Exploiting Dynamic Identification in VARs

Paul Beaudry, Fabrice Collard, Patrick Fève,
Alain Guay and Franck Portier*

May 2025

`version 6.4.2`

Abstract

Most macroeconomic models describe economic outcomes as the result of a combination of endogenous and exogenous dynamic forces, with the exogenous components typically modeled as restricted autoregressive processes. In this paper, we show if data is generated by this dual dynamic structure, then the full set of structural shocks and impulse responses can be identified. No additional identifying restrictions are needed. Building on this result, we demonstrate how it can be used to evaluate commonly used Structural VAR identification strategies, including impact restrictions, long-run restrictions, and proxy-VAR approaches.

Keywords: Structural VAR, Proxy-VAR, Dynamic Identification

1 Introduction

Macroeconomists are often interested in understanding how the economy responds to various types of shocks (see Ramey (2016) and Stock and Watson (2017) for comprehensive overviews of the recent literature). A dominant approach is the use of Structural Vector Autoregressive (SVAR) models.[1] The SVAR framework is built on two key premises. First, the macroeconomic model that generates the data features *(i)* some forcing processes (technology, oil price, monetary or fiscal policy, etc.) and *(ii)* a theory on how endogenous variables (consumption, employment, inflation, etc.) react to these forcing

*Beaudry: Vancouver School of Economics and NBER, paul.beaudry@ubc.ca, Collard: Toulouse School of Economics (CNRS, UTC) and CEPR, fabrice.collard@gmail.com, Fève: Toulouse School of Economics (UTC), patrick.feve@tse-fr.eu, Guay: Université du Québec à Montréal, CIREQ and CMP, guay.alain@uqam.ca, Portier: University College London and CEPR, f.portier@ucl.ac.uk. We thank participants at conferences and seminars for helpful comments and discussions. We are particularly thankful to Christian Gouriéroux for very insightful remarks and discussions. Collard and Fève acknowledge funding from the French National Research Agency (ANR) under the Investments for the Future (Investissements d'Avenir) program, grant ANR-17-EURE-0010 and from the French National Research Agency (ANR) under the project ANR-23-CE26-0011-01.

[1] Sims (1980) is a seminal paper in the SVAR litterature. See Section 4 in Stock and Watson (2016) and Kilian and Lütkepohl (2016) for a complete review.

processes. Second, the model solution can be represented (or well approximated) by a Vector Autoregressive (VAR) model.[2] Once the VAR specification is chosen (variables, order of differentiation and number of lags), estimation is straightforward and one can learn about the response of the macroeconomy to shocks by inspecting the implied Impulse Response Functions (IRFs). However, obtaining meaningful IRFs can only be achieved by imposing a set of identifying restrictions, allowing the VAR to be rewritten as a SVAR. As was put forward by Sims (1980), IRFs from a SVAR are potentially less prone to mis-specification than those obtained by estimating a fully specified macroeconomic model. However, mapping their implications into the language of theoretical models is not uncontroversial due to the reliance on specific identifying restrictions.

In this paper, we keep the two premises of the SVAR literature and exploit the fact that in macroeconomic models, forcing processes underlying exogenous driving forces are generally restricted. For example, in the vast majority of macroeconomic models, the underlying driving forces are assumed to be linearly independent autoregressive processes (most often of order one). Our main theoretical result is to show that the macroeconomic model IRFs can be recovered from a SVAR if one is willing to exploit these implicit restrictions on exogenous driving forces. Thanks to this result, most identifying restrictions commonly used in the SVAR literature are not needed for identification, and can be viewed as over-identifying restrictions. They can therefore be visually evaluated and formally tested. For instance, this will allows for the evaluation of the variety of identifying restrictions imposed in SVARs such as short-run, long-run, proxy-variables, sign restrictions, etc.

Exploiting Dynamic Restrictions. As noted above, a key contribution of this paper is to show that the restrictions commonly imposed on the forcing processes in most macroeconomic model are typically sufficient to identify the IRFs of the model, without requiring any additional assumption(s). Because our approach explicitly leverages restrictions on the dynamic structure of these forcing processes, we refer to the resulting SVAR as a "D-SVAR." By identification of model IRFs, we mean that, in general, there exists a unique[3] vector of mutually orthogonal innovations (or shocks) within the D-SVAR that satisfies the restrictions imposed by the dynamic structure of the macroeconomic model. Traditional SVAR identification strategies usually rely on imposing restrictions on the loading matrix of shocks while ensuring compatibility with the covariance matrix of the VAR innovations. In contrast, our approach focuses on exploiting restrictions tied to the dynamic propagation of shocks, rather than only focusing on the structure of their

[2]We use the generic term VAR, but our identification result equally applies to Vector Error Correction Models (VECM) and Vector Autoregressive Moving Average (VARMA) models.

[3]Uniqueness is defined up to the sign and/or permutation of the shocks.

contemporaneous impact. In such a setting, solely constraining the covariance matrix is no longer sufficient to ensure identification of the structural shocks. To derive sufficient conditions for local identification, we need to fully leverage the autocovariance structure of the data across different lags, as implied by its dynamic representation—whether it takes the form of a VAR, VARMA, or state-space model[4]—or, equivalently, make use of the information contained in the spectral density function.[5] This theoretical result will be formally established in Section 4. Under certain configurations of the identifying restrictions imposed on the exogenous driving forces, the D-SVAR model becomes overidentified, thereby permitting the use of a formal test to assess the validity of the overidentifying restrictions within the structural framework. For clarity of exposition—and because the applied macroeconomic literature predominantly focuses on VARs as opposed to VARMAs—we will mainly focus on the VAR case in the main text and will accordingly maintain the D-SVAR terminology throughout.

Testing Commonly Used SVAR Restrictions. It is important to emphasize that the shocks identified by our D-SVAR are initially unlabelled, as our identification strategy does not assign explicit economic interpretations—such as technology, monetary, or fiscal—to the structural shocks recovered. While standard SVAR approaches impose additional restrictions to label shocks —such as impact restrictions (Christiano et al. (1999)), long-run restrictions (Blanchard and Quah (1989)), or sign restrictions (Uhlig (2005))—our method instead provides a framework for testing the validity of these very restrictions. We view this as a strength rather than a limitation: shock identification in our framework does not rely on prior labelling, which is not necessary for identification.

A substantial part of this paper is devoted to illustrating how our D-SVAR approach can be used to test common restrictions. To this end, we build on three well-known examples from the literature. First, we revisit the analysis by Blanchard and Quah (1989), which used long run restrictions to distinguishes between demand and supply shocks. Next, we examine the proxy VAR method proposed by Gertler and Karadi (2015) for identifying monetary policy shocks using a high-frequency instrument (see also Kuttner (2001), Gürkaynak et al. (2005), Bernanke and Kuttner (2005), and Gürkaynak et al. (2007) for early work on high-frequency identification of such shocks). Finally, we assess the impact restrictions employed in Christiano et al. (1999, 2005) to identify monetary policy shocks. In all three cases, we assume that the data is generated by a structural

[4]Section D of the online appendix covers these cases which typically arise when some endogenous state variables are unobserved to the econometrician.

[5]Other statistical structural identification methods in VAR models have been proposed based on non-Gaussianity, captured for example by third– or fourth–order moments (see *e.g* Lanne et al., 2017; Gouriéroux et al., 2017; Guay, 2021). In contrast, in this paper, we rely solely on second-order moments, as characterized by the autocovariance structure of the data.

model where the forcing variables are linearly independent AR(1) processes. We show this results in the D-SVAR being overidentified and that this initial overidentification is not rejected by the data. We find that the identifying restrictions used in Blanchard and Quah (1989) are also not rejected by the data. In contrast, the restrictions underlying the proxy VAR approach of Gertler and Karadi (2015) are not supported as the instrument is likely correlated with at least two of the shocks identified be the D-SVAR. For the Christiano et al. (1999) analysis, the identifying restrictions cannot be formally rejected but visual inspection suggest that some doubt regarding its validity may remain.

Related Literature. Our identification result relates to several papers including, among others, McGrattan (2010), Pagan and Robinson (2019), Bai and Wang (2015) and Gourieroux and Jasiak (2022). While several of our theoretical results have precedents in the literature, our main contribution is to show why standard SVAR restrictions can be seen as over-identifying restrictions and therefore can be tested. McGrattan (2010) derives conditions for identification of an unrestricted state-space representation associated with a specific small-scale Real Business Cycle model.[6] In particular, the paper shows that, when the model features a permanent technology shock and a stationary labor wedge shock (in the form of a labor income tax), the unrestricted state-space representation is identified. In contrast, our paper establishes identification conditions for a broader class of macroeconomic models, including those whose solutions can be represented as VAR or VARMA processes. Pagan and Robinson (2019) note that SVARs may face difficulty to properly uncover the loading matrix of macroeconomic models because standard estimation of SVAR models avoids imposing the type of statistical restrictions commonly used in macroeconomic models —*e.g.* that structural shocks follow mutually orthogonal univariate autoregressive processes. The two authors discuss necessary conditions for identification in SVARs when the autoregressive matrix is diagonal and shocks are normalized. Our formal analysis shows more generally the conditions on the autoregressive matrix that allows to identify the structural shocks. Bai and Wang (2015) study identification in dynamic factor models similar to our unrestricted state-space representation. Their approach, in line with the conventional practice in the VAR literature, imposes restrictions on the loading matrix while leaving unrestricted the autoregressive matrix of factors. In this paper, we take the opposite viewpoint and determine which type of organisation of the autoregressive matrix allows to freely identify the loading matrix in the state-space representation. Finally, Gourieroux and Jasiak (2022) provide conditions for identification in multivariate undetermined convoluted systems when the shocks (the "*sources*" in their terminology) are linearly independent autoregressive processes of order

[6] See also Kascha and Mertens (2009) for simulation experiments in a similar setup.

one and when there is no intrinsic dynamics of the endogenous variables. They show that when the autoregressive parameters are distinct, the loading matrix (the *"mixing matrix"* in their terminology) is identified. Our paper departs from theirs in at least three important ways. First, we consider a larger class of dynamic models and makes connections with the macroeconomic literature. Second, we extend the identification problem to non diagonal autoregressive processes. Third, we provide conditions for partial identification when the practitioner seeks to identify only a subset of structural shocks.

Outline. The paper is organized as follows. Section 2 derives the D-SVAR representation and provides a heuristic explanation for its identification. Section 3 demonstrates how the D-SVAR framework can be used to test common identification strategies in SVARs, first using simulated data generated from an estimated New Keynesian model, and then by revisiting three influential SVAR applications: Blanchard and Quah (1989), Gertler and Karadi (2015), and Christiano et al. (1999). Section 4 presents a formal proof of local identification. Section 5 discusses estimation and inference within the D-SVAR framework. The final section concludes. All technical proofs are provided in the appendix.

2 A Primer on D-SVARs

The main idea behind the SVAR literature is to use a VAR model—which can be straightforwardly estimated—and recover a set of economically meaningful impulse response functions by imposing identifying restrictions. A key assumption underlying this approach is that the macroeconomic structure generating the data can be represented by a vector of endogenous variables Y_t in the form of a VAR. We further assume that the data generating process includes both endogenous propagation mechanisms and exogenous driving forces. Under this structure, the data generating process (omitting constant terms) can be written as:

$$\begin{aligned} Y_t &= \sum_{i=1}^{m} G_i Y_{t-i} + F Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t. \end{aligned} \tag{1}$$

Here, Y_t is an $n \times 1$ vector of endogenous variables, and Z_t is an $n \times 1$ vector of exogenous structural driving forces, unobserved by the econometrician. The structural innovations ε_t are assumed to be mean-zero with an identity covariance matrix. The matrices G_i capture the internal propagation dynamics of the system, while R governs the dynamics of the exogenous forces. Finally, the matrix F, assumed to be full rank, loads the structural shocks into the endogenous part of the system. This type of solution arises in a variety of

macroeconomic frameworks, including standard optimization-based DSGE models with rational expectations, as well as models built on more behavioral foundations.

In the setup described by (1), we are focusing on the case where the exogenous driving forces follow first-order autoregressive processes. However, it is important to emphasize that our framework can be readily extended to accommodate higher-order dynamics. In fact, as we will show later, our approach applies more broadly to settings where the model solution can be expressed in state-space form, encompassing both VAR and VARMA representations.[7] However, to keep the exposition clear and aligned with much of the existing SVAR literature, we will—for this and the next few sections—focus on the case where the macroeconomic model solution is a VAR that can also be written in the form given by (1).

By eliminating Z_t in (1), we can derive the VAR representation of the macroeconomic model solution:

$$Y_t = \left(G_1 + FRF^{-1}\right) Y_{t-1} + \left(G_2 - FRF^{-1}G_1\right) Y_{t-2} + \cdots - FRF^{-1}G_m Y_{t-m-1} + F\varepsilon_t. \quad (2)$$

We dub representation (2) the D-SVAR representation. The model solution can also be written as:

$$Y_t = \Gamma_1 Y_{t-1} + \Gamma_2 Y_{t-2} + \cdots + \Gamma_{m+1} Y_{t-m-1} + \nu_t, \quad (3)$$

where ν_t is a vector of non-structural innovations with covariance matrix Σ_ν. The advantage of VAR representation (3) is that the Γ_i matrices and the Σ_ν matrix can be freely estimated from data. The mapping between the matrices in (3) and (2) is simply given by:

$$\begin{cases} \Gamma_1 &= G_1 + FRF^{-1}, \\ \Gamma_2 &= G_2 - FRF^{-1}G_1, \\ \Gamma_3 &= G_3 - FRF^{-1}G_2, \\ \vdots & \\ \Gamma_{m+1} &= -FRF^{-1}G_m, \\ \Sigma_\nu &= FF'. \end{cases} \quad (4)$$

Identifying the D-SVAR in equation (2) involves recovering the matrices $\{G_i\}_{i=1}^m$, F, and R from the autoregressive matrices $\{\Gamma_j\}_{j=1}^{m+1}$ and the covariance matrix, Σ_ν, of non-structural innovations. Once the D-SVAR representation is recovered from Γ_j and Σ_ν, one can recover structural model shocks, IRFs, and forecast error variance decompositions.

However, because Σ_ν is symmetric, this system provides only $(m+2)n^2 - \frac{n(n-1)}{2}$ independent equations to identify $(m+2)n^2$ unknowns. This leads to the classic under-identification problem in SVARs. Early work in the SVAR literature addressed this by

[7] In DSGE models with rational expectations, the solution may take the form of a VARMA rather than a VAR, depending on which variables are observable. Our approach also covers this case—see Section D of the online appendix for a formal treatment, and Section H for a simulation experiment using a Real Business Cycle model.

imposing a lower triangular structure on the loading matrix F, effectively restricting the contemporaneous impact of shocks (see Sims (1980)). This assumption imposes exactly $\frac{n(n-1)}{2}$ restrictions—just enough to achieve exact identification of the model shocks and IRFs. In practice, the loading matrix F derived from most structural macroeconomic models do not satisfy this lower triangular representation. This highlights the importance of being able to test such structural restrictions, rather than taking them as given. Similarly, approaches based on long-run restrictions, sign restrictions, or instrumental variables may not be uncontroversial in certain settings—making the ability to formally test these identifying assumptions particularly valuable.

Our D-SVAR approach does not rely on restricting the loading matrix F, nor does it depend on external instruments. Instead, it complements these approaches by leveraging the common maintained assumption inherent to most macroeconomic models that the exogenous driving forces are themselves subject to structural restrictions. This assumption amounts to placing restrictions on the autoregressive matrix R. Consider, for example, the case where the exogenous variables Z_t are mutually orthogonal at all leads and lags—implying that R is a diagonal matrix. This assumption is commonly shared in the macroeconomic literature, as it aligns with the idea that these forces should be meaningfully exogenous. Our main theoretical result, presented in Section 4, includes this case—though it is more general. It shows that if the structural driving forces in (1) are linearly independent, then the D-SVAR system in (2) allows us to recover the matrices $\{G_i\}_{i=1}^{m}$, F, and R from $\{\Gamma_j\}_{j=1}^{m+1}$ and Σ_ν. A simple count of equations and unknowns in (2) when R is diagonal suggests that the order condition for identification is satisfied. Obviously, this order condition is only a necessary condition for identification. The formal proof in Section 4 shows that if R is diagonal (with distinct diagonal elements) is sufficient for local identification of the D-SVAR. While the diagonality of R is not required for identification, it is a common and convenient assumption that we will adopt in the next sections. Moreover, when R is diagonal, the D-SVAR becomes overidentified, allowing for a formal test of overidentifying restrictions (see Section 5.2). To avoid any confusion, let us emphasize once again that the identification result in Section 4 applies to a broader class of models than the specific structure given in (1). Nevertheless, for many practical applications, focusing on the structure in (1) will be more than sufficient.

From a practical standpoint, the main reason we view recovering the D-SVAR representation as valuable is that it enables the empirical testing of identification restrictions that are often imposed in the SVAR literature. These include impact restrictions, long-run restrictions, sign restrictions, and approaches based on instrumental variables, among others. On its own, the D-SVAR representation is not especially useful, as it does not assign economic labels to the shocks. Instead, its value lies in how it interacts with

standard identification schemes as it provides a framework for testing these schemes and, when the restrictions are not rejected by the data, for labeling the shocks accordingly. This process is illustrated in Section 3. Throughout these applications, we maintain a core assumption shared by much of the SVAR literature—namely, that the underlying VAR model from which we seek to extract structural information is correctly specified in terms of both the number of variables and the lag length.[8]

3 Assessing Identification Strategies with D-SVAR

In this section, we illustrate how the mapping defined in (4) can be used to derive the D-SVAR representation and, in turn, can be used to test common identification strategies employed to recover structural IRFs from a VAR. As noted earlier, our starting point assumes that the VAR model represents the solution of an underlying macroeconomic model where the number of variables in the VAR is equal to the number of exogenous driving forces.[9] Based on that, they are willing to impose identification restrictions that allow for the extraction of a set of structural impulse responses of interest.[10]

We begin with a simulated example based on a recursive or a proxy VAR setup. Starting with simulated data is useful for two reasons: it allows us to show *(i)* how effectively our D-SVAR framework can recover structural shocks in sample sizes comparable to those commonly used in the literature, and *(ii)* how the method distinguishes between a "good" and a "bad" instrument in such settings. We also show how our approach can be applied to evaluate Cholesky-type (recursive) identification restrictions. In the following subsection, we apply our framework to three well-known SVAR examples from the literature.

[8]Our D-SVAR approach can also be useful when starting from a fully specified macroeconomic model, but before estimating it and imposing all the models cross equation restrictions. Once the model is fully specified —and before it is estimated— it will generally have a VAR or VARMA solution, from which the D-SVAR representation can be obtained. The researcher immediately obtains an estimate of all the model impulse responses, before any cross equation restrictions are exploited. This set of impulse responses can then be compared with that obtained by full-information estimation imposing all the cross-equation restrictions. If the two sets of results are in line, this gives support for the model. However, if they are significantly different, it places into question the validity of the model.

[9]In this illustrative section, when we refer to an underlying macroeconomic model, we have in mind one whose solution can be expressed in state-space form, with exogenous driving forces that are linearly independent processes. In cases where the model admits a VAR representation, this corresponds to a structural model of the form given in (1), assuming the exogenous forces follow a first-order process.

[10]The case in which the solution to the macroeconomic model is a VARMA—often encountered when the data are generated by a DSGE model with unobserved state variables—is addressed in Section 4, where we present the general identification results.

3.1 A simulated data illustration of the use and performance of a D-VAR

To illustrate how our D-SVAR framework can we used to test common identifying restrictions, in this section we generated data from an estimated three-equation New Keynesian model featuring habit persistence, a hybrid Phillips curve and a Taylor type rule with monetary policy inertia.[11] This model has a solution given by:

$$\begin{aligned} Y_t &= G_1 Y_{t-1} + F Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t, \end{aligned} \tag{5}$$

where both Y_t and ε_t are three-dimensional vectors. Here, Y_t comprises the output gap (y), inflation (π), and the nominal interest rate (i). The vector Z_t consists of three exogenous shifters: a demand shifter, a cost-push shifter, and a monetary policy shifter, with their respective innovations denoted by ε_{dt}, ε_{st}, and ε_{mt}. The model solution also admits the following unrestricted VAR(2) representation:

$$Y_t = \Gamma_1 Y_{t-1} + \Gamma_2 Y_{t-2} + \nu_t, \tag{6}$$

where ν_t is a three-dimensional vector of reduced-form innovations. The endogenous dynamics in this system are driven by habit formation, inertial monetary policy, and partial indexation in price setting.

To create our simulated data, we generated 1,000 sequences of 300-period innovations ε_t and fed them into the data generating process described above, thereby constructing artificial series for Y_t. From these generated data, we estimated a VAR(2) to obtain estimates of Γ_1, Γ_2, and Σ_ν. We then used the following mapping:

$$\begin{cases} \Gamma_1 &= G_1 + FRF^{-1}, \\ \Gamma_2 &= -FRF^{-1}G_1, \\ \Sigma_\nu &= FF', \end{cases} \tag{7}$$

to recover estimates of G_1, F, and R using (7).[12] Given estimated $\widehat{F}$, we construct estimates of the New Keynesian model shocks using $\widehat{\varepsilon}_t = \widehat{F}^{-1}\widehat{\nu}_t$. Importantly, the only assumption we relied on to recover these components is that R is a diagonal matrix—a relatively minimal and commonly used condition in structural models.

In Figure 1, we plot the three elements of $\widehat{\varepsilon}_t$ against their corresponding true shocks ε_t for one arbitrary 300-period simulation.[13]

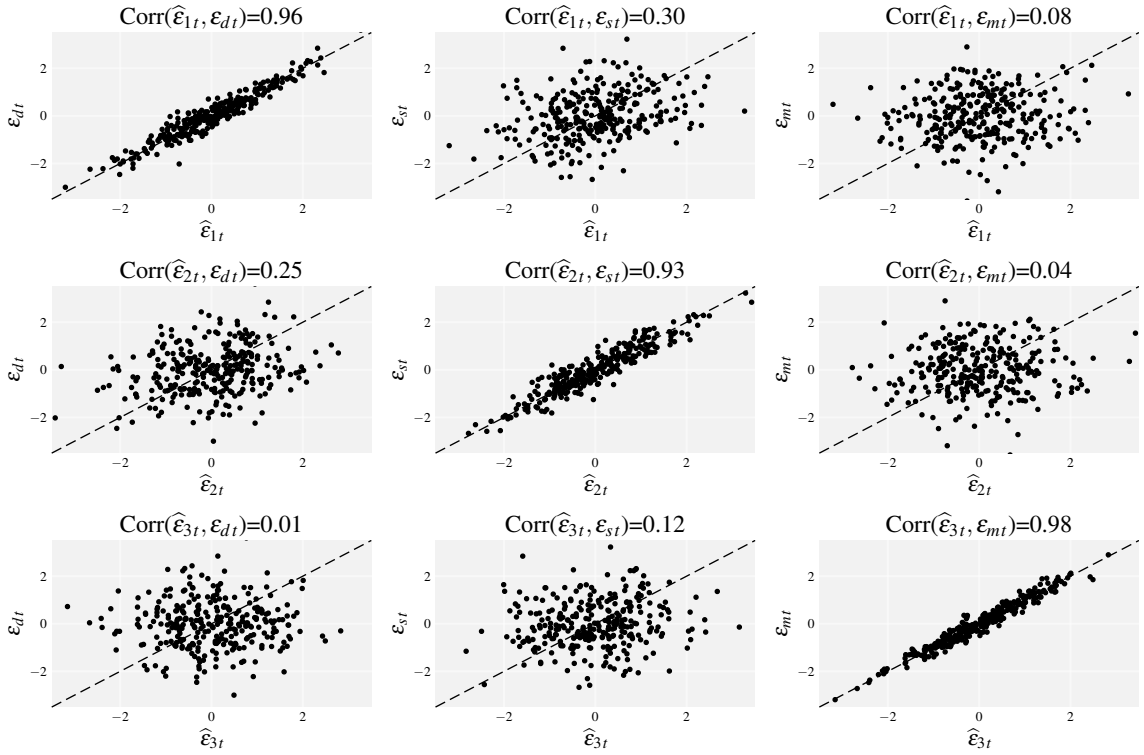
[11] See Appendix G for details of the model and its estimation.

[12] The distribution of the elements of the estimated matrices $\widehat{G}_1$, $\widehat{F}$, and $\widehat{R}$ relative to their theoretical counterparts is presented in Appendix G.3.

[13] By normalization, $\widehat{\varepsilon}_{1t}$ (respectively, $\widehat{\varepsilon}_{2t}$, $\widehat{\varepsilon}_{3t}$) is the shock most highly correlated with ε_{dt} (respectively, ε_{st}, ε_{mt}).

As can be seen from the figure, the $\widehat{\varepsilon}_t$ recovered through our procedure closely match the actual structural shocks used in generating the data. Each estimated shock is highly correlated with one of the structural shocks and nearly orthogonal to the other two.[14] In this finite-sample setting, our D-SVAR framework performs very well at retrieving the shocks of the theoretical model.[15]

Figure 1: New Keynesian Model and D-SVAR Shocks For an Arbitrary Simulation



Notes: The estimated New Keynesian model is simulated over 300 periods with random ε_{dt}, ε_{st}, ε_{mt} drawn in their estimated distribution. The macroeconomic times series generated are used to estimate a D-SVAR with innovations $\widehat{\varepsilon}_{1t}$, $\widehat{\varepsilon}_{2t}$, $\widehat{\varepsilon}_{3t}$. By normalization, $\widehat{\varepsilon}_{1t}$ (resp. $\widehat{\varepsilon}_{2t}$, $\widehat{\varepsilon}_{3t}$) is the shock that is the most correlated with ε_{dt} (resp. ε_{st}, ε_{mt}).

3.1.1 Evaluating the Proxy VAR Approach

In recent years, one widely used methods for recovering structural impulse responses from VAR models has been the use of instrumental variables. This approach is commonly referred to as the proxy VAR approach and relies on the availability of an external

[14] In that particular simulation, some off-diagonal correlations (e.g., corr($\widehat{\varepsilon}_{1t}, \varepsilon_{st}$)) are low but significant. However, over the 1,000 simulations, we cannot reject that these off-diagonal correlations are zero.

[15] This result holds on average across the 1,000 simulations. For instance, the average correlations of $\widehat{\varepsilon}_{3t}$ with the structural shocks ε_{dt}, ε_{dt}, and ε_{dt} are 0.21, -0.06, and 0.79, respectively, with standard deviations of 0.34, 0.34, and 0.26. Appendix G.4 reports the average impulse responses obtained from the D-SVAR, alongside those from the true New Keynesian model, to illustrate how well the D-SVAR recovers the true IRFs across simulations.

instrument, v_t, which is assumed to be correlated with one structural shock but uncorrelated with any other structural shock.[16] Under this assumption, v_t can be used to instrument the innovations associated with their estimated VAR to identify the macroeconomic model impulse responses of interest.

However, before applying this instrumental procedure, the D-SVAR representation offers a powerful advantage: it allows researchers to test the validity of the instrument. Indeed, the D-SVAR provides an estimate of the matrix F, which can then be used to recover the structural shocks via the mapping $\varepsilon_t = \widehat{F}^{-1}\nu_t$. With these estimated structural shocks in hand, one can directly test whether the proposed instrument v_t is correlated with exactly one of the shocks in ε_t—a critical requirement for identification.

To illustrate this idea, we use the elements of the structural shock vector $\varepsilon_t = \{\varepsilon_{dt}, \varepsilon_{st}, \varepsilon_{mt}\}$ to construct two candidate instruments. The first is a "good" instrument for the monetary policy shock, ε_t^m, defined as:

$$v_{1t} = \varepsilon_{mt} + \mu_{1t},$$

where μ_{1t} is a white noise process with standard deviation $\sigma_m/2.5$. By "good", we mean that v_{1t} is noisy but strongly correlated (in population) with the monetary policy shock, and orthogonal (in population) to the other two shocks.

The second is an "invalid" instrument, v_{2t}, defined as

$$v_{2t} = .5\,\varepsilon_{mt} + .45\,\frac{\sigma_m}{\sigma_d}\,\varepsilon_{dt} + .05\,\frac{\sigma_m}{\sigma_s}\,\varepsilon_{st} + \mu_{2t},$$

where μ_{2t} is also a white noise process with standard deviation $\sigma_m/2.5$. This instrument is invalid because, by construction, it is correlated with more than one structural shock. For instance, it has population correlations of 0.64 with the monetary shock, 0.57 with the demand shock, and 0.06 with the supply shock.

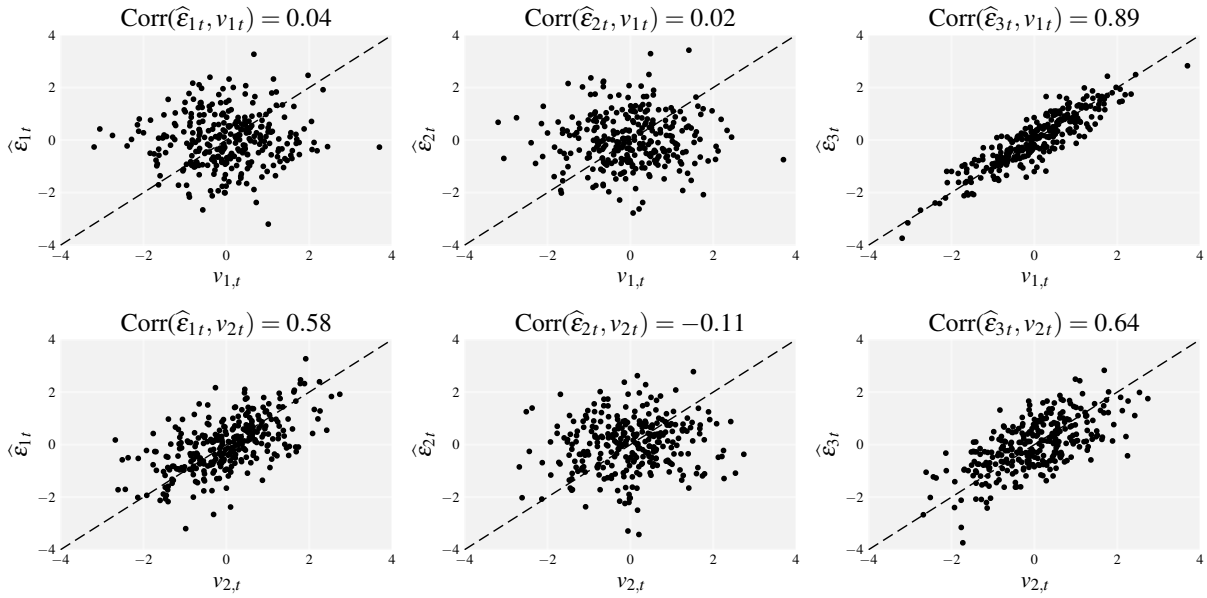
In Figure 2, we plot the two instruments, v_{1t} and v_{2t}, against the three structural shocks identified by the D-SVAR ($\widehat{\varepsilon}_t$) for the same simulation discussed earlier, and report the corresponding correlations. As shown in the figure, v_{1t} is correlated with only one component of $\widehat{\varepsilon}_t$, namely $\widehat{\varepsilon}_{3t}$, which is itself highly correlated with the true monetary policy shock. This suggests that v_{1t} is a potentially good instrument, as it satisfies the key requirement of being correlated with only one of the structural shocks identified by the D-SVAR. In contrast, v_{2t} is significantly correlated with two components of $\widehat{\varepsilon}_t$, indicating that it is not a valid instrument for identifying the monetary policy shock.[17] In Appendix G.4, we present this information from another perspective by comparing the average IRFs (over the 1,000 simulations) to the identified monetary policy

[16] See Beaudry and Saito (1998), Stock (2008), Stock and Watson (2012), and Mertens and Ravn (2013) among others for other applications of this approach.

[17] This result also holds on average across the 1,000 simulations. The average correlations between

shock. We show the IRFs from the proxy-VARs (using both instruments), from the true data-generating process (DGP), and from the D-SVAR shock most correlated with the true monetary shock. Both the proxy-VAR using the valid instrument and the D-SVAR produce IRFs that closely align with the true ones. In contrast, the proxy-VAR using the invalid instrument deviates substantially, confirming the poor quality of that instrument.

Figure 2: Instruments and D-SVAR Shocks For an Arbitrary Simulation



Notes: The estimated New Keynesian model is simulated over 300 periods with random ε_{dt}, ε_{st}, ε_{mt} drawn in their estimated distribution. The instruments v_{1t} and v_{2t} are constructed using these shocks. The macroeconomic times series generated are used to estimate a D-SVAR with innovations $\widehat{\varepsilon}_{1t}$, $\widehat{\varepsilon}_{2t}$, $\widehat{\varepsilon}_{3t}$.

This example illustrates how the validity of a proposed instrument can be assessed using the D-SVAR representation. The key advantage of the D-SVAR framework is that it enables recovery of estimates of the structural shocks in the macroeconomic model under the relatively mild assumption that the exogenous driving forces are linearly independent and follow first-order autoregressive processes.

A macroeconomist estimating a proxy-VAR with instrument v_1, alongside our D-SVAR, could then reasonably conclude: "*Given the set of variables and the lag structure I chose, my instrument is consistent with a model in which the exogenous forces are linearly independent AR(1) processes*"—a structure that characterizes most models used in modern macroeconomics.[18] More importantly, the D-SVAR approach allows us to

v_{1t} and the three D-SVAR shocks $\widehat{\varepsilon}_t$ are 0.07, 0.06, and 0.73, with standard deviations of 0.23, 0.40, and 0.27, respectively. For v_{2t}, the corresponding correlations are 0.58, 0.06, and 0.61, with standard deviations of 0.17, 0.37, and 0.30.

[18] If, however, one assumes that the R matrix is not diagonal, as is often the case in the international RBC literature, our D-SVAR framework remains valid. As we will show later, the model is still identified, and one can continue to assess the quality of the instrument under this broader setting.

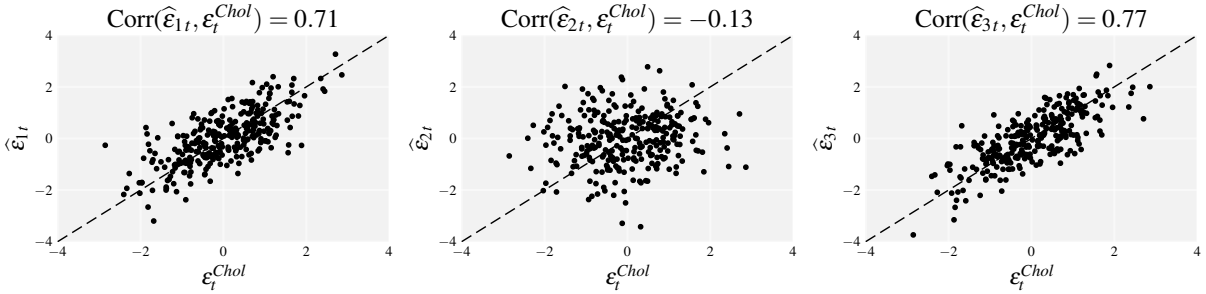
directly test whether a given instrument is correlated with only one structural shock—a necessary condition for it to serve as a valid instrument.

3.1.2 Evaluating the Recursive Ordering

In the New Keynesian model we used to generate the data, the matrix F is neither triangular nor block-recursive. This has important implications for identification. For instance, if we were to identify a monetary policy shock following an approach in the spirit of Christiano et al. (1999), which relies on having a monetary policy shock that does not have instantaneous impact on inflation and output, this should be rejected (see Carlstrom et al., 2009). To explore this, we estimate a VAR(2) using the simulated data and apply a Cholesky decomposition to construct a shock that has no contemporaneous impact on output and inflation. We denote this identified shock by ε_t^{Chol}. By construction, this shock is orthogonal to output and inflation on impact. However, we already know that it cannot correspond to the true model-based monetary policy shock ε_{dt}, since the matrix F in the New Keynesian model does not contain two zeros in its first row, *i.e.* the true monetary policy shock does affect output and inflation contemporaneously.

Figure 3 shows how this constructed shock, ε_t^{Chol}, correlates with the three shocks recovered via our D-SVAR procedure (for one representative simulation). It is immediately evident that ε_t^{Chol} is strongly correlated with two components of $\widehat{\varepsilon}_t$. Since a valid structural shock should be associated with only one element of $\widehat{\varepsilon}_t$, this would strongly suggest to reject this shock as a structural shock. An alternative, and equally informative, way to assess the issue is by comparing the impulse response functions (IRFs) associated with ε_t^{Chol} to those obtained from the D-SVAR.[19]

Figure 3: Cholesky and D-SVAR Shocks For an Arbitrary Simulation



Notes: The estimated New Keynesian model is simulated over 300 periods with random ε_{dt}, ε_{st}, ε_{mt} drawn in their estimated distribution. The macroeconomic times series generated are used to estimate a D-SVAR with innovations $\widehat{\varepsilon}_{1t}$, $\widehat{\varepsilon}_{2t}$, $\widehat{\varepsilon}_{3t}$, and to recover ε_t^{Chol} using a Cholesky decomposition. ε_t^{Chol} is the VAR innovation that does not move inflation and output on impact.

[19] In Appendix G.4, we report the average impulse responses (across 1,000 simulations) for ε_t^{Chol} alongside those from the true New Keynesian model. This comparison highlights how the recursive identification fails to capture the correct structural responses when its assumptions are violated.

3.2 D-SVAR in Action

In this section, we move beyond simulated data and apply our D-SVAR methodology to three well-known empirical SVAR applications.[20]

3.2.1 Revisiting Blanchard and Quah (1989): Long-Run Restrictions

We begin with a bivariate VAR model that includes the growth rate of output per capita, Δy_t, and a measure of the unemployment rate gap, u_t, defined as the difference between the actual and the estimated non-cyclical rate of unemployment. Blanchard and Quah (1989) (hereafter BQ) used this VAR framework to uncover the permanent, ε_{Pt}, and transitory, ε_{Tt}, components of output by imposing the restriction that the transitory component has no long-run effect on the level of output. Their findings suggested that, in the short run, ε_{Tt} closely resembles an aggregate demand shock, while ε_{Pt} behaves like an aggregate supply shock.

Motivated by Blanchard and Quah (1989), we estimate a VAR over the period 1960Q1–2007Q4 and apply both the original BQ identification and our D-SVAR methodology, assuming a diagonal R matrix.[21] The diagonal R assumption introduces one overidentifying restriction, which is not rejected by the data. Figure 4 presents scatter plots comparing the BQ permanent and transitory shocks with $\widehat{\varepsilon}_{1t}$ and $\widehat{\varepsilon}_{2t}$ respectively, revealing nearly perfect correlations: both identification strategies yield essentially the same shocks. This suggests that the BQ identifying restrictions are consistent with the assumptions underpinning our D-SVAR approach—namely, that the underlying macroeconomic model is an impulse-propagation system driven by linearly independent AR(1) forcing variables.

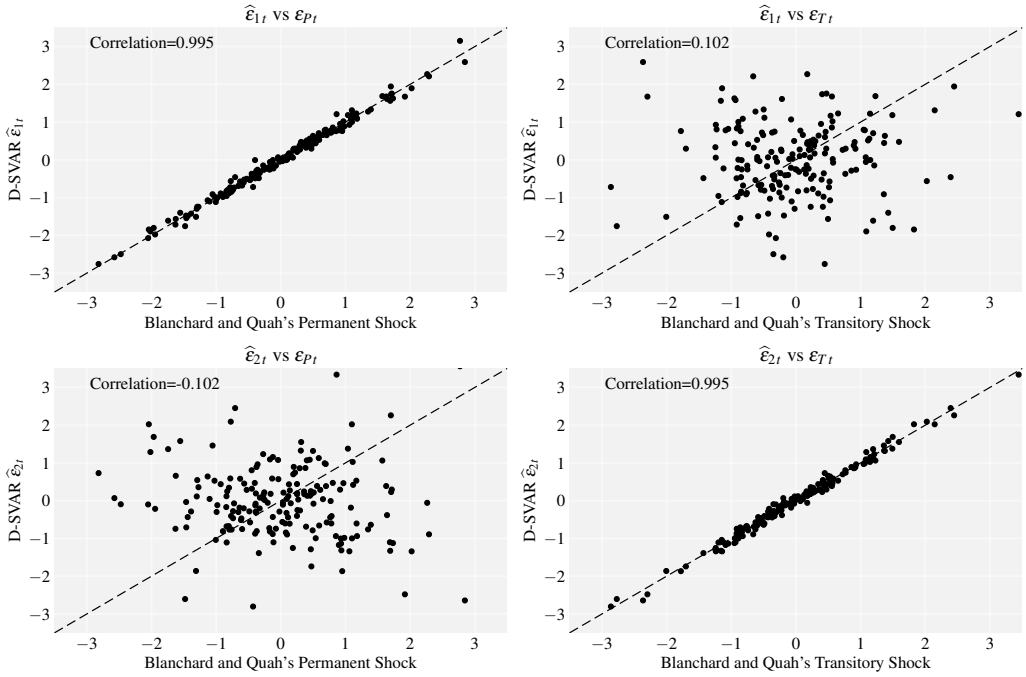
We then compute impulse responses for the two identifications. They are displayed in Figure 5. We uncover a familiar and interesting pattern. One of the identified shocks, $\widehat{\varepsilon}_{2t}$, raises output and lowers unemployment on impact. The output response is hump-shaped, eventually returning to zero in the long run. This shock accounts for more than 80% of unemployment volatility at all horizons, around 75% of output volatility on impact, but contributes virtually nothing to output fluctuations in the long run. The other shock, $\widehat{\varepsilon}_{1t}$, exerts a permanent effect on output but very little influence on unemployment. These two shocks closely resemble the transitory and permanent shocks identified by BQ. This is confirmed by the dashed lines in Figure 5, which show the IRFs under the original BQ identification. In this case, the IRFs are very similar to those obtained in the D-SVAR.

Interestingly, the D-SVAR enables us to formally test the BQ identification restriction. We cannot reject the hypothesis that only one shock has a permanent effect on output: a

[20] All data sources used in this section are described in Section F of the online appendix.

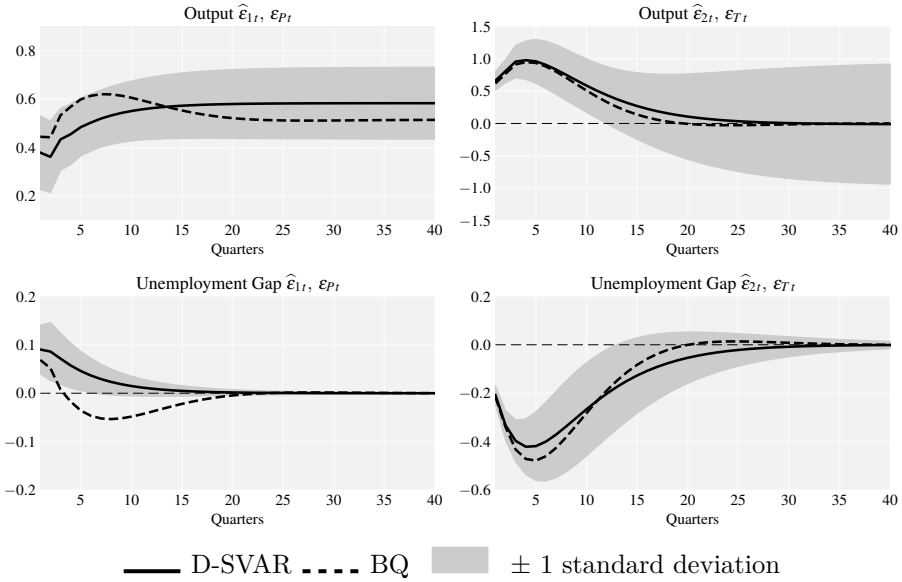
[21] We use two lags, as selected by the Bayesian Information Criterion (BIC), and estimate the D-SVAR model using the Asymptotic Least Squares method discussed in Corollary 1, treating the unrestricted VAR as the auxiliary model.

Figure 4: Correlation between D-SVAR and BQ shocks



Notes: Sample is 1960Q1-2007Q4. Estimation is done with $(\Delta y_t, u_t)$ using two lags, where y_t is the real GDP and u_t is the unemployment rate gap.

Figure 5: Comparing the D-SVAR with Blanchard and Quah (1989) Identification



Notes: Sample is 1960Q1-2007Q4. y_t is the real GDP, u_t is the unemployment rate gap. Estimation is done with $(\Delta y_t, u_t)$ using two lags. The grey area represents 68% confidence bands obtained from 1,000 Bootstrap replications.

formal test of the null hypothesis that the IRFs at the long-run horizon are equal yields a p-value of 0.40, indicating no statistically significant difference. However, the impact responses of the unemployment gap differ slightly across the two identification schemes.

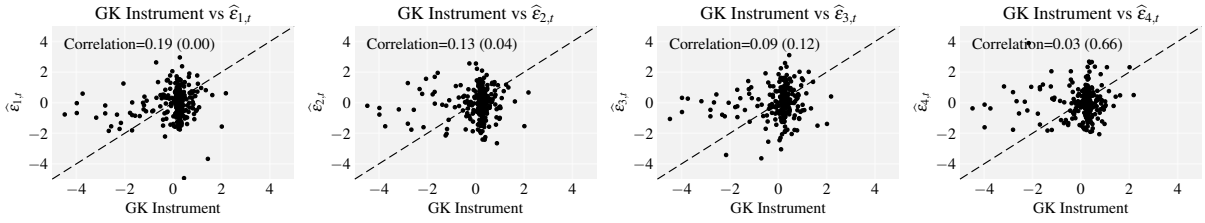
3.2.2 Revisiting Gertler and Karadi (2015): Proxy VAR

In this section, we revisit Gertler and Karadi (2015), who identified monetary policy shocks using a proxy-VAR approach using an external instrument. To serve as a valid instrument for identifying a structural shock, recall that two conditions must be met. First, the instrument must be relevant, that is the instrument must be contemporaneously correlated with the structural monetary policy shock. Second, the instrument must be uncorrelated with all other structural shocks in the model (exogeneity assumption). One of the key advantages highlighted by Gertler and Karadi (2015) is that the proxy-VAR approach does not require imposing a specific structure—or restrictions—on the loading matrix F. However, it does not offer a way to evaluate whether the identification restrictions are reasonable, This is where our D-SVAR framework can be used.

To explore whether Gertler and Karadi (2015) approach is likely isolating only a monetary shock, we first replicate their baseline VAR which includes the log of the consumer price index (CPI), log industrial production, the one-year government bond rate, and the excess bond premium from Gilchrist and Zakrajšek (2012). The data are monthly, spanning the period from July 1979 to June 2012. Following their specification, we estimate the VAR with twelve lags and compute the IRFs of these variables to a monetary policy shock identified via the proxy-VAR approach (using the surprise component of the three-month-ahead futures rate as the external instrument). These IRFs are reported by the black lines in Figure 7.

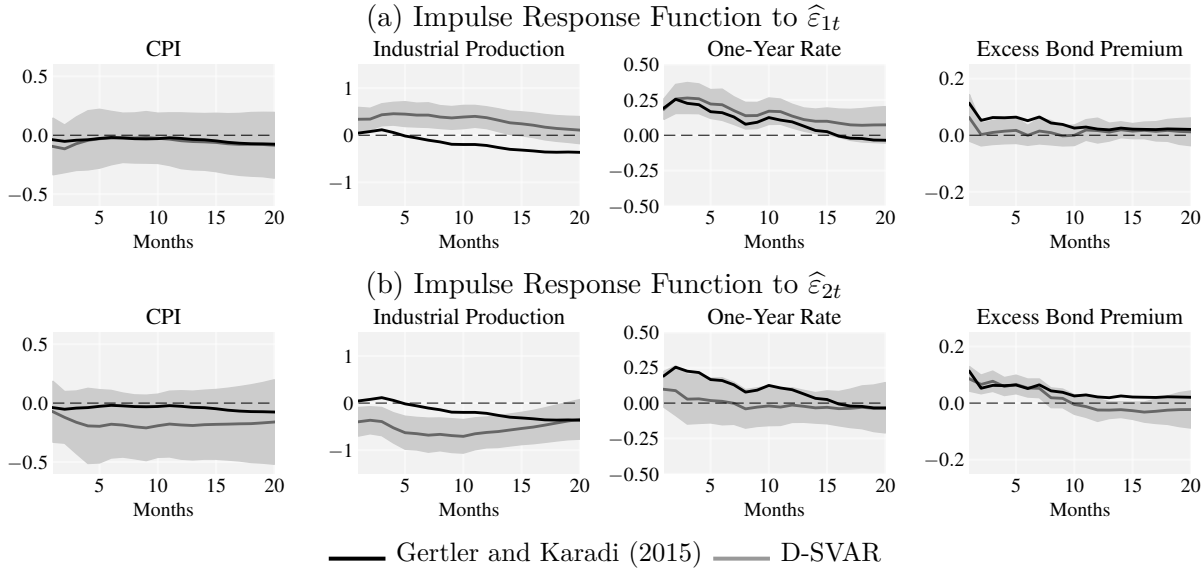
We then proceed to estimate our D-SVAR model. As in the Blanchard and Quah (1989) example discussed in Section 3.2.1, we identify four structural shocks under the assumption that the exogenous driving forces generating the data has a diagonal autoregressive matrix R. This assumption imposes six overidentifying restrictions, which are not rejected by the data. From the D-SVAR, we recover four unlabelled structural shocks,

Figure 6: GK instrument against D-SVAR Shocks



Notes: p-value of a zero correlation between parenthesis.

Figure 7: IRFs to Gertler and Karadi (2015) monetary policy shock and D-SVAR shocks $\widehat{\varepsilon}_{1t}$ and $\widehat{\varepsilon}_{2t}$



Notes: The black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shocks identified in the D-SVAR with diagonal R matrix. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

which we arbitrarily index as $\widehat{\varepsilon}_{1t}, \ldots, \widehat{\varepsilon}_{4t}$. In Figure 6, we plot the external instrument from Gertler and Karadi (2015) against each of these four D-SVAR shocks. If the Gertler and Karadi (2015) instrument is appropriate for this VAR, it should be highly correlated with only one of the shocks obtained from our D-VAR. The first observation we make is that the instrument is not highly correlated with any of the four structural shocks. Nonetheless, a formal correlation tests shows that the instrument is weakly but significantly correlated with the first two shocks only—$\widehat{\varepsilon}_{1t}$ and $\widehat{\varepsilon}_{2t}$ (p-values reported in Figure 6). Therefore, according our D-VAR, the instrument is likely identifying a combination of $\widehat{\varepsilon}_{1t}$ and $\widehat{\varepsilon}_{2t}$ [22], as opposed to identifying a pure monetary policy shock.

To help understand what type of shock the proxy-VAR is likely capturing, Figure 7 displays the IRFs of the four variables to both $\widehat{\varepsilon}_{1t}$ and $\widehat{\varepsilon}_{2t}$,[23] alongside the IRFs to the shock identified by Gertler and Karadi (2015).

The $\widehat{\varepsilon}_{1t}$ structural shock looks like a demand shock (industrial production goes up) to which monetary authorities would respond (interest rate goes up), without much response of inflation and the excess bond premium. The $\widehat{\varepsilon}_{2t}$ structural shock looks like a financial shock that would increase the excess bond premium and decrease industrial production, without much response of inflation and the interest rate. According our D-SVAR, what

[22] This is a case we highlighted in the simulation example of section 3.1.

[23] The IRFs associated with $\widehat{\varepsilon}_{3t}$ and $\widehat{\varepsilon}_{4t}$ are reported in Section I of the online Appendix.

Gertler and Karadi (2015) are most likely capturing is a combination of these two shocks rather that a structural monetary policy shock.

3.2.3 Revisiting Christiano et al. (1999): Recursive Ordering

Christiano et al. (1999) identified monetary policy shocks using a set of impact restrictions.[24] As shown in Section 2 using simulated data, our D-SVAR framework provides a natural way to evaluate the validity of such restrictions. To do so, we follow the approach of Christiano et al. (1999) and estimate a VAR(4), including real GDP, the unemployment rate, CPI inflation, commodity price inflation, and the federal funds rate in that order over the sample period 1965Q1 to 2007Q4.[25]

We then estimate the corresponding D-SVAR and recover five unlabelled structural shocks, denoted $\widehat{\varepsilon}_{1t}, \ldots, \widehat{\varepsilon}_{5t}$. As before, we assume the autoregressive matrix R is diagonal, which in this five-variable case implies ten overidentifying restrictions. These restrictions are not rejected by the data, supporting the validity of the D-SVAR representation. Figure 8 illustrates the correlation between the monetary policy shock identified using the impact restrictions of Christiano et al. (1999) and the shocks from the D-SVAR. We find a strong correlation with $\widehat{\varepsilon}_{1t}$ —so strong that we cannot reject the hypothesis that the correlation is exactly one (p-value = 0.986). Moreover, we cannot reject the null that this identified shock is orthogonal to the four other D-SVAR shocks.[26]

Figure 9 displays the IRFs to $\widehat{\varepsilon}_{1t}$ (in grey), alongside those to the monetary policy shock identified by Christiano et al. (1999). The comparison shows that the dynamics following the two shocks are remarkably similar. As with the Christiano et al. (1999) monetary policy shock, a tightening of monetary policy triggers a prolonged recession: output (respectively, unemployment) reaches its trough after roughly seven (respectively, nine) quarters, before gradually returning to its long-run level. Both output and unemployment display persistent, hump-shaped dynamics, consistent with the effects of a contractionary monetary policy shock as documented in Christiano et al. (1999). Also in line with Christiano et al. (1999), we observe a persistent price puzzle—prices. In our case, however, the puzzle is somewhat more pronounced and more persistent.[27]

The D-SVAR framework allows us to go a step further and formally test whether $\widehat{\varepsilon}_1$ satisfies the zero impact restrictions imposed by Christiano et al. (1999). Table 1 reports the p-values associated with testing whether the immediate (impact) response of $\widehat{\varepsilon}_1$ is

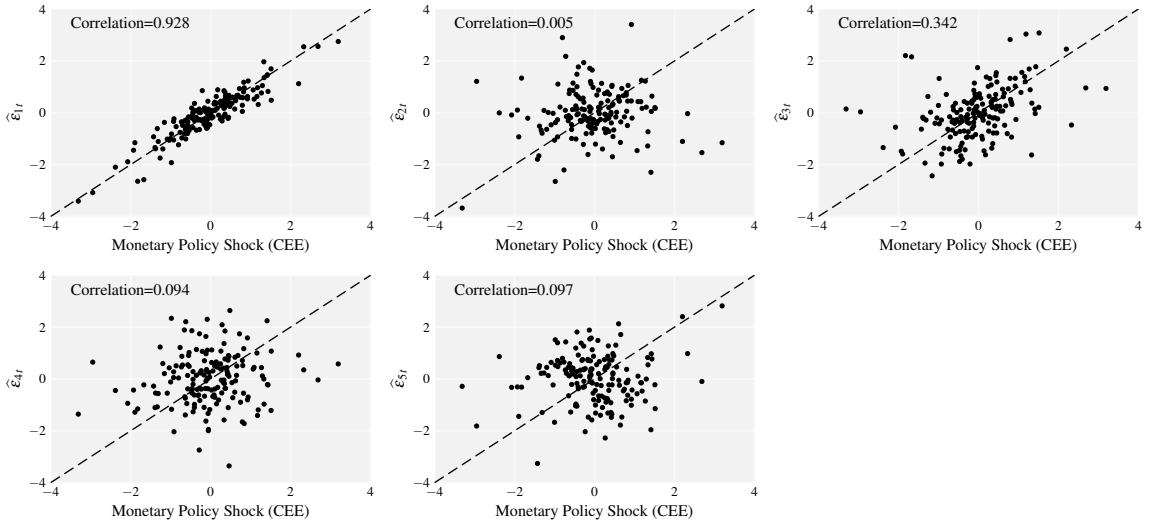
[24] See also Christiano et al. (2005).

[25] We end the sample in 2007Q4 to avoid complications related to the zero lower bound. Addressing that period would require explicitly modeling the non-linear dynamics of interest rates in the presence of occasionally binding constraints; see, for example, Mavroeidis (2021).

[26] See Table J.1 in Online Appendix J.

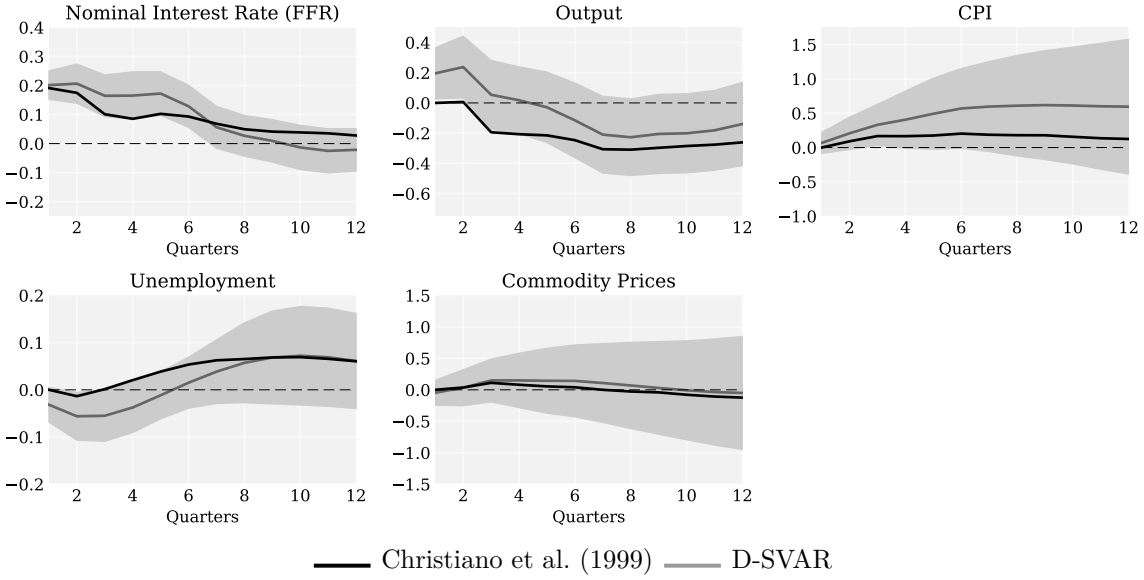
[27] Such a "price puzzle" is reminiscent of the results in Beaudry et al. (2024), and can be rationalized in a model with a flat Phillips curve and a cost channel.

Figure 8: Correlation between Christiano et al. (1999) Monetary Shock and D-SVAR Shocks



Notes: $\widehat{\varepsilon}_{1t}$ to $\widehat{\varepsilon}_{5t}$ are the structural shocks as obtained with the D-SVAR. Sample is 1965Q1-2007Q4.

Figure 9: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR shock $\widehat{\varepsilon}_{1t}$



Notes: On the five panel, the black line is the response to a monetary policy shock, as identified following Christiano et al. (1999). The grey line is the response to shock $\widehat{\varepsilon}_{1t}$ in the D-SVAR. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Table 1: Test of Zero Impact Restriction (p-values)

	Output	CPI	Unemp.	Com. Price
$\widehat{\varepsilon}_{1t}$	0.2470	0.6587	0.3890	0.8169

Notes: $\widehat{\varepsilon}_{1t}$ is the D-SVAR structural shock that is the closest to the Christiano et al. (1999) monetary policy shock. Sample is 1965Q1-2007Q4.

zero for output, the CPI, unemployment, and commodity prices. The results indicate that the data do not reject the restrictions proposed by Christiano et al. (1999): none of the individual p-values falls below 0.25, well above the conventional 5% significance threshold. Additionally, we do not reject the joint hypothesis that the impact responses of output and unemployment are simultaneously zero (p-value = 0.32). This exercise shows that the zero impact restrictions proposed by Christiano et al. (1999), which have been criticized for being potentially inconsistent with standard macroeconomic models (see, for example, Carlstrom et al. (2009)), are not rejected when evaluated through the lens of our D-SVAR framework. A joint test of these restrictions also fails to reject the null (p-value = 0.2).

That said, a closer look at the impulse responses associated with $\widehat{\varepsilon}_{1t}$ may raise some concerns. Specifically, the dynamics suggest a pattern more consistent with a demand shock that is countered by monetary policy, rather than a pure monetary policy shock. Since this alternative interpretation also cannot be statistically rejected, the results calls for caution when using simple impact restrictions to identify monetary shocks. Structural interpretations may be more ambiguous than they appear.

4 D-SVAR Identification: Theory

In Section 2, we introduced the D-SVAR framework and discussed the types of restrictions on the matrix R that are necessary for identifying structural shocks. However, that discussion was primarily heuristic. In this section, we formally establish a set of propositions that provide theoretical support for the local identification of the D-SVAR. We begin by fixing some notation, and then proceed to prove identification results based on both the covariance structure and the dynamic restrictions of the model.

4.1 Setup

Consider an economy whose Data Generating Process (DGP)—i.e., the solution to a macroeconomic model—is represented by the following state-space system (without loss

of generality, we abstract from constant terms):

$$\underbrace{K_t}_{n_k\times 1} = \underbrace{\Pi_{kk}}_{n_k\times n_k}\underbrace{K_{t-1}}_{n_k\times 1} + \underbrace{\Pi_{kz}}_{n_k\times n_z}\underbrace{Z_t}_{n_z\times 1}, \tag{8}$$

$$\underbrace{Y_t}_{n_y\times 1} = \underbrace{\Pi_{yk}}_{n_y\times n_k}\underbrace{K_{t-1}}_{n_k\times 1} + \underbrace{\Pi_{yz}}_{n_y\times n_z}\underbrace{Z_t}_{n_z\times 1}, \tag{9}$$

$$\underbrace{Z_t}_{n_z\times 1} = \underbrace{R}_{n_z\times n_z}\underbrace{Z_{t-1}}_{n_z\times 1} + \underbrace{\varepsilon_t}_{n_z\times 1}, \tag{10}$$

where the $(n_y \times 1)$ vector Y_t gathers all observed variables, the $(n_k \times 1)$ vector K_t collects possibly unobserved endogenous state variables (predetermined at time $t-1$), Z_t represents the $(n_z \times 1)$ vector of unobserved exogenous variables, and ε_t is the $(n_z \times 1)$ vector of the macroeconomic model shocks (the innovations to Z_t). In particular, ε_t satisfies $\mathbb{E}_{t-1}\varepsilon_t = 0$, where $\mathbb{E}_{t-1}$ denotes the expectation operator conditional on the information set of histories until period $t-1$—*i.e.* all past realizations and histories of $\{K_t, Y_t, Z_t\}$.[28]

This framework is general enough to represent the (log-)linear solution of most macroeconomic models, including DSGE models. In the case of DSGE models, this solution usually depends on a limited number of parameters, denoted by $\boldsymbol{\theta}$, which represent "deep" structural parameters (such as preferences, technology, institutions, policies, etc.), as well as the stochastic process of the exogenous forcing variables in the structural model. In this context, the matrices in the state-space representation (8)–(10) are functions of $\boldsymbol{\theta}$.

In this paper, however, we do not consider the identification of $\boldsymbol{\theta}$; instead, we focus on the identification of the state-space parameters that freely enter the matrices of the state-space representation. We denote this vector of parameters as $\boldsymbol{\psi} \in \boldsymbol{\Psi}$. It must be emphasized that System (8)–(10) imposes fewer restrictions than the (possibly underlying) macroeconomic model. The only restrictions we explore apply to the matrix R and the covariance matrix of the shocks ε_t in Equation (10). Finally, the shocks ε_t are zero-mean weak white noise processes with covariance matrix $\mathbb{E}(\varepsilon_t\varepsilon_t') = \Sigma_\varepsilon$.

System (8)–(10) can be expressed in the Fernández-Villaverde et al. (2007) ABCD representation as:

$$S_{t+1} = AS_t + B\varepsilon_{t+1}, \tag{11}$$

$$Y_{t+1} = CS_t + D\varepsilon_{t+1}, \tag{12}$$

where

$$\underbrace{S_t}_{n_s\times 1} = \begin{bmatrix} K_t \\ Z_t \end{bmatrix}, \quad \underbrace{A}_{n_s\times n_s} = \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0_{n_z\times n_k} & R \end{bmatrix}, \quad \underbrace{B}_{n_s\times n_z} = \begin{bmatrix} \Pi_{kz} \\ I_{n_z} \end{bmatrix},$$

$$\underbrace{C}_{n_y\times n_s} = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \end{bmatrix}, \quad \text{and} \quad \underbrace{D}_{n_y\times n_z} = \Pi_{yz},$$

[28] If some elements of K_t are observed, those elements should be reassigned to vector Y_t, and the matrices Π_{yk} and Π_{yz} should be adjusted accordingly.

with $n_s = n_k + n_z$.[29]

In the sequel, we consider the case where the number of observables is equal to the number of shocks –*i.e.* $n_y = n_s$. As noted earlier, the vector

$$\boldsymbol{\psi} = (\text{vec}(\Pi_{kk})', \text{vec}(\Pi_{kz})', \text{vec}(\Pi_{yk})', \text{vec}(\Pi_{yz})', \text{vec}(R)', \text{vech}(\Sigma_\varepsilon)')'$$

collects all the parameters of the state-space representation (8)–(10).

Under the assumptions defined in Section A of the Appendix, the process $\{Y_t\}$ described by (11)–(12) admits the following infinite Vector Moving Average (VMA) representation:

$$\begin{aligned} Y_t &= \left[C(I_{n_s} - AL)^{-1}BL + D\right]\varepsilon_t, \\ &= \left[\Pi_{yk}(I_{n_k} - \Pi_{kk}L)^{-1}\Pi_{kz}L + \Pi_{yz}\right](I - RL)^{-1}\varepsilon_t, \\ &= \left[\Pi_{yk}(I_{n_k} - \Pi_{kk}L)^{-1}\Pi_{kz}\Pi_{yz}^{-1}L + I_{n_z}\right]\Pi_{yz}(I - RL)^{-1}\varepsilon_t, = H(L, \boldsymbol{\psi})\varepsilon_t, \end{aligned} \quad (13)$$

where L denotes the lag operator. This VMA representation can be recovered through a VAR, a VARMA, or a state-space model, such that the IRFs can be obtained for the identified shocks ε_t.

By imposing restrictions only on the covariance matrix of the shocks, ε_t, and on the matrix R, which governs the dynamics of the exogenous process Z_t, we can recover the IRFs to the ε_t shocks. In the general context with unobserved state variables K_t, such restrictions allow for the local identification of the matrix R, the loading matrix Π_{yz}, and the expression

$$\left[\Pi_{yk}(I_{n_k} - \Pi_{kk}L)^{-1}\Pi_{kz}\Pi_{yz}^{-1}L + I_{n_z}\right], \quad (14)$$

but not the individual matrices within this expression. Nevertheless, this level of identification is sufficient to obtain the IRFs to the ε_t shocks. This result holds for all the cases we analyze below. When all endogenous state variables K_t are observed, however, the individual matrices can be fully and locally identified.

The following section presents key propositions on local identification under the assumption that restrictions are applied solely to the covariance matrix and the matrix R. Specifically, our setup assumes that the shocks, ε_t, are mutually orthogonal, and their covariance matrix is normalized to the identity matrix, such that $\mathbb{E}(\varepsilon_t\varepsilon_t') = I_{n_z}$. This normalization is a widely adopted identifying assumption in the SVAR literature (see Ramey (2016)). Within this framework, the matrix Π_{yz} encapsulates any scale effects of the shocks on all observed variables. In other words, it contains information about the volatility of the shocks.

[29] In the specific case where there are no endogenous state variables K_t, $n_k = 0$, $n_s = n_z$, and the matrices are defined accordingly. For example, $A = R$.

4.2 Dynamic Restrictions

This subsection focuses on the local identification of the IRFs to the ε_t shocks, relying solely on the second-order moments of the observables Y_t. In addition to the restriction $\mathbb{E}(\varepsilon_t \varepsilon_t') = I_{n_z}$, local identification is achieved by imposing structure on the matrix R. The following four cases, commonly encountered in macroeconomic models, are considered:

1. R is a diagonal matrix;
2. R is a lower (or equivalently upper) triangular matrix;
3. R is a symmetric matrix;
4. R is a block diagonal matrix with blocks corresponding to cases 1 and 2.

The identification problem is examined and proven for each of these cases.

4.2.1 R is a Diagonal Matrix

The case of a diagonal matrix is of particular interest. It implies, together with the restriction on the covariance matrix, that all processes in the Z_t vector are mutually orthogonal at any leads and lags. While this assumption may appear restrictive, it corresponds to common practice in the macroeconomics literature and aligns with economic theory. The following proposition derives the sufficient condition for local identification.

Proposition 1 *If R is a diagonal matrix with distinct diagonal elements ($r_{i,i} \neq r_{j,j}$, $\forall i \neq j$), then the IRFs to the macroeconomic model shocks ε_t (corresponding to the VMA representation (13)) are locally identified.*

Proof: See Appendix C.3.

In other words, the loading matrix Π_{yz} and the autoregressive matrix R are locally identifiable if all the autoregressive parameters of the n_z linearly independent forcing variables are distinct. As noted earlier, when the state variables K_t, only the combined expression in equation (14) is identified—its individual components cannot be separately recovered. Nonetheless, this level of identification is sufficient to compute the IRFs to the structural shocks ε_t. In contrast, when all endogenous state variables K_t are observed, the individual matrices can be fully and locally identified.

Therefore, in a "standard" macroeconomic model, where the structural shocks are mutually orthogonal at all leads and lags, dynamic identification follows naturally. Intuitively, identification is possible because, once the economic structure is fixed, differences in the persistence of shocks ensure that the impulse response functions uniquely reveal the dynamic behavior associated with each structural shock.

4.2.2 R is a Lower Triangular Matrix

We now consider the case of a lower triangular autoregressive matrix R, extending beyond the diagonal structure to a more general form. Let us define the matrix

$$\Lambda = \begin{bmatrix} \lambda_1 I_{n_{z1}} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{n_{z2}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_L I_{n_{zL}} \end{bmatrix},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$, and $n_{z\ell}$ are positive integers for $\ell = 1, \ldots, L$, such that $n_{z1} + \ldots + n_{zL} = n_z$. Here, $n_{z\ell} = m(\lambda_\ell)$ denotes the multiplicity of eigenvalue λ_ℓ.

Proposition 2 *If R is a lower triangular matrix with the same main diagonal as Λ and all elements on the first sub-diagonal are non-zero –i.e. $r_{i+1,i} \neq 0$ for $i = 1, \ldots, n_z - 1$, then the IRFs to the macroeconomic model shocks ε_t (corresponding to the VMA representation (13)) are locally identified.*

Proof: See Appendix C.4.

Proposition 2 shows that, unlike the strictly diagonal case, the lower triangular structure does not require all diagonal elements of R to be distinct for identification. If some diagonal elements are identical, they must be grouped together—i.e., appear consecutively within one of the blocks $\lambda_\ell I_{n_{z\ell}}$. The proposition also implies that while not all entries below the diagonal need to be non-zero, the elements on the first sub-diagonal must be non-zero to ensure identification. Intuitively, the key requirement for local identification is that each shock generates distinguishable dynamics, as captured, for example, by the IRFs. To guarantee this, some elements of the matrix R must differ, so that different shocks result in different dynamic patterns.

4.2.3 R is a Symmetric Matrix

For simplicity, the presentation is restricted to the case with two shocks. The matrix R is symmetric, with identical values on the diagonal and identical values on the anti-diagonal. While this configuration may seem unusual at first glance, it has significant practical relevance in the international macroeconomic literature, originating from the seminal work of Backus et al. (1992).[30] For this specification, the autoregressive matrix takes the following form:

$$R = \begin{bmatrix} \rho & \tau \\ \tau & \rho \end{bmatrix}, \quad \text{with } \tau \neq 0. \tag{15}$$

[30]This dynamic structure of the autoregressive matrix has been widely used, among others, by Backus et al. (1994), Baxter and Crucini (1995), Heathcote and Perri (2002), and Kehoe and Perri (2002).

The restriction $\tau \neq 0$ is critical. When $\tau = 0$, the matrix (15) reduces to a diagonal matrix with identical elements, which does not satisfy Proposition 1. Provided that $\tau \neq 0$, the following proposition holds.

Proposition 3 *For the 2×2 symmetric matrix R (eq. 15), the IRFs to the macroeconomic model shocks ε_t (corresponding to the VMA representation (13)) are locally identified.*

In particular, and in contrast to the diagonal case, it is shown in Appendix C.5 that the diagonal elements must necessarily be identical for the local identification of the IRFs.[31]

4.2.4 Partial Identification.

Finally, Proposition 4 provides conditions for partial identification.Even though some shocks may not be identified, it is still possible to identify a subset.[32]

Proposition 4 *In the case of a block diagonal organisation of the R matrix with one block corresponding to one of the preceding cases, the IRFs to the macroeconomic model corresponding shocks ε_t are locally identified.*

Proof : See Appendix C.6.

5 Estimation and Inference

This section discusses estimation and inference in the D-SVAR approach and shows how this approach can be used to evaluate SVAR models.

5.1 Estimation

To ease the exposition, let us consider the simplified state-space representation:[33]

$$Y_t = \Pi_{yz} Z_t, \tag{16}$$

$$Z_t = R Z_{t-1} + \varepsilon_t, \tag{17}$$

where $\mathbb{E}(\varepsilon_t \varepsilon_t') = I_{nz}$. In the sequel, we will restrict ourselves to cases where the parameter vector $\boldsymbol{\psi} = (vec(\Pi_{yz})', vec(R)')'$ is indeed locally identified, such that the conditions for the validity of Propositions 1 and 2 are satisfied. Vector $\boldsymbol{\psi}$ can thus be estimated either by

[31] Appendix K provides an illustration of the symmetric autoregressive matrix R with an application to the international transmission of shocks between the US and the Euro area.

[32] Online Appendix E provides a simple illustrative example.

[33] This discussion can be extended to VAR or VARMA representations.

Maximum Likelihood (ML) from (16)-(17) or equivalently by a two step Asymptotic Least Square (ALS) approach (See Corollary 1 below). Let us denote $\widehat{\boldsymbol{\psi}}_T$ the ML estimator of $\boldsymbol{\psi}$ for a sample size T. Absent any unobserved variables, the D-SVAR representation rewrites as a VAR(1) model:

$$Y_t = \left(\Pi_{yz} R \Pi_{yz}^{-1}\right) Y_{t-1} + \Pi_{yz}\varepsilon_t, \tag{18}$$

where the Π_{yz} is identified using the dynamic structure of unobserved forcing variables Z_t. So the loading matrix Π_{yz} is obtained without any restriction. Consider now the VAR(1) representation :

$$Y_t = \Gamma Y_{t-1} + u_t, \tag{19}$$

with $E(u_t) = 0$ and $E(u_t u_t') = \Sigma_u$. This implies that $u_t = \Pi_{yz}\varepsilon_t$.

Our D-SVAR representation imposes as many or more restrictions on the dynamic structure of the data Y_t than the VAR model. This offers an opportunity to use the information contained in the parameters of the estimated VAR model (19) to estimate $\boldsymbol{\psi}$ in the D-SVAR (16)-(17). Let us define $\eta = (vec(\Gamma)', vec(\Sigma_u)')'$ and the binding function $\widetilde{\eta}(\boldsymbol{\psi})$. We derive a version of the Corollary in Gouriéroux and Monfort (1995) (Chap. 10, Section 10.4.2, Corollary 10.2) adapted to our D-SVAR.

Corollary 1 *Let $\widehat{\eta}_T$ be a consistent and asymptotically normal estimator of η from (19) and let $\eta(\boldsymbol{\psi})$ be the binding function. The ALS estimator $\widehat{\boldsymbol{\psi}}_T$ obtained by solving*

$$\min_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \left[\widehat{\eta}_T - \widetilde{\eta}(\boldsymbol{\psi})\right]' S_T \left[\widehat{\eta}_T - \widetilde{\eta}(\boldsymbol{\psi})\right], \tag{20}$$

where S_T is an estimator of the inverse of the asymptotic covariance matrix of $\widehat{\eta}_T$, is a consistent estimator of $\boldsymbol{\psi}$ and is asymptotically equivalent to the ML estimator $\widehat{\boldsymbol{\psi}}_T$ of $\boldsymbol{\psi}$ obtained from (16)-(17).

Intuitively, Corollary 1 states that estimating the parameter vector $\boldsymbol{\psi}$ in one direct step (ML approach) or relying on a two step procedure using η as an auxiliary parameter (ALS approach) yields asymptotic equivalent estimator of $\boldsymbol{\psi}$. The two-step estimation uses the constraints $\Gamma = \left(\Pi_{yz} R \Pi_{yz}^{-1}\right)$ and $\Sigma_u = \Pi_{yz}\Pi_{yz}'$ to uncover the elements of the R and Π_{yz} matrices. This corollary therefore illustrates the strong connection between our approach and standard VAR modelling.

5.2 Inference

The D-SVAR offers the possibility to conduct statistical inference on SVAR models. Most notably, it allows for a formal specification test (a J-test) in cases of over-identification—*i.e.*, when the number of moment conditions (the dimension of the vector η) exceeds the

number of parameters to be estimated (the dimension of $\boldsymbol{\psi}$). This test is performed by multiplying the objective function in equation (20) by the number of observations. It provides a means to assess whether the dynamic restrictions implied by the D-SVAR are consistent with the data, thereby offering a way to evaluate the validity of the D-SVAR framework relative to a standard VAR model.

Beyond specification testing, the D-SVAR also provides a basis for evaluating identification strategies commonly used in SVAR analysis. In particular, it allows to test general null hypotheses on the loading matrix Π_{yz}. For example, one may be interested in the relevance of the timing imposed by short-run restrictions. In this case, the null hypothesis writes $H_0 : \mathcal{H}vec(\Pi_{yz}) = 0$, where the selection matrix $\mathcal{H}$ is such that the elements above the diagonal of Π_{yz} are all equals to zero. Likewise, matrix $\mathcal{H}$ can be adapted to test for long-run restrictions à la Blanchard and Quah (1989). A Wald statistic can then be computed with a consistent estimator of the appropriate covariance matrix. Likewise, one may be interested in testing for the dynamic response to a particular shock, as identified using two competing identification schemes. This can be achieved with a Wald statistic and using the appropriate asymptotic distribution or by bootstrapping techniques as proposed by Inoue and Kilian (2016).[34]

6 Conclusion

In this paper, we have shown that one can identify macroeconomic model shocks in a SVAR under the identifying assumption that the economy shares the dynamic structure of the vast majority of macroeconomic models. Put simply, if the economy is driven by exogenous variables that follow mutually orthogonal AR(1) processes (or more generally structured autoregressive processes), then a D-SVAR framework allows for the identification of the model's structural shocks without relying on zero-impact, long-run, or sign restrictions, or on external instruments. We have provided a formal proof of identification, outlined how estimation and inference can be conducted within the D-SVAR framework, and shown how it can be used to test identification restrictions commonly imposed in the SVAR literature. To illustrate our approach in practice, we applied it to three well-known SVAR studies: Blanchard and Quah (1989) (long-run restrictions), Gertler and Karadi (2015) (proxy VAR), and Christiano et al. (1999) (impact restrictions).

The value of our D-SVAR approach lies in its ability to serve as a benchmark for evaluating identifying assumptions in SVARs. Once a researcher has chosen a set of variables, a lag structure, and an identification scheme for a standard SVAR, she can also

[34] In particular, in the case when the number of structural impulse responses exceeds the number of the VAR parameters, the asymptotic distribution of a Wald statistic is degenerated but the bootstrap test can still be implemented following the transformation of the statistic.

estimate a D-SVAR using the same variables and lag length, while specifying a dynamic process for the unobserved exogenous driving forces. If the structural shock of interest is found to be strongly correlated with one and only one of the D-SVAR shocks, this supports the validity of the identifying restrictions under minimal assumptions about the Data Generating Process (DGP). On the other hand, if the shock correlates with multiple components, this casts doubt on the identifying assumptions—again, based only on minimal assumptions about the DGP.

References

Panos J. Antsaklis and Anthony N. Michel (1997): *Linear Systems*. McGraw-Hill College.

David K Backus, Patrick J Kehoe, and Finn E Kydland (1992): "International Real Business Cycles," *Journal of Political Economy*, 100(4), 745–775.

——— (1994): "Dynamics of the Trade Balance and the Terms of Trade: The J-Curve?," *American Economic Review*, 84(1), 84–103.

Jushan Bai and Peng Wang (2015): "Identification and Bayesian Estimation of Dynamic Factor Models," *Journal of Business & Economic Statistics*, 33(2), 221–240.

Marianne Baxter and Mario J Crucini (1995): "Business Cycles and the Asset Structure of Foreign Trade," *International Economic Review*, 36(4), 821–854.

Paul Beaudry, Chenyu Hou, and Franck Portier (2024): "Monetary Policy When the Phillips Curve Is Quite Flat," *American Economic Journal: Macroeconomics*, 16(1), 1–28.

Paul Beaudry and Makoto Saito (1998): "Estimating the Effects of Monetary Shocks: An Evaluation of Different Approaches," *Journal of Monetary Economics*, 42(2), 241–260.

Ben S. Bernanke and Kenneth N. Kuttner (2005): "What Explains the Stock Market's Reaction to Federal Reserve Policy?," *Journal of Finance*, 60(3), 1221–1257.

Olivier Jean Blanchard and Danny Quah (1989): "The Dynamic Effects of Aggregate Demand and Supply Disturbances," *American Economic Review*, 79(4), 655–673.

Charles T. Carlstrom, Timothy S. Fuerst, and Matthias Paustian (2009): "Monetary policy shocks, Choleski identification, and DNK models," *Journal of Monetary Economics*, 56(7), 1014–1021.

Lawrence J. Christiano, Martin Eichenbaum, and Charles L. Evans (1999): "Monetary Policy Shocks: What Have we Learned and to What End?," in *Handbook of Macroeconomics*, vol. 1, chap. 2, pp. 65–148. Elsevier.

——— (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113(1), 1–45.

Jesús Fernández-Villaverde, Juan F. Rubio-Ramírez, Thomas J. Sargent, and Mark W. Watson (2007): "ABCs (and Ds) of Understanding VARs," *American Economic Review*, 97(3), 1021–1026.

Jordi Galí (2015): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications*, no. 10495 in Economics Books. Princeton University Press, second edn.

F. R. Gantmacher (1959): *The Theory of Matrices. Vols. 1, 2*, Translated by K. A. Hirsch. Chelsea Publishing Co., New York.

Mark Gertler and Peter Karadi (2015): "Monetary Policy Surprises, Credit Costs, and Economic Activity," *American Economic Journal: Macroeconomics*, 7(1), 44–76.

Simon Gilchrist and Egon Zakrajšek (2012): "Credit Spreads and Business Cycle Fluctuations," *American Economic Review*, 102(4), 1692–1720.

Gene H. Golub and Charles F. Van Loan (2013): *Matrix Computations*. The Johns Hopkins University Press, fourth edn.

Christian Gourieroux and Joann Jasiak (2022): "Dynamic Deconvolution and Identification of Independent Autoregressive Sources," *Journal of Time Series Analysis*.

Christian Gouriéroux and Alain Monfort (1995): *Statistics and Econometric Models*, Cambridge Books. Cambridge University Press.

C. Gouriéroux, A. Monfort, and J.-P. Renne (2017): "Statistical Inference for Independent Component Analysis: Application to Structural VAR Models," *Journal of Econometrics*, 196, 111–126.

A. Guay (2021): "Identification of Structural Vector Autoregressions through Higher Unconditional Moments," *Journal of Econometrics*, 225, 27–46.

Refet S Gürkaynak, Brian Sack, and Eric Swanson (2005): "Do Actions Speak Louder Than Words? The Response of Asset Prices to Monetary Policy Actions and Statements," *International Journal of Central Banking*, 1(1).

Refet S Gürkaynak, Brian P Sack, and Eric T Swanson (2007): "Market-Based Measures of Monetary Policy Expectations," *Journal of Business & Economic Statistics*, 25(2), 201–212.

Jonathan Heathcote and Fabrizio Perri (2002): "Financial Autarky and International Business Cycles," *Journal of Monetary Economics*, 49(3), 601–627.

Roger A. Horn and Charles R. Johnson (2013): *Matrix Analysis*. Cambridge University Press, Cambridge; New York, 2nd edn.

Atsushi Inoue and Lutz Kilian (2016): "Joint Confidence Sets for Structural Impulse Responses," *Journal of Econometrics*, 192(2), 421–432.

Alejandro Justiniano, Giorgio Primiceri, and Andrea Tambalotti (2011): "Investment Shocks and the Relative Price of Investment," *Review of Economic Dynamics*, 14(1), 101–121.

Christian Kascha and Karel Mertens (2009): "Business Cycle Analysis and VARMA Models," *Journal of Economic Dynamics and Control*, 33(2), 267–282.

Patrick J. Kehoe and Fabrizio Perri (2002): "International Business Cycles with Endogenous Incomplete Markets," *Econometrica*, 70(3), 907–928.

Lutz Kilian and Helmut Lütkepohl (2016): *Structural Vector Autoregressive Analysis*. Cambridge University Press.

Andrzej Kocięcki and Marcin Kolasa (2018): "Global Identification of Linearized DSGE Models," *Quantitative Economics*, 9(3), 1243–1263.

Ivana Komunjer and Serena Ng (2011): "Dynamic Identification of Dynamic Stochastic General Equilibrium Models," *Econometrica*, 79(6), 1995–2032.

Kenneth N Kuttner (2001): "Monetary Policy Surprises and Interest Rates: Evidence from the Fed Funds Futures Market," *Journal of monetary economics*, 47(3), 523–544.

M. Lanne, M. Meitz, and P. Saikkonen (2017): "Identification and Estimation of Non-Gaussian Structural Vector Autoregressions," *Journal of Econometrics*, 196, 288–304.

Sophocles Mavroeidis (2021): "Identification at the Zero Lower Bound," *Econometrica*, 89(6), 2855–2885.

Ellen R. McGrattan (2010): "Measurement with Minimal Theory," *Quarterly Review*, pp. 2–13.

Karel Mertens and Morten O. Ravn (2013): "The Dynamic Effects of Personal and Corporate Income Tax Changes in the United States," *American Economic Review*, 103(4), 1212–47.

Adrian Pagan and Tim Robinson (2019): "Implications of Partial Information for Econometric Modeling of Macroeconomic Systems," CAMA Working Papers 2019-41, Centre for Applied Macroeconomic Analysis, Crawford School of Public Policy, The Australian National University.

V.A. Ramey (2016): "Macroeconomic Shocks and Their Propagation," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and H. Uhlig, vol. 2 of *Handbook of Macroeconomics*, chap. 0, pp. 71–162. Elsevier.

Christopher A Sims (1980): "Macroeconomics and Reality," *Econometrica*, 48(1), 1–48.

James Stock (2008): "What's New in Econometrics: Time Series," Lecture 7 short course lectures, NBER Summer Institute.

J.H. Stock and M.W. Watson (2016): "Dynamic Factor Models, Factor-Augmented Vector Autoregressions, and Structural Vector Autoregressions in Macroeconomics," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and H. Uhlig, vol. 2 of *Handbook of Macroeconomics*, chap. 0, pp. 415–525. Elsevier.

James H. Stock and Mark Watson (2012): "Disentangling the Channels of the 2007-09 Recession," *Brookings Papers on Economic Activity*, 43(1 (Spring)), 81–156.

James H. Stock and Mark W. Watson (2017): "Twenty Years of Time Series Econometrics in Ten Pictures," *Journal of Economic Perspectives*, 31(2), 59–86.

Harald Uhlig (2005): "What are the Effects of Monetary Policy on Output? Results from an Agnostic Identification Procedure," *Journal of Monetary Economics*, 52(2).

Appendix

A Assumptions, Definitions and Preliminary Propositions

Some assumptions need to be placed on the representation (11)-(12).

Assumption 1 *For any* $z \in \mathbb{C}$, $det(zI_{n_s} - A) = 0$ *implies* $|z| < 1$.

Assumption 1 restricts the class of matrices A to those with eigenvalues lying inside the unit circle. Under Assumption 1 and (11)-(12), the process $\{Y_t\}$ admits the following infinite Vector Moving Average (VMA) representation:

$$Y_t = \left[C(I_{n_s} - AL)^{-1}BL + D\right]\varepsilon_t = \sum_{j=0}^{\infty} h(j;\boldsymbol{\theta})\varepsilon_{t-j} = H(L,\boldsymbol{\psi})\varepsilon_t,$$

where L denotes the lag operator. For $z \in \mathbb{C}$, the transfer function (the z-transform of the impulse response function) is given by:

$$H(z;\boldsymbol{\psi}) = D + C\left(zI_{n_s} - A\right)^{-1}B = \sum_{j=0}^{\infty} h(j;\boldsymbol{\theta})z^{-j}.$$

For every $\boldsymbol{\psi} \in \boldsymbol{\Psi} \subset \mathbb{R}^{n_\psi}$ where $n_\psi = \dim(\psi)$, $\mathbb{E}(Y_t) = 0$ and

$$\mathbb{E}(Y_t Y_s') \equiv \Gamma(s-t;\boldsymbol{\psi}) = \sum_{j=0}^{\infty} h(j;\boldsymbol{\psi})\Sigma_\varepsilon h(j+s-t;\boldsymbol{\psi})',$$

for all $t, s \geq 1$.

For any weakly stationary process $\{Y_t\}$ implied by Assumption 1 and under the assumption that the shocks ϵ_t are Gaussian, the unconditional mean and auto-covariance function completely characterize the properties of the process. Here, we do not impose the Gaussianity of the shocks but the local identification is based only on the auto-covariance function. Let us therefore define the auto-covariance generating function as:

$$\Omega(z,\boldsymbol{\psi}) = \sum_{j=-\infty}^{\infty} \Gamma(j;\boldsymbol{\psi})z^{-j},$$

for any $z \in \mathbb{C}$. Evaluating $\Omega(z,\boldsymbol{\psi})$ at $z = \exp(-i\omega)$ for any $\omega \in [-\pi,\pi]$ and rescaling it by $(2\pi)^{-1}$ yields the spectral density of the observable $\{Y_t\}$ which is always positive semi-definite. To simplify, we hereafter refer to $\Omega(z,\boldsymbol{\psi})$ as the spectral density as well.

As will be clear later, it will prove useful to defined observational equivalence for this class of multivariate covariance stationary process. We closely follow Komunjer and Ng (2011) and define it with respect to the entire auto-covariance function of the observable (or the spectral density).

Definition 1 *Two sets of state-space parameters* $\boldsymbol{\psi}$ *and* $\widetilde{\boldsymbol{\psi}}$ *are observationally equivalent if* $\Omega(z;\boldsymbol{\psi}) = \Omega(z;\widetilde{\boldsymbol{\psi}})$, *for all* $z \in \mathbb{C}$ *or, equivalently,* $\Gamma(j;\boldsymbol{\psi}) = \Gamma(j;\widetilde{\boldsymbol{\psi}})$ *at all* $j \geq 0$.

In other words, two stationary state-space models are observationally equivalent is they share the same auto-covariance (spectral) properties. This then allows us to define local identification.

Definition 2 *The state-space representation (11)-(12)) is locally identifiable from the spectral density of Y_t (or equivalently from the auto-covariances of Y_t) at $\boldsymbol{\psi} \in \boldsymbol{\psi}$ if there exists an open neighborhood of $\boldsymbol{\psi}$ such that for every $\widetilde{\boldsymbol{\psi}}$ in this neighbourhood, $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\psi}}$ are observationally equivalent if and only if $\widetilde{\boldsymbol{\psi}} = \boldsymbol{\psi}$.*

In state-space system, the spectral density function can be simply obtained from the transfer function and the covariance matrix of the shocks, Σ_ε, as

$$\Omega(z, \boldsymbol{\psi}) = H(z; \boldsymbol{\psi}) \Sigma_\varepsilon H(z^{-1}; \boldsymbol{\psi})',$$

at $z = \exp(-i\omega)$ for any $\omega \in [-\pi, \pi]$. As explained in Komunjer and Ng (2011), this representation of the spectral density makes it clear that equivalent spectral density function can obtain because *(i)* for given Σ_ε, two distinct vectors of state-space parameters, $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\psi}}$, yield the same transfer function ($H(z; \boldsymbol{\psi}) = H(z; \widetilde{\boldsymbol{\psi}})$) or *(ii)* many pairs of $H(z; \boldsymbol{\psi})$ and Σ_ε give rise to the same spectral density.

In general, the state-space parameter $\boldsymbol{\psi}$ is not identifiable from the second order moments of the observable variables. As an illustration, let us consider the following two state-space representations (K_t is observable):

$$\mathcal{S} = \begin{cases} K_t &= \Pi_{kk} K_{t-1} + \Pi_{kz} Z_t, \\ Y_t &= \Pi_{yk} K_{t-1} + \Pi_{yz} Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t, \end{cases} \quad \text{and} \quad \widetilde{\mathcal{S}} = \begin{cases} K_t &= \Pi_{kk} K_{t-1} + \widetilde{\Pi}_{kz} \widetilde{Z}_t, \\ Y_t &= \Pi_{yk} K_{t-1} + \widetilde{\Pi}_{yz} \widetilde{Z}_t, \\ \widetilde{Z}_t &= \widetilde{R} \widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t, \end{cases}$$

where $\widetilde{Z}_t = U^{-1} Z_t$, $\widetilde{\Pi}_{kz} = \Pi_{kz} U$, $\widetilde{\Pi}_{yz} = \Pi_{yz} U$, $\widetilde{R} = U^{-1} R U$ and $\widetilde{\Sigma}_\varepsilon = U^{-1} \Sigma_\varepsilon U^{-1\prime}$ for some full rank matrix U. The two representations $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are observationally equivalent with respect to the spectral function since $\Omega(z, \boldsymbol{\psi}) = \Omega(z, \widetilde{\boldsymbol{\psi}})$ for all $z \in \mathbb{C}$ where the vectors $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\psi}}$ gather, respectively, the elements of the vectorization of matrices defining, respectively, $\mathcal{S}$ and $\widetilde{\mathcal{S}}$.

Following Komunjer and Ng (2011) (see Proposition 1-S), the following property obtains in the case of the $ABCD$ representation

Property: *Two distinct vectors of state-space parameters $\boldsymbol{\psi}, \widetilde{\boldsymbol{\psi}} \in \Psi$ are observationally equivalent respective to the transfer function and the spectral density if and only if there exists a full rank $n_s \times n_s$ matrix T and a full rank $n_z \times n_z$ matrix U such that $\widetilde{A} = TAT^{-1}$, $\widetilde{B} = TBU$, $\widetilde{C} = CT^{-1}$, $\widetilde{D} = DU$ and $\Sigma_{\widetilde{\varepsilon}} = U^{-1} \Sigma_\varepsilon U^{-1\prime}$.*

This property obtains as follows. First, the equalities $\widetilde{A} = TAT^{-1}$, $\widetilde{B} = TB$, $\widetilde{C} = CT^{-1}$ are necessary and sufficient for the equivalence of the transfer function $H(z; \widetilde{\boldsymbol{\psi}}) = H(z; \boldsymbol{\psi})$. Sufficiency follows directly from the observation of the transfer function in the ABCD representation

$$\begin{aligned} H(z; \widetilde{\boldsymbol{\psi}}) &= \widetilde{C}(z I_{n_s} - \widetilde{A})^{-1} \widetilde{B} + D, \\ &= CT^{-1}(z I_{n_s} - TAT^{-1})^{-1} TB + D, \\ &= CT^{-1} T (z I_{n_s} - A)^{-1} T^{-1} TB + D = H(z; \boldsymbol{\psi}). \end{aligned}$$

The necessary condition follows directly from a well known result in control theory under the condition of minimality of the state-space representation (See Theorem 3.10 in Antsaklis and Michel (1997) and Chapter 8 in Gouriéroux and Monfort (1995)). A system is minimal if and only if it is controllable and observable (see Appendix B for formal definitions), implying that, among all systems leading to the same output spectral density, it is driven by the minimal number of state variables. This equivalence class is function of a non-singular transformation that corresponds to a rotation of the state, i. e., $T^{-1} S_t$.

The equivalence of the spectral density is obtained for a full rank matrix U such that:

$$H(z; \boldsymbol{\psi}) \Sigma_\varepsilon H(z^{-1}; \boldsymbol{\psi})' = H(z; \boldsymbol{\psi}) U U^{-1} \Sigma_\varepsilon U^{-1\prime} U' H(z^{-1}; \boldsymbol{\psi})'.$$

Fixing $\widetilde{B} = BU$, $\widetilde{D} = DU$ and $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_{\varepsilon}U^{-1\prime}$, this yields $H(z;\widetilde{\boldsymbol{\psi}}) = H(z;\boldsymbol{\psi})U = DU + C\left[I - Az\right]^{-1}BzU$. Observational equivalence follows immediately.

We have established that the identification of $\boldsymbol{\psi}$ or a subset of $\boldsymbol{\psi}$ cannot be achieved without imposing additional restrictions on the covariance matrix Σ_{ε}. Specifically, we consider the case where the structural innovations are mutually orthogonal, and their covariance matrix is normalized to the identity matrix, such that $\mathbb{E}(\varepsilon_t\varepsilon_t') = I_{n_z}$. This is a common identifying assumption in the SVAR literature (see Ramey (2016)).

A direct implication of this assumption is that the only admissible matrix U that satisfies $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_{\varepsilon}U^{-1\prime} = I_{n_z}$ is an orthonormal matrix –*i.e.* $UU' = I_{n_z}$ (see Corollary 1 of Kocięcki and Kolasa (2018)).

We further adapt Assumption 1 to the initial state-space representation (8)–(10):

Assumption 1′ *In the presence of endogenous state variables ($n_k > 0$), for any $z \in \mathbb{C}$, $\det(zI_{n_s} - A) = \det(zI_{n_k} - \Pi_{kk})\det(zI_{n_z} - R) = 0$ implies $|z| < 1$, and the matrices Π_{kk} and R have no eigenvalues in common.*

When $n_k > 0$, since the matrix A is block triangular (see System (11)–(12)), it holds that $\det(zI_{n_s} - A) = \det(zI_{n_k} - \Pi_{kk})\det(zI_{n_z} - R)$. Both Π_{kk} and R have eigenvalues that lie within the unit circle and do not share any common eigenvalues. This assumption is necessary to disentangle the dynamics of the endogenous variables K_t from those of the exogenous process Z_t.

Assumption 2

(i) *When $n_k = 0$, $C = \Pi_{yz}$ is of full rank –i.e.* $\operatorname{rank}(\Pi_{yz}) = n_s$.

(ii) *When $n_k > 0$, the matrices Π_{kk}, Π_{yk}, and Π_{yz} are of full rank, and* $\operatorname{rank}(\Pi_{kz}R) \geq \min(n_k, n_z)$.

Without endogenous state variables ($n_k = 0$), Π_{yz} has rank n_z. In this case, the system is defined solely by equations (9)–(10), without the term $\Pi_{yk}K_{t-1}$ on the right-hand side, and satisfies $\det(zI_{n_s} - A) = \det(zI_{n_k} - R)$.

When $n_k > 0$, Assumption 2 ensures that no endogenous state variables are redundant. The condition $\operatorname{rank}(\Pi_{kz}R) \geq \min(n_k, n_z)$ implies that the matrices Π_{kz} and R may be rank-deficient. However, this condition is essential for obtaining a minimal representation.

The following proposition follows directly:

Proposition A.1 *Under Assumption 1′ and Assumption 2, the state-space representation (8)–(10) is minimal, and the transfer function $H(z;\boldsymbol{\psi})$ is invertible.*

Proof: See Appendix C.1.

Assumption 2 ensures that the dynamics of Z_t can be recovered from the observables Y_t. To see this, consider the transfer function of System (8)–(10):

$$\begin{aligned} H(z;\boldsymbol{\psi}) &= \left[\Pi_{yk}(zI_{n_k} - \Pi_{kk})^{-1}\Pi_{kz} + \Pi_{yz}\right](zI_{n_z} - R)^{-1}, \\ &= \left[\Pi_{yk}(zI_{n_k} - \Pi_{kk})^{-1}\Pi_{kz}\Pi_{yz}^{-1} + I_{n_z}\right]\Pi_{yz}(zI_{n_z} - R)^{-1}. \end{aligned}$$

Under *(ii)*, the $rank(H(z;\boldsymbol{\psi})) = n_z$ for every $|z| < 1$, then $H(z;\boldsymbol{\psi})$ is invertible as shown in Proposition A.1. When $n_y = 0$, $H(z;\boldsymbol{\psi}) = \Pi_{yz}(zI_{n_z} - R)^{-1}$ is full rank for every $|z| < 1$. In the following, we consider the case where $n_k > 0$.

Endowed with Assumption 1′ and Assumption 2, the next proposition shows that the rotation matrix T is block diagonal.

Proposition A.2 *Under Assumption 1′ and Assumption 2 we have*

- *When the endogenous state variables K_t are unobserved, the full rank matrix T has the form*

$$T = \begin{bmatrix} T_{11} & 0_{n_k \times n_z} \\ 0_{n_z \times n_k} & V \end{bmatrix}, \tag{A.1}$$

  with T_{11} a full rank $(n_k \times n_k)$ matrix.

- *When the endogenous state variables K_t are observed, the full rank matrix T has necessarily the following form:*

$$T = \begin{bmatrix} I_{n_k} & 0_{n_k \times n_z} \\ 0_{n_z \times n_k} & V \end{bmatrix}, \tag{A.2}$$

In both cases, V is an orthonormal $(n_z \times n_z)$ matrix such that $VV' = I_{n_z}$ and $V = U' = U^{-1}$ defined above.

Proof : See Appendix C.2.

Under the assumption that the endogenous state variables K_t are unobserved, the block diagonal structure of the matrix T implies that

$$\begin{aligned} H(z;\boldsymbol{\psi}) &= \left[\Pi_{yk}(zI_{n_k} - \Pi_{kk})^{-1}\Pi_{kz} + \Pi_{yz}\right](zI_{n_z} - R)^{-1}, \\ &= \left[\Pi_{yk}T_{11}^{-1}(zI_{n_k} - T_{11}\Pi_{kk}T_{11}^{-1})^{-1}T_{11}\Pi_{kz} + \Pi_{yz}V'\right](zI_{n_z} - R)^{-1}. \end{aligned}$$

Proposition A.1 also implies that, as long as the endogenous state variable K_t is observed, the matrices Π_{kk} and Π_{yk} can be identified using the observed spectral density function.[35] However, in both cases, this does not allow for proper identification of the loading matrix Π_{yz} based solely on the properties of Z_t.

Indeed, a direct implication of the proposition is that there exists at least one equivalent exogenous process to (10):

$$Z_t = VRV'Z_{t-1} + \varepsilon_t, \tag{A.3}$$

where observational equivalence in terms of the transfer function (and spectral density) holds if and only if the sub-matrix V in the matrix T, as defined in (A.2), is orthonormal ($VV' = I_{n_z}$). In this case, pre-multiplying (A.3) by V', we obtain:

$$\widetilde{Z}_t = R\widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t,$$

where $\widetilde{Z}_t = V'Z_t$, $\widetilde{\varepsilon}_t = V'\varepsilon_t$, and $\mathbb{E}(\widetilde{\varepsilon}_t\widetilde{\varepsilon}_t') = I_{n_z}$.

In other words, the sole knowledge of the spectral properties of Z_t is insufficient to identify Π_{yz}. Additional (dynamic) restrictions must be imposed on the autoregressive matrix R.

B Preliminaries: Controllability, Observability and Minimality

In the literature on linear systems, invariant finite-dimensional dynamical systems are described by the following state-space system:

$$S_{t+1} = AS_t + B\epsilon_t, \tag{B.4}$$

$$Y_t = CS_t + D\epsilon_t, \tag{B.5}$$

[35] The observed spectral density (or equivalently, the auto-covariance) function can be obtained by estimating a VAR, a VARMA, or through the maximum likelihood estimation of a state-space representation.

where the $n_s \times 1$ vector S_t denotes the state variables, the $n_y \times 1$ vector Y_t denotes the outputs, and the $n_\epsilon \times 1$ vector ϵ_t contains the inputs (see, for example, Antsaklis and Michel (1997) and Gouriéroux and Monfort (1995)). Note that this ABCD state-space representation differs from that of Fernández-Villaverde et al. (2007), but it allows us to define the concepts of controllability, observability, and minimality in accordance with the standard linear system literature.

The system (or the pair (A, B)) is considered controllable when any state S_t can be driven to the initial state in a finite number of steps for a given input sequence ϵ_t. To illustrate the concept of controllability, the state variables at time $t = k$ can be expressed as:

$$S_k = A^k S_0 + A^{k-1} B\epsilon_0 + A^{k-2}\epsilon_1 + \ldots + AB\epsilon_{k-2} + B\epsilon_{k-1},$$

where S_0 is the vector of initial states. The state S_k is said to be reachable at time k if, for some finite $t = 0 \leq k$, there exists an input sequence $\epsilon_t, t \in [0, k]$, that transfers the state from the origin at $t = 0$ to S_k at time $t = k$. This implies that the system is reachable if and only if the rank of the matrix $\mathbb{C} = \left[B, AB, A^2B, \ldots, A^{k-1}B\right]$, which contains the matrices related to the input sequence in the above system, is of full rank. Moreover, if the system is reachable, then it is also controllable.

According to the ABCD system representation (B.4) and (B.5), a formal definition of controllability is given by:

Definition 3 Controllability *: A system is controllable if and only if the controllability matrix* $\mathbb{C} = \left[B, AB, A^2B, \ldots, A^{n_s-1}B\right] \in \mathbb{R}^{n_s \times (n_z n_s)}$ *has full row rank –i.e.* $rank(\mathbb{C}) = n_s$.

If a system is state observable, its present state can be determined from the knowledge of the present and future outputs Y_t and inputs ϵ_t. The expression for output Y_k can be rewritten as:

$$Y_k = CA^k S_0 + CA^{k-1} B\epsilon_0 + CA^{k-2} B\epsilon_1 + \ldots + CAB\epsilon_{k-2} + CB\epsilon_{k-1} + D\epsilon_k.$$

Observability refers to the ability to uniquely determine the state of a system from knowledge of current and future outputs and inputs. Specifically, a system is said to be observable if it is possible to determine the state at time $t = 0$ from knowledge of future system outputs Y_k and system inputs ϵ_k for $k \geq 0$. A state is considered unobservable if it remains indistinguishable from the zero state under any possible output observation, mathematically expressed as $CA^k S = 0$ for all $k \geq 0$. Conversely, a system is completely state observable if the only state that satisfies this condition is the zero state, $S = 0$. The degree of observability is determined by the rank of the matrices CA^k for all k, with full rank indicating complete observability.

A formal definition of observability is given by:

Definition 4 Observability*: A system is observable if and only if the observability matrix* $\mathbb{O}$ *has full column rank –i.e.* $rank(\mathbb{O}) = n_s$*, where*

$$\mathbb{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n_s-1} \end{bmatrix} \in \mathbb{R}^{n_y n_s \times n_s}.$$

Theorem 1 *A state-space representation is minimal if and only if it is both controllable and observable.*

See Antsaklis and Michel (1997), Theorem 3.9, p. 395 for a proof of minimality under controllability and observability For a minimal state-space representation, the state vector is of the smallest possible dimension.

C Proofs of Preliminary and Main Propositions

C.1 Proof of Proposition A.1

Here we formal prove controllability and observability of the system given by:

- The state-space system is defined as:

$$A = \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0 & R \end{bmatrix}, \quad B = \begin{bmatrix} \Pi_{kz} \\ I_{n_z} \end{bmatrix}, \quad C = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \end{bmatrix}.$$

The total system dimension is $n_s = n_k + n_z$, where n_k and n_z are the dimensions of the Π_{kk}-block and R-block, respectively.

First, consider the case without endogenous state variables. In this scenario, $A = R$ and $C = \Pi_{yz}$, which is of full rank. This implies that S_t follows a VAR(1) process given by $S_t = RS_{t-1} + \epsilon_t$, where ϵ_t has the same dimension as S_t. This directly implies that the system is controllable according to the definition above. Moreover, since $Y_t = \Pi_{yz}S_t$ and Π_{yz} is of full rank, the system is also observable, which implies that the system is minimal.

In the case with unobserved endogenous state variables, the controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n_s-1}B \end{bmatrix}.$$

In the first step, we compute powers of A applied to B
1. First, we compute AB:

$$AB = AB = \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \Pi_{kz} \\ I_{n_z} \end{bmatrix} = \begin{bmatrix} \Pi_{kk}\Pi_{kz} + \Pi_{kz}R \\ R \end{bmatrix}.$$

2. Next, compute A^2B:

$$A^2B = A(AB) = \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \Pi_{kk}\Pi_{kz} + \Pi_{kz}R \\ R \end{bmatrix} = \begin{bmatrix} \Pi_{kk}^2\Pi_{kz} + \Pi_{kk}\Pi_{kz}R + \Pi_{kz}R^2 \\ R^2 \end{bmatrix}.$$

3. The general form of powers of A applied to B is:

$$A^kB = \begin{bmatrix} \Pi_{kk}^k\Pi_{kz} + \sum_{i=0}^{k-1} \Pi_{kk}^i\Pi_{kz}R^{k-1-i} \\ R^k \end{bmatrix}.$$

The controllability matrix takes the form:

$$\mathcal{C} = \begin{bmatrix} \Pi_{kz} & \Pi_{kk}\Pi_{kz} + \Pi_{kz}R & \dots & \\ I_{n_z} & R & R^2 & \dots \end{bmatrix}.$$

The block structure of A reveals that its eigenvalues are the union of the eigenvalues of Π_{kk} and R. Since Π_{kk} and R do not share any common eigenvalues, the dynamics of the two subsystems are distinguishable. The controllability of the G-subsystem (the upper-block of $\mathcal{C}$) is governed by the pair (Π_{kk}, Π_{kz}). Even if Π_{kz} is not of full rank, Π_{kk} retains partial controllability. However, any rank deficiency in Π_{kz} is compensated by the term $\Pi_{kz}R$, under the assumption that $\text{rank}(\Pi_{kz}R) \geq \min(n_k, n_z)$. The R-subsystem (the lower block of $\mathcal{C}$), on the other hand, is directly controllable through the identity matrix I_{n_z} in B. This ensures that all states of the R-subsystem are controllable. The coupling term $\Pi_{kz}R$ plays a crucial role in linking the Π_{kk}- and R-subsystems. Since Π_{kk} and R have distinct eigenvalues, the rank of $\Pi_{kz}R$ does not diminish the rank of the overall controllability matrix $\mathcal{C}$. In conclusion, the system is controllable because the G-subsystem is partially controllable through Π_{kz}, the R-subsystem is fully controllable through I_{n_z}, and the term $\Pi_{kz}R$ ensures interaction between the two subsystems. This implies that the system is fully controllable.

Now for the observability. The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n_s-1} \end{bmatrix}.$$

In the first step, we compute powers of A applied to C
1. First, compute CA:

$$CA = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \end{bmatrix} \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0 & R \end{bmatrix} = \begin{bmatrix} \Pi_{yk}\Pi_{kk} + \Pi_{yz}\Pi_{kz}R & \Pi_{yz}R \end{bmatrix}.$$

2. Next, compute CA^2:

$$CA^2 = CA \cdot A = \begin{bmatrix} \Pi_{yk}\Pi_{kk}^2 + \Pi_{yz}\Pi_{kz}\Pi_{kk}R + \Pi_{yz}\Pi_{kz}R^2 & \Pi_{yz}R^2 \end{bmatrix}.$$

The observability matrix takes the form:

$$\mathcal{O} = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \\ \Pi_{yk}\Pi_{kk} + \Pi_{yz}\Pi_{kz}R & \Pi_{yz}R \\ \Pi_{yk}\Pi_{kk}^2 + \Pi_{yz}\Pi_{kz}\Pi_{kk}R + \Pi_{yz}\Pi_{kz}R^2 & \Pi_{yz}R^2 \\ \vdots & \vdots \end{bmatrix}.$$

The left block of the system ensures the observability of the Π_{kk}-block through Π_{yk}, while the right block ensures the observability of the R-block through Π_{yz}. Additionally, the coupling term $\Pi_{kz}R$ guarantees interaction between the two blocks.

Given that $\text{rank}(\Pi_{kz}R) \geq \min(n_k, n_z)$ and that the eigenvalues of Π_{kk} and R are distinct, the rank of the observability matrix $\mathcal{O}$ is:

$$\text{rank}(\mathcal{O}) = n_k + n_z = n_s.$$

Therefore, the system is fully observable. By controllability and observability, the system is minimal.

Let us analyze the Rosenbrock Matrix (see Antsaklis and Michel (1997)). The transfer function $H(z, \boldsymbol{\psi})$ of the linear system (11) and (12) given by:

$$H(z, \boldsymbol{\psi}) = C(zI_{n_s} - A)^{-1}B + D.$$

The Rosenbrock matrix for the system is defined as:

$$\mathcal{R}(z) = \begin{bmatrix} zI_{n_s} - A & -B \\ C & D \end{bmatrix}.$$

For the system to be controllable and observable, the Rosenbrock matrix $\mathcal{R}(z)$ must have full rank for all $z \in \mathbb{C}$. If the Rosenbrock matrix $\mathcal{R}(z)$ has full rank for all $z \in \mathbb{C}$, the transfer function $T(z)$ is invertible, with no zeros or singularities. Conversely, if the Rosenbrock matrix loses rank at any z, the transfer function has zeros at those values, indicating potential non-invertible. The Rosenbrock matrix for the system is:

$$\mathcal{R}(z) = \begin{bmatrix} zI_{n_k} - \Pi_{kk} & -\Pi_{kz}R & -\Pi_{kz} \\ 0 & zI_{n_z} - R & -I_{n_z} \\ \Pi_{yk} & \Pi_{yz} & \Pi_{yz} \end{bmatrix}.$$

The rank of the Rosenbrock matrix $\mathcal{R}(z)$ can be determined as follows. The block $zI_{n_s} - A$ is composed of $zI_{n_k} - \Pi_{kk}$ and $zI_{n_z} - R$. Since Π_{kk} and R are distinct, the matrix $zI_{n_k} - \Pi_{kk}$ is full rank for z values outside the eigenvalues of Π_{kk}, and $zI_{n_z} - R$ is full rank for z values outside the eigenvalues of R.

The blocks B, C, and D ensure that controllability and observability are preserved by compensating for any potential rank deficiencies in the system. Consequently, the rank of the Rosenbrock matrix $\mathcal{R}(z)$ is given by:

$$\text{rank}(\mathcal{R}(z)) = n_k + n_z,$$

where n_k represents the dimension of Π_{kk}, and n_z represents the dimension of R. Since the rank of $\mathcal{R}(z)$ equals the total state dimension, the system is both controllable and observable, as demonstrated above. This also implies that the transfer function $T(z)$ is invertible.

C.2 Proof of Proposition A.2

Consider a general form for the matrix T such that $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$.

We first show that the relation $\tilde{A}T = TA$ implies that $T_{21} = 0$. Since $\tilde{A}$ is necessarily block upper triangular, $\tilde{R}T_{21} = T_{21}\Pi_{kk}$ which can be rewritten as:

$$[(I_{n_k} \otimes \tilde{R}) - (\Pi'_{kk} \otimes I_{n_z})]vec(T_{21}) = 0. \tag{C.6}$$

Matrix $\tilde{A}$ is *similar* to matrix A which implies that the eigenvalues of $\tilde{A}$ are the same than the eigenvalues of A.[36] Since A is block upper diagonal, for the set of eigenvalues of A denoted $\lambda(A)$, we have $\lambda(A) = \lambda(\Pi_{kk}) \cup \lambda(R)$. The same property holds for $\tilde{A}$ –*i.e.* $\lambda(\tilde{A}) = \lambda(\tilde{\Pi}_{kk}) \cup \lambda(\tilde{R})$. Since the matrices A and $\tilde{A}$ have the same block upper triangular structure and the same eigenvalues, this implies that the eigenvalues of the matrix $\tilde{R}$ are the same as the eigenvalues of the matrix R. Under Assumption 1′, R and Π_{kk} share no common eigenvalues, this also holds for $\tilde{R}$ and $\tilde{\Pi}_{kk}$. The expression $[(I_{n_k} \otimes \tilde{R}) - (\Pi'_{kk} \otimes I_{n_z})]$ is then of full rank, Equation (C.6) holds only for $T_{21} = 0$.[37]

Now, for the block upper triangular matrix $T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n_z \times n_k} & T_{22} \end{bmatrix}$, the equation $\widetilde{B} = TBU$ gives for the left lower block $I_{n_z} = T_{22}U$. Since U is orthonormal, this implies $T_{22} \equiv U^{-1} = U' = V$ an orthonormal matrix. The matrix T has the following form: $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & V \end{bmatrix}$.

We also have that $\widetilde{D} = DU$, where U is an orthonormal matrix. This implies that $\widetilde{\Pi}_{yz} = \Pi_{yz}U$. Moreover, since $\widetilde{C} = CT^{-1}$, we can write:

$$\widetilde{C} = [\widetilde{\Pi}_{yk} \quad \widetilde{\Pi}_{yz}] = \left[\Pi_{yk}T_{11}^{-1} \quad -\Pi_{yk}T_{11}^{-1}T_{12}T_{22}^{-1} + \Pi_{yz}T_{22}^{-1}\right].$$

Since $T_{22}^{-1} = V^{-1} = V'$, where V is an orthonormal matrix, and using $\widetilde{\Pi}_{yz} = \Pi_{yz}U$ from $\widetilde{D} = DU$, it necessarily follows that $T_{12} = 0$.

Finally, the result for the case where the state variables K_t are observed follows directly.

[36] If X is a square matrix and nonsingular, then A and $B = X^{-1}AX$ are similar and X is called a similarity transformation. If two matrices A and B are similar, they have the same eigenvalues –*i.e.* $\lambda(A) = \lambda(B)$, and the same number of independent eigenvectors but probably not the same eigenvectors (see Golub and Van Loan (2013) p. 349). Moreover, if X is an orthonormal matrix –*i.e.* $XX' = I$, A and B real matrices and $A = XBX'$, A is said to be real orthogonally similar to B (see Horn and Johnson (2013), p. 94).

[37] If λ is an eigenvalue of A and μ an eigenvalue of B, $\lambda - \mu$ is an eigenvalue of $(I \otimes A) - (B \otimes I)$ and all eigenvalues of $(I \otimes A) - (B \otimes I)$ is on this form. Thus $(I \otimes A) - (B \otimes I)$ has zero as an eigenvalue if and only if A has an eigenvalue λ and B has an eigenvalue μ such that $\lambda - \mu = 0$.

C.3 Proof of Proposition 1

Our strategy is to show that the only admissible permutation matrix in equation (A.3) is $V = I_{n_z}$ (up to changes of sign and/or permutation of the identity matrix) and hence that there indeed exists only one Z_t process that is compatible with the spectral properties of observables Y_t. The only admissible orthonormal matrix V such that all elements (i,i) of the diagonal matrix R are identified is $V = I$ when the diagonal element $r_{i,i} \neq r_{j,j}$ for $\forall i \neq j$. For any other orthonormal matrix V, it is easy to verify that the resulting $\tilde{R} = VRV'$ matrix is not diagonal. Since V is a square matrix and is non singular, V corresponds to a *similarity transformation*, the eigenvalues of matrix $\tilde{R}$ are the same as the eigenvalues of matrix R which implies that the diagonal elements of $\tilde{R}$ are the same as the diagonal elements of R. Moreover, for diagonal matrices $\tilde{R}$ and R, the system of equations $\tilde{R}V - VR = 0$ leads to

$$(\tilde{r}_{i,i} - r_{i,i})V_{i,i} = 0 \quad (\text{ for } i = 1, \ldots, n_z), \tag{C.7}$$

$$(\tilde{r}_{i,i} - r_{j,j})V_{i,j} = 0 \quad (\text{for } i \neq j \text{ and } i, j = 1, \ldots, n_z), \tag{C.8}$$

where $r_{i,j}$ and $\tilde{r}_{i,j}$ are respectively the element (i,j) of R and $\tilde{R}$.[38] The first set of equation implies that $\tilde{r}_{i,i} = r_{i,i}$ for diagonal elements of V which are different from zero. Since the diagonal elements of R (and then $\tilde{R}$) are different, the second set of equations implies necessarily that $V_{i,j} = 0$ for all $i \neq j$. Matrix V is necessarily diagonal and the only orthonormal matrix which is diagonal is the identity. The result also holds up to changes of sign and/or permutation of the identity matrix.

Assume now that some elements on the diagonal are the same. Denote the multiplicity of similar diagonal elements $r_{i,i}$ by $m(r_{i,i})$. One can first check that all diagonal elements are the same, in which case any orthonormal matrix V is admissible by equations (C.7) and (C.8). Now consider that a subgroup of elements has the same value. The elements in the V matrix corresponding to multiple values are not uniquely defined but only up to the post multiplication by an $m(r_{i,i}) \times m(r_{i,i})$ orthogonal matrix. Without lost of generality, suppose that the diagonal elements with the same value are ordered as the first $m(r_{1,1})$ elements on the diagonal and the other elements on the diagonal $r_{j,j}$ are different from $r_{1,1}$, then define the matrix V such that $V = \begin{bmatrix} V_{m(r_{1,1})\times m(r_{1,1})} & 0 \\ 0 & I_{(n_z - m(r_{1,1}))} \end{bmatrix}$, where $V_{m(r_{1,1})\times m(r_{1,1})}$ is an orthonormal matrix. Consequently, there exists an infinity of admissible V matrices such that $V_{m(r_{1,1})\times m(r_{1,1})}$ is orthonormal. This argument can be generalised to more than one diagonal element with multiplicity. A sufficient condition for local identification is therefore that matrix R be diagonal with distinct diagonal elements ($r_{i,i} \neq r_{j,j}$, $\forall i \neq j$).

In the case of unobserved endogenous state variables K_t, the transfer function $H(z;\boldsymbol{\psi})$ is then expressed as:

$$\begin{aligned} H(z;\boldsymbol{\psi}) &= \left[\Pi_{yk}(zI_{n_k} - \Pi_{kk})^{-1}\Pi_{kz} + \Pi_{yz}\right](zI_{n_z} - R)^{-1}, \\ &= \left[\Pi_{yk}T_{11}^{-1}(zI_{n_k} - T_{11}\Pi_{kk}T_{11}^{-1})^{-1}T_{11}\Pi_{kz} + \Pi_{yz}V'\right](zI_{n_z} - R)^{-1}, \\ &= \left[\Pi_{yk}T_{11}^{-1}(zI_{n_k} - T_{11}\Pi_{kk}T_{11}^{-1})^{-1}T_{11}\Pi_{kz}\Pi_{yz}^{-1} + I_{n_z}\right]\Pi_{yz}(zI_{n_z} - R)^{-1}, \end{aligned}$$

for $V = I$. This representation implies that the matrices Π_{yz} and R can be locally identified. However, only the expression

$$\left[\Pi_{yk}T_{11}^{-1}(zI_{n_k} - T_{11}\Pi_{kk}T_{11}^{-1})^{-1}T_{11}\Pi_{kz}\Pi_{yz}^{-1} + I_{n_z}\right]$$

can be identified, and the individual matrices within this term cannot be separately identified. Nevertheless, this is sufficient to retrieve the impulse response functions (IRFs) of the structural

[38] The system of equations $\tilde{R}V - VR = 0$ is of the well known form $AX - BX = C$ in control theory called the Sylvester equation. For $C = 0$, this corresponds to the homogeneous Sylvester equation (see Gantmacher (1959), chap VIII).

shocks. In contrast, when the endogenous state variables K_t are observed, all individual matrices can be locally identified.

C.4 Proof of Proposition 2.

By $\tilde{R} = VRV'$, one show that the only admissible matrix V which satisfies $\tilde{R}V = VR$ for lower triangular matrices $\tilde{R}$ and R is $V = I_{n_z}$. Since V is of full rank and orthonormal and $\tilde{R} = VRV'$, $\tilde{R}$ and R have the same eigenvalues. Moreover, the eigenvalues of a lower triangular matrix are the elements on the diagonal. By $\tilde{R}V = VR$, we have

$$[(I_{n_z} \otimes \tilde{R}) - (R' \otimes I_{n_z})]vec(V) = 0. \tag{C.9}$$

This implies that $vec(V)$ belong to the null space of $A = [(I_{n_z} \otimes \tilde{R}) - (R' \otimes I_{n_z})]$. Since $\tilde{R}$ and R have n_z common roots, the null space of A has a dimension equal to n_z. The system of equations (C.9) can be written more explicitly as:

$$\begin{bmatrix} (\tilde{R} - \mu_1 I) & -r_{2,1}I & -r_{3,1}I & \cdots & -r_{n_z,1}I \\ 0 & (\tilde{R} - \mu_2 I) & -r_{3,2}I & \cdots & -r_{n_z,2}I \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & (\tilde{R} - \mu_{n_z-1} I) & -r_{n_z,n_z-1}I \\ 0 & 0 & \cdots & 0 & (\tilde{R} - \mu_{n_z} I) \end{bmatrix} \begin{bmatrix} V_{[.,1]} \\ V_{[.,2]} \\ \vdots \\ V_{[.,n_z-1]} \\ V_{[.,nz]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

with $V_{[.,i]}$ a vector containing the i-th column of matrix V, μ_j for $j = 1, \ldots, n_z$ are the eigenvalues of R and the eigenvalues of $\tilde{R}$ and R are the same which are given by the elements on the diagonal. Under Proposition 1, the eigenvalues μ_j for $j = 1, \ldots, n_z$ are the same as the matrix Λ. Moreover, all sub-matrices $(\tilde{R} - \mu_j I)$ are lower triangular since $\tilde{R}$ is lower triangular and the entire matrix is a block upper triangular matrix.

This system of equations can be solved by forward substitution for the lower triangular block $(\tilde{R} - \mu_{n_z} I)$ to obtain the vector $V_{[.,n_z]}$ and by backward substitution for the upper block diagonal matrices. Thus, the lower triangular form of the system of equations $(\tilde{R} - \mu_{n_z} I)V_{[.,n_z]} = 0$ implies that the elements of $V_{[.,n_z]}$ are equal to zero except the last one V_{n_z,n_z}. Indeed the recursive form of the equations allows to rewrite the system of equations as $\sum_{j=1}^{i} \hat{r}_{i,j}^{n_z} V_{j,n_z} = 0$ for $i = 1, \ldots, n_z$, which imply that $V_{j,n_z} = 0$ except for V_{n_z,n_z} where $\hat{r}_{i,j}^{n_z}$ is the element (i, j) of the sub-matrix $(\tilde{R} - \mu_{n_z} I)$ and $\hat{r}_{n_z,n_z}^{n_z} = 0$ since R and $\tilde{R}$ share the same eigenvalues. According to Proposition 1, this holds under the weaker restriction that only the first sub-diagonal elements are different from zero which implies that $\hat{r}_{1,1}^{n_z} V_{1,n_z} = 0$ and $\sum_{j=i-1}^{i} \hat{r}_{i,j}^{n_z} V_{j,n_z} = 0$ for $i = 2, \ldots, n_z$.

Now, for the next upper block, we have

$$\begin{bmatrix} (\tilde{R} - \mu_{n_z-1} I) & -r_{nz,nz-1}I \end{bmatrix} \begin{bmatrix} V_{[.,n_z-1]} \\ V_{[.,n_z]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

using the preceding result above for $V_{[.,n_z]}$ yields $V_{j,n_z-1} = 0$ for all $j < n_z - 1$. We can continue to solve the following upper blocks by backward substitution to obtain that all elements of the matrix V above the diagonal are equal to zero: $(\tilde{R} - \mu_j I)V_{[.,j]} = 0$ for $j = n_z$ and $(\tilde{R} - \mu_j I)V_{[.,j]} = \sum_{i=j+1}^{n_z} (r_{i,j} I)V_{[.,i]}$ for $j = 1, \ldots n_z - 1$. The resulting matrix V has all elements above the diagonal equal to zero which implies that the only admissible orthonormal matrix is $V = I$ (up to changes of sign and/or permutation of the identity matrix).

C.5 Proof of Proposition 3.

Consider the following general 2×2 symmetric matrix $R = \begin{bmatrix} \rho_1 & \tau \\ \tau & \rho_2 \end{bmatrix}$ where ρ_1, ρ_2 and τ are real numbers and $\tau \neq 0$. The eigenvalues are the roots of the following characteristic equation:

$$\begin{vmatrix} \rho_1 - \lambda & \tau \\ \tau & \rho_2 - \lambda \end{vmatrix} = (\rho_1 - \lambda)(\rho_2 - \lambda) - \tau^2 = \lambda^2 - \lambda(\rho_1 + \rho_2) + \rho_1\rho_2 - \tau^2.$$

The two roots can be written as $\lambda_{1,2} = \frac{1}{2}\left[\rho_1 + \rho_2 \pm \sqrt{(\rho_1 - \rho_2)^2 + 4\tau^2}\right]$. Since $(\rho_1 - \rho_2)^2 + 4\tau^2 > 0$, the two eigenvalues are necessarily real. By the expressions of λ_1 and λ_2, there exists an infinity of values for ρ_1, ρ_2 and τ that gives the same eigenvalues. In other words, for any orthonormal matrix V and $\tilde{R} = VRV'$, the matrices $\tilde{R}$ and R are similar and therefore have the same eigenvalues. The matrix R is then not identifiable.

Now consider the case where the elements on the diagonal have the same value –*i.e.* $\rho_1 = \rho_2 = \rho$. The two eigenvalues are now given by $= \lambda_{1,2} = \rho \pm \tau$, so that $\lambda_1 + \lambda_2 = 2\rho$ and $\lambda_1 - \lambda_2 = 2\tau$. This implies that there does not exist another 2×2 symmetric matrix $\tilde{R}$ with $\tilde{\rho} \neq \rho$ and/or $\tilde{\tau} \neq \tau$ with $\tilde{\tau} \neq 0$ having the same eigenvalues as matrix R. In this particular case, matrix R is then locally identifiable.

C.6 Proof of Proposition 4.

We can consider cases with a block diagonal matrix R with blocks corresponding to the two preceding cases. For example $R = \begin{bmatrix} \tilde{R} & 0 \\ 0 & R_2 \end{bmatrix}$, where $\tilde{R}$ is a $(n_{z1} \times n_{z1})$ diagonal matrix with different elements on the diagonal and R_2 is any $(n_{z2} \times n_{z2})$ matrix with $z_1 + z_2 = z$. In this case, matrix V has the form $V = \begin{bmatrix} I & 0 \\ 0 & V_{n_{z2} \times n_{z2}} \end{bmatrix}$, where $V_{n_{z2} \times n_{z2}}$ is an orthonormal matrix. The first z_1 exogenous processes are then locally identified. Matrix $\tilde{R}$ could be also lower triangular.

Exploiting Dynamic Identification in VARs
Additional Online Material
(Not for Publication)

Paul Beaudry, Fabrice Collard, Patrick Fève,
Alain Guay and Franck Portier
May 2025
`version 6.4.2`

D The Case of Unobserved Endogenous State Variables

We investigate the the case of a vector X_t consisting of both observed, Y_t, and unobserved state variables, K_t –*i.e.* $X_t = (K_t, Y_t)'$. This implies, in particular, that Π_{yk} is not necessarily equal to the identity matrix. The standard RBC model studied in McGrattan (2010) is a typical example of such a situation as the capital stock may not be directly observed (or at least not without measurement errors) by the econometrician. In that case, the solution of the macroeconomic model admits the following state-space representation

$$\begin{aligned} K_t &= \Pi_{kk} K_{t-1} + \Pi_{kz} Z_t, \\ Y_t &= \Pi_{yk} K_{t-1} + \Pi_{yz} Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t. \end{aligned} \tag{D.1}$$

As for the previous case, we impose that Y_t has the same dimension as Z_t ($n_y = n_z$). We consider the case where Z_t consists of at least two elements and where there are not more state variables than elements in Z_t. For simplicity, we take K_t to be of the same order as Z_t and impose that Π_{kk} is of full rank. Moreover, we also assume that Π_{yz}, Π_{yk} and Π_{kz} are of full rank.

Making use of these assumptions, the dynamics of Y_t can be expressed as

$$Y_t = C_1 Y_{t-1} + C_2 Z_t + C_3 Z_{t-1}, \tag{D.2}$$

where $C_1 = \Pi_{yk}\Pi_{kk}\Pi_{yk}^{-1}$, $C_2 = \Pi_{yz}$ and $C_3 = \Pi_{yk}(\Pi_{kz} - \Pi_{kk}\Pi_{yk}^{-1}\Pi_{yz})$. The issue is then whether, when R is diagonal (or lower triangular), C_1, C_2, C_3 and R can be identified.[39]

Making use of the Z_t process in (D.2), it is easy to show that the vector of observed variables Y_t follows a VARMA(2,1) process of the form

$$Y_t = \left(DRD^{-1} + C_1\right) Y_{t-1} - DRD^{-1}C_1 Y_{t-2} + C_2\varepsilon_t + (D - DRD^{-1}C_2)\varepsilon_{t-1}.$$

where $D \equiv C_3 + C_2 R$ and D is invertible.

As in the previous case, counting the number of coefficients to uncover and the number of moments the VARMA(2,1) structure offers, we recover that imposing a lower triangular structure on R provides us with the right number of restrictions. Note that this does not generically guarantees identification, as the system that needs to be solved features a quadratic term implying that a pair of solutions generally arise. However, as long as R is sparser than a triangular matrix, the system features more equations than unknowns. The order condition is then clearly satisfied (in fact, it is over-identified). As before, checking the rank condition is non trivial, and we follow another strategy in the paper to formally prove identification. In Section H, we provide an application using simulated data generated by a RBC model.

[39] Note that while Π_{kk}, Π_{kz} and Π_{yk} cannot (in general) be identified separately. This is however not an issue as far as the identification of the structural impulse responses is concerned as all that is needed is the identification of R and Cs. Indeed, we will identify $\Pi_{yk}\Pi_{kk}\Pi_{yk}^{-1}$, $\Pi_{yk}\Pi_{kz}$, Π_{yz} and R.

E Partial Identification: an Example

Consider the following textbook 3-equation New Keynesian model (see Galí (2015)) featuring three shocks:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - E_t[\pi_{t+1}]) + z_{1,t}, \\ \pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + \kappa y_t + z_{2,t}, \\ i_t &= \phi_\pi \pi_t + z_{3,t}, \end{aligned}$$

where y_t, π_t and i_t denote respectively aggregate output, the rate of inflation and the nominal interest rate and $\mathbb{E}_t[\cdot]$ denotes the conditional expectation operator. Parameter $\beta \in (0,1)$ is the discount factor, $\kappa \geq 0$ denotes the slope of the Phillips curve and ϕ_π is the degree of aggressiveness of monetary policy to inflation. The forcing variable $z_{1,t}$ can be interpreted as a demand shifter to the IS curve, whereas $z_{2,t}$ is a cost-push shifter to the Phillips curve. $z_{3,t}$ is a monetary policy shifter to the Taylor rule. To ease exposition, we assume that the two forcing variables $z_{1,t}$ and $z_{2,t}$ are not serially correlated, while third one $z_{3,t}$ is. As will be clear later, the two serially uncorrelated shifters cannot be separately identified, whereas the third one one can be. Assuming the Taylor principle holds, the model can be solved forward and yields the state-space representation

$$\begin{aligned} Y_t &= \Pi_{yz} Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t, \end{aligned}$$

$$\text{where } \Pi_{yz} = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{33} & \pi_{32} & \pi_{33} \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \text{ and } \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}.$$

The vector $Y_t = (y_t, \pi_t, i_t)'$ collects the three endogenous variables and $Z_t = (z_{1,t}, z_{2,t}, z_{3,t})'$ is the vector of the three model forcing variables, ε_t is a zero mean weak white noise and $E(\varepsilon_t \varepsilon_t') = I_3$.

The model solution for output writes as the following moving average:

$$y_t = \pi_{11}\varepsilon_{1,t} + \pi_{12}\varepsilon_{2,t} + \pi_{13} \sum_{i=0}^{\infty} \rho^i \varepsilon_{3,t-i},$$

from which we get the output auto-covariance functions as

$$\begin{aligned} \gamma_y(0) &= \pi_{11}^2 + \pi_{12}^2 + \frac{\pi_{13}^2}{1-\rho^2} \quad \text{when} \quad h = 0, \\ \gamma_y(h) &= \frac{\pi_{13}^2 \rho^h}{1-\rho^2} \quad \text{when} \quad h > 0. \end{aligned}$$

The persistence parameter ρ can be directly identified by computing the ratio $\gamma_y(h+1)/\gamma_y(h)$ for $h > 0$, which is free from any parameter but π_{13}. Given ρ, the direct observation of $\gamma_y(h)$, for any $h > 0$, allows to recover parameter π_{13} (up to a sign term). In other words, the effect of a monetary policy shock on output is identified. Applying the same procedure on inflation and the nominal interest allows for the identification of π_{23} and π_{33} (up to a sign term). It is therefore possible to identify the monetary transmission mechanism for all variables in the D-SVAR.

Note that, on the contrary, the knowledge of $\gamma_y(0)$ only helps identify the sum $\pi_{11}^2 + \pi_{12}^2$, not π_{11} and π_{12} separately, In other words, the effect of two remaining shocks $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ cannot be separately identified. The reason for this result is that the dynamics of these two shocks do not bring any information to disentangle them. This result obviously extends to the inflation rate and the nominal interest rate. This example simply illustrates that one shock can be identified in so far as it displays a dynamic structure that differs from the other shocks in the economy.

F Data Appendix

F.1 Data from Section 3.2.1

- **Real GDP** is measured as the ratio of Gross Domestic Product in value (Table 1.1.5 from BEA) divided by the GDP price index (Table 1.1.4 from BEA), and is expressed in per capita term by dividing by the Civilian non-institutional population from 16 years of age and older residing in the 50 states and the District of Columbia (`CNP16OV` in the Federal Reserve DataBase (FRED, `http://fred.stlouisfed.org`)).
- The **unemployment gap** is measured as the difference between the average unemployment rate over a quarter (`UNRATE` from FRED) and the natural rate of unemployment –*i.e.* the rate of unemployment arising from all sources except fluctuations in aggregate demand (`NROU` from FRED).

F.2 Data from Section 3.2.2

The data from Section 3.2.2 are borrowed from Gertler and Karadi (2015) and are downloadable from `http://doi.org/10.3886/E114082V1`

- The **price level** is measured by the Consumer Price Index for all urban consumers (`CPIAUCSL` from FRED, All Items in U.S. City Average)
- **Economic activity** is measured by the industrial production index (`INDPRO` from FRED)
- The **nominal interest rate** is measured by the Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity (`GS1` fom FRED)
- The **credit spread** is measured by the Gilchrist and Zakrajšek (2012) excess bond premium.
- The **external instrument** corresponds to the three month ahead monthly fed funds futures (`FF4` from Gertler and Karadi (2015))

The price level and the economic activity index are both expressed in logs prior to estimation.

F.3 Data from Section 3.2.3

- The **nominal interest rate** is measured by the Effective Federal Funds Rate (`DFF` from FRED).
- **Output** is measured as the Real Gross Domestic Product expressed in Billions of Chained 2012 Dollars (`GDPC1` from FRED, Quarterly, Seasonally Adjusted Annual Rate).
- The **price level** is measured by the Consumer Price Index for all items for the United States (`CPALTT01USM661S` from FRED, Index 2015=100, Quarterly, Seasonally Adjusted).
- The **unemployment rate** corresponds to the quarterly average of the monthly unemployment rate in the US (`UNRATE` from FRED, Percent, Quarterly, Seasonally Adjusted)
- The **commodity price** is the Producer Price Index of all commodities (`PPIACO` from FRED, Index 1982=100, Quarterly).

Output, the Price level and the commodity price are all transformed by applying the log function prior to estimation.

G The New Keynesian Model of Section 3.1

G.1 Description of the Model

The set up is standard. The economy is populated by a large number of identical infinitely–lived households and economy consists of two sectors: one producing intermediate goods and the other final goods. The intermediate good is produced with labor and the final good with intermediate goods.

The Household: Household preferences are characterised by the lifetime utility function:[40]

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \omega_t \left(\frac{(c_{t+\tau} - h\bar{c}_{t+\tau-1})^{1-\gamma}}{1-\gamma} - \vartheta \frac{n_{t+\tau}^{1+\varphi}}{1+\varphi} \right), \tag{G.1}$$

where $0 < \beta < 1$ is a constant discount factor, c_t denotes individual consumption, $\bar{c}_t$ is Joneses' consumption and n_t labor.

In each and every period, the representative household faces a budget constraint of the form

$$B_t + P_t c_t \quad \leq \quad R_{t-1} B_{t-1} + \Pi_t + P_t w_t n_t, \tag{G.2}$$

where B_t are nominal bonds acquired during period t, P_t is the nominal price of the final good, R_{t-1} is the nominal interest rate, w_t denotes the real wage. The household consumes c_t and supplies n_t units of labor and claims the profits, Π_t, earned by the firms. ω_t will act as a demand shock and can be interpreted as a premium shock.

The first order conditions lead to

$$\vartheta n_t^{\varphi} = (c_t - h\bar{c}_{t-1})^{-\gamma} w_t, \tag{G.3}$$

$$\omega_t (c_t - h\bar{c}_{t-1})^{-\gamma} = \beta R_t \mathbb{E}_t \left[\frac{\omega_{t+1} (c_{t+1} - h\bar{c}_t)^{-\gamma}}{\pi_{t+1}} \right], \tag{G.4}$$

where $\pi_t = P_t / P_{t-1}$ denotes the gross inflation rate.

Final sector: The final good is produced by combining intermediate goods. This process is described by the following CES function

$$y_t = \left(\int_0^1 y_t(i)^{\eta_t} \mathrm{d}i \right)^{\frac{1}{\eta_t}}, \tag{G.5}$$

where $\eta_t \in (-\infty, 1)$. η_t determines the elasticity of substitution between the various inputs, which will be modelled as a stochastic process and will appear as a cost push shock in the New Keynesian Phillips curve. The producers in this sector are assumed to behave competitively and to determine their demand for each good, $y_t(i)$, $i \in (0,1)$ by maximising the static profit equation

$$\max_{\{X_t(i)\}_{i \in (0,1)}} P_t y_t - \int_0^1 P_t(i) y_t(i) \mathrm{d}i, \tag{G.6}$$

subject to (G.5), where $P_t(i)$ denotes the price of intermediate good i. This yields demand functions of the form:

$$y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{\frac{1}{\eta_t - 1}} y_t, \tag{G.7}$$

[40] $\mathbb{E}_t(.)$ denotes mathematical conditional expectations. Expectations are conditional on information available at the beginning of period t.

and the following general price index

$$P_t = \left(\int_0^1 P_t(i)^{\frac{\eta_t}{\eta_t - 1}} \mathrm{d}i \right)^{\frac{\eta_t - 1}{\eta_t}}. \tag{G.8}$$

The final good may be used for consumption — private or public — and investment purposes.

Intermediate Good Producers: Each firm i, $i \in (0,1)$, produces an intermediate good by means of capital and labor according to a constant returns–to–scale technology, represented by the production function

$$y_t(i) = n_t(i), \tag{G.9}$$

where $n_t(i)$ denotes the labor input used by firm i in the production process. a_t is an exogenous stationary stochastic technology shock. Assuming that each firm i operates under perfect competition in the input markets, the firm determines its production plan so as to minimise its total cost

$$\min_{\{h_t(i)\}} P_t w_t n_t(i),$$

subject to (G.9). This yields to the following expression for total costs:

$$P_t s_t y_t(i),$$

where the real marginal cost, s_t, is simply given by w_t.

Intermediate goods producers are monopolistically competitive, and therefore set prices for the good they produce. We follow Calvo [1983] in assuming that firms set their prices for a stochastic number of periods. In each and every period, a firm either gets the chance to adjust its price (an event occurring with probability $1-\alpha$) or it does not. When the firm does not reset its price, it just applies steady state inflation to the price it charged in the last period such that $P_t(i) = \pi_{t-1}^{\zeta} \overline{\pi}^{1-\zeta} P_{t-1}(i)$. When it gets a chance to do it, firm i resets its price, $\widetilde{P}_t(i)$, in period t in order to maximise its expected discounted profit flow this new price will generate. In period t, the profit is given by $\Pi(\widetilde{P}_t(i))$. In period $t+1$, either the firm resets its price, such that it will get $\Pi(\widetilde{P}_{t+1}(i))$ with probability α, or it does not and its $t+1$ profit will be $\Pi(X_{t,t+1}\widetilde{P}_t(i))$ with probability $(1-\alpha)$. Likewise in $t+2$. Expected profit flow generated by setting $\widetilde{P}_t(i)$ in period t writes

$$\max_{\widetilde{P}_t(i)} \mathbb{E}_t \sum_{\tau=0}^{\infty} \Phi_{t,t+\tau} \alpha^{\tau-1} \Pi(X_{t,t+\tau} \widetilde{P}_t(i)),$$

subject to the total demand it faces:

$$y_t(i) = \left(\frac{\widetilde{P}_t(i)}{P_t} \right)^{\frac{1}{\eta-1}} y_t,$$

where $X_{t,t+1} = \pi_t^{\zeta} \overline{\pi}^{1-\zeta} X_{t-1,t}$ and $\Pi(X_{t,t+\tau}\widetilde{P}_t(i)) = \left(X_{t,t+\tau}\widetilde{P}_t(i) - P_{t+\tau} s_{t+\tau} \right) y_{t+\tau}(i)$. $\Phi_{t+\tau}$ is an appropriate discount factor related to the way the household values future as opposed to current consumption, such that

$$\Phi_{t,t+\tau} \propto \beta^{\tau} \frac{\Lambda_{t+\tau}}{\Lambda_t} \text{ where } \Lambda_{t+\tau} \equiv \omega_{t+\tau} (c_{t+\tau} - h \overline{c}_{t+\tau-1})^{-\gamma}.$$

This leads to the price setting equation

$$\mathbb{E}_t \left[\sum_{\tau=0}^{\infty} (\beta\alpha)^{\tau} \frac{\Lambda_{t,t+\tau}}{\eta_{t+\tau} - 1} \left(\eta_{t+\tau} X_{t,t+\tau} \widetilde{P}_t(i) - P_{t+\tau} s_{t+\tau} \right) y_{t+\tau}(i) \right] = 0. \tag{G.10}$$

From the definition of the aggregate price (G.8) and the Calvo fairy assumption, the aggregate price level may be expressed as

$$P_t = \left(\sum_{j=0}^{\infty} (1-\alpha)\alpha^j (X_{t-j,t}\widetilde{P}_{t-j})^{\frac{\eta_t}{\eta_t - 1}} \right)^{\frac{\eta_t - 1}{\eta_t}}. \tag{G.11}$$

Monetary Authorities: Monetary authorities are assumed to follow a Taylor rule of the form (in log-linear deviations from deterministic steady state)

$$i_t = \rho_r i_{t-1} + (1-\rho_r)(\phi_\pi \pi_t + \phi_y y_t) + \epsilon_{i,t},$$

where $|\rho_r| < 1$ and $\phi_y, \phi_\pi > 0$.

Equilibrium: An equilibrium of this economy is a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty} = \{w_t, , P_t, R_t, \widetilde{P}_t\}_{t=0}^{\infty}$ and a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty} = \{\{\mathcal{Q}_t^H\}_{t=0}^{\infty}, \{\mathcal{Q}_t^F\}_{t=0}^{\infty}\}$ with

$$\begin{aligned} \{\mathcal{Q}_t^H\}_{t=0}^{\infty} &= \{c_t, B_t, n_t\}_{t=0}^{\infty}, \\ \{\mathcal{Q}_t^F\}_{t=0}^{\infty} &= \{y_t, y_t(i), n_t(i); i \in (0,1)\}_{t=0}^{\infty}, \end{aligned}$$

such that:

(i) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^H\}_{t=0}^{\infty}$ is a solution to the representative household's problem;

(ii) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^F\}_{t=0}^{\infty}$ is a solution to the representative firms' problem;

(iii) given a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{P}_t\}_{t=0}^{\infty}$ clears the markets. In particular, we have $\int_0^1 y_t(i)\mathrm{d}i = c_t$ and $\int_0^1 n_t(i)\mathrm{d}i = n_t$.

(iv) Prices satisfy (G.10) and (G.11).

Log-linearisation of the equilibrium around the deterministic steady state gives rise to the 3-equation New-Keynesian model:

$$y_t = \frac{h}{1+h} y_{t-1} + \frac{1}{1+h}\mathbb{E}_t[y_{t+1}] - \frac{1-h}{\gamma(1+h)}(i_t - \mathbb{E}_t[\pi_{t+1}]) + z_{dt}, \tag{G.12}$$

$$\pi_t = \frac{\zeta}{1+\beta\zeta}\pi_{t-1} + \frac{\beta}{1+\beta\zeta}\mathbb{E}_t[\pi_{t+1}] + \frac{(1-\alpha)(1-\beta\alpha)}{\alpha(1+\beta\zeta)}(\gamma+\varphi) y_t + z_{st}, \tag{G.13}$$

$$i_t = \rho\, i_{t-1} + (1-\rho)(\phi_\pi \pi_t + \phi_y y_t) + z_{mt}, \tag{G.14}$$

with $z_{jt} = \rho_j z_{jt-1} + \varepsilon_{jt}$, where $\varepsilon_{jt} \rightsquigarrow N(0, \sigma_j^2)$ with $j \in \{d, s, m\}$.

G.2 Estimation Results

Table G.1 below reports the prior distribution and posterior mean, mode and 90% Highest Posterior Density Interval (HPDI) for all the estimated parameters of the model.

Table G.1: New Keynesian Model, Priors and Posteriors

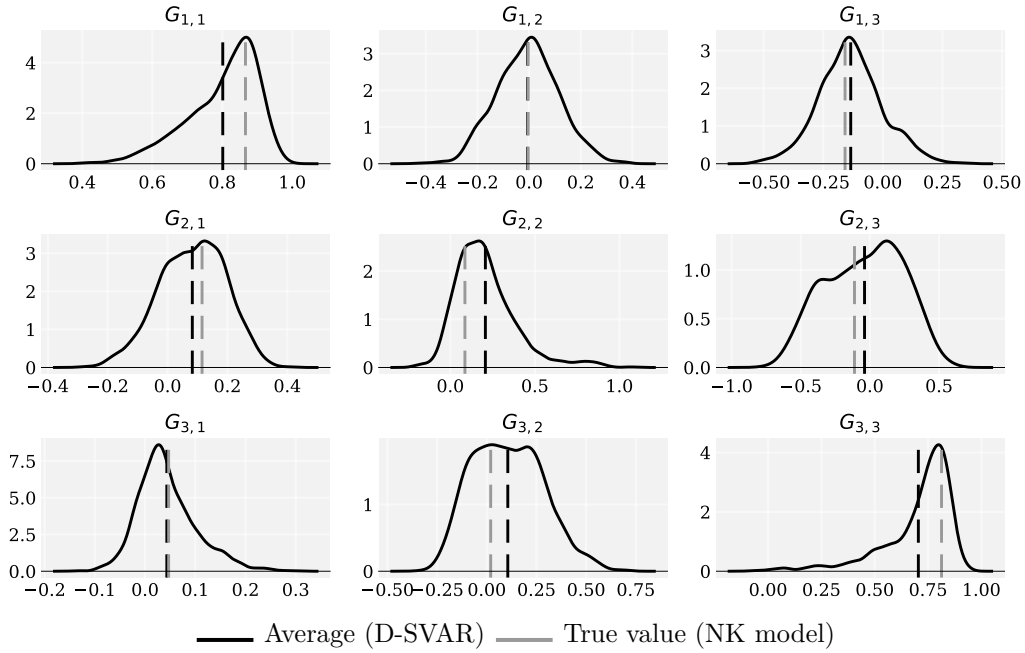
	Priors			Posteriors		
	distribution	Mean	std. dev.	Mode	Mean	90% HPDI
σ	Gamma	1.00	0.10	1.30	1.32	[1.12;1.50]
φ	Gamma	2.00	0.30	2.43	2.57	[1.92;3.22]
h	Beta	0.10	0.05	0.08	0.13	[0.03;0.23]
ζ	Gamma	0.10	0.05	0.04	0.05	[0.01;0.08]
ρ	Beta	0.90	0.05	0.85	0.84	[0.81;0.87]
θ	Beta	0.90	0.01	0.87	0.86	[0.83;0.89]
ϕ_π	Normal	1.50	0.10	1.33	1.36	[1.14;1.56]
ϕ_y	Normal	0.10	0.05	0.35	0.33	[0.27;0.41]
ρ_d	Beta	0.80	0.10	0.87	0.85	[0.83;0.88]
ρ_s	Beta	0.80	0.10	0.53	0.54	[0.45;0.63]
ρ_m	Beta	0.20	0.10	0.26	0.28	[0.17;0.38]
σ_d	Inv. Gamma	0.50	0.10	0.22	0.22	[0.21;0.24]
σ_s	Inv. Gamma	0.50	0.10	0.37	0.37	[0.32;0.43]
σ_m	Inv. Gamma	0.50	0.10	0.24	0.24	[0.21;0.26]

Notes: The estimated model is the New Keynesian (G.12)–(G.14). Sample is 1960Q1-2007Q4. Estimation is done with minus the unemployment gap, GDP deflator inflation and the Federal fund rate. Posterior distribution obtained from MCMC using 2 chains of 200,000 draws each.

G.3 Estimated D-SVAR: Densities

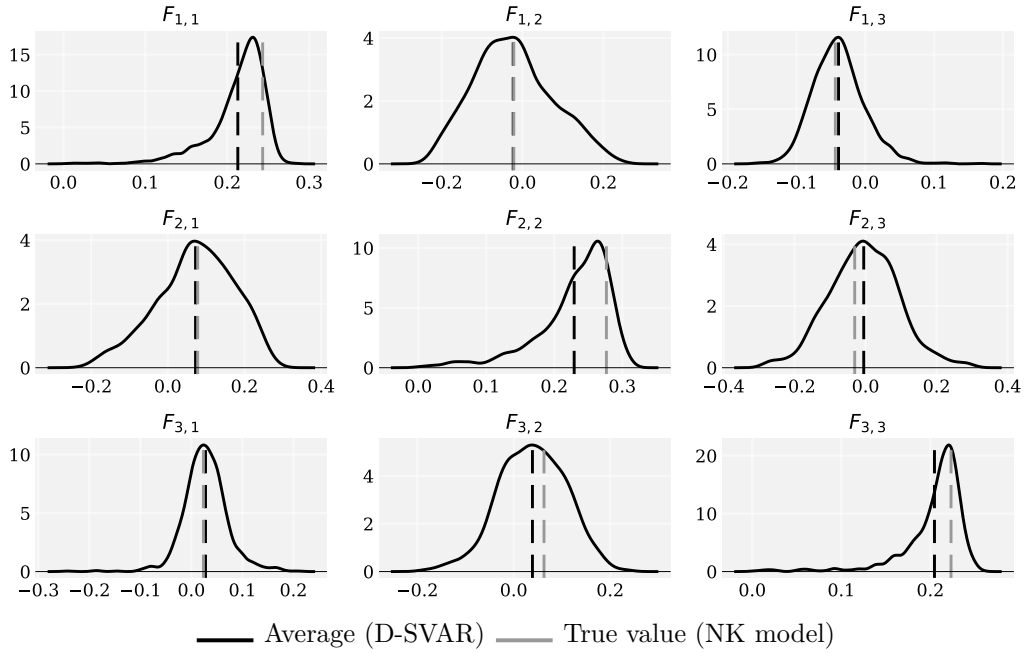
Below we report the densities over 1,000 simulations of the estimated elements of the matrices G, F and R. In each simulation, we simulate the estimated New Keynesian model for 300 periods and use the simulated data to estimate a D-SVAR and therefore recovering each time an estimation of the matrices G, F and R.

Figure G.1: Density of the Elements of Matrix G



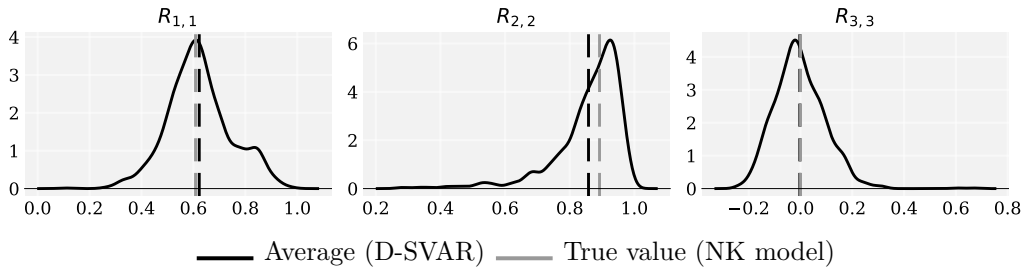
Notes: The estimated model is a D-SVAR with two lags. Data are generated by the estimated New Keynesian model. We report the density over 1,000 estimations.

Figure G.2: Density of the Elements of Matrix F



Notes: The estimated model is a D-SVAR with two lags. Data are generated by the estimated New Keynesian model. We report the density over 1,000 estimations.

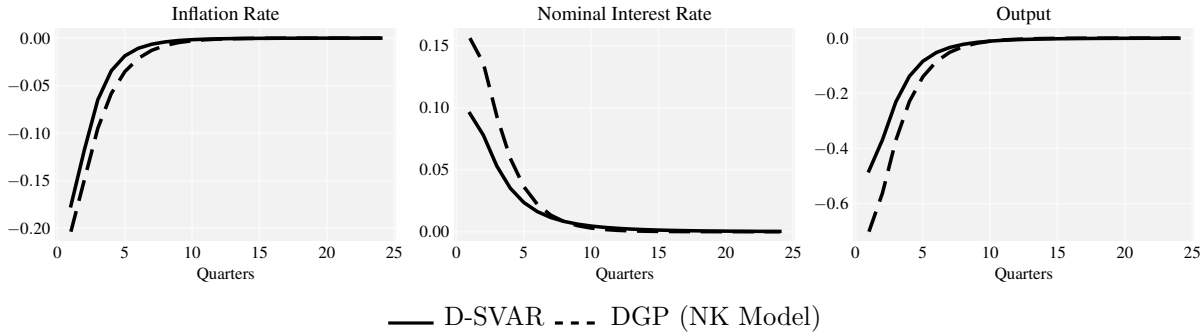
Figure G.3: Density of the Elements of Matrix R



Notes: The estimated model is a D-SVAR with two lags. Data are generated by the estimated New Keynesian model. We report the density over 1,000 estimations.

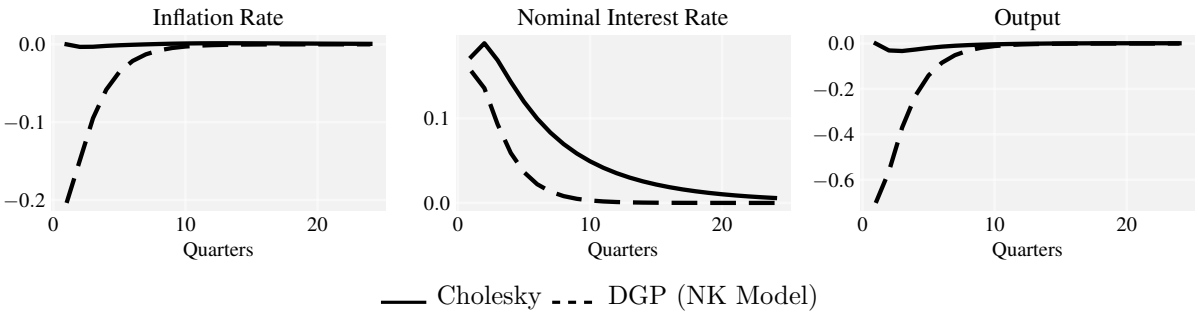
G.4 IRF Obtained From Various Identification Schemes with Data Simulated with the New Keynesian Model

Figure G.4: Impulse Responses to a Monetary Shock, New Keynesian Model and DSVAR



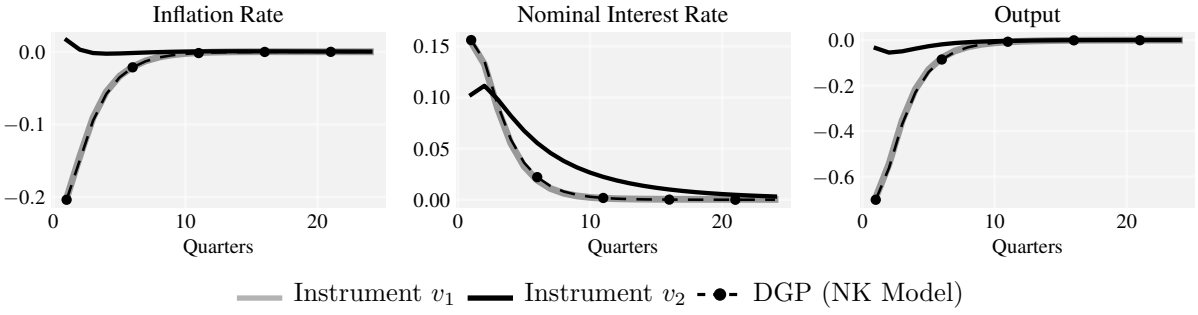
Notes: The New Keynesian model IRF to a monetary shock are obtained from the estimated model. The D-SVAR one are estimated on data generated by the New Keynesian model. They are the average over 1,000 simulations of the response to the shock that is, in each simulation, the most correlated with the monetary shock.

Figure G.5: Impulse Responses to a Monetary Shock, New Keynesian Model and Cholesky Identification



Notes: The New Keynesian model IRFs to a monetary shock are obtained from the estimated model. The Cholesky ones are estimated on data generated by the New Keynesian model. They are the average over 1,000 simulations of the response to the shock that, on impact, does not move output and inflation.

Figure G.6: Impulse Responses to a Monetary Shock, New Keynesian Model and Proxy VAR Identification



Notes: The New Keynesian model IRFs to a monetary shock are obtained from the estimated model. The "instrument v_i" ones are estimated on data generated by the New Keynesian model. They are the average over 1,000 simulations of the proxy VAR using v_1 or v_2 as an instrument.

H A Real Business Cycle Model

We consider a real business cycle model featuring a *catching up with the Joneses* mechanism and real frictions on the capital accumulation process that take the form of investment adjustment costs. The problem of the Central planner takes the form

$$\begin{aligned}
&\max \mathbb{E}_t \left[\sum_{\tau=0}^{+\infty} \beta^\tau \left(\log(C_{t+\tau} - b\overline{C}_{t+\tau-1}) - \vartheta_{t+\tau}^{-1} \frac{h_{t+\tau}^{1+\nu}}{1+\nu} \right) \right], \\
C_t + I_t &= A_t K_t^\alpha (\Gamma_t h_t)^{1-\alpha}, \\
K_{t+1} &= \zeta_t I_t \left(1 - \Phi\left(\frac{I_t}{I_{t-1}} \right) \right) + (1-\delta) K_t,
\end{aligned}$$

where C_t, $\bar{C}_t$, h_t, I_t, K_t and Y_t denote, respectively, consumption, Joneses' consumption, hours worked, investment, capital and output. $\beta \in (0,1)$ denotes the discount factor, $b \in (0,1)$ governs habit persistence, $\nu > 0$ is the inverse of the Frish elasticity, $\alpha \in (0,1)$ is capital elasticity and the function $\Phi(\cdot)$ is strictly increasing and convex and satisfies $\Phi(\gamma) = \Phi'(\gamma) = 0$. Furthermore, $\varphi \equiv \Phi''(\gamma)\gamma > 0$ governs the importance of investment adjustment costs. Γ_t denotes exogenous technological progress that evolves deterministically as $\Gamma_t = \gamma\Gamma_{t-1}$, $\gamma > 1$. Finally ϑ_t, A_t and ζ_t denote respectively a labor wedge, a technology and an investment specific shock, which are all assumed to follow a stationary AR(1) process of the form

$$\log(X_t) = \rho_X \log(X_{t-1}) + \varepsilon_t^X \text{ for } x \in \{\vartheta, A, \zeta\},$$

where $|\rho_X| < 1$ and $\varepsilon^X \rightsquigarrow N(0, \sigma_X^2)$. The (deflated for growth, $x_t = X_t/\Gamma_t$) optimal allocation of this economy is then characterised by the set of equations

$$\begin{aligned}
h_t^\nu &= (1-\alpha) \frac{y_t}{h_t} \frac{\gamma\vartheta_t}{\gamma c_t - b c_{t-1}}, \\
1 &= \frac{\beta}{\gamma} \mathbb{E}_t \left[\frac{\gamma c_t - b c_{t-1}}{\gamma c_{t+1} - b c_t} \left(\alpha \frac{y_{t+1}}{k_{t+1}} + (1-\delta) q_{t+1} \right) \right], \\
1 &= \zeta_t q_t \left(1 - \Phi\left(\gamma \frac{i_t}{i_{t-1}} \right) - \Phi'\left(\gamma \frac{i_t}{i_{t-1}} \right) \gamma \frac{i_t}{i_{t-1}} \right) + \frac{\beta}{\gamma} \mathbb{E}_t \left[\frac{\gamma c_t - b c_{t-1}}{\gamma c_{t+1} - b c_t} \zeta_{t+1} \Phi'\left(\gamma \frac{i_{t+1}}{i_t} \right) \gamma \frac{i_{t+1}}{i_t} \right], \\
y_t &= A_t k_t^\alpha h_t^{1-\alpha}, \\
y_t &= c_t + i_t \\
\gamma k_{t+1} &= \zeta_t i_t \left(1 - \Phi\left(\gamma \frac{i_t}{i_{t-1}} \right) \right) + (1-\delta) k_t,
\end{aligned}$$

where lowercase variable x denotes the deflated for growth variable X ($x_t = X_t/\Gamma_t$) for any $X \in \{Y, C, I, K\}$. The solution of a log-linearised version of the optimal allocation admits the state-space representation:

$$\begin{aligned}
Y_t &= \Pi_X X_t + \Pi_Z Z_t, \\
X_{t+1} &= M_X X_t + M_Z Z_t, \\
Z_{t+1} &= R Z_t + \varepsilon_t,
\end{aligned}$$

where $X_t = (\widehat{k}_t, \widehat{c}_{t-1}, \widehat{i}_{t-1})'$, $Y_t = (\widehat{c}_t, \widehat{y}_t, \widehat{i}_t, \widehat{h}_t, \widehat{q}_t)$ and $Z_t = (\widehat{a}_t, \widehat{\zeta}_t, \widehat{\vartheta}_t)'$ and $\varepsilon_t = (\varepsilon_t^A, \varepsilon_t^\zeta, \varepsilon_t^\vartheta)'$. As usual in the literature $\widehat{x}_t$ denotes the log-deviation of variable x from its deterministic steady state. We then assess the ability of the dynamic identification technique developed in the main text to recover the dynamics of the "true" structural model. The main difference from the New-Keynesian model investigated in the text is that the structural model features a latent variable unobserved by the econometrician —e.g. the capital stock.

Table H.1: Parametrisation

Preferences		
β	Discount Factor	0.990
b	Habit Persistence	0.650
ν	Inv. Frish Elasticity	1.000
Technology		
α	Capital Elasticity	0.330
φ	Investment Adjustment Costs	2.500
δ	Depreciation Rate	0.025
γ	Gross Rate of Growth	1.004
Shock Persistence		
ρ_A	Technology Shock	0.950
ρ_ζ	Investment Specific Shock	0.810
ρ_ϑ	Labor Wedge Shock	0.940
Shock Volatility (in %)		
ρ_A	Technology Shock	0.700
ρ_ζ	Investment Specific Shock	2.000
ρ_ϑ	Labor Wedge Shock	0.800

In order to perform this assessment we parametrise the model by borrowing values for the structural parameters from the RBC literature (see Table H.1). The parameters pertaining to the investment specific shock are directly borrowed from Justiniano et al. (2011). Those pertaining to the labor wedge shock are taken from Kascha and Mertens (2009).[41] We then run the following Monte-Carlo experiment. We use the RBC model as the DGP for output, investment and hours worked and simulate the estimated model 10,000 times over 250 periods. For each simulation, we estimate the following VARMA

$$\begin{pmatrix} y_t \\ i_t \\ h_t \end{pmatrix} = \Phi_1 \begin{pmatrix} y_{t-1} \\ i_{t-1} \\ h_{t-1} \end{pmatrix} + \Phi_2 \begin{pmatrix} y_{t-2} \\ i_{t-2} \\ h_{t-2} \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{pmatrix} + \Theta_1 \begin{pmatrix} u_{1,t-1} \\ u_{2,t-1} \\ u_{3,t-1} \end{pmatrix}$$

and use it as auxiliary model to recover the D-SVARMA representation by ALS as

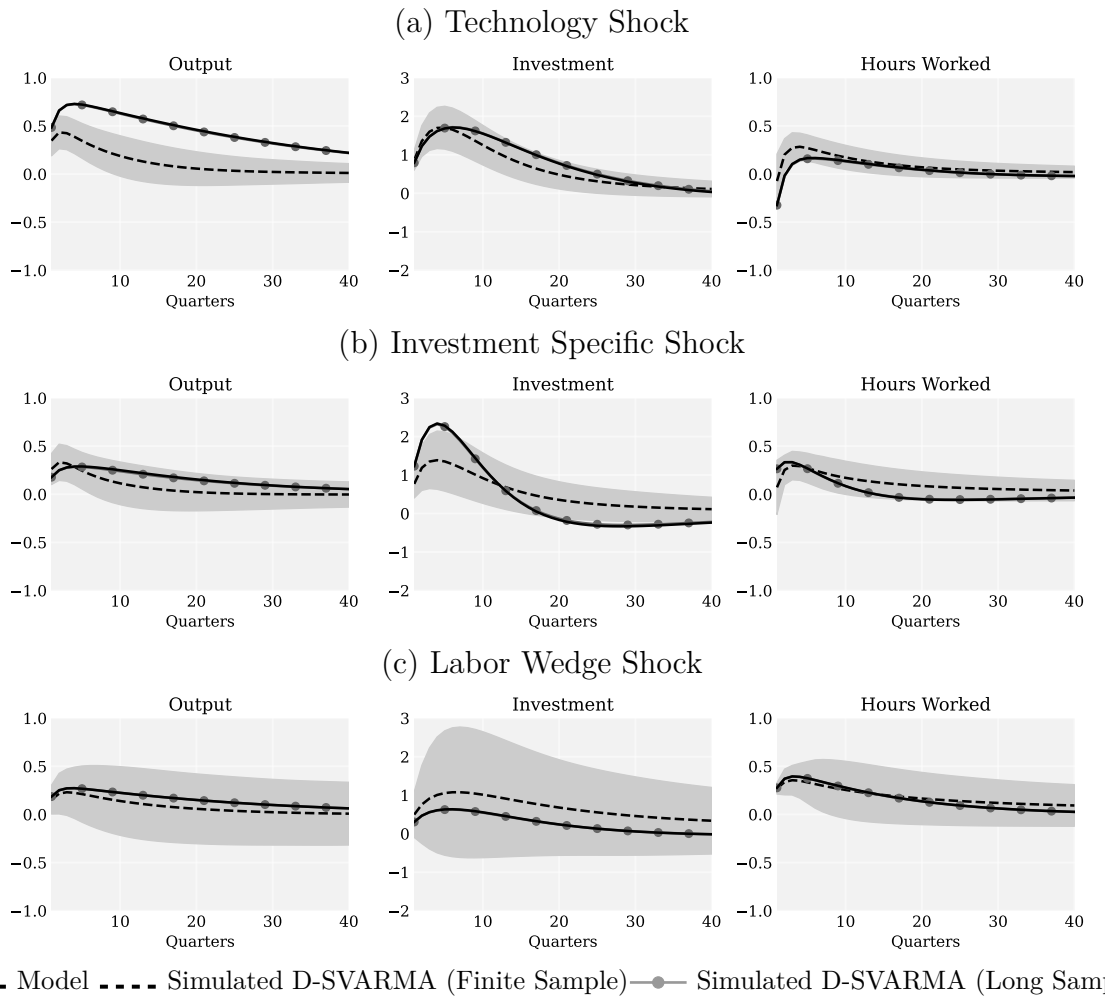
$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = G \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + F_0 \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} + F_1 \begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \\ z_{3,t-1} \end{pmatrix} \text{ where } \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} = R \begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \\ z_{3,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

We then compute the response of each variable to each shock. Figure H.1 reports for each shock the theoretical IRFs of output, investment and hours worked (plain dark line) alongside the average IRF as recovered from the simulated D-SVAR (dashed line). The shaded area corresponds to the 68% confidence band of each IRF in the simulated model.

Note that the three shocks are unlabelled in the D-SVARMA, so we order them by minimising the distance between the model structural shock and each D-SVARMA shock. Inspection of the figure suggests that the D-SVARMA allows to recover the three structural shocks: the impulse response functions share the same shape and the same properties in the model and in the D-SVARMA. In particular, the D-SVARMA is able to properly recover the short-run response

[41] Note that the model does not pretend to be an accurate representation of a specific economy, but is used as a *reasonable* DGP to conduct our assessment exercise.

Figure H.1: Impulse responses, RBC model vs Simulated D-SVARMA

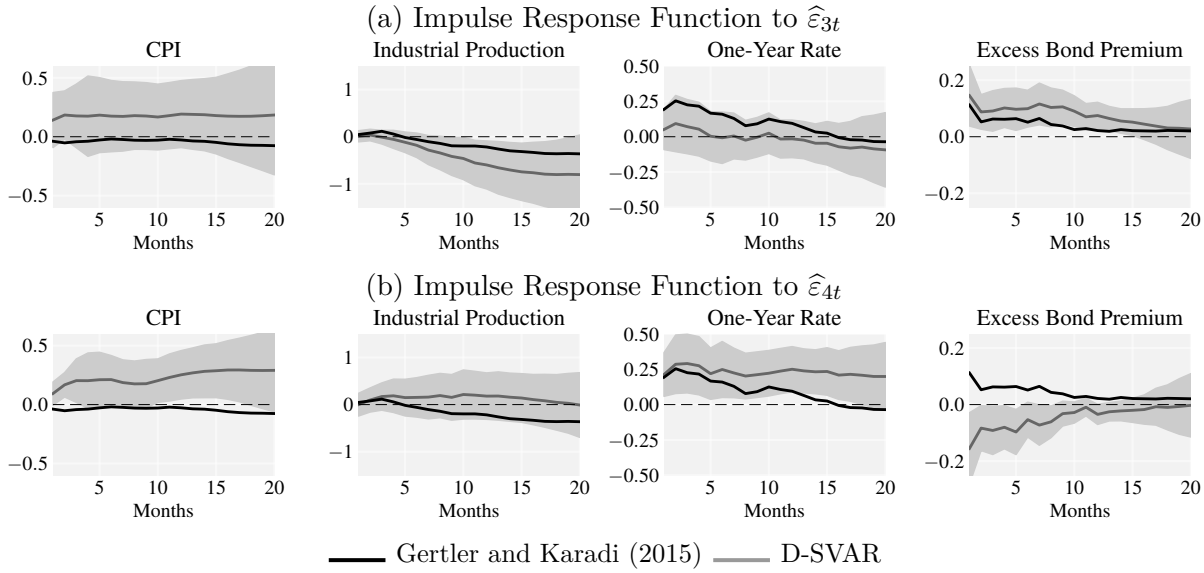


Notes: The estimated model is a D-SVARMA(2,1). Data are generated by the calibrated RBC model. Observables are y, i and h, but k is not observable. Short sample corresponds to the average of 10,000 estimations of length 250 periods. The shaded area represents 68% confidence bands, as computed from the 10,000 simulations. Long sample corresponds to one simulation of length 1,000,000 periods.

to the various shocks. Closer inspection however reveals that the match is not perfect. The D-SVARMA tends to underestimate the response of output, and slightly overestimate that of hours worked. Two sources of bias are usually taken as the main culprit for this type of imperfect match: truncation bias and small sample bias. The truncation bias occurs when the state-space representation of the solution is approximated by a finite VAR. This is not the case in our experiment. As explained above, the state-space representation actually admits a VARMA(2,1) representation, which is precisely the unrestricted model we estimate. The truncation bias is therefore inoperative in our case. The main culprit is the small sample bias. As we increase the sample size of our simulations, the bias recedes and eventually vanishes. The grey line with bullet markers in Figure H.1 corresponds to IRFs obtained from the D-SVARMA estimated on a sample of 1,000,000 periods. The match is then perfect. Therefore, our dynamic identification method asymptotically correctly recovers the theoretical shocks, even in the presence of a latent variable.

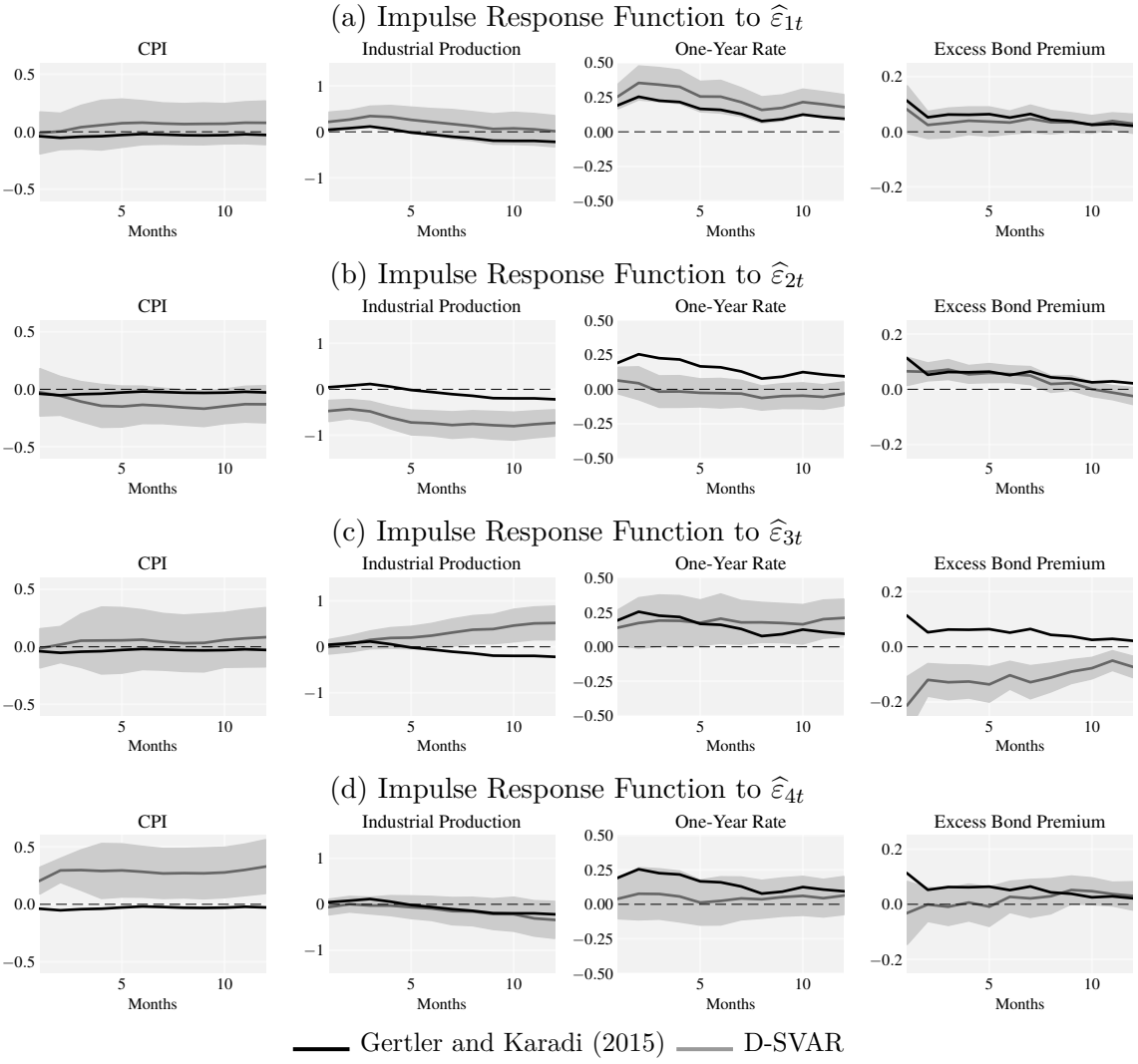
I Additional Material for Section 3.2.2 (Gertler and Karadi (2015))

Figure I.1: IRFs to Gertler and Karadi (2015) monetary policy shock and D-SVAR's shocks $\widehat{\varepsilon}_{3t}$ and $\widehat{\varepsilon}_{4t}$



Notes: The black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shocks identified in the D-SVAR with diagonal R matrix. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

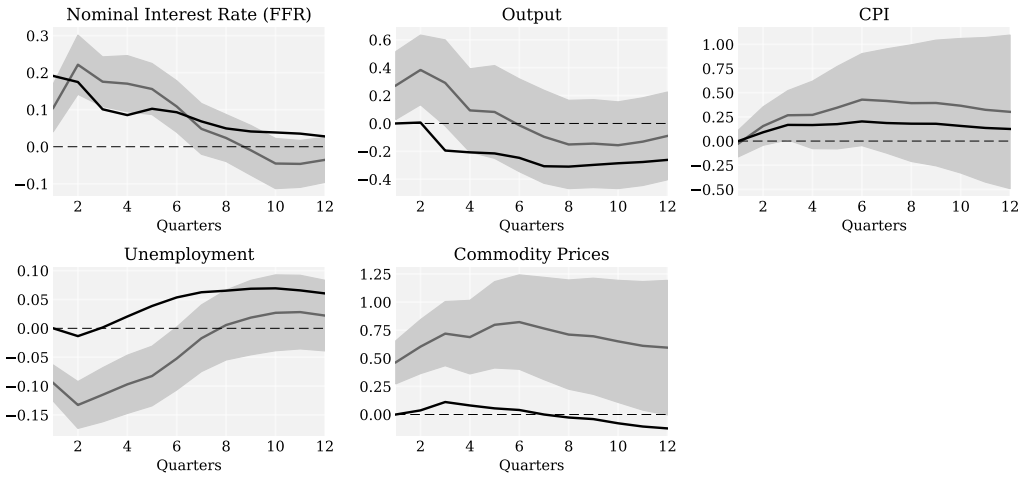
Figure I.2: IRFs to Gertler and Karadi (2015) monetary policy shock and D-SVAR's shocks with Lower Triangular R Matrix ($\widehat{\varepsilon}_{1t}$–$\widehat{\varepsilon}_{4t}$)



Notes: On the four panels, the black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shock in the D-SVAR. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

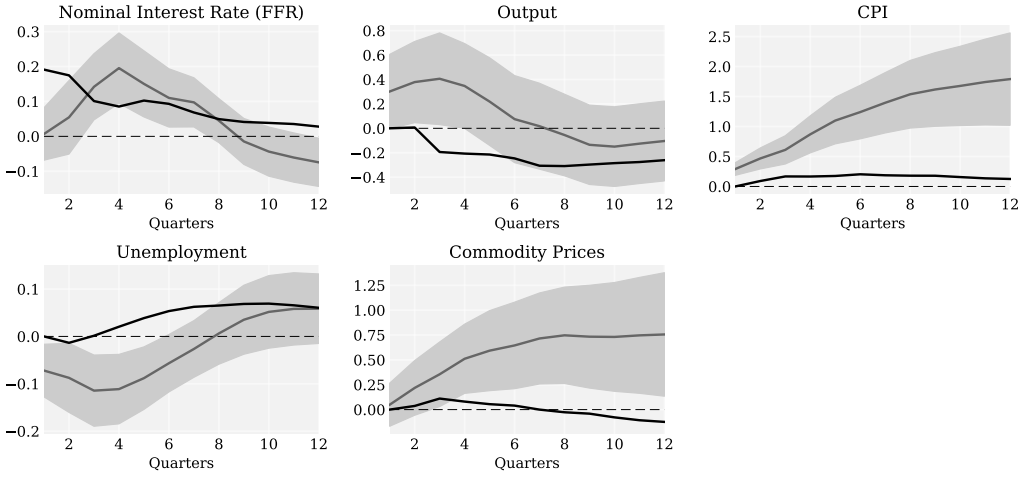
J Additional Material for Section 3.2.3 (Christiano et al. (1999))

Figure J.1: IRFs to Christiano et al. (1999) monetary policy shock and $\widehat{\varepsilon}_{2t}$



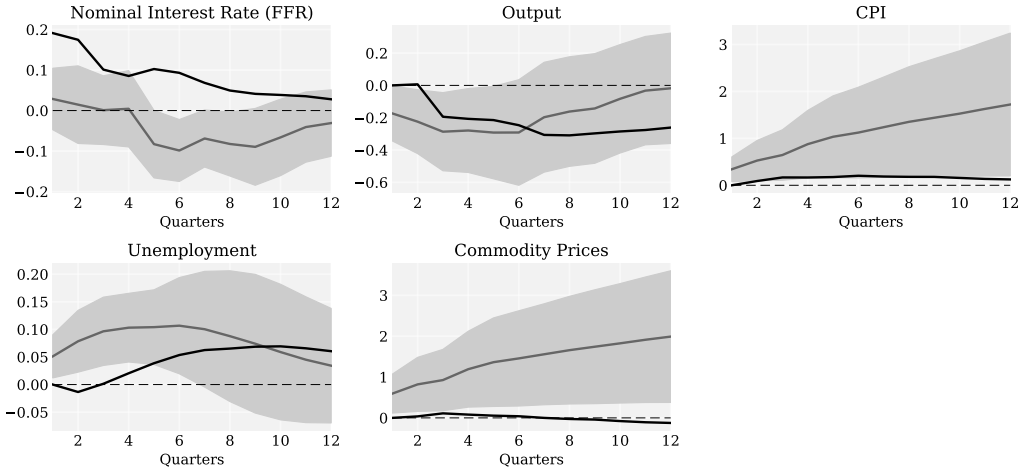
Notes: Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock $\widehat{\varepsilon}_{2t}$ in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure J.2: IRFs to Christiano et al. (1999) monetary policy shock and $\widehat{\varepsilon}_{3t}$



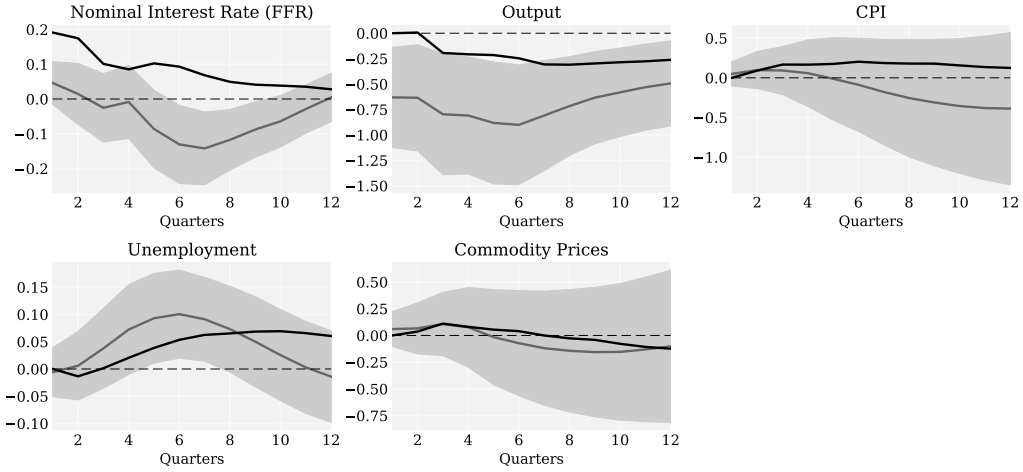
Notes: Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock $\widehat{\varepsilon}_{3t}$ in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure J.3: IRFs to Christiano et al. (1999) monetary policy shock and $\widehat{\varepsilon}_{4t}$



Notes: Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock $\widehat{\varepsilon}_{4t}$ in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure J.4: IRFs to Christiano et al. (1999) monetary policy shock and $\widehat{\varepsilon}_{5t}$



Notes: Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock $\widehat{\varepsilon}_{5t}$ in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Table J.1: Correlation between CEE's MP Shock and DSVAR Shocks

	$\widehat{\varepsilon}_{1t}$	$\widehat{\varepsilon}_{2t}$	$\widehat{\varepsilon}_{3t}$	$\widehat{\varepsilon}_{4t}$	$\widehat{\varepsilon}_{5t}$
Corr	0.928	-0.005	-0.342	0.094	0.097
p(0)	0.000	0.986	0.353	0.695	0.652
p(1)	0.665	0.013	0.015	0.035	0.011

Notes: Corr reports the correlation between the CEE Monetary Policy Shock and each of the shock as identified by our DSVAR. p(0) (resp. p(1)) reports the p-value associated to the Wald test of a zero (resp. unit) correlation between the CEE Monetary Policy Shock and each of the shock as identified by our DSVAR. This p-value is obtained by means of 1,000 bootstrap, so as to accommodate for the fact that both shocks are generated by an estimated VAR model and are hence contaminated by the uncertainty surrounding the estimated coefficients of the VAR.

Table J.2: Forecast Error Variance Decomposition of (in %)

	Christiano et al. (1999)	D-SVAR				
Horizon	ε_{Rt}	$\widehat{\varepsilon}_{1t}$	$\widehat{\varepsilon}_{2t}$	$\widehat{\varepsilon}_{3t}$	$\widehat{\varepsilon}_{4t}$	$\widehat{\varepsilon}_{5t}$
Nominal Interest Rate						
1	66.79	74.37	20.16	0.09	1.50	3.89
4	26.03	42.62	37.11	19.00	0.32	0.94
8	17.74	34.46	29.15	19.75	5.36	11.28
20	13.39	31.30	27.11	21.18	7.42	12.99
Output						
1	0.00	6.08	11.72	14.39	4.77	63.04
4	2.47	3.01	9.65	15.92	7.38	64.04
8	5.47	3.22	5.47	8.92	7.38	75.01
20	5.95	3.48	4.12	6.23	4.81	81.36
Price Index (CPI)						
1	0.00	2.16	0.30	40.50	55.86	1.17
4	1.80	9.12	4.80	40.62	44.78	0.68
8	1.04	8.83	4.36	46.40	39.72	0.70
20	0.23	5.44	1.66	46.05	45.27	1.58
Unemployment						
1	0.00	5.61	50.38	29.17	14.64	0.20
4	0.46	6.51	37.46	29.12	21.81	5.11
8	5.70	5.93	25.66	21.69	29.76	16.97
20	6.47	13.22	20.87	22.49	28.28	15.14
Commodity Price						
1	0.00	0.37	37.40	0.42	61.19	0.62
4	0.37	0.89	29.19	8.12	61.34	0.47
8	0.13	0.56	21.06	12.05	65.98	0.35
20	0.49	0.14	9.86	12.86	76.95	0.19

Notes: In this table we compare the variance decomposition as obtained by Christiano et al. (1999) for their monetary policy shocks and for the five shocks of the D-SVAR. Sample is 1965Q1-2007Q4.

K A Two-Country VAR

In this section, we consider the modelling of a 2-country VAR featuring the log-difference of US GDP and residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP for the 1995Q1-2019Q4 period as reported by OECD (https://stats.oecd.org/, VPVOBARSA, US Dollars, volume estimates at fixed PPP, seasonally adjusted). In this example, we illustrate how the D-SVAR allows to recover a shock structure à la Backus et al. (1992) involving dynamic symmetric spillovers. More precisely the shock process is assumed to take the form

$$Z_t \equiv \begin{pmatrix} z_t^{us} \\ z_t^{can} \end{pmatrix} = \begin{bmatrix} \rho & \nu \\ \nu & \rho \end{bmatrix} Z_{t-1} + \varepsilon_t \text{ with } \varepsilon_t \rightsquigarrow N(0_2, I_2),$$

where ν captures the dynamic spillovers. Both the AIC, BIC and Hannan-Quinn information criteria led us to select a VAR(2) specification. The D-SVAR identification then leads to a value of $\rho = 0.43$ and $\nu = 0.20$. The loading matrix Π_{yz} then takes the form

$$\Pi_{yz} = \begin{bmatrix} 0.498 & 0.053 \\ -0.229 & 0.360 \end{bmatrix}.$$

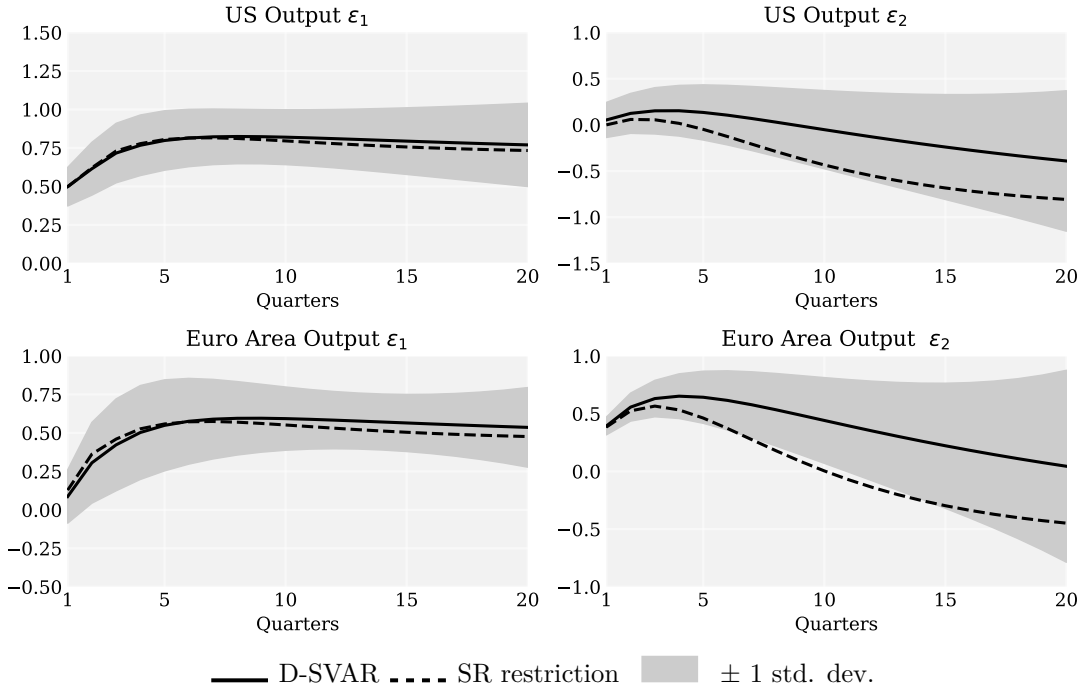
The forecast error variance decomposition of levels is reported in Table K.1.. While the US shock explain essentially all of US GDP, it accounts for less than 5%the Euro volatility in the very short-run and 56% at the 20 quarters horizon. In other words, in the very short-run, the US and Euro Area economies are essentially insulated from each other, while the shocks are transmitted in the medium run. Figure K.1 reports the IRF of US and Euro GDP to both shocks. These IRF confirm and illustrate the broad picture conveyed by forecast error volatility decomposition: US and Euro GDP only responds to their respective shocks on impact, and are essentially insulated from exogenous developments in the other economy in the very short-run. This is actually slightly contrasts with an identification of the US shock as the only shock that affects US GDP on impact. In that latter case, the US shock is transmitted faster to the Euro Area: the US shock accounts for about 10% of the Euro output volatility on impact and about 40% after 1 year (30% in our case). In the longer run, the Cholesky decomposition indicates that while the US economy is essentially not affected by Euro shocks, US shocks account for 60% of GDP volatility in the Euro Area.

Table K.1: Forecast Error Decomposition

	US GDP		Euro Area GDP	
Horizon	ε_t^{us}	ε_t^{euro}	ε_t^{us}	ε_t^{euro}
D-SVAR				
1	98.9	1.1	4.6	95.4
4	96.4	3.6	29.3	70.7
8	97.8	2.2	40.9	59.1
20	93.6	6.4	60.8	39.2
Short-Run Restriction				
1	100.0	0.0	10.0	90.0
4	99.6	0.4	38.3	61.7
8	96.7	3.3	56.5	43.5
20	68.7	31.3	67.0	33.0

Notes: The variance decomposition is obtained from a bivariate D-SVAR or a SVAR with a short-run restriction. Variables are the log-difference of US GDP and the residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP. Sample is 1995Q1-2019Q4.

Figure K.1: Impulse Response Functions: US vs Euro Area



Notes: These IRFs are obtained from a bivariate D-SVAR or a SVAR with a short-run restriction. Variables are the log-difference of US GDP and the residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP. Sample is 1995Q1-2019Q4.