Valuing Commercial Real Estate and its Real Options

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Abstract

Using an extension of standard option pricing theory for sporadically traded assets, this paper presents a new methodology for valuing commercial real estate (CRE) and real options on CRE. We apply this method to value multifamily apartments in Los Angeles, California over the 2001 - 2019 time period and compare it to a standard hedonic model. Our model performs better than the hedonic model both in- and out-of-sample, maintaining robustness despite changing market conditions. We also illustrate how to use our approach to value a European call option to purchase a CRE.

Keywords: Commercial real estate; real options; fair value; arbitrage-free value; hedonic model.

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1 Introduction

Although the commercial real estate (CRE) market is the fourth largest asset market[1] in the U.S., having a value of $22.5 trillion with $5.9 trillion debt outstanding[2](2023Q4) [4], little is known about its price dynamics. Understanding what drives CRE price fluctuations is an important research question for several reasons. First, it is a significant part of the macroeconomy: banks and thrifts hold 50% of CRE debt, with government sponsored enterprises accounting for an additional 17%. It is also an important allocation in institutional portfolios, typically representing 8% to 10%[3]. Second, it requires regular pricing: 30% of CRE outstanding debt ($1.7 trillion) is expected to mature near term (2024 to 2026). Since CRE debt is typically refinanced, the CRE value underlying this debt will need to be repriced to obtain a reliable estimate of credit exposure and default risk. An understanding of pricing fundamentals also helps decision-makers in making regulatory capital decisions, establishing loan limits as well as providing a benchmark in the resolution of lawsuits. Third, derivatives on CRE are prevalent and only increasing in prominence, expanding the need for regular pricing. As of November 2024, Commercial Mortgage-Backed Securities (CMBS) securities had a market value of $652.8 billion[4]. As the value of the underlying properties backing CMBS loans fluctuate, so does CMBS risk and return. Other derivatives include the right of first refusal option and the lease to own option, where the price is established at option origination. Thus, CRE is heavily integrated in financial markets, making its accurate and timely pricing paramount.

Despite its importance, CRE is a notoriously difficult asset class to price. The challenge is often due to how real estate is inherently heterogeneous in its property attributes, particularly location. It also exhibits high transaction costs, as well as illiquidity with no short sales permitted. Furthermore, the presence of market frictions further compounds issues in predicting changes of CRE prices.

A key challenge to pricing commercial real estate, and a focus of our paper, is that it is a sporadically traded asset class. Generally, there may be two approaches to estimating CRE values. The first, and the more traditional approach, is to use

[1]The three largest asset markets in the U.S. are equities, residential real estate, and Treasury securities respectively

[2]Federal Reserve, April 2024 Financial Stability Report

[3]https://www.invesco.com/us/en/insights/private-real-estate-income-returns.html

[4]https://www.spglobal.com/_assets/documents/ratings/research/101610491.pdf

a statistical model to determine current prices based on regressions using past CRE sales data and their characteristics. The second would be to use a pricing model based on the Black-Scholes-Merton option pricing theory.

The problem with the first approach is that its underlying economic theory can be limited to just the selection of a property's covariates, so the model will fail to capture a structural change in the market. In contrast, if constructed properly, the option pricing approach will capture these structural shifts. However, the main benefit of the option pricing approach is that the model will not only provide the current value of a CRE, but it can be used to price derivatives on the CREs if successful. One example would be options to purchase or lease agreements. The purpose of this paper is to construct and empirically test a valuation model for CREs based on the option pricing framework.

Option theory prices assets by synthetic construction in a continuous trading, frictionless, and competitive market setting. It is well-known how to modify this theory for market frictions (see Karatzas 1997[24], Jarrow 2006[18], Jarrow 2021[22]), so frictions in trading CREs are not a critical limitation when applying this theory. The key obstacle that remained is that CREs do not trade continuously, but only sporadically. However, this limitation has recently been overcome by Jarrow 2023[23], whose approach we adopt in this paper.

The idea underlying the extended pricing methodology is quite intuitive and readily applies to CRE markets. The approach can be explained as follows. Fix a particular CRE to be valued, call it apartment A (aptA). As with any asset, aptA's risks can be decomposed into market and idiosyncratic.[5] Next, suppose there is a real estate investment trust (REIT) trading whose market risks reflect those exhibited by aptA. Note that the REIT trades continuously, so we can value the REIT using standard option pricing theory. Then, the only remaining risks of aptA are idiosyncratic, which can be priced under the statistical probability measure.

Consequently, we can value aptA by first rescaling the REIT's value to make it the same magnitude as aptA's, and then adjust this rescaled value to account for aptA's unique characteristics. These adjustments can be based on past (sporadic) sales of aptA and comparable apartments, determined using a statistical model based on covariates. Although a statistical model is used for this adjustment, it

[5]For an analogy, think of Ross' APT for equity returns.

differs from the purely statistical valuation methodology, in that its usage is consistent with the underlying economic theory. Simply stated, this is the modeling analogy to pricing aptA by first looking at the average value of apartments in aptA's geographical area, and then adjusting aptA's value by its unique characteristics relative to that average.

We compare the performance of our model to the hedonic model, traditionally used in the existing literature to price both residential and commercial real estate. The price of real estate in the hedonic model is explained using property attributes, location characteristics, and features of the transaction. Most attributes used in the hedonic model are fixed effects that do not vary over time. While these attributes are useful in accounting for CRE heterogeneity and in estimating the current market price, they are not a source of price variation over time, hindering accurate forecasting of future CRE prices. Until recently, the lack of historical price data has prevented researchers from conducting meaningful out-of-sample comparisons. Consequently, the prior predictability evidence consists mainly of in-sample comparisons that exploit cross-sectional differences. In contrast to the traditional hedonic model, our proposed model includes time-varying attributes that allows us to capture any structural shifts in the real estate market.

In addition to this ability to maintain accurate valuations over time, our model not only estimates the value of current CRE, but also the price of derivatives on the CREs such as options to purchase or lease agreements. Although a hedonic pricing model can also price these options, a different hedonic model is required for each new derivative analyzed. In contrast, our model will work for both values of CREs and options. A distinguishing feature of our study is that we also investigate the robustness of our pricing model relative to the hedonic pricing model on an out-of-sample basis. As part of this process, we will also explore the extent to which our estimated parameters are stable over time.

We apply this approach to value multifamily CREs in Los Angeles, California over the 2001 - 2019 time period. This span captures the business cycles before, during, and after the credit crisis. We estimate our valuation model over the three sub-periods: (1) 07/31/2001 - 07/01/2006, (2) 01/01/2010 - 01/01/2015, and (3) 01/02/2015 - 12/31/2019, reflecting these different markets. In-sample, our valuation model, after the consideration of market frictions and financing constraints, works well in pricing multifamily CREs. Across the three periods, the mean pricing error is always equal to zero, and the median pricing error is

close to zero. The standard deviation of the pricing error is approximately 20 percent in each of the three periods. Furthermore, our valuation model is robust across the different markets in our observation period, each sub-period having similar pricing error distributions.

We also explored our model's out-of-sample performance and did a comparison of the in-sample and out-of-sample performances of our model with the standard hedonic model, where the latter was estimated using the same set of covariates employed as in our model. For the in-sample comparison, our model's R^2s were uniformly larger than the hedonic model's R^2s for the three sub-periods mentioned above (0.822 vs 0.757, 0.8756 vs 0.777, and 0.809 vs 0.772).

For the out-of sample comparison, we did a period-ahead prediction using our model, and a day-ahead prediction using the hedonic model. The difference in the prediction intervals gives the hedonic model a significant advantage. Despite this, our model exhibited a much smaller decrease in the out-of-sample R^2s from the in-sample R^2s. The hedonic model's R^2s decreased by over 50% whereas our model's R^2s decreased by only 19% in period 2 and 20% in period 3. More importantly, in the apples-to-apples out-of-sample R^2 comparison, for period 2 and period 3, our model's R^2 were twice as large (0.713 vs 0.277, 0.655 vs 0.349). This better out-of-sample prediction exhibited by our model is due to the fact that our model is constructed to capture changing market conditions via the inclusion of a market REIT index and default-free zero-coupon bond prices of various maturities.

To illustrate the use of our valuation method for pricing real options on CREs, we provide a case study. The case study is based on a European call option written on Regency Detroit Apartments in December 1992 with a maturity of one month and a strike price equal to the immediate purchase price of \$3,150,000. This option was priced at 1.59% of the purchase price's value, or \$50,000. We value similar hypothetical call options written on Regency Detroit Apartments on three different dates, one for each of our different observation periods. Using our model's estimated parameters, on these three dates, the options were valued at 1.46, 2.32, and 2.38 percent of the property's value, which ultimately supports the model's validity.

An outline of the paper is as follows. Section 2 gives a literature review, while section 3 presents the valuation model. Section 4 and 5 present the data and the results for the empirical testing of the valuation model. Section 6 provides a comparison of our valuation model with a typical hedonic model used in the

existing literature. Section 7 presents an illustration of how this model can be used to price real options on CRE, and section 8 concludes.

2 Literature Review

The hedonic price model is the predominant method for valuing both residential and commercial real estate in the existing literature (Rosen 1974[33]). The hedonic approach entails regressing sale prices on property characteristics across different properties at a single point in time. The resulting coefficients represent the marginal contribution of each attribute to the sales price. Prior studies have also used a repeat sales model which is a more specialized version of the hedonic model (see for example Bailey, Muth, and Nourse 1963[2], Case and Shiller 1989[7]). The repeat sales model focuses on the price changes of the same property over time. The primary advantage of this method over the hedonic methodology is that it controls for time-invariant unobserved property attributes that affect prices. However, the repeat sales model omits many transactions; only properties that have transacted at least twice are included in the estimation process. As such, the resulting repeat sales sample of properties is non-random and therefore may not be representative of the universe of all properties; sample selection bias exists (Hwang and Quigley (2004[21]). This also causes repeat-sales valuations to be subject to substantial revisions as new data becomes available (Clapham et al 2006[8]. These limitations have led academics to prefer the hedonic model for pricing real estate.

While most studies have focused on residential real estate, a few studies have examined commercial real estate. Hoag 1980[20] was the first to use the hedonic model to construct a CRE index for industrial real estate. Using NCREIF data, Miles, Cole and Guilkey 1990[27] extended this transaction-based index to several property types. Both authors demonstrate that CRE is more volatile than prices based on appraisals. Sagi 2021[35] constructs an equilibrium search-based illiquid asset pricing model. He finds that idiosyncratic risk is a primary driver of CRE prices. The premise underlying this transaction risk is the combination of an illiquid real estate market coupled with heterogenous valuation beliefs among market participants. In a related study, Koijen et al 2025[25] estimate a hedonic pricing model using institutional grade real estate transactions in conjunction with

buyer and seller characteristics for four CRE property types - apartments, office, retail, and industrial - to ascertain whether different valuations are associated with buyer and seller characteristics. They find that these characteristics are an important valuation driver in addition to a salient determinant of the likelihood that a sale will occur. Their result is consistent with the earlier work of Harding et al 2003[16][17] on housing prices. The authors' findings also complement the literature on the role of asymmetric information in buyer-seller negotiations and the resulting valuation consequences (Garmaise and Moskowitz 2004[12]; Kurlat and Stroebel 2015[26]; Stroebel 2016[36]). Using differences in property tax assessment ratios across jurisdictions to proxy for information quality, Garmaise and Moskowitz 2004[12] find that when a higher degree of asymmetric information about local market conditions exists, there is a greater likelihood that buyers are local. Uninformed buyers also favor properties that have longer income histories. Badarinza, Ramadorai, and Shimizu 2020[3] show that buyers prefer to transact with sellers from their own or proximate countries when transacting large cross-border CRE under asymmetric information.

In one of the few CRE studies to use a repeat sales model, Wheaton et al 2009[38] uses 86 repeat sales transactions over 1899 to 1999 to estimate CRE price appreciation in Manhattan. The authors find that CRE prices for Manhattan declined slightly over this period after adjusting for inflation.

In examining CRE, a few studies, most notably Plazzi et al. 2010[31] and more recently Boudry et al. 2024[5] have used an alternative valuation paradigm pricing CRE using a log-linearized version of the discounted rent model (Campbell and Shiller 1988[6]). Plazzi et al. 2010[31] finds that expected returns vary over time due to its income return and changes in the state of the economy. Boudry et al 2024[5] offer confirmatory evidence that cash flows constitute signals of CRE fundamental value. For a more detailed review of the real estate pricing literature please see Ghysels, et al 2013[14] and Hill 2013[19].

3 The Model

The model is based on Jarrow 2023[23]. Details with respect to the fundamental theorems of asset pricing invoked below can be found in Jarrow 2021[22].

3.1 The General Model

Given is a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} \coloneqq (\mathcal{F}_t)_{t\in[0,\mathbb{T}]}, \mathbb{P})$ where $\mathcal{F} = \mathcal{F}_{\mathbb{T}}$, the filtration $\mathbb{F}$ satisfies the usual hypotheses, and $\mathbb{P}$ is the statistical probability measure. Traded in the market over $[0, \mathbb{T}]$ are a REIT index, an entire term structure of default-free zero-coupon bonds, and a collection of CREs $i = 1, ..., I$. The markets for the stock, bonds, and CREs are assumed to be *competitive* and *frictionless*. Competitive means that traders act as price takers, and frictionless means that there are no transaction costs and no trading constraints. The stock and bonds are assumed to be trade continuously, the CREs only sporadically. The sporadic trading of CREs makes the CRE market incomplete.

We let S_t denote the REIT's time t market price, assumed to be a continuous semi-martingale adapted to $\mathbb{F}$. The REIT pays dividends, which without-loss-of-generality, are assumed to be reinvested in the REIT (see Jarrow [22], p. 35). The cumulative gains process (REIT stock price plus accumulated dividends) is $S_t^* \coloneqq S_t e^{\delta t}$ where $\delta \geq 0$ is the constant dividend yield per unit time.

We denote the time t value of a default-free zero-coupon bond maturing at time T by $p(t, T)$ for $0 \leq t \leq T \leq \mathbb{T}$. We assume that $p(t, T) > 0$ and it is $\mathbb{F}$ adapted for all $T \in [0, \mathbb{T}]$. The money market account value is given by $B_t = e^{\int_0^t r_s ds}$ where r_s is the default-free spot rate of interest. We assume that r_t is $\mathbb{F}$ adapted.

Let $\mathbb{F}^s = (\mathcal{F}_t^s)_{0\leq t\leq \mathbb{T}}$ be the filtration generated by S_t and $p(t, T)$ for all $T \in [0, \mathbb{T}]$. We assume that $\mathbb{F}^s \subset \mathbb{F}$ and $\mathbb{F}^s \neq \mathbb{F}$. This means that the CRE has additional randomness not reflected in the either the REIT index or the bonds.

Let $V_{t,i}$ denote the time t value of the i^{th} CRE. Let $\mathbb{F}^\xi$ corresponds to the filtration generated by $V_{t,i}$ for all i. This filtration $\mathbb{F}^\xi$ represents the randomness in CREs that is not in $\mathbb{F}^s$. We assume that $\mathbb{F} = \mathbb{F}^\xi \vee \mathbb{F}^s$.

Next, we assume that the market for the REIT, zero-coupon bonds, and the CREs is arbitrage free (satisfies no free lunch with vanishing risk - NFLVR) and satisfies no dominance so, by the first fundamental theorem of asset pricing, that there exists a martingale measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F}_T)$ such that $\frac{S_t^*}{B_t}$, $\frac{p(t,T)}{B_t}$ for all $T \in [0, \mathbb{T}]$, and $\frac{V_{t,i}}{B_t}$ for all i are $\mathbb{Q}$ - martingales with respect to $\mathbb{F}$.

This implies

$$S_t^* = \mathbb{E}^{\mathbb{Q}}\left(\frac{S_{\mathbb{T}}^*}{B_{\mathbb{T}}}\middle|\mathcal{F}_t\right)B_t, \quad p(t,T) = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B_{\mathbb{T}}}\middle|\mathcal{F}_t\right)B_t,$$

and

$$V_{t,i} = \mathbb{E}^{\mathbb{Q}} \left(\frac{V_{\mathbb{T},i}}{B_{\mathbb{T}}} \middle| \mathcal{F}_t \right) B_t.$$

By construction, this market is incomplete with respect to $\mathbb{F}$ because the CRE trades only sporadically and $\mathbb{F}^s \neq \mathbb{F}$.

In addition, we assume that the restricted market consisting of just the REIT and bonds is complete with respect to $\mathbb{F}^s$. That is, for any time $T \in [0, \mathbb{T}]$, the $\mathcal{F}_T^s$ - measurable payoff X_T integrable with respect to $\mathbb{Q}$ can be attained by trading in an admissible, self-financing trading strategy using just the REIT and bonds. This implies, by the second fundamental theorem of asset pricing, that given any martingale measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, its restriction $\mathbb{Q}^s := \mathbb{Q} \mid_{\mathcal{F}_{\mathbb{T}}^s}$ on $(\Omega, \mathcal{F}_{\mathbb{T}}^s)$ is unique and can be used to give the time 0 value of any $\mathcal{F}_T^s$ - measurable random payoff X_T via risk-neutral valuation, i.e.

$$X_t = \mathbb{E}^{\mathbb{Q}} \left(\frac{X_T}{B_T} \middle| \mathcal{F}_t \right) B_t = \mathbb{E}^{\mathbb{Q}^s} \left(\frac{X_T}{B_T} \middle| \mathcal{F}_t \right) B_t.$$

Finally, consider the time $T \in [0, \mathbb{T}]$, $\mathcal{F}_T$ - measurable payoff of the i^{th} CRE, $V_{T,i}$. First, compute its $\mathcal{F}_T^s$ conditional expected value with respect to $\mathbb{P}$, i.e.

$$\tilde{V}_{T,i} := \mathbb{E}^{\mathbb{P}}[V_{T,i} | \mathcal{F}_T^s].$$

Note that this quantity is a $\mathcal{F}_T^s$ - measurable random payoff, and thus can be replicated with an admissible, self-financing portfolio using just the REIT and bonds. One can think of this as a partial replication of $V_{T,i}$.

The hedging error in replicating $\tilde{V}_{T,i}$ instead of $V_{T,i}$ is

$$\varepsilon_{T,i} = V_{T,i} - \tilde{V}_{T,i}.$$

Our final assumption is that the CRE hedging errors for all i and all T are idiosyncratic, i.e. for all martingale measures $\mathbb{Q}$, $\mathbb{E}^{\mathbb{Q}}(\varepsilon_{T,i}) = \mathbb{E}^{\mathbb{P}}(\varepsilon_{T,i})$.

It is shown in Jarrow [23] that this implies $\mathbb{E}^{\mathbb{Q}}(\varepsilon_{T,i}) = 0$, i.e. the idiosyncratic component of the i^{th} CRE's hedging error has zero value. In conjunction, this implies that the arbitrage-free value for the i^{th} CRE satisfies

$$V_{t,i} = \mathbb{E}^{\mathbb{Q}} \left(\frac{V_{T,i}}{B_T} \middle| \mathcal{F}_t \right) B_t = \mathbb{E}^{\mathbb{Q}^s} \left(\mathbb{E}^{\mathbb{P}} \left(\frac{V_{T,i}}{B_T} \middle| \mathcal{F}_T^s \right) \middle| \mathcal{F}_t \right) B_t. \tag{1}$$

The same logic applies to extend this valuation formula to any derivative on the i^{th} CRE. Let $g(V_{\tau,i})$ be the payoff at time $\tau \in [t,T]$ to such a derivative with $g(\cdot) : \mathbb{R} \to \mathbb{R}$ a Borel measurable function. Then, the time t value of the derivative, denoted C_t, is

$$C_t = \mathbb{E}^{\mathbb{Q}} \left(\frac{g(V_{\tau,i})}{B_\tau} \middle| \mathcal{F}_t \right) B_t = \mathbb{E}^{\mathbb{Q}^s} \left(\mathbb{E}^{\mathbb{P}} \left(\frac{g(V_{\tau,i})}{B_\tau} \middle| \mathcal{F}_\tau^s \right) \middle| \mathcal{F}_t \right) B_t. \tag{2}$$

These are the valuation formulas implemented and estimated below.

3.2 The Empirical Specification

To empirically implement the CRE valuation model, we need to impose more structure on the CRE's values and cash flows at future dates.

Fix a selling horizon for the i^{th} CRE as time T. Assume that there are discrete payment dates over this horizon $t = 1, ..., T$ where cash flows $c_i > 0$ are received from owning the property. The cash flows are assumed to be constant, although this assumption is easily modified.

We impose the following assumption on the value of the i^{th} CRE at the selling time T. For the subsequent analysis, the selling time is always fixed at T years into the future.

Assumption 1 (the CRE's time T Selling Value) For all $t + T \in [0, \mathbb{T}]$,

$$V_{t+T,i} = \frac{V_{0-,i}}{S_{0-}} S_{t+T} e^{\xi_{t+T,i}}$$

where $V_{0-,i}$ corresponds to the value of the i^{th} CRE at some date before time 0, S_{0-} corresponds to the value of the REIT at the same time $0-$ before time 0, and $\xi_{t+T,i}(\omega) \geq 0$ is a $\mathcal{F}_{t+T}^{\xi}$ measurable random variable.

In this assumption, the constant $a_i \coloneqq \frac{V_{0-,i}}{S_{0-}}$ corresponds to a rescaling factor needed so that the value $V_{t+T,i}$ of the i^{th} CRE is of the same rough magnitude as $V_{0-,i}$. Note that we do not include the dividend in the REIT's future value (S_{t+T}^*) because the value of the CRE property at the future selling date $t + T$ is unrelated to the dividends paid on the REIT.

Given the above model structure, given the selling time is T years into the

future, we have that

$$V_{t,i} = \mathbb{E}^{\mathbb{Q}} \left(\frac{V_{t+T,i}}{B_{t+T}} + \sum_{t=1}^{t+T} \frac{c_i}{B_{t+T}} \middle| \mathcal{F}_t \right) B_t$$

$$= \mathbb{E}^{\mathbb{Q}} \left(\frac{a_i S_{t+T} e^{\xi_{t+T,i}}}{B_{t+T}} \middle| \mathcal{F}_t \right) B_t + c_i \sum_{k=1}^{T} p(t, t+k)$$

$$= \mathbb{E}^{\mathbb{Q}^s} \left(a_i \frac{S_{t+T}}{B_{t+T}} \mathbb{E}^{\mathbb{P}} \left(e^{\xi_{t+T,i}} \middle| \mathcal{F}^s_{t+T} \right) \middle| \mathcal{F}_t \right) B_t + c_i \sum_{k=1}^{T} p(t, t+k) \tag{3}$$

where $a_i \coloneqq \frac{V_{0-,i}}{S_{0-}}$.

We see here that the i^{th} CRE's value is equal to the value of the REIT, modified by the idiosyncratic component of the CRE's value, plus the present value of the CRE's known cash flows over $[t, t+T]$. We add the following assumption to facilitate the empirical estimation of the CRE's value. The key limitation of this assumption is that the idiosyncratic component is assumed to be a deterministic function of the holding horizon.

Assumption 2 *(the CRE's Idiosyncratic Value Adjustment)* For all $t+T \in [0, \mathbb{T}]$,

$$e^{\xi_{t+T,i}} = \mu_T e^{\Theta_T \cdot X_{0,i}}$$

with constants μ_T, Θ_T where Θ_T is a n-vector. These parameters depend on the selling horizon T. The n-vector $X_{0,i}$ represents a set of known covariates at time 0 that characterize the i^{th} CRE. These covariates are constants related to the property, e.g. number of floors and square footage per floor, among others. In this expression, $\Theta_T \cdot X_{0,i}$ is the inner product.

Given assumption 2, the valuation formula simplifies to:

$$V_{t,i} = a_i S_t e^{-\delta T} \mu_T e^{\Theta_T \cdot X_{0,i}} + c_i \sum_{k=1}^{T} p(t, t+k) \tag{4}$$

where $a_i \coloneqq \frac{V_{0-,i}}{S_{0-}}$.

The "observable" quantities in this expression are the following:

(1) the selling horizon T;

(2) the value of the i^{th} CRE before time 0, $V_{0-,i}$;

(3) the values of the REIT before time 0 and at times t in our observation period, S_{0-},S_t;

(4) the dividend yield per year on the REIT, δ;

(5) the i^{th} CRE's covariates at time 0;

(6) the i^{th} CRE's cash flows, c_i; and

(7) at times t in our observation period, zero-coupon bond prices $p(t, t+k)$ for $k = 1, ..., T$.

The only "observable" that may not be readily available is the value of the i^{th} CRE before time 0. This occurs if the CRE was not sold prior to the start of the estimation period, time 0. Estimation of this quantity in this circumstance is discussed below.

The parameters to be estimated on the right side are μ_T, Θ_T.

3.3 The Empirical Estimation Methodology

Given noise in the data and model error, the cross-sectional and time-series estimating expression for $i = 1, ..., I$ and $t \in [0, T]$ is

$$V_{t,i} = a_i S_t e^{-\delta T} \mu_T e^{\Theta_T \cdot X_{0,i}} + c_i \sum_{k=1}^{T} p(t, t+k) + \varepsilon_{t,i} \tag{5}$$

where $a_i \coloneqq \frac{V_{0-,i}}{S_{0-}}$ and $\varepsilon_{t,i}$ are the error terms with zero mean and constant variance under the statistical measure $\mathbb{P}$.

There are three steps in estimating $V_{t,i}$. First, we select the selling horizon T. For our main results, we have chosen $T = 5$, but the results of other time horizons can be found in Appendix A.1. Second, we estimate a_i given only sporadic trading in CREs. Third, we estimate μ_T, Θ_T.

3.3.1 Estimating a_i

Given the i^{th} CRE in our sample, there could be a single or multiple observations of $V_{t,i}$ at different times. If there are multiple observations of the selling price, we use the first selling time, say at time s, and set $V_{0-,i} = V_{s,i}$. Then, this observation is removed from the sample. We compute $a_i \coloneqq \frac{V_{s,i}}{S_s}$.

If there is only one observation of $V_{t,i}$ in the sample, then we compute $V_{0-,i}$ using comparable CREs. The procedure is as follows. Let the only observation for the i^{th} CRE be at time t. From among the remaining CREs, we select five, indexed $j = 1, ..., 5$, that are located in the same 3-digit ZIP code and have the smallest value for

$$\sum_{k=1}^{m} \frac{1}{m} \left| X_{0,i}^{k} - X_{0,j}^{k} \right|$$

at times $s_1, ..., s_5 \leq t$ for a collection of the m covariates. Five covariates were chosen based on their common use in identifying comparables in practice: number of units, number of floors, building age, Walk Score, and Star Rating. All covariates were z-standardized prior to the matching process in order to give them equal weighting in the process.

Let these CRE market values be denoted V_{j,s_j} for $j = 1, ..., 5$. Next, we find the s_j date closest to time t. Call this time s^*. Using an inflation adjustment, we bring all the remaining market values up to time s^*. This yields the market values V_{j,s^*} for $j = 1, ..., 5$. We compute

$$V_{0-,i} = median\{V_{j,s^*} : j = 1, ..., 5\}$$

used to calculate $a_i \coloneqq \frac{V_{0-,i}}{S_{0-}}$ with its corresponding REIT price at the time.

3.3.2 Estimating μ_T, Θ_T

Denote the model error (residual) as:

$$\varepsilon_{t,i} = V_{t,i} - V_{t,i}(\mu_T, \Theta_T)$$

where $V_{t,i}(\mu_T, \Theta_T)$ corresponds to the remaining terms on the right side of expression (5). We estimate the parameters μ_T, Θ_T minimizing the sum of squared errors subject to an unbiasedness constraint, i.e.

$$\min_{\mu_T, \Theta_T} \sum_{t \in [0,\mathbb{T}], i=1,...,I} \varepsilon_{t,i}^2$$

subject to

$$\sum_{t \in [0,\mathbb{T}], i=1,...,I} \varepsilon_{t,i} = 0.$$

This is a non-linear regression minimizing the residuals between the model estimates and the observed values. We estimate μ_T and Θ_T using a Constrained Optimization by Linear Approximation (COBYLA)[32] local optimizer with an Improved Stochastic Ranking Evolution Strategy (ISRES)[34] global optimizer. Both are derivative-free optimization methods that are able to handle the above equality constraint.

4 Data

To test our theoretical commercial real estate (CRE) pricing paradigm, we use 3703 multifamily sale transactions for the Los Angeles market from CoStar over the 2001 to 2022 period[6]. We chose multifamily properties in Los Angeles since it has the longest history of CRE transactions. Our time frame covers a complete real estate market cycle starting from the boom, bust, and subsequent recovery period in the commercial property market. It also allows us to ascertain how well our proposed pricing paradigm predicts the price of a multifamily building out-of-sample. In aggregate, the 3703 transactions are valued at $49 billion.

For each transaction, we collect building characteristics on each building transacted including the number of units, total building square feet, the land area, the number of floors, when the building was built, the composition of units in the building e.g., studio, 1-bedroom, 2-bedroom, etc., and if the building was a historic landmark. To control for building quality, we use building class (A, B, and C) in addition to the star rating that ranges from 1 (poor) to 5 (excellent) reported in CoStar. Since building class is based on local relative comparisons (e.g. an A quality building in one location might only qualify as a B quality building in another location), we also include the Costar star rating system[7], which evaluates a building separate from its immediate vicinity based on specific standards for each property type. These standards include architectural design, structure/systems, unit and on-site amenities, landscaping, and certifications (LEED, energy star). In addition to this, we use information from CoStar on the condition of the building: whether it is adequate, excellent, good, needs improvement or poor. Since some

[6]Many studies have used CoStar in analyzing commercial real estate (see for example Eichholtz, et al 2010 [11]). CoStar (https://www.costar.com/) maintains a comprehensive database of over 20 million sale and lease transactions of various property types.

[7]https://www.costar.com/sites/costar.com.na/files/2023-09/costarbuildingratingsystem-definition.pdf

apartment buildings are not market based, we include affordability indicators if the building has affordable units, if units are rent controlled, rent restricted, rent subsidized, or rent stabilized.

To account for the general desirability of a location we use the Federal Housing Finance Agency (FHFA) constant quality house price index[8] based on 3-digit ZIP code, as well as the Walk Score[9] metric for walkability. The FHFA housing price index is a weighted, repeat sale index that measures the average price change for houses that have sold at least twice. This index proxies for submarket conditions that are not time invariant including changes in unobservable neighborhood quality and local public amenities such as school quality and crime. This housing market index should also exhibit similar co-movement with CRE prices (Gyourko, J. 2009[15]). The Walk Score metric which ranges from 0 to 100 measures locational efficiency with higher scores indicating greater walkability. Prior studies such as Pivo and Fisher 2011[30] find that higher Walk Scores are correlated with higher apartment values.

Although we collect information on the characteristics of the buyer and seller such as whether they are local and whether they are institutional or individual, we do not include this information in our analysis. The reason for this is that in predicting prices of apartment buildings out-of-sample, information on buyer characteristics is unknown ex-ante. Stated differently, the bargaining power of the buyer relative to the seller is unobservable prior to the completion of the transaction. It is only observable after the price is established. Our price estimates therefore omit the measured effects of buyer and seller attributes.

For each building, the net-operating income (NOI) serves as the CRE's cash flows. We use the pro forma NOI when available in the data (8.3 % of the sample). The remainder, and the majority of the sample, uses the trailing 12-month NOI at the time of the sale.

For estimation purposes, we divide the sample into three time periods reflecting structural changes during the sample period. These are from 07/31/2001 to 07/01/2006 (before the financial crisis), 01/01/2010 to 01/01/2015 (immediately following the recession, where delinquency rates were elevated, as seen in figure 1), and 01/02/2015 to 12/31/2019 (return to normalcy of delinquency rates). Table 1 reports the descriptive statistics of the CRE during the three periods. The

[8]https://www.fhfa.gov/data/hpi
[9]https://www.walkscore.com/

price in denominated in price per apartment unit in accordance with common convention in commercial real estate. Additional summary statistics on the other covariates can be found on Appendix A.2.

	Variable	Mean	Standard Deviation	Minimum	Maximum
Period 1 (07/31/2001 - 07/01/2006) $N = 806$					
	Price per unit (in dollars)	103800.50	50470.08	25425.95	429752.07
	Building Sqft	56024.63	66994.52	10082.00	704104.0
	Floors	2.60	1.19	1.00	21.00
	Building Age	39.13	17.94	-18.00	94.00
Period 2 (01/01/2010 - 01/01/2015) $N = 501$					
	Price per unit (in dollars)	157423.08	95636.82	32142.86	754285.71
	Building Sqft	58020.02	78693.07	11692.00	632192.0
	Floors	2.84	1.90	1.00	35.00
	Building Age	47.88	22.82	-1.00	108.00
Period 3 (01/02/2015 - 12/31/2019) $N = 1006$					
	Price per unit (in dollars)	269302.29	141974.50	48648.65	1842105.26
	Building Sqft	63356.68	78719.82	9874.00	550872.0
	Floors	2.80	1.42	1.00	17.00
	Building Age	48.81	24.21	-1.00	111.00

Table 1: Summary statistics for LA apartment buildings during the three sample periods. Summary statistics on all covariates used in the paper can be found on Appendix A.2.

The REIT used for this estimation is Essex Property Trust, Inc (ESS) obtained from the Center for Research in Security Prices (CRSP). ESS is a multifamily REIT which has a West Coast investment focus: Los Angeles (33%), San Diego (10%), San Jose/Santa Clara (20%), the San Francisco Bay area (19%) and Seattle (17%). During the sample period, ESS has issued dividends quarterly which increased annually at a relatively consistent growth rate. In the estimation and simulation, we projected the dividend per share by using the trailing 5-year growth rate average.

In addition, we used the Federal Reserve's daily H.15 Selected Interest Rates data which are yields for U.S. Treasury securities with maturities $m \in \{\frac{1}{12}, \frac{3}{12}, \frac{6}{12}, 1, 2, 3, 5, 7, 10, 20, 30\}$ measured in years. From these yields, we computed a piecewise linear forward rate curve that exactly matches the U.S. Treasury security prices implied by these yields. The change points for the piecewise

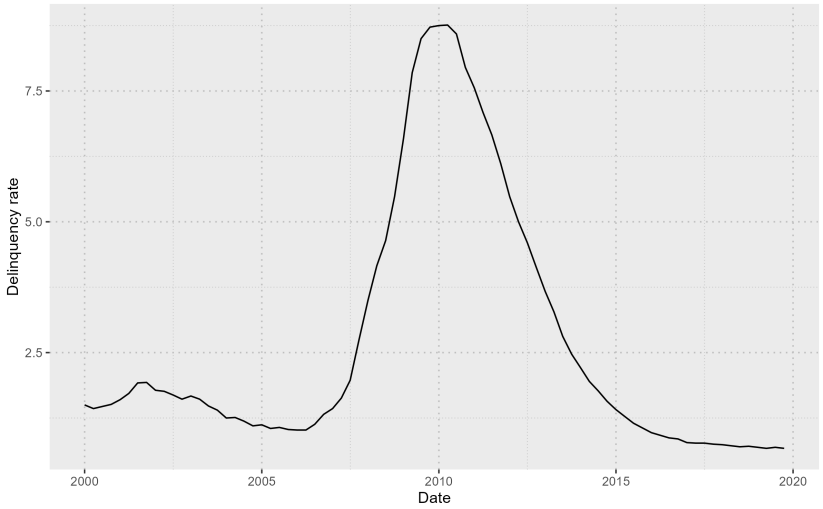


Figure 1: Delinquency rates on commercial real estate loans during the sample period



Figure 2: Price of REIT ESS during the sample period

linear curve occur at the same maturities $m \in \{\frac{1}{12}, \frac{3}{12}, \frac{6}{12}, 1, 2, 3, 5, 7, 10, 20, 30\}$, where the curve is assumed constant over the initial period $[0, \frac{1}{12}]$. The details of this computation are contained in Appendix A.5. For the default-free spot rate we use the yield on the 1-month Treasury Bill.

5 Results

For our optimization process, we minimized the percent residuals, rather than the actual dollar amount, in order to mitigate the effect of highly skewed prices of CRE in the sample. To further account for outliers, which may lack appropriate comparables when estimating the scaling parameter a_i, we run a two-step process: first estimating for the whole sample, then estimating again in a truncated sample, dropping those beyond the 5% and 95% levels. This truncation is done because, as explained below, our collection of covariates cannot capture very high- and low-valued CREs specific characteristics. Results from the first step can be found in Appendix A.3.

Table 2 provides the main results of our estimation, broken down by the aforementioned three periods of interest. We report both the dollar/unit and percent error, along with the absolute value of the latter. Performance is relatively uniform across all three periods, which implies that the valuation methodology is relatively robust to structural shifts in the CRE market.

As seen, the mean CRE pricing error per unit is zero dollars. And, the median percentage pricing errors are trivial at 0.49, -0.77, and -1.07 percent in periods 1, 2, and 3, respectively. In terms of dollar pricing errors, the medians are 252.33, -470.95, and $1,841.37$ dollars for periods 1, 2, and 3, respectively. The increase in the dollar medians is likely due to inflation, which increased over these three periods.

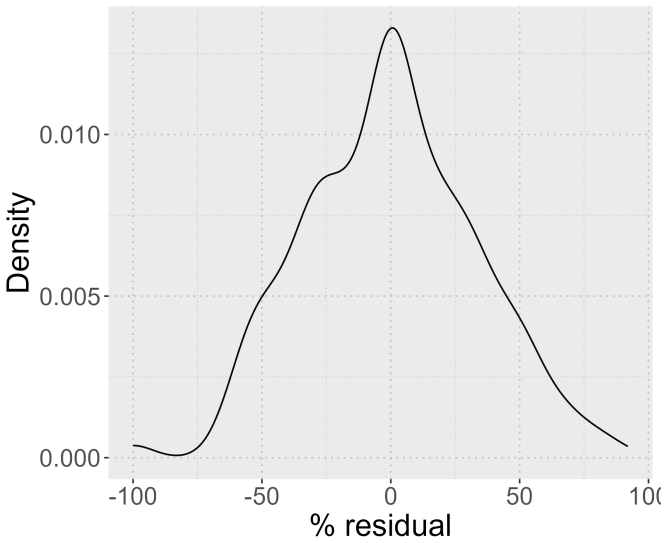
The Q25 quartile percentage errors are -24.77, -23.64, and -23.28 percent in periods 1, 2, and 3, respectively. The Q75 quartile percentage errors are 23.09, 23.55, and 21.37 percent in periods 1, 2, and 3, respectively. These percentage errors suggest that the pricing error distribution is symmetric, as confirmed in Figure 2 below.

5.1 Industry Usage of the Valuation Model

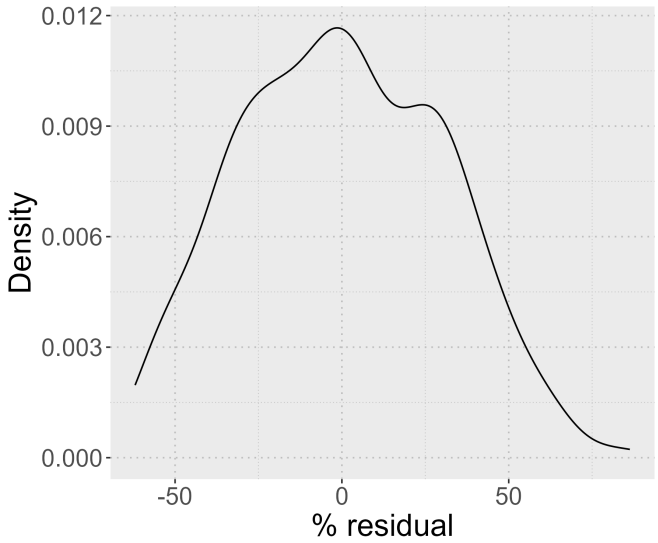
From a practical perspective, i.e. using this model in industry, our valuation estimates are very promising. Indeed, recall that the model valuation is based on markets that are assumed to be frictionless and competitive. However, CRE markets are are far from being so. CRE markets have significant frictions in terms of search costs, lawyer fees to execute a transaction, broker fees, and financing

	Measure	Mean	Q25	Median	Q75	St. Dev.
Period 1: 07/31/2001 - 07/01/2006						
	Residual (dollars/unit)	-2705.77	-16151.20	252.33	12830.56	30320.33
	Percent residual	0.00	-24.77	0.49	23.09	33.38
	Absolute percent residual	26.30	8.93	24.07	39.87	20.54
$N = 806$						
$R^2 = 0.821991$						
Period 2: 01/01/2010 - 01/01/2015						
	Residual (dollars/unit)	-4492.81	-22620.16	-470.95	18628.76	46314.55
	Percent residual	0.00	-23.64	-0.77	23.55	30.29
	Absolute percent residual	25.09	11.09	23.64	36.03	16.94
$N = 501$						
$R^2 = 0.8747557$						
Period 3: 01/02/2015 - 12/31/2019						
	Residual (dollars/unit)	-7152.62	-45656.94	-1841.37	37012.34	85566.22
	Percent residual	0.00	-23.28	-1.07	21.37	32.27
	Absolute percent residual	26.64	11.63	22.55	39.93	18.20
$N = 1006$						
$R^2 = 0.8090582$						

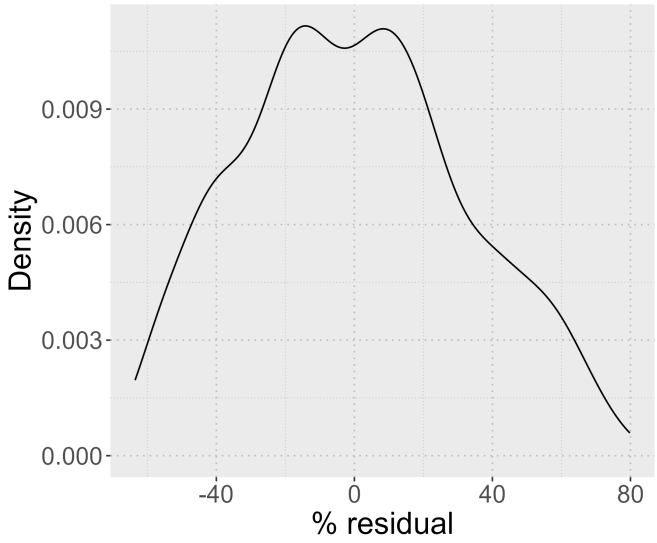
Table 2: Model errors (residuals) during the three periods. The model was estimated using non-linear optimization to minimize percent residuals of the CRE value, along with one equality constraint to prevent bias. We provide three measures of these residuals: the actual dollar amount per unit, the percentage deviation in the price per unit, and the absolute percentage deviation.



(a) Period 1



(b) Period 2



(c) Period 3

Figure 3: Distribution of percent residuals in each period, with kernel density smoothing

constraints. In addition, although reasonably competitive, there are bargaining games played between buyers and sellers as well. In conjunction, these frictions imply the existence of transaction cost price bands around the arbitrage-free values computed above (see Karatzas 1997[24], Jarrow 2006[18], Jarrow 2021[22]). In this

regard, a buyer or seller can use our fair value estimate as a starting point, and adjust it to account for these frictions when offering or accepting bids for the property.

We are not surprised that the heterogeneity of apartment characteristics and other factors we include in our valuation model cannot completely account for the variation in multifamily prices. Our model does not include buyer-seller characteristics nor the attributes of tenants. Buyer and seller characteristics such as asymmetric information and bargaining power in addition to other characteristics e.g., local vs non-local investor, institutional vs individual buyer, etc are plausible drivers of unexplained price variation (Harding et al. 2003[16][17]; Koijen et al 2025[25], Sagi 2021[35]). Heterogeneity in beliefs about future fundamentals and heterogeneity in preferences among market participants means that investors differ in their willingness to pay for the same NOI. The reason we omit buyer-seller characteristics in our study is that our model is designed to price properties regardless of whether a transaction occurs. In the absence of a transaction, the buyer (and their characteristics) is unknown ex-ante when predicting price. A similar logic process applies to tenant attributes. Measuring the heterogeneity of parties is challenging at best. We also note that while the empirical estimation of our model with respect to this study does not include information on lease contracts, it can be included in our model if this information is available. Lease terms matter since any opportunity for NOI growth comes when current leases expire.

We also recognize that our RMSE estimates for CRE properties should exceed those for residential properties based on our theory. As Gavazza 2011[13] notes, assets that trade in thinner markets should have a higher price dispersion. His observation is equivalent to the transaction cost bands around arbitrage-free values computed in our model. This band arises from the frictions associated with CRE markets in addition to buyer-seller bargaining games. Nowak and Smith 2017[29] find that the average out-of-sample error for single family homes is 5% to 10%.

In addition, recall that the valuation model estimates the difference between the specific CRE's value and the "average" CRE's value (based on the scaled REIT value) using a collection of covariates. The covariates we use were selected to capture many of the characteristics of a specific CRE.

Table 3 provides the estimates for the coefficients μ_T and Θ_T of the covariates in model. It is important to note that since these covariates were scaled prior to

Variable	Period 1 (07/31/2001 - 07/01/2006)	Period 2 (01/01/2010 - 01/01/2015)	Period 3 (01/02/2015 - 12/31/2019)
μ_T	0.4578812	0.2301493	0.2718529
Building Sqft	-0.1449902	-0.0338377	-0.0714425
Land Sqft	-0.1348676	-0.1263473	0.0012950
Floors	0.4081228	0.6626529	0.0391000
I(Floors 6 to 10)	0.1073309	-0.2221716	-0.1809877
I(Floors 10+)	-0.1154857	-0.0287634	-0.0090186
I(Three star)	-0.5445726	0.1272575	-0.2493223
I(Four star)	-0.1495825	0.3547986	-0.1324334
I(Five star)	-0.2576476	-0.1796376	-0.3332020
I(Class A)	-0.2462028	0.2552530	-0.2674985
I(Class B)	3.9625832	0.1684785	0.0179746
Floor Ratio	-0.5462106	0.5523006	0.3796263
Age	-0.1403717	-0.6589712	-0.2802934
% studio	-0.3489838	-1.0121463	0.0278912
% 1BR	-0.3829515	-0.0863833	-0.0408294
% 2BR	0.0034835	0.0740501	-0.0797333
I(Affordable units)	0.0000270	-0.0416269	-0.0198995
I(Controlled units)	-0.2249079	0.2427342	0.2424030
I(Restricted units)	-0.1235381	0.3922444	0.1526333
I(Subsidized units)	-0.0437006	-0.1648579	-0.1168315
I(Stabilized units)	-0.2637212	-0.1170091	-0.0906019
Housing price index	-0.3417083	-0.1699200	-0.1020432
% Δ Housing price index	-0.1738408	0.0685111	-0.0780533
Number of units	-0.2759281	0.0864681	0.0641138
Walk score	-0.2414520	0.1524279	-0.0318817
I(Historic units)	-0.2370837	-0.0444301	-0.1643994
I(Lease-hold units)	-0.1347894	0.0057786	-0.0755462
Year	-0.3731216	-0.2055709	0.1466052
I(Poor building condition)	-0.2186395	-0.0300048	-0.0786076

Table 3: Estimated coefficients (μ_T, Θ_T) for all three periods. Each period was estimated separately using the aforementioned non-linear estimation with one equality constraint. Note that the model was estimated after scaling these covariates. Thus, their relative magnitudes should be interpreted with caution.

the estimation process, and because many of them are correlated, one should be cautious when interpreting these magnitudes or comparing them amongst each other. The coefficients on the 3-digit zip-code indicator variables can be found in Appendix A.4. Although the covariates capture many of a specific CRE's

characteristics as distinct from the "average" CRE, we are limited in the covariates we could include by electronically available data. Clearly, a specific CRE will have important characteristics not captured by this collection. These differences (along with market frictions) account for the errors in our fair values. Nonetheless in practice, a buyer or seller armed with our fair values, will know the characteristics included in this fair value estimate. They will also know, in contrast to our model, what important characteristics are excluded in a specific CRE property. With this expert knowledge, they can adjust our fair values to get an improved estimate.

To illustrate the use of our CRE valuation model, we consider a double escrow deal that occurred on February 28, 2003. In a double escrow deal, the buyer purchases the property then subsequently resells it on the same day at a higher price. In essence, the buyer acts as a wholesaler. The double escrow involves two separate transactions, where the process starts with the wholesaler getting the seller to sign either a purchase agreement or an option agreement (see section 7). Typically, the wholesaler funds the original transaction prior to flipping it to the subsequent buyer, although some financing options are available including flash (same day) funding, unsecured short-term personal loan, or hard money loans. Funds from the second transaction cannot be used for the first purchase. Another drawback is that the wholesaler must manage deal logistics e.g., coordinate the complexities of handling both transactions simultaneously. However, the riskiest part of this strategy depends on the ultimate buyer not backing out of the deal or their inability to obtain financing. Typically, these types of deals involve distressed properties, motivated sellers, and/or other special situations. The presence of double escrow transactions is evidence that information asymmetries are present in CRE, which our model does not explicitly include.

Date	Units	Seller	Buyer	Price
2/28/2003	149	Whittier 26	GRW Partnership	$10,100,000
2/28/2003	149	GRW Partnership	Regatta Investors	$11,350,000

Table 4: Details on the double-escrow transaction on February 28, 2003. This yielded a profit of $1,250,000 or $8389 per unit for GRW Partnership

The specifics of this double escrow deal are contained in Table 4. The unit price of these transactions were $67,785 and $76,174, respectively, yielding a profit of $8,389 per unit. Our estimate of the fair value of this CRE is $55,900 per unit, in a frictionless, competitive, and arbitrage-free market.

Although our fair value does not include these market frictions, it can still be used to create such double escrow deals (before-the-fact) or understand the transactions (after-the-fact). First, the initial purchase is only 21% larger than our fair value, and the second 36% larger. The first is within one standard deviation of our value, and the second within two standard deviations (see Table 2). Hence, they are plausible sale prices, well within the observed error distribution around our fair value. These are aggressive offers by a clever CRE trader whose knows the market.

Second, this sale occurred prior to the credit crisis, and as noted below in section 7.2, the REIT index was probably exhibiting a price bubble. Our fair value estimate is equivalent to the fundamental value of the CRE, and when a bubble exists, the market price exceeds the fundamental value (see Jarrow 2021[22]), which is consistent with the larger sales prices executed. Again, a knowledgeable CRE trader experiencing a bubble in CRE prices, would quote prices above our fair value.

And finally, our covariates do not include an indicator for small and medium-sized multifamily (SMMF) properties. SMMF properties are typically defined to be those with less than 49 units. According to An et al 2021[1], these properties are the most affordable housing segment attracting lower income households. As such, the authors find that SMMF properties transact at a significant price discount to larger unit multifamily properties, which includes the CRE in our example. Hence, this observation also suggests the a CRE trader aware of this fact, and knowing our fair value, would require a larger sales price as observed. In conjunction, these three observations demonstrate how a knowledgeable CRE trader, starting with our fair value, could adjust it to reflect market realities not included within it.

6 Comparison with a Hedonic Model

As noted previously, the hedonic model is the standard approach for valuing CRE in both the academic literature and industry. The purpose of this section is to compare the performance of a hedonic CRE valuation model with our approach. To compute the hedonic model, we use a standard linear regression employing the same covariates as our baseline model in order to provide an apples-to-apples comparison. Then, we compare the out-of-sample prediction of both models.

For our valuation model, we use the coefficients from our estimated model in period 1 to predict the CRE values in period 2. Here, the coefficients are fixed from period 1, and only the covariates change to their updated values in period 2. We repeat this exercise using period 2's model coefficients to predict the CRE values in period 3.

For the hedonic model, for each time t in our observation period, we re-estimate the coefficients, and subsequently use the updated coefficients at time t to predict the CRE values at time t+1. In essence, this is almost an in-sample analysis, allowing the model's parameters to be updated daily to predict the next day's value. Thus, the hedonic model should have a significant advantage in predicting CRE values.

We compare the two model's predictive performances using an out-of-sample R^2 defined as:

$$R^2 = 1 - \frac{\sum_{t=1}^{T}(\frac{\hat{V}_t - V_t}{V_t})^2}{\sum_{t=1}^{T}(\frac{\bar{V}_t - V_t}{V_t})^2} \tag{6}$$

where $\hat{V}_t$ is the predicted CRE value and $\bar{V}_t$ is the average realized value over the out-of-sample period. This is just one minus the sum of squared prediction errors divided by the total sum of squared errors.

Table 5 shows the results from the out-of-sample analysis. First, as expected, the out-of-sample R^2 for our model decline from their in-sample values as given in Table 2. The period 2 value declines from 0.875 to 0.713 and 0.809 to 0.655 in period 3. Second, the in-sample R^2 for the hedonic model for periods 1 - 3 are 0.7574418, 0.777013, and 0.772082, respectively as seen in Table 6. While the hedonic model may be only slightly inferior at fitting within the sample, its predictive power is noticeably worse than our model, highlighting the contrast between a statistical model and one grounded in economic theory.

	Period 1 (07/31/2001 - 07/01/2006)	Period 2 (01/01/2010 - 01/01/2015)	Period 3 (01/02/2015 - 12/31/2019)
Baseline model	-	0.7130617	0.6550251
Hedonic model	0.4588303	0.2769854	0.3487526

Table 5: Out-of-sample R^2 results calculated using equation (6) for each period. Our baseline model performs noticeably better at creating more accurate predictions of CRE prices.

For the out-of-sample comparison, our model performs better than the hedonic model for all the periods in our sample. Surprisingly perhaps, the R^2 for our

	Period 1 (07/31/2001 - 07/01/2006)	Period 2 (01/01/2010 - 01/01/2015)	Period 3 (01/02/2015 - 12/31/2019)
Baseline model	0.821991	0.8747557	0.8090582
Hedonic model	0.7574418	0.777013	0.772082

Table 6: In-sample R^2 results calculated using equation (6) for each period. The smaller difference here may lead one to believe the hedonic model is sufficient if one does not conduct the out-of-sample analysis.

valuation model is more than twice as large as the hedonic model's R^2. Despite the rolling re-estimation in the hedonic model, it is unable to capture changing market conditions. In contrast, our model captures changing market conditions well due to the inclusion of a REIT index and zero-coupon bond prices. As documented, our model is robust to changing market conditions. As a robustness test, Appendix A.6 contains an alternative R^2 measure that also confirms the results in Table 5.

7 Real Options and CREs

As noted in the introduction, a key motivation for developing our valuation methodology is to price real options on CRE. This section illustrates this methodology via a case study where we price a European call option on a CRE contained in our sample. Examples of such options are rare, but one representative case study was found in Regency Detroit Apartments which was purchased in December 15, 1992 (before our observation period). The purchase agreement included an option to purchase the CRE one month after it was signed to allow time for due diligence. The option was exercised December 15, 1992. The strike price of the option was equal to the immediate purchase price of \$3,150,000 on the date that the contract was signed. The maturity of the option was 30 days, and the cost of the option was \$50,000. We value similar options on Regency Detroit Apartments, as if the CRE was being sold on the first day of the three time periods in our observation period: 07/31/2001, 01/01/2010, 01/02/2015.

To value any derivative on a CRE, we need to specify stochastic evolutions for the REIT and default-free spot rate of interest. For the purposes of this illustration, we select simple processes so that the parameters are easily estimated. It is a straightforward, albeit significant effort, to generalize and estimate derivative

values for more complex evolutions. This is left for future research.

7.1 Empirical Evolutions

We assume that the REIT stock price satisfies a constant elasticity of variance (CEV) process given by

$$dS_t = [\kappa(\nu - S_t) - \delta]dt + \alpha S_t^{\beta} dW_1(t) \tag{7}$$

where $\kappa > 0, \nu > 0$, $\alpha > 0$, β are constants and recall that $\delta \geq 0$ is the constant dividend yield. Here W_1 is a standard Brownian motion (BM) under $\mathbb{P}$ with $W_1(0) = 0$. We assume that W_1 is $\mathbb{F}$ adapted. Here, $\beta = 1$ corresponds to geometric BM, and if $\beta > 1$, then a price bubble exists as defined in Jarrow 2021[22].

We assume that the term structure follows the 1- factor model

$$dr_t = \varsigma_t dt + \sigma dW_2(t) \tag{8}$$

where W_2 is a standard BM adapted to $\mathbb{F}$ with $W_2(0) = 0$ under $\mathbb{P}$, $\sigma > 0$ is a constant, and ς_t is a $\mathbb{F}$ adapted stochastic drift process. Here, W_1 and W_2 are correlated with parameter $\rho \in [0, 1]$ defined by

$$cov_{\mathbb{P}}(dW_1, dW_2) = \rho dt.$$

7.2 Parameter Estimation

For the REIT parameters, we use the REIT stock price data described previously. For the spot rates r_t, we use daily 1 month Treasury yields obtained from the Federal Reserve's daily H.15 Selected Interest Rates. The initial forward rate curve $f(0, t)$ is estimated by fitting a piecewise linear curve to the U.S. Treasury security yields available from the Federal Reserve's daily H.15 Selected Interest Rates. The forward rate curve is assumed constant for $t \in [0, \frac{1}{12}]$. The details of this construction are in Appendix A.5.

To estimate the parameters, we discretize the continuous time processes over $[t, t + \Delta]$ where $\Delta = \frac{1}{250}$ years, and we use a simple and sequential method of moments estimation procedure.

For the CEV process, we note that

$$var_{\mathbb{P}}(\Delta S_t) = \left(\alpha S_t^{\beta}\sqrt{\Delta}\right)^2 \approx \Delta S_t^2$$

where the right side is an observable estimate of the left side. Taking natural logarithms yields

$$ln\left(\Delta S_t^2\right) \approx 2ln\left(\alpha\sqrt{\Delta t}\right) + 2\beta ln\left(S_t\right).$$

Running the linear regression

$$ln(\Delta S_t^2) = b_0 + b_1 ln(S_t)$$

over the sample period yields the parameters estimates

$$\alpha = \frac{e^{\frac{b_1}{2}}}{\sqrt{\Delta t}} \tag{9}$$

$$\beta = \frac{b_0}{2} \tag{10}$$

Given (α, β), to estimate the parameters of the term structure evolution, we again discretize over $[t, t+\Delta]$. We have at any time t,

$$var_{\mathbb{P}}(\Delta r_t) = \sigma^2\Delta \approx \left(\Delta r_t\right)^2. \tag{11}$$

Again, the right side of the above expression gives an observable estimate of the left side at time t. The average of the right side over the sample period, therefore, gives an unbiased estimate of the left side. We use this observation to estimate σ.

Similarly, the sample covariance of the left side of the following expression gives an unbiased estimate of the right side.

$$cov_{\mathbb{P}}\left(\frac{\Delta S_t}{S_t^{\beta}}, \Delta r\right) = \alpha\sigma\rho\Delta \tag{12}$$

Given (α, σ), this is used to estimate ρ.

The parameter estimates for both the REIT and spot rate process are are contained in Table 6. Of note, period 1 leading up to the subprime mortgage crisis exhibits a $\beta > 1$, consistent with the existence of a price bubble in the

REIT index. During and after the crisis, given that $\beta < 1$, these estimates are also imply that the bubble no longer existed. Although weak evidence in this regard, these estimates are consistent with the literature alleging that the crash of a housing price bubble caused the credit crisis due to the huge losses on the voluminous outstanding positions in credit default swaps (CDS) and collateralized debt obligations (see Crouhy, Jarrow, and Turnbull 2008[10]).

Parameters	Period 1 (07/31/2001 - 07/01/2006)	Period 2 (01/01/2010 - 01/01/2015)	Period 3 (01/02/2015 - 12/31/2019)
α	0.038882	19.055036	40.655887
β	1.204037	-0.075747	-0.116779
σ	0.007187	0.007254	0.007295
ρ	0.031328	0.078202	0.016801

Table 7: Parameters used in the simulation to value the options. This is the result from the discretized estimates of equations (9), (10), (11), and (12).

7.3 The Valuation Formula

Consider a European call option on the i^{th} CRE at time t with maturity date $t+\tau$ where $\tau \leq T$ and strike price K. Note that the time to the option's maturity τ is less than or equal to time to the fixed selling horizon T. Using the risk neutral valuation formula expression (2), the call's time t price is

$$C_t = \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(max\left\{\frac{V_{t+\tau,i}-K,0}{B_{t+\tau}}\right\}\left|\mathcal{F}^s_{t+\tau}\right.\right)\middle|\mathcal{F}_t\right)B_t$$

where $K = V_{t,i}$ and

$$V_{t,i} = a_i S_t e^{-\delta T}\mu_T e^{\Theta_T \cdot X_{0,i}} + c_i \sum_{k=1}^{T} p(t,t+k).$$

Assuming for simplicity that for small τ the zero-coupon discount curve is approximately constant, i.e. $p(t+\tau,\mathcal{T}) \approx p(t,\mathcal{T})$, we have

$$V_{t+\tau,i} - K = a_i e^{-\delta T}\mu_T e^{\Theta_T \cdot X_{0,i}}\left(S_{t+\tau}-S_t\right) + c_i\left[\sum_{k=1}^{T} p(t+\tau,t+\tau+k) - \sum_{k=1}^{T} p(t,t+k)\right]$$

$$\approx a_i e^{-\delta T}\mu_T e^{\Theta_T \cdot X_{0,i}}\left(S_{t+\tau}-S_t\right).$$

This implies that

$$C_t = \mathbb{E}^{\mathbb{Q}^s}\left(\mathbb{E}^{\mathbb{P}}\left(max\left\{\frac{a_i e^{-\delta T}\mu_T e^{\Theta_T \cdot X_{0,i}}\left(S_{t+\tau}-S_t\right),0}{B_{t+\tau}}\right\}\left|\mathcal{F}^s_{t+\tau}\right.\right)\right|\mathcal{F}_t\right)B_t.$$

This simplifies to

$$C_t = \frac{V_{0-,i}}{S_{0-}}\mu_T e^{\Theta_T \cdot X_{0,i}}\mathbb{E}^{\mathbb{Q}^s}\left(max\left\{\frac{S_{t+\tau}-S_t}{B_{t+\tau}},0\right\}\right|\mathcal{F}_t\right)B_t. \tag{13}$$

For the case study, the maturity of the option is $\tau = \frac{1}{12}$ years. The CRE is valued at time t using the fixed selling horizon $T = 5$ years.

7.4 The Computation and Results

This section presents the results from computing the valuation formula for our case study.

We use simulation to evaluate the expectation in expression (13) under the risk neutral measure $\mathbb{Q}^s$. The procedure is as follows.Let the starting date for the simulation be denote time 0. Using the parameters from the previous section, we simulate the evolution of the spot rate r_t and the REIT stock price S_t for $t \in [0, \tau]$. To do so, we first simulate two correlated Brownian motions $\tilde{W}_1$ and $\tilde{W}_2$ under the risk neutral measure $\mathbb{Q}^s$.

Given τ^* is the number of trading days in τ years, we multiply a $\tau^* \times 2$ matrix of standard correlation ρ and multiply it by $\sqrt{\Delta}$. This generates the required discretized changes in the two Brownian motions: $\Delta\tilde{W}_1$ and $\Delta\tilde{W}_2$ under $\mathbb{Q}^s$.

Using these simulated changes in the Brownian motions, along with the previously estimated parameters $(\alpha, \beta, \sigma, \rho)$, we discretize the REIT and spot price evolutions under $\mathbb{Q}^s$ (see Musiela and Rutkowski [28], p. 396 for the spot rate evolution under $\mathbb{Q}^s$). For $t \in [0, \tau]$, this yields

$$\Delta S_t = S_t(r_t - \delta)\Delta + \alpha S_t^{\beta}\Delta\tilde{W}_1(t)$$

$$\Delta r_t = (\frac{\partial f(0,t)}{\partial \mathcal{T}} + \sigma^2)\Delta + \sigma\Delta\tilde{W}_2(t).$$

which are the discretized analogues of equations (7) and (8). As noted earlier, when computing the forward rate curve $f(0, \mathcal{T})$, we assume it is constant for

$\mathcal{T} \in [0, \frac{1}{12}]$, which implies that $\frac{\partial f(0,t)}{\partial \mathcal{T}} = 0$.

We sum up those discretized changes for the τ years to get the simulated values of S_τ, B_τ for a single iteration. We repeat the above process for 10,000 iterations. This gives a vector of 10,000 realizations of S_τ, B_τ.

We perform these simulations for three initial starting dates $t_1, t_2, t_3 > 0$, the beginning date for each of the three different periods.These dates are 07/31/2001, 01/01/2010, 01/02/2015. With some abuse of notation, we let $t_i = 0$ for each $i = 1, 2, 3$ for the subsequent explanation.

We have characteristics $X_{0,i}$ of the Detroit property i from the original data and compute $a_i = \frac{V_{0-,i}}{S_{0-}}$ as previously described for the comparables. For each sub-period, we estimate the parameters $\hat{\Theta}_T$ as previously explained.

Recall that the maturity of the option is $\tau = \frac{1}{12}$, the selling horizon $T = 5$, and the strike price equals the CRE value at time 0. Using expression (13), with the vector of simulated realizations S_τ and B_τ, we calculate

$$E := max\left\{\frac{S_\tau - S_0}{B_\tau}, 0\right\}$$

generating a vector of 10,000 values. The average values of these realizations for the three dates are 0.7456214, 1.925812, and 5.008604. The call's price is then computed as

$$C_0 = a_i \hat{\mu}_T e^{\hat{\Theta}_T \cdot X_{0,i}} E,$$

which gives call prices \$4,229.23, \$2,564.63, and \$6,345.92 for periods one, two, and three, respectively. These correspond to 1.46, 2.32, and 2.38 percents of the property's value.

Although the time periods differ considerably, for comparison, we note that the actual value of the call option on this CRE as a percent of the property value in December 1992 was (\$50,000/\$3,150,000) = 1.59%. Clearly, this is consistent with our valuation model's results.

8 Conclusion

This paper introduced an alternative approach to pricing commercial real estate. Because of its roots in the option theory framework, the model is able to incorporate structural shifts in the market, allowing more accurate valuations and

subsequent predictions. Furthermore, it is able to appropriately price derivatives on these properties, as shown in our example with real options. Thus, our approach can be useful to practitioners, both those that transact in commercial real estate, as well as its derivatives.

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Appendix

A.1. Other specifications for selling horizon T

	Measure	Mean	Q25	Median	Q75	St. Dev.
Period 1: 07/31/2001 - 07/01/2006						
	Residual (dollars/unit)	-2057.56	-15973.44	-395.66	12472.92	26467.60
	Percent residual	0.00	-27.32	-0.88	23.22	34.24
	Absolute percent residual	28.31	12.26	26.24	41.65	19.24
$N = 806$						
Period 2: 01/01/2010 - 01/01/2015						
	Residual (dollars/unit)	-7898.66	-20899.76	-512.52	17178.00	56006.81
	Percent residual	0.00	-25.67	-0.57	26.34	37.01
	Absolute percent residual	29.79	12.31	26.08	42.00	21.92
$N = 501$						
Period 3: 01/02/2015 - 12/31/2019						
	Residual (dollars/unit)	-8923.95	-45972.90	-366.26	36274.05	85498.57
	Percent residual	0.00	-26.08	-0.21	23.47	35.05
	Absolute percent residual	28.73	12.35	25.37	43.58	20.06
$N = 1006$						

Table 8: Results from $T = 7$. This is analogous to Table 2, with the model estimated using non-linear optimization to minimize percent residuals of the CRE value, along with one equality constraint to prevent bias.

	Measure	Mean	Q25	Median	Q75	St. Dev.
Period 1: 07/31/2001 - 07/01/2006						
	Residual (dollars/unit)	-3825.73	-17151.55	-3361.81	9826.45	28821.92
	Percent residual	0.00	-33.91	-8.12	26.73	50.09
	Absolute percent residual	37.56	15.34	31.08	52.43	33.10
$N = 806$						
Period 2: 01/01/2010 - 01/01/2015						
	Residual (dollars/unit)	-5010.44	-23587.86	-2916.83	13233.11	41896.43
	Percent residual	0.00	-34.75	-4.35	28.30	47.90
	Absolute percent residual	36.40	15.34	31.68	51.03	31.10
$N = 501$						
Period 3: 01/02/2015 - 12/31/2019						
	Residual (dollars/unit)	-8454.03	-42537.80	-3206.26	31366.09	75907.28
	Percent residual	0.00	-27.48	-2.33	25.23	37.56
	Absolute percent residual	30.73	12.99	26.60	45.72	21.57
$N = 1006$						

Table 9: Results from $T = 10$. This is analogous to Table 2, with the model estimated using non-linear optimization to minimize percent residuals of the CRE value, along with one equality constraint to prevent bias.

A.2. Summary statistics of all covariates

Variable	Mean	Standard Deviation	Minimum	Maximum
Building Sqft	57.8465	74.5379	9.874	704.104
Land Sqft	64.3742	121.5024	1.481	2088.436
Floors	2.7541	1.4807	1	35
I(Floors 6 to 10)	0.0406	0.2806	0	3
I(Floors 10+)	0.0296	0.6203	0	25
I(Three star)	0.2455	0.4305	0	1
I(Four star)	0.0693	0.254	0	1
I(Five star)	0.0075	0.0863	0	1
I(Class A)	0.0512	0.2205	0	1
I(Class B)	0.2247	0.4175	0	1
Floor Ratio	1.3731	1.6475	0.0568	53.1621
Age	45.2834	22.6159	0	111
% studio	0.2155	0.3356	0	1
% 1BR	0.4105	0.3083	0	1
% 2BR	0.3383	0.3188	0	1
I(Affordable units)	0.0208	0.1426	0	1
I(Controlled units)	0.0132	0.1143	0	1
I(Restricted units)	0.03	0.1707	0	1
I(Subsidized units)	0.0199	0.1396	0	1
I(Stabilized units)	0.004	0.0629	0	1
Housing price index	945.647	532.8201	0	3903.98
% Δ Housing price index	10.2419	9.0269	-20.62	41.54
Number of units	65.7347	78.1562	25	1526
Walk score	77.6283	14.7068	10	99
I(Historic units)	0.0035	0.0593	0	1
I(Lease-hold units)	0.0031	0.0555	0	1
Year	2011.9938	6.5792	2001	2022
I(Poor building condition)	0.0119	0.1086	0	1
zip 900	0.3311	0.4707	0	1
zip 902	0.0737	0.2614	0	1
zip 903	0.015	0.1216	0	1
zip 904	0.0119	0.1086	0	1
zip 905	0.0155	0.1234	0	1
zip 906	0.0397	0.1954	0	1
zip 907	0.0313	0.1743	0	1
zip 908	0.0442	0.2055	0	1
zip 910	0.0132	0.1143	0	1
zip 911	0.0159	0.1251	0	1
zip 912	0.0274	0.1632	0	1
zip 913	0.14	0.347	0	1
zip 914	0.0918	0.2889	0	1
zip 915	0.0075	0.0863	0	1
zip 916	0.0521	0.2223	0	1
zip 917	0.0662	0.2487	0	1
zip 918	0.0071	0.0838	0	1
zip 935	0.0163	0.1268	0	1

Table 10: Summary statistics of all covariates used in the main model

A.3. First step results

Measure	Mean	Q25	Median	Q75	St. Dev	
Period 1: 07/31/2001 - 07/01/2006						
	Residual (dollars/unit)	-4453.45	-21740.03	-1714.91	15559.22	38136.07
	Percent residual	0.00	-30.53	-2.68	26.79	44.08
	Absolute percent residual	34.32	14.10	29.13	49.91	27.63
$N = 830$						
Period 2: 01/01/2010 - 01/01/2015						
	Residual (dollars/unit)	-7813.10	-28470.16	-3548.84	19593.52	65151.07
	Percent residual	0.00	-29.72	-4.08	25.87	44.40
	Absolute percent residual	33.20	13.83	28.14	46.07	29.45
$N = 557$						
Period 3: 01/02/2015 - 12/31/2019						
	Residual (dollars/unit)	-12155.44	-60840.12	-10829.61	38499.28	109012.29
	Percent residual	0.00	-30.75	-6.08	24.84	44.02
	Absolute percent residual	33.95	13.73	27.81	48.20	28.00
$N = 1118$						

Table 11: Model errors (residuals) during the three periods in the first iteration prior to trimming for outliers. The model estimated using non-linear optimization to minimize percent residuals of the CRE value, along with one equality constraint to prevent bias

A.4. Coefficients of 3-digit ZIP code indicators

Variable	Period 1 (07/31/2001 - 07/01/2006)	Period 2 (01/01/2010 - 01/01/2015)	Period 3 (01/02/2015 - 12/31/2019)
zip 900	-0.2817603	-0.2220169	-0.0774550
zip 902	0.1071530	0.7105005	0.2329049
zip 903	0.0938454	-0.8360283	-0.3745572
zip 904	0.0084782	0.0376256	0.0692243
zip 905	0.0325766	0.0669886	0.0158611
zip 906	-2.0444845	-0.0725003	0.0370659
zip 907	-0.0004897	0.0471001	-0.0441345
zip 908	0.0017282	-0.0032188	-0.0190114
zip 910	0.0000716	0.0860716	0.0235334
zip 911	0.0776637	-0.3269844	-0.0787250
zip 912	0.0719657	0.1062049	-0.0417572
zip 913	0.1084141	0.0881618	-0.0324081
zip 914	0.0095443	0.1390660	-0.0083889
zip 915	0.0005657	0.0370269	0.0174703
zip 916	-0.1054385	0.0337809	0.0061542
zip 917	0.0155794	-0.0573394	-0.0695493
zip 918	-0.1189472	0.0181036	0.0265900
zip 935	-0.0234803	-0.0065104	-0.0161287

Table 12: Estimated coefficients on 3-digit ZIP code indicators. Each period was estimated separately using the aforementioned non-linear estimation with one equality constraint.

A.5. Forward rate curve smoothing

Defining $p_t(m)$ as the price at time t of a zero-coupon bond paying \$1 at time $t+m$, the forward rate $f_t(m)$ fulfills:

$$p_t(m) = e^{-\int_0^m f_t(s)ds}$$

We estimate the forward rate $f_t(m)$ at time t using yields $y_t(m)$ observed at time t for maturities (in years) $m \in \{\frac{1}{12}, \frac{3}{12}, \frac{6}{12}, 1, 2, 3, 5, 7, 10, 20, 30\}$. We approximate $f(m)$ using a piece-wise linear function g with knots at each maturity. For a given time $s \in [a, b]$:

$$g(a, b, s, x) = f(a) + \frac{x - f(a)}{b - a} \times (s - a)$$

The price function h is solved via:

$$h(a, b, s, x) = p(a)e^{-f(a)(s-a)-\frac{1}{2}\frac{g(a,b,s,x)-f(a)}{b-a}\times(s-a)^2}$$

We then construct a recursive algorithm. For $m = \frac{1}{12}$, set $f(0) = f(\frac{1}{12}) = y(\frac{1}{12})$. For subsequent $m = \frac{3}{12}, \frac{6}{12}, 1$, first calculate $p(m)$ through:

$$p(m) = e^{-y(m)m}$$

Then, solve for $f(m)$ using the relationship:

$$h(m_{j-1}, m_j, m_j, f(m_j)) = p(m_j)$$

For $m > 1$, the par yield $y(m)$ of an m-year bond with semi-annual coupon payments is defined as:

$$1 = \sum_{i=1}^{2m} \frac{y(m)}{2} p\left(\frac{i}{2}\right) + p(m)$$

Solve for $f(m_j), \forall m_j > 1$ recursively. After solving for $f(m_k), k < j$, we can calculate $p(m_k)$ and the value of coupon payments $\forall i \leq 2m_{j-1}$. For $2m_{j-1} < i < 2m_j$, we calculate prices using $h(m_{j-1}, m_j, \frac{i}{2}, x)$ for any x, which the one that satisfies the above equation is $f(m_j)$.

A.6. Further out-of-sample results

Table 5 shows the results of the hedonic model when the out-of-sample R^2 is constructed as follows:

$$R^2_{OOS} = 1 - \frac{\sum_{t=1}^{T}(\hat{V}_{t+1} - V_{t+1})^2}{\sum_{t=1}^{T}(\bar{V}_t - V_{t+1})^2}$$

where $\hat{V}_t$ is the predicted CRE value and $\bar{V}_t$ is the average value over the out-of-sample period.

Period 1	Period 2	Period 3
0.4089054	0.467745	0.4249241

Table 13: Results from the hedonic model when constructing the out-of-sample R^2 using dollar value deviation, rather than percent deviation. The results are still inferior to the performance of our baseline model.