Evaluating the Effects of the Federal Reserve's Conventional and Unconventional Monetary Policies Using a Shadow Rate Affine Term Structure Model*

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Abstract

This paper proposes a novel factor rotation within a shadow rate term structure model of the U.S. Treasury yield curve, combined with external instruments, to decompose the U.S. monetary policy shocks into three orthogonal components affecting the short rate, the expected path of future short rates, and the term premium. By including these shocks in a monthly VAR applied to U.S. data from 1991 to 2019, we find that unconventional monetary policies, which alter the expected path of future short rates or the term premium, have a greater impact on economic activity than conventional policies targeting the short rate. In addition, the path and term premium shocks affect commodity prices and credit conditions more than conventional monetary policy, underscoring their significant roles in influencing financial markets and the transmission of monetary policy shocks.

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1 Introduction

In the decade following December 2008, the federal funds rate was constrained by the Zero lower bound (ZLB), prompting the Federal Reserve Bank to employ a range of unconventional monetary policies, including forward guidance and large-scale asset purchases, to address the Great Financial Crisis (GFC) and its aftermath. An important question remains: how effective have these unconventional policies been compared to conventional policies in influencing the economy? The answer has implications beyond the ZLB period, as the Fed has recently pursued Quantitative Tightening (QT) to exit the Quantitative Easing (QE) regime. Moreover, these unconventional tools may remain an integral part of the Fed's toolkit, potentially rivaling conventional policies in importance.

Numerous studies in the existing literature have sought to evaluate the effectiveness of unconventional monetary policies. For example, Gürkaynak et al. (2005), Gertler and Karadi (2015), and Nakamura and Steinsson (2018) use high frequency data on short-term Treasury yields and medium-term federal funds futures to extract monetary policy shock to the expected path of the future short rate. Similarly, Rogers et al. (2018), Gilchrist et al. (2019), and Swanson (2021, 2023) derive shocks from large-scale asset purchases (LSAPs) using long-term Treasury yields. We follow their approaches of using multiple asset prices to delineate different dimensions of monetary policy. However, In contrast to these studies, we further decompose these monetary policy shocks by employing a shadow rate no-arbitrage affine term structure model (SR-ATSM) for the U.S. Treasury yield curve. We introduce a novel factor rotation which yields three types of orthogonal shocks: the shock to the current rate, or action shock; the shock to the expected path of future short rates, or path shock, after controlling for the action shock; and the shock to the term premium, or term premium shock, after controlling for the first two shocks.

We use the daily changes in these three components around FOMC announcements to construct instruments, which are then included in a structural vector autoregression (VAR) framework to estimate the impacts of monetary policy shocks, following Gertler and Karadi (2015), Stock and Watson (2018), and Swanson (2023). We find that while both conventional and unconventional monetary policies have significant effects on the macroeconomy,

the unconventional ones that alter the expected path of future short rates or the term premium have greater impacts on the economic activity and financial markets. For example, at its peak, the shock to the expected path of future short rates affects industrial production growth or commodity prices twice as much as a shock to the current short rate. This finding corroborates Gertler and Karadi (2015) and Swanson (2021) which also document the important role of financial markets in the transmission of monetary policy shocks. Although Swanson (2021) documents substantial and significant impacts of unconventional monetary policy shocks on financial markets, including Treasury yields and corporate bond yields, comparable to those of conventional policy, Swanson (2023) finds that the effects of the former on real economic activity are much smaller. Our approach of dissecting bond yield movements into structural components can potentially shed further light on these findings and help reconcile them.

Our paper relates to two strands of research. The first strand of the literature uses term structure models to fit the Treasury yield curve and extract information on the economy and monetary policy stance from the yield curve (see Campbell, 1995 for an introduction). Duffie and Kan (1996), Dai and Singleton (2000), and Joslin et al. (2011) build on the dynamic term structure model and develop the affine term structure model with the no-arbitrage condition to fit the movement of the Treasury yield curve and extract the term premium. Studies such as Ang and Piazzesi (2003), Cochrane and Piazzesi (2005), Wright (2011), and Bauer and Rudebusch (2017) have long documented the close relationship between the Treasury yield curve and macroeconomics variables. Rudebusch and Wu (2008) and Ang et al. (2011) use term structure models to analyze the effect of conventional monetary policy. We contribute to this literature by developing a model that allows us to derive orthogonalized components of the yield curve movements with clear economic interpretation in one step, thereby facilitating the identification of different types of monetary policy shocks.

The second strand of the research focuses on employing high-frequency data on interest rates to estimate the effects and transmission of monetary policy shocks. The use of high-frequency data in the literature has been pivotal in extracting monetary policy shocks due to its ability to mitigate endogeneity. (see Ramey, 2016). Early examples, such as Kuttner (2001), Cochrane and Piazzesi (2002), and Bernanke and Kuttner (2005), focus on shocks to the federal funds rate. Subsequent studies, including Gürkaynak

et al. (2005), Gertler and Karadi (2015), Miranda-Agrippino and Ricco (2021, 2023), and Swanson (2021, 2023), consider policy shocks related to forward guidance and LSAPs.

In contrast to prior research work, our approach provides a structural decomposition of monetary policy shocks via the term structure model of Treasury yield curve. Moreover, thanks to the adoption of a continuous-time term structure model, our shock to the current short rate reflects the variation in the instantaneous risk-free rate and is not restricted by the shortest maturity of the yields used in the estimation. Furthermore, our component of the expected future short rate corresponds to the trajectory spanning the entire yield curve, not tied to any specific maturity. Finally, the derived term premium shock is a more direct measure of the risk premium changes than the LSAPs shock used in the previous literature. The term premium of the Treasury yield curve generally reflects the perception of market risk (Piazzesi, 2010) and has been shown to be closely related to inflation uncertainty (Wright (2011)). Additionally, the variable shadow rate during the ZLB period provides a source of continuing changes of the current short rate and preserves the stability of the model (Wu and Xia, 2016, 2020), including a more reliable estimate of term premium in low-interest-rate environment.[1] As a result, our approach yields three sets of monetary policy shocks that are present throughout the entire sample period, encompassing both the pre- and post-ZLB periods.[2]

The approach developed in this paper is closely related to Kaminska et al. (2021) (hereafter KMS), who also use an affine term structure model to decompose yield curve movements and derive monetary policy instruments. However, there are three key differences between our paper and theirs. First, we propose a novel factor rotation to estimate the term structure model and extract the orthogonal policy shocks jointly in one step, whereas KMS employs a two-step procedure to derive the policy shocks. Second, the empirical

[1] When the short end of the Treasury yield curve remains nearly intact at the ZLB and the unconventional monetary policies are aimed at medium- and long-term yields, methods that identify the effect of monetary policy from the long end of the yield curve without considering the ZLB tend to underestimate variations in the expected path of future short rates or the term premium, which in turn may downplay the effects of unconventional monetary policies.

[2] In contrast, for example, in previous studies the LSAPs shock is often set to zero before 2008.

analysis in KMS is restricted to the period prior to 2009, excluding the subsequent years when the federal funds rate was stuck at the ZLB and unconventional monetary policies became the dominant policy tools. By incorporating a shadow rate into the ATSM, we extend the analysis in KMS to include the critical ZLB period, providing a more comprehensive evaluation of the effects of both conventional and unconventional monetary policies. Third, the policy shocks employed in KMS do not account for the "Fed information effect" documented in Nakamura and Steinsson (2018) or the "Fed response to news" channel suggested in Bauer and Swanson (2023a,b). In contrast, we use Tealbook data to control for the Fed's perceptions and projections of economic conditions in deriving the instruments, following Miranda-Agrippino and Ricco (2021). Our approach, using these policy shocks, eliminates several empirical puzzles present in KMS, highlighting the advantages of our approach.

The rest of this paper proceeds as follows. Section 2 describes the modeling framework, including the shadow rate dynamic term structure model, the factor rotation, the construction of our monetary policy instruments, and the structural VAR with external instruments. Section 3 presents the data and estimation procedures. Section 4 discusses the empirical results. Section 5 concludes.

2 Methodology

2.1 Shadow Rate Dynamic Term Structure Model

We present a continuous-time version of SR-ATSM to extract information from the yield curve of Treasury bonds. First, following the convention in the literature we posit that the yield curve is driven by three latent factors in X_t which follows a Ornstein-Uhlenbeck process under the physical measure (P-measure)[3]:

$$dX_t = A_x(\gamma_x - X_t)dt + \Sigma_x dz_t \tag{1}$$

[3]See, for example, Dai and Singleton (2000), Joslin et al. (2011), and Kim and Orphanides (2012).

where dz_t follows a standard Brownian motion. The model is equipped with a shadow rate that is affine in latent factors:

$$s_t = \delta_0 + \delta_1' X_t \tag{2}$$

The actual short-term nominal interest rate (i) is the maximum of the shadow rate and the effective lower bound ($\underline{i}$), which is set to zero for the U.S. Treasury yield curve: $i_t = max(\underline{i}, s_t)$.

The no-arbitrage requirement ensures the existence of the risk-neutral measure (Q-measure) for X_t:

$$dX_t = A_x^Q(\gamma_x^Q - X_t)dt + \Sigma_x dz_t^Q \tag{3}$$

Using the essentially affine model specification, the risk price λ_t is also affine in latent factors, having the form:

$$\begin{aligned} \Sigma_x \lambda_t &= \lambda_0 + \lambda_1' X_t \\ \lambda_0 &= A_x \gamma_x - A_x^Q \gamma_x^Q \\ \lambda_1 &= A_x^Q - A_x \end{aligned} \tag{4}$$

Consequently, the bond excess returns are:

$$\frac{dP_t}{P_t} - s_t dt = b\Sigma_x \lambda_t dt + b\Sigma_x dz_t \tag{5}$$

Where, the term premium or risk premium is given by $b\Sigma_x\lambda_t$. The expression for b is shown in Appendix A.

The observed Treasury yields can then be approximated as a function of the latent factors and parameters. We sketch the derivation below and refer readers to Han and Ma (2024), Krippner (2015) and Wu and Xia (2016) for further details.

The Appendix A shows that the m-period ahead *shadow* forward rate can be expressed as:

$$f_{t,m} = E_t^Q[s_{t+m}] - CAT_m \tag{6}$$

where CAT_m is a convexity adjustment term:

$$CAT_m = \frac{1}{2}\Big[\int_0^m \delta_1' e^{-A_x^Q u} du\Big]\Sigma\Sigma'\Big[\int_0^m e^{-A_x^{Q'} u}\delta_1 du\Big] \tag{7}$$

And $E_t^Q[s_{t+m}]$ is the conditional expectation of the shadow rate in m periods under the Q-measure:

$$E_t^Q[s_{t+m}] = \delta_0 + \delta_1 E_t^Q[X_{t+m}] \tag{8}$$

Equation (3) yields the conditional expectation of X_t m-period ahead under the Q-measure:

$$E_t^Q[X_{t+m}] = \gamma_x^Q + e^{-A_x^Q m}(X_t - \gamma_x^Q) \tag{9}$$

Similarly, the conditional expectation of X_t m-period ahead under the P-measure is:

$$E_t[X_{t+m}] = \gamma_x + e^{-A_x m}(X_t - \gamma_x) \tag{10}$$

Denote by $\underline{f}_{t,m}$ the effective forward rate when the actual short rate is subject to the effective lower bound. Han and Ma (2024) show, under certain conditions analogous to those outlined in Wu and Xia (2016), that the following approximation holds:

$$\underline{f}_{t,m} \approx E_t^Q(max(\underline{i}, s_{t+m} - CAT_m)) \tag{11}$$

As a result, the actual forward rate subject to the effective lower bound can be derived as:

$$\underline{f}_{t,m} \approx \underline{i} + (f_{t,m} - \underline{i}) \cdot \mathbf{\Phi}\left(\frac{f_{t,m} - \underline{i}}{\sigma_m}\right) + \boldsymbol{\phi}\left(\frac{f_{t,m} - \underline{i}}{\sigma_m}\right) \tag{12}$$

where $\mathbf{\Phi}(\cdot)$ and $\boldsymbol{\phi}(\cdot)$ are the cumulative distribution function and probability density function of a standard normal distribution, σ_m^2 is the conditional second moment of s_{t+m} under the Q-measure and has the form:

$$\sigma_m^2 = Var_t^Q(s_{t+m}) = \int_0^m \delta_1' e^{-A_x^Q u} \Sigma\Sigma' e^{-A_x^{Q'} u} \delta_1 du \tag{13}$$

Given the actual forward rate, the observed Treasury yield subject to the ZLB can be formulated as:

$$\underline{y}_{t,m} = \frac{1}{m} \int_0^m \underline{f}_{t,u} du \tag{14}$$

2.2 The Factor Rotation

The latent factors may be arbitrarily rotated in the affine model (see Dai and Singleton (2000)). The Appendix B lays out the invariant affine transformation that supports the factor rotation and makes clear the number of restrictions required to anchor the factors. We propose the following rotation that keeps the maximum flexibility of the model while achieving our identification purpose (Han-Ma rotation hereinafter):

$$\delta_1 = (1,0,0)' \quad \gamma_x^Q = (0,0,0)'$$
$$A_x^Q = \begin{pmatrix} a_{11}^Q & a_{12}^Q & a_{13}^Q \\ 0 & a_{22}^Q & a_{23}^Q \\ a_{31}^Q & a_{32}^Q & a_{33}^Q \end{pmatrix} \quad A_x = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
$$\Sigma_x = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

The remaining parameters in $\{\delta_0, \gamma_x\}$ are unrestricted. As a result, there are yet 22 freely estimated parameters in our model: δ_0, eight parameters in A_x^Q, three in Σ_x, seven in A_x, and three in γ_x[4].

With the Han-Ma rotation, we are able to extract three monetary policy shocks (instruments) orthogonal to each other at the time when the policy announcement is made: shock to the current (shadow) short rate, shock to the expected path of future short rate, and shock to the term premium. Specifically, the Han-Ma rotation allows the first factor in X_t to summarize solely the policy shock to the current short rate because according to equation (2) we have:

$$s_t = \delta_0 + (1,0,0) \begin{pmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \end{pmatrix} = \delta_0 + x_{1t} \tag{15}$$

Moreover, equation (15) and (10) imply that the expected path of future

[4]We also tried the rotation where a_{21}^Q and a_{31}^Q are unrestricted and found our results are robust to the change.

short rate under the P-measure is determined by the first two factors in X_t:

$$\begin{aligned} E_t s_{t+m} &= \delta_0 + \delta_1' E_t X_{t+m} \\ &= \bar{\delta_0} + \begin{pmatrix} PL_{1m} & PL_{2m} & 0 \end{pmatrix} \begin{pmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \end{pmatrix} \\ &= \bar{\delta_0} + PL_{1m} x_{1t} + PL_{2m} x_{2t} \end{aligned} \tag{16}$$

where $\bar{\delta_0}$ absorbs all relevant constants including δ_0, and PL_{1m} and PL_{2m} are the coefficients that can be calculated from δ_1 and equation (10). Therefore, x_{2t} is the factor that affects the expected path of future short rate after controlling for the shock to the current short rate, making it a good measure to reflect changes in the expectation component of the yield curve or the so-called forward guidance.

Finally, equations (19) and (5) imply that the expected term premium for the m-period yield is:

$$TP_t(m) = E_t[\frac{dP_t}{P_t} - s_t dt] = b(m)\Sigma_x \lambda_t \cdot dt \tag{17}$$

substituting $b(m)$ and λ_t into the equation provides:

$$\begin{aligned} TP_t(m) &= -\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x[(A_x\gamma_x - A_x^Q\gamma_x^Q + (A_x^Q - A_x)X_t] \cdot dt \\ &= TP_m^c \cdot dt - \delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x(A_x^Q - A_x)X_t \cdot dt \end{aligned} \tag{18}$$

where TP_m^c includes all time-invariant components. According to our rotation, $(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x(A_x^Q - A_x)$ has full rank and nonzero elements at each position. Combined with $\delta_1 = (1, 0, 0)'$, the loading before X_t becomes a nonzero vector $(TL_{1m}\ TL_{2m}\ TL_{3m})$. Then, the expected term premium can be represented as:

$$TP_t(m) = (TP_m^c + TL_{1m}x_{1t} + TL_{2m}x_{2t} + TL_{3m}x_{3t}) \cdot dt \tag{19}$$

It is obvious that all three factors in X_t contribute to the term premium while the last one does not affect the current or expected future short rate. Furthermore, the diagonalization of Σ_x ensures the three shocks are orthogonal to each other at the time when policy shocks are revealed. Hence, changes in x_{3t} represents the term premium shock unrelated to shocks to current short rate or expected future short rate.

2.3 Extracting the Instruments for Monetary Policy Shocks

After obtaining the parameter estimates and filtered estimates of the latent factors from the SR-ATSM, we can measure changes of the yield curve around each FOMC (Federal Open Market Committee) announcement. Specifically, we decompose changes in the yield curve into three components: changes in the contemporaneous short(shadow) rate, changes in the expected path of future short rate, and changes in the term premium. In Kaminska et al. (2021), the authors estimate the term structure model first and then employ the principal component to impose an additional rotation to the yield curve. Our approach, instead, allows us to obtain these orthogonal components directly from the model estimates, as illustrated in the previous section.

We first use the monthly Treasury yield data to estimate the model, and then use the daily yield data to extract the monetary policy instruments. For each month, we select the yields at the end of the month for the estimation. Thus, we have the filtered estimates of factors $X_{t|t}$ containing information in the yield curve until the end of month t. Then, for the FOMC announcement at date t_a, we denote the last end of month date by t_{a-1}, the next end of month date by t_{a+1}, and the day before the announcement by t_{-1}. Prior to the FOMC announcement, the filtered estimate $X_{t_{-1}|t_{-1}}$ using information up to t_{-1} can be computed using the filtered estimate $X_{t_{a-1}|t_{a-1}}$ and the yields at t_{-1}. As a result, the expected value of the factors at t_a conditional on the information up to one day before the FOMC announcement are:

$$X_{t_a|t_{-1}} = E_{t_{-1}}[X_{t_a}] = \gamma_x + e^{-A_x d_t}(X_{t_{-1}|t_{-1}} - \gamma_x)$$

where d_t is the one-day interval, and $X_{t_a|t_{-1}}$ represents the market expectation of yields at t_a based on all available information up to t_{-1}. On the other hand, the Treasury yields at t_a and corresponding filtered factors $X_{t_a|t_a}$ should incorporate all information from the FOMC announcement. Thus, the difference between $X_{t_a|t_{-1}}$ and $X_{t_a|t_a}$ reflects the shock from the FOMC announcement not anticipated by the market.

Equipped with the decomposition of yield curve, we are able to define our three instruments for monetary policy shocks as follows: the action instru-

ment related to changes in the contemporaneous short rate ($it^a_{t_a}$):

$$it^a_{t_a} \equiv (1,0,0) \cdot \Big(X_{t_a|t_a} - X_{t_a|t_{-1}} \Big),$$

the path instrument that measures surprise in the expected path of future short rate ($it^p_{t_a}$):

$$it^p_{t_a} \equiv (0,1,0) \cdot \Big(X_{t_a|t_a} - X_{t_a|t_{-1}} \Big),$$

and the term premium instrument reflecting the effect of policy shocks on risk premium ($it^r_{t_a}$):

$$it^r_{t_a} \equiv (0,0,1) \cdot \Big(X_{t_a|t_a} - X_{t_a|t_{-1}} \Big).$$

2.4 Constructing Informationally-Robust Instruments

As documented in the literature, the monetary policy shocks obtained from changes in the yield curve around the FOMC announcements may contain the Fed's information relevalation of the state of the economy (Miranda-Agrippino and Ricco, 2021) or its thresponses to publicly available economic news (Bauer and Swanson, 2023a). Using these unadjusted shocks as instruments to analyze the response of the economy to policy shocks can lead to some puzzling results (Bauer and Swanson, 2023a; Swanson, 2023). Therefore, we follow Miranda-Agrippino and Ricco (2021) to remove these information contents to construct the informationally-robust instruments exploiting the Tealbook dataset.

Explicitly, the Tealbook dataset provides quarterly projections for real GDP, inflation, and unemployment rate which can be used to control for the state of the economy. For each FOMC announcement and corresponding instruments $\{it^a_{t_a}, it^p_{t_a}, it^r_{t_a}\}$, we first eliminate the component related to the response to economic fundamentals by:

$$it_{t_a} = a_0 + \sum_{j=-1}^{3} b_j F_{t_a,q+j} + \sum_{j=-1}^{2} \tilde{b}_j (F_{t_a,q+j} - F_{t_a-1,q+j}) + \tilde{it}_{t_a} \qquad (20)$$

where it_{t_a} represents any instrument from $\{it^a_{t_a}, it^p_{t_a}, it^r_{t_a}\}$ at announcement t_a, $F_{t_a,q+j}$ is the vector of the latest Tealbook forecast for real GDP, inflation, and unemployment rate at $q+j$ quarterly horizon and q is the quarter where t_a is in. The variable $(F_{t_a,q+j} - F_{t_a-1,q+j})$ denotes revisions to projections from last announcement date to current one. $\tilde{it}_{t_a}$, as a residual, is the part of the instrument orthogonal to these economic contents. Then, for each month t, we add all $\tilde{it}_{t_a}$ within the month to construct monthly time series for $\tilde{it}^a_t$, $\tilde{it}^p_t$, and $\tilde{it}^r_t$. If there is no FOMC announcement in the month, the value of the instruments is set to zero.

Next, we exclude the autoregressive part of the monthly series that might be anticipated by the market in advance via the regression:

$$\tilde{it}_t = \phi_0 + \sum_{j=1}^{12} \phi_{t-j} \tilde{it}_{t-j} + irmp_t \tag{21}$$

The resulting $\{irmp^a_t, irmp^p_t, irmp^r_t\}$ are the informationally-robust action, path, and term premium instruments we rely on to analyze the impact of monetary policy shocks. Again, the value of instruments is assigned to zero if there is no FOMC announcement in the month.

2.5 Structural VAR with External Instruments

We build on a structural VAR framework including the growth rate of industrial production (IP), the inflation rate based on Consumer Price Index (CPI), log of the commodity price index published by Commodity Research Bureau (CRB price), the excess bond premium (EBP) from (Gilchrist and Zakrajšek, 2012), the shadow federal funds rate provided by Wu and Xia (2016), the yields of two-year Treasury bond and ten-year Treasury bond. We include both the IP and CPI to analyze the response of macro-economy to monetary policy shocks (e.g., Bernanke and Gertler, 1995; Gertler and Karadi, 2015). The CRB price is used to explain how prices respond to these shocks as advocated in Swanson (2023). The EBP captures variations in credit supply and aids in explaining the transmission mechanism of monetary policy. The shadow federal funds rate contributes to identifying the effect of conventional monetary policy that focuses on federal funds rate. Adopting the shadow federal funds rate instead of observed federal funds rate is aimed

at disentangling the effects of conventional and unconventional monetary policies, especially during the ZLB episodes. The two-year Treasury yield consists of expected short rates in the near future and thus, helps to reveal the expected path of future short rate. Lastly, the ten-year Treasury yield contains important information of the term premium as well as the expected rate path.

The structural VAR system has a standard representation as:

$$Y_t = \Phi_0 + \sum_{j=1}^{p} \Phi_j Y_{t-j} + v_t \tag{22}$$

where Y_t is a vector containing the seven variables discussed above, Φ_0 is a vector of constants, Φ_js are coefficient matrices, and the number of lags, p, is set to six following Swanson (2023). v_t is the reduced form shocks with $var(v_t) = \Omega_v$. Following Stock and Watson (2012, 2018) and Mertens and Ravn (2013), the structural shocks in ϵ_t are assumed to be serially uncorrelated and standardized to have unit variance:

$$var(\epsilon_t) = I \qquad E[\epsilon_t \epsilon_{t-j}] = 0, j \neq 0 \tag{23}$$

The structural shocks are posited to be mapped to reduced form shocks via:

$$\begin{aligned} v_t &= B\epsilon_t \\ \Omega_v &\equiv var(v_t) = Bvar(\epsilon_t)B' = BB' \end{aligned} \tag{24}$$

In this paper, we focus on three structural shocks, which are the action shock, path shock, and term premium shock. Without loss of generality, we assume the first three shocks in ϵ_t correspond to the action shock, path shock, and term premium shocks respectively, and denote their corresponding columns in B by s_1, s_2, and s_3.

Drawing on the idea of Stock and Watson (2012) and Gertler and Karadi (2015), we apply the method in Swanson (2023) using our informationally-robust instruments to identify the effect of monetary policy shocks. Take the action shock for an example, in the first-stage regression, we use the shadow federal funds rate (ffr) as the reference variable:

$$ffr_t = \alpha + \sum_{j=1}^{p} \bar{\phi}_j Y_{t-j} + \beta irmp_t^a + \eta_t \tag{25}$$

where ffr_t is the shadow federal funds rate at t, Y_{t-j} and p are the same as in the VAR system above, $irmp_t^a$ is the informationally-robust action instrument. In the second-stage regression, the fitted shadow federal funds rate, $\widehat{ffr}_t$, is used to identify s_1 through:

$$Y_t = \widetilde{\Phi}_0 + \sum_{j=1}^{p} \widetilde{\Phi}_j Y_{t-j} + s_1 \widehat{ffr}_t + \widetilde{v}_t \tag{26}$$

As a result, the contemporaneous effect of the action shock on variables in Y_t is summarized in s_1. Similarly, s_2 is identified exploiting the two-year Treasury yield as a reference variable while the ten-year Treasury yield is used to estimate s_3 in a similar fasion. Equipped with $\{s_1, s_2, s_3\}$, the impulse responses of macro variables to monetary policy shocks can be computed using equation (22).

3 Estimation

3.1 Data

The Treasury yields data is downloaded from the Federal Reserve Board constructed by the method of Gürkaynak et al. (2007). To estimate the SR-ATSM, we select the Treasury yields at the end of each month with typical maturities of 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, and 10 years, covering the entire yield curve. The Tealbook date is from the website of Federal Reserve Bank of Philadelphia. In the structural VAR, the Industrial Production and CPI data are from the Federal Reserve Economic Data (FRED) database and the year-over-year growth rates of IP and CPI are used. The CRB commodity price index is obtained from Bloomberg. The updated version of EBP constructed by Gilchrist and Zakrajšek (2012) is from the website of Federal Reserve Board. The Wu-Xia shadow rate (Wu and Xia, 2016) is downloaded from Cynthia Wu's website. We use the FOMC announcements documented in Swanson (2021) to study the effect of conventional and unconventional monetary policies. Because the documented announcements span from 1991 to 2019, Our estimation of SR-ATSM and

subsequent VAR use monthly observations in the same period[5].

3.2 Estimating SR-ATSM

Our SR-ATSM can be written in a state-space form:

$$Y_t^o = H_t(X_t) + e_t \qquad e_t \overset{\text{iid}}{\sim} N(0, \sigma_e^2 I_M) \tag{27}$$

$$X_{t+\Delta t} = \tilde{\mu} + F X_t + \tilde{v}_{t+\Delta t} \qquad \tilde{v}_{t+\Delta t} \overset{\text{iid}}{\sim} N(0, Q) \tag{28}$$

where Y_t^o is an $M \times 1$ vector including observed yields subject to ZLB at time t, X_t stores all three state variables. $H_t(X_t)$ is the nonlinear function in equation (14) approximating observed yields with various maturities. e_t contains measurement errors with a diagonal variance-covariance matrix. According to the physical process of X_t in equation (1), $\tilde{\mu} = \gamma_x + e^{-A_x \Delta t}$, $F = e^{-A_x \Delta t}$, and $Q = \int_0^{\Delta t} e^{-A_x u} \Sigma_x \Sigma_x' e^{-A_x' u} du$. The time interval Δt in this paper is one month. Interested readers are referred to Fisher and Gilles (1996) for a detailed derivation of these results.

Because of the nonlinearity in the measurement equation (27), we employ the extended Kalman filter (EKF) to update the state variables. We denote the estimate of $H(X_t)$ using information up to time $t-1$ by $H(X_{t|t-1})$ and the first-order derivative of $H(X_t)$ with respect to X_i evaluated at $X_{t|t-1}$ by $\tilde{H}_t$ which has the form:

$$\tilde{H}_t = \left(\frac{\partial H(X_{t|t-1})}{\partial x_1} | X_{t|t-1} \quad \frac{\partial H(X_{t|t-1})}{\partial x_2} | X_{t|t-1} \quad \frac{\partial H(X_{t|t-1})}{\partial x_3} | X_{t|t-1} \right)'$$

where $\frac{\partial H(X_{t|t-1})}{\partial x_i} | X_{t|t-1}$ is a $M \times 1$ vector that each item represents the first-order derivative of the fitted m-period yield with respect to x_i. The specific expression for $\frac{\partial H(X_{t|t-1})}{\partial X_i} | X_{t|t-1}$ is given by equations (12)-(14). Then, the updating of state variables proceeds:

$$\begin{aligned} \tilde{V}_t &= \tilde{H}_t' P_{t|t-1} \tilde{H} + \sigma_e^2 I_M \\ P_{t|t-1} &= F P_{t-1|t-1} F' + Q_v \\ X_{t|t} &= X_{t|t-1} + P_{t|t-1} \tilde{H}_t \tilde{V}_t^{-1} e_t \end{aligned}$$

[5]In constructing the informationally-robust instruments, we rely on the Tealbook data which is released several years after the forecast. By the time of our estimation, the Tealbook data is updated to 2017. Therefore, our VAR model and subsequent impulse response analysis are estimated on data from 1991 to 2017.

As a result, the log-likelihood function of our SR-ATSM can be written in the form:

$$LLK = -\frac{1}{2}\sum_{t=1}^{T}\left[Mlog(2\pi) + log(|\tilde{V}_t|) + e_t'\tilde{V}_t^{-1}e_t\right] \tag{29}$$

Using the observed yield data, the parameters in $\{\delta_0, A_x^Q, \Sigma_x, A_x, \gamma_x, \sigma_e^2\}$ are selected to maximize the log-likelihood and generate filtered state variables $X_{t|t}$. Then, as described in Subsection 2.3 and 2.4, we collect the three state variables in $X_{t|t}$, Treasury yields for each announcement day and the day before, and the Tealbook data to construct our monthly series for informationally-robust instruments, which are used to estimate the impact of monetary policy shocks on macro-economy through the structural VAR and impulse response analyses.

4 Empirical Results

4.1 Factor Estimates

Figure 1 plots filtered estimate of our shadow short rate, which is the action factor, x_{1t}, plus a constant. To benchmark this estimate, we also plot the well-established Wu-Xia shadow rate series [6]. First, the short rate declined sharply in each of the two recessions in the sample period, as expected. Toward the end of the Great Recession, it fell below zero, marking the onset of the ZLB period, which ends around 2016 when the shadow rate turned to positive again. The shadow rate estimates track the movement of the observed federal funds rate closely when the latter is well above zero. However, when the actual federal funds rate was stuck at an ultra-low level near zero, the shadow rate approach can still yield a variable short rate reflective of monetary policy stance, during this so-called ZLB period.

Our shadow short rate tracks Wu-Xia shadow rate very well and the correlation between the two is over 0.99. However, there are several notable differences between the two: Wu-Xia rate shot up sharply and much more than ours around 1995; our rate went up briefly immediately after 2010 before

[6]The shaded areas indicate the NBER recessions in this and all following figures.

sharply declining again and was more negative than Wu-Xia rate during 2013-14.

Figure 2 plots the filtered estimate of the orthogonal path factor, x_{2t}. By definition, this factor influences the expected changes of future short rate but has zero impact on the current short rate, similar to the way the path factor is defined and extracted in Gürkaynak et al. (2005).[7] Note that it seems to have trended downward from the early 2000s to the post-Great Recession ZLB period, consistent with the notion that the Fed had to ramp up its efforts to stimulate the economy through forward guidance when the actual federal funds rate was stuck at the ZLB. However, this factor started to rebound around 2012-13 and has gradually returned to the level prior to the Great Recession, indicating the gradual monetary policy normalization during this time period.

Cyclically, the path factor in general declines sharply preceding each of the two recessions in 2001 and 2008, reflecting the market's forecast of the upcoming recessions, which often led to the so-called "inverted yield curve." In the middle of each recession, however, the path factor often rises, perhaps implying easing economic pressure, and it often rises sharply immediately after each recession.

It is worth noting that the estimated eigenvalue corresponding to the path factor is the largest among the three eigenvalues of matrix A_x in equation (1). Therefore, the path factor is the most persistent factor among the three factors. This result makes sense because the path factor essentially corresponds to the long run forecast of the future short rate, which has been found to bear a sizable slow-moving component (see Bauer and Rudebusch (2020) and Han and Ma (2024)).

Figure 3 plots the orthogonal term premium factor, which influences term premium but has no impact on either the current short rate or the expected future short rate. To clearly distinguish this series from the actual term premium, which is a function of all three factors as shown in equation (19), we also plot, in Figure 4, the term premium estimate of the ten-year treasury bond.

The orthogonal term premium factor in Figure 3 has been trending up

[7]Since the estimate of PL_{2m} is positive it moves in the same direction as the expected future short rate.

over our sample period. Interestingly, the term premium in Figure 4 has been trending down during the same period. Taken together, it suggests that the orthogonal term premium factor that has no effect on the short rate or the expected future short rate tend to drive up the term premium, suggesting a secular force that increases the risk perception of US treasury bond over time. However, the forces that influence the current short rate as well as the expected future short rate have worked in the opposite direction and seemed to have prevailed, resulting in a secular decline in the overall term premium since the 1990s, resembling the estimate in Wright (2011).

4.2 Impulse Responses to the Monetary Policy Shocks

Figure 5-7 report the estimated impulse responses of the macroeconomic and financial variables in the VAR to the three types of monetary policy shocks discussed above.[8]. The solid line represents the impulse response function, while the dashed lines depict the 90% confidence band, which is based on the wild bootstrap method used in Mertens and Ravn (2013), Gertler and Karadi (2015) and Swanson (2023). In each figure, the impact of a policy shock on the reference variable is normalized to 25 basis points(bp).

Figure 5 plots the impulse responses to the action shock. In response to a 25bp increase in the shadow federal funds rate, IP growth decreases by approximately 0.08% upon impact and continues to decline for about five months, reaching a peak decrease of 0.3% before it starts to rise. The declines, however, are not statistically significant.

The CPI inflation rate drops by approximately 0.16% upon impact and continues to decline, reaching 0.28% at its lowest point. These declines are all statistically significant. The CPI inflation starts to rise after a few months, with the increase becoming significant around the 20-month horizon. Meanwhile, commodity prices decline by 0.014% upon impact, with the decline reaching 0.02% at its lowest point after four months. These declines are all statistically significant. The excess bond premium (EBP) rises by approximately 0.02% in response to the shock and continues to increase for a few months before gradually falling, though these increases are not statistically

[8]In Appendix C, we report the results obtained by substituting our shadow short rate estimate for the Wu-Xia shadow rate in the VAR and find that the patterns of impulse responses remains largely unchanged

significant. These results are broadly consistent with findings in past studies (e.g., Gertler and Karadi (2015), Kaminska et al. (2021), and Swanson (2023)).

The shadow federal funds rate increases by about 0.44% at its peak around the 11-month horizon, with the magnitude of the increase similar to that documented in Swanson (2023). The impulse response of the two-year Treasury yield is smaller than that of the shadow federal funds rate but follows a similar pattern. However, in contrast to Kaminska et al. (2021), the responses of the ten-year Treasury yield to the action shock is positive, though relatively mild in scale, aligning with Swanson (2023).

Figure 6 plots the impulse responses to a tightening path shock. In response to a 25bp increase in the path shock, IP growth drops by 0.22% upon impact, with the decline becoming statistically significant after six months. The magnitude of the decline far exceeds that of the response to the action shock, reaching nearly 1% at its lowest point.

The CPI inflation, however, initially increases for the first four months before significantly declining. The inertia of inflation in response to tightening monetary policy is somewhat consistent with the findings documented in Bernanke and Gertler (1995). Commodity prices falls significantly below the baseline level in response to the shock, with the magnitude of the decline being twice as large as that of the responses to the action shock. The EBP rises substantially and significantly, exceeding 0.2% at its peak, in contrast to its small and insignificant response to the action shock. Both the decline in commodity prices and the increase in the EBP reflect financial markets' perception of elevated future interest rates and expectations of an economic slowdown. Forward guidance, aimed at influencing the expected path of the future short rate, is found to have significantly larger impacts on the economy than conventional monetary policy targeting the federal funds rate.

Following a tightening path shock, the shadow federal funds rate increases slightly for a few months before declining, with the changes remaining largely insignificant. This is consistent with the notion that the path shock should have minimal impact on the current short rate by construction. The two-year Treasury yield also gradually declines after the initial increase. The ten-year Treasury yield jumps up upon impact before declining, though it remains significantly above the baseline for nearly 50 months. These findings suggests that an unexpected upward revision of the future short rate path results in

significant credit contraction, as reflected by the increase in the EBP, and a large, persistent rise in the Treasury bond term premium. These substantial impacts on financial markets facilitate the transmission of the monetary policy shock, leading to significant declines in real economic activity.

Figure 7 plots the impulse responses to an increase in the orthogonal term premium shock. In response to a 25bp increase in this shock, IP growth rises for the first few months and fluctuates around the baseline level in the following years. CPI inflation drops below the baseline to approximately -0.1%, though the response is not statistically significant. Moreover, commodity prices decline, and the EBP rises in response to a positive term premium shock, suggesting credit contraction. These results align with the findings in Gertler and Karadi (2015), which suggests that monetary polices influence the macro-economy and financial markets through their impact on term premium.

Regarding Treasury yields, the ten-year yield, as the reference variable, returns to the baseline after the initial increase. Interestingly, the federal funds rate declines following a positive term premium shock, indicating a subsequent lower policy rate in response to the initial credit contraction. The opposing movements of the short rate and the term premium likely explain the insignificant response of the two-year yield.

In summary, both action and path shocks have significant effects on the economy in terms of IP growth and CPI inflation. Furthermore, path and term premium shocks have larger and more significant effects on commodity prices and the EBP than the action shock. Together, our results underscore the importance of unconventional monetary policies in influencing the economy, reinforcing and complementing the findings of Swanson (2023), which emphasizes the efficacy of the conventional monetary policy.

5 Conclusion

We build on the existing approaches of Swanson (2021, 2023), Kaminska et al. (2021), and etc. that use multiple asset prices to delineate different dimensions of monetary policy in efforts to assess the effectiveness of the conventional and unconventional monetary policies implemented by the Fed-

eral Reserve Bank during the period from 1991 to 2019. By working with a shadow rate affine term structure model and through a novel factor rotation, we structurally decompose the unexpected monetary policy shocks around the FOMC announcements, using a set of exogenous policy instruments, into three types of orthogonal shocks: the action shock, the path shock, and the term premium shock. We document significantly stronger effects of the path shock and the term premium shock, reflecting largely the unconventional monetary policies, on the macroeconomy and financial markets than the action shock corresponding to the conventional monetary policy. For instance, at its peak, the path shock affects industrial production growth or commodity prices twice as much as the action shock. The term premium shock has also been shown to have significant impact on commodity prices and credit supply. Our findings highlight the important role of the unconventional monetary policies.

References

Ang, A., Boivin, J., Dong, S., and Loo-Kung, R. (2011). Monetary policy shifts and the term structure. *The Review of Economic Studies*, 78(2):429–457.

Ang, A. and Piazzesi, M. (2003). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary economics*, 50(4):745–787.

Bauer, M. D. and Rudebusch, G. D. (2017). Resolving the spanning puzzle in macro-finance term structure models. *Review of Finance*, 21(2):511–553.

Bauer, M. D. and Rudebusch, G. D. (2020). Interest rates under falling stars. *American Economic Review*, 110(5):1316–54.

Bauer, M. D. and Swanson, E. T. (2023a). An alternative explanation for the "fed information effect". *American Economic Review*, 113(3):664–700.

Bauer, M. D. and Swanson, E. T. (2023b). A reassessment of monetary policy surprises and high-frequency identification. *NBER Macroeconomics Annual*, 37(1):87–155.

Bernanke, B. S., Boivin, J., and Eliasz, P. (2005). Measuring the effects of monetary policy: a factor-augmented vector autoregressive (favar) approach. *The Quarterly journal of economics*, 120(1):387–422.

Bernanke, B. S. and Gertler, M. (1995). Inside the black box: the credit channel of monetary policy transmission. *Journal of Economic perspectives*, 9(4):27–48.

Bernanke, B. S. and Kuttner, K. N. (2005). What explains the stock market's reaction to federal reserve policy? *The Journal of finance*, 60(3):1221–1257.

Campbell, J. Y. (1995). Some lessons from the yield curve. *Journal of economic perspectives*, 9(3):129–152.

Christiano, L. J., Eichenbaum, M., and Evans, C. L. (1999). Monetary policy shocks: What have we learned and to what end? *Handbook of macroeconomics*, 1:65–148.

Cieslak, A. and Povala, P. (2016). Information in the term structure of yield curve volatility. *The Journal of Finance*, 71(3):1393–1436.

Clarida, R., Gali, J., and Gertler, M. (1999). The science of monetary policy: a new keynesian perspective. *Journal of economic literature*, 37(4):1661–1707.

Cochrane, J. H. and Piazzesi, M. (2002). The fed and interest rates—a high-frequency identification. *American economic review*, 92(2):90–95.

Cochrane, J. H. and Piazzesi, M. (2005). Bond risk premia. *American economic review*, 95(1):138–160.

Dai, Q. and Singleton, K. J. (2000). Specification analysis of affine term structure models. *The journal of finance*, 55(5):1943–1978.

Duffee, G. R. (2011). Information in (and not in) the term structure. *The Review of Financial Studies*, 24(9):2895–2934.

Duffie, D. and Kan, R. (1996). A yield-factor model of interest rates. *Mathematical finance*, 6(4):379–406.

Duffie, D. and Singleton, K. J. (2003). *Credit risk: pricing, measurement, and management*. Princeton university press.

Fisher, M. and Gilles, C. (1996). Estimating exponential-affine models of the term structure. *Unpublished working paper. Federal Reserve Bank of Atlanta*.

Gertler, M. and Karadi, P. (2015). Monetary policy surprises, credit costs, and economic activity. *American Economic Journal: Macroeconomics*, 7(1):44–76.

Gilchrist, S., Yue, V., and Zakrajšek, E. (2019). Us monetary policy and international bond markets. *Journal of Money, Credit and Banking*, 51:127–161.

Gilchrist, S. and Zakrajšek, E. (2012). Credit spreads and business cycle fluctuations. *American economic review*, 102(4):1692–1720.

Gürkaynak, R. S., Sack, B., and Wright, J. H. (2007). The us treasury yield curve: 1961 to the present. *Journal of monetary Economics*, 54(8):2291–2304.

Gürkaynak, R. S., Sack, B. P., and Swanson, E. T. (2005). Do actions speak louder than words? the response of asset prices to monetary policy actions and statements. *International Journal of Central Banking*, 1:55–93.

Gürkaynak, R. S. and Wright, J. H. (2012). Macroeconomics and the term structure. *Journal of Economic Literature*, 50(2):331–367.

Han, Y. and Ma, J. (2024). Estimating the interest rate trend in a shadow rate term structure model. *Working Paper*.

Joslin, S., Singleton, K. J., and Zhu, H. (2011). A new perspective on gaussian dynamic term structure models. *The Review of Financial Studies*, 24(3):926–970.

Kaminska, I., Mumtaz, H., and Šustek, R. (2021). Monetary policy surprises and their transmission through term premia and expected interest rates. *Journal of Monetary Economics*, 124:48–65.

Kim, D. H. and Orphanides, A. (2012). Term structure estimation with survey data on interest rate forecasts. *Journal of Financial and Quantitative Analysis*, 47(1):241–272.

Krippner, L. (2015). *Zero lower bound term structure modeling: A practitioner's guide*. Springer.

Kuttner, K. N. (2001). Monetary policy surprises and interest rates: Evidence from the fed funds futures market. *Journal of monetary economics*, 47(3):523–544.

Mertens, K. and Ravn, M. O. (2013). The dynamic effects of personal and corporate income tax changes in the united states. *American economic review*, 103(4):1212–1247.

Miranda-Agrippino, S. and Ricco, G. (2021). The transmission of monetary policy shocks. *American Economic Journal: Macroeconomics*, 13(3):74–107.

Miranda-Agrippino, S. and Ricco, G. (2023). Identification with external instruments in structural vars. *Journal of Monetary Economics*, 135:1–19.

Nakamura, E. and Steinsson, J. (2018). High-frequency identification of monetary non-neutrality: the information effect. *The Quarterly Journal of Economics*, 133(3):1283–1330.

Piazzesi, M. (2010). Affine term structure models. In *Handbook of financial econometrics: Tools and Techniques*, pages 691–766. Elsevier.

Ramey, V. A. (2016). Macroeconomic shocks and their propagation. *Handbook of macroeconomics*, 2:71–162.

Rogers, J. H., Scotti, C., and Wright, J. H. (2018). Unconventional monetary policy and international risk premia. *Journal of Money, Credit and Banking*, 50(8):1827–1850.

Romer, C. D. and Romer, D. H. (2004). A new measure of monetary shocks: Derivation and implications. *American economic review*, 94(4):1055–1084.

Rudebusch, G. D. and Wu, T. (2008). A macro-finance model of the term structure, monetary policy and the economy. *The Economic Journal*, 118(530):906–926.

Stock, J. H. and Watson, M. W. (2012). Disentangling the channels of the 2007-2009 recession. Technical report, National Bureau of Economic Research.

Stock, J. H. and Watson, M. W. (2018). Identification and estimation of dynamic causal effects in macroeconomics using external instruments. *The Economic Journal*, 128(610):917–948.

Swanson, E. T. (2021). Measuring the effects of federal reserve forward guidance and asset purchases on financial markets. *Journal of Monetary Economics*, 118:32–53.

Swanson, E. T. (2023). The macroeconomic effects of the federal reserve's conventional and unconventional monetary policies. Technical report, National Bureau of Economic Research.

Wright, J. H. (2011). Term premia and inflation uncertainty: Empirical evidence from an international panel dataset. *American Economic Review*, 101(4):1514–1534.

Wu, J. C. and Xia, F. D. (2016). Measuring the macroeconomic impact of monetary policy at the zero lower bound. *Journal of Money, Credit and Banking*, 48(2-3):253–291.

Wu, J. C. and Xia, F. D. (2020). Negative interest rate policy and the yield curve. *Journal of Applied Econometrics*, 35(6):653–672.

Figure 1: **Shadow Short Rate**

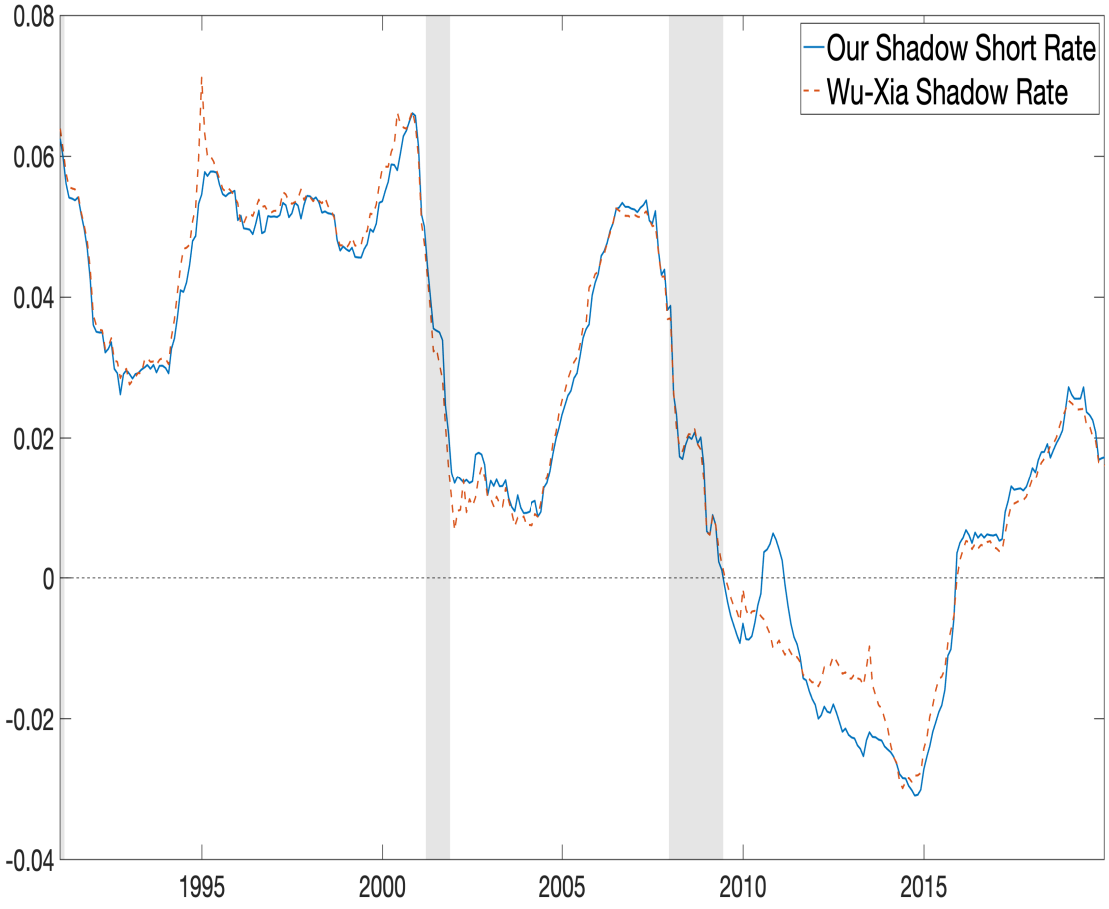


Figure 2: **Orthogonal Path Factor**

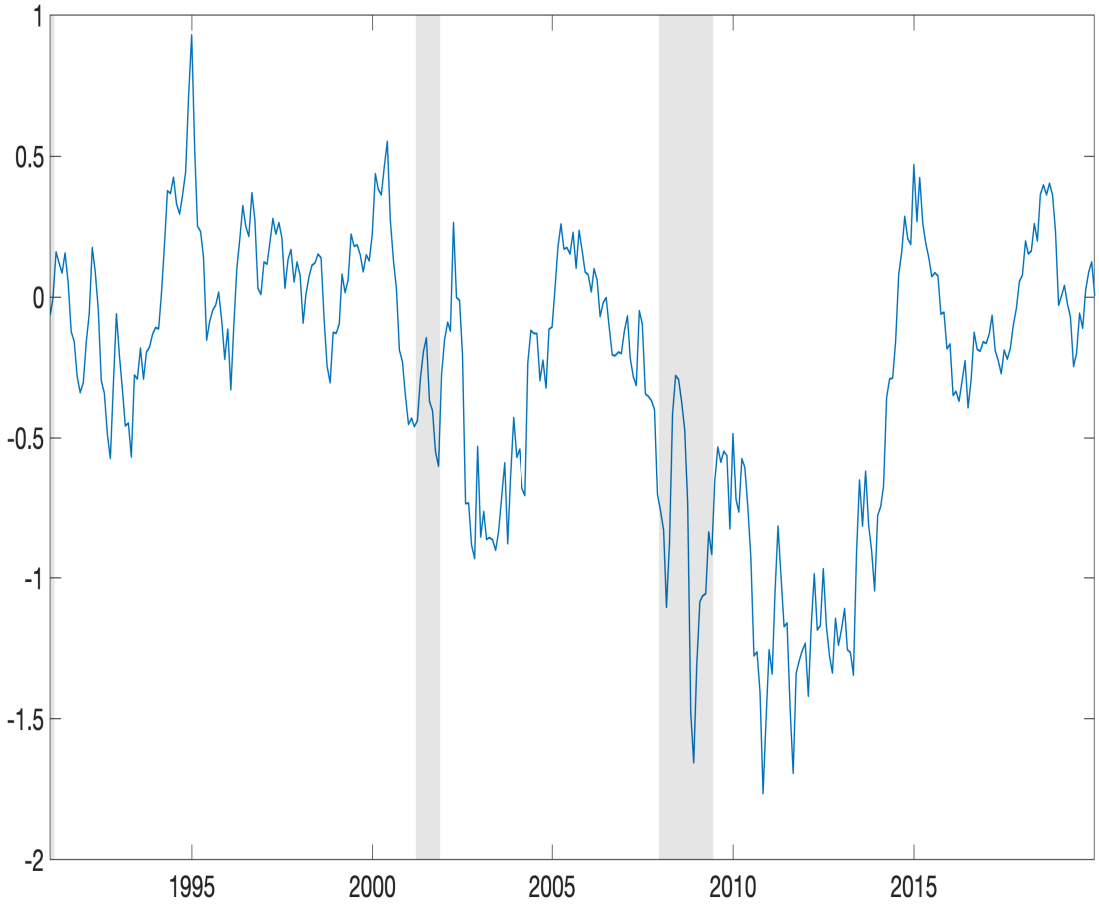


Figure 3: **Orthogonal Term Premium Factor**

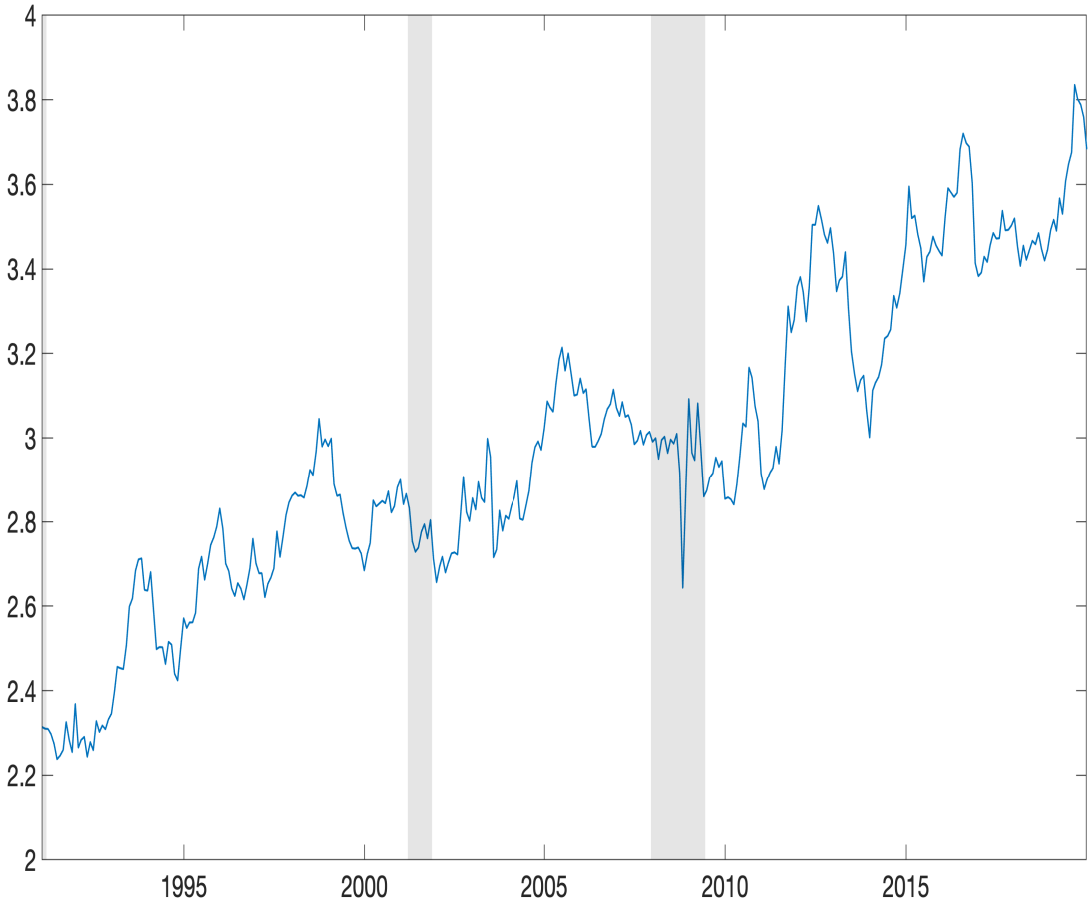


Figure 4: **Ten-year Term Premium Estimate**

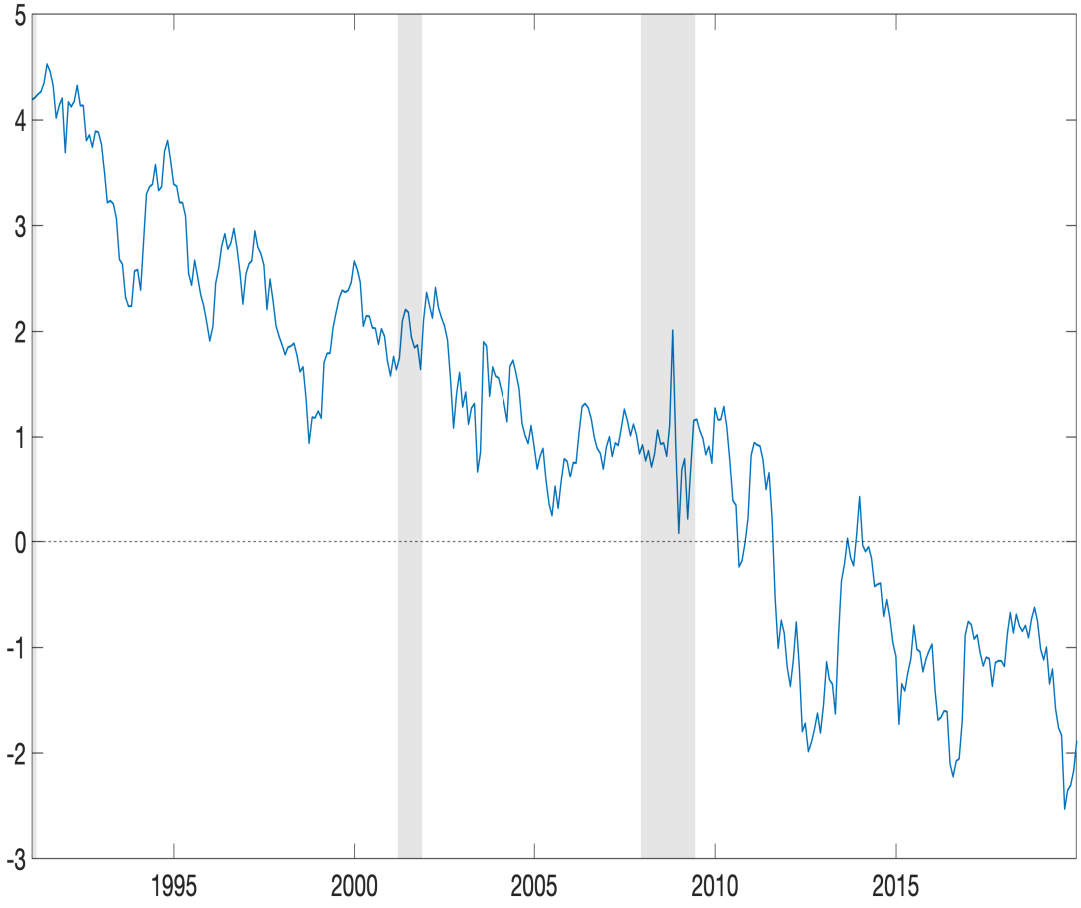


Figure 5: **Impulse Responses to an Action Shock**

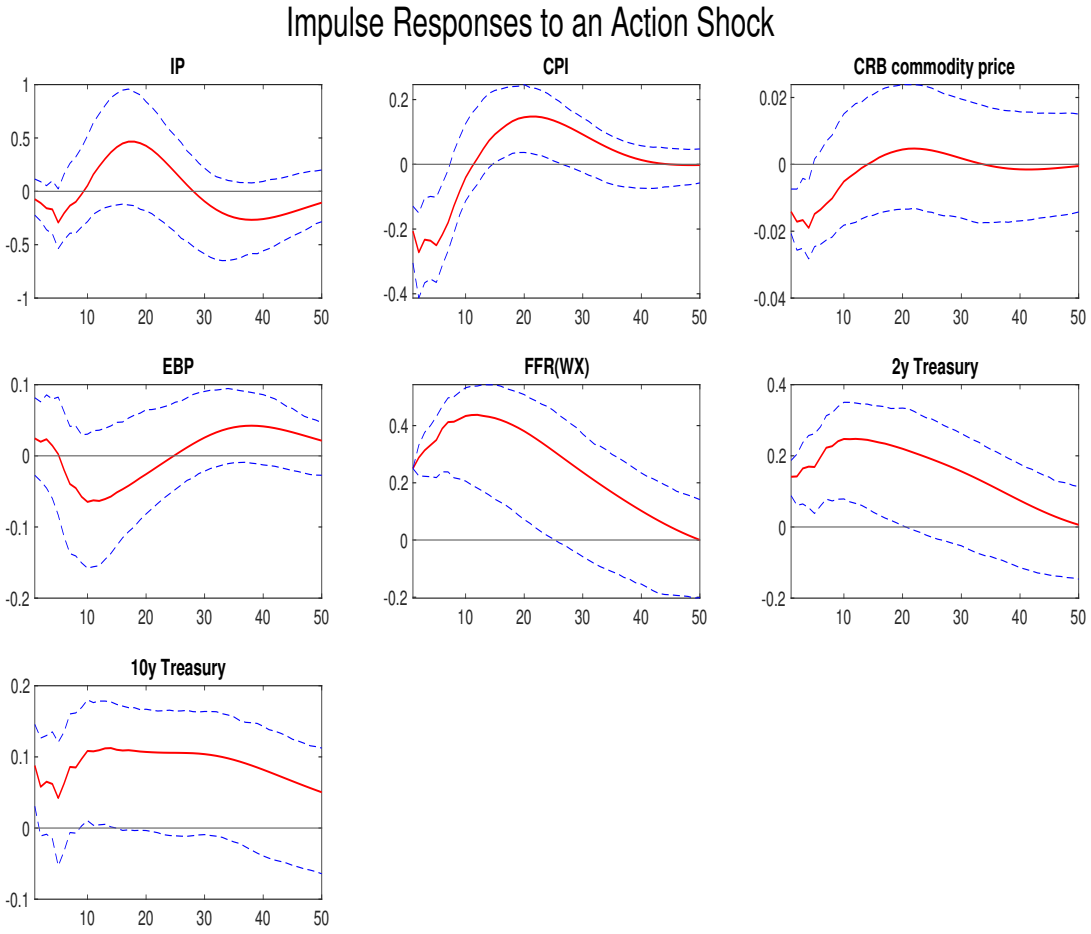


Figure 6: **Impulse Responses to a Path Shock**

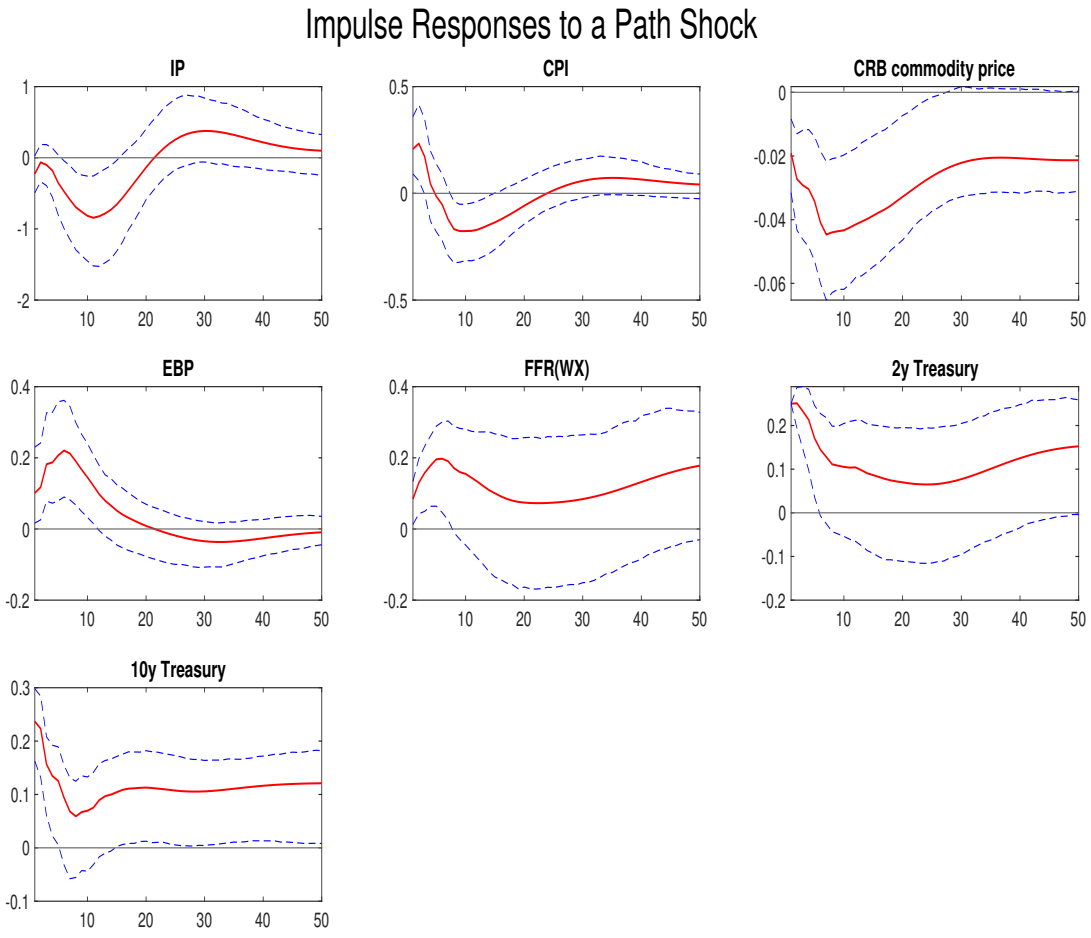
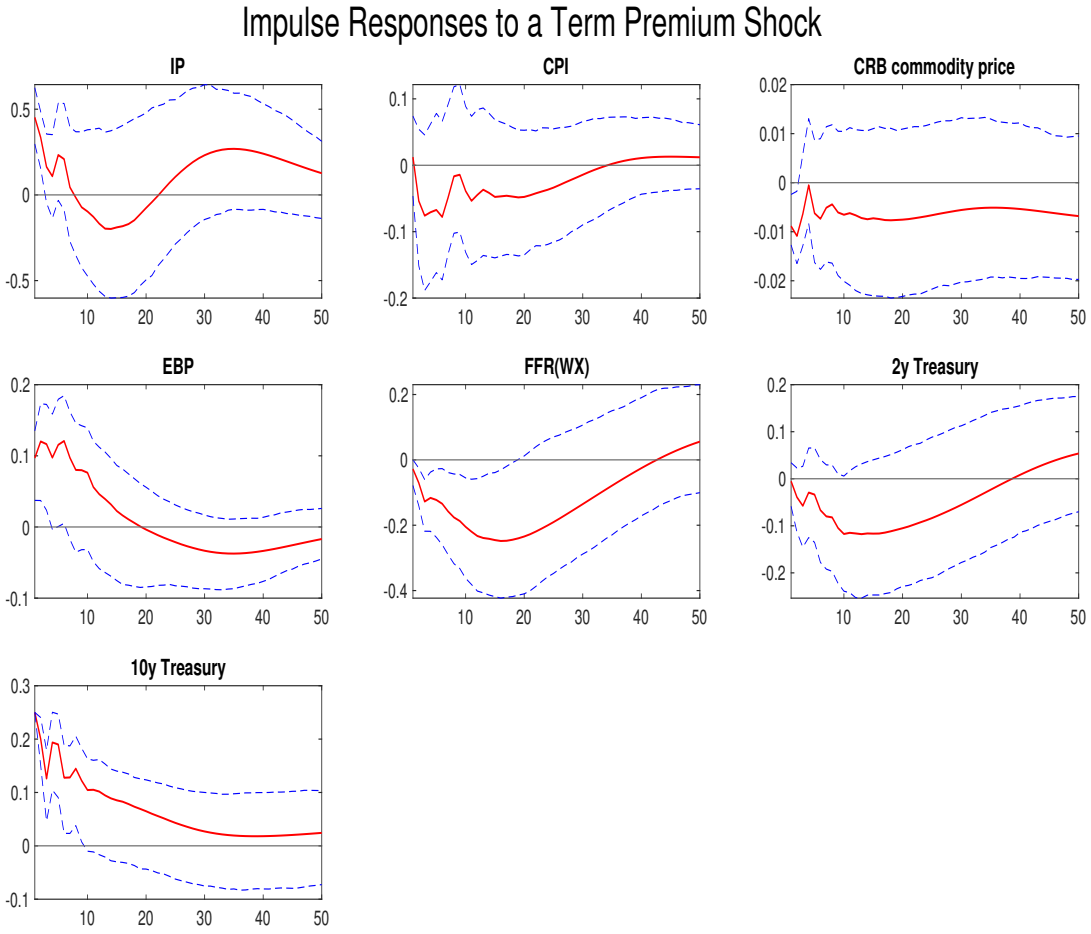


Figure 7: **Impulse Responses to a Term Premium Shock**



Appendix A Derivation of Forward Rates

As shown in Duffie and Kan (1996), the bond price takes an exponential affine form:

$$P_t(m) = e^{a(m)+b(m)\cdot X_t} \tag{A.1}$$

where $a(m)$ and $b(m)$ are solutions to ODEs:

$$\begin{aligned}\frac{db}{dm} &= -bA_x^Q - \delta_1' \\ \frac{da}{dm} &= bA_x^Q\gamma_x^Q + \frac{1}{2}b\Sigma_x\Sigma_x'b' - \delta_0\end{aligned}$$

The closed-form solution for b is:

$$b(m) = -\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m}) \tag{A.2}$$

And plugging $b(m)$ into the second ODE yields:

$$\frac{da}{dm} = -\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})A_x^Q\gamma_x^Q + \frac{1}{2}\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x\Sigma_x'(I - e^{-A_x^{Q'} m})(A_x^{Q'})^{-1}\delta_1 - \delta_0 \tag{A.3}$$

Moreover, $\frac{db}{dm}$ can be computed:

$$\begin{aligned}\frac{db}{dm} &= -\frac{d}{dm}\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m}) \\ &= \frac{d}{dm}\delta_1'(A_x^Q)^{-1}e^{-A_x^Q m} \\ &= \delta_1'(A_x^Q)^{-1}\frac{d}{dm}e^{-A_x^Q m} \\ &= \delta_1'(A_x^Q)^{-1}(-A_x^Q)e^{-A_x^Q m} \\ &= -\delta_1'e^{-A_x^Q m}\end{aligned}$$

Then, the m-period ahead forward rate can be computed as:

$$
\begin{aligned}
f_{t,m} &= -\frac{d}{dm} log P_t(m) \\
&= -\frac{d}{dm} a(m) - \frac{d}{dm} b(m) \cdot X_t \\
&= \delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})A_x^Q \gamma_x^Q - \frac{1}{2}\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x\Sigma_x'(I - e^{-A_x^{Q'} m})(A_x^{Q'})^{-1}\delta_1 \\
&+ \delta_0 + \delta_1' e^{-A_x^Q m} \cdot X_t \\
&= \delta_0 + \delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})A_x^Q \gamma_x^Q + \delta_1' e^{-A_x^Q m} \cdot X_t \\
&- \frac{1}{2}\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x\Sigma_x'(I - e^{-A_x^{Q'} m})(A_x^{Q'})^{-1}\delta_1 \\
&= \delta_0 + \delta_1'\gamma_x^Q - \delta_1' e^{-A_x^Q m}\gamma_x^Q + \delta_1' e^{-A_x^Q m} \cdot X_t - \frac{1}{2}\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x\Sigma_x'(I - e^{-A_x^{Q'} m})(A_x^{Q'})^{-1}\delta_1 \\
&= \delta_0 + \delta_1'[\gamma_x^Q + e^{-A_x^Q m}(X_t - \gamma_x^Q)] - \frac{1}{2}\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x\Sigma_x'(I - e^{-A_x^{Q'} m})(A_x^{Q'})^{-1}\delta_1
\end{aligned}
$$

Equation (8) and (9) has indicated:

$$
\begin{aligned}
E_t^Q[s_{t+m}] &= \delta_0 + \delta_1' E_t^Q[X_{t+m}] \\
&= \delta_0 + \delta_1'[\gamma_x^Q + e^{-A_x^Q m}(X_t - \gamma_x^Q)]
\end{aligned} \tag{A.4}
$$

Therefore, $f_{t,m}$ can be written as:

$$
f_{t,m} = E_t^Q[s_{t+m}] - \frac{1}{2}\delta_1'(A_x^Q)^{-1}(I - e^{-A_x^Q m})\Sigma_x\Sigma_x'(I - e^{-A_x^{Q'} m})(A_x^{Q'})^{-1}\delta_1 \tag{A.5}
$$

where the second term is our convexity adjustment term (CAT), which can be represented in another form as in equation (7):

$$
CAT_m = \frac{1}{2}\Big[\int_0^m \delta_1' e^{-A_x^Q u} du\Big]\Sigma_x\Sigma_x'\Big[\int_0^m e^{-A_x^{Q'} u}\delta_1 du\Big]
$$

In the end, we have the m-period ahead shadow forward rate in equation (6):

$$
f_{t,m} = E_t^Q[s_{t+m}] - CAT_m
$$

Appendix B Invariant Affine Transformation

Let X_t be the factor that determines yields in an affine term structure model. It can be rotated to $\tilde{X}_t$ as $\tilde{X}_t = C + DX_t$ with an invertible matrix D and,

$$\begin{aligned} d\tilde{X}_t &= D \cdot dX_t \\ X_t &= D^{-1}(\tilde{X}_t - C) \end{aligned}$$

Plugging the transformation into the P-process (equation (1)) and the Q-process (equation (3)) yields:

$$d\tilde{X}_t = DA_x(\gamma_x + D^{-1}C - D^{-1}\tilde{X}_t)dt + D\Sigma_x dz_t \tag{B.1}$$

$$d\tilde{X}_t = DA_x^Q(\gamma_x^Q + D^{-1}C - D^{-1}\tilde{X}_t)dt + D\Sigma_x dz_t^Q \tag{B.2}$$

Substituting $\tilde{X}_t$ into equation (2) produces:

$$\begin{aligned} s_t &= \delta_0 + \delta_1' D^{-1}(\tilde{X}_t - C) \\ &= \delta_0 - \delta_1' D^{-1}C + \delta_1' D^{-1}\tilde{X}_t \end{aligned} \tag{B.3}$$

Therefore, the new rotation builds on the factor $\tilde{X}_t$ becomes:

$$d\tilde{X}_t = \tilde{A}_x(\tilde{\gamma}_x - \tilde{X}_t)dt + \tilde{\Sigma}_x dz_t \tag{B.4}$$

$$d\tilde{X}_t = \tilde{A}_x^Q(\tilde{\gamma}_x^Q - \tilde{X}_t)dt + \tilde{\Sigma}_x dz_t^Q \tag{B.5}$$

$$s_t = \tilde{\delta}_0 + \tilde{\delta}_1'\tilde{X}_t \tag{B.6}$$

where

$$\begin{aligned} \tilde{A}_x^Q &= DA_x^Q D^{-1} & \tilde{\gamma}_x^Q &= D\gamma_x^Q + C \\ \tilde{A}_x &= DA_x D^{-1} & \tilde{\gamma}_x &= D\gamma_x + C \\ \tilde{\delta}_1' &= \delta_1' D^{-1} & \tilde{\delta}_0 &= \delta_0 - \delta_1' D^{-1}C \\ \tilde{\Sigma}_x &= D\Sigma_x \end{aligned}$$

Therefore, the factors can be rotated without altering the fit of the yield curve and the parameters in C and D can be chosen to anchor the factors.

Appendix C Impulse Responses Using Our Shadow Short Rate

Figure C.1: **Impulse Responses to an Action Shock**

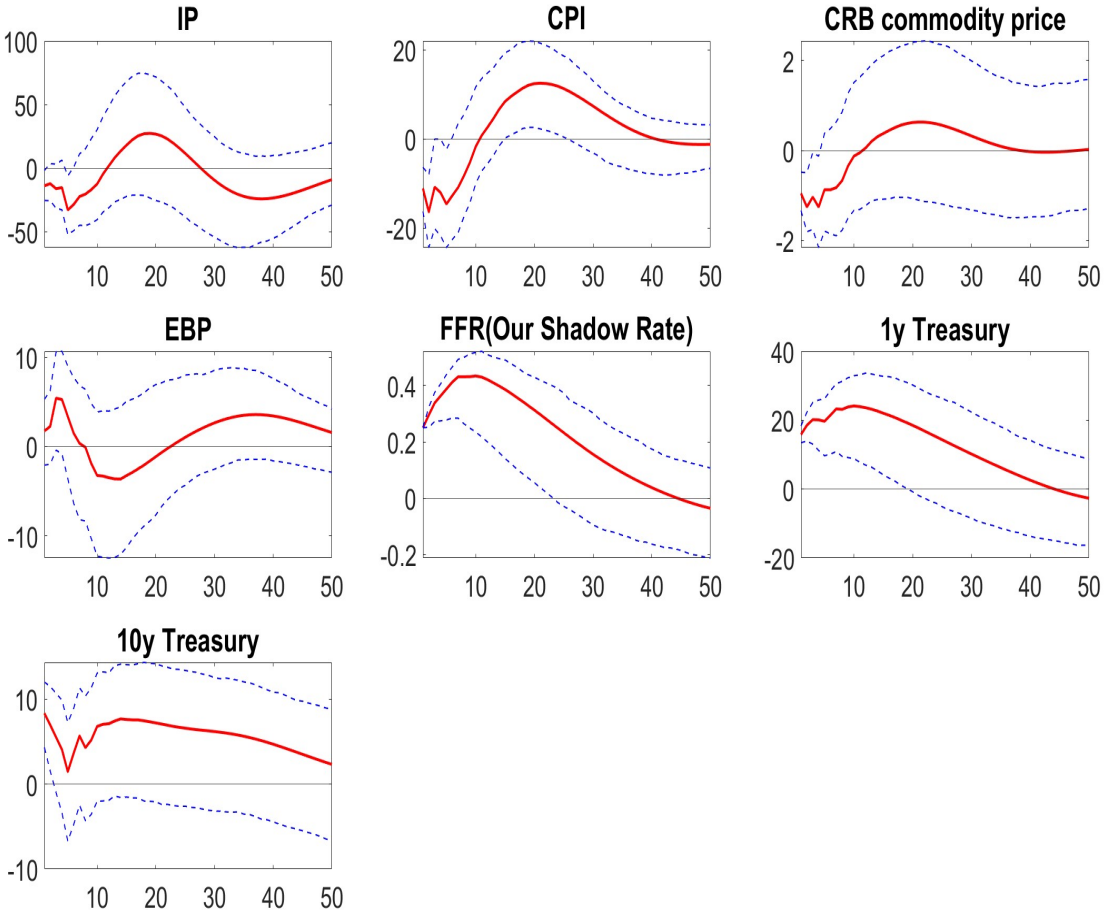


Figure C.2: **Impulse Responses to a Path Shock**

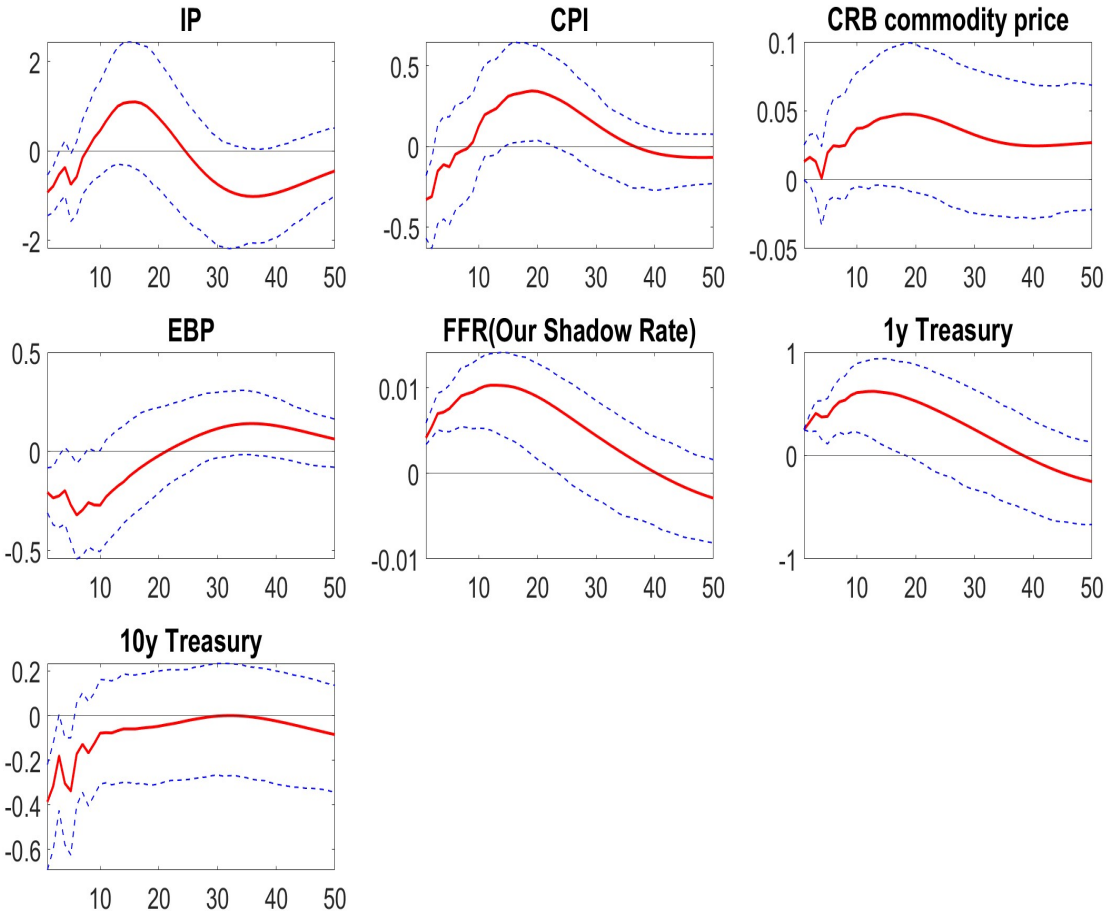


Figure C.3: **Impulse Responses to a Risk Premium Shock**

