# Exploiting Dynamic Identification in VARs

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**Abstract**

Most macroeconomic models view economic outcomes as being generated by a combination of endogenous and exogenous dynamic forces. where the exogenous forces are modelled as following restricted autoregressive processes. In this paper we show that this dual dynamic structure is sufficient to identify the entire set of the model shocks and impulse responses. No extra restrictions are needed. We then show how this result can be used to evaluate common Structural VAR restrictions such as impact restrictions, long run restrictions and proxy-VAR approaches.

**Keywords:** Structural VAR, Proxy-VAR, Dynamic Identification

## 1 Introduction

Macroeconomists are often interested in knowing how the economy reacts to different types of shocks (see Ramey (2016) and Stock and Watson (2017) for very detailed accounts of the recent macroeconomic literature). A dominant approach is the use of Structural Vector Autoregressive (SVAR) models.[1] This approach relies on two premises. First, the macroeconomic model that generates the data features *(i)* some forcing processes (technology, oil price, monetary or fiscal policy, etc.) and *(ii)* a theory on how endogenous variables (consumption, employment, inflation, etc.) react to these forcing

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[1] Sims (1980) is a seminal paper in the SVAR litterature. See Section 4 in Stock and Watson (2016) and Kilian and Lütkepohl (2016) for a complete review.

processes. Second, the model solution can be represented (or well approximated) by a Vector Autoregressive (VAR) model[2]. Once the VAR specification is chosen (variables, order of differentiation and number of lags), estimation is straightforward and one can learn about the response of the macroeconomy to shocks by inspecting the implied Impulse Response Functions (IRFs). However, obtaining meaningful IRFs can only be achieved by imposing a set of identifying restrictions, allowing the VAR to be rewritten as a SVAR. These identifying restrictions can be controversial. As was put forward by Sims (1980), IRFs from a SVAR are potentially less prone to mis-specification than those obtained by estimating a fully specified macroeconomic model. But mapping their implications into the language of models is not uncontroversial because of the underlying identifying assumption one needs to impose.

In this paper, we keep the two premises of the SVAR literature and exploit the fact that in macroeconomic models, forcing processes underlying exogenous driving forces are generally restricted. For example, in the vast majority of macroeconomic models, the underlying driving forces are assumed to be independent autoregressive processes (most often of order one).[3] Our main theoretical result is to show that the macroeconomic model impulse responses can be recovered from a SVAR if one is willing to exploit these restrictions on exogenous driving forces. Thanks to this result, most identifying restrictions commonly used in the SVAR literature are not needed for identification, and can be viewed as over-identifying restrictions. They can therefore be visually evaluated and formally tested. For instance, this will allows for the evaluation of the variety of identifying restrictions imposed in SVARs such as short-run, long-run, proxy-variables, sign restrictions, etc.

**Exploiting Dynamic Restrictions.** As noted above, the commonly used restrictions on macroeconomic models forcing processes are enough identify the model IRFs without the need of any of the additional assumption. Because it explicitly makes use of restrictions on the dynamic structure of forcing processes, we dub the particular SVAR derived a "D-SVAR". By *identification of model IRFs*, we mean that there generically exists a unique[4] vector of mutually orthogonal innovations (shocks) in the D-SVAR that satisfies the restrictions imposed by the dynamic structure of the macroeconomic model. Identification in SVARs usually relies on restrictions imposed on the loading matrix of shocks, under the requirement of compatibility with covariance matrix of the VAR innovations. Our approach focuses instead on exploiting constraints on the dynamic propagation of

[2]We use the generic term VAR, but our identification result equally applies to Vector Error Correction Models (VECM) and Vector Autoregressive Moving Average (VARMA) models.

[3]As we will prove, our results do not depend on the independence and AR(1) assumption, but much of the discussion will proceed with this assumption for the ease of exposition.

[4]Uniqueness is up to the sign and/or a permutation of the shocks.

shocks rather than the loading of the shocks. Consequently, identification solely through the covariance matrix is no longer sufficient, we need to fully exploit the auto-covariance function at different order of the dynamic representation (that can be a VAR, a VARMA or a State-Space model), or equivalently, the spectral density, to obtain sufficient conditions for local identification. This theoretical result will be shown in Section 4. Importantly all identification results are obtained from the Wold representation of the model solution, whether this representation obtains from a VAR, VARMA, or a linear state space model. For expositional purposes, and because the applied macroeconomic literature considers almost exclusively VARs, we will mainly focus on the case of a VAR model in the main text and, accordingly, will maintain the D-SVAR terminology .

**Testing Commonly Used SVAR Restrictions.** It is worth mentioning that the shocks identified by our D-SVAR are the macroeconomic model shocks but they remain unlabelled (nothing in the identification assigns a name –e.g. technology, monetary, fiscal, etc. – to the identified shocks). While standard SVAR restrictions (impact as in Christiano et al. (1999), long-run as in Blanchard and Quah (1989) or sign restrictions as in Uhlig (2005)) can be used to label the identified shock *ex-post*, our approach can be used to test these latter restrictions. We believe this is a strength of our approach, not a weakness: the naming of the shocks is not needed for identification. A substantial part of the paper will use three examples drawn from the literature to illustrate our approach. We first present the Blanchard and Quah (1989) analysis of demand versus supply shocks which is based on long run restrictions. Then we examine the proxy VAR strategy presented in Gertler and Karadi (2015) to identify monetary shocks using a high frequency instrument (see Kuttner (2001), Gürkaynak et al. (2005), Bernanke and Kuttner (2005) and Gürkaynak et al. (2007) for early work on High Frequency Identification of monetary policy shocks). Finally, we examine the validity of the impact restrictions used in Christiano et al. (1999, 2005) to identify monetary policy shocks. We will show that the identifying restrictions used in Blanchard and Quah (1989) are readily accepted by the data under the assumption the D-SVAR represention is true, while those associated with the proxy-VAR of Gertler and Karadi (2015) are rejected. For the Christiano et al. (1999)) anaylsis, the identifying restrictions cannot be rejected but some features put them into question.

**Related Literature.** Our identification result relates to several papers including, among others, McGrattan (2010), Pagan and Robinson (2019), Bai and Wang (2015) and Gourieroux and Jasiak (2022). While several of our theoretical results have precedents in the literature, our main contribution is to show why standard SVAR restrictions can be seen

as over-identifying restrictions and therefore can be tested.

McGrattan (2010) derives conditions for identification of an unrestricted state-space representation associated with a specific small-scale Real Business Cycle model.[5] In particular, the paper shows that, when the model includes a permanent technology shock and a stationary labor wedge shock (in the form of a labor income tax), the unrestricted state-space representation is identified. Our paper provides conditions for identification in a broader class of macroeconomic models, admitting a VAR or a VARMA representation of the solution. Pagan and Robinson (2019) note that SVARs may face difficulty to properly uncover the loading matrix of macroeconomic models because standard estimation of SVAR models avoids imposing the type of statistical restrictions commonly used in macroeconomic models —*e.g.* that structural shocks follow mutually orthogonal univariate autoregressive processes. The two authors discuss conditions for local identification in SVARs when the autoregressive matrix is diagonal and shocks are normalized. Our formal analysis shows more generally the conditions on the autoregressive matrix that allows to identify the structural shocks. Bai and Wang (2015) study identification in dynamic factor models similar to our unrestricted state space representation. Their approach, in line with the conventional way of identifying shocks in the VAR literature, imposes restrictions on the loading matrix while leaving unrestricted the autoregressive matrix of factors. In this paper, we take the opposite viewpoint and determine which type of organisation of the autoregressive matrix allows to freely identify the loading matrix in the state-space representation. Finally, Gourieroux and Jasiak (2022) provide conditions for identification in multivariate undetermined convoluted systems when the forcing processes shocks (the "*sources*" in their terminology) are linearly independent autoregressive processes of order one and when there is no intrinsic dynamics of the endogenous variables. They show that when the autoregressive parameters are distinct, the loading matrix (the "*mixing matrix*" in their terminology) is identified. Our paper departs from theirs in at least three dimensions. First, we consider a larger class of dynamic models and makes connections with the macroeconomic literature. Second, we extend the identification problem to non diagonal autoregressive processes. Third, we determine conditions for partial identification when the practitioner seeks to identify only a subset of structural shocks.

**Outline.** The paper is structured as follows. Section 2 overviews the main theoretical results of the paper. It shows how the D-SVAR representation can be derived and explains heuristically why it is identified. Section 3 shows how one can test for some common identification in SVARs, first on simulated data generated by an estimated New Keynesian

[5]See also Kascha and Mertens (2009) for simulation experiments in a similar setup.

model, and then by revisiting three well-known SVAR studies (Blanchard and Quah (1989), Gertler and Karadi (2015) and Christiano et al. (1999)). Section 4 formally proves local identification. Section 5 discusses estimation and inference in the D-SVAR setup. A last section concludes. All proofs are reported in the appendix.

# 2 A Primer on D-SVARs

The main idea behind the SVAR literature is to use a VAR model that can be readily estimated and to recover a set of meaningful impulse responses using identifying restrictions. The underlying assumption in this approach is that the solution of the macroeconomic model is a VAR.[6] We start from the same premise that the model solution for a set of variables $X_t$ is a VAR. We further assume that the macroeconomic model embeds endogenous propagation mechanisms and exogenous driving forces, so that its solution can be written as (dropping contants):

$$\begin{aligned} Y_t &= \sum_{i=1}^{m} G_i Y_{t-i} + F Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t, \end{aligned} \tag{1}$$

where $Y_t$ is a $n \times 1$ vector of endogenous variables, $Z_t$ is a $n \times 1$ vector of exogenous structural driving forces with the structural innovations $\varepsilon_t$ that are assumed to have zero mean and their covariance matrix is identity. The matrices $G_i$ capture the endogenous propagation mechanism of the model, while the matrix $R$ expresses the exogenous dynamics and is generally assumed to be a diagonal matrix. Note that such solution can arise from different types of models including standard optimization based DSGE models with rational expectations or from models that rely more on behavioural assumptions.

With the setup given by (1), we are focusing on the case where the forcing processes are first order autorefressions, but the setup can be extended to higher order. In fact, our approach will later be shown to apply to more general setups where the model solution can be written in the state space form, and can therefore have a VAR or a VARMA representation. However, to keep things simpler for now and to be in line with much of the SVAR literature, we will focus here and in the next few sections on the case where the macroeconomic model solution is a VAR and can also be written in the form given by (1).[7]

[6] If the empirical model used in a SVAR exercise is not viewed as the solution an underlying structural model, then it is unclear what structural information can be extracted from it.

[7] In the case of DSGE model under rational expectations, the solution may be a VARMA instead a VAR, depending on variables observability. Our approach also applies in such cases, but given that much of the applied SVAR literature focuses of rmodel solutions that are thought to be well captured by a VAR, we spend considerable amount of time on this case.

By eliminating $Z_t$ in (1), we can derive the VAR representation of the macroeconomic model solution:

$$Y_t = \left(G_1 + FRF^{-1}\right) Y_{t-1} + \left(G_2 - FRF^{-1}G_1\right) Y_{t-2} + \cdots - FRF^{-1}G_m Y_{t-m-1} + F\varepsilon_t. \quad (2)$$

We dub the representation given by (1) the D-SVAR representation. Alternatively, the model solution can be written as:

$$Y_t = \Gamma_1 Y_{t-1} + \Gamma_2 Y_{t-2} + \cdots + \Gamma_{m+1} Y_{t-m-1} + \nu_t, \quad (3)$$

where $\nu_t$ is a vector of non-structural innovations with covariance matrix $\Sigma_\nu$. The advantage of VAR representation (3) is that the $\Gamma_i$ matrices and the $\Sigma_\nu$ matrix can be estimated from data. The link between the matrices in (3) and (2) is given by:

$$\begin{cases} \Gamma_1 &= G_1 + FRF^{-1}, \\ \Gamma_2 &= G_2 - FRF^{-1}G_1, \\ \Gamma_3 &= G_3 - FRF^{-1}G_2, \\ \vdots & \\ \Gamma_{m+1} &= -FRF^{-1}G_m, \\ \Sigma_\nu &= FF'. \end{cases} \quad (4)$$

Identifying the D-SVAR (2) means recovering matrices $\{G_i\}_{i=1}^m$, $F$ and $R$ from $\{\Gamma_j\}_{j=1}^{m+1}$ and $\Sigma_\nu$. Note that if the D-SVAR representation can be recovered from $\{\Gamma_j\}_{j=1}^{m+1}$ and $\Sigma_\nu$, one can then compute the model shocks, impulse responses functions and variance decompositions. However, because $\Sigma_\nu$ is symmetric, this system only provides with $(m+2)n^2 - \frac{n(n-1)}{2}$ independent equations for $(m+2)n^2$ unknowns. This is the well-known problem of the identification in SVARs. In the initial literature on SVARs, a restriction commonly imposed was to assume that $F$ is lower triangular, which amounts to restrict the effect of shocks on impact (see Sims (1980)). This puts exactly $\frac{n(n-1)}{2}$ restrictions, so the model shocks and IRFs can be identified. But most often when the loading matrix $F$ is obtained by solving a macroeconomic model, $F$ does not comply with the lower triangular assumption, so it would be valuable to be able to test such restrictions. Likewise, using long-run restrictions, sign restrictions or instrumental variables may not be uncontroversial in many setting and so again it would be valuable to test such restrictions.

Our D-SVAR approach does not rely on restricting the loading matrix $F$, nor does it rely in obtaining outside instruments. Instead, it will complement these approaches by exploiting the commonly held view that the forcing processes are most often restricted, which implicitly restricts the autoregressive matrix $R$. Let's take as an example the case in which the exogenous variables $Z_t$ are mutually orthogonal at all leads and lags, so that $R$ is a diagonal matrix. This is the assumption shared by a vast majority of the

macroeconomic literature as, for many, it is viewed as the requirement for considering these forces as meaningful exogenous forces. The main theoretical result of the paper (in Section 4) can be applied to that case, although it is more general. It shows that – if one views the structural driving forces in (1) as linearly independent– then (2) can be used to recover the matrices $\{G_i\}_{i=1}^m$, $F$ and $R$ from $\{\Gamma_j\}_{j=1}^{m+1}$ and $\Sigma_\nu$. Note that in terms of simply counting the number of equations and the number of variables in (2) when $R$ is diagonal, the order condition for identification will be satisfied. Obviously, this order condition is only a necessary condition for identification. The proof in Section 4 will show that if $R$ is diagonal (with distinct diagonal elements), which is almost a definitional characteristic of the exogenous process, this is sufficient for identification of the D-SVAR representation. Diagonality of $R$ is by no means necessary as we show later, but is a common assumption that we will maintain in the next section. At the risk of repetition, let us emphasize again that the identification result in Section 4 covers a greater range of models than that captured (1) but, for many applications, focusing on (1) will be sufficient.

In terms of practical value, the reason we view obtaining the D-SVAR representation (that is, getting $\{G_i\}_{i=1}^m$, $F$ and $R$ from $\{\Gamma_j\}_{j=1}^{m+1}$ and $\Sigma_\nu$) as useful is that it allows for identification restrictions commonly used in the SVAR literature to be tested instead of being a matter of faith. This includes testing impact restrictions, long run restrictions, sign restrictions and instrumental variables approaches among others. By itself, the D-SVAR representation is not overly useful as it does not provide names to the different shocks. Instead, its usefulness arises mainly when combined with common identification schemes since it allows such restrictions to be tested and, if accepted, shocks to be named, as illustrated in Section 3. It is important to note that throughout these exercises, we will be maintaining the assumption inherent to the SVAR literature that VAR model from which we want to extract structural information is well specified in terms of number of variables and lag length. In this sense, we see our approach as being useful once a researcher has settled on a VAR model which is believed to properly capture the solution of a macroeconomic model, but before they have actually exploited their identification restrictions. This is the point at which an identification strategy can be tested[8] before being imposed.[9]

[8]If the restrictions are rejected, this could be either because they are conceptually invalid or because the VAR model used is not sufficiently rich to allow one to extract the macroeconomic model shocks. For example, in a proxy VAR case, if the actual data generating process is driven by four forces and the VAR only contains three variables, then the validity of an instrument could be rejected even if it is conceptually a good instrument.

[9]Our D-SVAR approach can also be useful when starting from a fully specified macroeconomic model, but before estimating it and imposing all the models cross equation restrictions. Once the model is is fully specified, and before it is estimated, it will generally have a VAR or VARMA solution, from which the D-SVAR representation can be obtained. The researcher immediately obtains an estimate of all the model impulse responses, before any cross equation restrictions are exploited. This set of impulse responses can then be compared with that obtained by full-information estimation imposing all the cross-

# 3 Illustrating How a D-SVAR approach Can Be Used to Evaluate Different Identification Restrictions

In this section we will illustrate how one can use the mapping given by (4) to get the D-SVAR representation and use it to test common strategies imposed to identify macroeconomic model impulse responses from a VAR. As noted previously, our starting point is at the stage where a researcher is satisfied of their VAR model as a solution of an underlying macroeconomic model [10] and accordingly are willing to impose on it identification restrictions that allow them to extract a set of the structural impulse responses of interest.[11]

In the first subsection, we provide a Choleski or proxy VAR example based on simulated data. The advantage of beginning with a simulated data example is that it allows to both show *(i)* how well our D-SVAR can recover the macroeconomic model shocks in samples similar in length to that commonly used in the literature and *(i)* how our procedure will be able to differentiate between a "good" and a "bad" instrument with such samples. In addition, we will also show our our procedure can be alternatively used to evaluate Choleski-type restriction. In the subsequent subsection we apply our approach to three well know SVAR examples drawn from the literature.

## 3.1 A Simulated Data Example based on a New Keynesian Model as the Data Generating Process.

To illustrate how our D-SVAR can be used to test commonly used identifying restrictions, we have generated data from an estimated three equation New Keynesian model with habit persistence and a hybrid Phillips curve[12]. Our goral is to recover the IRF to a monetary policy shock. This model has a solution given by:

$$\begin{aligned} Y_t &= G_1 Y_{t-1} + F Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t, \end{aligned} \tag{5}$$

where $Y_t$ and $\varepsilon_t$ are three dimensional vectors. $Y$ is composed of output gap $y$, inflation $\pi$ and nominal interest rate $i$. The three exogenous shifters in $Z$ are a demand shifter, a cost-push shifter and a monetary policy shifter. The innovations to these shifters are

equation restrictions. If the two sets of results are in line, this gives support for the model. However, if they are significantly different, it places into question the validity of the model.

[10] When we refer to an underlying macroeconomic model, we are referring in this illustrative section to a model whose solution can be written in state space form and where the exogenous driving forces are independent processes. For the case where the model allows a VAR representation, this implies a structural model of the form given by (1) when the exogenous driving forces obey a first order process.

[11] The case where the macroeconomic model solution is a VARMA, which is often the case when the data are thought to be driven by DSGE model with hidden state variables, will be discussed in Section 4 when we derive the general identification result.

[12] See details of the model and the estimation in Appendix G

denoted $\varepsilon^d$, $\varepsilon^s$ and $\varepsilon^m$. The model solution can also be written as a VAR with two lags as given by:

$$Y_t = \Gamma_1 Y_{t-1} + \Gamma_2 Y_{t-2} + \nu_t, \tag{6}$$

where $\nu_t$ is also a three dimensional vector. The endogenous dynamics of this model are due to habit, sluggish monetary policy and partial price indexation.

We feed in 1,000 times a sequence of 300 periods $\varepsilon_t$ into this data generating process, to create a series for $Y_t$. From the generated data, we estimated a VAR with two lags in order to get estimates of $\Gamma_1, \Gamma_2, \Sigma_\nu$. We then used the mapping

$$\begin{cases} \Gamma_1 &= G_1 + FRF^{-1}, \\ \Gamma_2 &= -FRF^{-1}G_1, \\ \Sigma_\nu &= FF', \end{cases} \tag{7}$$

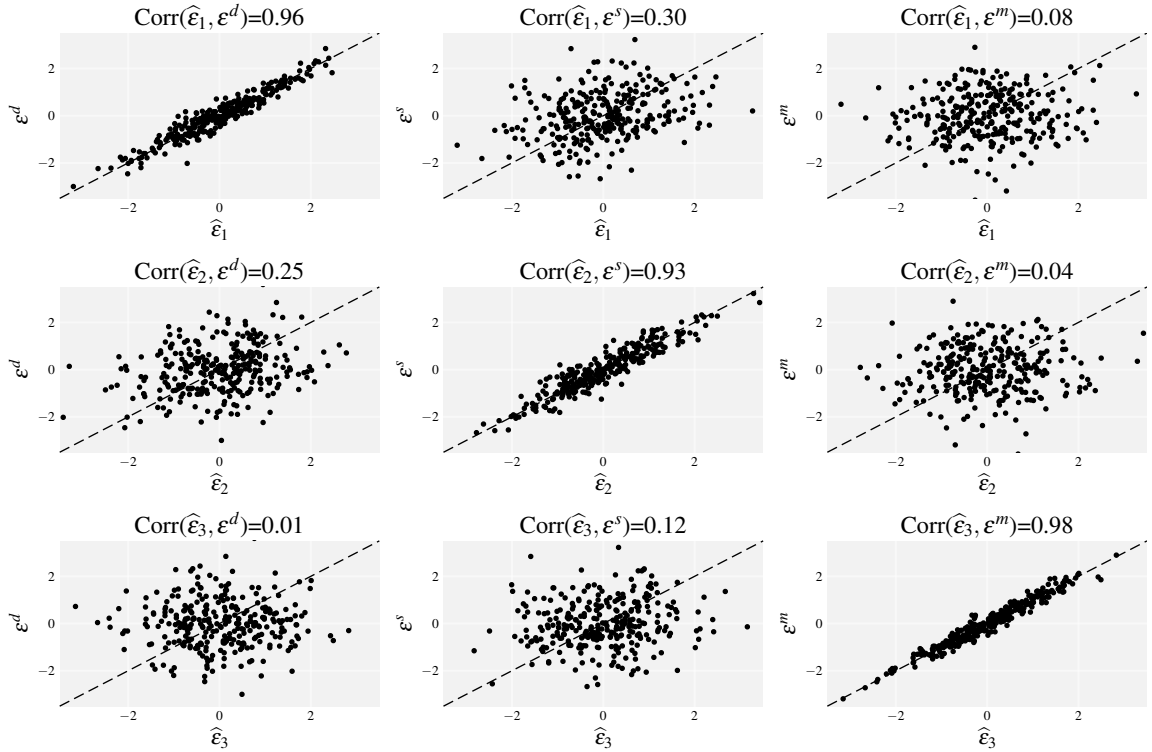
to recover estimates of $G_1$, $F$ and $R$ using (4). The distribution of the element of the estimated matrices $\widehat{G}_1$, $\widehat{F}$ and $\widehat{R}$ in comparison to their theoretical counterpart is presented in Appendix G.3. Given estimated $\widehat{F}$, we construct estimates of the New Keynesian model shocks using $\widehat{\varepsilon}_t = \widehat{F}^{-1}\widehat{\nu}_t$. Recall that to recover these elements we are only exploiting the rather minimal condition that $R$ be a diagonal matrix. In Figure 1, we begin by plotting the three elements of $\widehat{\varepsilon}_t$ against the corresponding elements in $\varepsilon_t$ for one arbitrary simulation of 300 periods.[13] As can be seen on this figure, the $\widehat{\varepsilon}_t$ obtained through our procedure are very close to the actual structural shocks used in generating the data. Indeed, each is highly correlated with one of the structural shocks and close to orthogonal for the two other ones.[14] In that sense, in that finite sample, our D-SVAR can be seen as working quite well in recuperating the model shocks.[15]

[13] By normalization, $\widehat{\varepsilon}_1$ (resp. $\widehat{\varepsilon}_2$, $\widehat{\varepsilon}_3$) is the shock that is the most correlated with $\varepsilon^d$ (resp. $\varepsilon^s$, $\varepsilon^m$).

[14] In that particular simulation, off-diagonal correlations of Figure 1 are sometimes low but significant, as for example corr$(\widehat{\varepsilon}_1, \varepsilon^s)$. Over the 1,000 simulations, we cannot reject that these off-diagonal correlations are indeed zero.

[15] This result holds on average over the 1,000 simulations. For example, the correlations of $\widehat{\varepsilon}_3$ with the three structural shocks $\varepsilon^d, \varepsilon^s, \varepsilon^m$ are on average .21, -.06 and .79, with standard deviations over the 1,000 simulations .34, .34 and .26. In Appendix G.4, we report the average impulse responses (over the 1,000 simulations) associated to the D-SVAR together with those obtained with the New Keynesian model, to illustrate how well the true IRFs are obtained with the D-SVAR over relatively short samples (300 observations).

Figure 1: New Keynesian Model Shocks and the D-SVAR Shocks For One Arbitrary Simulation




Notes: The estimated New Keynesian model is simulated over 300 periods with random $\varepsilon^d$, $\varepsilon^s$, $\varepsilon^m$ drawn in their estimated distribution. The macroeconomic times series generated are used to estimate a D-SVAR with innovations $\widehat{\varepsilon}_1$, $\widehat{\varepsilon}_2$, $\widehat{\varepsilon}_3$. By normalization, $\widehat{\varepsilon}_1$ (resp. $\widehat{\varepsilon}_2$, $\widehat{\varepsilon}_3$) is the shock that is the most correlated with $\varepsilon^d$ (resp. $\varepsilon^s$, $\varepsilon^m$).

### 3.1.1 Evaluating a Proxy VAR Approach with our Simulated Data

Over the last several years, one of the most common ways to recover structural impulse responses from VAR models is by the use of instrumental variables. This is generally referred to as a proxy VAR approach and relies on the researcher having an instrument $v_t$ which is believed to be correlated with one structural shock but not correlated with any other structural shock. Under this belief, the researcher can then use $v_t$ to instrument the innovations associated with their estimated VAR to identify the macroeconomic model impulse responses of interest. However, before performing this instrumental procedure, obtaining the D-SVAR representation from the VAR allows for the validity of the instrument to be tested. To see how this can be easily performed, recall that the D-SVAR representation provides an estimate of the matrix $F$ and therefore the macroeconomic model shocks $\varepsilon_t$ can be obtained from the VAR innovations $\nu_t$ though $\varepsilon_t = \widehat{F}^{-1}\nu_t$. With estimates of the macroeconomic model shocks in hand, one can then test whether an instrument $v_t$ is correlated with one, and precisely one, of the set of model shocks $\varepsilon_t$

To this end, we used the elements of vector $\varepsilon_t = \{\varepsilon_t^d, \varepsilon_t^s, \varepsilon_t^m\}$ to create two potential instruments. The first is a good instrument for the monetary policy shock $\varepsilon_t^m$, and is generating as:

$$v_{1t} = \varepsilon_t^m + \mu_{1t},$$

where $\mu_{1t}$ is iid noise with variance $\sigma_m^2/2.5$. By "good" instrument, we mean that $v_1$ is noisy but perfectly correlated with the monetary shock (in population) and orthogonal (in population) to the two other ones. We also created an invalid instrument $v_{2t}$ constructed as:

$$v_{2t} = .5\,\varepsilon_t^m + .45\,\frac{\sigma_m^2}{\sigma_d^2}\,\varepsilon_t^d + .05\,\frac{\sigma_m^2}{\sigma_s^2}\,\varepsilon_t^s + \mu_{2t},$$

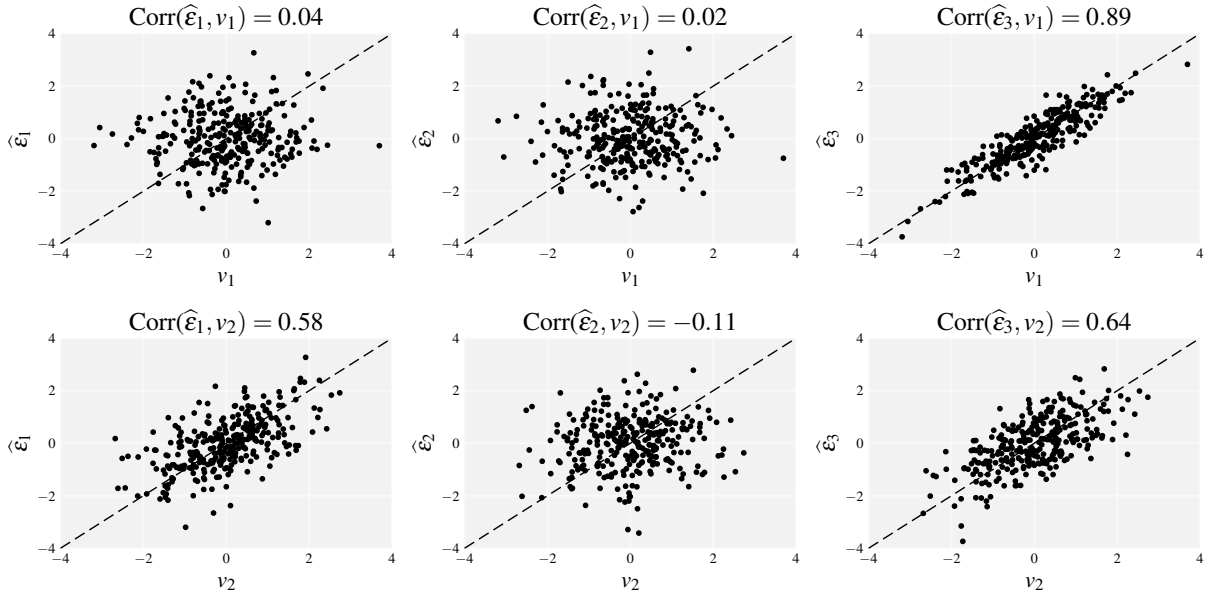
where $\mu_{2t}$ is again iid noise with variance $\sigma_m^2/2.5$. This instrument is invalid because, by construction, it is correlated with two structural shocks not one (correlations are .64 with the monetary shock and .57 with the demand shock, while it is .06 with the supply shock).

In Figure 2, we plot the instruments $v_{1t}$ and $v_{2t}$ against the three shocks identified by the D-SVAR ($\widehat{\varepsilon}_t$) for the same particular simulation, and indicate the correlations. As can be seen, in Figure 2, $v_{1t}$ is only correlated with one element in $\widehat{\varepsilon}_t$ ($\widehat{\varepsilon}_3$), which itself is highly correlated with the monetary policy shock). This indicates that it is potentially a good instrument as it correlated with only one of the D-SVAR structural shocks. At the same time, $v_{2t}$ is correlated with two elements of $\widehat{\varepsilon}_t$ and therefore cannot be an appropriate instrument for the monetary policy shock.[16] In Appendix G.4, we

[16] This result again holds on average over the 1,000 simulations. The correlations of $v_1$ with the three $\widehat{\varepsilon}$ shocks are on average .07, .06 and .73, with standard deviations over the 1,000 simulations .23, .40 and .27. For $v_2$, correlations are .58, .06 and .61 with standard deviations .17, .37 and .30.

report the same information in a different way by showing the average impulse responses (over the 1,000 simulations) to the identified monetary policy shock for the proxy-VARs together with the DGP ones and to the D-SVAR shock that is the most correlated with the true monetary policy shock. As can seen in this Appendix, the proxy-VAR with good instrument and the D-SVAR impulse responses are aligned with the true ones, while this is not the case for the proxy-VAR with the bad instrument.

Figure 2: Instruments and D-SVAR Shocks For One Arbitrary Simulation of the Estimated New Keynesian Model




Notes: The estimated New Keynesian model is simulated over 300 periods with random $\varepsilon^d$, $\varepsilon^s$, $\varepsilon^m$ drawn in their estimated distribution. The instruments $v_1$ and $v_2$ are constructed using these shocks. The macroeconomic times series generated are used to estimate a D-SVAR with innovations $\widehat{\varepsilon}_1$, $\widehat{\varepsilon}_2$, $\widehat{\varepsilon}_3$.

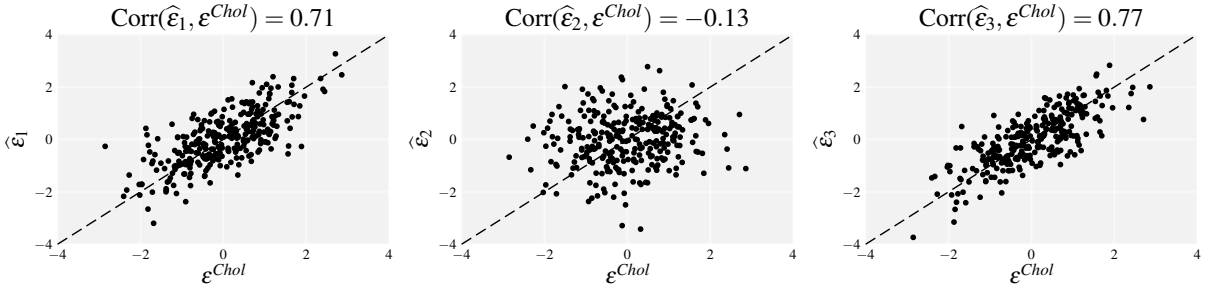
This example is meant to illustrate how the validity of a proposed instrument can be tested by the exploitation of the D-SVAR representation. The D-SVAR representation allows one to recover estimates of the macroeconomic model shocks under the rather minimal assumption that the exogenous driving forces are linearly independent and of order one. A macroeconomist estimating a proxy-VAR with instrument $v_1$ and also estimating our D-SVAR would then conclude: "given the set of variables and lag structure I have chosen, my instrument is compatible with a model economy in which forcing variables are independent AR(1)", which is the case of most of the model macroeconomists work with.[17] More importantly, it allows us to examine whether a given instrument is correlated with only one of these shocks as is required for it to be a valid instrument.

[17]If one believes that the underlying model of the world has AR(2) independent forcing variables, or that the $R$ matrix is not diagonal (as for example in the international RBC literature), our D-SVAR is still identified, as we will prove later on, so that one could still test for the quality of the instrument.

### 3.1.2 Evaluating a Recursive Ordering with our Simulated Data

In the New Keynesian model we are using to generate data, the matrix $F$ is not triangular nor bloc recursive. In this case, if we were to identify a monetary policy shock using an identification in the spirit of Christiano et al. (1999), which relies on having a monetary policy shock that does not have instantaneous impact on inflation and output, this should be rejected. To examine this possibility, we take our simulated data and use a Choleski decomposition to isolate a shock that has no impact effect on output and inflation. We will denoted this identified shock $\varepsilon^{Chol}$. We know already that this shock will not be the model shock $\varepsilon^m$ as in the New Keynesian model $F$ does not have two zeros on its first line. Figure 3 shows how this constructed shock correlates with the three shocks we retrieved using our D-SVAR methodology (for one particular simulation). One can clearly see a strong correlation of $\varepsilon^{Chol}$ with two of the $\widehat{\varepsilon}$. Given that $\varepsilon^{Chol}$ is strongly correlated with two shocks, and not only one, this would appropriately suggest to rejection this shock as a structural shock. Another way to approach this issue, would be to compare the impulse response associated with $\varepsilon^{Chol}$ with those obtained using our D-SVAR. [18]

Figure 3: Choleski and D-SVAR Shocks For One Arbitrary Simulation of the Estimated New Keynesian Model



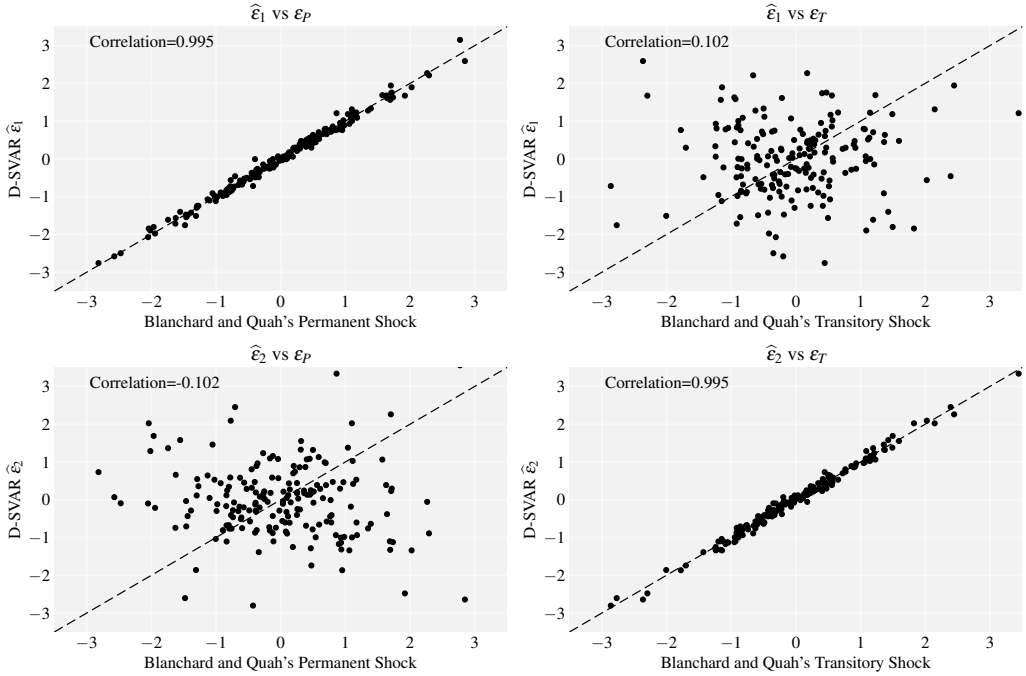

Notes: The estimated New Keynesian model is simulated over 300 periods with random $\varepsilon^d$, $\varepsilon^s$, $\varepsilon^m$ drawn in their estimated distribution. The macroeconomic times series generated are used to estimate a D-SVAR with innovations $\widehat{\varepsilon}_1$, $\widehat{\varepsilon}_2$, $\widehat{\varepsilon}_3$, and to revover $\varepsilon^{Chol}$ using a Choleski decomposition. $\varepsilon^{Chol}$ is the VAR innovation that does not move inflation and output on impact.

## 3.2 Applying our D-SVAR Testing Methodology to Well Known Examples

It this section we move away from simulated data and instead apply our D-SVAR methodology to three well know applications of SVAR.

[18] In Appendix G.4, we report the average impulse responses (over the 1,000 simulations) associated with $\varepsilon^{Chol}$ together with those obtained with the New Keynesian model, to illustrate how this recursive identification fails to reproduce the structural response when the recursive identification is inappropriate.

Figure 4: Correlation between D-SVAR and BQ shocks




Notes: Sample is 1960Q1-2007Q4. Estimation is done with $(\Delta y, u)$ using two lags, where $y$ is the real GDP and $u$ is the unemployment rate gap.

### 3.2.1 Revisiting Blanchard and Quah (1989) Long Run Restrictions SVARs

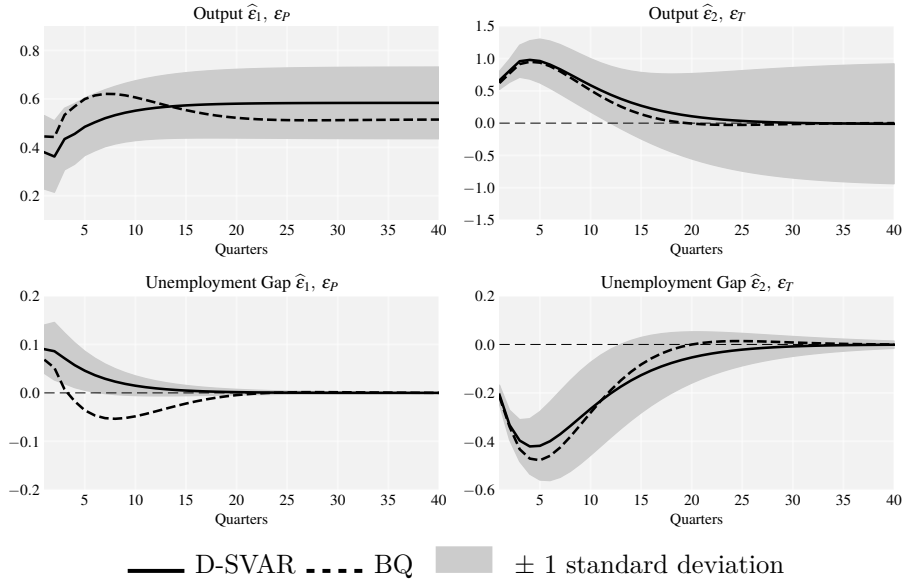
Here start with a bivariate VAR featuring the growth rate of output per capita, $\Delta y_t$, and a measure of the unemployment rate gap, $u_t$, computed as the gap between the actual and non-cyclical rate of unemployment. Blanchard and Quah (1989) (BQ hereafter) used this VAR to uncover the permanent, $\varepsilon_P$, and transitory, $\varepsilon_T$, component of output by imposing that the latter has no long-run effect on the level of output. They then showed that in the short run, $\varepsilon_T$ was very much looking like an aggregate demand shock and $\varepsilon_P$ like an aggregate supply shock.

Motivated by Blanchard and Quah (1989), we estimate a VAR for the 1960Q1-2007Q4 period and perform both the BQ identification and the D-SVAR one with diagonal $R$ matrix.[19] As we can see on Figure 4, the two procedures essentially recover the same shocks, which suggests that BQ identifying restrictions cannot be rejected under the assumptions made for the D-SVAR, namely that the macroeconomic model of the data is an impulse-propagation model with independent AR(1) forcing variables. We then compute impulse responses for the two identifications. They are displayed in Figure 5.

We uncover an interesting and familiar pattern. There is a shock, $\widehat{\varepsilon}_2$, that increases output and decreases unemployment on impact. Then the response of output is hump-

[19]We use two lags of data as selected by BIC and we use an Asymptotic Least Squares estimation method, as discussed in Corollary 1, using the unrestricted VAR as auxiliary model.

Figure 5: Comparing the D-SVAR with Blanchard and Quah (1989) Identification




Notes: Sample is 1960Q1-2007Q4. $y$ is the real GDP, $u$ is the unemployment rate gap. Estimation is done with ($\Delta y$,$u$) using two lags. The grey area represents 68% confidence bands obtained from 1,000 Bootstrap replications.

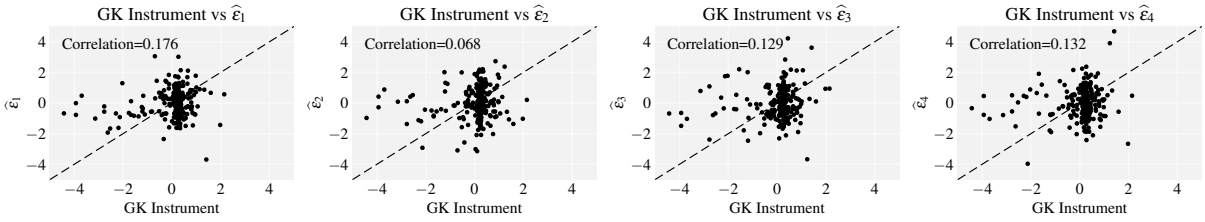
shaped and goes back to almost zero in the long-run. This shock explains more than 80% of unemployment volatility at any horizon, about 75% of the volatility of output on impact, but about 0% in the long-run. The other shock, $\widehat{\varepsilon}_1$, exerts a permanent effect on output and little effect in unemployment. The two shocks look pretty much like the permanent and temporary shocks of BQ. This is confirmed by the dash lines on Figure 5, which correspond to the BQ identification. Responses are indeed very similar. Figure 4 shows scatter plots of the BQ's permanent and transitory shocks against $\widehat{\varepsilon}_1$ and $\widehat{\varepsilon}_2$: the correlation between the D-SVAR and BQ shocks is almost perfect in both cases.

The estimation of the D-SVAR allows us to test the BQ identification restriction. If one restricts the data to be generated by a model in which the two forcing variables are linearly independent AR(1) processes, then, as can be seen on Figure 5, one cannot reject at 0% that only one shock has a permanent effect on output. Figure 5 seems to indicate that the impact response of the unemployment gap differs across the two identifications. However, a formal test of the null hypothesis of equality between the two IRF does not reject the null hypothesis, with a p-value of 40%.

### 3.2.2 Revisiting Gertler and Karadi (2015) Proxy VAR

In this section, we revisit Gertler and Karadi (2015), who identified a monetary policy shock relying on a proxy-VAR approach with an external instrument (see Beaudry and Saito (1998), Stock (2008), Stock and Watson (2012) and Mertens and Ravn (2013) among others). The instrumental variable needs to satisfy two assumptions to identify a

Figure 6: GK instrument against the four $\widehat{\varepsilon}$




given structural shock: *(i)* the instrument must be relevant –*i.e.* the contemporaneous correlation between the monetary policy shock and the external instrument must be non-zero; *(ii)* the instrument must be exogenous –*i.e.* the instrument must be uncorrelated with the other macroeconomic model shocks. According to Gertler and Karadi (2015), an advantage of proxy VARs is that it does not impose a special organization (and thus restrictions) of the loading matrix $F$. This is also the case in our D-SVAR, and it is therefore interesting to compare the two approaches.

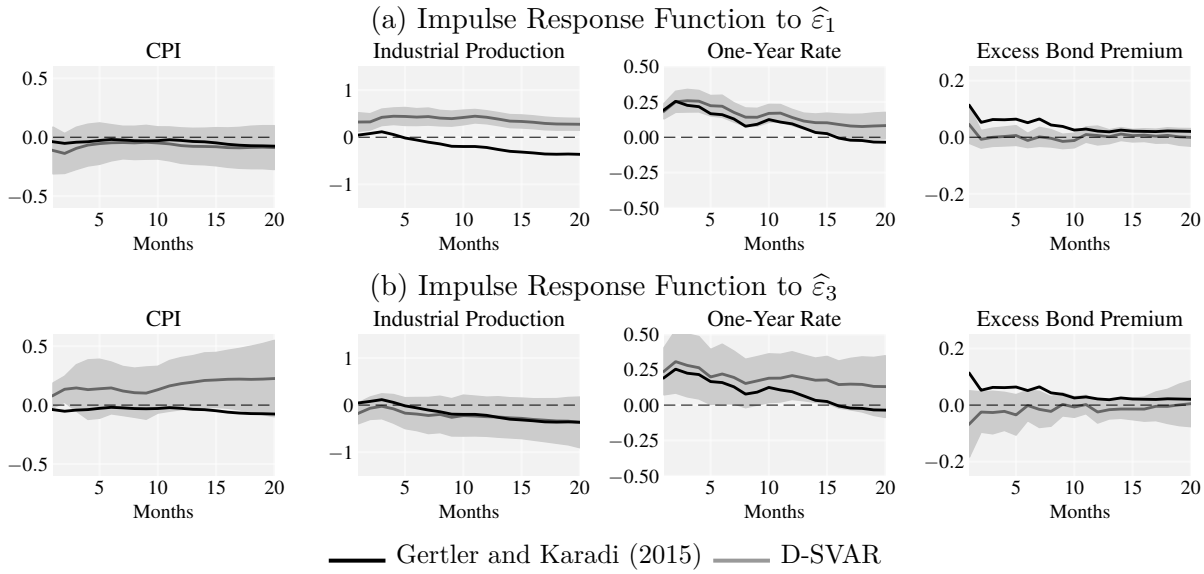
We first replicate Gertler and Karadi (2015) by estimating a VAR featuring the log consumer price index, the log industrial production, the one year government bond rate, and Gilchrist and Zakrajšek (2012) excess bond premium. The economy is observed at the monthly frequency from July 1979 to June 2012. Following Gertler and Karadi (2015), the VAR includes twelve lags. We first recover the impulse response functions of these variables to a monetary policy shock identified relying on the proxy VAR approach where, like Gertler and Karadi (2015), the external instrument is the surprise in the three month ahead futures rate. The identified contractionary monetary policy shock shifts the one-year rate upward, decreases economic activity after one year, raises the excess bond premium persistently, which signals the presence of financial frictions, and leads to a very small negative response of the CPI, without exhibiting any price puzzle (see black lines in Figure 7).

Then we proceed to estimating our D-SVAR. As in the Blanchard and Quah (1989) example of Section 3.2.1, the autoregressive matrix $R$ is assumed to be diagonal. We recover four unlabelled structural shocks that we arbitrarily index $\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_4$. On Figure 6, we plot the Gertler and Karadi (2015) instrument against our four D-SVAR structural shocks. As it can be seen and confirmed by tests, the instrument is significantly correlated with two of the shocks ($\widehat{\varepsilon}_1$ and $\widehat{\varepsilon}_3$). Through the lens of model with four AR(1) shocks, the instrument identifies a shock that is a combination of $\widehat{\varepsilon}_1$ and $\widehat{\varepsilon}_3$, and not a single exogenous monetary policy shocks.

Figure 7 plots the responses of the four variables to the two shocks $\widehat{\varepsilon}_1$ and $\widehat{\varepsilon}_3$, together with the responses to the Gertler and Karadi (2015) shock.

The shock $\widehat{\varepsilon}_1$ gives a very similar response of the one-year government bond rate as

Figure 7: IRFs to Gertler and Karadi (2015) monetary policy shock and D-SVAR's shocks $\widehat{\varepsilon}_1$ and $\widehat{\varepsilon}_3$




Notes: The black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shocks identified in the D-SVAR with diagonal $R$ matrix. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

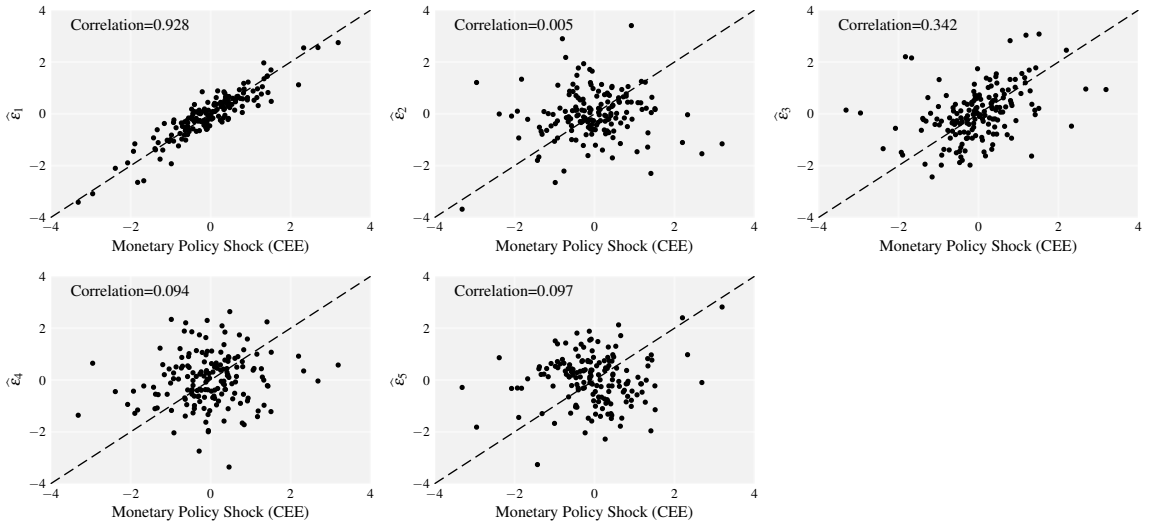
compared to Gertler and Karadi (2015). Although the response is more persistent, the response of the one-year government bond rate is also pretty similar. Inspection of the IRF for the other variables reveals that none of these two shocks gives similar responses for the four variables. The credit channel narrative of Gertler and Karadi (2015) requires an increase in the excess bond premium, which is obtained with $\widehat{\varepsilon}_1$, but comes with a persistent increase in industrial production. For the shock $\widehat{\varepsilon}_3$, there is indeed an increase in the one-year rate, a decrease in industrial production but an increase in inflation and a decrease in the excess bond premium, which does not square with the credit channel narrative of Gertler and Karadi (2015). According to this analysis, one is lead to reject the identifying assumption of Gertler and Karadi (2015) as their obtained shock is not a valid instrument.

### 3.2.3 Revisiting Christiano et al. (1999) Impact Restrictions SVAR

Christiano et al. (1999) identified a monetary policy shock by means of impact restrictions.[20] As we saw in Section 2 using simulated data, we can use our D-SVAR to evaluate those restrictions. To do so, we follow the method of Christiano et al. (1999) and estimate a four lags VAR featuring real GDP, the unemployment rate, CPI inflation, commodity

[20] See also Christiano et al. (2005).

Figure 8: Correlation between Christiano et al. (1999) Monetary Shock and D-SVAR Shocks




Notes: $\widehat{\varepsilon}_1$ to $\widehat{\varepsilon}_5$ are the structural shocks as obtained with the D-SVAR. Sample is 1965Q1-2007Q4.

price inflation and the federal funds rate, in that order, for the period 1965Q1-2007Q4.[21] We then proceed to estimating our D-SVAR, and recover five unlabelled shocks, $\widehat{\varepsilon}_1, \ldots, \widehat{\varepsilon}_5$. As illustrated in Figure 8, the Christiano et al. (1999) monetary shock identified using their impact restriction correlates strongly with $\widehat{\varepsilon}_1$ (one cannot reject that this correlation is one (p-value=0.986)), while we cannot reject that it is orthogonal to the four other D-SVAR shocks (p-value<0.035).[22]

Figure 9 plots the responses to $\widehat{\varepsilon}_1$ (grey lines), together with the responses to the Christiano et al. (1999) monetary policy shock. The figure indicates that the dynamics following a Christiano et al. (1999) monetary policy shock and those following $\widehat{\varepsilon}_1$ are quite similar. Just like for Christiano et al. (1999) monetary policy shock, a tightening of monetary policy leads to a prolonged recession with output (resp. unemployment) reaching it trough after about seven (resp. nine) quarters, and eventually reverting back in the longer run. Accordingly, both output and unemployment exhibit persistent hump shaped dynamics to the shock, just like in the aftermaths of a Christiano et al. (1999) monetary policy shock. Just like in Christiano et al. (1999), prices exhibit a persistent price puzzle, although it is more severe.[23]

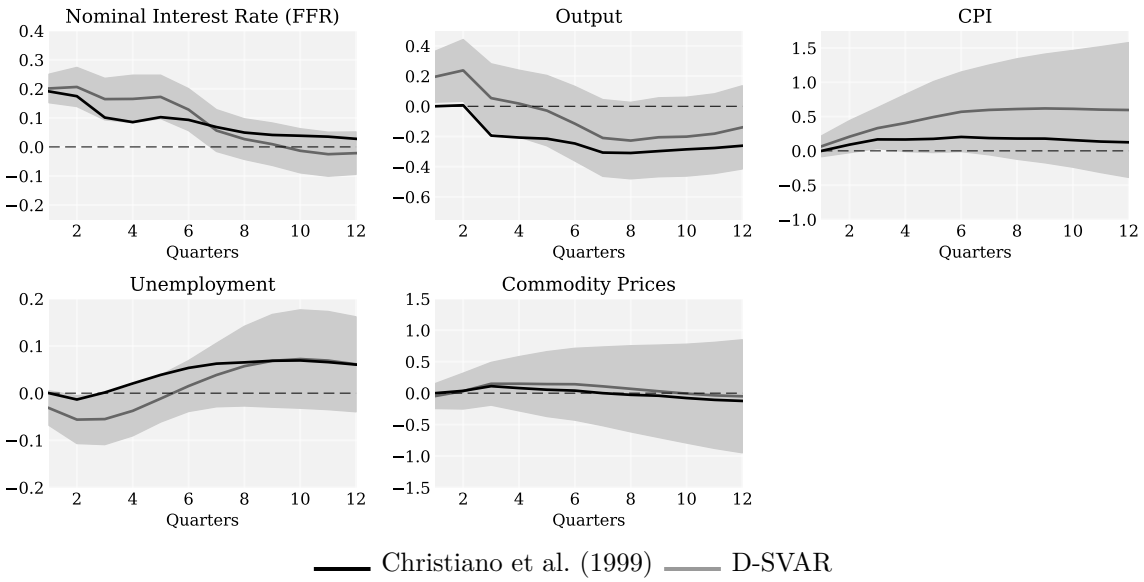
Under the D-SVAR, we can go a step further and test if $\widehat{\varepsilon}_1$ satisfies the zero impact restriction imposed by Christiano et al. (1999). Table 1 reports the p-value associated

[21] We end the sample period in 2007Q4 to avoid dealing with the zero lower bound, which would require an explicit non-linear modelling of the dynamics of the nominal interest rate to account for the presence of an occasionally binding constraint (see *e.g.* Mavroeidis (2021)).

[22] See Table J.1 in Online Appendix J.

[23] Such a "price puzzle" is reminiscent of the results in Beaudry et al. (2024), and can be rationalized in a model with a flat Phillips curve and a cost channel.

Figure 9: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock $\widehat{\varepsilon}_1$




Notes: On the five panel, the black line is the response to a monetary policy shock, as identified following Christiano et al. (1999). The grey line is the response to shock $\widehat{\varepsilon}_1$ in the D-SVAR. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Table 1: Test of Zero Impact Restriction (p-values)

| | Output | CPI | Unemp. | Com. Price |
|---|---|---|---|---|
| $\widehat{\varepsilon}_1$ | 0.2470 | 0.6587 | 0.3890 | 0.8169 |

Notes: $\widehat{\varepsilon}_1$ is the D-SVAR structural shock that is the closest to the Christiano et al. (1999) monetary policy shock. Sample is 1965Q1-2007Q4.

to the zero impact effect of $\widehat{\varepsilon}_1$ on, respectively, output, the CPI, unemployment and the commodity price. The results do not reject the restriction imposed by Christiano et al. (1999) as none of the p-values lies below 25%, well above the 5% standard level. Furthermore, we do not reject (p-value is 32%) that the responses of output and unemployment are jointly zero. This exercise shows that the Christiano et al. (1999) zero restrictions, which have been criticized as generally not compatible with a macroeconomic model (*e.g.* Carlstrom et al. (2009)), are not rejected by our D-SVAR approach. However, it should be noted that if one focuses on the impulse responses associated with $\widehat{\varepsilon}_1$, one may come away thinking that this shock may look more like a demand shock that is countered by monetary policy than a pure monetary shock. Since this alternative interpretation cannot be rejected either, it suggests some need for caution in concluding that a simple impact restriction can be used to identify monetary shocks.

# 4 D-SVAR Identification: Being More Formal

In Section 2, we presented our D-SVAR and noted that sufficient restrictions of the matrix $R$ should allow for identification from a VAR. However, this discussion was only suggestive. This section presents a set of propositions that support the local identification of the D-SVAR. We start by fixing some notations, we then prove identification relying on covariance matrix and dynamic restrictions.

## 4.1 Setup

Consider an economy whose Data Generating Process (DGP), which is the macroeconomic model solution, is described by the following state-space representation (we abstract from constant vectors without any loss of generality):

$$\underbrace{K_t}_{n_k\times 1} = \underbrace{\Pi_{kk}}_{n_k\times n_k}\underbrace{K_{t-1}}_{n_k\times 1} + \underbrace{\Pi_{kz}}_{n_k\times n_z}\underbrace{Z_t}_{n_z\times 1}, \tag{8}$$

$$\underbrace{Y_t}_{n_y\times 1} = \underbrace{\Pi_{yk}}_{n_y\times n_k}\underbrace{K_{t-1}}_{n_k\times 1} + \underbrace{\Pi_{yz}}_{n_y\times n_z}\underbrace{Z_t}_{n_z\times 1}, \tag{9}$$

$$\underbrace{Z_t}_{n_z\times 1} = \underbrace{R}_{n_z\times n_z}\underbrace{Z_{t-1}}_{n_z\times 1} + \underbrace{\varepsilon_t}_{n_z\times 1}, \tag{10}$$

where the $(n_y \times 1)$ vector $Y_t$ gathers all observed variables, the $(n_k \times 1)$ vector $K_t$ collects possibly unobserved endogenous state variables (predetermined at time $t-1$), $Z_t$ represents the $(n_z \times 1)$ vector of unobserved exogenous variables, and $\varepsilon_t$ is the $(n_z \times 1)$ vector of the macreconomic model shocks (the innovations to $Z_t$). In particular, $\varepsilon_t$ satisfies $\mathbb{E}_{t-1}\varepsilon_t = 0$, where $\mathbb{E}_{t-1}$ denotes the expectation operator conditional on the information set of histories until period $t-1$ –*i.e.* all past realizations and histories of $\{K_t, Y_t, Z_t\}$.[24]

This framework is general enough to represent the (log-)linear solution of most macroeconomic models, including DSGE models. For DSGE models, this solution usually depends on a limited number of parameters, denoted by $\boldsymbol{\theta}$, which represent "deep" structural parameters (such as preferences, technology, institutions, policies, etc.), as well as the stochastic process of the exogenous forcing variables in the structural model. In this context, the matrices in the state-space representation (8)–(10) are functions of $\boldsymbol{\theta}$.

In this paper, we do not consider the identification of $\boldsymbol{\theta}$; instead, we focus on the identification of the state-space parameters that freely enter the matrices of the state-space representation. We denote this vector of state-space parameters as $\boldsymbol{\psi}$. It must be emphasized that System (8)–(10) imposes fewer restrictions than the (possibly underlying) macroeconomic model. The only restrictions we explore apply to the matrix $R$ and the

[24] If some elements of $K_t$ are observed, those elements should be reassigned to vector $Y_t$, and the matrices $\Pi_{yk}$ and $\Pi_{yz}$ should be adjusted accordingly.

covariance matrix of the shocks $\varepsilon_t$ in Equation (10). Finallly, the shocks $\varepsilon_t$ are zero-mean weak white noise processes with covariance matrix $\mathbb{E}(\varepsilon_t\varepsilon_t') = \Sigma_\varepsilon$.

System (8)–(10) can be expressed in the Fernández-Villaverde et al. (2007) ABCD representation as:

$$S_{t+1} = AS_t + B\varepsilon_{t+1}, \tag{11}$$

$$Y_{t+1} = CS_t + D\varepsilon_{t+1}, \tag{12}$$

where

$$\underbrace{S_t}_{n_s\times 1} = \begin{bmatrix} K_t \\ Z_t \end{bmatrix}, \quad \underbrace{A}_{n_s\times n_s} = \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0_{n_z\times n_k} & R \end{bmatrix}, \quad \underbrace{B}_{n_s\times n_z} = \begin{bmatrix} \Pi_{kz} \\ I_{n_z} \end{bmatrix},$$

$$\underbrace{C}_{n_y\times n_s} = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \end{bmatrix}, \quad \text{and} \quad \underbrace{D}_{n_y\times n_z} = \Pi_{yz},$$

with $n_s = n_k + n_z$.[25]

In the sequel, we consider the case where the number of observables is equal to the number of shocks –*i.e.* $n_y = n_s$. As noted earlier, the vector

$$\boldsymbol{\psi} = (\text{vec}(\Pi_{kk})', \text{vec}(\Pi_{kz})', \text{vec}(\Pi_{yk})', \text{vec}(\Pi_{yz})', \text{vec}(R)', \text{vech}(\Sigma_\varepsilon)')'$$

collects all the parameters of the state-space representation (8)–(10).

Under the assumptions defined in the Appendix, the process $\{Y_t\}$ described by (11)–(12) admits the following infinite Vector Moving Average (VMA) representation:

$$\begin{aligned} Y_t &= \left[C(I_{n_s} - AL)^{-1}BL + D\right]\varepsilon_t, \\ &= \left[\Pi_{yk}(I_{n_k} - \Pi_{kk}L)^{-1}\Pi_{kz}L + \Pi_{yz}\right](I - RL)^{-1}\varepsilon_t, \\ &= \left[\Pi_{yk}(I_{n_k} - \Pi_{kk}L)^{-1}\Pi_{kz}\Pi_{yz}^{-1}L + I_{n_z}\right]\Pi_{yz}(I - RL)^{-1}\varepsilon_t, = H(L, \boldsymbol{\psi})\varepsilon_t, \end{aligned} \tag{13}$$

where $L$ denotes the lag operator. This VMA representation can be recovered through a VAR, a VARMA, or an estimation method based on the state-space representation, such that the impulse response functions can be obtained for the identified shocks $\varepsilon_t$.

Imposing restrictions solely on the covariance matrix of the $\varepsilon_t$ shocks and on the matrix $R$, which governs the dynamics of the exogenous process $Z_t$, enables the retrieval of the impulse response functions to the $\varepsilon_t$ shocks. In the general context with unobserved state variables $K_t$, such restrictions allow for the local identification of the matrix $R$, the loading matrix $\Pi_{yz}$, and the expression

$$\left[\Pi_{yk}(I_{n_k} - \Pi_{kk}L)^{-1}\Pi_{kz}\Pi_{yz}^{-1}L + I_{n_z}\right], \tag{14}$$

[25] In the specific case where there are no endogenous state variables $K_t$, $n_k = 0$, $n_s = n_z$, and the matrices are defined accordingly. For example, $A = R$.

but not the individual matrices within this expression. Nevertheless, this level of identification is sufficient to obtain the IRFs to the $\varepsilon_t$ shocks. This holds for every cases analyze below. When all endogenous state variables $K_t$ are observed, the individual matrices can be fully and locally identified.

The following section presents key propositions on local identification under the assumption that restrictions are applied solely to the covariance matrix and the matrix $R$, which governs the dynamics of the exogenous process $Z_t$. Specifically, the scenario considered assumes that the $\varepsilon_t$ shocks are mutually orthogonal, and their covariance matrix is normalized to the identity matrix, such that $\mathbb{E}(\varepsilon_t \varepsilon_t') = I_{n_z}$. This normalization is a widely adopted identifying assumption in the SVAR literature (see Ramey (2016)). Within this framework, the matrix $\Pi_{yz}$ encapsulates any scale effects of the shocks on all observed variables. In other words, it contains information about the volatility of the shocks.

## 4.2 Covariance Matrix and Dynamic Restrictions

This subsection focuses on the local identification of the IRFs to the $\varepsilon_t$ shocks, relying solely on the second-order moments of the observables $Y_t$. In addition to the restriction $\mathbb{E}(\varepsilon_t \varepsilon_t') = I_{n_z}$, local identification is achieved by imposing structure on the matrix $R$. The following four cases, commonly encountered in macroeconomic models, are considered:

1. $R$ is a diagonal matrix;
2. $R$ is a lower (or equivalently upper) triangular matrix;
3. $R$ is a symmetric matrix;
4. $R$ is a block diagonal matrix with blocks corresponding to cases 1 and 2.

The identification problem is examined and proven for each of these cases.

### 4.2.1 $R$ is a Diagonal Matrix

The case of a diagonal matrix is of particular interest. It implies, together with the restriction on the covariance matrix, that all processes in the $Z_t$ vector are mutually orthogonal at any leads and lags. While this assumption may appear restrictive, it corresponds to common practice in the macroeconomics literature and aligns with economic theory. The following proposition derives the sufficient condition for local identification.

**Proposition 1** *If $R$ is a diagonal matrix with distinct diagonal elements ($r_{i,i} \neq r_{j,j}$, $\forall i \neq j$), then the IRFs to the macroeconomic model shocks $\varepsilon_t$ (corresponding to the VMA representation (13)) are locally identified.*

**Proof:** See Appendix C.3.

In other words, the loading matrix $\Pi_{yz}$ and the autoregressive matrix $R$ are locally identifiable if all the autoregressive parameters of the $n_z$ linearly independent forcing variables are distinct. As mentioned earlier, in the case of unobserved state variables $K_t$, only the expression (14) is identified as a whole, but its individual components cannot be separately identified. However, this is sufficient to obtain the IRFs the $\varepsilon_t$ shocks. In contrast, when all endogenous state variables $K_t$ are observed, the individual matrices can be fully and locally identified.

Hence, in a "standard" macroeconomic model featuring mutually orthogonal shocks at any leads and lags, dynamic identification is readily achieved. Intuitively, identification arises because, given an economic structure, differences in the persistence of shocks imply that the impulse response functions to each shock provide unique information about the dynamics.

### 4.2.2 $R$ is a Lower Triangular Matrix

The case of a lower triangular autoregressive matrix $R$ is now considered, extending beyond the diagonal structure to a more general form. Let us define the matrix

$$\Lambda = \begin{bmatrix} \lambda_1 I_{n_{z1}} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{n_{z2}} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_L I_{n_{zL}} \end{bmatrix},$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$, and $n_{z\ell}$ are positive integers for $\ell = 1, \ldots, L$, such that $n_{z1} + \ldots + n_{zL} = n_z$. Here, $n_{z\ell} = m(\lambda_\ell)$ denotes the multiplicity of eigenvalue $\lambda_\ell$.

**Proposition 2** *If $R$ is a lower triangular matrix with the same main diagonal as $\Lambda$ and all elements on the first sub-diagonal are non-zero –i.e. $r_{i+1,i} \neq 0$ for $i = 1, \ldots, n_z - 1$, then the IRFs to the macroeconomic model shocks $\varepsilon_t$ (corresponding to the VMA representation (13)) are locally identified.*

**Proof:** See Appendix C.4.

Proposition 2 highlights that, unlike the strictly diagonal case, the lower triangular case does not require all diagonal elements to be distinct for identification. If some diagonal elements are identical, they must appear in consecutive order *–i.e.* grouped within one of the matrices $\lambda_\ell I_{n_{z\ell}}$. The proposition further implies that, while not all elements below the diagonal must be non-zero, those on the first sub-diagonal must be non-zero. Again, intuitively, the condition for proper local identification is that all shocks

must lead to distinguishable dynamics, as captured, for example, by the IRFs. To ensure this, some elements of $R$ must be distinct, guaranteeing that different shocks generate different dynamic responses.

### 4.2.3 $R$ is a Symmetric Matrix

For simplicity, the presentation is restricted to the case with two shocks. The matrix $R$ is symmetric, with identical values on the diagonal and identical values on the anti-diagonal. While this configuration may seem unusual at first glance, it has significant practical relevance in the international macroeconomic literature, originating from the seminal work of Backus et al. (1992).[26] For this specification, the autoregressive matrix takes the following form:

$$R = \begin{bmatrix} \rho & \tau \\ \tau & \rho \end{bmatrix}, \quad \text{with } \tau \neq 0. \tag{15}$$

The restriction $\tau \neq 0$ is critical. When $\tau = 0$, the matrix (15) reduces to a diagonal matrix with identical elements, which does not satisfy Proposition 1. Provided that $\tau \neq 0$, the following proposition holds.

**Proposition 3** *For the $2 \times 2$ symmetric matrix $R$ (eq. 15), the IRFs to the macroeconomic model shocks $\varepsilon_t$ (corresponding to the VMA representation (13)) are locally identified.*

In particular, and in contrast to the diagonal case, it is demonstrated in Appendix C.5 that the diagonal elements must necessarily be identical for the local identification of the IRFs.[27]

### 4.2.4 Partial Identification.

Finally, Proposition 4 provides conditions for partial identification.Even though some shocks may not be identified, it is still possible to identify a subset. [28]

**Proposition 4** *In the case of a block diagonal organisation of the $R$ matrix with one block corresponding to one of the preceding cases, the IRFs to the macroeconomic model corresponding shocks $\varepsilon_t$ are locally identified.*

**Proof :** See Appendix C.6.

[26] This dynamic structure of the autoregressive matrix has been widely used, among others, by Backus et al. (1994), Baxter and Crucini (1995), Heathcote and Perri (2002), and Kehoe and Perri (2002).

[27] Appendix K provides an illustration of the symmetric autoregressive matrix $R$ with an application to the international transmission of shocks between the US and the Euro area.

[28] Online Appendix E provides a simple illustrative example.

# 5 Estimation and Inference

This section discusses estimation and inference in the D-SVAR approach and shows how this approach can be used to evaluate SVAR models.

## 5.1 Estimation

To ease the exposition, let us consider the simplified state space representation:[29]

$$Y_t = \Pi_{yz} Z_t, \tag{16}$$

$$Z_t = R Z_{t-1} + \varepsilon_t, \tag{17}$$

where $\mathbb{E}(\varepsilon_t \varepsilon_t') = I_{nz}$. In the sequel, we will restrict ourselves to cases where the parameter vector $\boldsymbol{\psi} = (vec(\Pi_{yz})', vec(R)')'$ is indeed locally identified, such that the conditions for the validity of Propositions 1 and 2 are satisfied. Vector $\boldsymbol{\psi}$ can thus be estimated either by Maximum Likelihood (ML) from (16)-(17) or equivalently by a two step Asymptotic Least Square (ALS) approach (See Corollary 1 below). Let us denote $\widehat{\boldsymbol{\psi}}_T$ the ML estimator of $\boldsymbol{\psi}$ for a sample size $T$. Absent any unobserved variables, the D-SVAR representation rewrites as a VAR(1) model:

$$Y_t = \left(\Pi_{yz} R \Pi_{yz}^{-1}\right) Y_{t-1} + \Pi_{yz} \varepsilon_t, \tag{18}$$

where the $\Pi_{yz}$ is identified using the dynamic structure of unobserved forcing variables $Z_t$. So the loading matrix $\Pi_{yz}$ is obtained without any restriction. Consider now the VAR(1) representation :

$$Y_t = \Gamma Y_{t-1} + u_t, \tag{19}$$

with $E(u_t) = 0$ and $E(u_t u_t') = \Sigma_u$. This implies that $u_t = \Pi_{yz} \varepsilon_t$.

Our D-SVAR representation imposes as many or more restrictions on the dynamic structure of the data $Y_t$ than the VAR model. This offers an opportunity to use the information contained in the parameters of the estimated VAR model (19) to estimate $\boldsymbol{\psi}$ in the D-SVAR (16)-(17). Let us define $\eta = (vec(\Gamma)', vec(\Sigma_u)')'$ and the binding function $\widetilde{\eta}(\boldsymbol{\psi})$. We derive a version of the Corollary in Gouriéroux and Monfort (1995) (Chap. 10, Section 10.4.2, Corollary 10.2) adapted to our D-SVAR.

**Corollary 1** *Let $\widehat{\eta}_T$ be a consistent and asymptotically normal estimator of $\eta$ from (19) and let $\eta(\boldsymbol{\psi})$ be the binding function. The ALS estimator $\widehat{\boldsymbol{\psi}}_T$ obtained by solving*

$$\min_{\boldsymbol{\psi}} \left[\widehat{\eta}_T - \widetilde{\eta}(\boldsymbol{\psi})\right]' S_T \left[\widehat{\eta}_T - \widetilde{\eta}(\boldsymbol{\psi})\right], \tag{20}$$

[29] This discussion can be extended to a VAR(p) or VARMA(p,q) representation.

*where $S_T$ is an estimator of the inverse of the asymptotic covariance matrix of $\widehat{\eta}_T$, is a consistent estimator of $\boldsymbol{\psi}$ and is asymptotically equivalent to the ML estimator $\widehat{\boldsymbol{\psi}}_T$ of $\boldsymbol{\psi}$ obtained from (16)-(17).*

Intuitively, Corollary 1 states that estimating the parameter vector $\boldsymbol{\psi}$ in one direct step (ML approach) or relying on a two step procedure using $\eta$ as an auxiliary parameter (ALS approach) yields asymptotic equivalent estimator of $\boldsymbol{\psi}$. The two-step estimation uses the constraints $\Gamma = \left(\Pi_{yz} R \Pi_{yz}^{-1}\right)$ and $\Sigma_u = \Pi_{yz}\Pi_{yz}'$ to uncover the elements of the $R$ and $\Pi_{yz}$ matrices. This corollary therefore illustrates the strong connection between our approach and standard VAR modelling.

Consider now the macroeconomic model that underlies System (16) and (17). This macroeconomic model imposes cross-equation restrictions on the elements of the $\Pi_{yz}(\boldsymbol{\theta})$ matrix together with those contained in the $R(\boldsymbol{\theta})$ matrix. Let us define the binding function $\widetilde{\boldsymbol{\psi}}(\boldsymbol{\theta})$ that expresses the vector of state-space parameters, $\boldsymbol{\psi}$, as a function of $\boldsymbol{\theta}$. Under the conditions provided in Komunjer and Ng (2011), vector of parameters $\boldsymbol{\theta}$ is also identifiable and can thus be estimated by ML, a usual practice in the applied macroeconomic literature. Let us denote by $\widehat{\boldsymbol{\theta}}_T$ the ML estimator of $\boldsymbol{\theta}$. Because our D-SVAR imposes less restrictions on the state-space representation (16) and (17) than the macroeconomic model and provided dim $\boldsymbol{\theta}$ < dim $\boldsymbol{\psi}$, vector $\boldsymbol{\psi}$ can be used as an auxiliary parameter to estimate $\boldsymbol{\theta}$. The following corollary, again adapted from Gouriéroux and Monfort (1995), holds.

**Corollary 2** *Let $\widehat{\boldsymbol{\psi}}_T$ be the ML estimator of $\boldsymbol{\psi}$ from the unconstrained state-space version of the representation (16) and (17) and $\widetilde{\boldsymbol{\psi}}(\boldsymbol{\theta})$ the binding function. The estimator $\widetilde{\boldsymbol{\theta}}_T$ obtained by solving*

$$\min_{\boldsymbol{\theta}} \left[\widehat{\boldsymbol{\psi}}_T - \widetilde{\boldsymbol{\psi}}(\boldsymbol{\theta})\right]' S_T \left[\widehat{\boldsymbol{\psi}}_T - \widetilde{\boldsymbol{\psi}}(\boldsymbol{\theta})\right]$$

*where $S_T$ converges to the inverse of the asymptotic covariance matrix of $\widehat{\boldsymbol{\psi}}_T$ which is given by the information matrix $\mathcal{I}(\boldsymbol{\psi})$ of the log-likelihood function, is asymptotically equivalent to the ML estimator $\widehat{\boldsymbol{\theta}}_T$ of $\boldsymbol{\theta}$ obtained from the constrained state-space version of the representation (16) and (17).*

Estimating $\boldsymbol{\theta}$ is one step or by a two step procedure using $\boldsymbol{\psi}$ as an auxiliary parameter yields asymptotically equivalent estimator of $\boldsymbol{\theta}$. Our unrestricted state-space representation features less restrictions than the macroeconomic model and thus contains potentially useful information about the relevance of the structural restrictions imposed by the macroeconomic model. This corollary illustrates the tight relationship between the macroeconomic model and the D-SVAR.

### 5.2 Inference

The D-SVAR offers the possibility to conduct statistical inference on SVAR models. First and foremost, a specification test ($J$-test) can be performed in the case of over-identification –*i.e.* when the dimension of the vector $\eta$ is greater than the dimension of $\boldsymbol{\psi}$, by multiplying the objective function (20) by the number of observations. This allows to assess whether the dynamic restrictions imposed by the D-SVAR model are satisfied and hence evaluate the reliability of our D-SVAR approach as compared to a VAR model.

The D-SVAR also offers the opportunity to assess the relevance of various identification schemes used in SVAR modelling. In particular, it allows to test general null hypotheses on the loading matrix $\Pi_{yz}$. For example, one may be interested in the relevance of the timing imposed by short-run restrictions. In this case, the null hypothesis writes $H_0 : Hvec(\Pi_{yz}) = 0$, where the selection matrix $H$ is such that the elements above the diagonal of $\Pi_{yz}$ are all equals to zero. Likewise, matrix $H$ can be adapted to test for long-run restrictions à la Blanchard and Quah (1989). A Wald statistic can then be computed with a consistent estimator of the appropriate variance covariance matrix. Likewise, one may be interested in testing for the dynamic response to a particular shock, as identified using two competing identification schemes. This can be achieved with a Wald statistic and using the appropriate asymptotic distribution or by bootstrapping techniques as proposed by Inoue and Kilian (2016).[30]

## 6 Conclusion

In this paper, we have shown that one can identify macroeconomic model shocks in a SVAR under the identifying assumption that the economy shares the dynamic structure of the vast majority of macroeconomic models. To put it loosely, if the economy is moved by exogenous variables that follow mutually orthogonal AR(1) processes (or more general specification of the autoregressive matrix), then a D-SVAR will allow for the identification of the macreconomic model shocks, without the need for zero-impact, long-run or sign restrictions nor instruments. We have given a formal proof for identification and have shown how to conduct estimation and inference with D-SVAR. It can be then used to test SVARs identifying restrictions. We have illustrated our methodology with three well-known SVARs: Blanchard and Quah (1989) (long run restrictions), Gertler and Karadi (2015) (proxy-VAR) and Christiano et al. (1999) (impact zero restrictions).

We see the interest in using our D-SVAR approach as follows. Once a researcher has decided of variables, lag structure and identifying restrictions of a SVAR, they can

[30] In particular, in the case when the number of structural impulse responses exceeds the number of the VAR parameters, the asymptotic distribution of a Wald statistic is degenerated but the bootstrap test can still be implemented following the transformation of the statistic.

also identify our D-SVAR, keeping same variables and number of lags, and choosing a process for the latent forcing variables of the underlying structural model. Finding that the researcher structural shocks of interest are correlated with one and only one $\widehat{\varepsilon}$ of the D-SVAR is reassuring, as it amounts to accepting their identifying restrictions under minimal assumptions about the Data Generating Process. Finding that the shocks are correlated with more than one of the $\widehat{\varepsilon}$ suggests that identifying restrictions are likely to be rejected, again under minimal assumptions about the DGP.

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# Appendix

# A Assumptions, Definitions and Preliminary Propositions

Some assumptions need to be placed on the representation (11)-(12).

**Assumption 1** *For any* $z \in \mathbb{C}$, $det(zI_{n_s} - A) = 0$ *implies* $|z| < 1$.

Assumption 1 restricts the class of matrices $A$ to those with eigenvalues lying inside the unit circle. Under Assumption 1 and (11)-(12), the process $\{Y_t\}$ admits the following infinite Vector Moving Average (VMA) representation:

$$Y_t = \left[C(I_{n_s} - AL)^{-1}BL + D\right]\varepsilon_t = \sum_{j=0}^{\infty} h(j;\boldsymbol{\theta})\varepsilon_{t-j} = H(L,\boldsymbol{\psi})\varepsilon_t,$$

where $L$ denotes the lag operator. For $z \in \mathbb{C}$, the transfer function (the $z$-transform of the impulse response function) is given by:

$$H(z;\boldsymbol{\psi}) = D + C\left(zI_{n_s} - A\right)^{-1}B = \sum_{j=0}^{\infty} h(j;\boldsymbol{\theta})z^{-j}.$$

For every $\boldsymbol{\psi} \in \boldsymbol{\psi}$, $\mathbb{E}(Y_t) = 0$ and

$$\mathbb{E}(Y_t Y_s') \equiv \Gamma(s-t;\boldsymbol{\psi}) = \sum_{j=0}^{\infty} h(j;\boldsymbol{\psi})\Sigma_\varepsilon h(j+s-t;\boldsymbol{\psi})',$$

for all $t, s \geq 1$.

For any weakly stationary process $\{Y_t\}$ implied by Assumption 1 and under the assumption that the shocks $\epsilon_t$ are Gaussian, the unconditional mean and auto-covariance function completely characterize the properties of the process. Here, we don't impose the Gaussianity of the shocks but the local identification is based only on the auto-covariance function. Let us therefore define the auto-covariance generating function as:

$$\Omega(z,\boldsymbol{\psi}) = \sum_{j=-\infty}^{\infty} \Gamma(j;\boldsymbol{\psi})z^{-j},$$

for any $z \in \mathbb{C}$. Evaluating $\Omega(z,\boldsymbol{\psi})$ at $z = \exp(-i\omega)$ for any $\omega \in [-\pi,\pi]$ and rescaling it by $(2\pi)^{-1}$ yields the spectral density of the observable $\{Y_t\}$ which is always positive semi-definite. To simplify, we hereafter refer to $\Omega(z,\boldsymbol{\psi})$ as the spectral density as well.

As will be clear later, it will prove useful to defined observational equivalence for this class of multivariate covariance stationary process. We closely follow Komunjer and Ng (2011) and define it with respect to the entire auto-covariance function of the observable (or the spectral density).

**Definition 1** *Two sets of state-space parameters $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\psi}}$ are observationally equivalent if $\Omega(z;\boldsymbol{\psi}) = \Omega(z;\widetilde{\boldsymbol{\psi}})$, for all $z \in \mathbb{C}$ or, equivalently, $\Gamma(j;\boldsymbol{\psi}) = \Gamma(j;\widetilde{\boldsymbol{\psi}})$ at all $j \geq 0$.*

In other words, two stationary state-space models are observationally equivalent is they share the same auto-covariance (spectral) properties. This then allows us to define local identification.

**Definition 2** *The state-space representation (11)-(12)) is locally identifiable from the spectral density of $Y_t$ (or equivalently from the auto-covariances of $Y_t$) at $\boldsymbol{\psi} \in \boldsymbol{\psi}$ if there exists an open neighborhood of $\boldsymbol{\psi}$ such that for every $\widetilde{\boldsymbol{\psi}}$ in this neighbourhood, $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\psi}}$ are observationally equivalent if and only if $\widetilde{\boldsymbol{\psi}} = \boldsymbol{\psi}$.*

In state space system, the spectral density function can be simply obtained from the transfer function and the covariance matrix of the shocks, $\Sigma_\varepsilon$, as

$$\Omega(z,\boldsymbol{\psi}) = H(z;\boldsymbol{\psi})\Sigma_\varepsilon H(z^{-1};\boldsymbol{\psi})',$$

at $z = \exp(-i\omega)$ for any $\omega \in [-\pi,\pi]$. As explained in Komunjer and Ng (2011), this representation of the spectral density makes it clear that equivalent spectral density function can obtain because *(i)* for given $\Sigma_\varepsilon$, two distinct vectors of state space parameters, $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\psi}}$, yield the same transfer function ($H(z;\boldsymbol{\psi}) = H(z;\tilde{\boldsymbol{\psi}})$) or *(ii)* many pairs of $H(z;\boldsymbol{\psi})$ and $\Sigma_\varepsilon$ give rise to the same spectral density.

In general, the state-space parameter $\boldsymbol{\psi}$ is not identifiable from the second order moments of the observable variables. As an illustration, let us consider the following two state-space representations ($K_t$ is observable):

$$\mathcal{S} = \begin{cases} K_t &= \Pi_{kk}K_{t-1} + \Pi_{kz}Z_t, \\ Y_t &= \Pi_{yk}K_{t-1} + \Pi_{yz}Z_t, \\ Z_t &= RZ_{t-1} + \varepsilon_t, \end{cases} \quad \text{and} \quad \widetilde{\mathcal{S}} = \begin{cases} K_t &= \Pi_{kk}K_{t-1} + \widetilde{\Pi}_{kz}\widetilde{Z}_t, \\ Y_t &= \Pi_{yk}K_{t-1} + \widetilde{\Pi}_{yz}\widetilde{Z}_t, \\ \widetilde{Z}_t &= \widetilde{R}\widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t, \end{cases}$$

where $\widetilde{Z}_t = U^{-1}Z_t$, $\widetilde{\Pi}_{kz} = \Pi_{kz}U$, $\widetilde{\Pi}_{yz} = \Pi_{yz}U$, $\widetilde{R} = U^{-1}RU$ and $\widetilde{\Sigma}_\varepsilon = U^{-1}\Sigma_\varepsilon U^{-1\prime}$ for some full rank matrix $U$. The two representations $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are observationally equivalent with respect to the spectral function since $\Omega(z,\boldsymbol{\psi}) = \Omega(z,\widetilde{\boldsymbol{\psi}})$ for all $z \in \mathbb{C}$ where the vectors $\boldsymbol{\psi}$ and $\widetilde{\boldsymbol{\psi}}$ gather, respectively, the elements of the vectorization of matrices defining, respectively, $\mathcal{S}$ and $\widetilde{\mathcal{S}}$.

Following Komunjer and Ng (2011) (see Proposition 1-S), the following property obtains in the case of the $ABCD$ representation

**Property:** *Two distinct vectors of state-space parameters $\boldsymbol{\psi}, \widetilde{\boldsymbol{\psi}} \in \Psi$ are observationally equivalent respective to the transfer function and the spectral density if and only if there exists a full rank $n_s \times n_s$ matrix $T$ and a full rank $n_z \times n_z$ matrix $U$ such that $\tilde{A} = TAT^{-1}$, $\widetilde{B} = TBU$, $\widetilde{C} = CT^{-1}$, $\widetilde{D} = DU$ and $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime}$.*

This property obtains as follows. First, the equalities $\tilde{A} = TAT^{-1}$, $\widetilde{B} = TB$, $\widetilde{C} = CT^{-1}$ are necessary and sufficient for the equivalence of the transfer function $H(z;\widetilde{\boldsymbol{\psi}}) = H(z;\boldsymbol{\psi})$. Sufficiency follows directly from the observation of the transfer function in the ABCD representation

$$\begin{aligned} H(z;\widetilde{\boldsymbol{\psi}}) &= \widetilde{C}(zI_{n_s} - \tilde{A})^{-1}\widetilde{B} + D, \\ &= CT^{-1}(zI_{n_s} - TAT)^{-1})^{-1}TB + D, \\ &= CT^{-1}T(zI_{n_s} - A)^{-1}T^{-1}TB + D = H(z;\boldsymbol{\psi}). \end{aligned}$$

The necessary condition follows directly from a well known result in control theory under the condition of minimality of the state-space representation (See Theorem 3.10 in Antsaklis and Michel (1997) and Chapter 8 in Gouriéroux and Monfort (1995) ). A system is minimal if and only if it is controllable and observable (see Appendix B for formal definitions), implying that, among all systems leading to the same output spectral density, it is driven by the minimal number of state variables. This equivalence class is function of a non-singular transformation that corresponds to a rotation of the state, i. e., $T^{-1}S_t$.

The equivalence of the spectral density is obtained for a full rank matrix $U$ such that:

$$H(z;\boldsymbol{\psi})\Sigma_\varepsilon H(z^{-1};\boldsymbol{\psi})' = H(z;\boldsymbol{\psi})UU^{-1}\Sigma_\varepsilon U^{-1\prime}U'H(z^{-1};\boldsymbol{\psi})'.$$

Fixing $\widetilde{B} = BU$, $\widetilde{D} = DU$ and $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime}$, this yields $H(z;\widetilde{\boldsymbol{\psi}}) = H(z;\boldsymbol{\psi})U = DU + C\left[I - Az\right]^{-1}BzU$. Observational equivalence follows immediately.

We have established that the identification of $\boldsymbol{\psi}$ or a subset of $\boldsymbol{\psi}$ cannot be achieved without imposing additional restrictions on the covariance matrix $\Sigma_\varepsilon$. Specifically, we consider the case where the structural innovations are mutually orthogonal, and their covariance matrix is normalized to the identity matrix, such that $\mathbb{E}(\varepsilon_t\varepsilon_t') = I_{n_z}$. This is a common identifying assumption in the SVAR literature (see Ramey (2016)).

A direct implication of this assumption is that the only admissible matrix $U$ that satisfies $\Sigma_{\widetilde{\varepsilon}} = U^{-1}\Sigma_\varepsilon U^{-1\prime} = I_{n_z}$ is an orthonormal matrix –*i.e.* $UU' = I_{n_z}$ (see Corollary 1 of Kocięcki and Kolasa (2018)).

We further adapt Assumption 1 to the initial state-space representation (8)–(10):

**Assumption 1′** *In the presence of endogenous state variables ($n_k > 0$), for any $z \in \mathbb{C}$,* $\det(zI_{n_s} - Az) = 0$ *implies* $|z| < 1$, *and the matrices* $\Pi_{kk}$ *and* $R$ *have no eigenvalues in common.*

When $n_k > 0$, since the matrix $A$ is block triangular (see System (11)–(12)), it holds that $\det(zI_{n_s} - A) = \det(zI_{n_k} - \Pi_{kk})\det(zI_{n_z} - R)$. Both $\Pi_{kk}$ and $R$ have eigenvalues that

lie within the unit circle and do not share any common eigenvalues. This assumption is necessary to disentangle the dynamics of the endogenous variables $K_t$ from those of the exogenous process $Z_t$.

**Assumption 2**

*(i)* *When* $n_k = 0$, $C = \Pi_{yz}$ *is of full rank –i.e.* $\text{rank}(\Pi_{yz}) = n_s$.

*(ii)* *When* $n_k > 0$, *the matrices* $\Pi_{kk}$, $\Pi_{yk}$, *and* $\Pi_{yz}$ *are of full rank, and* $\text{rank}(\Pi_{kz}R) \geq \min(n_k, n_z)$.

Without endogenous state variables ($n_k = 0$), $\Pi_{yz}$ has rank $n_z$. In this case, the system is defined solely by equations (9)–(10), without the term $\Pi_{yk}K_{t-1}$ on the right-hand side, and satisfies $\det(zI_{n_s} - A) = \det(zI_{n_k} - R)$.

When $n_k > 0$, Assumption 2 ensures that no endogenous state variables are redundant. The condition $\text{rank}(\Pi_{kz}R) \geq \min(n_k, n_z)$ implies that the matrices $\Pi_{kz}$ and $R$ may be rank-deficient. However, this condition is essential for obtaining a minimal representation.

The following proposition follows directly:

**Proposition A.1** *Under Assumption 1′ and Assumption 2, the state-space representation (8)–(10) is minimal, and the transfer function* $H(z; \boldsymbol{\psi})$ *is invertible.*

**Proof:** See Appendix C.1.

Assumption 2 ensures that the dynamics of $Z_t$ can be recovered from the observables $Y_t$. To see this, consider the transfer function of System (8)–(10):

$$\begin{aligned} H(z;\boldsymbol{\psi}) &= \left[\Pi_{yk}(zI_{n_k} - \Pi_{kk})^{-1}\Pi_{kz} + \Pi_{yz}\right](zI_{n_z} - R)^{-1}, \\ &= \left[\Pi_{yk}(zI_{n_k} - \Pi_{kk})^{-1}\Pi_{kz}\Pi_{yz}^{-1} + I_{n_z}\right]\Pi_{yz}(zI_{n_z} - R)^{-1}. \end{aligned}$$

Under *(ii)*, the $rank(H(z;\boldsymbol{\psi})) = n_z$ for every $|z| > 1$, then $H(z;\boldsymbol{\psi})$ is invertible as shown in Proposition A.1. When $n_y = 0$, $H(z;\boldsymbol{\psi}) = \Pi_{yz}(zI_{n_z} - R)^{-1}$ is full rank for every $|z| > 1$. In the following, we consider the case where $n_k > 0$.

Endowed with Assumption 1′ and Assumption 2, the next proposition shows that the rotation matrix $T$ is block diagonal.

**Proposition A.2** *Under Assumption 1′ and Assumption 2 we have*

- *When the endogenous state variables* $K_t$ *are unobserved, the full rank matrix* $T$ *has the form*

$$T = \begin{bmatrix} T_{11} & 0_{n_k \times n_z} \\ 0_{n_z \times n_k} & V \end{bmatrix}, \tag{A.1}$$

*with* $T_{11}$ *a full rank* $(n_k \times n_k)$ *matrix.*

- *When the endogenous state variables $K_t$ are observed, the full rank matrix $T$ has necessarily the following form:*

$$T = \begin{bmatrix} I_{n_k} & 0_{n_k \times n_z} \\ 0_{n_z \times n_k} & V \end{bmatrix}, \tag{A.2}$$

*In both cases, $V$ is an orthonormal $(n_z \times n_z)$ matrix such that $VV' = I_{n_z}$ and $V = U' = U^{-1}$ defined above.*

**Proof :** See Appendix C.2.

Under the assumption that the endogenous state variables $K_t$ are unobserved, the block diagonal structure of the matrix $T$ implies that

$$\begin{aligned} H(z;\boldsymbol{\psi}) &= \left[\Pi_{yk}(zI_{n_k} - \Pi_{kk})^{-1}\Pi_{kz} + \Pi_{yz}\right](zI_{n_z} - R)^{-1}, \\ &= \left[\Pi_{yk}T_{11}^{-1}(zI_{n_k} - T_{11}\Pi_{kk}T_{11}^{-1})^{-1}T_{11}\Pi_{kz} + \Pi_{yz}V'\right](zI_{n_z} - R)^{-1}. \end{aligned}$$

Proposition A.1 also implies that, as long as the endogenous state variable $K_t$ is observed, the matrices $\Pi_{kk}$ and $\Pi_{yk}$ can be identified using the observed spectral density function.[31] However, in both cases, this does not allow for proper identification of the loading matrix $\Pi_{yz}$ based solely on the properties of $Z_t$.

Indeed, a direct implication of the proposition is that there exists at least one equivalent exogenous process to (10):

$$Z_t = VRV'Z_{t-1} + \varepsilon_t, \tag{A.3}$$

where observational equivalence in terms of the transfer function (and spectral density) holds if and only if the sub-matrix $V$ in the matrix $T$, as defined in (A.2), is orthonormal ($VV' = I_{n_z}$). In this case, pre-multiplying (A.3) by $V'$, we obtain:

$$\widetilde{Z}_t = R\widetilde{Z}_{t-1} + \widetilde{\varepsilon}_t,$$

where $\widetilde{Z}_t = V'Z_t$, $\widetilde{\varepsilon}_t = V'\varepsilon_t$, and $\mathbb{E}(\widetilde{\varepsilon}_t\widetilde{\varepsilon}_t') = I_{n_z}$.

In other words, the sole knowledge of the spectral properties of $Z_t$ is insufficient to identify $\Pi_{yz}$. Additional (dynamic) restrictions must be imposed on the autoregressive matrix $R$.

[31] The observed spectral density (or equivalently, the auto-covariance) function can be obtained by estimating a VAR, a VARMA, or through the maximum likelihood estimation of a state-space representation.

# B Preliminaries: Controllability, Observability and Minimality

In the literature on linear systems, invariant finite-dimensional dynamical systems are described by the following state-space system:

$$S_{t+1} = AS_t + B\epsilon_t, \tag{B.4}$$

$$Y_t = CS_t + D\epsilon_t, \tag{B.5}$$

where the $n_s \times 1$ vector $S_t$ denotes the state variables, the $n_y \times 1$ vector $Y_t$ denotes the outputs, and the $n_\epsilon \times 1$ vector $\epsilon_t$ contains the inputs (see, for example, Antsaklis and Michel (1997) and Gouriéroux and Monfort (1995)). Note that this ABCD state-space representation differs from that of Fernández-Villaverde et al. (2007), but it allows us to define the concepts of controllability, observability, and minimality in accordance with the standard linear system literature.

The system (or the pair $(A, B)$) is considered controllable when any state $S_t$ can be driven to the initial state in a finite number of steps for a given input sequence $\epsilon_t$. To illustrate the concept of controllability, the state variables at time $t = k$ can be expressed as:

$$S_k = A^k S_0 + A^{k-1}B\epsilon_0 + A^{k-2}\epsilon_1 + \ldots + AB\epsilon_{k-2} + B\epsilon_{k-1},$$

where $S_0$ is the vector of initial states. The state $S_k$ is said to be reachable at time $k$ if, for some finite $t = 0 \leq k$, there exists an input sequence $\epsilon_t, t \in [0, k]$, that transfers the state from the origin at $t = 0$ to $S_k$ at time $t = k$. This implies that the system is reachable if and only if the rank of the matrix $\mathbb{C} = \left[B, AB, A^2B, \ldots, A^{k-1}B\right]$, which contains the matrices related to the input sequence in the above system, is of full rank. Moreover, if the system is reachable, then it is also controllable.

According to the ABCD system representation (B.4) and (B.5), a formal definition of controllability is given by:

**Definition 3 Controllability** *: A system is controllable if and only if the controllability matrix* $\mathbb{C} = [B, AB, A^2B, \ldots, A^{n_s-1}B] \in \mathbb{R}^{n_s \times (n_z n_s)}$ *has full row rank –i.e. rank*$(\mathbb{C}) = n_s$.

If a system is state observable, its present state can be determined from the knowledge of the present and future outputs $Y_t$ and inputs $\epsilon_t$. The expression for output $Y_k$ can be rewritten as:

$$Y_k = CA^k S_0 + CA^{k-1}B\epsilon_0 + CA^{k-2}B\epsilon_1 + \ldots + CAB\epsilon_{k-2} + CB\epsilon_{k-1} + D\epsilon_k.$$

Observability refers to the ability to uniquely determine the state of a system from knowledge of current and future outputs and inputs. Specifically, a system is said to be

observable if it is possible to determine the state at time $t = 0$ from knowledge of future system outputs $Y_k$ and system inputs $\epsilon_k$ for $k \geq 0$. A state is considered unobservable if it remains indistinguishable from the zero state under any possible output observation, mathematically expressed as $CA^kS = 0$ for all $k \geq 0$. Conversely, a system is completely state observable if the only state that satisfies this condition is the zero state, $S = 0$. The degree of observability is determined by the rank of the matrices $CA^k$ for all $k$, with full rank indicating complete observability.

A formal definition of observability is given by:

**Definition 4 Observability**: *A system is observable if and only if the observability matrix $\mathbb{O}$ has full column rank –i.e. $rank(\mathbb{O}) = n_s$, where*

$$\mathbb{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n_s-1} \end{bmatrix} \in \mathbb{R}^{n_y n_s \times n_s}.$$

**Theorem 1** *A state-space representation is minimal if and only if it is both controllable and observable.*

See Antsaklis and Michel (1997), Theorem 3.9, p. 395 for a proof of minimality under controllability and observability For a minimal state-space representation, the state vector is of the smallest possible dimension.

# C Proofs of Preliminary and Main Propositions

## C.1 Proof of Proposition A.1

Here we formal prove controllability and observability of the system given by:

- The state-space system is defined as:

$$A = \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0 & R \end{bmatrix}, \quad B = \begin{bmatrix} \Pi_{kz} \\ I_{n_z} \end{bmatrix}, \quad C = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \end{bmatrix}.$$

The total system dimension is $n_s = n_k + n_z$, where $n_k$ and $n_z$ are the dimensions of the $\Pi_{kk}$-block and $R$-block, respectively.

First, consider the case without endogenous state variables. In this scenario, $A = R$ and $C = \Pi_{yz}$, which is of full rank. This implies that $S_t$ follows a VAR(1) process given by $S_t = RS_{t-1} + \epsilon_t$, where $\epsilon_t$ has the same dimension as $S_t$. This directly implies that the system is controllable according to the definition above. Moreover, since $Y_t = \Pi_{yz}S_t$ and $\Pi_{yz}$ is of full rank, the system is also observable, which implies that the system is minimal.

In the case with unobserved endogenous state variables, the controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n_s-1}B \end{bmatrix}.$$

In the first step, we compute powers of $A$ applied to $B$

1. First, we compute $AB$:

$$AB = AB = \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \Pi_{kz} \\ I_{n_z} \end{bmatrix} = \begin{bmatrix} \Pi_{kk}\Pi_{kz} + \Pi_{kz}R \\ R \end{bmatrix}.$$

2. Next, compute $A^2B$:

$$A^2B = A(AB) = \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \Pi_{kk}\Pi_{kz} + \Pi_{kz}R \\ R \end{bmatrix} = \begin{bmatrix} \Pi_{kk}^2\Pi_{kz} + \Pi_{kk}\Pi_{kz}R + \Pi_{kz}R^2 \\ R^2 \end{bmatrix}.$$

3. The general form of powers of $A$ applied to $B$ is:

$$A^kB = \begin{bmatrix} \Pi_{kk}^k\Pi_{kz} + \sum_{i=0}^{k-1} \Pi_{kk}^i\Pi_{kz}R^{k-1-i} \\ R^k \end{bmatrix}.$$

The controllability matrix takes the form:

$$\mathcal{C} = \begin{bmatrix} \Pi_{kz} & \Pi_{kk}\Pi_{kz} + \Pi_{kz}R & \dots & \\ I_{n_z} & R & R^2 & \dots \end{bmatrix}.$$

The block structure of $A$ reveals that its eigenvalues are the union of the eigenvalues of $\Pi_{kk}$ and $R$. Since $\Pi_{kk}$ and $R$ do not share any common eigenvalues, the dynamics of the two subsystems are distinguishable. The controllability of the $G$-subsystem is governed by the pair $(\Pi_{kk}, \Pi_{kz})$. Even if $\Pi_{kz}$ is not of full rank, $\Pi_{kk}$ retains partial controllability. However, any rank deficiency in $\Pi_{kz}$ is compensated by the term $\Pi_{kz}R$, under the assumption that $\text{rank}(\Pi_{kz}R) \geq \min(n_k, n_z)$. The $R$-subsystem, on the other hand, is directly controllable through the identity matrix $I_{n_z}$ in $B$. This ensures that all states of the $R$-subsystem are controllable. The coupling term $\Pi_{kz}R$ plays a crucial role in linking the $\Pi_{kk}$- and $R$-subsystems. Since $\Pi_{kk}$ and $R$ have distinct eigenvalues, the rank of $\Pi_{kz}R$ does not diminish the rank of the overall controllability matrix $\mathcal{C}$. In conclusion, the system is controllable because the $G$-subsystem is partially controllable through $\Pi_{kz}$, the $R$-subsystem is fully controllable through $I_{n_z}$, and the term $\Pi_{kz}R$ ensures interaction between the two subsystems. This implies that the system is fully controllable.

Now for the observability. The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n_s-1} \end{bmatrix}.$$

In the first step, we compute powers of $A$ applied to $C$

1. First, compute $CA$:

$$CA = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \end{bmatrix} \begin{bmatrix} \Pi_{kk} & \Pi_{kz}R \\ 0 & R \end{bmatrix} = \begin{bmatrix} \Pi_{yk}\Pi_{kk} + \Pi_{yz}\Pi_{kz}R & \Pi_{yz}R \end{bmatrix}.$$

2. Next, compute $CA^2$:

$$CA^2 = CA \cdot A = \begin{bmatrix} \Pi_{yk}\Pi_{kk}^2 + \Pi_{yz}\Pi_{kz}\Pi_{kk}R + \Pi_{yz}\Pi_{kz}R^2 & \Pi_{yz}R^2 \end{bmatrix}.$$

The observability matrix takes the form:

$$\mathcal{O} = \begin{bmatrix} \Pi_{yk} & \Pi_{yz} \\ \Pi_{yk}\Pi_{kk} + \Pi_{yz}\Pi_{kz}R & \Pi_{yz}R \\ \Pi_{yk}\Pi_{kk}^2 + \Pi_{yz}\Pi_{kz}\Pi_{kk}R + \Pi_{yz}\Pi_{kz}R^2 & \Pi_{yz}R^2 \\ \vdots & \vdots \end{bmatrix}.$$

The left block of the system ensures the observability of the $\Pi_{kk}$-block through $\Pi_{yk}$, while the right block ensures the observability of the $R$-block through $\Pi_{yz}$. Additionally, the coupling term $\Pi_{kz}R$ guarantees interaction between the two blocks.

Given that $\text{rank}(\Pi_{kz}R) \geq \min(n_k, n_z)$ and that the eigenvalues of $\Pi_{kk}$ and $R$ are distinct, the rank of the observability matrix $\mathcal{O}$ is:

$$\text{rank}(\mathcal{O}) = n_k + n_z = n_s.$$

Therefore, the system is fully observable. By controllability and observability, the system is minimal.

Let us analyze the Rosenbrock Matrix (see Antsaklis and Michel (1997)). The transfer function $H(z, \boldsymbol{\psi})$ of the linear system (11) and (11) given by:

$$H(z, \boldsymbol{\psi}) = C(zI_{n_s} - A)^{-1}B + D.$$

The Rosenbrock matrix for the system is defined as:

$$\mathcal{R}(z) = \begin{bmatrix} zI_{n_s} - A & -B \\ C & D \end{bmatrix}.$$

For the system to be controllable and observable, the Rosenbrock matrix $\mathcal{R}(z)$ must have full rank for all $z \in \mathbb{C}$. If the Rosenbrock matrix $\mathcal{R}(z)$ has full rank for all $z \in \mathbb{C}$, the transfer function $T(z)$ is invertible, with no zeros or singularities. Conversely, if the Rosenbrock matrix loses rank at any $z$, the transfer function has zeros at those values, indicating potential non-invertible. The Rosenbrock matrix for the system is:

$$\mathcal{R}(z) = \begin{bmatrix} zI_{n_k} - \Pi_{kk} & -\Pi_{kz}R & -\Pi_{kz} \\ 0 & zI_{n_z} - R & -I_{n_z} \\ \Pi_{yk} & \Pi_{yz} & \Pi_{yz} \end{bmatrix}.$$

The rank of the Rosenbrock matrix $\mathcal{R}(z)$ can be determined as follows. The block $zI_{n_s} - A$ is composed of $zI_{n_k} - \Pi_{kk}$ and $zI_{n_z} - R$. Since $\Pi_{kk}$ and $R$ are distinct, the matrix $zI_{n_k} - \Pi_{kk}$ is full rank for $z$ values outside the eigenvalues of $\Pi_{kk}$, and $zI_{n_z} - R$ is full rank for $z$ values outside the eigenvalues of $R$.

The blocks $B$, $C$, and $D$ ensure that controllability and observability are preserved by compensating for any potential rank deficiencies in the system. Consequently, the rank of the Rosenbrock matrix $\mathcal{R}(z)$ is given by:

$$\text{rank}(\mathcal{R}(z)) = n_k + n_z,$$

where $n_k$ represents the dimension of $\Pi_{kk}$, and $n_z$ represents the dimension of $R$. Since the rank of $\mathcal{R}(z)$ equals the total state dimension, the system is both controllable and observable, as demonstrated above. This also implies that the transfer function $T(z)$ is invertible.

## C.2 Proof of Proposition A.2

Consider a general form for the matrix $T$ such that $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$.

We first show that the relation $\tilde{A}T = TA$ implies that $T_{21} = 0$. Since $\tilde{A}$ is necessarily block upper triangular, $\tilde{R}T_{21} = T_{21}\Pi_{kk}$ which can be rewritten as:

$$[(I_{n_k} \otimes \tilde{R}) - (\Pi'_{kk} \otimes I_{n_z})]vec(T_{21}) = 0. \tag{C.6}$$

Matrix $\tilde{A}$ is *similar* to matrix $A$ which implies that the eigenvalues of $\tilde{A}$ are the same than the eigenvalues of $A$.[32] Since $A$ is block upper diagonal, for the set of eigenvalues of $A$ denoted $\lambda(A)$, we have $\lambda(A) = \lambda(\Pi_{kk}) \cup \lambda(R)$. The same property holds for $\tilde{A}$ –*i.e.* $\lambda(\tilde{A}) = \lambda(\tilde{\Pi}_{kk}) \cup \lambda(\tilde{R})$. Since the matrices $A$ and $\tilde{A}$ have the same block upper triangular structure and the same eigenvalues, this implies that the eigenvalues of the matrix $\tilde{R}$ are the same as the eigenvalues of the matrix $R$. Under Assumption 1′′, $R$ and $\Pi_{kk}$ share no common eigenvalues, this also holds for $\tilde{R}$ and $\tilde{\Pi}_{kk}$. The expression $[(I_{n_k} \otimes \tilde{R}) - (\Pi'_{kk} \otimes I_{n_z})]$ is then of full rank, Equation (C.6) holds only for $T_{21} = 0$.[33]

Now, for the block upper triangular matrix $T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n_z \times n_k} & T_{22} \end{bmatrix}$, the equation $\widetilde{B} = TBU$ gives for the left lower block $I_{n_z} = T_{22}U$. Since $U$ is orthonormal, this implies

[32] If $X$ is a square matrix and nonsingular, then $A$ and $B = X^{-1}AX$ are similar and $X$ is called a similarity transformation. If two matrices A and B are similar, they have the same eigenvalues –*i.e.* $\lambda(A) = \lambda(B)$, and the same number of independent eigenvectors but probably not the same eigenvectors (see Golub and Van Loan (2013) p. 349). Moreover, if $X$ is an orthonormal matrix –*i.e.* $XX' = I$, $A$ and $B$ real matrices and $A = XBX'$, $A$ is said to be real orthogonally similar to $B$ (see Horn and Johnson (2013), p. 94).

[33] If $\lambda$ is an eigenvalue of $A$ and $\mu$ an eigenvalue of $B$, $\lambda - \mu$ is an eigenvalue of $(I \otimes A) - (B \otimes I)$ and all eigenvalues of $(I \otimes A) - (B \otimes I)$ is on this form. Thus $(I \otimes A) - (B \otimes I)$ has zero as an eigenvalue if and only if $A$ has an eigenvalue $\lambda$ and $B$ has an eigenvalue $\mu$ such that $\lambda - \mu = 0$.

$T_{22} \equiv U^{-1} = U' = V$ an orthonormal matrix. The matrix $T$ has the following form: $T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & V \end{bmatrix}$.

We also have that $\widetilde{D} = DU$, where $U$ is an orthonormal matrix. This implies that $\widetilde{\Pi}_{yz} = \Pi_{yz}U$. Moreover, since $\widetilde{C} = CT^{-1}$, we can write:

$$\widetilde{C} = [\widetilde{\Pi}_{yk} \quad \widetilde{\Pi}_{yz}] = \left[\Pi_{yk}T_{11}^{-1} \quad -\Pi_{yk}T_{11}^{-1}T_{12}T_{22}^{-1} + \Pi_{yz}T_{22}^{-1}\right].$$

Since $T_{22}^{-1} = V^{-1} = V'$, where $V$ is an orthonormal matrix, and using $\widetilde{\Pi}_{yz} = \Pi_{yz}U$ from $\widetilde{D} = DU$, it necessarily follows that $T_{12} = 0$.

Finally, the result for the case where the state variables $K_t$ are observed follows directly.

## C.3 Proof of Proposition 1

Our strategy is to show that the only admissible permutation matrix in equation (A.3) is $V = I_{n_z}$ (up to changes of sign and/or permutation of the identity matrix) and hence that there indeed exists only one $Z_t$ process that is compatible with the spectral properties of observables $Y_t$. The only admissible orthonormal matrix $V$ such that all elements $(i,i)$ of the diagonal matrix $R$ are identified is $V = I$ when the diagonal element $r_{i,i} \neq r_{j,j}$ for $\forall i \neq j$. For any other orthonormal matrix $V$, it is easy to verify that the resulting $\tilde{R} = VRV'$ matrix is not diagonal. Since $V$ is a square matrix and is non singular, $V$ corresponds to a *similarity transformation*, the eigenvalues of matrix $\tilde{R}$ are the same as the eigenvalues of matrix $R$ which implies that the diagonal elements of $\tilde{R}$ are the same as the diagonal elements of $R$. Moreover, for diagonal matrices $\tilde{R}$ and $R$, the system of equations $\tilde{R}V - VR = 0$ leads to

$$(\tilde{r}_{i,i} - r_{i,i})V_{i,i} = 0 \quad (\text{ for } i = 1, \ldots, n_z), \tag{C.7}$$

$$(\tilde{r}_{i,i} - r_{j,j})V_{i,j} = 0 \quad (\text{for } i \neq j \text{ and } i, j = 1, \ldots, n_z), \tag{C.8}$$

where $r_{i,j}$ and $\tilde{r}_{i,j}$ are respectively the element $(i,j)$ of $R$ and $\tilde{R}$.[34] The first set of equation implies that $\tilde{r}_{i,i} = r_{i,i}$ for diagonal elements of $V$ which are different from zero. Since the diagonal elements of $R$ (and then $\tilde{R}$) are different, the second set of equations implies necessarily that $V_{i,j} = 0$ for all $i \neq j$. Matrix $V$ is necessarily diagonal and the only orthonormal matrix which is diagonal is the identity. The result also holds up to changes of sign and/or permutation of the identity matrix.

Assume now that some elements on the diagonal are the same. Denote the multiplicity of similar diagonal elements $r_{i,i}$ by $m(r_{i,i})$. One can first check that all diagonal elements

[34] The system of equations $\tilde{R}V - VR = 0$ is of the well known form $AX - BX = C$ in control theory called the Sylvester equation. For $C = 0$, this corresponds to the homogeneous Sylvester equation (see Gantmacher (1959), chap VIII).

are the same, in which case any orthonormal matrix $V$ is admissible by equations (C.7) and (C.8). Now consider that a subgroup of elements has the same value. The elements in the $V$ matrix corresponding to multiple values are not uniquely defined but only up to the post multiplication by an $m(r_{i,i}) \times m(r_{i,i})$ orthogonal matrix. Without lost of generality, suppose that the diagonal elements with the same value are ordered as the first $m(r_{1,1})$ elements on the diagonal and the other elements on the diagonal $r_{j,j}$ are different from $r_{1,1}$, then define the matrix $V$ such that $V = \begin{bmatrix} V_{m(r_{1,1})\times m(r_{1,1})} & 0 \\ 0 & I_{(n_z - m(r_{1,1}))} \end{bmatrix}$, where $V_{m(r_{1,1})\times m(r_{1,1})}$ is an orthonormal matrix. Consequently, there exists an infinity of admissible $V$ matrices such that $V_{m(r_{1,1})\times m(r_{1,1})}$ is orthonormal. This argument can be generalised to more than one diagonal element with multiplicity. A sufficient condition for local identification is therefore that matrix $R$ be diagonal with distinct diagonal elements ($r_{i,i} \neq r_{j,j},\ \forall i \neq j$).

In the case of unobserved endogenous state variables $K_t$, the transfer function $H(z;\boldsymbol{\psi})$ is then expressed as:

$$\begin{aligned} H(z;\boldsymbol{\psi}) &= \left[\Pi_{yk}(zI_{n_k} - \Pi_{kk})^{-1}\Pi_{kz} + \Pi_{yz}\right](zI_{n_z} - R)^{-1}, \\ &= \left[\Pi_{yk}T_{11}^{-1}(zI_{n_k} - T_{11}\Pi_{kk}T_{11}^{-1})^{-1}T_{11}\Pi_{kz} + \Pi_{yz}V'\right](zI_{n_z} - R)^{-1}, \\ &= \left[\Pi_{yk}T_{11}^{-1}(zI_{n_k} - T_{11}\Pi_{kk}T_{11}^{-1})^{-1}T_{11}\Pi_{kz}\Pi_{yz}^{-1} + I_{n_z}\right]\Pi_{yz}(zI_{n_z} - R)^{-1}, \end{aligned}$$

for $V = I$. This representation implies that the matrices $\Pi_{yz}$ and $R$ can be locally identified. However, only the expression

$$\left[\Pi_{yk}T_{11}^{-1}(zI_{n_k} - T_{11}\Pi_{kk}T_{11}^{-1})^{-1}T_{11}\Pi_{kz}\Pi_{yz}^{-1} + I_{n_z}\right]$$

can be identified, and the individual matrices within this term cannot be separately identified. Nevertheless, this is sufficient to retrieve the impulse response functions (IRFs) of the structural shocks. In contrast, when the endogenous state variables $K_t$ are observed, all individual matrices can be locally identified.

## C.4 Proof of Proposition 2.

By $\tilde{R} = VRV'$, one show that the only admissible matrix $V$ which satisfies $\tilde{R}V = VR$ for lower triangular matrices $\tilde{R}$ and $R$ is $V = I_{n_z}$. Since V is of full rank and orthonormal and $\tilde{R} = VRV'$, $\tilde{R}$ and $R$ have the same eigenvalues. Moreover, the eigenvalues of a lower triangular matrix are the elements on the diagonal. By $\tilde{R}V = VR$, we have

$$[(I_{n_z} \otimes \tilde{R}) - (R' \otimes I_{n_z})]vec(V) = 0. \tag{C.9}$$

This implies that $vec(V)$ belong to the null space of $A = [(I_{n_z} \otimes \tilde{R}) - (R' \otimes I_{n_z})]$. Since $\tilde{R}$ and $R$ have $n_z$ common roots, the null space of $A$ has a dimension equal to $n_z$. The

system of equations (C.9) can be written more explicitly as:

$$\begin{bmatrix} (\tilde{R}-\mu_1 I) & -r_{2,1}I & -r_{3,1}I & \cdots & -r_{n_z,1}I \\ 0 & (\tilde{R}-\mu_2 I) & -r_{3,2}I & \cdots & -r_{n_z,2}I \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & (\tilde{R}-\mu_{n_z-1}I) & -r_{n_z,n_z-1}I \\ 0 & 0 & \cdots & 0 & (\tilde{R}-\mu_{n_z}I) \end{bmatrix} \begin{bmatrix} V_{[.,1]} \\ V_{[.,2]} \\ \vdots \\ V_{[.,n_z-1]} \\ V_{[.,nz]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

with $V_{[.,i]}$ a vector containing the $i$-th column of matrix $V$, $\mu_j$ for $j = 1, \ldots, n_z$ are the eigenvalues of $R$ and the eigenvalues of $\tilde{R}$ and $R$ are the same which are given by the elements on the diagonal. Under Proposition 1, the eigenvalues $\mu_j$ for $j = 1, \ldots, n_z$ are the same as the matrix $\Lambda$. Moreover, all sub-matrices $(\tilde{R}-\mu_j I)$ are lower triangular since $\tilde{R}$ is lower triangular and the entire matrix is a block upper triangular matrix.

This system of equations can be solved by forward substitution for the lower triangular block $(\tilde{R}-\mu_{n_z}I)$ to obtain the vector $V_{[.,n_z]}$ and by backward substitution for the upper block diagonal matrices. Thus, the lower triangular form of the system of equations $(\tilde{R}-\mu_{n_z}I)V_{[.,n_z]} = 0$ implies that the elements of $V_{[.,n_z]}$ are equal to zero except the last one $V_{n_z,n_z}$. Indeed the recursive form of the equations allows to rewrite the system of equations as $\sum_{j=1}^{i} \hat{r}_{i,j}^{n_z} V_{j,n_z} = 0$ for $i = 1, \ldots, n_z$, which imply that $V_{j,n_z} = 0$ except for $V_{n_z,n_z}$ where $\hat{r}_{i,j}^{n_z}$ is the element $(i,j)$ of the sub-matrix $(\tilde{R}-\mu_{n_z}I)$ and $\hat{r}_{n_z,n_z}^{n_z} = 0$ since $R$ and $\tilde{R}$ share the same eigenvalues. According to Proposition 1, this holds under the weaker restriction that only the first sub-diagonal elements are different from zero which implies that $\hat{r}_{1,1}^{n_z} V_{1,n_z} = 0$ and $\sum_{j=i-1}^{i} \hat{r}_{i,j}^{n_z} V_{j,n_z} = 0$ for $i = 2, \ldots, n_z$.

Now, for the next upper block, we have

$$\begin{bmatrix} (\tilde{R}-\mu_{n_z-1}I) & -r_{nz,nz-1}I \end{bmatrix} \begin{bmatrix} V_{[.,n_z-1]} \\ V_{[.,n_z]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

using the preceding result above for $V_{[.,n_z]}$ yields $V_{j,n_z-1} = 0$ for all $j < n_z - 1$. We can continue to solve the following upper blocks by backward substitution to obtain that all elements of the matrix $V$ above the diagonal are equal to zero: $(\tilde{R}-\mu_j I)V_{[.,j]} = 0$ for $j = n_z$ and $(\tilde{R}-\mu_j I)V_{[.,j]} = \sum_{i=j+1}^{n_z} (r_{i,j}I)V_{[.,i]}$ for $j = 1, \ldots n_z - 1$. The resulting matrix V has all elements above the diagonal equal to zero which implies that the only admissible orthonormal matrix is $V = I$ (up to changes of sign and/or permutation of the identity matrix).

## C.5 Proof of Proposition 3.

Consider the following general $2 \times 2$ symmetric matrix $R = \begin{bmatrix} \rho_1 & \tau \\ \tau & \rho_2 \end{bmatrix}$ where $\rho_1$, $\rho_2$ and $\tau$ are real numbers and $\tau \neq 0$. The eigenvalues are the roots of the following characteristic equation:

$$\begin{vmatrix} \rho_1 - \lambda & \tau \\ \tau & \rho_2 - \lambda \end{vmatrix} = (\rho_1 - \lambda)(\rho_2 - \lambda) - \tau^2 = \lambda^2 - \lambda(\rho_1 + \rho_2) + \rho_1\rho_2 - \tau^2.$$

The two roots can be written as $\lambda_{1,2} = \frac{1}{2}\left[\rho_1 + \rho_2 \pm \sqrt{(\rho_1 - \rho_2)^2 + 4\tau^2}\right]$. Since $(\rho_1 - \rho_2)^2 + 4\tau^2 > 0$, the two eigenvalues are necessarily real. By the expressions of $\lambda_1$ and $\lambda_2$, there exists an infinity of values for $\rho_1$, $\rho_2$ and $\tau$ that gives the same eigenvalues. In other words, for any orthonormal matrix $V$ and $\tilde{R} = VRV'$, the matrices $\tilde{R}$ and $R$ are similar and therefore have the same eigenvalues. The matrix $R$ is then not identifiable.

Now consider the case where the elements on the diagonal have the same value –*i.e.* $\rho_1 = \rho_2 = \rho$. The two eigenvalues are now given by $= \lambda_{1,2} = \rho \pm \tau$, so that $\lambda_1 + \lambda_2 = 2\rho$ and $\lambda_1 - \lambda_2 = 2\tau$. This implies that there does not exist another $2 \times 2$ symmetric matrix $\tilde{R}$ with $\tilde{\rho} \neq \rho$ and/or $\tilde{\tau} \neq \tau$ with $\tilde{\tau} \neq 0$ having the same eigenvalues as matrix $R$. In this particular case, matrix $R$ is then locally identifiable.

## C.6 Proof of Proposition 4.

We can consider cases with a block diagonal matrix $R$ with blocks corresponding to the two preceding cases. For example $R = \begin{bmatrix} \tilde{R} & 0 \\ 0 & R_2 \end{bmatrix}$, where $\tilde{R}$ is a $(n_{z1} \times n_{z1})$ diagonal matrix with different elements on the diagonal and $R_2$ is any $(n_{z2} \times n_{z2})$ matrix with $z_1 + z_2 = z$. In this case, matrix $V$ has the form $V = \begin{bmatrix} I & 0 \\ 0 & V_{n_{z2} \times n_{z2}} \end{bmatrix}$, where $V_{n_{z2} \times n_{z2}}$ is an orthonormal matrix. The first $z_1$ exogenous processes are then locally identified. Matrix $\tilde{R}$ could be also lower triangular.

# Exploiting Dynamic Identification in VARs

## Additional Online Material

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# D The Case of Unobserved Endogenous State Variables

We investigate the the case of a vector $X_t$ consisting of both observed, $Y_t$, and unobserved state variables, $K_t$ –*i.e.* $X_t = (K_t, Y_t)'$. This implies, in particular, that $\Pi_{yk}$ is not necessarily equal to the identity matrix. The standard RBC model studied in McGrattan (2010) is a typical example of such a situation as the capital stock may not be directly observed (or at least not without measurement errors) by the econometrician. In that case, the solution of the macroeconomic model admits the following state space representation

$$\begin{aligned} K_t &= \Pi_{kk} K_{t-1} + \Pi_{kz} Z_t, \\ Y_t &= \Pi_{yk} K_{t-1} + \Pi_{yz} Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t. \end{aligned} \tag{D.1}$$

As for the previous case, we impose that $Y_t$ has the same dimension as $Z_t$ ($n_y = n_z$). We consider the case where $Z_t$ consists of at least two elements and where there are not more state variables than elements in $Z_t$. For simplicity, we take $K_t$ to be of the same order as $Z_t$ and impose that $\Pi_{kk}$ is of full rank. Moreover, we also assume that $\Pi_{yz}$, $\Pi_{yk}$ and $\Pi_{kz}$ are of full rank.

Making use of these assumptions, the dynamics of $Y_t$ can be expressed as

$$Y_t = C_1 Y_{t-1} + C_2 Z_t + C_3 Z_{t-1}, \tag{D.2}$$

where $C_1 = \Pi_{yk}\Pi_{kk}\Pi_{yk}^{-1}$, $C_2 = \Pi_{yz}$ and $C_3 = \Pi_{yk}(\Pi_{kz} - \Pi_{kk}\Pi_{yk}^{-1}\Pi_{yz})$. The issue is then whether, when $R$ is diagonal (or lower triangular), $C_1$, $C_2$, $C_3$ and $R$ can be identified.[35]

Making use of the $Z_t$ process in (D.2), it is easy to show that the vector of observed variables $Y_t$ follows a VARMA(2,1) process of the form

$$Y_t = \left(DRD^{-1} + C_1\right) Y_{t-1} - DRD^{-1}C_1 Y_{t-2} + C_2\varepsilon_t + (D - DRD^{-1}C_2)\varepsilon_{t-1}.$$

[35] Note that while $\Pi_{kk}$, $\Pi_{kz}$ and $\Pi_{yk}$ cannot (in general) be identified separately. This is however not an issue as far as the identification of the structural impulse responses is concerned as all that is needed is the identification of $R$ and $Cs$. Indeed, we will identify $\Pi_{yk}\Pi_{kk}\Pi_{yk}^{-1}$, $\Pi_{yk}\Pi_{kz}$, $\Pi_{yz}$ and $R$.

where $D \equiv C_3 + C_2 R$ and $D$ is invertible.

As in the previous case, counting the number of coefficients to uncover and the number of moments the VARMA(2,1) structure offers, we recover that imposing a lower triangular structure on $R$ provides us with the right number of restrictions. Note that this does not generically guarantees identification, as the system that needs to be solved features a quadratic term implying that a pair of solutions generally arise. However, as long as $R$ is sparser than a triangular matrix, the system features more equations than unknowns. The order condition is then clearly satisfied (in fact, it is over-identified). As before, checking the rank condition is non trivial, and we follow another strategy in the paper to formally prove identification. In Appendix H, we provide an application using simulated data generated by a RBC model.

# E Partial Identification: an Example

Consider the following textbook 3-equation New Keynesian model (see Galí (2015)) featuring three shocks:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - E_t[\pi_{t+1}]) + z_{1,t}, \\ \pi_t &= \beta \mathbb{E}_t[\pi_{t+1}] + \kappa y_t + z_{2,t}, \\ i_t &= \phi_\pi \pi_t + z_{3,t}, \end{aligned}$$

where $y_t$, $\pi_t$ and $i_t$ denote respectively aggregate output, the rate of inflation and the nominal interest rate and $\mathbb{E}_t[\cdot]$ denotes the conditional expectation operator. Parameter $\beta \in (0,1)$ is the discount factor, $\kappa \geq 0$ denotes the slope of the Phillips curve and $\phi_\pi$ is the degree of aggressiveness of monetary policy to inflation. The forcing variable $z_{1,t}$ can be interpreted as a demand shifter to the IS curve, whereas $z_{2,t}$ is a cost-push shifter to the Phillips curve. $z_{3,t}$ is a monetary policy shifter to the Taylor rule. To ease exposition, we assume that the two forcing variables $z_{1,t}$ and $z_{2,t}$ are not serially correlated, while third one $z_{3,t}$ is. As will be clear later, the two serially uncorrelated shifters cannot be separately identified, whereas the third one one can be. Assuming the Taylor principle holds, the model can be solved forward and yields the state-space representation

$$\begin{aligned} Y_t &= \Pi_{yz} Z_t, \\ Z_t &= R Z_{t-1} + \varepsilon_t, \end{aligned}$$

$$\text{where } \Pi_{yz} = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{33} & \pi_{32} & \pi_{33} \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \text{ and } \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}.$$

The vector $Y_t = (y_t, \pi_t, i_t)'$ collects the three endogenous variables and $Z_t = (z_{1,t}, z_{2,t}, z_{3,t})'$ is the vector of the three model forcing variables, $\varepsilon_t$ is a zero mean weak white noise and $E(\varepsilon_t \varepsilon_t') = I_3$.

The model solution for output writes as the following moving average:

$$y_t = \pi_{11}\varepsilon_{1,t} + \pi_{12}\varepsilon_{2,t} + \pi_{13}\sum_{i=0}^{\infty}\rho^i\varepsilon_{3,t-i},$$

from which we get the output auto-covariance functions as

$$\gamma_y(0) = \pi_{11}^2 + \pi_{12}^2 + \frac{\pi_{13}^2}{1-\rho^2} \quad \text{when} \quad h = 0,$$
$$\gamma_y(h) = \frac{\pi_{13}^2\rho^h}{1-\rho^2} \quad \text{when} \quad h > 0.$$

The persistence parameter $\rho$ can be directly identified by computing the ratio $\gamma_y(h+1)/\gamma_y(h)$ for $h > 0$, which is free from any parameter but $\pi_{13}$. Given $\rho$, the direct observation of $\gamma_y(h)$, for any $h > 0$, allows to recover parameter $\pi_{13}$ (up to a sign term). In other words, the effect of a monetary policy shock on output is identified. Applying the same procedure on inflation and the nominal interest allows for the identification of $\pi_{23}$ and $\pi_{33}$ (up to a sign term). It is therefore possible to identify the monetary transmission mechanism for all variables in the D-SVAR.

Note that, on the contrary, the knowledge of $\gamma_y(0)$ only helps identify the sum $\pi_{11}^2+\pi_{12}^2$, not $\pi_{11}$ and $\pi_{12}$ separately, In other words, the effect of two remaining shocks $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ cannot be separately identified. The reason for this result is that the dynamics of these two shocks do not bring any information to disentangle them. This result obviously extends to the inflation rate and the nominal interest rate. This example simply illustrates that one shock can be identified in so far as it displays a dynamic structure that differs from the other shocks in the economy.

# F Data Appendix

## F.1 Data from Section 3.2.1

- **Real GDP** is measured as the ratio of Gross Domestic Product in value (Table 1.1.5 from BEA) divided by the GDP price index (Table 1.1.4 from BEA), and is expressed in per capita term by dividing by the Civilian non-institutional population from 16 years of age and older residing in the 50 states and the District of Columbia (`CNP16OV` in the Federal Reserve DataBase (FRED, http://fred.stlouisfed.org)).

- The **unemployment gap** is measured as the difference between the average unemployment rate over a quarter (`UNRATE` from FRED) and the natural rate of

unemployment –*i.e.* the rate of unemployment arising from all sources except fluctuations in aggregate demand (`NROU` from FRED).

## F.2 Data from Section 3.2.2

The data from Section 3.2.2 are borrowed from Gertler and Karadi (2015) and are downloadable from `http://doi.org/10.3886/E114082V1`

- The **price level** is measured by the Consumer Price Index for all urban consumers (`CPIAUCSL` from FRED, All Items in U.S. City Average)
- **Economic activity** is measured by the industrial production index (`INDPRO` from FRED)
- The **nominal interest rate** is measured by the Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity (`GS1` fom FRED)
- The **credit spread** is measured by the Gilchrist and Zakrajšek (2012) excess bond premium.
- The **external instrument** corresponds to the three month ahead monthly fed funds futures (`FF4` from Gertler and Karadi (2015))

The price level and the economic activity index are both expressed in logs prior to estimation.

## F.3 Data from Section 3.2.3

- The **nominal interest rate** is measured by the Effective Federal Funds Rate (`DFF` from FRED).
- **Output** is measured as the Real Gross Domestic Product expressed in Billions of Chained 2012 Dollars (`GDPC1` from FRED, Quarterly, Seasonally Adjusted Annual Rate).
- The **price level** is measured by the Consumer Price Index for all items for the United States (`CPALTT01USM661S` from FRED, Index 2015=100, Quarterly, Seasonally Adjusted).
- The **unemployment rate** corresponds to the quarterly average of the monthly unemployment rate in the US (`UNRATE` from FRED, Percent, Quarterly, Seasonally Adjusted)

- The **commodity price** is the Producer Price Index of all commodities (`PPIACO` from FRED, Index 1982=100, Quarterly).

Output, the Price level and the commodity price are all transformed by applying the log function prior to estimation.

# G The New Keynesian Model of Section 3.1

## G.1 Description of the Model

The set up is standard. The economy is populated by a large number of identical infinitely–lived households and economy consists of two sectors: one producing intermediate goods and the other final goods. The intermediate good is produced with labor and the final good with intermediate goods.

**The Household:** Household preferences are characterised by the lifetime utility function:[36]

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \omega_t \left( \frac{(c_{t+\tau} - h\bar{c}_{t+\tau-1})^{1-\gamma}}{1-\gamma} - \vartheta \frac{n_{t+\tau}^{1+\varphi}}{1+\varphi} \right), \tag{G.1}$$

where $0 < \beta < 1$ is a constant discount factor, $c$ denotes consumption and $n$ labor.

In each and every period, the representative household faces a budget constraint of the form

$$B_t + P_t c_t \quad \leq \quad R_{t-1} B_{t-1} + \Pi_t + P_t w_t n_t, \tag{G.2}$$

where $B_t$ are nominal bonds acquired during period $t$, $P_t$ is the nominal price of the final good, $R_{t-1}$ is the nominal interest rate, $w_t$ denotes the real wage. The household consumes $c_t$ and supplies $n_t$ units of labor and claims the profits, $\Pi_t$, earned by the firms. $\omega_t$ will act as a demand shock and can be interpreted as a premium shock.

The first order conditions lead to

$$\vartheta n_t^{\varphi} = (c_t - h\bar{c}_{t-1})^{-\gamma} w_t, \tag{G.3}$$

$$\omega_t (c_t - h\bar{c}_{t-1})^{-\gamma} = \beta R_t \mathbb{E}_t \left[ \frac{\omega_{t+1} (c_{t+1} - h\bar{c}_t)^{-\gamma}}{\pi_{t+1}} \right], \tag{G.4}$$

where $\pi_t = P_t / P_{t-1}$ denotes the gross inflation rate.

[36] $\mathbb{E}_t(.)$ denotes mathematical conditional expectations. Expectations are conditional on information available at the beginning of period $t$.

**Final sector:** The final good is produced by combining intermediate goods. This process is described by the following CES function

$$y_t = \left( \int_0^1 y_t(i)^{\eta_t} \mathrm{d}i \right)^{\frac{1}{\eta_t}}, \tag{G.5}$$

where $\eta_t \in (-\infty, 1)$. $\eta_t$ determines the elasticity of substitution between the various inputs, which will be modelled as a stochastic process and will appear as a cost push shock in the New Keynesian Phillips curve. The producers in this sector are assumed to behave competitively and to determine their demand for each good, $y_t(i)$, $i \in (0,1)$ by maximising the static profit equation

$$\max_{\{X_t(i)\}_{i\in(0,1)}} P_t y_t - \int_0^1 P_t(i) y_t(i) \mathrm{d}i, \tag{G.6}$$

subject to (G.5), where $P_t(i)$ denotes the price of intermediate good $i$. This yields demand functions of the form:

$$y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{\frac{1}{\eta_t - 1}} y_t, \tag{G.7}$$

and the following general price index

$$P_t = \left( \int_0^1 P_t(i)^{\frac{\eta_t}{\eta_t - 1}} \mathrm{d}i \right)^{\frac{\eta_t - 1}{\eta_t}}. \tag{G.8}$$

The final good may be used for consumption — private or public — and investment purposes.

**Intermediate Good Producers:** Each firm $i$, $i \in (0,1)$, produces an intermediate good by means of capital and labor according to a constant returns–to–scale technology, represented by the production function

$$y_t(i) = n_t(i), \tag{G.9}$$

where $n_t(i)$ denotes the labor input used by firm $i$ in the production process. $a_t$ is an exogenous stationary stochastic technology shock. Assuming that each firm $i$ operates under perfect competition in the input markets, the firm determines its production plan so as to minimise its total cost

$$\min_{\{h_t(i)\}} P_t w_t n_t(i),$$

subject to (G.9). This yields to the following expression for total costs:

$$P_t s_t y_t(i),$$

where the real marginal cost, $s_t$, is simply given by $w_t$.

Intermediate goods producers are monopolistically competitive, and therefore set prices for the good they produce. We follow Calvo [1983] in assuming that firms set their prices for a stochastic number of periods. In each and every period, a firm either gets the chance to adjust its price (an event occurring with probability $1-\alpha$) or it does not. When the firm does not reset its price, it just applies steady state inflation to the price it charged in the last period such that $P_t(i) = \pi_{t-1}^{\zeta}\overline{\pi}^{1-\zeta}P_{t-1}(i)$. When it gets a chance to do it, firm $i$ resets its price, $\widetilde{P}_t(i)$, in period $t$ in order to maximise its expected discounted profit flow this new price will generate. In period $t$, the profit is given by $\Pi(\widetilde{P}_t(i))$. In period $t+1$, either the firm resets its price, such that it will get $\Pi(\widetilde{P}_{t+1}(i))$ with probability $\alpha$, or it does not and its $t+1$ profit will be $\Pi(X_{t,t+1}\widetilde{P}_t(i))$ with probability $(1-\alpha)$. Likewise in $t+2$. Expected profit flow generated by setting $\widetilde{P}_t(i)$ in period $t$ writes

$$\max_{\widetilde{P}_t(i)} \mathbb{E}_t \sum_{\tau=0}^{\infty} \Phi_{t,t+\tau}\alpha^{\tau-1}\Pi(X_{t,t+\tau}\widetilde{P}_t(i)),$$

subject to the total demand it faces:

$$y_t(i) = \left(\frac{\widetilde{P}_t(i)}{P_t}\right)^{\frac{1}{\eta-1}} y_t,$$

where $X_{t,t+1} = \pi_t^{\zeta}\overline{\pi}^{1-\zeta}X_{t-1,t}$ and $\Pi(X_{t,t+\tau}\widetilde{P}_t(i)) = \left(X_{t,t+\tau}\widetilde{P}_t(i) - P_{t+\tau}s_{t+\tau}\right)y_{t+\tau}(i)$. $\Phi_{t+\tau}$ is an appropriate discount factor related to the way the household values future as opposed to current consumption, such that

$$\Phi_{t,t+\tau} \propto \beta^{\tau}\frac{\Lambda_{t+\tau}}{\Lambda_t} \text{ where } \Lambda_{t+\tau} \equiv \omega_{t+\tau}(c_{t+\tau} - h\overline{c}_{t+\tau-1})^{-\gamma}.$$

This leads to the price setting equation

$$\mathbb{E}_t\left[\sum_{\tau=0}^{\infty}(\beta\alpha)^{\tau}\frac{\Lambda_{t,t+\tau}}{\eta_{t+\tau}-1}\left(\eta_{t+\tau}X_{t,t+\tau}\widetilde{P}_t(i) - P_{t+\tau}s_{t+\tau}\right)y_{t+\tau}(i)\right] = 0. \tag{G.10}$$

From the definition of the aggregate price (G.8) and the Calvo fairy assumption, the aggregate price level may be expressed as

$$P_t = \left(\sum_{j=0}^{\infty}(1-\alpha)\alpha^j(X_{t-j,t}\widetilde{P}_{t-j})^{\frac{\eta_t}{\eta_t-1}}\right)^{\frac{\eta_t-1}{\eta_t}}. \tag{G.11}$$

**Monetary Authorities:** Monetary authorities are assumed to follow a Taylor rule of the form (in log-linear deviations from deterministic steady state)

$$i_t = \rho_r i_{t-1} + (1-\rho_r)(\phi_\pi\pi_t + \phi_y y_t) + \epsilon_{i,t},$$

where $|\rho_r| < 1$ and $\phi_y, \phi_\pi > 0$.

**Equilibrium:** An equilibrium of this economy is a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty} = \{w_t, , P_t, R_t, \widetilde{P}_t\}_{t=0}^{\infty}$ and a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty} = \{\{\mathcal{Q}_t^H\}_{t=0}^{\infty}, \{\mathcal{Q}_t^F\}_{t=0}^{\infty}\}$ with

$$\begin{aligned}\{\mathcal{Q}_t^H\}_{t=0}^{\infty} &= \{c_t, B_t, n_t\}_{t=0}^{\infty},\\ \{\mathcal{Q}_t^F\}_{t=0}^{\infty} &= \{y_t, y_t(i), n_t(i); i \in (0,1)\}_{t=0}^{\infty},\end{aligned}$$

such that:

*(i)* given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^H\}_{t=0}^{\infty}$ is a solution to the representative household's problem;

*(ii)* given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^F\}_{t=0}^{\infty}$ is a solution to the representative firms' problem;

*(iii)* given a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{P}_t\}_{t=0}^{\infty}$ clears the markets. In particular, we have $\int_0^1 y_t(i)\mathrm{d}i = c_t$ and $\int_0^1 n_t(i)\mathrm{d}i = n_t$.

*(iv)* Prices satisfy (G.10) and (G.11).

Log-linearisation of the equilibrium around the deterministic steady state gives rise to the 3-equation New-Keynesian model:

$$y_t = \frac{h}{1+h}y_{t-1} + \frac{1}{1+h}\mathbb{E}_t[y_{t+1}] - \frac{1-h}{\gamma(1+h)}(i_t - \mathbb{E}_t[\pi_{t+1}]) + z_{d,t}, \tag{G.12}$$

$$\pi_t = \frac{\zeta}{1+\beta\zeta}\pi_{t-1} + \frac{\beta}{1+\beta\zeta}\mathbb{E}_t[\pi_{t+1}] + \frac{(1-\alpha)(1-\beta\alpha)}{\alpha(1+\beta\zeta)}(\gamma+\varphi)y_t + z_{s,t}, \tag{G.13}$$

$$i_t = \rho_r i_{t-1} + (1-\rho_r)(\phi_\pi \pi_t + \phi_y y_t) + z_{r,t}, \tag{G.14}$$

with $z_{j,t} = \rho_j z_{j,t-1} + \varepsilon_{j,t}$, where $\varepsilon_{j,t} \rightsquigarrow N(0, \sigma_j^2)$ with $j \in \{d, s, r\}$.

## G.2 Estimation Results

Table G.1 below reports the prior distribution and posterior mean, mode and 90% Highest Posterior Density Interval (HPDI) for all the estimated parameters of the model.

Table G.1: New Keynesian Model, Priors and Posteriors

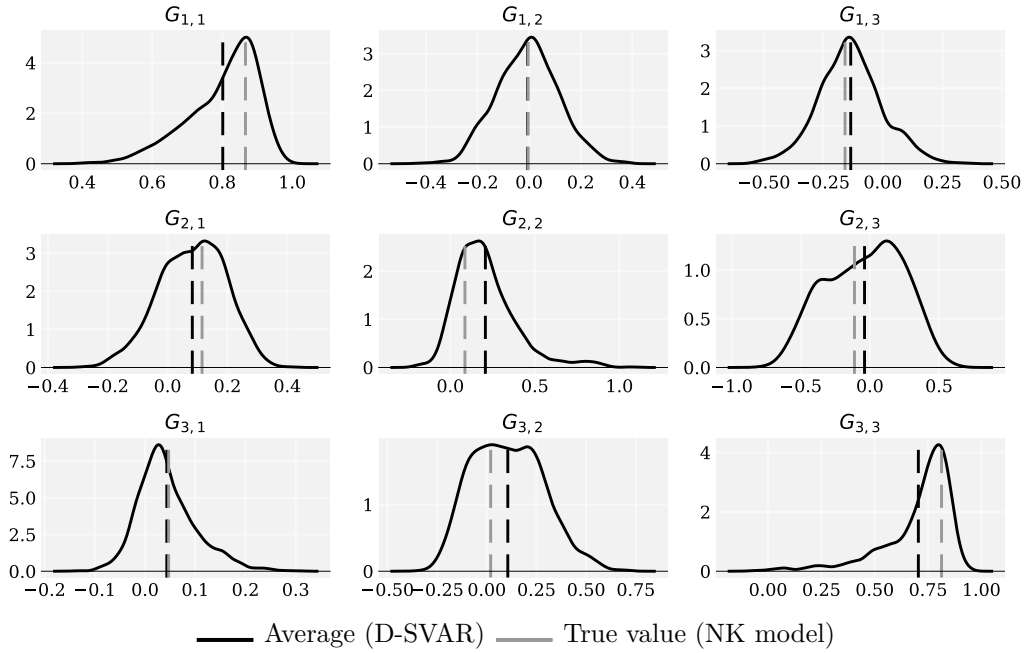
| | Priors | | | Posteriors | | |
|---|---|---|---|---|---|---|
| | distribution | Mean | std. dev. | Mode | Mean | 90% HPDI |
| $\sigma$ | Gamma | 1.00 | 0.10 | 1.30 | 1.32 | [1.12;1.50] |
| $\varphi$ | Gamma | 2.00 | 0.30 | 2.43 | 2.57 | [1.92;3.22] |
| $h$ | Beta | 0.10 | 0.05 | 0.08 | 0.13 | [0.03;0.23] |
| $\zeta$ | Gamma | 0.10 | 0.05 | 0.04 | 0.05 | [0.01;0.08] |
| $\rho$ | Beta | 0.90 | 0.05 | 0.85 | 0.84 | [0.81;0.87] |
| $\theta$ | Beta | 0.90 | 0.01 | 0.87 | 0.86 | [0.83;0.89] |
| $\phi_\pi$ | Normal | 1.50 | 0.10 | 1.33 | 1.36 | [1.14;1.56] |
| $\phi_y$ | Normal | 0.10 | 0.05 | 0.35 | 0.33 | [0.27;0.41] |
| $\rho_d$ | Beta | 0.80 | 0.10 | 0.87 | 0.85 | [0.83;0.88] |
| $\rho_s$ | Beta | 0.80 | 0.10 | 0.53 | 0.54 | [0.45;0.63] |
| $\rho_r$ | Beta | 0.20 | 0.10 | 0.26 | 0.28 | [0.17;0.38] |
| $\sigma_d$ | Inv. Gamma | 0.50 | 0.10 | 0.22 | 0.22 | [0.21;0.24] |
| $\sigma_s$ | Inv. Gamma | 0.50 | 0.10 | 0.37 | 0.37 | [0.32;0.43] |
| $\sigma_r$ | Inv. Gamma | 0.50 | 0.10 | 0.24 | 0.24 | [0.21;0.26] |

Notes: The estimated model is the New Keynesian (G.12)–(G.14). Sample is 1960Q1-2007Q4. Estimation is done with minus the unemployment gap, GDP deflator inflation and the Federal fund rate. Posterior distribution obtained from MCMC using 2 chains of 200,000 draws each.

## G.3 Estimated D-SVAR: Densities

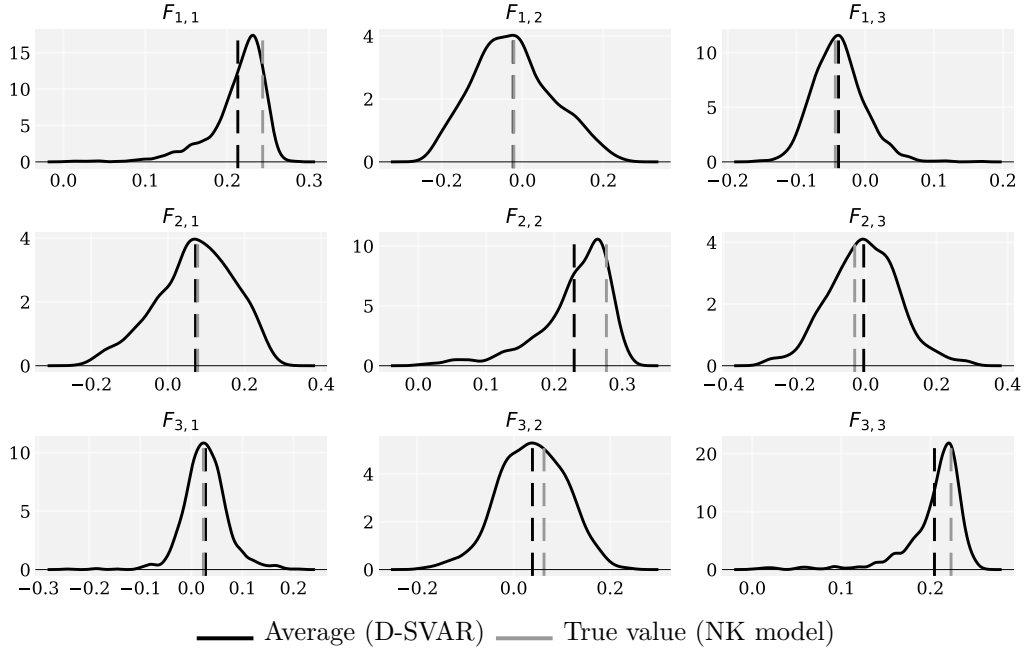
Below we report the densities over 1,000 simulations of the estimated elements of the matrices $G$, $F$ and $R$. In each simulation, we simulate the estimated New Keynesian model for 300 periods and use the simulated data to estimate a D-SVAR and therefore recovering each time an estimation of the matrices $G$, $F$ and $R$.

Figure G.1: Density of the Elements of Matrix $G$



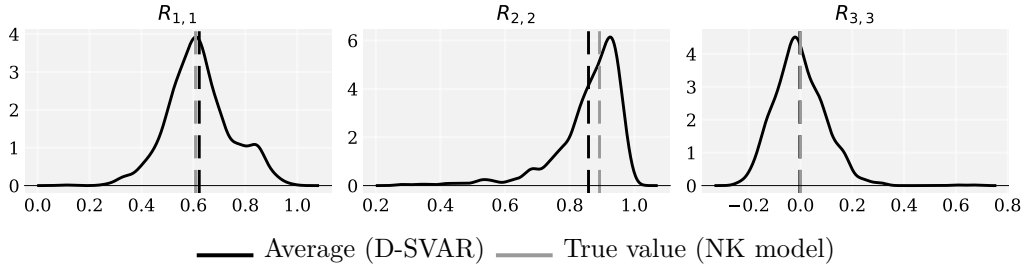

Notes: The estimated model is a D-SVAR with two lags. Data are generated by the estimated New Keynesian model. We report the density over 1,000 estimations.

Figure G.2: Density of the Elements of Matrix $F$




Notes: The estimated model is a D-SVAR with two lags. Data are generated by the estimated New Keynesian model. We report the density over 1,000 estimations.

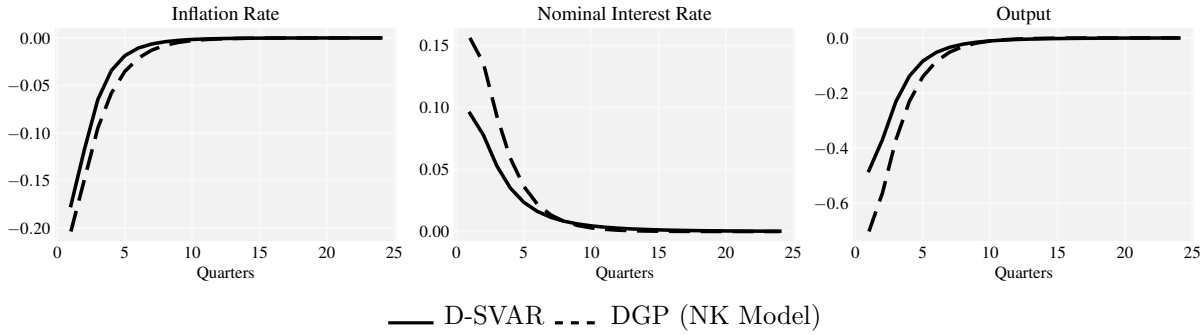
Figure G.3: Density of the Elements of Matrix $R$




Notes: The estimated model is a D-SVAR with two lags. Data are generated by the estimated New Keynesian model. We report the density over 1,000 estimations.

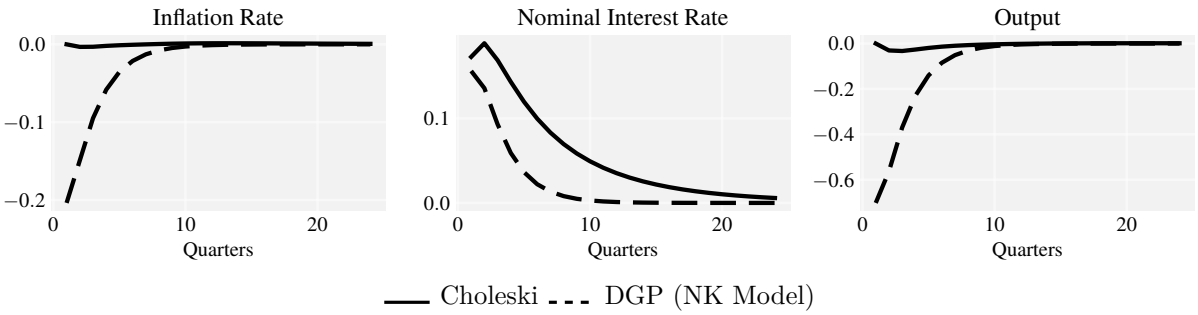
## G.4 IRF Obtained From Various Identification Schemes with Data Simulated with the New Keynesian Model

Figure G.4: Impulse Responses to a Monetary Shock, New Keynesian Model and DSVAR



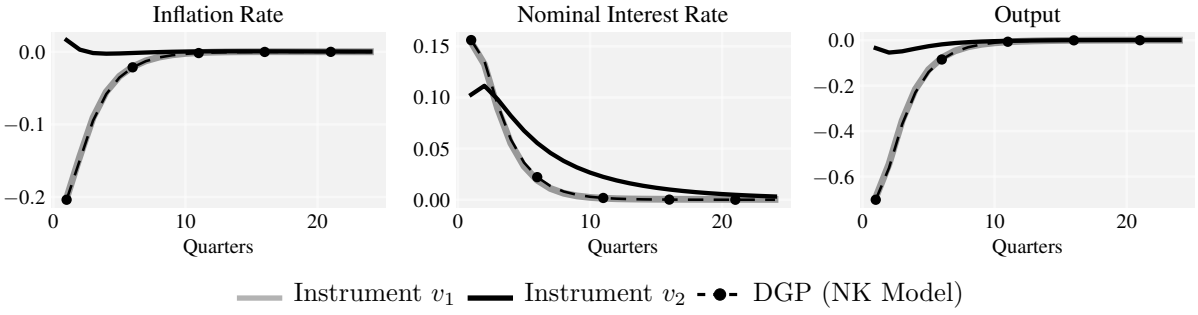

Notes: The New Keynesian model IRF to a monetary shock are obtained from the estimated model. The D-SVAR one are estimated on data generated by the New Keynesian model. They are the average over 1,000 simulations of the response to the shock that is, in each simulation, the most correlated with the monetary shock.

Figure G.5: Impulse Responses to a Monetary Shock, New Keynesian Model and Choleski Identification




Notes: The New Keynesian model IRFs to a monetary shock are obtained from the estimated model. The Choleski ones are estimated on data generated by the New Keynesian model. They are the average over 1,000 simulations of the response to the shock that, on impact, does not move output and inflation.

Figure G.6: Impulse Responses to a Monetary Shock, New Keynesian Model and Proxy VAR Identification




Notes: The New Keynesian model IRFs to a monetary shock are obtained from the estimated model. The "instrument $v_i$" ones are estimated on data generated by the New Keynesian model. They are the average over 1,000 simulations of the proxy VAR using $v_1$ or $v_2$ as an instrument.

# H A Real Business Cycle Model

We consider a real business cycle model featuring a *catching up with the Joneses* mechanism and real frictions on the capital accumulation process that take the form of investment adjustment costs. The problem of the Central planner takes the form

$$\begin{aligned}
&\max \mathbb{E}_t \left[ \sum_{\tau=0}^{+\infty} \beta^\tau \left( \log(C_{t+\tau} - b\overline{C}_{t+\tau-1}) - \vartheta_{t+\tau}^{-1} \frac{h_{t+\tau}^{1+\nu}}{1+\nu} \right) \right], \\
C_t + I_t &= A_t K_t^\alpha (\Gamma_t h_t)^{1-\alpha}, \\
K_{t+1} &= \zeta_t I_t \left( 1 - \Phi \left( \frac{I_t}{I_{t-1}} \right) \right) + (1-\delta) K_t,
\end{aligned}$$

where $\beta \in (0,1)$ denotes the discount factor, $b \in (0,1)$ governs habit persistence, $\nu > 0$ is the inverse of the Frish elasticity, $\alpha \in (0,1)$ is capital elasticity and the function $\Phi(\cdot)$ is strictly increasing and convex and satisfies $\Phi(\gamma) = \Phi'(\gamma) = 0$. Furthermore, $\varphi \equiv \Phi''(\gamma)\gamma > 0$ governs the importance of investment adjustment costs. $\Gamma_t$ denotes exogenous technological progress that evolves deterministically as $\Gamma_t = \gamma \Gamma_{t-1}$, $\gamma > 1$. Finally $\vartheta_t$, $A_t$ and $\zeta_t$ denote respectively a labor wedge, a technology and an investment specific shock, which are all assumed to follow a stationary AR(1) process of the form

$$\log(X_t) = \rho_X \log(X_{t-1}) + \varepsilon_t^X \text{ for } x \in \{\vartheta, A, \zeta\},$$

where $|\rho_X| < 1$ and $\varepsilon^X \rightsquigarrow N(0, \sigma_X^2)$. The (deflated for growth, $x_t = X_t/\Gamma_t$) optimal allocation of this economy is then characterised by the set of equations

$$\begin{aligned}
h_t^\nu &= (1-\alpha) \frac{y_t}{h_t} \frac{\gamma \vartheta_t}{\gamma c_t - b c_{t-1}}, \\
1 &= \frac{\beta}{\gamma} \mathbb{E}_t \left[ \frac{\gamma c_t - b c_{t-1}}{\gamma c_{t+1} - b c_t} \left( \alpha \frac{y_{t+1}}{k_{t+1}} + (1-\delta) q_{t+1} \right) \right], \\
1 &= \zeta_t q_t \left( 1 - \Phi \left( \gamma \frac{i_t}{i_{t-1}} \right) - \Phi' \left( \gamma \frac{i_t}{i_{t-1}} \right) \gamma \frac{i_t}{i_{t-1}} \right) + \frac{\beta}{\gamma} \mathbb{E}_t \left[ \frac{\gamma c_t - b c_{t-1}}{\gamma c_{t+1} - b c_t} \zeta_{t+1} \Phi' \left( \gamma \frac{i_{t+1}}{i_t} \right) \gamma \frac{i_{t+1}}{i_t} \right], \\
y_t &= A_t k_t^\alpha h_t^{1-\alpha}, \\
y_t &= c_t + i_t \\
\gamma k_{t+1} &= \zeta_t i_t \left( 1 - \Phi \left( \gamma \frac{i_t}{i_{t-1}} \right) \right) + (1-\delta) k_t,
\end{aligned}$$

where lowercase variable $x$ denotes the deflated for growth variable $X$ ($x_t = X_t/\Gamma_t$) for any $X \in \{Y, C, I, K\}$. The solution of a log-linearised version of the optimal allocation admits the state space representation:

$$\begin{aligned}
Y_t &= \Pi_X X_t + \Pi_Z Z_t, \\
X_{t+1} &= M_X X_t + M_Z Z_t, \\
Z_{t+1} &= R Z_t + \varepsilon_t,
\end{aligned}$$

where $X_t = (\widehat{k}_t, \widehat{c}_{t-1}, \widehat{i}_{t-1})'$, $Y_t = (\widehat{c}_t, \widehat{y}_t, \widehat{i}_t, \widehat{h}_t, \widehat{q}_t)$ and $Z_t = (\widehat{a}_t, \widehat{\zeta}_t, \widehat{\vartheta}_t)'$ and $\varepsilon_t = (\varepsilon_t^A, \varepsilon_t^\zeta, \varepsilon_t^\vartheta)'$. As usual in the literature $\widehat{x}_t$ denotes the log-deviation of variable $x$ from its deterministic steady state. We then assess the ability of the dynamic identification technique developed in the main text to recover the dynamics of the "true" structural model. The main difference from the New-Keynesian model investigated in the text is that the structural model features a latent variable unobserved by the econometrician —e.g. the capital stock.

Table H.1: Parametrisation

| | | |
|---|---|---|
| *Preferences* | | |
| $\beta$ | Discount Factor | 0.990 |
| $b$ | Habit Persistence | 0.650 |
| $\nu$ | Inv. Frish Elasticity | 1.000 |
| *Technology* | | |
| $\alpha$ | Capital Elasticity | 0.330 |
| $\varphi$ | Investment Adjustment Costs | 2.500 |
| $\delta$ | Depreciation Rate | 0.025 |
| $\gamma$ | Gross Rate of Growth | 1.004 |
| *Shock Persistence* | | |
| $\rho_A$ | Technology Shock | 0.950 |
| $\rho_\zeta$ | Investment Specific Shock | 0.810 |
| $\rho_\vartheta$ | Labor Wedge Shock | 0.940 |
| *Shock Volatility (in %)* | | |
| $\rho_A$ | Technology Shock | 0.700 |
| $\rho_\zeta$ | Investment Specific Shock | 2.000 |
| $\rho_\vartheta$ | Labor Wedge Shock | 0.800 |

In order to perform this assessment we parametrise the model by borrowing values for the structural parameters from the RBC literature (see Table H.1). The parameters pertaining to the investment specific shock are directly borrowed from Justiniano et al. (2011). Those pertaining to the labor wedge shock are taken from Kascha and Mertens (2009).[37] We then run the following Monte-Carlo experiment. We use the RBC model as the DGP for output, investment and hours worked and simulate the estimated model 10,000 times over 250 periods. For each simulation, we estimate the following VARMA

$$\begin{pmatrix} y_t \\ i_t \\ h_t \end{pmatrix} = \Phi_1 \begin{pmatrix} y_{t-1} \\ i_{t-1} \\ h_{t-1} \end{pmatrix} + \Phi_2 \begin{pmatrix} y_{t-2} \\ i_{t-2} \\ h_{t-2} \end{pmatrix} + \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ u_{3,t} \end{pmatrix} + \Theta_1 \begin{pmatrix} u_{1,t-1} \\ u_{2,t-1} \\ u_{3,t-1} \end{pmatrix}$$

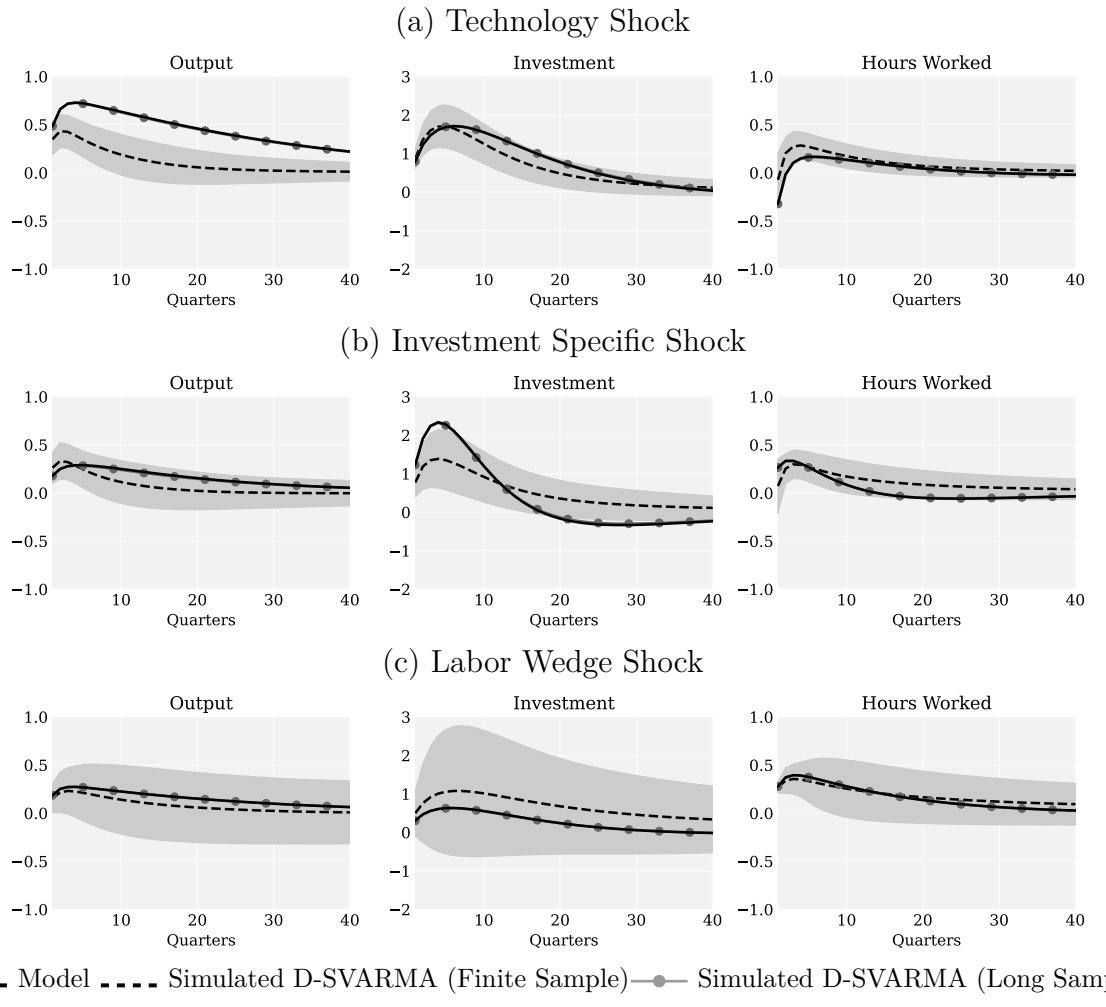
[37]Note that the model does not pretend to be an accurate representation of a specific economy, but is used as a *reasonable* DGP to conduct our assessment exercise.

and use it as auxiliary model to recover the D-SVARMA representation by ALS as

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = G \begin{pmatrix} y_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + F_0 \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} + F_1 \begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \\ z_{3,t-1} \end{pmatrix} \text{ where } \begin{pmatrix} z_{1,t} \\ z_{2,t} \\ z_{3,t} \end{pmatrix} = R \begin{pmatrix} z_{1,t-1} \\ z_{2,t-1} \\ z_{3,t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{pmatrix}$$

We then compute the response of each variable to each shock. Figure H.1 reports for each shock the theoretical IRFs of output, investment and hours worked (plain dark line) alongside the average IRF as recovered from the simulated D-SVAR (dashed line). The shaded area corresponds to the 68% confidence band of each IRF in the simulated model.

Figure H.1: Impulse responses, RBC model vs Simulated D-SVARMA



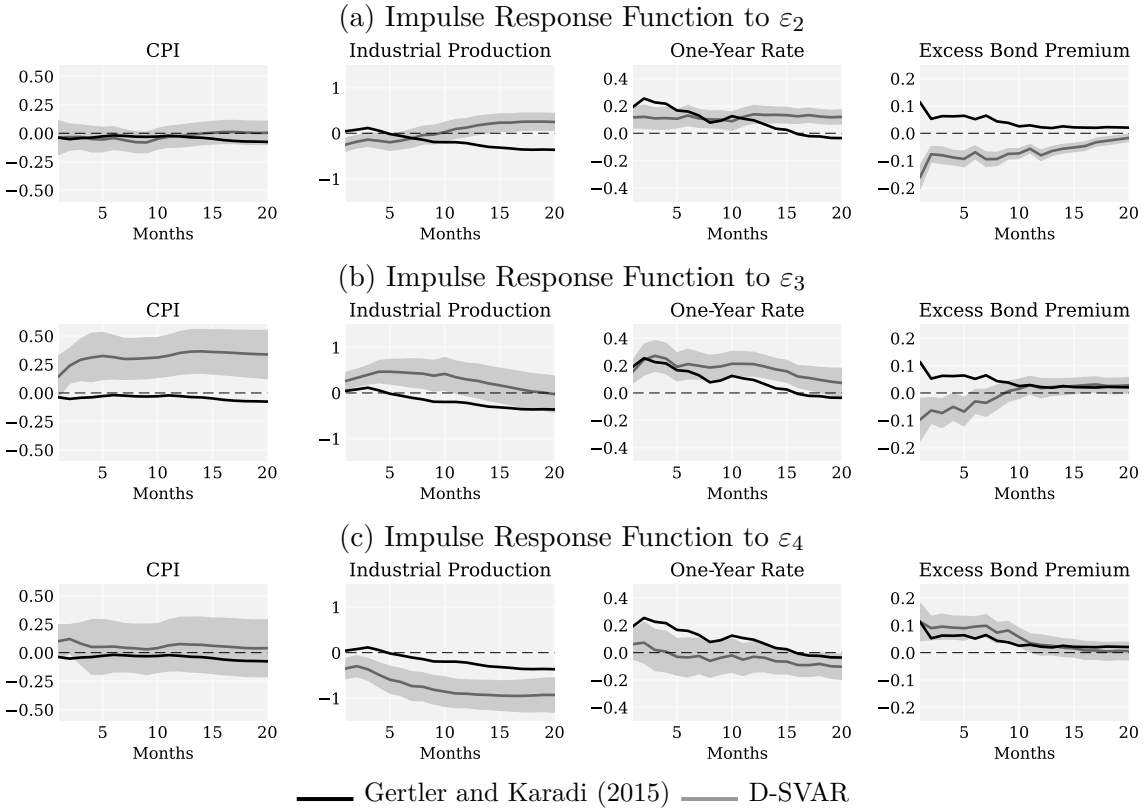

Notes: The estimated model is a D-SVARMA(2,1). Data are generated by the calibrated RBC model. Observables are $y$, $i$ and $h$, but $k$ is not observable. Short sample corresponds to the average of 10,000 estimations of length 250 periods. The shaded area represents 68% confidence bands, as computed from the 10,000 simulations. Long sample corresponds to one simulation of length 1,000,000 periods.

Note that the three shocks are unlabelled in the D-SVARMA, so we order them by minimising the distance between the model structural shock and each D-SVARMA shock. Inspection of the figure suggests that the D-SVARMA allows to recover the three structural shocks: the impulse response functions share the same shape and the same properties

in the model and in the D-SVARMA. In particular, the D-SVARMA is able to properly recover the short-run response to the various shocks. Closer inspection however reveals that the match is not perfect. The D-SVARMA tends to underestimate the response of output, and slightly overestimate that of hours worked. Two sources of bias are usually taken as the main culprit for this type of imperfect match: truncation bias and small sample bias. The truncation bias occurs when the state space representation of the solution is approximated by a finite VAR. This is not the case in our experiment. As explained above, the state space representation actually admits a VARMA(2,1) representation, which is precisely the unrestricted model we estimate. The truncation bias is therefore inoperative in our case. The main culprit is the small sample bias. As we increase the sample size of our simulations, the bias recedes and eventually vanishes. The grey line with bullet markers in Figure H.1 corresponds to IRFs obtained from the D-SVARMA estimated on a sample of 1,000,000 periods. The match is then perfect. Therefore, our dynamic identification method asymptotically correctly recovers the theoretical shocks, even in the presence of a latent variable.

# I Additional Material for Section 3.2.2 (Gertler and Karadi (2015))

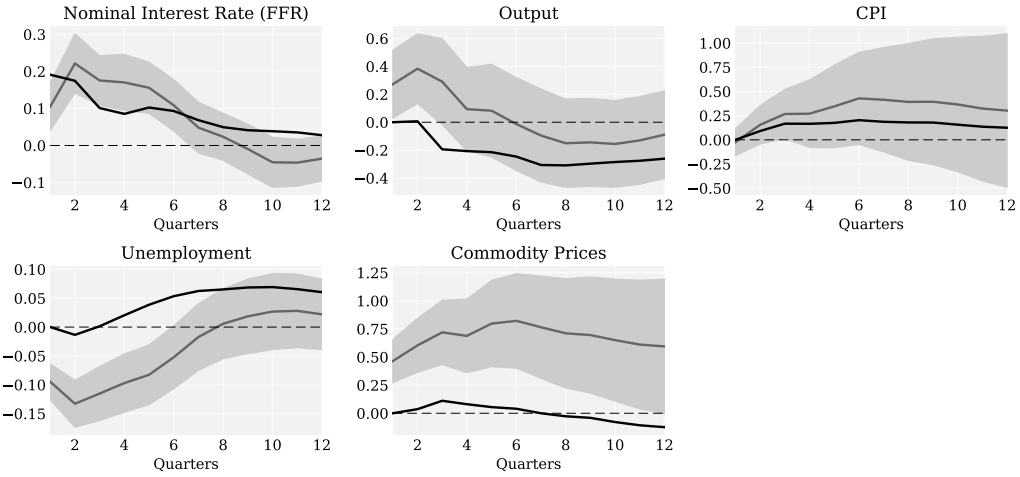
Figure I.1: IRFs to Gertler and Karadi (2015) monetary policy shock and D-SVAR's shocks with Lower Triangular $R$ Matrix ($\varepsilon_2$–$\varepsilon_4$)




Notes: On the four panels, the black line is the response to a monetary policy shock, as identified following Gertler and Karadi (2015). The grey line is the response to shock in the D-SVAR. Shaded area represent $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1979M7-2012M6.

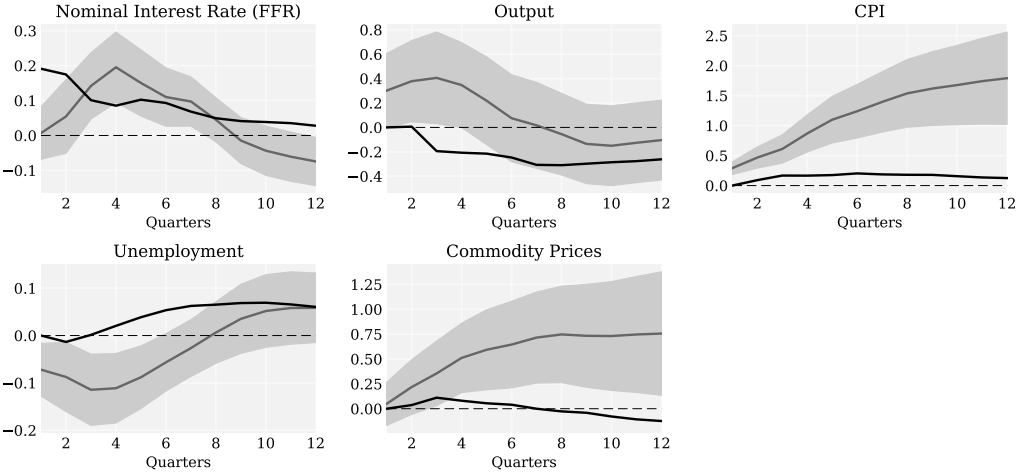
# J Additional Material for Section 3.2.3 (Christiano et al. (1999))

Figure J.1: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock $\varepsilon_2$



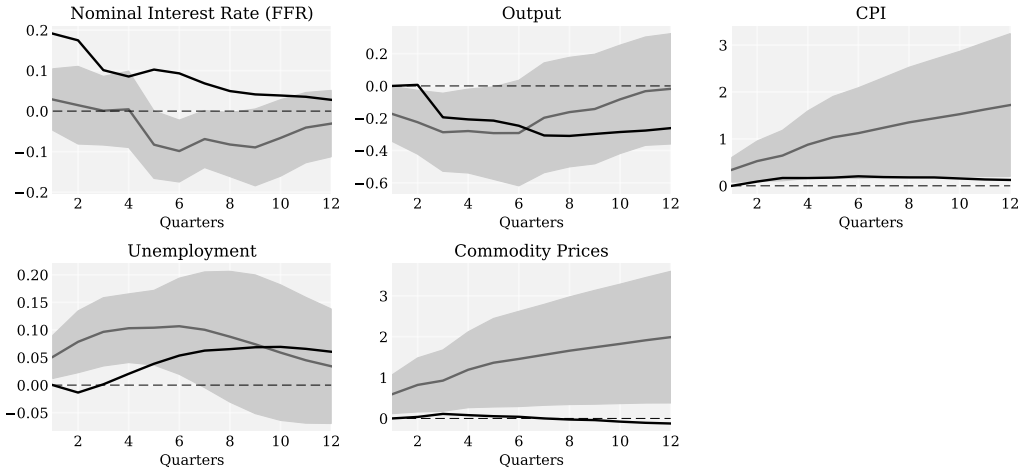

Notes: Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock $\varepsilon_4$ in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure J.2: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock $\varepsilon_3$



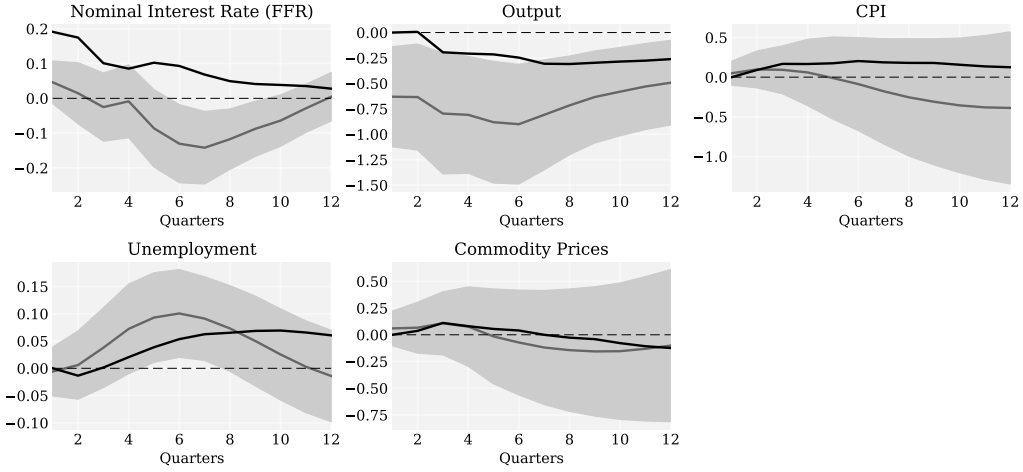

Notes: Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock $\varepsilon_4$ in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure J.3: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock $\varepsilon_4$




Notes: Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock $\varepsilon_4$ in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Figure J.4: IRFs to Christiano et al. (1999) monetary policy shock and D-SVAR's shock $\varepsilon_5$




Notes: Black line: response to a monetary policy shock, as identified following Christiano et al. (1999). Grey line: response to shock $\varepsilon_4$ in the D-SVAR. Shaded area: $\pm$ 1 standard deviation around average D-SVAR response obtained from 1,000 Bootstrap replications. Sample is 1965Q1-2007Q4.

Table J.1: Correlation between CEE's MP Shock and DSVAR Shocks

| | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ | $\varepsilon_5$ |
|---|---|---|---|---|---|
| Corr | 0.928 | -0.005 | -0.342 | 0.094 | 0.097 |
| p(0) | 0.000 | 0.986 | 0.353 | 0.695 | 0.652 |
| p(1) | 0.665 | 0.013 | 0.015 | 0.035 | 0.011 |

Notes: Corr reports the correlation between the CEE Monetary Policy Shock and each of the shock as identified by our DSVAR. p(0) (resp. p(1)) reports the p-value associated to the Wald test of a unit (resp. zero) correlation between the CEE Monetary Policy Shock and each of the shock as identified by our DSVAR. This p-value is obtained by means of 1,000 bootstrap, so as to accommodate for the fact that both shocks are generated by an estimated VAR model and are hence contaminated by the uncertainty surrounding the estimated coefficients of the VAR.

Table J.2: Forecast Error Variance Decomposition of (in %)

| | Christiano et al. (1999) | D-SVAR | | | | |
|---|---|---|---|---|---|---|
| Horizon | $\varepsilon_R$ | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ | $\varepsilon_5$ |
| *Nominal Interest Rate* | | | | | | |
| 1 | 66.79 | 74.37 | 20.16 | 0.09 | 1.50 | 3.89 |
| 4 | 26.03 | 42.62 | 37.11 | 19.00 | 0.32 | 0.94 |
| 8 | 17.74 | 34.46 | 29.15 | 19.75 | 5.36 | 11.28 |
| 20 | 13.39 | 31.30 | 27.11 | 21.18 | 7.42 | 12.99 |
| *Output* | | | | | | |
| 1 | 0.00 | 6.08 | 11.72 | 14.39 | 4.77 | 63.04 |
| 4 | 2.47 | 3.01 | 9.65 | 15.92 | 7.38 | 64.04 |
| 8 | 5.47 | 3.22 | 5.47 | 8.92 | 7.38 | 75.01 |
| 20 | 5.95 | 3.48 | 4.12 | 6.23 | 4.81 | 81.36 |
| *Price Index (CPI)* | | | | | | |
| 1 | 0.00 | 2.16 | 0.30 | 40.50 | 55.86 | 1.17 |
| 4 | 1.80 | 9.12 | 4.80 | 40.62 | 44.78 | 0.68 |
| 8 | 1.04 | 8.83 | 4.36 | 46.40 | 39.72 | 0.70 |
| 20 | 0.23 | 5.44 | 1.66 | 46.05 | 45.27 | 1.58 |
| *Unemployment* | | | | | | |
| 1 | 0.00 | 5.61 | 50.38 | 29.17 | 14.64 | 0.20 |
| 4 | 0.46 | 6.51 | 37.46 | 29.12 | 21.81 | 5.11 |
| 8 | 5.70 | 5.93 | 25.66 | 21.69 | 29.76 | 16.97 |
| 20 | 6.47 | 13.22 | 20.87 | 22.49 | 28.28 | 15.14 |
| *Commodity Price* | | | | | | |
| 1 | 0.00 | 0.37 | 37.40 | 0.42 | 61.19 | 0.62 |
| 4 | 0.37 | 0.89 | 29.19 | 8.12 | 61.34 | 0.47 |
| 8 | 0.13 | 0.56 | 21.06 | 12.05 | 65.98 | 0.35 |
| 20 | 0.49 | 0.14 | 9.86 | 12.86 | 76.95 | 0.19 |

Notes: In this table we compare the variance decomposition as obtained by Christiano et al. (1999) for their monetary policy shocks and for the five shocks of the D-SVAR. Sample is 1965Q1-2007Q4.

# K A Two-Country VAR

In this section, we consider the modelling of a 2-country VAR featuring the log-difference of US GDP and residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP for the 1995Q1-2019Q4 period as reported by OECD (https://stats.oecd.org/, VPVOBARSA, US Dollars, volume estimates at fixed PPP, seasonally adjusted). In this example, we illustrate how the D-SVAR allows to recover a shock structure à la Backus et al. (1992) involving dynamic symmetric spillovers. More precisely the shock process is assumed to take the form

$$Z_t \equiv \begin{pmatrix} z_t^{us} \\ z_t^{can} \end{pmatrix} = \begin{bmatrix} \rho & \nu \\ \nu & \rho \end{bmatrix} Z_{t-1} + \varepsilon_t \text{ with } \varepsilon_t \rightsquigarrow N(0_2, I_2),$$

where $\nu$ captures the dynamic spillovers. Both the AIC, BIC and Hannan-Quinn information criteria led us to select a VAR(2) specification. The D-SVAR identification then leads to a value of $\rho = 0.43$ and $\nu = 0.20$. The loading matrix $\Pi_{yz}$ then takes the form

$$\Pi_{yz} = \begin{bmatrix} 0.498 & 0.053 \\ -0.229 & 0.360 \end{bmatrix}.$$

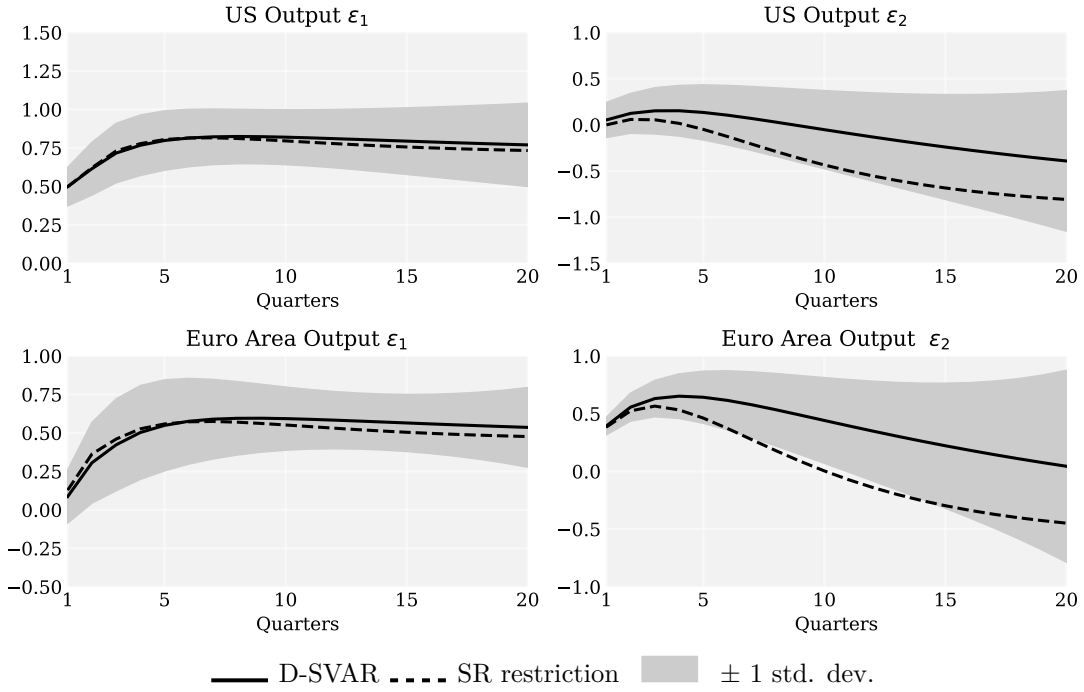
The forecast error variance decomposition of levels is reported in Table K.1.. While the US shock explain essentially all of US GDP, it accounts for less than 5%the Euro volatility in the very short-run and 56% at the 20 quarters horizon. In other words, in the very short-run, the US and Euro Area economies are essentially insulated from each other, while the shocks are transmitted in the medium run. Figure K.1 reports the IRF of US and Euro GDP to both shocks. These IRF confirm and illustrate the broad picture conveyed by forecast error volatility decomposition: US and Euro GDP only responds to their respective shocks on impact, and are essentially insulated from exogenous developments in the other economy in the very short-run. This is actually slightly contrasts with an identification of the US shock as the only shock that affects US GDP on impact. In that latter case, the US shock is transmitted faster to the Euro Area: the US shock accounts for about 10% of the Euro output volatility on impact and about 40% after 1 year (30% in our case). In the longer run, the Cholesky decomposition indicates that while the US economy is essentially not affected by Euro shocks, US shocks account for 60% of GDP volatility in the Euro Area.

Table K.1: Forecast Error Decomposition

| | US GDP | | Euro Area GDP | |
|---|---|---|---|---|
| Horizon | $\varepsilon_t^{us}$ | $\varepsilon_t^{euro}$ | $\varepsilon_t^{us}$ | $\varepsilon_t^{euro}$ |
| D-SVAR | | | | |
| 1 | 98.9 | 1.1 | 4.6 | 95.4 |
| 4 | 96.4 | 3.6 | 29.3 | 70.7 |
| 8 | 97.8 | 2.2 | 40.9 | 59.1 |
| 20 | 93.6 | 6.4 | 60.8 | 39.2 |
| Short-Run Restriction | | | | |
| 1 | 100.0 | 0.0 | 10.0 | 90.0 |
| 4 | 99.6 | 0.4 | 38.3 | 61.7 |
| 8 | 96.7 | 3.3 | 56.5 | 43.5 |
| 20 | 68.7 | 31.3 | 67.0 | 33.0 |

Notes: The variance decomposition is obtained from a bivariate D-SVAR or a SVAR with a short-run restriction. Variables are the log-difference of US GDP and the residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP. Sample is 1995Q1-2019Q4.

Figure K.1: Impulse Response Functions: US vs Euro Area




Notes: These IRFs are obtained from a bivariate D-SVAR or a SVAR with a short-run restriction. Variables are the log-difference of US GDP and the residual of the cointegration relationship (1,-0.63) between the Euro Area and US real GDP. Sample is 1995Q1-2019Q4.