain model-free. This flexibility arises from the fact that our asymptotic analysis is developed for situations where the number of time periods T remains fixed while the number of test assets N tends to infinity.

Third, we acknowledge the possibility that the specified factor asset-pricing model may be incorrectly defined, wherein some systematic risk factors could be mistakenly excluded from the model under consideration. To ensure that our testing procedure and its statistical properties are not adversely affected by the issue of omitted significant risk factors, we follow Giglio and Xiu (2021) and propose pre-estimating the strong factors'

[1] In their paper, DelloPreite et al. (2023) formally show that when unsystematic risk is priced, the stochastic discount factor contains an additional element (the so-called "unsystematic risk component"), which exhibits characteristics akin to those of a weak risk factor.

space through Principal Component Analysis (PCA). Through a suitable adjustment, we show that our test statistics remain asymptotically valid, ensuring their accuracy in terms of both size and power.

A fourth important aspect of our paper, which sets our methodology apart from existing approaches, is its ability to address singularity in risk exposures, which occurs when one subset of risk exposures is (asymptotically) collinear with the complementary subset of risk exposures. In this context, we are able to statistically distinguish between *weakness* and *singularity* of a given risk factor.

Finally, our methodology is applicable to any type of weakness in the risk exposure of a given factor, whether it manifests as *sparsity* (where risk exposures are sizeable for some assets and negligible for others) or *diffuseness* (where risk exposures exhibit a similar, modest magnitude across assets).

Our asymptotic analysis leverages several key insights related to the Fama-MacBeth two-pass OLS estimator under the large-N setting. In particular, when N diverges and T is kept fixed, we show that the two-pass OLS-estimated risk premium associated with the observed risk factor being tested converges to zero for all degrees of factor's weakness, except when the risk factor is *strong*. This holds true irrespective of whether the candidate risk factor is priced or not. The intuition is simple: when a factor is weak, its risk exposures tend to be small to such an extent that their estimation errors are (at the very least) of comparable order of magnitude as the beta themselves. Therefore, the estimation noise affecting the corresponding estimated risk premium becomes dominant, ultimately causing the estimator to infer that the true risk premium is null. Instead, whenever the factor is strong, the well-known (downward-biased) results characterizing the two-pass OLS estimator are restored (see Shanken (1992)). Paradoxically, it is precisely within these estimation errors that crucial information regarding the weak nature of the factor is inherently embedded. Indeed, we show that this result turns out to be extremely advantageous, because it allows us to develop a novel test on factor's strength able to differentiate between factors that are either weak or moderately semi-strong (i.e., when the sum of their squared betas is constant or only slowly diverges with N) and factor that are strong or semi-strong (i.e., when the sum of their squared betas diverges faster with N), without requiring any estimation of the true risk premium.

Unfortunately, the risk premium estimator lacks the statistical power to effectively differentiate between weak and the mildest forms of semi-strong factors. This is pri-

marily because the estimator relies on the behavior of the factor's sum of squared estimated betas, which diverges too slowly in both cases, rendering them practically indistinguishable from each other.[2]

However, one of our significant contributions lies in the examination of the statistical properties of the two-pass OLS estimator for the model "intercept", i.e, the so-called *zero-beta rate*. In contrast to the two-pass OLS estimator for the factor risk premium, the estimator of the zero-beta rate relies on the first-moment of the estimated factor's betas. This distinction becomes crucial in our analysis, because we show that the sample mean of the estimated zero-beta rate exhibits distinct limiting behavior in the case of weak factors compared to that of mild semi-strong factors. Therefore, we have now statistical power to discriminate between the two cases. This result marks a significant departure from existing methods, which solely focus on estimated risk premia and, hence, on the analysis of the sum of squared risk exposures only.

Further key practical advantages of our methodology include reliance on conventional limiting distributions (such as normal and chi-square distributions) and on closed-form standard errors employed in constructing our test statistics, eliminating the need for nonlinear numerical optimization or resampling methods like bootstrap.

Finally, all our results are valid *conditionally* on a given sample of size T, which may potentially be very small. This means that, when considering rolling estimates across time, these results can be practically interpreted as *locally* time-varying. This carries significant empirical implications, because now any factor could be weak or strong depending on the specific period we consider.

To illustrate our theoretical results, we initially explore a simplified (albeit unrealistic) scenario where a single risk factor explains the cross-section of (gross) returns. For the sake of clarity, we exclusively focus on *tradable* risk factors, acknowledging that valid mimicking portfolios can always be constructed in the case of non-tradable risk factors. The findings are then generalized to the more realistic case in which the asset pricing model contains a known (priced) set of strong *observed* factors. In this case, however, correct inference on weak factors hinges on the complete knowledge of the true set of strong factors, making our methodology vulnerable to model misspecification whenever any strong factors, or a combination of them, are mistakenly omitted from the asset

[2]This lack of power when dealing with weak factors is common to all existing approaches that tests for factors' strength, because all of them rely on second-moment properties of the estimated risk exposures (see Connor and Korajczyk (2022) as an example).

pricing model. Therefore, we further strengthen our methodology by acknowledging the potential influence of *unknown* strong risk factors in pricing the cross-section of expected returns. In doing so, we rely on the *local* PCA of Zaffaroni (2022), which permits capturing the full spectrum of priced strong factors even when T remains fixed.

The paper is organized as follows. Section 2 frames the conditional asset pricing setting, together with the main regularity assumptions. The testing methodology is described in Section 3, starting from the exemplified case of no-observed systematic factors, then the case of observed systematic factors, and lastly the case when the systematic factors are unknown. Extensions to the case of collinear risk exposures, and global misspecification are analyzed in Sections 5 and 6, respectively. Section 7 contains Monte Carlo experiments and Section 8 illustrates the empirical applications. Section 9 Concludes. Proofs are relegated to the Online Appendix.

1.1 Relevant Literature

Weak factors have originally attracted interest in the literature when manifesting in their most extreme form, characterized by a perfect lack of correlation with the test assets, a situation commonly referred to as *spurious* or *useless* factors. Jagannathan and Wang (1998) and Kan and Zhang (1999) show that the uselessness of one or more risk factors not only distorts inferences on their corresponding risk premia when using the two-pass Fama-MacBeth methodology, but also affects inference for the risk premia of strong factors in the model, thereby compromising the assessment of the overall pricing performance of the model.

Recognizing the sensitivity of the two-pass estimates to the collinearity of betas, Kleibergen (2009) extends these considerations to the case of *local-to-zero* risk exposures, that is when the risk exposures of weak factors are close to or equal to zero as T increases. The intuition is straightforward: as T becomes large, nearly-zero risk exposures imply singularity of the covariance matrix in the cross-sectional step of the Fama-MacBeth regression, rendering the risk premium associated with the useless risk factor unidentified. Similar statistical issues arise when the beta-matrix is nearly of *reduced rank*, indicating that one (or more) risk exposure is almost constant or collinear with the remaining set of risk exposures. A notable example is the risk exposure to the market factor, which typically approaches unity for most test assets, as documented in

Ahn, Perez, and Gadarowski (2013). In these cases, useless factors could even result in seemingly high values of the cross-sectional R-squared fit (see Kleibergen and Zhan (2015)), potentially misrepresenting the true strength of the cross-sectional relationship.

Guided by these considerations, researchers have explored inferential procedures to identify and consequently eliminate useless risk factors (see Kleibergen (2009), Gospodinov, Kan, and Robotti (2014), Bryzgalova (2016), Burnside (2016), Gospodinov, Kan, and Robotti (2017), Ahn, Horenstein, and Wang (2018), Kleibergen and Zhan (2020), Bryzgalova, Huang, and Julliard (2023) among others). Common to all existing methodologies is their reliance on non-conventional limiting distributions, boundeness of the results, the inclusion of nuisance parameters, and a potential lack of statistical power, often in various combinations. Moreover, these methodologies are all designed under either the large-T or large-N-T sampling scheme. In contrast, our methodology offers an inferential procedure that is asymptotically valid when only N is large and it exhibits conventional distributions with closed-form standard errors.

Given the crucial role of factor strength in evaluating the reliability of a given factor model, Chudik, Pesaran, and Tosetti (2011) categorize common factors as *weak*, *semi-strong*, or *strong* based on the limiting behavior of the maximum eigenvalue of the asset return covariance matrix as the number of assets N increases - specifically, whether it remains constant, grows slower than N, or equals the N-rate. Numerous advancements have emerged within this framework, with the predominant focus in the literature being on semi-strong factors as an alternative to strong factors.[3] In particular, two closely related challenges have been analyzed - namely, how to consistently estimate the risk premia in the presence of weak factors, and how to statistically assess (i.e., test for) the strength of such risk factors.

In terms of estimation, Pesaran and Smith (2023), highlight the crucial role of factor strengths in estimating risk premia with the conventional two-pass Fama-MacBeth estimator. They show that semi-strong factors can lead to sub-optimal convergence rates (hence, their use can only be justified in large sample sizes), while the inclusion of weak factors in the analysis can adversely impact the estimates of either the strong or semi-strong factors included in the model. The same problem is documented in Anatolyev and Mikusheva (2022), who propose a risk premia estimator based on sample-splitting

[3]More precisely, Chudik, Pesaran, and Tosetti (2011) distinguish between the case of weak and semi-weak common factors. For the sake of simplicity, we will categorize semi-weak as a specialized cases of semi-strong factors, where the maximum eigenvalues diverge at some rate with N.

instrumental variables regressions that is robust to weak included factors. Giglio, Xiu, and Zhang (2023) explore the challenge of weak factors in relation to the selection of test assets, arguing that the strength or weakness of a factor should not be inherently tied to the factor itself but rather to the set of test assets used in estimation. This motivates the introduction of their supervised principal component analysis (SPCA) that addresses both the issue of choosing the right set of test assets and simultaneously mitigates bias in estimating the risk premium of the factor of interest, irrespective of its strength and regardless of the strength of all the other factors (either observed or latent) considered in the asset pricing model. The importance of factor strength has also been extensively documented in the literature of approximate factor models. Starting with the work of Onatski (2012), who addressed the issue of inconsistent principal component estimates when loadings are extremely weak, Bai and Ng (2023) established conditions for consistent and asymptotically normal PC estimates in the case of semi-strong factors.

Concerning the assessment of factor strength, existing literature has primarily concentrated on providing statistical tests that distinguish between semi-strong and strong factors. Reliable testing procedures include those proposed by Bailey, Kapetanios, and Pesaran (2016, 2021) and Uematsu and Yamagata (2023b,a), which are valid under the condition of sparsity in the risk exposures. Connor and Korajczyk (2022) explore also the case in which the structure of risk exposures is diffuse.

However, delving into the analysis of weak factors - specifically discerning whether a factor is weak as opposed to semi-strong - has been proven to be a significantly more challenging task in asset pricing. This is due to the fact that, to some extent, the data is indistinguishable from the idiosyncratic error component of returns. Indeed, when a factor is weak, the estimation error in the risk exposures has the same order of magnitude as the true risk exposure, with the former entirely dominating the latter in the extreme case of spurious factors. Despite theoretical challenges, detecting weak factors is economically and empirically compelling as it can lead to asset pricing models with higher Sharpe ratios and lower pricing errors (see, for example, Lettau and Pelger (2020b) and Kozak, Nagel, and Santosh (2019)). Lettau and Pelger (2020a,b) propose a new estimator of latent factors based on a regularized version of PCA that imposes a penalty on pricing errors. They establish the asymptotic distribution under both weak and strong factor model assumptions, and show that thei approach outperforms

conventional PCA, particularly in detecting weak factors with high Sharpe ratios that PCA may miss.

Our approach can be seen as a valuable complement to existing methods for estimating risk factors and their premia under semi-strong assumptions, pioneering in its ability to statistically test for factor weakness against any form of semi-strongness. By distinguishing between weak and semi-strong factors, our methodology helps identify scenarios where prevailing methods are effective and those where they are not. It also shows asymptotic validity when the sample size is large and the time period T is fixed, providing conventional limiting distribution and closed-form standard errors.

Finally, our methodology, contributes to the extensive literature based on the two-pass methodology for estimating risk premia (Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973)), complementing the large-N analysis of Kim and Skoulakis (2018) and Raponi, Robotti, and Zaffaroni (2020) in the context of weak factors.[4]

2 Conditional Asset Pricing Set-Up

We assume that the N asset returns are governed by the following conditional asset pricing factor model across the T time periods:[5]

$$R_{it} \;=\; \alpha_{it-1} + \boldsymbol{\beta}'_{\mathrm{f},it-1}\mathbf{f}_t + \boldsymbol{\beta}'_{\mathrm{g},it-1}\mathbf{g}_t + \epsilon_{it}, \quad \text{for } i = 1, \cdots, N, \;\; t = 1, \ldots, T \qquad (1)$$

where R_{it} represents the excess return (gross return minus risk-free rate) of stock i at time t, α_{it-1} is a potentially time-varying and asset-specific intercept, $\boldsymbol{\beta}_{\mathrm{f},it-1} = (\beta_{\mathrm{f}_1,it-1}, \ldots, \beta_{\mathrm{f}_K,it-1,})'$ is the vector of time-varying risk exposures on K *strong* risk factors $\mathbf{f}_t = (\mathrm{f}_{1t}, \ldots, \mathrm{f}_{Kt})'$, $\boldsymbol{\beta}_{\mathrm{g},it-1} = (\beta_{\mathrm{g}_1,it-1}, \ldots, \beta_{\mathrm{g}_L,it-1,})'$ is the vector of time-varying risk exposures on L risk factors $\mathbf{g}_t = (\mathrm{g}_{1t}, \ldots, \mathrm{g}_{Lt})'$, whose strength we wish to test for,

[4]For the large-T properties of the two-pass estimator, see Shanken (1992), Jagannathan and Wang (1998), and Shanken and Zhou (2007). The double asymptotic theory of the two-pass estimator is investigated in Bai and Zhou (2015) and Gagliardini, Ossola, and Scaillet (2016). Turning to large-N/fixed-T asymptotic results, building on Litzenberger and Ramaswamy (1979), Shanken (1992) proposes a large-N estimator of the so-called *ex-post* risk premium. Its consistency and asymptotic normality have been established Kim and Skoulakis (2018) and Raponi, Robotti, and Zaffaroni (2020).

[5]For simplicity, we assume that the data is observed over the T periods, with the time index t ranging from 1 to T but, in practice, our methodology applies to any arbitrary set of T periods ranging from, say, $s - T + 1$ until s for any given time period s. For example, we admit a situation where the full data set is made by T_0 time observations (with $T_0 > T$) but one applies our methodology to rolling windows of data of length T setting $s = T, T + 1, \cdots, T_0$.

and ϵ_{it} is the asset-specific error component. In particular, we envisage the possibility that $\mathbf{g}_t$ is either *weak*, *semi-strong*, or *strong*, as we now define.

To specify factors' strength, we slightly generalize on Lettau and Pelger (2020b) and qualify a factor $\mathbf{g}_t$ as *locally-weak* over the period of length T whenever

$$\sum_{i=1}^{N} \parallel \boldsymbol{\beta}_{\mathrm{g},it-1} \parallel^2 \asymp O(1) < \infty \text{ for every } 1 \leq t \leq T, \tag{2}$$

where, for any matrix $\mathbf{A}$, we use the notation $\parallel \mathbf{A} \parallel = (tr(\mathbf{A}'\mathbf{A}))^{\frac{1}{2}}$ to denote the Froebenius norm, $tr(\cdot)$ represents the trace operator, where the symbol $\asymp$ denotes asymptotic equivalence.[6] Definition (2) extends the usual definition of weakness, making it *local* with respect to a given time period of length T.[7]

Alternatively, we consider $\mathbf{g}_t$ to be a *locally semi-strong* factor over the period of length T whenever

$$\sum_{i=1}^{N} \parallel \boldsymbol{\beta}_{\mathrm{g},it-1} \parallel^2 \asymp O(N^{\rho}) \text{ for some } 0 < \rho < 1, \text{ for every } 1 \leq t \leq T. \tag{3}$$

Definition (3) generalizes the (unconditional) definition of semi-strong factors of Chudik, Pesaran, and Tosetti (2011) to a *local* setting.[8]

Finally, we qualify $\mathbf{f}_t$ as a *locally strong* factor whenever $\rho = 1$ in (3). Therefore, the risk exposures to the factors $\mathbf{f}_t$ are always assumed to satisfy $\sum_{i=1}^{N} \parallel \boldsymbol{\beta}_{\mathrm{f},it-1} \parallel^2 \asymp O(N)$ for every $1 \leq t \leq T$.

Throughout the paper, we will assume that the risk factors $\mathbf{g}_t$ are always *observed*, whereas the risk factors $\mathbf{f}_t$ can be either observed or latent. For the sake of simplicity, we will assume that all risk factors $\mathbf{f}_t$ and $\mathbf{g}_t$ are tradable, i.e., portfolio returns, implying that should a non-tradable risk factor be allowed for, one considers its mimicking-portfolio equivalent (see Huberman, Kandel, and Stambaugh (1987)), which is asymptotically equivalent (when N becomes large) to the the PCA-estimated risk factors for

[6] When two (positive) sequences a_N and b_N satisfy $a_N/b_N \to c$, for some constant $c > 0$, then a_N is asymptotically equivalent to b_N, and we write $a_N \asymp O(b_N)$. When the sequences are random, with $a_N/b_N \to_p c$, we write $a_N \asymp O_p(b_N)$.

[7] When (2) applies for every t, then the (unconditional) definition of weakness is re-obtained (see Onatski (2012) or Lettau and Pelger (2020b)).

[8] Definition (3) can be further generalized by replacing $O(N^{\rho})$ with $O(N^{\rho}G(N))$, for some slowly-varying function $G(\cdot)$, a simple example being the $G(\cdot) = log(\cdot)$.

the case of latent $\mathbf{f}_t$ (see Theorem XXX of Zaffaroni (2022)).[9] In the simplest case, made for illustrative purposes, we will assume there are no observed strong risk factors $\mathbf{f}_t$ correlated with the R_{it}, and thus omitted from (1).

Using matrix notation, the conditional asset pricing model in (1) can be re-written as

$$\mathbf{R}_t = \boldsymbol{\alpha}_{t-1} + \mathbf{B}_{\mathrm{f},t-1}\mathbf{f}_t + \mathbf{B}_{\mathrm{g},t-1}\mathbf{g}_t + \boldsymbol{\epsilon}_t, \tag{4}$$

where $\mathbf{R}_t$ denotes the $N\times 1$ vector of asset excess returns at time t, $\boldsymbol{\alpha}_{t-1} \equiv [\alpha_{1t-1}, \ldots, \alpha_{Nt-1}]'$, $\mathbf{B}_{\mathrm{f},t-1} \equiv (\boldsymbol{\beta}_{\mathrm{f},1t-1}, \ldots, \boldsymbol{\beta}_{\mathrm{f},Nt-1})'$, $\mathbf{B}_{\mathrm{g},t-1} \equiv (\boldsymbol{\beta}_{\mathrm{g},1t-1}, \ldots, \boldsymbol{\beta}_{\mathrm{g},Nt-1})'$, and $\boldsymbol{\epsilon}_t \equiv (\epsilon_{1t}, \ldots, \epsilon_{Nt})'$.

When conditional no-arbitrage and full diversification of the mean-variance frontier hold (see Chamberlain and Rothschild (1983, Corollary 1) and Hansen and Richard (1987) for an extension to a conditional asset pricing setup), exact pricing follows. That is:

$$\mathbb{E}\left[R_{it}|I_{t-1}\right] = \boldsymbol{\gamma}'_{\mathrm{f,t-1}}\boldsymbol{\beta}_{\mathrm{f},it-1} + \boldsymbol{\gamma}'_{\mathrm{g,t-1}}\boldsymbol{\beta}_{\mathrm{g},it-1} \tag{5}$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator, I_t represents the information set available up to time t, $\boldsymbol{\gamma}_{\mathrm{f,T-1}}$ and $\boldsymbol{\gamma}_{\mathrm{g,T-1}}$ denote the risk premia associated with the risk factors $\mathbf{f}_t$ and $\mathbf{g}_t$, respectively.[10]Our analysis extends to the case when exact pricing does not necessarily hold, that is when he APT of Ross (1976) holds, when one replaces (5) with $\mathbb{E}\left[R_{i,t}|I_{t-1}\right] = a_{i,t-1} + \boldsymbol{\gamma}'_{\mathrm{f,t-1}}\boldsymbol{\beta}_{\mathrm{f},it-1} + \boldsymbol{\gamma}'_{\mathrm{g,t-1}}\boldsymbol{\beta}_{\mathrm{g},it-1}$, for some non-zero pricing errors $a_{i,t-1}$. Nonetheless, thanks to the discipline imposed by no-arbitrage constraint on the $a_{i,t-1}$, our asymptotic analysis is maintained without affecting the limiting properties of our methodology.[11] Therefore, hereafter we impose the exact pricing condition in (5) without any loss of generality. This implies that the model in (4) is equal to:

[9]The stochastic discount factor (SDF), written as a linear projection in the test assets, will be identical whenever a non-tradable factor is replaced by its mimicking-portfolio return.

[10]When the R_{it} are gross returns, one replaces (5) with

$$\mathbb{E}\left[R_{it}|I_{t-1}\right] = \gamma_{\mathrm{z},t-1} + \boldsymbol{\gamma}'_{\mathrm{f,t-1}}\boldsymbol{\beta}_{\mathrm{f},it-1} + \boldsymbol{\gamma}'_{\mathrm{g,t-1}}\boldsymbol{\beta}_{\mathrm{g},it-1} \tag{6}$$

where $\gamma_{z,t-1}$ denotes the zero-beta rate. The extension of our methodology to the case when $\gamma_{z,t-1}$ is allowed for, i.e., not zero, is described in Section XXX of the Online Appendix.

[11]By asymptotic no-arbitrage $\sum_{i=1}^{N} a_{i,t-1}^2 = O(1)$ for every $1 \leq t \leq T$, when the ϵ_{it} are mildly cross-correlated (see Chamberlain and Rothschild (1983)), implying that the cross-sectional averaging embedded in the two-pass estimator of the risk premia would dissipate the $a_{i,t-1}$ without any asymptotic effect, i.e., $\sum_{i=1}^{N} w_i a_{i,t-1} \to 0$ for any sequence of weighs w_i satisfying $\sum_{i=1}^{N} w_i^2 \to 0$.

$$\mathbf{R}_t = \mathbf{B}_{\mathrm{f},t-1}\boldsymbol{\delta}_{\mathrm{f,t-1}} + \mathbf{B}_{\mathrm{g},t-1}\boldsymbol{\delta}_{\mathrm{g,t-1}} + \boldsymbol{\epsilon}_t, \tag{7}$$

where $\mathbb{1}_N$ denotes a $N \times 1$ vector of ones, and where we set

$$\boldsymbol{\delta}_{\mathrm{f,t-1}} \equiv \boldsymbol{\gamma}_{\mathrm{f,t-1}} + \mathbf{f}_t - \mathbb{E}\left[\mathbf{f}_t | I_{t-1}\right], \quad \boldsymbol{\delta}_{\mathrm{g,t-1}} \equiv \boldsymbol{\gamma}_{\mathrm{g,t-1}} + \mathbf{g}_t - \mathbb{E}\left[\mathbf{g}_t | I_{t-1}\right], \tag{8}$$

which are known as the vector of *ex-post* risk premia.[12] When the risk factors are *return spreads* of tradable portfolios, such as when considering returns over the risk-free rate (or the zero-beta rate, in the absence of the risk-free asset) or the difference between long- and short-tradable portfolios, then $\boldsymbol{\gamma}_{\mathrm{f},t-1} = \mathbb{E}\left[\mathbf{f}_t | I_{t-1}\right]$ and $\boldsymbol{\gamma}_{\mathrm{g},t-1} = \mathbb{E}\left[\mathbf{g}_t | I_{t-1}\right]$, implying the asset pricing restriction $\boldsymbol{\alpha}_{t-1} = \mathbf{0}_N$ (an $N \times 1$ vector of zeros) in (4).

In order to estimate the asset pricing model (7), it is essential to specify the time-variation of the risk exposures $\boldsymbol{\beta}_{\mathrm{f},it-1}$ and $\boldsymbol{\beta}_{\mathrm{g},it-1}$. Two alternative approaches could be considered in this case. First, one can postulate a parametric specification for the loadings, by representing them as a function of certain observable variables. For example, one could assume

$$\boldsymbol{\beta}_{\mathrm{f},it-1} = \mathbf{b}_{\mathrm{f},i} + \boldsymbol{\Lambda}_{\mathrm{f},i}\mathbf{z}_{t-1} + \boldsymbol{\Delta}_{\mathrm{f}}\mathbf{z}_{it-1} \text{ and } \boldsymbol{\beta}_{\mathrm{g},it-1} = \mathbf{b}_{\mathrm{g},i} + \boldsymbol{\Lambda}_{\mathrm{g},i}\mathbf{z}_{t-1} + \boldsymbol{\Delta}_{\mathrm{g}}\mathbf{z}_{it-1}, \tag{9}$$

where $\mathbf{b}_{\mathrm{f},i}, \boldsymbol{\Lambda}_{\mathrm{f},i}, \boldsymbol{\Delta}_{\mathrm{f}}$, and $\mathbf{b}_{\mathrm{g},i}, \boldsymbol{\Lambda}_{\mathrm{g},i}, \boldsymbol{\Delta}_{\mathrm{g}}$ are matrices with K and L rows, respectively, and where $\mathbf{z}_t$ and $\mathbf{z}_{i,t-1}$ are observable common and asset-specific variables, respectively, often referred to as instruments.[13]

While it may seem appealing, the specification in (9) could lead to invalid inference on the asset pricing model in (7), whenever one postulate a wrong or incomplete relationship in the specification of the loadings in (9). Moreover, this approach could be particularly problematic if the objective is to test for factors' strength. For example,

[12] The notion of *ex-post* risk premia was originally coined by Shanken (1992) to denote a noisy version of the ex-ante risk premia ($\boldsymbol{\gamma}_{\mathrm{f,t-1}}$ and $\boldsymbol{\gamma}_{\mathrm{g,t-1}}$) due to the unexpected factor outcomes ($\mathbf{f}_t - \mathbb{E}\left[\mathbf{f}_t | I_{t-1}\right]$ and $\mathbf{g}_t - \mathbb{E}\left[\mathbf{g}_t | I_{t-1}\right]$, respectively), which arises whenever one considers the fixed-T sampling scheme. When model (7) is averaged across the time observations (as it is typically done before applying the two-pass methodology), and constancy of risk premia and conditional expectations is assumed, the vector of ex-post risk premia of the factors $\mathbf{f}_t$ and $\mathbf{g}_t$ are defined as $\boldsymbol{\delta}_{\mathrm{f}} = \boldsymbol{\gamma}_{\mathrm{f}} + \bar{\mathbf{f}} - \mathbb{E}\left[\mathbf{f}_t\right]$ and $\boldsymbol{\delta}_{\mathrm{g}} = \boldsymbol{\gamma}_{\mathrm{g}} + \bar{\mathbf{g}} - \mathbb{E}\left[\mathbf{g}_t\right]$, respectively. The ex-post risk premia equal $\bar{\mathbf{f}}$ and $\bar{\mathbf{g}}$ when the $\mathbf{f}_t$ and $\mathbf{g}_t$ are tradable spread returns.

[13] See, for example, Ferson and Harvey (1991) and Lettau and Ludvigson (2001) for the use of common instruments. One can refer to Avramov and Chordia (2006) and Gagliardini, Ossola, and Scaillet (2016), among others, for the use of asset-specific instruments.

consider the simplest case when $\boldsymbol{\Delta}_{\mathrm{g}}$ in (9) is a zero matrix. In this case, one could rewrite the conditional model (7) as a model with constant risk exposures on the transformed risk factors $(\mathbf{f}_t \otimes \mathbf{g}_{t-1})$, where $\otimes$ defines the Kronecker product. This would completely cloud the analysis, which aims to assess the strength of $\mathbf{g}_{t-1}$. Alternatively, consider the case when $\boldsymbol{\Lambda}_{\mathrm{g},i}$ in (9) are zero matrices, so that now $\mathbf{g}_t$'s strength depends on the cross-sectional properties of the observable variables $\mathbf{z}_{it-1}$. This would again complicate the evaluation of $\mathbf{g}_t$'s strength, as the results may vary based, for example, on whether the $\mathbf{z}_{it-1}$ are standardized or not.

A different approach, which we adopt in this paper, involves instead not modeling the time variation of $\boldsymbol{\beta}_{\mathrm{f},it-1}$ and $\boldsymbol{\beta}_{\mathrm{g},it-1}$ explicitly. Instead, we rely on some smoothness assumptions regarding the behavior of the betas within a specified time interval, enabling us to replace the time-varying model (7), with an equivalent model having (locally) constant risk exposures. The main assumption required to accomplish this is outlined below.

Assumption 1 (Smoothness of the Risk Exposures)**.** *For every* $1 \leq t \leq T$*,*

$$\| \mathbf{B}_{\mathrm{f},t} - \mathbf{B}_{\mathrm{f}} \|^2 = o(\| \mathbf{B}_{\mathrm{f},t} \|) \text{ and } \| \mathbf{B}_{\mathrm{g},t} - \mathbf{B}_{\mathrm{g}} \|^2 = o(\| \mathbf{B}_{\mathrm{g},t} \|), \tag{10}$$

for some (locally) constant matrices of risk exposures $\mathbf{B}_{\mathrm{f}}$ *and* $\mathbf{B}_{\mathrm{g}}$ *that depend on the time interval* T*.*[14]

To understand (10), recall that the $\mathbf{f}_t$ are strong factors, hence satisfying $\| \mathbf{B}_{\mathrm{f},t} \| \asymp O(N^{\frac{1}{2}})$. This implies that (10) is equivalent to $\| \mathbf{B}_{\mathrm{f},t} - \mathbf{B}_{\mathrm{f}} \|^2 = o(N^{\frac{1}{2}})$, indicating that, while the time-variation in the betas may be pronounced, it gradually diminishes as N becomes larger.[15] A similar argument holds when we examine $\mathbf{g}_t$, with the main distinction being that the rate at which time-variation dissipates can now be faster whenever a weaker strength of $\mathbf{g}_t$ is posited.

When (10) holds, then model (7), with time-varying risk exposures, can be replaced by an asset pricing model with *locally-constant* risk exposures:

$$\mathbf{R}_t = \mathbf{B}_{\mathrm{f}} \boldsymbol{\delta}_{\mathrm{f,t-1}} + \mathbf{B}_{\mathrm{g}} \boldsymbol{\delta}_{\mathrm{g,t-1}} + \boldsymbol{\epsilon}_t, \tag{11}$$

[14] We do not explicitly write $\mathbf{B}_{\mathrm{f}}$ and $\mathbf{B}_{\mathrm{g}}$ as a function of T to simplify the exposition.

[15] Section **??** of the Online Appendix reports some numerical illustrations of relevant examples of time-varying betas that satisfy (10).

without affecting the statistical analysis of our testing procedure (see Section **??** of the Online Appendix for details). Therefore, in the following, we will consistently refer to the conditional factor asset pricing model (11) in developing our methodology, building on the the locally constant risk exposures assumption in (10). Given that T can be taken almost arbitrarily small in our asymptotic analysis, it becomes evident that condition (10) is quite mild in practice. This is also especially advantageous as it remains robust to model misspecification, unlike the parametric approach in (9), which may result in unreliable inferences if the specifications for $\boldsymbol{\beta}_{\mathrm{f},it-1}$ or $\boldsymbol{\beta}_{\mathrm{g},it-1}$ are incorrect. Finally, although we assume locally constant betas in (11), both the risk premia and the zero-beta rate are still allowed to be time-varying.

Given the locally-constant model in (11), we then formalize the definition of factor strength as follows.

Assumption 2 (Local Factor Strength)**.** *For some $0 \leq \rho \leq 1$, the $N \times L$ matrix $\mathbf{B}_g$, function of the time interval T, satisfies*

$$\| \mathbf{B}_{\mathrm{g}} \|^2 \asymp O\left(N^{\rho}\right) \text{ and } \left|\mathbf{B}_{\mathrm{g}}' \mathbb{1}_N\right| \asymp o\left(N^{\frac{\rho+1}{2}}\right), \tag{12}$$

where $|\cdot|$ is understood to apply element-by-element.

The first condition in (12) merges the definitions in (2) and (3), since it implies that $\| \mathbf{B}_{\mathrm{g}} \|^2 = \sum_{i=1}^{N} \| \boldsymbol{\beta}_{\mathrm{g},i}' \boldsymbol{\beta}_{\mathrm{g},i} \|^2 \asymp O\left(N^{\rho}\right)$. To understand the second condition in (12), assume for simplicity the case $L = 1$, so that g_t is a scalar. Then, by Cauchy-Schwarz inequality, it follows that

$$\left| \frac{1}{N} \sum_{i=1}^{N} \beta_{\mathrm{g},i} \right| \leq \left(\frac{1}{N} \sum_{i=1}^{N} (\beta_{\mathrm{g},i})^2 \right)^{\frac{1}{2}},$$

implying that $|\sum_{i=1}^{N} \beta_{\mathrm{g},i}| = O\left(N^{\frac{(\rho+1)}{2}}\right)$. Therefore, (12) only slightly strengths the effect of Cauchy-Schwarz inequality (by imposing an exact rate instead of an upper bound), and sharps the bound (by replacing $O(\cdot)$ with $o(\cdot)$), hereby simplifying a technical aspect of our methodology.

Notice that, whenever $\rho < 1$, then $\left|\mathbf{B}_{\mathrm{g}}' \mathbb{1}_N\right|$ achieves a faster rate than $\| \mathbf{B}_{\mathrm{g}} \|^2$, with the effect becoming more pronounced as the value of ρ decreases. As ρ approaches 0, the risk exposures remain bounded when considering their individual values, but their

sum in levels diverges. This straightforward feature, resulting from the Cauchy-Schwarz inequality, is the crucial insight that enables us to develop our testing methodology of factors' weaknesses.[16]

2.1 Special Case: Unconditional Asset Pricing Set-Up

Our methodology can be easily applied to the more commonly used unconditional asset pricing framework, as in Lettau and Pelger (2020b) and Giglio, Xiu, and Zhang (2023) among others, where risk exposures and risk premia remain constant over time. This yields to the following factor asset-pricing model:

$$\mathbf{R}_t = \boldsymbol{\alpha} + \mathbf{B}_{\mathrm{f}}\mathbf{f}_t + \mathbf{B}_{\mathrm{g}}\mathbf{g}_t + \boldsymbol{\epsilon}_t, \tag{13}$$

which, after imposing the corresponding exact-pricing condition with constant parameters, $\mathbb{E}\left[R_{it}\right] = \boldsymbol{\gamma}_{\mathrm{f}}'\boldsymbol{\beta}_{\mathrm{f},i} + \boldsymbol{\gamma}_{\mathrm{g}}'\boldsymbol{\beta}_{\mathrm{g},i}$, reduces to

$$\mathbf{R}_t = \mathbf{B}_{\mathrm{f}}\boldsymbol{\delta}_{\mathrm{f}} + \mathbf{B}_{\mathrm{g}}\boldsymbol{\delta}_{\mathrm{g}} + \boldsymbol{\epsilon}_t, \tag{14}$$

setting

$$\boldsymbol{\delta}_{\mathrm{f}} \equiv \boldsymbol{\gamma}_{\mathrm{f}} + \mathbf{f}_t - \mathbb{E}\left[\mathbf{f}_t\right], \quad \boldsymbol{\delta}_{\mathrm{g}} \equiv \boldsymbol{\gamma}_{\mathrm{g}} + \mathbf{g}_t - \mathbb{E}\left[\mathbf{g}_t\right]. \tag{15}$$

In this setting, Assumption (1) becomes now redundant, since it will always be satisfied, while Assumption (2) applies unconditionally over time, because the risk exposures are now constant and do not depend on the sample period of length T.[17] Therefore, our methodology extends immediately to the unconditional asset pricing framework, where we still continue to adopt the large-N and fixed-T set-up.

3 Methodology

In this section, we introduce the primary results of our paper. To provide the main intuition behind our methodology, we begin our analysis by examining the simplest

[16] Assumption 2, together with further regularity conditions on the cross-sectional dependence of the error term $\boldsymbol{\epsilon}_t$, implies that $\mathbf{B}_{\mathrm{g}}'\boldsymbol{\epsilon}_t \asymp O_p\left(N^{\frac{\rho}{2}}\right)$.

[17] Details are available upon request.

case in which no strong factors are present in the asset pricing model (11). In this case $\mathbf{B}_{\mathrm{f}} = \mathbf{0}_{N\times K}$, where $\mathbf{0}_{N\times K}$ denotes an $N \times K$ matrix of zeros. We then show how our results extend to the case of non-zero $\mathbf{B}_{\mathrm{f}}$, exploring the scenario where the $\mathbf{f}_t$ are unobserved and estimated using PCA.[18]

3.1 Benchmark Case: Zero Factors

In this section, we consider the simple case of (4), assuming the absence of strong risk factors in the model. Formally, we assume all elements of $\mathbf{B}_{\mathrm{f}}$ to be zero, resulting in the following factor model for excess returns:

$$\mathbf{R}_t = \boldsymbol{\alpha}_{t-1} + \mathbf{B}_{\mathrm{g}}\mathbf{g}_t + \boldsymbol{\epsilon}_t. \tag{16}$$

Imposing the asset pricing restriction (5) and applying the standard first-pass Fama-Macbeth procedure - by means of N time-series OLS regressions of excess returns $\mathbf{R}_t$ onto $\mathbf{g}_t$ and an intercept - gives the estimated risk exposures:

$$\hat{\mathbf{B}}_{\mathrm{g}} = \mathbf{R}'\mathbb{J}_T\mathbf{G}\left(\mathbf{G}'\mathbb{J}_T\mathbf{G}\right)^{-1} = \mathbf{B}_{\mathrm{g}} + \boldsymbol{\epsilon}\mathbf{P}_{\mathrm{g}}, \tag{17}$$

where $\mathbf{R} = (\mathbf{R}_1, \cdots, \mathbf{R}_T)'$, $\mathbf{G} = (\mathbf{g}_1, \cdots, \mathbf{g}_T)'$, $\boldsymbol{\epsilon} = (\boldsymbol{\epsilon}_1, \cdots, \boldsymbol{\epsilon}_T)'$, and where we set $\mathbb{J}_T = \mathbb{I}_T - \mathbb{1}_T\mathbb{1}_T'/T$, with $\mathbb{1}_T$ denoting the T-dimensional vector of ones, $\mathbb{I}_T$ denoting a $T \times T$ identity matrix, and where $\mathbf{P}_{\mathrm{g}} = \mathbb{J}_T\mathbf{G}\left(\mathbf{G}'\mathbb{J}_T\mathbf{G}\right)^{-1}$. Next, consider the specification in (11) with $\mathbf{B}_{\mathrm{f}} = \mathbf{0}_{N\times K}$ but where now, importantly, we allow for the zero-beta rate $\gamma_{\mathrm{z},t-1}$, although its true value is in fact zero, i.e., $\gamma_{\mathrm{z},t-1} = 0$, when the R_{it} are excess returns, yielding

$$\mathbf{R}_t = \gamma_{\mathrm{z},t-1}\mathbb{1}_N + \mathbf{B}_g\boldsymbol{\delta}_{\mathrm{g,t-1}} + \boldsymbol{\epsilon}_t. \tag{18}$$

In fact, our methodoloy crucially rely on estimating the zero-beta rate, i.e., the intercept in the second-pass regression, even though its true value is known and equal to zero, as it be explained below. Averaging both sides of the equation across T, one gets:

[18] In the Online Appendix we also consider the case of observed $\mathbf{f}_t$.

$$\begin{aligned}\overline{\mathbf{R}} &= \bar{\gamma}_{\mathrm{z}}\mathbb{1}_N + \hat{\mathbf{B}}_{\mathrm{g}}\bar{\boldsymbol{\delta}}_{\mathrm{g}} + \bar{\boldsymbol{\epsilon}} + (\mathbf{B}_{\mathrm{g}} - \hat{\mathbf{B}}_{\mathrm{g}})\bar{\boldsymbol{\delta}}_{\mathrm{g}} \\ &= \hat{\mathbf{X}}_{\mathrm{g}}\overline{\boldsymbol{\Gamma}}_{\mathrm{g}} + \bar{\boldsymbol{\eta}}, \end{aligned} \tag{19}$$

setting $\hat{\mathbf{X}}_{\mathrm{g}} = (\mathbf{1}_N, \hat{\mathbf{B}}_{\mathrm{g}})$, with $\bar{\boldsymbol{\eta}} = T^{-1}\sum_{t=1}^{T}\boldsymbol{\eta}_t$, where $\boldsymbol{\eta}_t = \boldsymbol{\epsilon}_t + (\mathbf{B}_{\mathrm{g}} - \hat{\mathbf{B}}_{\mathrm{g}})\boldsymbol{\delta}_{\mathrm{g,t-1}}$, and where $\overline{\boldsymbol{\Gamma}}_{\mathrm{g}} = (\bar{\gamma}_{\mathrm{z}}, \bar{\boldsymbol{\delta}}_{\mathrm{g}}')' = T^{-1}\sum_{t=1}^{T}\boldsymbol{\Gamma}_{\mathrm{g},t-1}$, with $\boldsymbol{\Gamma}_{\mathrm{g},t-1} = (\gamma_{\mathrm{z},t-1}, \boldsymbol{\delta}_{\mathrm{g,t-1}}')' = (0, \boldsymbol{\delta}_{\mathrm{g,t-1}}')'$.[19] Applying a second-pass cross-sectional OLS regression to (19), yields:

$$\hat{\overline{\boldsymbol{\Gamma}}}_{\mathrm{g}} = \begin{bmatrix} \hat{\bar{\gamma}}_{\mathrm{z}} \\ \hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}} \end{bmatrix} = \left(\hat{\mathbf{X}}_{\mathrm{g}}'\hat{\mathbf{X}}_{\mathrm{g}}\right)^{-1}\hat{\mathbf{X}}_{\mathrm{g}}'\overline{\mathbf{R}}. \tag{20}$$

The following theorems establish the asymptotic behavior of $\hat{\overline{\boldsymbol{\Gamma}}}_{\mathrm{g}}$, which is examined separately for $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}$ (in Theorem 1 below) and $\hat{\bar{\gamma}}_{\mathrm{z}}$ (in Theorem 2 below). Understanding this behavior is essential for the development of our testing methodology, although recall that we are not interested in estimation of the (average) risk premia and the zero-beta rate. Their two-pass estimator $\hat{\overline{\boldsymbol{\Gamma}}}_{\mathrm{g}}$ is affected, generally speaking, by an estimation bias due to keeping T fixed, and is redundant as its true value $\overline{\boldsymbol{\Gamma}}_{\mathrm{g}} = (0, \bar{\mathbf{g}}')'$ is known, because of the tradability of $\mathbf{g}_t$ and our choice of using excess returns as test assets.

Theorem 1 (Asymptotic Properties of $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}$ - Benchmark Case)**.** *Under Assumptions 1-5, as N increases:*
(i)

$$\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}} \to_p \begin{cases} \mathbf{0}_L & \text{when } \rho < 1, \\ \left(\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} + s_2\mathbf{P}_{\mathrm{g}}'\mathbf{P}_{\mathrm{g}}\right)^{-1}\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}}\bar{\boldsymbol{\delta}}_{\mathrm{g}} & \text{when } \rho = 1. \end{cases}$$

(ii)

$$\sqrt{N}\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}} \to_d \begin{cases} \mathcal{N}\left(\mathbf{0}_L, \pi\mathbf{G}'\mathbb{J}_T\mathbf{G}\right) & \text{when } \rho < 1/2, \\ \pm\infty & \text{when } 1/2 < \rho \leq 1, \end{cases}$$

[19] More specifically, one gets $\bar{\boldsymbol{\delta}}_{\mathrm{g}} = (T^{-1}\sum_{t=1}^{T}\boldsymbol{\gamma}_{\mathrm{g,t-1}}) + \bar{\mathbf{g}} - (T^{-1}\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{g}_t|I_{t-1}\right]) = \bar{\mathbf{g}}$, the latter equality holding because of the tradability of $\mathbf{g}_t$.

where we set $\pi \equiv (m_4 + (T-3)\, s_4)/(T^2 s_2^2)$*, with* s_2, s_4*,* m_4*, and* $\mathbf{\Sigma}_{\beta_{\mathrm{g}}}$ *are defined in Assumptions 3-5.*

Proof. See Online Appendix.

Theorem 1 describes the asymptotic behaviour of the two-pass OLS estimator $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}$ for the risk premia $\bar{\boldsymbol{\delta}}_{\mathrm{g}}$ under the assumption of no strong factors in the model. The theorem shows that, whenever the factors in $\mathbf{g}_t$ are either weak or semi-strong (i.e., $0 \leq \rho < 1$), the two-pass OLS estimator converges to zero uniformly, that is regardless of the true population value $\bar{\boldsymbol{\delta}}_{\mathrm{g}}$.[20]

The intuition behind this finding is straightforward: when $\rho < 1$, the influence of estimation errors dominates that of the underlying signal, as the squared average of the true risk exposures approaches zero. Consequently, the two-pass OLS estimator assigns negligible risk premia to the corresponding risk factors $\mathbf{g}_t$. On the other hand, when $\mathbf{g}_t$ is a strong factor (i.e. $\rho = 1$), then $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}$ converges to a *downward-biased* version of the true risk premia, which restores a well-known feature of the two-pass OLS estimator under the fixed-T set-up (see Shanken (1992)).[21]

Theorem 1 also shows that the rate of convergence (towards zero) of $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}$ depends on the factors' strength: the classical square-N rate is achieved whenever $\rho \leq 1/2$, while it becomes slower as ρ increases beyond $1/2$. This implies that, by exploiting the results of Theorem 1 (iii), one can construct a test for factors' strength that is able to discriminate the case of $\rho \leq 1/2$ from the case of $\rho > 1/2$.

However, the results of Theorem 1 do not offer any guidance on how to discriminate between the case of weak factors ($\rho = 0$) and the case of semi-strong or strong factors ($0 < \rho \leq 1$). This challenge is successfully resolved through Theorems 2 and 3 below, where we establish the behaviour of the two-pass OLS estimator for the zero-beta rate $\hat{\bar{\gamma}}_{\mathrm{z}}$.

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[20] We established, in the proof, the rate of convergence of $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}$ under the various possible factors' strength. In particular

$$\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}} \asymp \begin{cases} O_p(N^{\rho-1}) + O_p(N^{-\frac{1}{2}}) & \text{when } \rho < 1, \\ \left(\mathbf{\Sigma}_{\beta_{\mathrm{g}}} + s_2 \mathbf{P}_{\mathrm{g}}' \mathbf{P}_{\mathrm{g}}\right)^{-1} \mathbf{\Sigma}_{\beta_{\mathrm{g}}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} + O_p(N^{-\frac{1}{2}}) & \text{when } \rho = 1. \end{cases}$$

[21] Quantification of this bias-adjustment, although possible, is entirely irrelevant given the objective of the paper.

Theorem 2 (Asymptotic properties of $\hat{\bar{\gamma}}_{\mathrm{z}}$ - Benchmark Case)**.** *Under Assumptions 1-5, as N increases:*
(i)

$$\hat{\bar{\gamma}}_{\mathrm{z}} \to_p \begin{cases} 0 & \text{when } \rho < 1, \\ s_2 \boldsymbol{\mu}'_{\beta_{\mathrm{g}}} (\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} + s_2 \mathbf{P}'_{\mathrm{g}} \mathbf{P}_{\mathrm{g}})^{-1} \mathbf{P}'_{\mathrm{g}} \mathbf{P}_{\mathrm{g}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} & \text{when } \rho = 1. \end{cases}$$

(ii)

$$\sqrt{N} \hat{\bar{\gamma}}_{\mathrm{z}} \to_d \begin{cases} \mathcal{N}\left(0, \frac{s_2}{T}\right) & \text{when } \rho = 0, \\ \pm\infty & \text{when } 0 < \rho \leq, . \end{cases}$$

where s_2, $\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}}$, and $\boldsymbol{\mu}_{\beta_{\mathrm{g}}}$ are defined in Assumptions 3-5.

Proof. See Online Appendix.

The results of Theorem 2 are considerably sharper than that of Theorem 1. This is because $\hat{\bar{\gamma}}_{\mathrm{z}}$, being the estimator of the intercept in the cross-sectional regression (19), provides information on the limiting behavior of the *sample mean* of the risk exposures $\hat{\mathbf{B}}_{\mathrm{g}}$, in contrast to $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}$, which instead depends on the sum of the *squared* risk exposures. Recall that the true value of the zero-beta rate is zero in this setting, as the test assets are excess returns, i.e. $\bar{\gamma}_{\mathrm{z}} = 0$, but this is irrelevant for our methodology, based on the limiting behaviour of the estimator $\hat{\bar{\gamma}}_{\mathrm{z}}$ under different level of factor strength.[22]

Theorem 2 shows that the two-pass OLS estimator for the zero-beta rate is now consistent when $\rho < 1$ and asymptotically biased when $\rho = 1$ - with this latter result being already well-established in the literature (see Shanken (1992)).

[22] We established, in the proof, the rate of convergence of $\hat{\bar{\gamma}}_{\mathrm{z}}$ under the various possible factors' strength. In particular

$$\hat{\bar{\gamma}}_{\mathrm{z}} \asymp \begin{cases} o\left(N^{(\rho-1)/2}\right) + O_p(N^{-\frac{1}{2}}) & \text{when } \rho < 1, \\ s_2 \boldsymbol{\mu}'_{\beta_{\mathrm{g}}} (\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} + \sigma^2 \mathbf{P}'_{\mathrm{g}} \mathbf{P}_{\mathrm{g}})^{-1} \mathbf{P}'_{\mathrm{g}} \mathbf{P}_g \bar{\boldsymbol{\delta}}_{\mathrm{g}} + O_p(N^{-\frac{1}{2}}) & \text{when } \rho = 1. \end{cases}$$

The crucial finding arises from part (iii) of Theorem 2, which examines the convergence rates of the estimator: the root-N rate is now achieved only when $\rho = 0$, while it becomes slower when $\rho > 0$. This implies that, when $\rho > 0$, we have now statistical power to test for the case of weak $\mathbf{g}_t$.

To construct the test for weak factors, we build on a refinement of part (ii) of Theorem 2. Specifically, we build on the fact that

$$\begin{aligned} \hat{\bar{\gamma}}_{\mathrm{z}} - \bar{\gamma}_{\mathrm{z}} = \hat{\bar{\gamma}}_{\mathrm{z}} &= \frac{\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \hat{\mathbf{B}}_{\mathrm{g}}' \mathbb{1}_N}{N} + \frac{\mathbf{Q}_{\mathrm{g}}' \boldsymbol{\epsilon} \mathbb{1}_N}{N} + o_p(N^{-\frac{1}{2}}) \\ &= \frac{\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \hat{\mathbf{B}}_{\mathrm{g}}' \mathbb{1}_N}{N} + O_p(N^{-\frac{1}{2}}) + o_p(N^{-\frac{1}{2}}), \end{aligned} \tag{21}$$

with $\mathbf{Q}_{\mathrm{g}} = \mathbb{1}_T/T - \mathbf{P}_{\mathrm{g}} \bar{\boldsymbol{\delta}}_{\mathrm{g}}$, and where

$$\frac{\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \hat{\mathbf{B}}_{\mathrm{g}}' \mathbb{1}_N}{N} = \frac{\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \mathbf{B}_{\mathrm{g}}' \mathbb{1}_N}{N} + \frac{\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \mathbf{P}_{\mathrm{g}}' \boldsymbol{\epsilon} \mathbb{1}_N}{N} = o\left(N^{(\rho-1)/2}\right) + O_p(N^{-\frac{1}{2}}). \tag{22}$$

Then, combining (21) with (22), it is easy to see that the asymptotic behaviour of $\sqrt{N}\hat{\bar{\gamma}}_{\mathrm{z}}$ is equivalent to that of $N^{-1/2}(\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \mathbf{B}_{\mathrm{g}}' \mathbb{1}_N)$. This represents the key insight of our new test statistic, which will based on the sample analogue of $N^{-1/2}(\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \mathbf{B}_{\mathrm{g}}' \mathbb{1}_N)$.

Using all the above results, we are now ready to introduce our main testing procedure for factors' weakness, which we present in the following Theorem.

Theorem 3 (Testing for Weak factors - Benchmark Case)**.** *Under Assumptions 1-5, as N increases:*

(i) When $\rho = 0$,

$$\frac{\mathbb{1}_N' \hat{\mathbf{B}}_{\mathrm{g}}}{\sqrt{N}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} \to_d \mathcal{N}\left(0, s_2 \bar{\boldsymbol{\delta}}_{\mathrm{g}}' \left(\mathbf{G}' \mathbb{J}_T \mathbf{G}\right)^{-1} \bar{\boldsymbol{\delta}}_{\mathrm{g}}\right).$$

(ii) When $\rho > 0$,

$$\frac{\mathbb{1}_N' \hat{\mathbf{B}}_{\mathrm{g}}}{\sqrt{N}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} \to_p \pm\infty.$$

Proof. See Online Appendix.

Theorem 3 offers a new test for identifying weak factors, which ensures both correct size and power and remains asymptotically valid as N increases. In contrast to existing methodologies for assessing factor strength, which can only distinguish between semi-strong and strong factors, our results provide a complementary perspective. This is extremely relevant because, in most, if not all cases, empirical findings typically rely

on at least a mild form of semi-strongness (see, Pesaran and Smith (2023), Anatolyev and Mikusheva (2022), and Giglio, Xiu, and Zhang (2023) among others).

To make use of all the limiting distributions derived in this section (from Theorem 1 to Theorem 3), one must consistently estimate several quantities (s_2, s_4, m_4, $\boldsymbol{\mu}_{\beta_g}$, and $\boldsymbol{\Sigma}_{\beta_g}$). Asymptotically valid estimators for these parameters will be presented in the more general case of the following section (see Theorem 5 in Section 5).

Finally, note that the statistic $\mathbb{1}_N'\hat{\mathbf{B}}_g\bar{\boldsymbol{\delta}}_g/\sqrt{N}$ is feasible, not requiring further estimates beyond $\hat{\mathbf{B}}_g$, because $\bar{\boldsymbol{\delta}}_g = \bar{\mathbf{g}}$ by tradability of $\mathbf{g}_t$.

3.2 General Case: Multifactor Model

In this section, we extend all the theoretical results of sections 3.1 to a more general case, wherein we still want to assess the degree of weakness of certain observed risk factors $\mathbf{g}_t$, while simultaneously acknowledging the potential influence of *unknown* strong risk factors in pricing the cross-section of expected returns.

Acknowledging our agnosticism about the potential contribution of other strong risk factors to pricing the cross-section of stock returns, we build on the recent idea of Giglio and Xiu (2021) and implement PCA to (pre-)estimate the latent strong factor space. We then use these estimates to develop an asymptotically valid test statistic to assess whether (and to which degree) the observed factors in $\mathbf{g}_t$ are weak, or else are either semistrong or strong. Notice that, given our fixed-T set-up, the traditional double-asymptotic framework used to characterize PCA becomes inapplicable in our context. Therefore, we build on Zaffaroni (2022), which instead establishes the limiting distribution of PCA in the case of large-N and fixed-T.

Consider the general factor model (11), with locally-constant risk exposures and where now the $\mathbf{f}_t$ are assumed latent strong factors.

$$\mathbf{R}_t = \mathbf{B}_f\mathbf{f}_{t-1} + \mathbf{B}_g\mathbf{g}_{t-1} + \boldsymbol{\epsilon}_t. \tag{23}$$

As the $\mathbf{f}_t$ are latent, to make our methodology feasible we need to replace them with their PCA estimates, implying a slight re-parameterization of the model. Let $(\mathbf{R}\mathbf{R}')\,\hat{\mathbf{F}}/NT = \hat{\mathbf{F}}\hat{\boldsymbol{\Lambda}}$, where $\hat{\boldsymbol{\Lambda}}$ denotes the $K \times K$ diagonal matrix of the largest K eigenvalues of $(\mathbf{R}\mathbf{R}')/NT$, and $\hat{\mathbf{F}} = (\hat{\mathbf{f}}_1, \cdots, \hat{\mathbf{f}}_T)'$ represents the $T \times K$ matrix of the K eigenvectors associated to the eigenvalues in $\hat{\boldsymbol{\Lambda}}$.

Applying the first-pass OLS estimator to (23) yields the estimated betas for the estimated strong factor $\hat{\mathbf{f}}_t$:

$$\hat{\mathbf{B}}_{\mathrm{f}}^{*} = \mathbf{R}'\mathbb{J}_T\hat{\mathbf{F}}\left(\hat{\mathbf{F}}'\mathbb{J}_T\hat{\mathbf{F}}\right)^{-1}, \tag{24}$$

where $\mathbf{P}_{\hat{\mathrm{f}}} = \mathbb{J}_T\hat{\mathbf{F}}(\hat{\mathbf{F}}'\mathbb{J}_T\hat{\mathbf{F}})^{-1}$, and the weak factor $\mathbf{g}_t$:

$$\hat{\mathbf{B}}_{\mathrm{g}}^{*} = \mathbf{R}'\mathbb{J}_T\hat{\mathbf{Z}}\left(\hat{\mathbf{Z}}'\mathbb{J}_T\hat{\mathbf{Z}}\right)^{-1}\left[\mathbf{0}_{K_f}\ \mathbf{I}_{K_g}\right]' \tag{25}$$

where $\hat{\mathbf{G}}_{\perp} = (\mathbb{J}_T - \mathbb{J}_T\hat{\mathbf{F}}(\hat{\mathbf{F}}'\mathbb{J}_T\hat{\mathbf{F}})^{-1}\hat{\mathbf{F}}'\mathbb{J}_T)\mathbf{G}$ and $\mathbf{P}_{\hat{\mathrm{g}}\perp} = \hat{\mathbf{G}}_{\perp}(\hat{\mathbf{G}}_{\perp}'\hat{\mathbf{G}}_{\perp})^{-1}$. To obtain (24), we use only $\hat{\mathbf{f}}_t$ as regressors while for (25), we use both $\hat{\mathbf{f}}_t$ and $\mathbf{g}_t$ as regressors.

To understand how the OLS estimates $\hat{\mathbf{B}}_g^*$ and $\hat{\mathbf{B}}_f^*$ are affected by replacing the $\mathbf{f}_t$ by $\hat{\mathbf{f}}_t$, recall that

$$\hat{\mathbf{F}} - \mathbf{F}\hat{\mathbf{H}} \to_p \mathbf{0}_{T\times K} \text{ as } N \to \infty,$$

for some (unobserved) non-singular matrix $\hat{\mathbf{H}}$ (see Lemma 19 in the Online Appendix). Therefore, we re-arrange (23) as

$$\begin{aligned}\mathbf{R}_t =& \mathbf{B}_{\mathrm{f}}(\mathbf{f}_t - \hat{\mathbf{H}}^{-1\prime}\hat{\mathbf{f}}_t + \hat{\mathbf{H}}^{-1\prime}\hat{\mathbf{f}}_t) + \mathbf{B}_{\mathrm{g}}\mathbf{g}_t + \boldsymbol{\epsilon}_t, &(26)\\ =& \mathbf{B}_{\mathrm{f}}^{*}\hat{\mathbf{f}}_t + \mathbf{B}_{\mathrm{g}}\mathbf{g}_t + \boldsymbol{\epsilon}_t^{*}, &(27)\end{aligned}$$

setting $\mathbf{B}_{\mathrm{f}}^{*} = \mathbf{B}_{\mathrm{f}}\hat{\mathbf{H}}^{-1\prime}$, which is estimated, together with $\mathbf{B}_{\mathrm{g}}$, by (24)-(25), and $\boldsymbol{\epsilon}_t^{*} = \boldsymbol{\epsilon}_t + \mathbf{B}_{\mathrm{f}}(\mathbf{f}_t - \hat{\mathbf{H}}^{-1\prime}\hat{\mathbf{f}}_t)$.

Regarding the second-pass step, using the usual rearrangements, we obtain the following feasible representation of the CSR regression:

$$\overline{\mathbf{R}} - \hat{\mathbf{B}}_{\mathrm{f}}^{*}\bar{\boldsymbol{\delta}}_{\mathrm{f}}^{*} = \hat{\mathbf{X}}_{\mathrm{g}}^{*}\overline{\boldsymbol{\Gamma}}_{\mathrm{g}} + \bar{\boldsymbol{\eta}}^{*}, \tag{28}$$

where $\bar{\boldsymbol{\delta}}_{\mathrm{f}}^{*} = \bar{\hat{\mathbf{f}}}$, $\hat{\mathbf{X}}_{\mathrm{g}}^{*} = (\mathbb{1}_N, \hat{\mathbf{B}}_{\mathrm{g}}^{*})$, $\bar{\boldsymbol{\eta}}^{*} = \boldsymbol{\epsilon}'\mathbf{l}^{*} + \mathbf{B}_{\mathrm{g}}\mathbf{G}'\mathbf{l}^{*} + \mathbf{B}_{\mathrm{f}}\hat{\mathbf{H}}'^{-1}(\hat{\mathbf{H}}'\mathbf{F}' - \hat{\mathbf{F}}')\mathbf{l}^{*} - \hat{\mathbf{B}}_{\mathrm{g}}^{*}\bar{\boldsymbol{\delta}}_{\mathrm{g}}$, $\mathbf{l}^{*} = \left(\frac{\mathbb{1}_T}{T} - \mathbf{P}_{\hat{\mathrm{f}}}\bar{\boldsymbol{\delta}}_{\mathrm{f}}^{*}\right)$, and where, as before, $\overline{\boldsymbol{\Gamma}}_{\mathrm{g}}$ denotes the average across time of $\boldsymbol{\Gamma}_{\mathrm{g,t-1}} = (0, \boldsymbol{\delta}_{\mathrm{g,t-1}}')'$. A nice feature of PCA is that the (ex-post) risk premia estimator $\bar{\boldsymbol{\delta}}_{\mathrm{f}}^{*} = \bar{\hat{\mathbf{f}}}$ coincides with the risk-premia estimator obtained with the two-pass methodology.[23]

[23] Proof of that...

Estimating (28) by CSR OLS gives:

$$\hat{\overline{\mathbf{\Gamma}}}_{\mathrm{g}}^{*}=\left[\begin{array}{c}\hat{\bar{\gamma}}_{\mathrm{z}}^{*}\\ \hat{\boldsymbol{\delta}}_{\mathrm{g}}^{*}\end{array}\right]=\left(\hat{\mathbf{X}}_{\mathrm{g}}^{*\prime}\hat{\mathbf{X}}_{\mathrm{g}}^{*}\right)^{-1}\hat{\mathbf{X}}_{\mathrm{g}}^{\prime *}\left(\overline{\mathbf{R}}-\hat{\mathbf{B}}_{\mathrm{f}}^{*}\bar{\boldsymbol{\delta}}_{\mathrm{f}}^{*}\right). \tag{29}$$

The following theorems establish the asymptotic properties of the two-pass OLS estimator in (29).

Theorem 4 (Asymptotic properties of $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}^{*}$ - Latent Strong Factors)**.** *Under Assumptions 1-9, as N increase:*
(i)

$$\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}^{*}\rightarrow_{p}\begin{cases}\mathbf{0}_{L} & \text{when } \rho<1,\\ (\mathbf{\Sigma}_{\beta_{\mathrm{g}}}+s_{2}\mathbf{P}_{\mathrm{g}}^{\prime}\mathbf{P}_{\mathrm{g}})^{-1}\mathbf{\Sigma}_{\beta_{\mathrm{g}}}\bar{\boldsymbol{\delta}}_{\mathrm{g}} & \text{when } \rho=1.\end{cases}$$

(ii)

$$\sqrt{N}\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}^{*}\rightarrow_{d}\begin{cases}\mathcal{N}\left(\mathbf{0}_{L},\left(\frac{s_{4}}{s_{2}^{2}}\right)\mathbf{l}'\mathbf{l}\left(\mathbf{G}_{\perp}^{\prime}\mathbf{G}_{\perp}\right)+\left(\frac{m_{4}-3s_{4}}{s_{2}^{2}}\right)\left(\mathbf{G}_{\perp}^{\prime}\,diag\left(\mathbf{l}\odot\mathbf{l}\right)\mathbf{G}_{\perp}\right)\right), & \text{when } \rho<1/2,\\ \pm\infty & \text{when } 1/2<\rho\leq 1.\end{cases}$$

where we set $\mathbf{l}=\left(\frac{\mathbb{1}_{T}}{T}-\mathbf{P}_{\mathrm{f}}\bar{\boldsymbol{\delta}}_{\mathrm{f}}\right)$ *and* $\mathbf{\Sigma}_{\beta_{\mathrm{g}}}$ *and* $\mathbf{\Sigma}_{\beta_{\mathrm{f}}}$ *are defined in Assumptions 1-9, while* s_{2},s_{4}*, and* m_{4} *are defined in Assumptions 3-9.*

Proof. See Online Appendix.

The same conclusions from the previous section hold even in the presence of latent strong factors.[24] Indeed, $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}^{*}$ still converges to zero whenever $\rho<1$, while it converges to a non-zero quantity, i.e., the true risk premia $\bar{\boldsymbol{\delta}}_{\mathrm{g}}$ but plagued by a bias, when the factor exhibits strength ($\rho=1$). Similarly, under the null hypothesis of $\rho<1/2$, the rate of convergence (toward zero) still provides the statistical power to test whether $\rho<1/2$

[24]We established, in the proof, the rate of convergence of $\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}^{*}$ under the various possible factors' strength. In particular

$$\hat{\bar{\boldsymbol{\delta}}}_{\mathrm{g}}^{*}\asymp\begin{cases}O_{p}(N^{\delta-1})+O_{p}(N^{-\frac{1}{2}}) & \text{when } \delta<1,\\ (\mathbf{\Sigma}_{\beta_{\mathrm{g}}}+s_{2}\mathbf{P}_{\mathrm{g}}^{\prime}\mathbf{P}_{\mathrm{g}})^{-1}\mathbf{\Sigma}_{\beta_{\mathrm{g}}}\bar{\boldsymbol{\delta}}_{\mathrm{g}}+O_{p}(N^{-\frac{1}{2}}) & \text{when } \rho=1.\end{cases}$$

versus $\rho > 1/2$. The use of PCAs is essential here because even partially omitting some of the strong risk factors $\mathbf{f}_t$, would have made the impossible to use the limiting distribution theory in part (iii), as the asymptotic distribution of $\sqrt{N}\hat{\bar{\boldsymbol{\delta}}}^*_{\mathrm{g}}$ would depend on features of the omitted risk factors, such as their risk premia.[25]

The following theorem presents the limiting properties of the two-pass estimator for the zero-beta rate.

Theorem 5 (Asymptotic Properties of $\hat{\bar{\gamma}}^*_{\mathrm{z}}$ - Latent Strong Factors)**.** *Under Assumptions 1-9, as N increase:*
(i)

$$\hat{\bar{\gamma}}^*_{\mathrm{z}} \to_p \begin{cases} 0 & \text{when } \rho < 1, \\ s_2 \boldsymbol{\mu}'_{\beta_{\mathrm{g}}} (\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} + s_2 \mathbf{P}'_{\mathrm{g}} \mathbf{P}_{\mathrm{g}})^{-1} \mathbf{P}'_{\mathrm{g}} \mathbf{P}_{\mathrm{g}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} & \text{when } \rho = 1. \end{cases}$$

(ii)

$$\sqrt{N}\hat{\bar{\gamma}}^*_{\mathrm{z}} \to_d \left\{ \begin{array}{cc} \mathcal{N}\left(0, s_2 \left(\frac{1}{T} + \overline{\boldsymbol{\delta}}'_{\mathrm{f}} \left(\mathbf{F}'_{\perp}\mathbf{F}_{\perp}\right)^{-1} \overline{\boldsymbol{\delta}}_{\mathrm{f}}\right) + S^*\right) & \text{when } \delta = 0, \\ \pm\infty & \text{when } 0 < \delta \leq 1. \end{array} \right\}.$$

where

$$\begin{aligned} S^* = & \left(\left(\mathbf{H}^{-1}\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right) \otimes \mathbf{l}\right)' \mathbf{V}_{\hat{\mathrm{F}}} \left(\left(\mathbf{H}^{-1}\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right) \otimes \mathbf{l}\right) \\ & + \left(\left(\mathbf{H}^{-1}\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right) \otimes \mathbf{l}\right)' \mathbf{A} \left(\left(\mathbf{H}^{-1}\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right) \otimes \mathbb{I}_T\right) \mathbf{l} + \mathbf{l}' \left(\left(\mathbf{H}^{-1}\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right) \otimes \mathbb{I}_T\right)' \mathbf{A}' \left(\left(\mathbf{H}^{-1}\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right) \otimes \mathbf{l}\right), \end{aligned}$$

$\mathbf{V}_{\hat{\mathrm{F}}}$ *is the asymptotic covariance matrix of* $\hat{\mathbf{F}}$ *defined in XXX, and and* $\mathbf{A}$ *is given in Lemma 19.*

Proof. See Online Appendix.

Building on the same arguments of the previous sections, Theorem 5 can be further refined to construct a simple feasible test for weak factors.[26]

[25] Our concern for the effect of omitting strong risk factors is different in our analysis from the concerns associated with estimating risk premia, such as, e.g., in Giglio and Xiu (2021), as our aim here is not accurate estimation of risk premia.

[26] We established, in the proof, the rate of convergence of $\hat{\bar{\gamma}}^*_{\mathrm{z}}$ under the various possible factors'

Theorem 6 (Testing for Weak Factors - Latent Strong Factors)**.** *Under Assumptions 1-9, as N increase:*

(i) When $\rho = 0$,

$$\frac{\mathbb{1}_N' \hat{\mathbf{B}}_{\mathrm{g}}^*}{\sqrt{N}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} \to_d \mathcal{N}\left(0, s_2 \bar{\boldsymbol{\delta}}_{\mathrm{g}}' \left(\mathbf{G}_\perp' \mathbf{G}_\perp\right)^{-1} \bar{\boldsymbol{\delta}}_{\mathrm{g}} + c^*\right),$$

(ii) When $\rho > 0$,

$$\frac{\mathbb{1}_N' \hat{\mathbf{B}}_{\mathrm{g}}^*}{\sqrt{N}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} \to_p \pm\infty.$$

where c^* is defined in Lemma 29.

Proof. See Online Appendix.

To put the theoretical insights of Theorems 4, 5 and 6 into practice with the data, we first need to estimate the several quantities, that is s_2, s_4, m_4, and c^*. Let $\hat{\boldsymbol{\epsilon}}^* = [\hat{\boldsymbol{\epsilon}}_1^*, ..., \hat{\boldsymbol{\epsilon}}_T^*]'$ be the $T \times N$ matrix of the estimated residuals from the first-pass time-series regressions, where $\hat{\boldsymbol{\epsilon}}_t^* = [\hat{\epsilon}_{1t}^*, ..., \hat{\epsilon}_{Nt}^*]'$. Then, $\hat{\boldsymbol{\epsilon}}^* = \mathbb{M}^* \boldsymbol{\epsilon}$, where

$$\mathbb{M}^* \equiv \mathbb{J}_T - \mathbf{P}_{\hat{\mathbf{Z}}} \hat{\mathbf{Z}}' \mathbb{J}_T, \tag{30}$$

and $\mathbf{P}_{\hat{\mathbf{Z}}} \equiv \mathbb{J}_T \hat{\mathbf{Z}} (\hat{\mathbf{Z}} \mathbb{J}_T \hat{\mathbf{Z}})^{-1}$, with $\hat{\mathbf{Z}} = [\hat{\mathbf{F}}, \mathbf{G}]$.[27] Finally, define the $T^2 \times N$ matrix $\hat{\mathcal{E}}^* = [\hat{\mathcal{E}}_1^*, ..., \hat{\mathcal{E}}_N^*]'$, with $\hat{\mathcal{E}}_i^* = \hat{\boldsymbol{\epsilon}}_i^* \otimes \hat{\boldsymbol{\epsilon}}_i^*$.

Theorem 7 (Consistent estimation of s_2, s_4, m_4, and c^* - Latent Strong Factors)**.** *Assume that the factor asset-pricing model* (23) *holds. Then, when $\rho < 1$ under As-*

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strength. In particular

$$\hat{\hat{\gamma}}_{\mathrm{z}}^* \asymp \begin{cases} o(N^{(\delta-1)/2}) + O_p(N^{-\frac{1}{2}}) & \text{when } \delta < 1, \\ s_2 \boldsymbol{\mu}_{\beta_{\mathrm{g}}}' (\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} + s_2 \mathbf{P}_{\mathrm{g}}' \mathbf{P}_{\mathrm{g}})^{-1} \mathbf{P}_{\mathrm{g}}' \mathbf{P}_{\mathrm{g}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} + O_p(N^{-\frac{1}{2}}) & \text{when } \rho = 1. \end{cases}$$

[27] See Lemmas 17 and 16 in the Online Appendix.

sumptions 1-9, as N increase:

$$\hat{s}_2^* = \frac{tr(\frac{\hat{\boldsymbol{\epsilon}}^*\hat{\boldsymbol{\epsilon}}^{*\prime}}{N})}{T-(K+L+1)} \to_p s_2, \tag{31}$$

$$\begin{bmatrix} \hat{s}_4^* \\ \hat{m}_4^* \end{bmatrix} = \mathbb{A}^{*-1} \begin{bmatrix} tr\left(\frac{\hat{\mathcal{E}}^*\hat{\mathcal{E}}^{*\prime}}{N}\right) \\ tr\left(\mathbb{N}'\frac{\hat{\mathcal{E}}^*\hat{\mathcal{E}}^{*\prime}}{N}\mathbb{N}\right) \end{bmatrix} \to_p \begin{bmatrix} s_4 \\ m_4 \end{bmatrix}, \tag{32}$$

$$\hat{c}^* = \left(\left(\frac{\hat{\mathbf{B}}_{\mathrm{f}}^{*\prime}\mathbb{1}_N}{N}\right) \otimes \bar{\boldsymbol{\delta}}_{\mathrm{g}}'\mathbf{P}_{\hat{\mathrm{g}}\perp}'\right) \hat{\mathbf{V}}_{\hat{\mathrm{f}}} \left(\left(\frac{\mathbb{1}_N'\hat{\mathbf{B}}_{\mathrm{f}}^*}{N}\right) \otimes \mathbf{P}_{\hat{\mathrm{g}}\perp}\bar{\boldsymbol{\delta}}_{\mathrm{g}}\right) \tag{33}$$

$$-2s_2 \left(\left(\frac{\hat{\mathbf{B}}_{\mathrm{f}}^{*\prime}\mathbb{1}_N}{N}\right) \otimes \bar{\boldsymbol{\delta}}_{\mathrm{g}}'\mathbf{P}_{\hat{\mathrm{g}}\perp}'\right) \hat{\mathbf{A}} \left(\left(\frac{\mathbb{1}_N'\hat{\mathbf{B}}_{\mathrm{f}}^*}{N}\right) \otimes \mathbb{I}_T\right) \mathbf{P}_{\hat{\mathrm{g}}\perp}\bar{\boldsymbol{\delta}}_{\mathrm{g}} \to_p c^*. \tag{34}$$

$$\tag{35}$$

where $\mathbb{N}$ is a $T^2 \times T$ matrix of zeros except ones on the $(t+T(t-1),t)$-th elements, for every $t=1,\ldots,T$ and where $\mathbb{A}^*$ is a 2×2 matrix defined as

$$\mathbb{A}^* = \begin{bmatrix} tr(\mathbb{M}^*)^2 + 2tr(\mathbb{M}^*) - 3tr(\mathbb{M}^* \odot \mathbb{M}^*) & tr(\mathbb{M}^* \odot \mathbb{M}^*) \\ 3tr(\mathbb{M}^* \odot \mathbb{M}^*) - 3tr((\mathbb{M}^* \odot \mathbb{M}^*)^2) & tr((\mathbb{M}^* \odot \mathbb{M}^*)^2) \end{bmatrix},$$

and $\hat{\mathbf{V}}_{\hat{\mathrm{f}}}$ is defined in Lemma 29, $\hat{\mathbf{A}}$ is given in Lemma XX, and c^* is defined in Lemma 30.

Proof. See Online Appendix.

3.3 Diagnostic Regression Statistics

While the primary focus of our paper is not directly related to model estimation, we deem it worthwhile to explore the behavior of conventional metrics, traditionally employed in evaluating the significance and the goodness of fit of linear regression models, and assess whether (and how) their behaviours vary when factors exhibit different degrees of strength. Therefore, in the following we analyze the limiting distributions of both the R-squared and the F-test corresponding to the CSR OLS, across different degrees of factors' strength. For simplicity, we assume the baseline scenario of Section 3.1, i.e., no strong factors $\mathbf{f}_t$. These findings can be readily generalized to the multi-factor

case of Section 3.2, which we detail in the Online Appendix for the sake of brevity.

Let

$$R^2 = 1 - \frac{\hat{\bar{\boldsymbol{\eta}}}' \hat{\bar{\boldsymbol{\eta}}}}{\overline{\mathbf{R}}' \mathbb{J}_N \overline{\mathbf{R}}} \tag{36}$$

be the cross-sectional R-squared of the CSR OLS in (19), with $\hat{\bar{\boldsymbol{\eta}}} = \overline{\mathbf{R}} - \hat{\mathbf{X}}_{\mathrm{g}} \hat{\bar{\boldsymbol{\Gamma}}}_{\mathrm{g}}$ denoting the second-pass residuals. Let

$$\mathcal{F} = \frac{R^2/L}{\left(1 - R^2\right)/\left(N - L - 1\right)}. \tag{37}$$

denote the F-statistics to test the null hypothesis that all the risk premia (except the zero-beta rate intercept) are jointly equal to zero. The following theorem outlines the asymptotic properties of the above measures for different levels of factor strength.

Theorem 8 (Diagnostic Regression Statistics - Benchmark Case)**.** *Under Assumptions 1-5, as N increases:*
(i)

$$R^2 \to_p \begin{cases} 0 & \text{when } \rho < 1, \\ \frac{\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} \left(s_2 (\mathbf{G}' \mathbb{J}_T \mathbf{G})^{-1} + \boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} \right)^{-1} \boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} \bar{\boldsymbol{\delta}}_{\mathrm{g}}}{\bar{\boldsymbol{\delta}}_{\mathrm{g}}' \boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} \bar{\boldsymbol{\delta}}_{\mathrm{g}} + \frac{s_2}{T}}, & \text{when } \rho = 1. \end{cases} \tag{38}$$

(ii)

$$\mathcal{F} \to_d \begin{cases} \left(\frac{T}{L}\,\pi\right)\, \chi^2_L & \text{when } \rho < 1/2, \\ +\infty & \text{when } \rho \geq 1/2. \end{cases} \tag{39}$$

$$\tag{40}$$

recalling $\pi = (m_4 + (T-3)s_4)/(T^2 s_2^2)$, $\tilde{\boldsymbol{\Sigma}}_{\beta_{\mathrm{g}}} = \boldsymbol{\Sigma}_{\beta_{\mathrm{g}}} - \boldsymbol{\mu}_{\beta_{\mathrm{g}}} \boldsymbol{\mu}_{\beta_{\mathrm{g}}}'$, with $\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}}$ and $\boldsymbol{\mu}_{\beta_{\mathrm{g}}}$ defined in Assumptions 3-5, and where $\mathbf{Q}_{\mathrm{g}} = \mathbb{1}_T/T - \mathbf{P}_{\mathrm{g}} \bar{\boldsymbol{\delta}}_{\mathrm{g}}$.

Proof. See Online Appendix.

As expected, the results of Theorem 8 aligning with the findings of Theorem 1, by effectively distinguishing between the locally weak/ mild semi-strong case ($\rho < 1/2$) and the more pronounced semi-strong/strong case ($\rho > 1/2$). Therefore, under the null

hypothesis of $\rho < 1/2$, these statistics can represent valuable tools for assessing the significance and goodness of fit of the model.

4 Simulation

In this section, we conduct a Monte Carlo simulation experiment to evaluate the finite sample performance of our proposed methodology. We initially focus on the scenario involving observable strong factors (as detailed in Section 5 above) and then extend the analysis to the context of latent strong factors (as in Section 3.2). In all of our simulation experiments, we consider balanced panels with a time-series dimension of T = 12, 24, 36 time periods and cross-sections of N =100, 500, 1,000, 3,000 stocks. Additionally, we assume the presence of a single ($L = 1$) weak factor g_t and a single ($K = 1$) strong factor f_t, the latter being observed or latent depending on the specific setting. All results are based on 1,000 Monte Carlo replications. The return-generating process is given by

$$\mathbf{R}_t = \gamma_0 \mathbb{1}_N + \mathbf{B}_{\mathrm{f}}(\gamma_{\mathrm{f}} + f_t - E[f_t]) + \mathbf{B}_{\mathrm{g}}(\gamma_{\mathrm{g}} + g_t - E[g_t]) + \boldsymbol{\epsilon}_t \tag{41}$$

in which we set $\gamma_0 = 0.05$, $\gamma_{\mathrm{f}} = 0.04/12$, and $\gamma_{\mathrm{g}} = 0.08/12$. The strong and the weak factors have been calibrated using $f_t \sim \mathcal{N}(0.04/12, 0.16/\sqrt{12})$ and $g_t \sim \mathcal{N}(0.08/12, 0.16/\sqrt{12})$, respectively. For the calibration of the betas of the strong factor we use $\mathbf{B}_{\mathrm{f}} \sim \mathcal{N}(1, 0.4/3)$, while we generate $\mathbf{B}_{\mathrm{g}} \sim \mathcal{N}(1, 0.5/3)/N^{\rho}$, by considering different values of the parameter $\rho = \{0, 0.1, 0.3, 0.5, 0.7, 0.9, 1\}$ to analyze the behavior of the test statistics under different degrees of weakness. Finally, $\boldsymbol{\epsilon}_t \sim \mathcal{N}(0_N, \sigma^2 I_N)$, with $\sigma^2 = 0.20/\sqrt{12}$. The rejection rates of the various test statistics are computed using our asymptotic results derived in Sections 5 and 3.2 and are presented in the following sections.

4.1 Simulation Evidence: Strong Observable Factors

We start from the case of strong observed factors. Using the data generating process described in the above section, we first estimate the parameters $\bar{\gamma}_0$ and $\bar{\delta}_{\mathrm{g}}$ using our estimator in (240) and then apply our theoretical results of Theorem 2(iii) and Theorem 4.

In Table 1, we consider the empirical rejection rates of the t-tests for $\hat{\bar{\delta}}_{\mathrm{g}}$ as derived

in part (iii) of Theorem 2, where the parameters s_2, s_4 and m_4 have been estimated using the estimators provided in Theorem 5. The null hypothesis is that $\rho < 1/2$, against the alternative that $\rho > 1/2$. The results are displayed for various significance levels (10%, 5%, and 1%) and for different numbers of time-series and cross-sectional observations, spanning different degrees of the factor g_t's strength, from a weak factor case ($\rho = 0$) to a strong factor one ($\rho = 1$).The rejection rates are compared with the critical values from a standard normal distribution. As expected, the t-test exhibits a good performance across all cases where ρ is less than 1/2, regardless of the sizes of N and T. However, when ρ exceeds 1/2, the t-test begins to over-reject, which becomes especially pronounced for N greater than 1,000 and as T increases. This tendency is confirmed even more clearly and visually if we look at Figure 1, which plots the distribution of the 1,000 simulated t-statistics for the case of T=24 and N=3,000, with ρ ranging from 0 to 1. The figure clearly shows that the test's reliability diminishes with stronger factors, starting from $\rho > 1/2$.

Next, we evaluate the in-sample statistical performance of our test of weakness presented in Theorem 4, to assess its effectiveness in discriminating between weak factors ($\rho = 0$) and non-weak factors ($\rho > 0$). As before, we analyze the empirical rejection rates of the t-test utilizing the estimated quantities proposed in Theorem 5, and examining their performance under different numbers of time-series and cross-sectional observations, spanning a range of the factor g_t's strengths from 0 to 1. The findings are summarized in Table 2, revealing that, aside from the case of weak factors, the test demonstrates exceptional statistical power in all cases where ρ is different from zero. The numerical findings of Table 2 are complemented by Figure 2, which reports the distribution of the weak factor test across the 1,000 simulations, specifically for the case of N=3,000 and T=24. The figure clearly offers an intuitive understanding of how the test's performance varies with different factor strengths.

In conclusion, the analysis conducted in this section corroborates the reliability of our methodology to test for the presence of weak factors in a given asset pricing model, providing valuable insights into its applicability in empirical research where the dynamics of factor strength are often complex and can vary across different scenarios and settings.

4.2 Simulation Evidence: Strong Latent Factors

- Shows rejection rates on the test for γ_{g} and on the test of weakness under the strong latent factors assumption
- Use the Giglio, Xiu and Zhang methodology to generate a factor g_t which is strong for a subset of N_2 assets and semistrong for a subest N_1 of assets. Then, apply our methodology in each subset and check whether we are also able to confirm that g_t is semistrong in N_1 and strong in N_2.
- Using the same data generating process for g_t, apply our test for weakness to both the subset in N_1 and N_2 to check whether, in both cases, we reject H_0.
- Finally, as a third exercise, use our gdp to generate g_t as a weak factor. Then, apply their method to show that their test is inconclusive.

5 Empirical Results

In this section, we apply our methodology in a practical context to investigate and assess the strength (or weakness) of many commonly used factors in asset pricing models. Our objective is to empirically examine the variation in the strength of these factors over time and across different industries. This analysis will not only serve to validate our theoretical framework but also yield interesting insights into the dynamic nature of factor strength.

5.1 Data

Our main dataset is the Feng, Giglio, and Xiu (2020) data, which contains 150 risk factors at the monthly frequency for the period from July 1976 to December 2017. As test assets, we use monthly return data on individual stocks from the Center for Research in Security Prices (CRSP) database, while the monthly Fama-French factors are downloaded from the Kenneth French website.

5.2 The strength of the five Fama-French factors

In this section, our aim is to evaluate the strength of the well-known Fama-French five factors, namely the market factor (MKT), size (SMB), value (HML), profitability (RMW) and investment (CMA). As a first exercise, we empirically apply our methodology under the assumption of strong observed factors (see Section 5). For this analysis, we implement a rolling exercise using balanced panels over fixed-time windows of two years (i.e., $T = 24$) and obtain the time series of test statistics by shifting the time window month by month over the 1965-2022 period. To ensure a realistic approach in our testing methodology - which also considers the historical context of these factors' discovery - we start with the assessment of the market factor's weakness, assuming initially an empty set F of strong factors in the model. Subsequently, we investigate the degree of weakness of the SMB and HML factors individually, under the assumption that the market represents the sole strong observable factor. Lastly, we analyze the remaining two factors, RMW and CMA, conducting the test of weakness for each one while conditioning on the set F of strong observable factors, now defined as the Fama-French three-factor model.

The results are depicted in the heatmap of Figure 3. Each row of the graph represents one of the five Fama-French factors, while the horizontal axis plots the rolling monthly time series of the t-statistics associated with our test of weakness. The null hypothesis is that ρ is equal to zero, hence indicating that the factor is weak. In the heatmap, we employ a color-coded scheme to identify the different degrees of statistical significance of the t-statistics: gray cells indicate a non-significant t-statistic, suggesting inability to reject the null hypothesis of the factor being weak; yellow, orange, and red cells, on the other hand, denote significance at the 10%, 5%, and 1% confidence levels, respectively, confirming evidence of some factor's strength over the analyzed time interval. From the figure, we observe distinct patterns among the Fama-French factors. While MKT and SMB always exhibit strength, HML, RMW, and CMA show more varied behavior over time, being characterized as weak in approximately 10%, 21%, and 29% of the time, respectively.

In the second step of our analysis, we seek to explore whether the observed time-varying strength of factors could also be linked to specific characteristics of the test assets. To delve deeper into this aspect, we conduct a subsequent exercise where we first group the assets based on their industry classification before repeating our analysis. The

results of this industry-specific investigation are displayed in Figure 4. In this figure, each heatmap corresponds to a different industry, utilizing the 12-industry classification system defined by Fama and French. From the analysis of the figure, a distinct pattern emerges across the various industries analyzed, with the percentage of weak factors in each industry quantified in Table 3. Each industry exhibits unique strengths in different factors, suggesting that the impact of each factor varies significantly across industries. While market and size factors almost always have a strong impact in each industry, there are other factors (especially RMW and CMA) that are found to be weak in a significant number of cases, particularly in industries 1, 5, and 7.

5.3 Testing for the weakness of the factor zoo

In this section, we implement our methodology to identify the presence of weak factors within the zoo of factors proposed by Feng, Giglio, and Xiu (2020). Our approach involves testing each factor in the zoo against a dynamically defined set of strong factors, the latter being determined by the performance of the Fama-French five factors, as identified in the preceding section. Specifically, at each point in time, we establish the set of strong factors - which may range from a null model to the full Fama-French five-factor model - by selecting only the Fama-French factors that have been found to be non-weak in the previous exercise. Upon defining this set, we then systematically apply our test of weakness to the remaining factors in the zoo, assessing each one individually. This process allows us to evaluate the degree of strength of each factor within the asset pricing model under consideration, ensuring a thorough and time-sensitive analysis of factor strength.

The main findings of our analysis are summarized in the heatmap of Figure 5. This figure illustrates the t-statistics associated with the test of weakness, calculated for each individual factor (along the vertical axis) and for each month (on the horizontal axis). As before, the heatmap visually represents the degree of statistical significance of these t-statistics: gray identifies non-significant t-statistics - indicating inability to reject the null hypothesis of the factor being weak; yellow denotes significance at the 10% level, orange identifies significance at the 5% level, and red indicates a high level of significance at the 1% level.

Interestingly, the analysis of the factors themselves does not show any significant

pattern in strength, in the sense that, at first glance, no single factor seems to consistently outperform others. Indeed, on average, each factor is found to be weak around 34% of the time, with a median and third quartile occurrence of 34.9% and 38%, respectively, suggesting a uniform pattern of the weakness of factors across the zoo.[28] This uniform pattern persists even when the 145 factors are grouped into six different categories (namely, Momentum, Value versus Growth, Investment, Profitability, Intangibles and Trading Frictions).[29] The first row of Table 4, which reports the average percentage of occurrences of weak factors, confirms this tendency, showing that the proportion of time each category is identified as weak remains roughly similar across categories.

A distinct structure appears to emerge, instead, when considering the distribution of the t-statistics over time. Indeed, many of the red and gray data points in Figure 5 seem to cluster within specific time intervals. This suggests that there are periods where certain factors either become particularly strong (weak), or they lose their strength (weakness) altogether. The presence of such clusters over time indicates the potential to identify temporal dynamics affecting the strength or weakness of many factors, highlighting the importance of considering time-varying elements in the asset pricing analyses.

To better understand this temporal pattern, we present in Figure 6 the percentage of factors (denoted by the vertical black bars) that have been identified as weak in every given time periods, together with the NBER periods of business cycle recessions (represented by a gray band). From the figure, an interesting negative relationship becomes evident between periods of factor weakness and economic recessions. Specifically, during times of crisis, the percentage of weak factors decreases substantially, suggesting that factors which are typically weak may gain importance or exhibit increased strength during turbulent periods.

To statistically assess the impact of this negative correlation, we employ a panel regression model with factor-specific fixed effects, that is: $\text{weak}_{jt} = \alpha_j + \beta \text{crisis}_{jt} + \epsilon_{jt}$. The dependent variable is the *weak* dummy, assigned a value of 1 when factor j is deemed weak (according to our test) at time t and 0 otherwise. The independent variable is a *crisis* dummy, set to 1 during time periods t identified as financial crises by the NBER,

[28] Specific statistics relative to each factor are reported in Table XX in the Appendix.

[29] classification of Hou, Xue, and Zhang (2020) - list of variables in Online Appendix

and 0 in all other periods. The outcomes of this regression analysis are presented in Panel (1) of Table 5. The results confirm a strongly significant negative relationship between recession periods and the probability of a factor being weak. Specifically, being in a crisis period reduces the probability of a factor being identified as weak by almost 11%.

To further investigate the temporal pattern of weak factors and how markets may adapt to these factors once they become publicly known (see McLean and Pontiff (2016)), we also include the year of publication (YoP) of each factor as a key explanatory variable in our regression analysis. Specifically, we define YoP as a post-publication dummy, being equal to one if factor j at time t is after the year of publication and zero if t is before. The influence of the year of publication is quantified in Panels 2 and 3 of Table 5, where we regress the weak dummy variable solely on the YoP dummy in Panel 2, and on both the crisis and YoP dummies in Panel 3. In both scenarios, the year of publication shows a negative and highly significant effect: once a factor becomes known to market participants, it decreases the likelihood of the factor being classified as weak by nearly 20%. This is in contrast with the results of McLean and Pontiff (2016), who argued that average returns of factors tend to be lower after their publication.

In line with the findings of the previous section, we also investigate whether the time-varying pattern in the factors' strength might also relate to specific attributes of test assets. Therefore, in the following we undertake a follow-up exercise in which we initially categorize the assets according to their industry classification before repeating the analysis.[30] The main results are depicted in Figure 7, which reports the 12 industry-specific heatmaps of the t-statistics resulting from the test of weakness. Complementing this visual representation, Table 4 provides a summary of the percentage of weak factors, grouped by factor categories.[31] The results reveal a significant heterogeneity in factor strength across the twelve industries. Interestingly, industries related to energy (Industry 4), utilities (Industry 8), and finance (Industry 11) emerge as sectors where all factors strongly contribute to pricing the cross-section of asset returns. This is evidenced by their relatively low percentage of weak factors, as shown in Table 4, displaying an overall average percentage of weak factors of approximately 25%.

[30] In this case, the dynamic set of strong factors, specific for each industry, is established by selecting the Fama-French factors which are identified as non-weak in a given industry by the exercise of the previous section.

[31] Statistics relative to each specific factor are reported in Table XX in the Appendix.

Conversely, industries related to durable and non-durable goods (Industries 1 and 2), chemicals (Industry 5), and telecommunications (Industry 7) exhibit the highest percentages of weak factors, averaging around 70%, with Industry 5 reaching a peak of 76%.

Despite the clear heterogeneity in factor strength across different sectors, a more homogeneous picture emerges when examining factors strength within the same industry. Indeed, as shown in Table 4, none of the factors categories distinctly outperforms the others in terms of factor strength, when focusing on a given industry. Rather, all categories appear to contribute equally on average.

However, what becomes evident once more is a temporal pattern, in which almost all factors become stronger or weaker in specific periods of time. Building on this insight and following the previous analysis presented in Figure 6, we further explore this relationship in Figure 8, where we plot the percentage of weak factors in each industry and for every time period, together with the NBER periods of business cycle recessions. In line with our previous findings, the figure reveals a predominantly negative relationship in most industries, indicating that the weakness of factors tends to inversely correlate with economic downturns.

To statistically quantify this negative relationship and to further investigate the impact of the publication year on factor strength, we present the main regression results in Table 6, which essentially replicates the analysis of Table 5 within each industry. The results confirm that being in a recession period significantly reduces the probability of a factor being weak in most industries, with effects ranging from a minimum of 3.2% (in Industry 9) to 14% (in Industry 2). The only exceptions are Industries 5, 8, and 10. The year of publication also exhibits a negative effect on factor strength across all industries, with the average impact varying from 1.3% (in Industry 9) to nearly 20% (in Industry 5). Interestingly, Industry 8 (Utilities) is the only sector where we identify a positive effect of both recession and year of publication on the probability of the factor being weak.

To summarize, the empirical analysis of this section highlights the importance of considering time-varying dynamics in asset pricing and confirms the dynamic nature of factor strength, significantly influenced by assets specific components, as well as by economic cycles and the evolution of market knowledge.

6 Conclusion

This paper provides a guidance to the correct asset pricing model by providing a testing framework for spurious factors. Another crucial dimension of mispsecification is *missing* factors. Some recent papers including Anatolyev and Mikusheva (2022) and Giglio and Xiu (2021) make a substantial contribution to the literature by addressing the issue of missing factors. Still, the methods proposed in those papers are not directly applicable to the data structure of our interest, a large number of asset returns over a short horizon. We leave this issue for future research.

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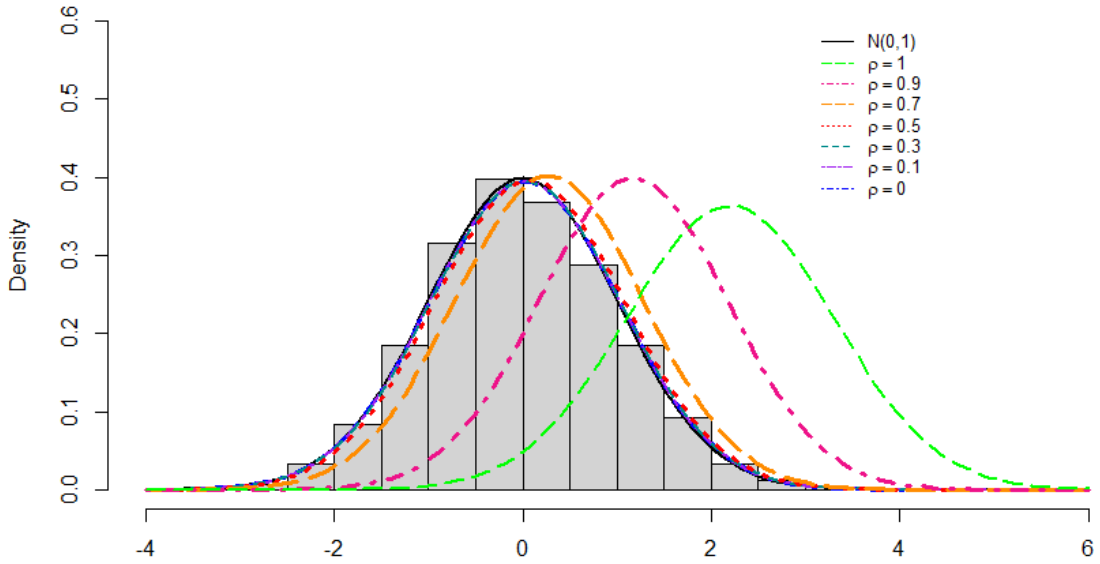


Figure 1: Distribution of simulated t-statistics for $\hat{\bar{\delta}}_g$. The figure plots the distribution of the 1,000 simulated t-statistics of $\hat{\bar{\delta}}_g$, as derived in part (iii) of Theorem 2, for the case of T=24 and N=3,000, with ρ ranging from 0 to 1. Each simulation uses the data generating process defined in (41).

	N=100			N=500			N=1000			N=3000		
	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
	Panel A: T=12											
ρ=0	0.112	0.059	0.011	0.109	0.057	0.012	0.103	0.053	0.009	0.103	0.054	0.015
ρ=0.1	0.114	0.063	0.012	0.108	0.058	0.011	0.102	0.054	0.007	0.105	0.053	0.015
ρ=0.3	0.109	0.062	0.013	0.109	0.058	0.012	0.102	0.055	0.007	0.104	0.052	0.016
ρ=0.5	0.111	0.057	0.014	0.111	0.060	0.012	0.105	0.056	0.007	0.103	0.051	0.015
ρ=0.7	0.113	0.054	0.014	0.117	0.055	0.010	0.107	0.051	0.007	0.115	0.061	0.017
ρ=0.9	0.107	0.054	0.013	0.118	0.061	0.018	0.128	0.077	0.018	0.191	0.122	0.046
ρ=1	0.103	0.056	0.012	0.140	0.077	0.018	0.202	0.133	0.038	0.389	0.276	0.131
	Panel B: T=24											
ρ=0	0.115	0.053	0.014	0.096	0.045	0.011	0.116	0.059	0.009	0.099	0.054	0.013
ρ=0.1	0.113	0.050	0.014	0.095	0.049	0.010	0.113	0.056	0.009	0.100	0.054	0.013
ρ=0.3	0.106	0.051	0.014	0.097	0.048	0.010	0.112	0.056	0.009	0.100	0.054	0.013
ρ=0.5	0.104	0.048	0.012	0.098	0.047	0.010	0.112	0.057	0.007	0.096	0.055	0.013
ρ=0.7	0.088	0.047	0.013	0.082	0.043	0.009	0.126	0.055	0.009	0.110	0.049	0.012
ρ=0.9	0.091	0.042	0.013	0.125	0.061	0.010	0.192	0.121	0.035	0.327	0.215	0.075
ρ=1	0.096	0.043	0.010	0.198	0.116	0.031	0.345	0.248	0.095	0.703	0.590	0.377
	Panel C: T=36											
ρ=0	0.094	0.044	0.011	0.113	0.054	0.013	0.096	0.053	0.010	0.093	0.049	0.009
ρ=0.1	0.097	0.035	0.009	0.110	0.056	0.013	0.096	0.053	0.011	0.091	0.047	0.009
ρ=0.3	0.092	0.035	0.008	0.108	0.053	0.013	0.096	0.050	0.012	0.096	0.046	0.010
ρ=0.5	0.080	0.037	0.007	0.103	0.053	0.011	0.095	0.048	0.012	0.094	0.047	0.011
ρ=0.7	0.073	0.036	0.004	0.109	0.053	0.012	0.107	0.058	0.010	0.125	0.061	0.014
ρ=0.9	0.084	0.036	0.003	0.181	0.101	0.027	0.255	0.155	0.051	0.526	0.406	0.182
ρ=1	0.094	0.036	0.001	0.316	0.196	0.058	0.548	0.412	0.157	0.918	0.856	0.676

Table 1: Empirical rejection Rates of $\hat{\bar{\delta}}_{\mathrm{g}}$. The table presents the empirical rejection rates of t-tests for $\hat{\bar{\delta}}_{\mathrm{g}}$, as derived from part (iii) of Theorem 2, over 1,000 Monte Carlo replications. The null hypothesis is that $\rho < 1/2$, against the alternative that $\rho > 1/2$. The results are displayed for various significance levels (10%, 5%, and 1%) and for different numbers of time-series and cross-sectional observations, spanning different degrees of the factor g_t's strength, from a weak factor case ($\rho = 0$) to a strong factor one ($\rho = 1$).The rejection rates are compared with the critical values from a standard normal distribution. For the derivation of the t-statistic, the parameters s_2, s_4 and m_4 have been estimated using the estimators provided in Theorem 5. Each simulation uses the data generating process defined in (41).

	N=100			N=500			N=1000			N=3000		
	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
	Panel A: T=12											
ρ=0	0.109	0.057	0.012	0.096	0.042	0.011	0.102	0.045	0.014	0.094	0.041	0.008
ρ=0.1	0.877	0.819	0.655	0.921	0.876	0.747	0.922	0.883	0.753	0.940	0.906	0.809
ρ=0.3	0.985	0.976	0.939	0.997	0.996	0.991	0.999	0.997	0.988	0.999	0.999	0.999
ρ=0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=0.7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=0.9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Panel B: T=24											
ρ=0	0.111	0.059	0.013	0.102	0.048	0.007	0.095	0.052	0.011	0.096	0.048	0.009
ρ=0.1	0.998	0.988	0.956	0.998	0.994	0.975	0.999	0.997	0.984	0.999	0.998	0.989
ρ=0.3	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=0.7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=0.9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Panel C: T=36											
ρ=0	0.145	0.077	0.012	0.109	0.055	0.012	0.112	0.053	0.013	0.101	0.056	0.008
ρ=0.1	1.000	0.999	0.993	1.000	1.000	0.998	1.000	1.000	0.999	1.000	1.000	1.000
ρ=0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=0.7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=0.9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
ρ=1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 2: Empirical rejection rates of the test of factor's weakness. The table shows the in-sample statistical performance of the test of weakness presented in Theorem 4, over 1,000 Monte Carlo replications. The null hypothesis is that $\rho = 0$ (weak factor) against the alternative that $\rho > 0$. The empirical rejection rates of the t-test are obtained using the estimated quantities proposed in Theorem 5. The performance of the test is analyzed under different numbers of time-series and cross-sectional observations, spanning a range of the factor g_t's strengths from 0 to 1. The rejection rates are compared with the critical values from a standard normal distribution. Each simulation uses the data generating process defined in (41).

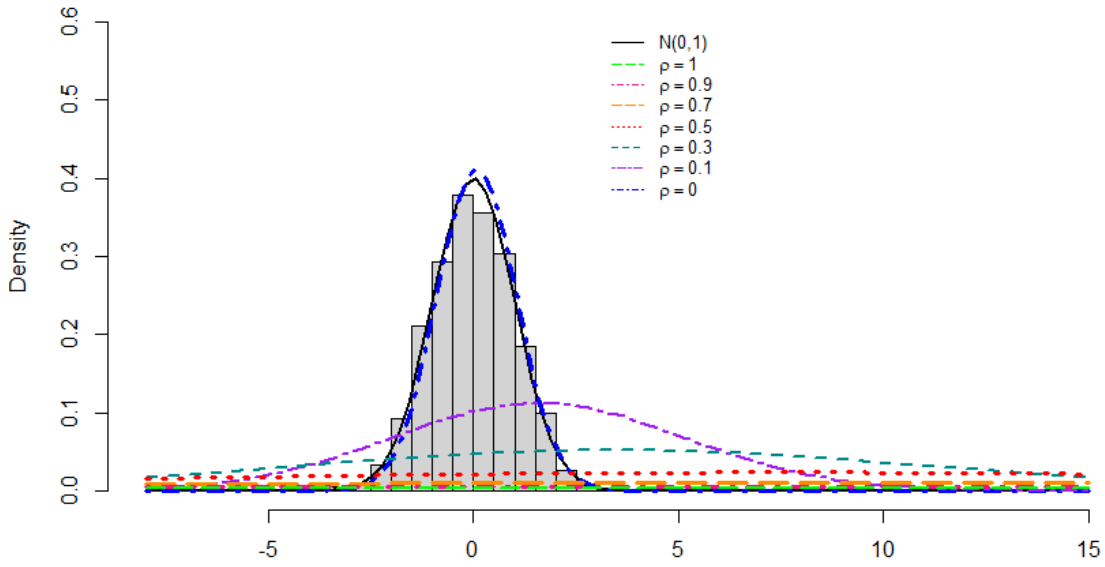


Figure 2: Distribution of test statistic of factor weakness. The figure plots the distribution of the 1,000 simulated t-statistics of the test on factor weakness, as derived in part Theorem 4, for the case of T=24 and N=3,000, with ρ ranging from 0 to 1. Each simulation uses the data generating process defined in (41).

	MKT	SMB	HML	RMW	CMA	N (average)
Ind 1 – NoDur	0.00	0.00	26.51	51.96	41.72	280
Ind 2 – Durbl	0.00	0.15	21.84	58.28	60.69	123
Ind 3 – Manuf	0.00	0.00	15.66	46.99	44.88	519
Ind 4 – Enrgy	0.90	12.80	17.02	18.52	12.50	208
Ind 5 – Chems	0.00	4.37	32.68	75.30	60.99	100
Ind 6 – BusEq	0.00	0.00	14.76	33.43	30.72	591
Ind 7 – Telcm	0.00	13.55	40.06	60.84	53.92	101
Ind 8 – Utils	1.05	20.18	8.73	30.72	14.61	153
Ind 9 – Shops	0.00	0.00	15.06	43.07	37.95	409
Ind 10 – Hlth	0.00	1.81	19.88	29.52	47.14	301
Ind 11– Money	0.00	4.37	8.58	28.16	18.83	1,452
Ind 12 – Other	0.00	0.00	18.52	32.83	32.08	721
All	0.00	0.00	9.94	21.08	28.46	2,948

Table 3: Weakness of Fama-French factors by industry (% values). The table reports the percentage of times in which each of the five Fama-French factors has been identified as weak in a particular industry, according to our test of weakness in Theorem 4. The assets have been first grouped using the 12-industry Fama-French classification. The results are then computed for each industry, using a rolling exercise of balanced panels over fixed-time windows of two years (i.e., T=24) and obtaining the time series of the test statistic by shifting the time window month by month over the period 1965-2022. The last row of the table (All) presents the aggregated results, where the analysis is repeated by pooling all the assets together. The last column reports also the number of assets that is available for the analysis, averaged across each rolling window of two years. To test for the weakness of the MKT factor, we assume a null set of strong factors F. In testing SMB and HML, MKT is used as the set of strong factors. For RMW and CMA, the set of strong factors is defined as F=$\{MKT, SMB, HML\}$.

	Momentum	Value vs Growth	Investment	Profitability	Intangibles	Trading Frictions
All	31.6	32.2	34.5	35.3	35.9	31.5
Ind 1 – NoDur	64.9	65.6	69.1	65.5	66.4	61.7
Ind 2 – Durbl	72.3	69.2	70.6	67.2	69.4	68.0
Ind 3 – Manuf	45.1	42.6	47.3	44.9	41.8	42.9
Ind 4 – Enrgy	26.9	23.8	26.1	23.5	24.2	26.9
Ind 5 – Chems	75.6	75.6	79.2	77.6	76.0	72.1
Ind 6 – BusEq	43.5	42.4	43.8	42.2	40.1	33.8
Ind 7 – Telcm	74.4	70.9	70.4	69.9	71.9	70.4
Ind 8 – Utils	31.7	27.4	33.4	3.5	28.3	33.5
Ind 9 – Shops	49.6	51.4	51.3	46.5	46.9	47.1
Ind 10 – Hlth	45.8	47.2	49.2	47.6	48.4	45.6
Ind 11– Money	24.7	27.1	28.4	26.7	29.2	23.1
Ind 12 – Other	39.3	41.1	43.0	40.8	42.1	36.9

Table 4: Weakness of the factor zoo by industry and factor category (% values). The table reports the percentage of times in which each factor of the Feng, Giglio, and Xiu (2020) zoo has been identified as weak in a particular industry, according to our test of weakness in Theorem 4. The assets have been first grouped using the 12-industry Fama-French classification. The results are then computed for each industry, using a rolling exercise of balanced panels over fixed-time windows of two years (i.e., T=24) and obtaining the time series of the test statistic by shifting the time window month by month over the period 1976-2017. The occurrence of weakness has been further averaged across the six categories (Momentum, Value vs Growth, Investment, Profitability, Intangibles, and Trading Frictions) proposed by Hou, Xue, and Zhang (2020) (see Appendix **??**). The first row of the table (All) presents the aggregated results, where the analysis is repeated by pooling all the assets together. To test for the weakness of the zoo, we establish the set of strong observed factors (F) at each point in time, by selectingonly the Fama-French factors that have been classified as non-weak in Section 5.2.

	(1)	(2)	(3)
Crisis	-0.106***		-0.119***
	(-19.69)		(-22.47)
YoP		-0.186***	-0.191***
		(-45.92)	(-47.21)

Table 5: Factor weakness, recessions and publication years: regression results. The table reports the estimation results of panel regressions with factor-specific fixed effects. The dependent variable is the *weak* dummy, which assigns a value of one when factor j is identified as weak at time t (according to our test in Theorem 4, and zero otherwise. We consider three different model specifications. Model (1) estimates $\text{weak}_{jt} = \alpha_j + \beta\text{crisis}_{jt} + \epsilon_{jt}$, where the only dependent variable is the *crisis* dummy, set to one if period t identifies an NBER recession period and zero otherwise. Model (2) considers the publication year of the factor (YoP) as explanatory variable, being equal to 1 if factor j at time t has been already published and zero if t is before the publication period. In Model (3), we consider the effect of both the *crisis* and YoP dummies. The t-statistic for each coefficient is reported in parenthesis. Asterisks denote statistical significance.

	Industry 1			Industry 2		
	(1)	(2)	(3)	(1)	(2)	(3)
Crisis	-0.106***		-0.119***	-0.137***		-0.140***
	(-19.69)		(-22.47)	(-26.44)		(-27.04)
YoP		-0.186***	-0.191***		-0.040***	-0.046***
		(-45.92)	(-47.21)		(-10.12)	(-11.59)
	Industry 3			Industry 4		
	(1)	(2)	(3)	(1)	(2)	(3)
Crisis	-0.113***		-0.122***	-0.054***		-0.061***
	(-20.16)		(-21.97)	(-11.04)		(-12.54)
YoP		-0.131***	-0.136***		-0.100***	-0.103***
		(-30.71)	(-31.94)		(-26.89)	(-27.55)
	Industry 5			Industry 6		
	(1)	(2)	(3)	(1)	(2)	(3)
Crisis	0.025***		0.011*	-0.063***		-0.065***
	(5.22)		(2.45)	(-11.38)		(-11.72)
YoP		-0.196***	-0.195***		-0.026***	-0.028***
		(-54.72)	(-54.52)		(-6.01)	(-6.62)
	Industry 7			Industry 8		
	(1)	(2)	(3)	(1)	(2)	(3)
Crisis	-0.046***		-0.055***	0.020***		0.022***
	(-8.91)		(-10.82)	(3.84)		(4.36)
YoP		-0.134***	-0.136***		0.038***	0.039***
		(-34.46)	(-35.01)		(9.77)	(9.98)
	Industry 9			Industry 10		
	(1)	(2)	(3)	(1)	(2)	(3)
Crisis	-0.032***		-0.033***	0.081***		0.077***
	(-5.71)		(-5.87)	(14.39)		(13.60)
YoP		-0.013**	-0.014***		-0.068***	-0.065***
		(-3.03)	(-3.34)		(-15.78)	(-15.06)
	Industry 11			Industry 12		
	(1)	(2)	(3)	(1)	(2)	(3)
Crisis	-0.035***		-0.045***	-0.076***		-0.084***
	(-6.80)		(-9.09)	(-13.71)		(-15.17)
YoP		-0.146***	-0.148***		-0.109***	-0.113***
		(-38.26)	(-38.74)		(-25.86)	(-26.67)

Table 6: Factor weakness, recessions and publication years: regression results by industry. The table reports the estimation results of panel regressions with factor-specific fixed effects for each industry. The dependent variable is the *weak* dummy, which assigns a value of one when factor j is identified as weak at time t (according to our test in Theorem 4, and zero otherwise. We consider three different model specifications. Model (1) estimates $\text{weak}_{jt} = \alpha_j + \beta \text{crisis}_{jt} + \epsilon_{jt}$, where the only dependent variable is the *crisis* dummy, set to one if period t identifies an NBER recession period and zero otherwise. Model (2) considers the publication year of the factor (YoP) as explanatory variable, being equal to 1 if factor j at time t has been already published and zero if t is before the publication period. In Model (3), we consider the effect of both the *crisis* and YoP dummies. The t-statistic for each coefficient is reported in parenthesis. Asterisks denote statistical significance.

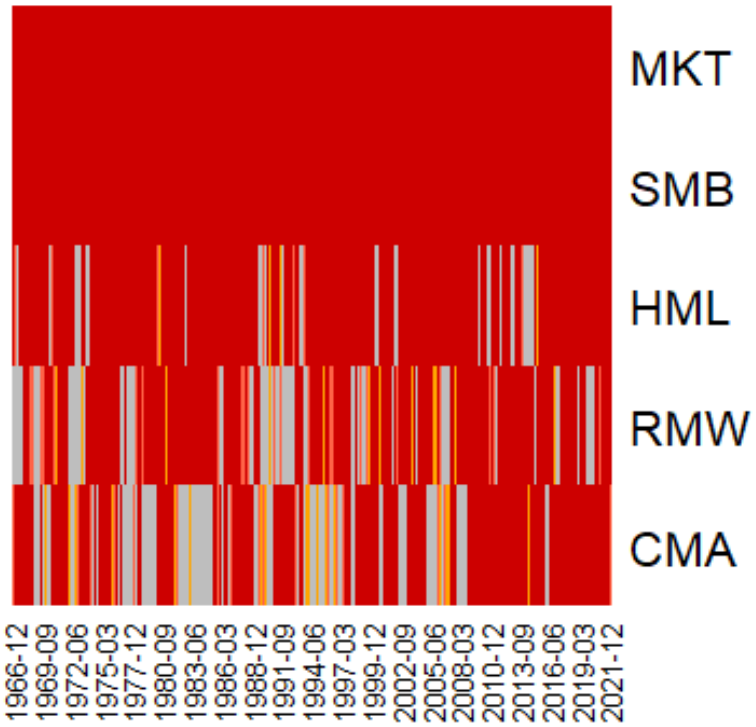
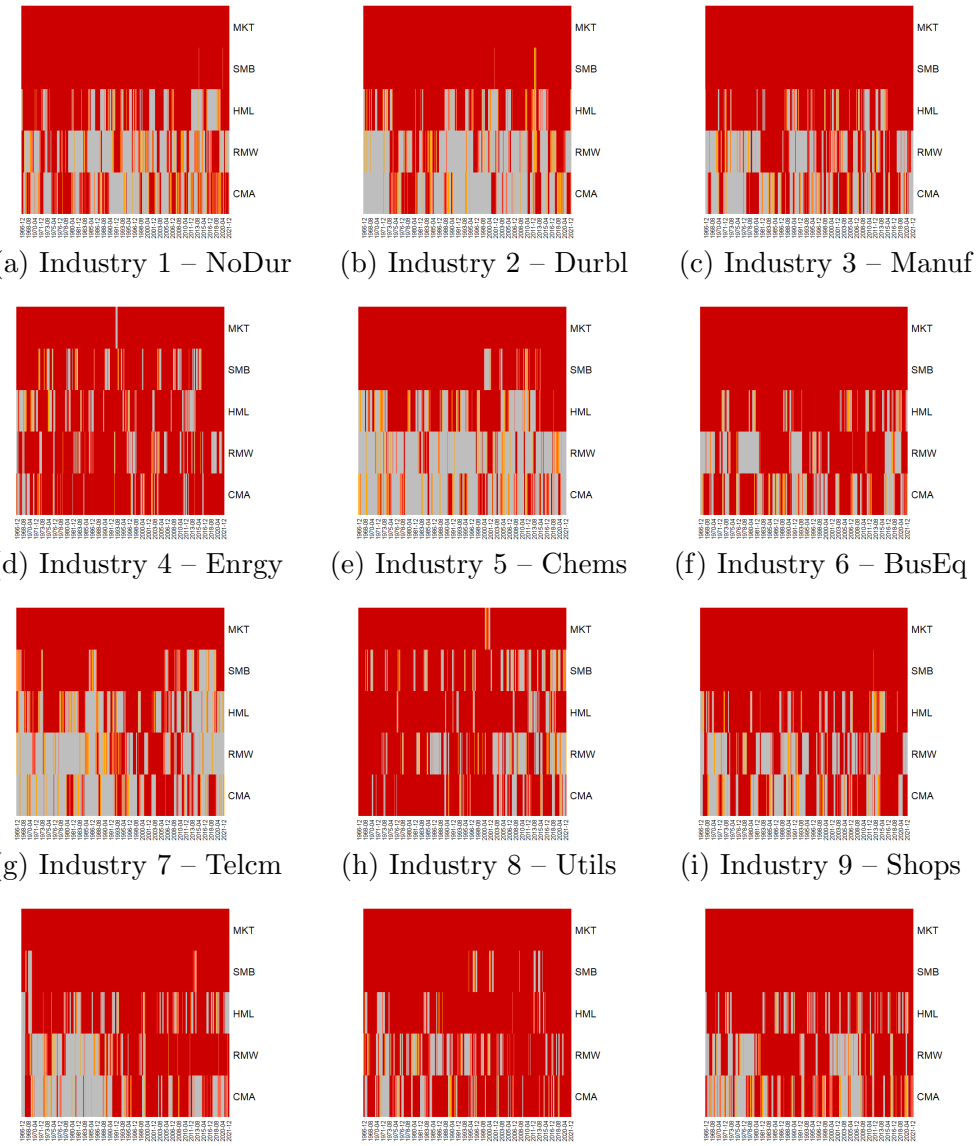


Figure 3: The strength of the five Fama-French factors over time. The heatmap shows the strength of the five Fama-French factors for each rolling sample of T=24 months, over the period 1965-2022. Each cell in the graph represents the t-statistic associated with the test of factor weakness presented in Theorem 4. The null hypothesis is that $\rho = 0$ (that is, the factor is weak), against the alternative that $\rho \neq 0$. The heatmap employs a color-coded scheme to identify the degree of statistical significance: gray cells denote a non-significant t-statistic, for which one can not reject the hypothesis of the factor being weak; yellow, orange and red, on the other hand, denote significance at 10%, 5% and 1% confidence levels, respectively. To test for the weakness of the MKT factor, we assume a null set of strong factors F. In testing SMB and HML, MKT is used as the set of strong factors. For RMW and CMA, the set of strong factors is defined as F=$\{MKT, SMB, HML\}$.



(a) Industry 1 – NoDur (b) Industry 2 – Durbl (c) Industry 3 – Manuf

(d) Industry 4 – Enrgy (e) Industry 5 – Chems (f) Industry 6 – BusEq

(g) Industry 7 – Telcm (h) Industry 8 – Utils (i) Industry 9 – Shops

(j) Industry 10 – Hlth (k) Industry 11 – Money (l) Industry 12 – Other

Figure 4: The strength of the five Fama-French factors over time and by industry. The twelve heatmaps in the figure show the industry-specific strength of the five Fama-French factors, for each rolling sample of T=24 months, over the period 1965-2022. The assets have been classified in each industry according their SIC code, following the 12-industry classification of Fama and French. Each cell in each heatmap represents the t-statistic associated with the test of factor weakness presented in Theorem 4. The null hypothesis is that $\rho = 0$ (that is, the factor is weak), against the alternative that $\rho \neq 0$. The color coding indicates the level of statistical significance to reject the null hypothesis of the factor being weak: gray for non-significant, yellow for 10% significance level, orange for 5%, and red for 1%. To test for the weakness of the MKT factor, we assume a null set of strong factors F. In testing SMB and HML, MKT is used as the set of strong factors. For RMW and CMA, the set of strong factors is defined as F=$\{MKT, SMB, HML\}$.

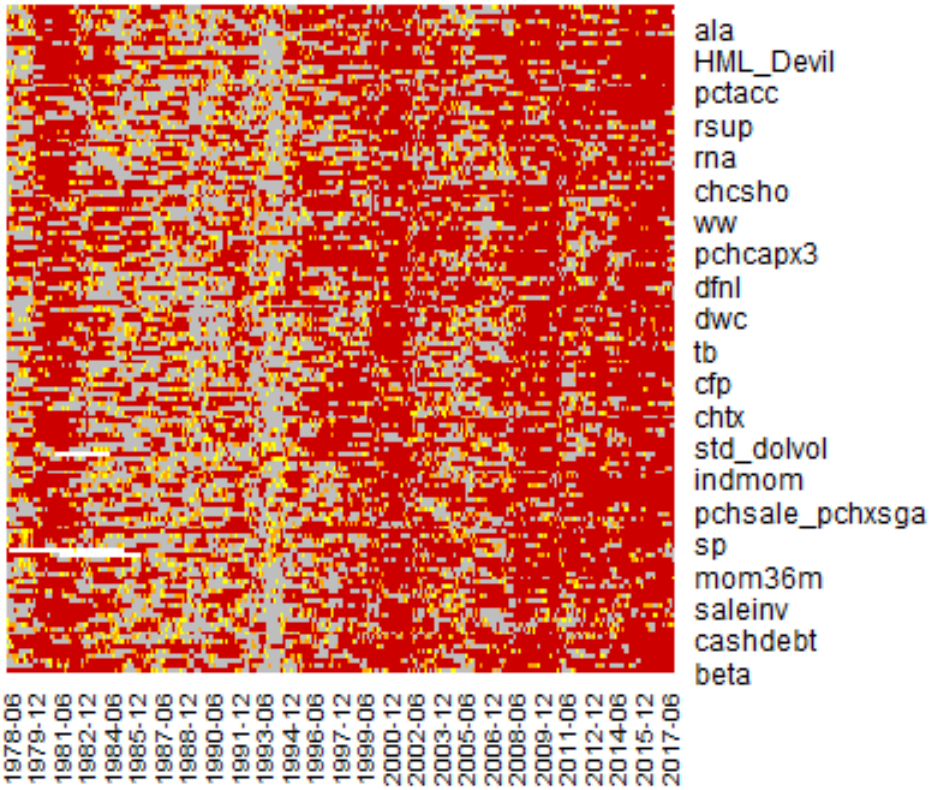


Figure 5: The strength of the factor zoo over time. This figure displays the t-statistics of the test of weakness for each factor in the zoo (vertical axis), and across different rolling samples of T=24 months, over the period 1976-2017. The color coding indicates the level of statistical significance to reject the null hypothesis of the factor being weak: gray for non-significant, yellow for 10% significance level, orange for 5%, and red for 1%. To test for the weakness of the zoo, we establish the set of strong observed factors (F) at each point in time, by selectingonly the Fama-French factors that have been classified as non-weak in Section 5.2. The full list of factors is reported in Appendix **??**.

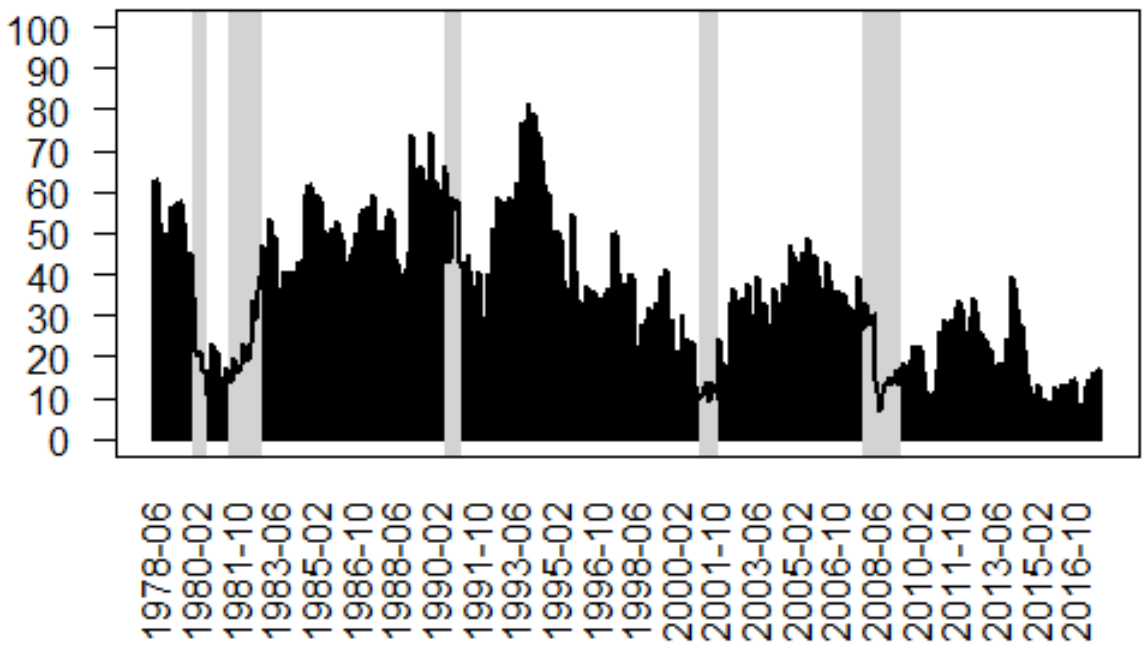
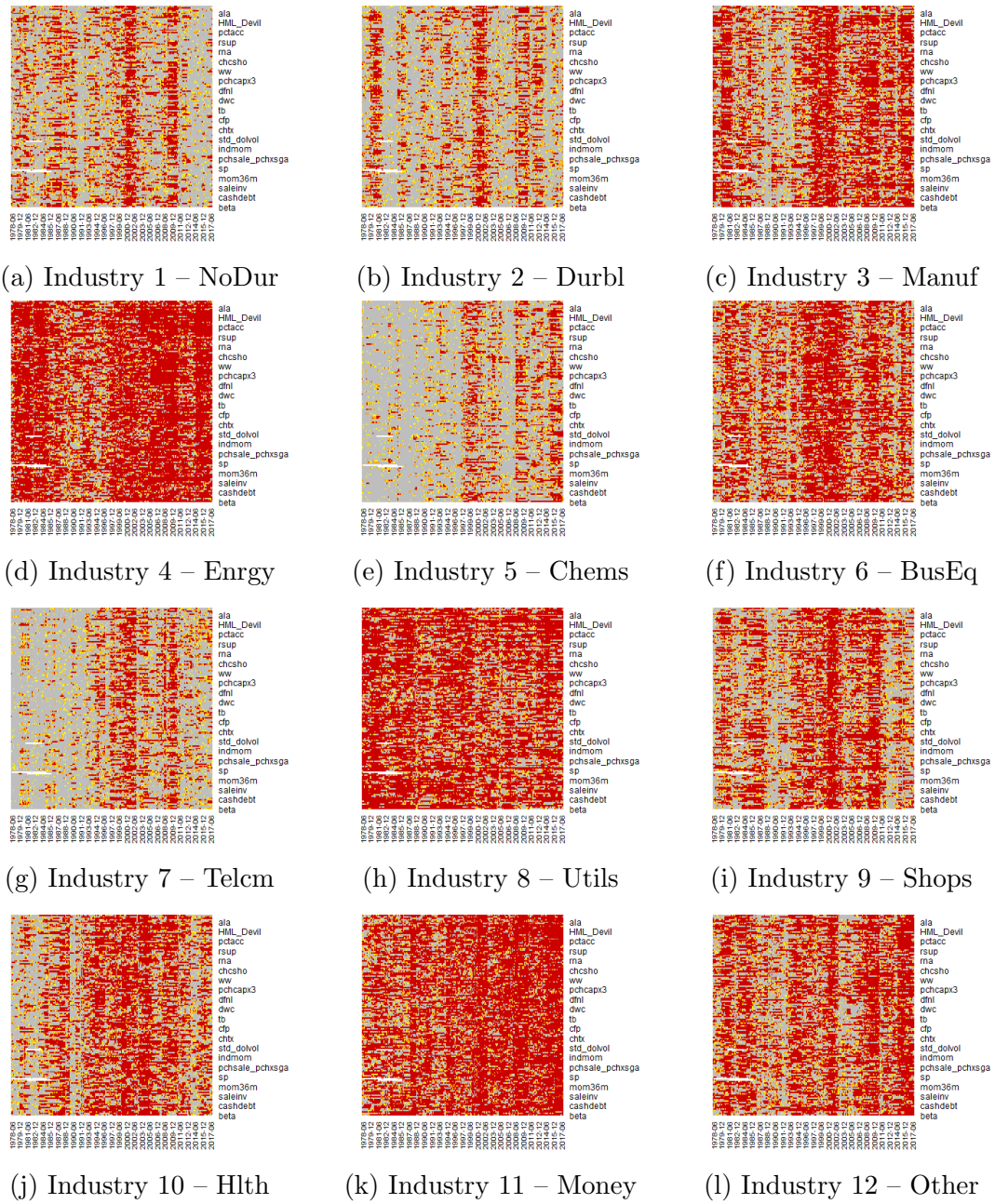


Figure 6: Weak factors and recession periods. The figure shows the percentage of factors (denoted by the vertical black bars) that have been identified as weak according to our test of weakness in Theorem 4, for every given rolling time periods. The gray bands in the figure depicts the main NBER periods of business cycle recessions. The analysis considers rolling samples of T=24 months, over the period 1976-2017.



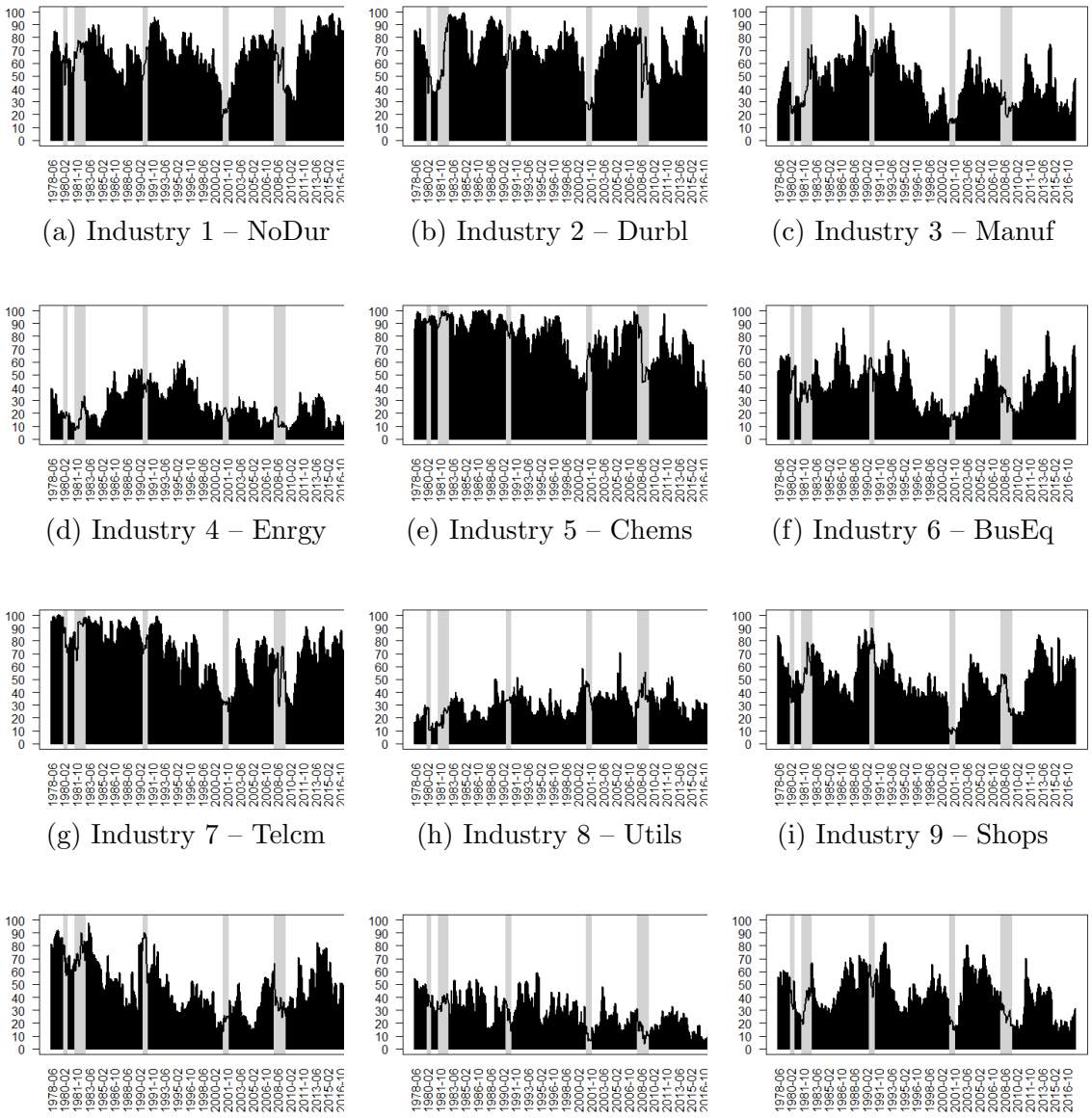
(a) Industry 1 – NoDur (b) Industry 2 – Durbl (c) Industry 3 – Manuf

(d) Industry 4 – Enrgy (e) Industry 5 – Chems (f) Industry 6 – BusEq

(g) Industry 7 – Telcm (h) Industry 8 – Utils (i) Industry 9 – Shops

(j) Industry 10 – Hlth (k) Industry 11 – Money (l) Industry 12 – Other

Figure 7: The strength of the factor zoo over time and by industry. Each heatmap in the figure displays the industry-specific t-statistics of the test of weakness for each factor in the zoo (vertical axis), and across different rolling samples of T=24 months, over the period 1976-2017. The assets have been classified in each industry according their SIC code, following the 12-industry classification of Fama and French. The color coding indicates the level of statistical significance to reject the null hypothesis of the factor being weak: gray for non-significant, yellow for 10% significance level, orange for 5%, and red for 1%. To test for the weakness of the zoo, we establish the set of strong observed factors (F) at each point in time, by selectingonly the Fama-French factors that have been classified as non-weak in Section 5.2. The full list of factors is reported in Appendix **??**.



(a) Industry 1 – NoDur (b) Industry 2 – Durbl (c) Industry 3 – Manuf

(d) Industry 4 – Enrgy (e) Industry 5 – Chems (f) Industry 6 – BusEq

(g) Industry 7 – Telcm (h) Industry 8 – Utils (i) Industry 9 – Shops

(j) Industry 10 – Hlth (k) Industry 11 – Money (l) Industry 12 – Other

Figure 8: Weak factors and recession periods by industry. Each sub-figure shows the percentage of factors (denoted by the vertical black bars) that have been identified as weak according to our test of weakness in Theorem 4, for every given rolling time periods and in each specific industry. The assets have been classified in each industry according their SIC code, following the 12-industry classification of Fama and French. The gray bands in the figure depicts the main NBER periods of business cycle recessions. The analysis considers rolling samples of T=24 months, over the period 1976-2017.

7 Appendix

why assumptions here? are also in OA.

7.1 General Assumption

Next, we introduce certain properties of residuals. We let $\mathbf{1}_{\{\cdot\}}$ denote the indicator function.

Assumption 3. *As $N \to \infty$, it holds that*
(i)

$$E\left[\epsilon_{it}\right] = 0, \ \frac{1}{N}\sum_{i=1}^{N} \epsilon_{it} \to_p 0,$$

where $t = 1, \cdots, T$,
(ii)

$$E\left[\epsilon_{is}\epsilon_{it}\right] = \sigma_i^2 \mathbf{1}_{\{s=t\}}, \ \frac{1}{N}\sum_{i=1}^{N}\left(\sigma_i^2 - s_2\right) = o\left(\frac{1}{\sqrt{N}}\right), \ \frac{1}{N}\sum_{i=1}^{N} \epsilon_{is}\epsilon_{it} \to_p s_2 \mathbf{1}_{\{s=t\}},$$

where $s, t = 1, \cdots, T$,
(iii)

$$E[\epsilon_{iq}\epsilon_{ir}\epsilon_{is}\epsilon_{it}] = m_{4i}\mathbf{1}_{\{q=r=s=t\}} + \sigma_i^4\left(\mathbf{1}_{\{(q=r,s=t)|(q=s,r=t)|(q=t,r=s)\}} - \mathbf{1}_{\{q=r=s=t\}}\right),$$

$$\frac{1}{N}\sum_{i=1}^{N}\sigma_i^4 \to_p s_4, \ \frac{1}{N}\sum_{i=1}^{N} m_{4i} \to_p m_4,$$

and

$$\frac{1}{N}\sum_{i=1}^{N}\epsilon_{iq}\epsilon_{ir}\epsilon_{is}\epsilon_{it} \to_p m_4\mathbf{1}_{\{q=r=s=t\}} + s_4\left(\mathbf{1}_{\{(q=r,s=t)|(q=s,r=t)|(q=t,r=s)\}} - \mathbf{1}_{\{q=r=s=t\}}\right),$$

where $q, r, s, t = 1, \cdots, T$.

Assumption 3(i) states the first moment and the cross-sectional average of residuals. Assumption 3(ii) specifies the second moments of residuals. The second condition limits the cross-sectional heterogeneity of the return conditional variance. The last condition describes the cross-sectional average of the product of residuals. Assumption 3(iii)

details out the fourth moments of residuals. The first and second limits show the cross-sectional average of squared variance and the kurtosis, respectively. The last property shows that the cross-sectional average of the products of four residuals goes to m_4 if all four residuals are identical, goes to s_4 if two distinct pairs are multiplied, and goes to zero otherwise.

7.2 Assumptions for No-Strong Factors

The following assumption explicitly articulates the rescaled limit of the summation of squared factor loadings on the weak factors and that of the interaction between the residuals and the factor loadings on the weak factors.

Assumption 4. *As N increases, it holds that*

$$N^{-\rho}\sum_{i=1}^{N}\boldsymbol{\beta}_{\mathrm{g},i}\boldsymbol{\beta}_{\mathrm{g},i}'\to_p \boldsymbol{\Sigma}_{\beta_{\mathrm{g}}},\ \ N^{\left(\frac{1-\rho}{2}\right)}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i\boldsymbol{\beta}_{\mathrm{g},i}'\to_p 0_{T\times L}.$$

Note that we do not enforce the positive definiteness of $\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}}$. The second limit suggests that the rescaled factor loadings on the weak factors are orthogonal to the residuals in a cross-sectional context.

Then, we introduce asymptotic distributions of primitives for the case without any strong factor. Let $U_\epsilon = \lim \frac{1}{N}\sum_{i=1}^{N} E\left[\mathrm{vec}\left(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'-\sigma_i^2\mathbb{I}_T\right)\mathrm{vec}\left(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'-\sigma_i^2\mathbb{I}_T\right)'\right]$, and $\otimes$ and $\mathrm{vec}(\cdot)$ denote the Kronecker product and vec operators, respectively.[32]

Assumption 5. *As N increases, it holds that*

$$\begin{bmatrix} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i \\ \frac{1}{\sqrt{N}}\sum_{i=1}^{N} vec\left(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'-\sigma_i^2\mathbb{I}_T\right) \\ N^{\left(\frac{1-\rho}{2}\right)}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} vec\left(\boldsymbol{\epsilon}_i\boldsymbol{\beta}_{\mathrm{g},i}'\right)\end{bmatrix}\to_d \mathcal{N}\left(0_{T+T^2+L},\begin{bmatrix} s_2\mathbb{I}_T & 0_{T\times T^2} & 0_{T\times LT} \\ 0_{T^2\times T} & U_\epsilon & 0_{T^2\times LT} \\ 0_{LT\times T} & 0_{LT\times T^2} & \left(s_2\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}}\right)\otimes\mathbb{I}_T\end{bmatrix}\right).$$

7.3 Further Assumption for Observed Strong-Factors Case

We introduce assumptions on LLN related to the factor loadings on the strong factors.

[32]The details on U_ϵ under Assumption 3 is given in Lemma 6.

Assumption 6. *As N increases,*
(i)

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\beta}_{\mathrm{f},i} \to_p \boldsymbol{\mu}_{\beta_{\mathrm{f}}},$$

(ii)

$$\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\beta}_{\mathrm{f},i}-\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right)\left(\boldsymbol{\beta}_{\mathrm{f},i}-\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right)' \to_p \boldsymbol{\Sigma}_{\beta_{\mathrm{f}}},$$

where $\boldsymbol{\Sigma}_{\beta_{\mathrm{f}}}$ is a $(K\times K)$ positive definite matrix,
(iii)

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i\left(\boldsymbol{\beta}_{\mathrm{f},i}-\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right)' \to_p 0_{T\times K}.$$

Assumption 6(i) gives the first cross-sectional moment of factor loadings on strong factors. In Assumption 6(ii), the cross-sectional variance of the factor loadings on the strong factors is defined. Finally, Assumption 6(iii) specifies that the residuals and factor-loadings on the strong factors are cross-sectionally orthogonal.

7.4 Further Assumptions for Latent Strong-Factors Case

We impose a mild regularity condition on $\mathcal{E}$, $\mathcal{B}$, $\mathcal{E}_{\mathcal{B}}$. Recall that $\mathcal{E}$ is a $(T^2\times N)$ matrix, $[\mathcal{E}_1\ \ldots\ \mathcal{E}_N]$, where $\mathcal{E}_i=\boldsymbol{\epsilon}_i\otimes\boldsymbol{\epsilon}_i$, that $\mathcal{B}$ is a $(N\times K^2)$ matrix, $[\mathcal{B}_1\ \ldots\ \mathcal{B}_N]'$, where $\mathcal{B}_i=\boldsymbol{\beta}_{\mathrm{f},i}\otimes\boldsymbol{\beta}_{\mathrm{f},i}$ and that $\mathcal{E}_{\mathcal{B}}$ is a $(TK\times N)$ matrix, $[\mathcal{E}_{\mathcal{B}1}\ \ldots\ \mathcal{E}_{\mathcal{B}N}]$, where $\mathcal{E}_{\mathcal{B}i}=\boldsymbol{\epsilon}_i\otimes\boldsymbol{\beta}_{\mathrm{f},i}$.

Assumption 7. *Let $\mathcal{A}=[\mathcal{E}'\ \mathcal{B}\ \mathcal{E}'_{\mathcal{B}}]$. Then, as N increases, $\frac{1}{N}\mathcal{A}'\mathcal{A}$ converges to a finite matrix.*

Next, we specify the cross-sectional independence between the factor loadings on the strong factors and those on the weak factors. It is important to note that this restriction on cross-sectional independence is a conventional practice in factor analysis, employed for identification purposes.

Assumption 8. *As N increases, it holds that*

$$N^{\left(\frac{1-\rho}{2}\right)}\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\beta}_{\mathrm{f},i}-\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right)\boldsymbol{\beta}'_{\mathrm{g},i} \to_p 0_{K\times L}.$$

Subsequently, we broaden Assumption 5 to encompass terms associated with factor loadings on the strong factors and weak factors. The ensuing assumption supersedes Assumption 5.

Assumption 9. *As N increases,*

$$\left[\begin{array}{c} \left[\begin{array}{c} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\boldsymbol{\beta}_{\mathrm{f},i}-\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right) \\ \frac{1}{\sqrt{N}}\sum_{i=1}^{N} vec\left(\boldsymbol{\epsilon}_i\left(\boldsymbol{\beta}_{\mathrm{f},i}-\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right)'\right) \\ \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i \\ \frac{1}{\sqrt{N}}\sum_{i=1}^{N} vec\left(\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i'-\sigma_i^2\mathbb{I}_T\right) \end{array}\right] \\ \left[\begin{array}{c} N^{\left(\frac{1-\rho}{2}\right)}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} vec\left(\left(\boldsymbol{\beta}_{\mathrm{f},i}-\boldsymbol{\mu}_{\beta_{\mathrm{f}}}\right)\boldsymbol{\beta}_{\mathrm{g},i}'\right) \\ N^{\left(\frac{1-\rho}{2}\right)}\frac{1}{\sqrt{N}}\sum_{i=1}^{N} vec\left(\boldsymbol{\epsilon}_i\boldsymbol{\beta}_{\mathrm{g},i}'\right) \end{array}\right] \end{array}\right] \rightarrow_d \mathcal{N}\left(0_{\mathcal{T}},\Omega^{+}\right),$$

where $\mathcal{T}=T+K+KT+T^2+LK+LT$,

$$\Omega=\left[\begin{array}{cccc} \boldsymbol{\Sigma}_{\beta_{\mathrm{f}}} & 0_{K\times KT} & 0_{K\times T} & 0_{K\times T^2} \\ 0_{KT\times K} & (s_2\boldsymbol{\Sigma}_{\beta_{\mathrm{f}}})\otimes\mathbb{I}_T & 0_{KT\times T} & 0_{KT\times T^2} \\ 0_{T\times K} & 0_{T\times KT} & s_2\mathbb{I}_T & 0_{T\times T^2} \\ 0_{T^2\times K} & 0_{T^2\times KT} & 0_{T^2\times T} & U_\epsilon \end{array}\right],$$

and

$$\Omega^{+}=\left[\begin{array}{ccc|cc} & \Omega & & 0_{K\times LK} & 0_{K\times LT} \\ & & & 0_{KT\times LK} & 0_{KT\times LT} \\ & & & 0_{(T+T^2)\times LK} & 0_{(T+T^2)\times LT} \\ \hline 0_{LK\times K} & 0_{LK\times KT} & 0_{LK\times(T+T^2)} & \boldsymbol{\Sigma}_{\beta_{\mathrm{g}}}\otimes\boldsymbol{\Sigma}_{\beta_{\mathrm{f}}} & 0_{LK\times LT} \\ 0_{LT\times K} & 0_{LT\times KT} & 0_{LT\times(T+T^2)} & 0_{LT\times L} & \left(s_2\boldsymbol{\Sigma}_{\beta_{\mathrm{g}}}\right)\otimes\mathbb{I}_T \end{array}\right].$$