Bounding High Dimensional Comparative Statics

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Abstract

A core purpose of economic modeling is to conduct comparative static analyses. Often one is interested in characterizing the comparative static in terms of the underlying structural parameters. Yet, except in highly-stylized models, the theoretically implied relationships are usually intractable, and empirically demanding, requiring complete identification of the model. I derive novel, sharp bounds on comparative statics that are analytically simpler and are enumerated under partial identification of the model. The model can be of arbitrary dimensionality; I require only that the Jacobian is diagonally dominant — intuitively, there is limited feedback in the model. I show that the theory can be applied in canonical models across many fields in economics. I demonstrate application in canonical models across economics, and address existing methodological limitations in the research on peer effects, the gains from trade liberalizations, and price-cost pass-through.

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1 Introduction

A core purpose of modeling in economics is to conduct comparative statics: the analysis of the effect of an exogenous shock on an endogenous state. Except in highly stylized models (a specification becoming increasingly untenable in modern quantitative economics), the equations linking these variables are very complicated. Even when linearized in the outcome and shock, the equations are generally highly non-linear in the parameters of the model. Moreover, even when the shock and outcome of interest are associated with a single node (agent, location, product, etc), complete identification of the model is required, as the comparative static depends on the properties, and responses, of all other nodes in the model.

This complexity raises at least two important challenges in models of high dimensionality — those of many endogenous states or many heterogeneous nodes. The first is that dependence of the comparative statics on the underlying structural parameters is obscured. This is a major downside and underpins why economists often resort to stylized models (Thisse, 2010; Baqaee and Farhi, 2024). The second is that identifying the comparative static is empirically demanding. For example, in a standard production network model (Carvalho and Tahbaz-Salehi, 2019), the effect of a shock to a firm's productivity on its own output requires knowledge of the entire input-output network between all firms. Availability of network data such as this is often scarce and costly to collect (Banerjee et al., 2019; Breza et al., 2020).

In this paper, I develop a new tool to help solve these challenges. I derive *sharp* upper and lower bounds on the (linearized) comparative statics. Relative to the exact relationship, they have a simpler functional dependence on, and do not require complete information of, the underlying parameters of the model. In particular, the bounds only depend on a set of low dimensional sufficient statistics. This thus reduces the burden of each of the aforementioned challenges. The trade-off of is that the comparative static is only partially identified — a bound — with the width of the bound depending on the value of the sufficient statistics, therefore its usefulness depends on the application. Nonetheless, because I prove the bounds are sharp, this means if one only knew the values of the sufficient statistics and nothing more, the bounds are the most one can say about the magnitude and sign of the comparative static.

The comparative static bounds I derive are valid in a wide range of models. For my

main result (theorem 1), the only substantive assumption imposed is that the Jacobian (with respect to the endogenous states) of the model is diagonally dominant. This can be intuitively understood as implying that the endogenous feedback in the system is greater within a node than between nodes (Arrow and Hahn 1971 p.242). Notably, the number of nodes is not restricted, and neither is their heterogeneity beyond this assumption, thus permitting its application in high dimensional models. I show that diagonal dominance is satisfied in many canonical models across economics under relatively common assumptions, such as those that also imply equilibrium uniqueness (see table 1). I provide details on how to derive the bounds in each of these models.

I offer a number of variants on this result. For instance, instead of relying on diagonal dominance, one variant utilizes the spectral radius of a suitably transformed Jacobian; this coincides with the adjacency matrix in network games, of which there is already much interest (Ballester et al., 2006; Bramoullé et al., 2016). Another variant utilizes the sign-pattern of the Jacobian, corresponding to the case where the system has only positive feedback between the nodes (the Jacobian is an M-matrix, Horn and Johnson, 1991), which is present for instance in the baseline production network model (Carvalho and Tahbaz-Salehi, 2019), and in network games under strategic complements (Ballester et al., 2006).

The precise way in which the bounds achieve a simpler form is that they do not require a matrix inversion of the Jacobian, which, following from the implicit function theorem, is a necessary step in the calculation of the point value of the comparative static. This matrix inversion arises because it captures how a shock propagates throughout the system, with its value exactly corresponding to the resulting effect on the endogenous states. The inverse, for example, corresponds to the Leontief Inverse in the case of production networks (Carvalho and Tahbaz-Salehi, 2019). Matrix inverses are very complicated objects in high dimensional models (and these matrices can be massive, with dimensionality equal to the number of firms in the production network model), thus avoiding needing to take the inverse, as my bounds do, offers substantial simplification.

The advantages of the bounds are twofold.[1] First, the analytic simplicity itself

[1]In principle, there is also a third advantage: a reduction in the computational cost of computing the comparative static, as inverting very large matrices is very costly. However, the size of the Jacobian in typical models is not too burdensome with modern computing power. The primary value of the bounds, therefore, is their analytical simplicity and identification by low dimensional sufficient statistics.

may permit the derivation of new and insightful properties about the comparative statics with regards to their dependence on parameters of the model. I use this to generalize results in the Industrial Organization on when price-cost pass-through is greater than one, from symmetric to asymmetric oligopolies. In the appendix, I consider the Macroeconomic literature on fiscal multipliers and derive novel sufficient conditions for these to be greater than one.

Second, the bounds only require partial knowledge of the Jacobian to compute. Specifically, only knowledge of each of the diagonal elements (describing the endogenous feedback within a node), and the sum of the magnitude of the off-diagonal elements for each column (describing the endogenous feedback between nodes). I refer to the ratio of the two as the inverse diagonally dominant (iDD) degree. These pieces of information, along with analogous statistics on the direct effect of the shock, constitute the low dimensional statistics sufficient for calculation of the bounds. Importantly, any information about the off-diagonal elements beyond this is not needed. Intuitively, diagonal dominance ensures that the feedback between nodes is sufficiently small relative to feedback within a node, allowing one to ignore any specific bilateral information about the feedback, and to still be able to say something about the comparative static.

I demonstrate this value in two prevalent applications. First, to the literature on peer effects — the causal effect of a peer's outcome on one's own outcome. A common challenge is that social network data is often unavailable yet point identification of peer effects require complete knowledge of the network. I show, however, that partial identification, specifically a lower bound on the magnitude, requires very limited knowledge of the network: only whether an individual has or does not have any peers. I demonstrate this using the Add Health dataset and find that the bound is close to the exact value.

Second, to the international trade literature. A primary question is what are the gains in welfare from decreasing international trade costs? The standard method to answering this requires knowledge of the bilateral trade flows between all countries. This is very challenging in historical settings as such data is limited. I show however that partial identification of the welfare gains requires only the own-import share, which can be calculated from GDP and total imports for the country of interest. This data is much more readily available and I calculate the gains over the past 800 years for the UK using data from the Bank of England. This result is analogous to the

seminal sufficient statistic of Arkolakis et al. (2012), which requires the own-import share before and *after* the trade liberalization, while mine only requires it before. This is a significant advantage because identifying the own-import share after the trade liberalization is challenging, for instance requiring no confounding shocks.

The outline of this paper is as follows. In section 2, I outline the general framework and derive the comparative statics. In section 3, I review diagonal dominance and present the bounds on the comparative statics. In section 4, I demonstrate these results in the applications and canonical models. In section 5, I conclude.

Literature. To my knowledge, no such general method exists to bounding both the magnitude and sign of comparative statics in high dimensional models using low dimensional sufficient statistics. There is a large literature that has sought to determine the sign (but not the magnitude) of a comparative static under the most general assumptions, such the traditional qualitative economics (Bassett et al., 1967; Hale et al., 1999) or monotone comparative statics (Milgrom and Shannon, 1994; Villas-Boas, 1997; Quah, 2007; Barthel and Sabarwal, 2018).[2] The appeal there, as in the present paper, is that the implications about the comparative statics can be robust to, or agnostic about, specific (quantitative) model assumptions.

The production of low-dimensional sufficient statistics is potentially quite powerful in settings where the researcher has incomplete information on the model parameters. This has been emphasized in the networks literature, because the collection of network data is often prohibitively costly. Banerjee et al. (2019) identify network central individuals by using "gossip" questions, Alidaee et al. (2020); Breza et al. (2020) provide a means to quantify a network model using aggregate relational data, all of which circumvent requiring the collection of bilateral network data.[3] Relatedly, in the international trade and economic geography literature, Allen et al. (2024) characterize equilibrium uniqueness in high dimensional models using low dimensional sufficient statistics.

There is naturally a close parallel to the econometrics literature on partial identi-

[2]A related strand of the literature is regarding Le Chatelier's principle, which ranks comparative statics in the short vs long run under the most general assumptions (Milgrom, 2006; Dekel et al., 2023).

[3]Similar, but distinct, are the network games where players have incomplete information about each other (Galeotti et al., 2010) or where the planner has incomplete information about the players (Galeotti et al., 2020)

fication (Manski, 2009; Tamer, 2010). Typically, the research question there concerns what is the sharp identified set for a parameter of interest, under the available data and the assumptions on the data-generating process. This is closely related to mine, where the parameter of interest is the comparative static, the available data forms the low dimensional sufficient statistics, and the assumptions are placed on the structure of the Jacobian.

My methodology utilizes the assumption of a diagonally dominant Jacobian. This assumption has a long history in economics (McKenzie, 1960), often being invoked to guarantee uniqueness or stability of equilibria (Gale and Nikaido, 1965; Hadar, 1965; Dixit, 1986; Kolstad and Mathiesen, 1987; Allen et al., 2024; Adão et al., 2023) or for an invertible demand system (Cheng, 1985; Berry et al., 2013). Diagonal dominance has been used in the characterization of comparative statics in specific models, such as establishing the sign of comparative statics in an oligopoly model (Dixit, 1986) or a trade model (Jones et al., 1993), or for ranking comparative statics in a spatial model (Allen et al., 2020).[4] My results significantly generalize the application of diagonal dominance, by providing bounds on the comparative static magnitude and being valid in any model for which diagonal dominance is satisfied.

The theory developed in this paper applies a result from the linear algebra literature in Ostrowski (1952), who provides bounds on the inverse of diagonally dominant matrices. I innovate on this result by proving that the bounds are sharp (lemma 1), and deriving its implications for comparative static analysis. The bounds from Ostrowski (1952) have not been used in economics, to my knowledge.[5]

2 Model

Consider a system of $i \in \{1, ..., N\} \equiv \mathcal{N}$ nodes. The nodes may represent products, countries, agents, etc. Each node has an endogenous state, $y_i \in \mathbb{R}$ (e.g. the price of good i), with the system being subject to an exogenous shock, $x \in \mathbb{R}$, (e.g. a demand

[4] Christensen (2019) establishes the sign of a comparative static using a related condition: mean positive dominance.

[5] There is a related set of matrices called M-matrices with many established properties (Johnson, 1982; Horn and Johnson, 1991), that have had frequent use in economics (such as in production networks, Carvalho and Tahbaz-Salehi, 2019). An important difference with diagonally dominant matrices is that M-matrices require a restriction on the sign of the elements of the matrix, whereas a diagonally dominant matrix does not. I discuss these in remark 5.

shifter of one of the goods).[6] The state of all nodes are determined jointly by the following equations of state

$$\forall i \in \mathcal{N}: \quad 0 = f_i(\boldsymbol{y}, x) \tag{1}$$

where $\boldsymbol{y} = \{y_j\}_{j \in \mathcal{N}}$. The function $f_i : \mathbb{R}^{N+1} \to \mathbb{R}$ is assumed differentiable, and is typically derived from the equilibrium conditions in the underlying economic model. For example, $f_i(\boldsymbol{y}, x)$ might be the excess demand for good i, with y_i the price of, and x a demand shifter for, good i. A solution to equation (1) corresponds to an equilibrium, and I denote this by $\boldsymbol{y}^*(x)$; the solution needn't be unique. I denote by ∇_{ij} the partial derivative of f_i with respect to an the j endogenous state, y_j,

$$\nabla_{ij} \equiv \frac{\partial f_i(\boldsymbol{y}, x)}{\partial y_j} \tag{2}$$

and refer to this simply as the Jacobian throughout the paper. The dependence of ∇ on $\{\boldsymbol{y}, x\}$ is suppressed. Intuitively, the Jacobian describes the endogenous feedback between nodes in the system (how the equation of state for node i responds to a change in node j's state). For example, if equation (1) is the system of excess demands in a pure exchange economy, then ∇_{ij} is the cross price elasticity of demand for good i with respect to price j.

Comparative statics of the system are considered by taking an infinitesimal perturbation in x about a solution $\boldsymbol{y}^*(x)$, and examining the resulting change in state $\boldsymbol{y}$. Only first order effects are considered in this paper. Applying the implicit function theorem to equation (1), the infinitesimal change in the state is given by

$$\frac{\partial y_i}{\partial x} = -\sum_{j \in \mathcal{N}} \left\{\nabla^{-1}\right\}_{ij} \frac{\partial f_j}{\partial x} \tag{3}$$

where $\frac{\partial f_j}{\partial x}$ is the partial derivative of f_j with respect to the exogenous shock, x, with its dependence on $\{\boldsymbol{y}, x\}$ suppressed. I refer to $\frac{\partial f_j}{\partial x}$ as the vector of direct effects.

The comparative static $\frac{\partial y_i}{\partial x}$ depends on the inverse of Jacobian, ∇^{-1}. Intuitively, the matrix inverse shows up in the comparative static as the inverse appropriately aggregates all the endogenous feedback in the system. That is, the effect of the shock x on the state in i incorporates not only the direct effect of x on y_i, but also the

[6] x may also represent an aggregate shock, or a shock to multiple nodes at once.

indirect effect via the changes in states of all other nodes. States in other nodes $y_{j\neq i}$ respond to a change in state y_i, and this in turn causes y_i to change again. This feedback between nodes is precisely what the Jacobian describes, and the aggregation of all this feedback throughout the system is described by the inverse of the Jacobian.

This can be seen using a Neumann expansion of the inverse[7]

$$\left\{\nabla^{-1}\right\}_{ij} = \left\{\left(I - \tilde{\nabla}\right)^{-1} D^{-1}\right\}_{ij} = (I_{ij} + \tilde{\nabla}_{ij} + \sum_{k\in\mathcal{N}} \tilde{\nabla}_{ik}\tilde{\nabla}_{kj} + \cdots)\frac{1}{\nabla_{jj}} \tag{4}$$

where $\tilde{\nabla}_{ij} \equiv -\frac{\nabla_{ij}}{\nabla_{ii}}(1 - I_{ij})$ and $D_{ij} \equiv \nabla_{ii} I_{ij}$. The first equality factors out the diagonal matrix D so that the matrix inverse is then applied to $I - \tilde{\nabla}$, on which the Neumann expansion can be applied, leading to the second equality.

The term I_{ij} reflects the direct effect of the shock on state i. The term $\tilde{\nabla}_{ij}$ reflects the effect of the shock on state j, which then causes a change in state i (e.g. a price of one good responding to the change in the price of another good). The term $\tilde{\nabla}_{ik}\tilde{\nabla}_{kj}$ reflects the effect of the shock on state j, which causes a change in state k, which then causes a change in state i. All states k change due to states j hence the sum of $k \in \mathcal{N}$. The next terms in the sum have higher order chains of state responses. The full effect of the shock on the state in i is the sum of all these terms. For example, equation (4) corresponds to a Leontief Inverse in the case of a production network model (see Carvalho and Tahbaz-Salehi, 2019 equation 5).

Matrix inverses are generally very complicated (except under high symmetry or $N = 2$). Moreover, even when one is interested in identifying the comparative static only between a single pair of nodes, the inverse implies that complete identification of the entire $N \times N$ Jacobian matrix is required. This makes the analysis of comparative statics challenging in two regards: the former in understanding its properties and economic mechanisms, and the latter in the empirical burden required for identification.

In the next section, I derive new results that allow one to place bounds on the comparative static that are only a function of the non-inverted Jacobian. Thus, potentially alleviating the two issues discussed above.

[7] This expansion assumes the sum converges, which is guaranteed under assumption 1 (assumption 1 implies the spectral radius of $\tilde{\nabla}$ is less than one).

3 Theoretical Results

In subsection 3.1, I review diagonal dominance, and present the relevant properties. In subsection 3.2, I present my main results on bounding comparative statics. In subsection 3.3, I present extensions on the theory.

3.1 Diagonal Dominance

In deriving bounds on the comparative static that require no matrix inversion of the Jacobian, the only assumption about the Jacobian that I use is presented in assumption 1: diagonal dominance.[8]

Assumption 1. (Diagonal Dominance). *At $\boldsymbol{y}^*(x)$, the Jacobian is strictly column diagonally dominant,*

$$\forall i \in \mathcal{N}: \quad |\nabla_{ii}| > \sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|$$

$|z|$ denotes the absolute value of $z \in \mathbb{R}$. Formally, assumption 1 is referred to as strict column diagonal dominance in linear algebra (see e.g. Horn and Johnson, 2012 definition 6.1.9.); I refer to it simply as diagonal dominance for convenience (in section 3.3, I consider alternative forms of diagonal dominance). Note that assumption 1 is sufficient for ∇ to be non-singular by the Levy–Desplanques theorem (Horn and Johnson, 2012, theorem 6.1.10.a), and thus ∇^{-1} in the comparative static, equation (3), is well-defined under assumption 1. Under the model in section 2, diagonal dominance has the interpretation of the feedback within a node, $|\nabla_{ii}|$, being greater than the feedback between nodes, $\sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|$. For example, in the case where equation (1) is the system of excess demands in a pure exchange economy, assumption 1 implies that the own price elasticity of demand is greater than the sum of all cross-price elasticities of demand (all in absolute terms), for each price.

Assumption 1 is valid in a wide range of economic models. I show this in section 4 with a summary in table 1. Application of assumption 1 has a long history in economics (McKenzie, 1960). Notably in the characterization of the sign ("qualitative" properties) of comparative statics, (Bassett et al., 1967; Dixit, 1986; Jones et al., 1993)

[8] There is an invariance in the system described by equation (1): one can rearrange the order of i and the order of the arguments y_j in $f(\boldsymbol{y}, x)$. Correspondingly, the order of the rows and columns of the Jacobian can be rearranged without loss of generality. Assumption 1 only needs to apply to one arrangement.

but also in stability analyses (Hadar, 1965; Hale et al., 1999) and the invertibility of demand (Cheng, 1985; Berry et al., 2013). It is well-known that assumption 1 implies invertibility of the Jacobian, which therefore guarantees local uniqueness (Mas-Colell et al., 1995, proposition 17.D.1). Moreover, if one assumes that the diagonal elements are all positive ($\forall i : \nabla_{ii} > 0$) in addition to diagonal dominance, then the Jacobian is a P-matrix (all principal minors are positive), which implies global uniqueness within any closed rectangular region for which both of these assumptions hold (Gale and Nikaido, 1965 theorem 4).[9] The diagonal being positive can be implied by invoking the equilibrium to be stable (Hahn, 1982 theorem T.1.7c).[10] The condition in Allen et al. (2024) for equilibrium uniqueness implies that the Jacobian is strictly row diagonally dominant in the case of one type of agent (see remark 3 for results using row diagonal dominance).[11] Milgrom and Roberts (1990) also appeals to diagonal dominance for equilibrium uniqueness. As summarized in table 1, sufficient conditions for diagonal dominance in many models are also sufficient for equilibrium uniqueness.

In order to present the bounds, I first need to define an object that describes the (inverse) intensity of the diagonal dominance in the matrix. I refer to this as the *inverse diagonally dominant (iDD) degree*.[12]

Definition 1. (Inverse Diagonally Dominant (iDD) Degree). *For any matrix* ∇*, define*

[9]Intuition for this result can be understood as follows. Multiple solutions to equation (1) can occur when the gradient of function f_i — the Jacobian — "changes sign", thus allowing the function to cross zero at multiple points. Diagonal dominance of the Jacobian essentially implies we can ignore the cross derivatives in determining the sign. A positive diagonal of the Jacobian implies f_i always moves in the same direction, and thus does not change sign (whereas a negative diagonal means f_i reverses direction and thus does change sign). Hence the combination of these two is sufficient for a single crossing of f_i at zero, and thus a unique solution.

[10]For instance, imposing a tâtonnement condition for off-equilibrium dynamics, $\dot{y}_i = -f_i(\boldsymbol{y}, x)$, where $\dot{y}_i$ is the change in state over time. A solution is locally stable iff all eigenvalues of ∇ have positive real parts (Horn and Johnson, 1991, section 2.0). This is implied if ∇ is diagonally dominant and has positive diagonal elements, as then ∇ is positive definite.

[11]The equations of state in Allen et al. (2024) using their Remark 3 and one type ($H = 1$) are $0 = y_i - F_i(\boldsymbol{y})$ (omitting my exogenous variable $\mathbf{x}$). The Jacobian of this system is $\nabla_{ij} \equiv I_{ij} - \frac{\partial F_i}{\partial y_j}$. Their sufficient condition for uniqueness is $\sum_{j \in \mathcal{N}} \left| \frac{\partial F_i}{\partial y_j} \right| < 1$, which implies ∇ is strictly row diagonally dominant: $\sum_{j \in \mathcal{N} \setminus i} |\nabla_{ij}| = \sum_{j \in \mathcal{N} \setminus i} \left| \frac{\partial F_i}{\partial y_j} \right| < 1 - \left| \frac{\partial F_i}{\partial y_i} \right| \leq \left| 1 - \frac{\partial F_i}{\partial y_i} \right| = |\nabla_{ii}|$.

[12]The iDD degree is inversely related to the *diagonally dominant degree*, $|\nabla_{ii}| - \sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|$, from the linear algebra literature (Liu et al., 2010; Zhao et al., 2013). When applied to a canonical network model, the iDD degree is proportional to the weighted *degree centrality* (number of other nodes a node is connected to, Bramoullé et al., 2016). See section B.4, equation (102).

the iDD degree of node $i \in \mathcal{N}$ *by*

$$\delta_i \equiv \frac{\sum_{j \in \mathcal{N} \setminus i} |\nabla_{ji}|}{|\nabla_{ii}|} \tag{5}$$

and denote the maximal iDD degree across all other nodes by

$$\delta_{-i} \equiv \max_{j \in \mathcal{N} \setminus i} \delta_j \tag{6}$$

with ∇ *evaluated at* $\boldsymbol{y} = \boldsymbol{y}^*(x)$.

Note that $\delta_i \in [0, 1)$ under diagonal dominance of ∇ (assumption 1). When $\forall i : \delta_i = 0$, then ∇ is a diagonal matrix and is therefore maximally diagonally dominant. As any δ_i increases up from 0, the intensity of diagonal dominance diminishes.

Under the model in section 2, δ_i can be understood as summarizing the exposure of node i to endogenous feedback from other nodes in the system, relative to the feedback within node i. If $\forall i : \delta_i = 0$, the equation of state for a given i, f_i, in equation (1) does not directly depend on any endogenous state except the state in i, y_i. This is true for the equations of state for all i. Thus, under $\forall i : \delta_i = 0$, there is no endogenous feedback between any nodes in the system. As δ_i increases, the feedback between nodes increases.

With this definition, I am now in a position to present the relevant linear algebraic properties implied by diagonal dominance. Specifically, the bounds on, and the sign of, elements in the inverse of a diagonally dominant matrix. These are presented in lemma 1. The bounds and sign are from the linear algebra literature and apply generally without the matrix needing to have the interpretation of a Jacobian. I contribute to this literature by proving these bounds are sharp, which as far as I can tell, is novel. In theorem 1 in section 3.2, I derive implications from lemma 1 for the comparative statics, which is the key result of this paper. In what follows, I denote the sign operator, $\text{sgn}(z) \in \{-1, 0, 1\}$, which takes these values if $z < 0$, $z = 0$ or $z > 0$, respectively, for any $z \in \mathbb{R}$.

Lemma 1. (Bounds on Inverse Diagonally Dominant Matrices). *Suppose* ∇ *satisfies*

assumption 1, then

$$\forall i \in \mathcal{N}: \quad \left|\left\{\nabla^{-1}\right\}_{ii}\right| \in \left[\frac{1}{|\nabla_{ii}|}\frac{1}{1+\delta_i\delta_{-i}}, \frac{1}{|\nabla_{ii}|}\frac{1}{1-\delta_i\delta_{-i}}\right] \tag{7}$$

$$\forall i, \in \mathcal{N}, j \in \mathcal{N}\backslash i: \quad \left|\left\{\nabla^{-1}\right\}_{ij}\right| \leq \delta_j \left|\left\{\nabla^{-1}\right\}_{ii}\right| \tag{8}$$

$$\forall i \in \mathcal{N}: \quad \operatorname{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right) = \operatorname{sgn}\left(\nabla_{ii}\right) \tag{9}$$

with ∇ *evaluated at* $\boldsymbol{y} = \boldsymbol{y}^*(x)$. Conditional on $\nabla_{ii}, \delta_i, \delta_j, \delta_{-i}$, equations (7) and (8) are sharp.

Proof. Equations (7) and (9): Ostrowski (1952) equation (14). Equation (8): Ostrowski (1952) equation (13). Ostrowski (1952) assumes strict row diagonal dominance, whereas assumption 1 is strict column diagonal dominance. Thus, in applying their bounds, one must replace the matrix with its transpose, notably in their equations (1) and (13). See appendix A.1 for proof of the bounds being sharp. □

There are two key features to note of all bounds and sign in lemma 1. 1) They only depend on the Jacobian in its non-inverted form.[13] 2) Only the diagonal elements, $\{\nabla_{ii}\}_{i\in\mathcal{N}}$, and the iDD degrees, $\{\delta_i\}_{i\in\mathcal{N}}$, are required. Notably, no information beyond this on the off-diagonal elements, $\nabla_{i,j\neq i}$, are required. Moreover, conditional on this information, the bounds are sharp: there exists a ∇ consistent with $\{\nabla_{ii}\}_{i\in\mathcal{N}}$ and $\{\delta_i\}_{i\in\mathcal{N}}$ that satisfies the bounds with equality.

3.2 Comparative Statics Bounds

Using the results of lemma 1, I now present the bounds for the comparative statics in theorem 1. I present a number of variants on theorem 1 in section 3.3. To present these bounds, I use one more piece of notation which summarizes the direct effect of the shock on all nodes other than i

$$\left|\frac{\partial f_{-i}}{\partial x}\right| \equiv \sum_{j\in\mathcal{N}\backslash i}\left|\frac{\partial f_j}{\partial x}\right| \tag{10}$$

Theorem 1. (Comparative Static Bounds under Diagonal Dominance). *Suppose* ∇

[13] The righthand-side of equation (8) can be substituted out using the bounds in equation (7).

satisfies assumption 1. If

$$\left|\frac{\partial f_i}{\partial x}\right| > \left|\frac{\partial f_{-i}}{\partial x}\right| \delta_{-i} \tag{11}$$

then, the magnitude of the comparative static satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left[\frac{\left|\frac{\partial f_i}{\partial x}\right| - \left|\frac{\partial f_{-i}}{\partial x}\right| \delta_{-i}}{1+\delta_i\delta_{-i}}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \left|\frac{\partial f_{-i}}{\partial x}\right| \delta_{-i}}{1-\delta_i\delta_{-i}}\right] \tag{12}$$

and its sign

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right) \tag{13}$$

Otherwise, the comparative static satisfies

$$\frac{\partial y_i}{\partial x} \in \frac{1}{|\nabla_{ii}|}\left[\frac{-sgn\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} - \left|\frac{\partial f_{-i}}{\partial x}\right| \delta_{-i}}{1-\delta_i\delta_{-i}}, \frac{-sgn\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} + \left|\frac{\partial f_{-i}}{\partial x}\right| \delta_{-i}}{1-\delta_i\delta_{-i}}\right] \tag{14}$$

with both ∇ *and* $\frac{\partial \boldsymbol{f}}{\partial x}$ *evaluated at* $\boldsymbol{y} = \boldsymbol{y}^*(x)$*. Conditional on* $\nabla_{ii}, \delta_i, \delta_{-i}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$*, the bounds in equations (12) and (14) are sharp.*

Proof. See appendix A.2 for the derivation of equations (12) and (13). See appendix A.1 for proof that equation (12) is sharp. □

Under diagonal dominance of the Jacobian (assumption 1), theorem 1 gives upper and lower bounds on the the comparative static $\frac{\partial y_i}{\partial x}$. If equation (11) is also satisfied, then the sign is also determined.

There are two key features to note, just like with lemma 1. 1) The bounds in, equations (12) and (14), and the sign, equation (13), only depend on the Jacobian in the non-inverted form. 2) For the comparative static on the state in node i, only the corresponding diagonal element of the Jacobian, ∇_{ii}, the associated iDD degree and the maximum across all other nodes, δ_i, δ_{-i}, and the direct effect on that node and the summary across all others, $\frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$ are required. That is, these five objects are sufficient statistics for the comparative static bounds and sign. The sufficient statistics are low dimensional in the sense that only five pieces of information about ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ are required ($\nabla_{ii}, \delta_i, \delta_{-i}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$), while the combined number of elements in ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ is $N^2 + N$.

Moreover, conditional on this information $(\nabla_{ii}, \delta_i, \delta_{-i}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|)$, the bounds in both equations (12) and (14) are sharp: there exists a ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ consistent with this information that satisfies the bounds with equality. Another way of saying this is, if one weren't to have any additional information on ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ beyond $\nabla_{ii}, \delta_i, \delta_{-i}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$, (such as regarding the individual values of $\nabla_{i,j\neq i}$ or $\frac{\partial f_{j\neq i}}{\partial x}$), the bounds in theorem 1 are the most one can say about the magnitude of the comparative static.

To understand the form of the bounds, figure 1 provides a visualization. The comparative static $\frac{\partial y_i}{\partial x}$ is on the y-axis, and $\delta_i\delta_{-i}$ on the x-axis. Each series corresponds to a different value of $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right|$.[14] The bounds are shown in the case of $-\text{sgn}\left(\nabla_{ii}\right)\text{sgn}\left(\frac{\partial f_i}{\partial x}\right) = 1$, which effectively centers the bounds on a positive value of the comparative static (multiply all lines by negative one for the case $-\text{sgn}\left(\nabla_{ii}\right)\text{sgn}\left(\frac{\partial f_i}{\partial x}\right) = -1$).

Its instructive to first start with the special case where the shock only directly affects node i, $\forall k \neq i : \frac{\partial f_k}{\partial x} = 0$, which corresponds to the case $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| = 0$ given by the black-solid line in the figure. This condition implies equation (11) holds, with the bounds on the comparative static magnitude given by equation (12), and the sign given by equation (13), which is positive. Moreover, assume either there is no feedback from node i to all other nodes, $\delta_i = 0$, or $\delta_{-i} = 0$, which implies there is no feedback from other nodes onto node i. Then, the magnitude of the bounds from equation (12) imply point identification of the comparative static, $\left|\frac{\partial y_i}{\partial x}\right| = \frac{\left|\frac{\partial f_i}{\partial x}\right|}{|\nabla_{ii}|}$, and corresponds to the point on the black-solid line intersecting the y-axis. This is precisely equal to the case of one-node model. The matrix inverse has reduced to taking the reciprocal of the diagonal element. The comparative static is equal to the direct effect $\frac{\partial f_i}{\partial x}$ multiplied by this reciprocal, which captures the within-node feedback of the state to the direct effect.

As the feedback between between node i and the other nodes increases from zero, $\delta_i\delta_{-i} > 0$, the comparative static bounds in equation (12) become

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{\left|\frac{\partial f_i}{\partial x}\right|}{|\nabla_{ii}|}\left[\frac{1}{1+\delta_i\delta_{-i}}, \frac{1}{1-\delta_i\delta_{-i}}\right]$$

Intuitively, a "window" around the no-feedback case, $\frac{\left|\frac{\partial f_i}{\partial x}\right|}{|\nabla_{ii}|}$, opens up, with a greater

[14] I'm assuming $\left|\frac{\partial f_i}{\partial x}\right| \neq 0$ for figure 1 to permit cleaner exposition. However this is not required for Theorem 1.

width the greater the feedback is, i.e. a greater value of $\delta_i\delta_{-i}$. We see the upper and lower bounds in figure 1 for the black-solid line starting to diverge as $\delta_i\delta_{-i}$ increases. The window reflects the impact on the comparative static of the between-node feedback, enveloping the impacts that would arise across all permissible configurations of feedback, given by the set of ∇ consistent with the sufficient statistics $\nabla_{ii}, \delta_i, \delta_{-i}$. As the iDD degree gets bigger, the set of permissible configurations of ∇ increases, and the window widens. In the limit of $\delta_i\delta_{-i} \to 1$, the permissible feedback can be so strong that the upper bound on the magnitude approaches infinity. The lower bound asymptotes to $0.5 \cdot \frac{\left|\frac{\partial f_i}{\partial x}\right|}{|\nabla_{ii}|}$, that is, half of the value under the no feedback case.

Assumption 1 allows the maximum amplification in the system arising from the between-node feedback to be infinite (note that if $\forall j : \delta_j = 1$, then ∇ is possibly singular). Yet, it limits the maximum attenuation possible to one half. Hence, the sign of the comparative static relative to the no-feedback case cannot be overturned, and thus continues to be positive, and can be determined without knowledge of the off-diagonal elements of ∇ (i.e. via equation 13).

If the shock also directly affects other nodes, $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| > 0$, then the bounds across all values of feedback in the system (values of $\delta_i\delta_{-i}$) are wider. This reflects the additional impact on the state in i due to the changes in state in other nodes due to these direct effects, and the resulting feedback that ensues. The red-dashed and blue-dotted line in figure 1 show this for $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right|$ equal to 0.5 and 1 respectively. As long as the direct effects on other nodes are not too strong, i.e. equation (11) holding, then this additional impact on the state y_i is not enough to cause a change in the sign of the comparative static — the lower bound in the case of $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| = 1$ is precisely zero for all values of $\delta_i\delta_{-i}$.

The turning point is when the direct effects on other nodes are large enough such that $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| > 1$, violating equation (11). Then, a reversal of sign is possible. The bounds now include zero, and the sign can no longer be determined without more information about ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$. The bounds in equation (14) are used for this case. The green-dot-dash line in figure 1 demonstrates this in the case where $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \left|\frac{\partial f_i}{\partial x}\right| = 1.5$. Indeed we see that the lower bound is now below zero.

The condition for the sign, equation (13), implies that the sign of the comparative static is the same as the sign of the direct effect of the shock $\frac{\partial f_i}{\partial x}$ if the feedback within node i is negative, $\nabla_{ii} < 0$. To understand why, let's consider an increase in x and suppose the direct effect on node i is positive, $\frac{\partial f_i}{\partial x} > 0$. Then, the direct effect of the

shock causes f_i to increase. In equilibrium, $f_i = 0$, thus, we require the states $\boldsymbol{y}$ to adjust endogenously such that f_i ultimately falls back down. Ignore the endogenous responses of $y_{j\neq i}$ for a moment. If $\nabla_{ii} < 0$, then an increase in y_i would bring f_i back down, and thus restore equilibrium. That is, the comparative static would be positive, and therefore the same sign as the direct effect, in this case. But what about the effect of the endogenous changes in $y_{j\neq i}$? Diagonal dominance effectively imposes that feedback from the node i onto itself has the greatest effect on f_i. In fact, so much so that its effect cannot be overturned by the feedback between nodes. Thus we can effectively ignore the changes in $y_{j\neq i}$ when determining the sign of the change in y_i. The intuition above therefore continues to hold even when we allow for the resulting endogenous changes in $y_{j\neq i}$

Summary. The power of theorem 1 is that the comparative static can be partially identified (i.e. a bound) without needing to invert the Jacobian. This has two implications. The first is that the analytic form of the bound is simpler than the exact expression, therefore providing a tool to derive properties about the comparative static. The second is that all the information contained in the Jacobian or the direct effects is not used in the bound; specifically only some lower dimensional sufficient statistics are needed. This therefore provides a method of learning about the value of the comparative static under incomplete information of the Jacobian or direct effects. In section 4, I provide applications demonstrating the value of these implications.

3.3 Variants

In this section I provide a number of variants on theorem 1.

Remark 1. **Conditioning on More Information.** Theorem 1 only use partial knowledge of the vector of direct effects, $\left\{\frac{\partial f_k}{\partial x}\right\}_{k\in\mathcal{N}}$, specifically direct effect on i, $\frac{\partial f_i}{\partial x}$, and the summary on all other nodes $\left|\frac{\partial f_{-i}}{\partial x}\right|$. In proposition 1 in the appendix, I present the bounds conditional on knowledge of all the direct effects.

Remark 2. **Conditioning on Less Information.** Theorem 1 requires knowledge of the maximal degree of all other nodes, δ_{-i}. In some applications where one only has information about i, this information may not be available. One can derive bounds that do not condition on δ_{-i} by utilizing the fact that $\delta_{-i} < 1$ under assumption 1.

I present this result in proposition 2 in the appendix. I utilize this result in section 4.3.

Remark 3. **Row Diagonal Dominance**. Assumption 1 is column diagonal dominance of the Jacobian, ∇. Alternatively, one could assume the Jacobian is row diagonally dominant: this is equivalent to the transpose of the Jacobian satisfying assumption 1. Lemma 1 holds in this case with ∇ replaced by its transpose (with the iDD degree, definition 1, now defined in terms of the transpose of the Jacobian). Theorem 1 however is more limited in the case of a row diagonally dominant matrix: it applies for the comparative static of node i when the shock only directly affects node i, $\left|\frac{\partial f_{-i}}{\partial x}\right| = 0$. I state this result formally in proposition 3 in the appendix. I use this result in section 4.2..

Remark 4. **Generalized Diagonal Dominance**. The solutions to the equations of state, equation (1), are invariant up to a transformation, $g_i\left[f_i\left(\boldsymbol{y}, x\right)\right] = 0$, for any injective, continuously differentiable function g_i. The Jacobian associated with these transformed equations of state is $g_i'\nabla_{ij}$, where g' is the derivative of g. Assumption 1 applied to this transformed system is then $\forall i \in \mathcal{N} : |g_i'|\,|\nabla_{ii}| > \sum_{j\in\mathcal{N}\setminus i}\left|g_j'\right|\,|\nabla_{ji}|$, a condition referred to as generalized diagonal dominance of ∇ (Gao and Wang, 1992). This result provides extra scope for applying theorem 1: in the case where ∇ does not satisfy assumption 1, but there exists a $\mathbf{g}'$ such that $g_i'\nabla_{ij}$ does satisfy assumption 1, then theorem 1 can still be applied with $\nabla_{ij} \to g_i'\nabla_{ij}$ (also in definition 1) and $\frac{\partial f_i}{\partial x} \to g_i'\frac{\partial f_i}{\partial x}$. I discuss an application of this in footnote 28 in the oligopoly model (section B.3).

A notable application of generalized diagonal dominance is presented in proposition 4, where g_i' is equal to the left-eigenvector of a scaled version of the Jacobian, call this matrix A, with associated eigenvalue equal to the spectral radius of A. I show that ∇ is generalized diagonally dominant if A is irreducible and has a spectral radius below one. I apply this proposition to the network game model (section B.4), where interest is often in this spectral radius. Thus these bounds may be more useful.

Remark 5. **Signed Diagonal Dominance**. While maintaining assumption 1, one can impose additional assumptions on the Jacobian and as a result get stronger versions of the magnitude bounds than those from theorem 1. A notable case is if one assumes that the signs of the off-diagonal elements of the Jacobian are opposite to the diagonal elements (and an analogous condition for the direct effects), which I refer to as *signed*

diagonal dominance.[15] I present the implied comparative static bounds in proposition 5 in the appendix. This corresponds to the case where the system only has positive feedback between nodes and is very common in economic models. It always holds in the production network (section 4.1) and New Keynesian (section B.1) models, and I apply proposition 5 there. It also holds, for example, in the industrial organization model (section B.3) under CES demand, and the network game model (section B.4) under pure strategic complements.

4 Applications

I demonstrate the theoretical results of this paper in a range of models across the economics discipline. Table 1 summarizes these, with a sufficient condition for diagonal dominance presented for each model, along with the economic relevance of this sufficient condition for the model. As can be seen, diagonal dominance is typically sufficient for uniqueness and/or stability of the equilibrium, and therefore, very regularly assumed in their respective applications.

I start in section 4.1 by providing a step-by-step guide of how to apply the results of this paper, using the canonical production network model (Carvalho and Tahbaz-Salehi, 2019) as an example. Then, in sections 4.2. 4.4 and 4.3, I show how the theory can be used to solve difficult problems in the literature: identifying peer effects with limited network data, and calculating the gains from trade liberalization with no bilateral trade data, and proving when price-cost pass-through is greater than one in asymmetric oligopolies, respectively. Finally, in section 4.5, I present a summary of the remaining models covered in table 1, with full details in appendix B.

4.1 Step-by-Step Guide using a Production Network Model

Consider the workhorse production network from Carvalho and Tahbaz-Salehi (2019), section 2.1 (for early applications see Long and Plosser 1983; Acemoglu et al. 2012). This model is not only highly tractable and thus useful for expositional purposes, but it also highly relevant, underlying many macroeconomic frameworks in the recent literature, as attention has begun to focus on the macroeconomic implications of

[15] In the case where the diagonal elements are positive and off-diagonals negative, and under assumption 1, the Jacobian is an M-matrix (see Horn and Johnson, 1991, chapter 2.5).

production networks.[16]

Nodes in this framework correspond to firms, and interactions between nodes are described the by the supply chain network. For the comparative static on firm production, I show that assumption 1, and the additional conditions required for proposition 5, are always satisfied. Thus, one can apply the stronger bounds from proposition 5; I also apply the bounds from theorem 1 for exposition. The bounds for the comparative statics with respect to a government expenditure shock, a firm-level fiscal multiplier, depend only the expenditure share on intermediate inputs by firms, and the share of expenditure on each firm by households. Notably, no supply chain data is required for the bounds. The bounds reveal a highly succinct necessary condition for the fiscal multiplier on a firm to be greater than one: the expenditure share on intermediate goods by the firm is greater than the share of household expenditure on that firm.

There are $i \in \mathcal{N}$ firms, each with production function

$$q_i = \zeta_i l_i^{1-\alpha_i} \prod_{j \in \mathcal{N}} x_{ji}^{A_{ji}}$$

where q_i is output, l_i is labor, and x_{ji} are intermediate inputs purchased from firm j. $\{A_{ji}\}_{i \in \mathcal{N}, j \in \mathcal{N}}$ is the input-output matrix: the share of firm i total expenditure spent on goods from firm j. Following the typical convention in the literature, I assume $\forall i \in \mathcal{N} : A_{ii} = 0$ (the firm doesn't buy from itself). $\sum_{j \neq i} A_{ji} \equiv \alpha_i \in (0, 1)$ is the expenditure share on intermediate inputs, with $1 - \alpha_i$ the labor share. $\zeta_i \equiv (1 - \alpha_i)^{-(1-\alpha_i)} \prod_{j \in \mathcal{N}} (A_{ji})^{-A_{ji}}$ is a normalization constant. The firms takes prices $\{p_j\}_{j \in \mathcal{N}}$ and the wage $w \equiv 1$ (which serves as the numeraire) as given, due to perfect competition.

There is a representative household who supplies labor to all the firms, consumes their goods according to a Cobb-Douglas utility function, and pays taxes T to the government. Demand from the household for good i is therefore $c_i = \frac{1}{p_i} \beta_i (w - T)$, where $\beta_i \in [0, 1]$ is the share of household expenditure on good i, and $\sum_{i \in \mathcal{N}} \beta_i = 1$.

The government exogenously demands g_i of the goods from firm i. The total

[16] For example. see Acemoglu et al. (2012, 2016); Baqaee (2018) for the macro amplification of micro shocks; Afrouzi and Bhattarai (2023); Ghassibe (2021); Rubbo (2023) for applications in monetary economics; Flynn et al. (2023) for fiscal multipliers; and Baqaee and Farhi (2021); Bonadio et al. (2021) for applications to covid 2019.

demand facing firm i is therefore

$$q_i = c_i + g_i + \sum_{j \in \mathcal{N}} x_{ij} \tag{15}$$

The government balances its budget, $T = \sum_i p_i g_i$.

Suppose we are interested in the comparative static of the endogenous firm output q_i with respect to the exogenous government expenditure g_j — a firm-level fiscal multiplier. Then, we can use the equilibrium conditions to derive the following equations of state, which solve for $\{q_i\}_{i \in \mathcal{N}}$ in terms of $\{g_i\}_{i \in \mathcal{N}}$[17]

$$0 = f_i(\mathbf{q}, \mathbf{g}) = q_i - \beta_i \left(1 - \sum_j g_j\right) - g_i - \sum_{j \in \mathcal{N}} A_{ij} q_j \tag{16}$$

This is an example where I explicitly include multiple shocks in the equations of state. Applying the implicit function theorem, equation (3), we derive the comparative static

$$\frac{\partial q_i}{\partial g_j} = -\sum_{k \in \mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \frac{\partial f_k}{\partial g_j}, \qquad \nabla_{ik} = I_{ik} - A_{ik}, \qquad \frac{\partial f_k}{\partial g_j} = -I_{kj} + \beta_k \tag{17}$$

The Jacobian ∇ is always diagonally dominant (definition 1) in this model

$$\sum_{i \neq j} |\nabla_{ij}| = \sum_{i \neq j} A_{ij} = \alpha_j < 1 = |\nabla_{ii}|$$

Thus, theorem 1 applies to this model. The iDD degree (definition 1) used in the theorem is given by

$$\delta_i \equiv \frac{\sum_{i \neq j} |\nabla_{ij}|}{|\nabla_{ii}|} = \alpha_i$$

i.e. it is equal to the expenditure share on intermediate inputs, α_i. The maximal iDD degree is $\delta_{-i} = \max_{j \neq i} \alpha_j \equiv \alpha_{-i}$. The direct effect summary on all other nodes is

$$\left|\frac{\partial f_{-i}}{\partial g_j}\right| = \sum_{k \neq i} \left|\frac{\partial f_k}{\partial g_j}\right| = \begin{cases} j = i & \sum_{k \neq i} \beta_k = 1 - \beta_i \\ j \neq i & 1 - \beta_j + \sum_{k \notin \{i,j\}} \beta_k = 2(1 - \beta_j) - \beta_i \end{cases}$$

[17] The firms' first order conditions imply the equilibrium price is $p_i = w \equiv 1$ and the equilibrium intermediate good expenditure share is $x_{ji} = A_{ji} q_i$. Inserting these, along with household demand, and the government budget balance, into equation (15), yields equation (16).

The direct effects $\frac{\partial f_i}{\partial g_j}$ satisfy equation (11) of theorem 1 for the comparative statics, $\frac{\partial q_i}{\partial g_j}$, with $j = i$, as

$$\delta_{-i} \left| \frac{\partial f_{-i}}{\partial g_j} \right| = \alpha_{-i} (1 - \beta_i) < 1 - \beta_i = \left| \frac{\partial f_i}{\partial g_i} \right|$$

Thus, for $j = i$, $\frac{\partial q_i}{\partial g_i}$, theorem 1 establishes the sign, equation (13), in addition to an upper and lower bound, equation (12). For $j \neq i$, $\frac{\partial q_i}{\partial g_j}$, theorem 1 establishes upper and lower bounds on the comparative static, equation (14), but the bounds include zero so the sign is not determined.

Applying theorem 1, the bounds on the comparative static for $j = i$ are

$$\frac{\partial q_i}{\partial g_i} \in \left[\frac{(1 - \beta_i)(1 - \alpha_{-i})}{1 + \alpha_i \alpha_{-i}}, \frac{(1 - \beta_i)(1 + \alpha_{-i})}{1 - \alpha_i \alpha_{-i}} \right] \tag{18}$$

noting that the sign, which is positive, has been determined by equation (13)

$$\operatorname{sgn}\left(\frac{\partial q_i}{\partial g_i} \right) = -\operatorname{sgn}(\nabla_{ii}) \operatorname{sgn}\left(\frac{\partial f_i}{\partial g_i} \right) = - \cdot 1 \cdot -1 = 1$$

The bounds for $j \neq i$ are

$$\left| \frac{\partial q_i}{\partial g_j} \right| \in \left[\frac{\beta_i - (2 - 2\beta_j - \beta_i) \alpha_{-i}}{1 - \alpha_i \alpha_{-i}}, \frac{\beta_i + (2 - 2\beta_j - \beta_i) \alpha_{-i}}{1 - \alpha_i \alpha_{-i}} \right] \tag{19}$$

Theorem 1 implies that the bounds in equations (18) and (19) are sharp, conditional on $\nabla_{ii}, \delta_i, \delta_{-i}, \frac{\partial f_i}{\partial g_j}, \left| \frac{\partial f_{-i}}{\partial g_j} \right|$. This information set corresponds to $\alpha_i, \alpha_{-i}, \beta_i, \beta_j$ in this model. That is, if one only knew the values of $\alpha_i, \alpha_{-i}, \beta_i, \beta_j$ and nothing else, notably no knowledge of the input-output matrix $\{A_{ij}\}_{i \in \mathcal{N}, j \in \mathcal{N} \setminus i}$, equations (18) and (19) are the most one can say about the magnitude of the comparative statics, $\frac{\partial q_i}{\partial g_j}$.

If one conditions on more information, one could in principle say more about the comparative static (remark 5). A notable example in this model is conditioning on the fact that trade flows are nonnegative, $\forall i, j \neq i : A_{ij} \geq 0$. This implies that the off-diagonal elements of the Jacobian are all nonpositive, $\forall i, j \neq i : \nabla_{ij} = -A_{ij} \leq 0$, while the diagonal elements are positive $\forall i : \nabla_{ii} = 1 - A_{ii} = 1 > 0$.[18] Analogously,

[18]The off-diagonal elements of ∇ being all negative, while the diagonal elements all positive, translates into the feedback in this model being entirely positive: an increase in q_j causes q_i to increase, for any i, j. In terms of the economics, when firm j increases their output $q_j \uparrow$, they buy

note that the direct effect matrix $\frac{\partial f_i}{\partial g_j}$ satisfies the opposite sign pattern. Utilizing this additional information allows one to derive even stronger bounds than those given in theorem 1. Specifically one can use proposition 5 (see appendix A.3.5), because, under this additional information, ∇_{ij} and $\frac{\partial f_i}{\partial g_j}$ satisfy equations (53) and (54), respectively.

The result is a strengthening of the $j = i$ comparative static bounds from equation (18) to

$$\frac{\partial q_i}{\partial g_i} \in \left[(1 - \beta_i)(1 - \alpha_{-i}), \frac{1 - \beta_i}{1 - \alpha_i \alpha_{-i}} \right] \tag{20}$$

Comparing equation (20) to (18), it is immediate that the bound range in (20) is smaller. To be clear, the source of this gain is the additional information used in constructing the bounds (notably, the nonnegativity of the input-output matrix, A).

The power of the bounds, from either theorem 1 or proposition 5, can be understood by comparing them to the exact comparative static, equation (17). First, they are identified with less information. As already highlighted, the bounds require no knowledge of the input-output matrix A (beyond α_i, α_{-i}, and the assumption that $\forall i : A_{ii} = 0$), whereas the exact comparative static requires full knowledge of A. Second, they are much simpler analytically. In particular, the bounds do not require a matrix inversion of $I - A$, unlike in the exact comparative static.

Not requiring knowledge of the full input-output matrix A is potentially quite powerful, because network data is expensive to collect (Breza et al., 2020). The results of this paper provide a way to learn about the comparative statics without needing to collect it. In particular, one only needs knowledge of the low dimensional sufficient statistics, which in this model correspond to the expenditure share by households on each firm, β_i and, the total expenditure share on intermediate inputs by each firm, α_i. These are firm level variables, rather than firm-firm level (e.g. input-output data), and are therefore much less costly to collect.

Not having a matrix inversion in the expression significantly increases tractability. This potentially allows one to derive properties regarding the comparative static. For example, a necessary condition for the firm-level fiscal multiplier to be greater than

more goods from firm i according to the input share A_{ij}, and thus firm i output increases, $q_i \uparrow$. This is the only source of feedback in this model.

one, $\frac{\partial q_i}{\partial g_i} > 1$, can be simply derived from using the upper bound in equation (20)[19]

$$\alpha_i > \beta_i \tag{21}$$

This result implies that in a production network with any input-output matrix, $\{A_{ij}\}_{i\in\mathcal{N},j\in\mathcal{N}\setminus i}$, the fiscal multiplier on firm i, $\frac{\partial q_i}{\partial g_i}$, can be greater than one only if the firm's expenditure share on intermediate inputs, α_i, is greater than the expenditure share by households on the firm, β_i. Discerning this property from the comparative static exact expression, equation (17), is less straightforward, if possible at all.

4.2 Peer Effects

An important question in the social network literature is whether and to what extent there are peer effects? That is, whether the outcome (for instance, school performance) of your peers causally affects your outcome. The standard model conceptualizing this is the linear-in-means model (see Bramoullé et al. 2016 chapter 5 for a review, and appendix B.4 for the microfoundation using network games)

$$y_i = \beta a_i + \phi \sum_{j\in\mathcal{N}} G_{ij} y_j \tag{22}$$

where y_i is the outcome of individual i; a_i is an exogenous characteristic (for instance age) and $\beta \in \mathbb{R}$ the direct effect of the characteristic on the outcome; G_{ij} is the (normalized) adjacency matrix equal to $1/n_i$ if j is a peer of i with n_i the total number of i's peers, and equal to zero otherwise ($G_{ii} \equiv 0$ by convention); and $\phi \in \mathbb{R}$ is the peer effect, which is typically assumed to satisfy $|\phi| < 1$ so that the solution for y_i is well-defined.

Without further assumptions, point identification of ϕ requires knowledge of the entire adjacency matrix G: one can then identify ϕ by regressing y_i on a_i and $\sum_{j\in\mathcal{N}} G_{ij} y_j$, with the latter instrumented by $\sum_{j\in\mathcal{N}} G_{ij} a_j$ (Bramoullé et al., 2009). A challenge in the social network literature is that G is very often not fully known, as network data either doesn't exist or is too costly to collect. A growing number of recent papers have therefore developed methods which can be used to identify ϕ in cases where G isn't fully known by making additional assumptions, for example

[19] Proof: $\frac{\partial q_i}{\partial g_i} > 1 \implies \frac{1-\beta_i}{1-\alpha_i\alpha_{-i}} > 1 \implies \alpha_i\alpha_{-i} > \beta_i \implies \alpha_i > \beta_i$. The last implication used $\alpha_{-i} \leq 1$, in the spirit of remark 2.

on the network formation process (Breza et al., 2020; Auerbach, 2022), or assuming that one observes many instances of the network (de Paula et al., 2024; Lewbel et al., 2023).

The results of the present paper provide an alternative strategy to learning about ϕ with incomplete knowledge of G without needing to impose any additional assumptions; specifically, partial identification of ϕ. The Jacobian of the system given by equation (22) is $\nabla_{ij} = I_{ij} - \phi G_{ij}$. This does not satisfy assumption 1 but it does satisfy row diagonal dominance (remark 3) under $|\phi| < 1$, and hence proposition 3 can be used without any further assumptions. This yields the following bounds on the comparative static of y_i with respect to a_i

$$\frac{\partial y_i}{\partial a_i} \in \left[\frac{\beta}{1+\phi^2}, \frac{\beta}{1-\phi^2}\right] \tag{23}$$

(See appendix B.4 for the full details and derivation.) Notably, these bounds do not depend on G. One can invert these bounds to get the following bound on ϕ which also does not depend directly on G

$$|\phi| \geq \sqrt{\left|1 - \left|\beta / \frac{\partial y_i}{\partial a_i}\right|\right|} \tag{24}$$

Thus, one can put a lower bound on the magnitude of the peer effect with only knowledge on $\frac{\partial y_i}{\partial a_i}$ and β. These two objects can be identified with very minimal knowledge about G: specifically, which individuals have peers, and which do not — i.e. connected vs isolated individuals. Given this, $\frac{\partial y_i}{\partial a_i}$ is identified from regressing y_i on a_i for the connected individuals, and β from regressing y_i on a_i for the isolated individuals.

I demonstrate this using the Add Health data (https://addhealth.cpc.unc.edu), which has been used abundantly for the analysis of peer effects (see Calvó-Armengol et al. 2009 for early work). The data comes from surveys of adolescents who were in grades 7-12 during the 1994-95 school year. The data contains variables on school performance, student characteristics, and, importantly, friendship networks. I follow the same specification in Bramoullé et al. (2009), up to the limitations of the public-use sample.[20] Note that 12% of students in the public-use sample are isolated:

[20] Bramoullé et al. (2009) use the restricted version, which provides the full friendship network G, whereas the public-use version only provides network averages of certain variables, i.e. $\sum_{j \in \mathcal{N}} G_{ij} x_j$

they do not report having any friends.

The results are presented in table 2. Column (1) presents the results using the data on G, which gives point identification of ϕ equal to 0.778. Column (2) estimates β equal to -0.061, and column (3) estimates $\frac{\partial y_i}{\partial a_i}$ equal to -0.125. Plugging these into the bounds in equation (24) gives a lower bound on $|\phi|$ equal to 0.713. This bound is very close to the point estimate of 0.778, yet no information on the network G has been used except for whether students have or do not have friends, whereas the point estimate uses entire knowledge of the friendship network G.

4.3 Gains from Trade Liberalization

One of the primary questions in the International Trade literature is what are the welfare gains from reducing international trade costs? This is often operationalized by using a quantitative model to calculate the a comparative static of the form: the change in domestic welfare (the endogenous state) given a change in trade costs (the exogenous shock) with the rest of the world. The standard method (hat algebra Dekle et al. 2008) used to calibrate the model in order to calculate this comparative static requires the full matrix of bilateral trade flows between countries before the shock, $\{X_{ij}\}$, for origin i and destination j countries (the nodes). This can be infeasible in settings where the full matrix is not known because of data limitations.

An important example where this limitation is prevalent is in economic history, as very rarely do we have the full trade matrix, yet the gains from trade is still an important question of interest (Federico and Tena, 1991; Findlay and O'Rourke, 2007). In these settings, we nonetheless do often have a subset of the trade matrix for some countries $i \in \mathcal{N}$: GDP, $Y_i \equiv \sum_{j \in \mathcal{N}} X_{ij}$, and total imports, $IM_i \equiv \sum_{j \in \mathcal{N} \setminus i} X_{ji}$, which notably allows one to calculate the own-import share, X_{ii}/Y_i. The seminal paper of Arkolakis et al. (2012) proved that, in a wide class of standard trade models, knowledge of only the own-import share before and *after* a trade shock, $\overline{\tau}$, is sufficient

for some variables x (including GPA and age). I therefore use GPA for y_i rather than recreational activities, and do not include exogenous network effects in the model, i.e. terms of the form $\sum_{j \in \mathcal{N}} G_{ij} a_j$ (as then $\sum_{j \in \mathcal{N}} \left\{G^2\right\}_{ij} a_j$ is required for an IV but this variable cannot be created using the public-use sample).

to calculate the change in welfare, W_i[21]

$$\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} = -\frac{1}{\phi}\frac{\partial \ln \left(X_{ii}/Y_i\right)}{\partial \ln \overline{\tau}} \tag{25}$$

where $\phi > 0$ is the trade elasticity (equal to one plus the elasticity of substitution between countries' goods, in many model microfoundations). Although highly succinct, the limitation of equation (25) is the requirement of X_{ii}/Y_i after the shock. One cannot in general simply use the observed X_{ii}/Y_i after the trade cost $\overline{\tau}$ change because there may be confounding shocks (say, from concurrent changes in productivity). One also cannot apply equation (25) to counterfactual liberalizations.[22]

Using the results of this paper, I show that one is, however, able to put bounds on the first order change in welfare due to a proportional increase in country i trade costs with the rest of the world, $\overline{\tau}$,[23] using only X_{ii}/Y_i before the shock, if one further assumes demand is CES (assumption R3' in Arkolakis et al. 2012) and trade costs are quasi-symmetric (trade costs are symmetric up to an origin and destination shifter), both of which are fairly standard modeling restrictions (Allen et al., 2020). Moreover, because the bounds are sharp, this is the most one can same about the welfare change without more information about the trade matrix. See appendix B.2 for the full details and derivation.

In this class of models, the point value of the welfare change is

$$\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} = -2\frac{1+2\phi}{1+\phi}\left\{\nabla^{-1}\right\}_{ii}\left(1-\frac{X_{ii}}{Y_i}\right), \quad \nabla_{ij} = I_{ij} + \frac{\phi}{1+\phi}\frac{X_{ij}}{Y_i} \tag{26}$$

where I_{ij} the identity matrix. As noted above, one can see from equation (26) that the full matrix of bilateral trade flows $\{X_{ij}\}$ is needed for point identification of the welfare change. The Jacobian ∇ in this model, as given in equation (26), captures feedback between countries (the nodes) that arises due to demand substitution: if prices fall in one country, other countries consume more from them. This feedback is

[21] Equation (25) holds with arbitrary changes in τ, as noted in Arkolakis et al. (2012). I present the differential form to allow cleaner comparison to my bounds, which hold only in differential form.

[22] This limitation does not apply in important special case where the country is in autarky after the trade cost change (i.e. a trade cost change of infinite). Then, the change in welfare is equal to $1-\left(X_{ii}/Y_i\right)^{-1/\phi}$.

[23] The effect of a shock to trade costs between an arbitrary pair of countries can be characterized if one also observes the trade flow between this pair of countries. See equation (87) in the appendix for this, which also allows for a more general supply curve and $\phi > -0.5$.

stronger if (for example) the elasticity of substitution is higher, and if countries buy more from each other, hence the dependence of the Jacobian in equation (26) on the trade elasticity ϕ and the trade flows (relative to GDP), $\frac{X_{ij}}{Y_i}$, respectively.

The Jacobian in this model satisfies assumption 1 because $\phi > 0$, and one can apply the results of proposition 2 (so that knowledge of δ_{-i}, and therefore $\frac{X_{jj}}{Y_j}$ for any $j \in \mathcal{N}$ but i, is not required — see remark 2)

$$\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} \in -\frac{2\phi+1}{|\phi|} 2\delta_i \left[\frac{1}{1+\delta_i}, \frac{1}{1-\delta_i}\right], \quad \delta_i = \left|\frac{\phi}{1+\phi}\right| \frac{1-\frac{X_{ii}}{Y_i}}{1+\frac{\phi}{1+\phi}\frac{X_{ii}}{Y_i}} \tag{27}$$

The bounds on the welfare change given in equation (27) only require knowledge of $\frac{X_{ii}}{Y_i}$ before the shock; this value after the shock is not needed (in contrast to equation 25), and no other information about the trade flow matrix is needed (in contrast to equation 26).

To demonstrate this result, I calculate the elasticity of welfare with respect to a reduction in trade costs, $-\frac{\partial \ln W_i}{\partial \ln \overline{\tau}}$, for the UK over the past 800 years using data from the Bank of England ("A millennium of macroeconomic data" from https://www.bankofengland.co.uk/statistics/research-datasets). Point identification, equation (26), is impossible for this period as bilateral trade data for all countries is limited to after 1961 (the introduction of the UN and the Comtrade dataset). However, data on total imports and GDP for some countries goes much further back than this (Federico and Tena-Junguito, 2017), making partial identification, equation (27), feasible. Figure 2a presents the results for the UK. The bounds are wider in more recent times, with a width of about 30% of the value of the comparative static (note the log scale), while the bounds become much narrower as we go further back in time. This is convenient given that it is precisely further back in time when we do not have the trade data and so would want to rely on the bounds for identification.

The reason why the bounds are wider in more recent times is because the value of the iDD degree δ_i for $i = UK$ is greater, as shown in figure 2b. δ_i is greater in recent times because it is an increasing function of this the export share, $\frac{EX_i}{Y_i}$ (see equation 27), and the latter has been increasing in recent times, as also shown in figure 2b. Intuitively, the gains from trade depend on the exact trade configuration between countries. If a country has a large export share $\frac{EX_i}{Y_i}$, then there are many possible configurations of bilateral trade flows (i.e. many possible matrices $\{X_{ij}\}$), leading to a wide range of values for the gains from trade, consistent with the value of $\frac{EX_i}{Y_i}$.

When the export share is small, the scope for the gains from trade is less, and so the exact pattern of trade has less relevance, causing the bounds to be narrow. Hence, far back in time, my method can tightly pin down the gains from a finite trade cost shock, despite bilateral trade data not being available.

4.4 Cost-Price Pass-Through

There has been longstanding interest in the literature concerning the pass-through of cost shocks to prices. This dates back to Marshall (1890) in the context of tax incidence, with more recent work leveraging pass-through as a sufficient statistic for various welfare analyses (Chetty, 2009). In imperfectly competitive settings, one property that has been explored is when pass-through is above unity, which implies a number of qualitative properties (Anderson et al., 2001; Weyl and Fabinger, 2013; Bagnoli and Bergstrom, 2005; Stern, 1987).[24] However, theoretical characterizations have typically been limited to symmetric models, as the dimensionality of the problem is then reduced, permitting a tractable analysis. Under symmetric imperfect competition and linear cost, pass-through exceeds unity iff demand is log-convex (Seade, 1985; Bulow and Pfleiderer, 1983; Weyl and Fabinger, 2013). Using the results of the current paper, I offer a generalization to asymmetric models.

Consider a Bertrand oligopoly with differentiated products. There are $i \in \mathcal{N}$ firms (the nodes) each producing one good with constant unit costs, c_i (exogenous shocks), and with demand $D_i(\mathbf{p})$, where $\mathbf{p} = \{p_i\}_{i\in\mathcal{N}}$ are the product prices (endogenous states). $D_i(\mathbf{p})$ is assumed to be twice-continuously differentiable. The log profits of firm i are

$$\pi_i(\mathbf{p}, c_i) = \ln\left[(p_i - c_i) D_i(\mathbf{p})\right]$$

and the firm chooses p_i to maximize $\pi_i(\mathbf{p}, c_i)$, taking all other firms' prices as given. The pass-through to p_i of a change in c_i is given by (see appendix B.3 for details)

$$\frac{\partial p_i}{\partial c_i} = \left\{\nabla^{-1}\right\}_{ii}, \qquad \nabla_{ij} = \frac{c_i}{p_i} I_{ij} - \left(1 - \frac{c_i}{p_i}\right)^2 \frac{\partial^2 \ln D_i(p)}{\partial \ln p_i \partial \ln p_j} \tag{28}$$

Pass-through is greater than one, $\frac{\partial p_i}{\partial c_i} > 1$, iff $\{\nabla^{-1}\}_{ii} > 1$. Except in the cases of a monopoly or duopoly, $\{\nabla^{-1}\}_{ii}$ is a very complicated object in general. The literature

[24] Armstrong and Vickers (2023) theoretically analyze the case where pass-through is negative, known as Edgeworth's paradox.

typically imposes complete symmetry in demand and costs across all firms to make this tractable. The results of section 3 allow one to proceed without resorting to symmetry.

Recalling that log-convexity of demand is the relevant condition in the symmetric case, a sufficient condition for demand to be log-convex in the general case is the Hessian of log demand, $\frac{\partial^2 \ln D_i(p)}{\partial \ln p_i \partial \ln p_j}$, is row diagonally dominant with positive diagonal. This condition also is sufficient for ∇ to be row diagonally dominant, hence one can use the bounds from proposition 3 (see remark 3). This implies pass-through is always greater than unity under this condition. Thus, log-convexity of demand is tightly related to above unity pass-through in asymmetric models, just like in symmetric models.

4.5 Summary of Other Models

In section B.1, I consider a New Keynesian macroeconomic model. I utilize the framework of Auclert et al. (2024) due to its large temporal dimensionality. Nodes in this framework correspond to time periods, and interactions between nodes are described by the matrix of intertemporal marginal propensities to consume (MPC): the spending in each period from an income shock in some period. For the comparative statics of GDP on government expenditure — the fiscal multiplier — assumption 1 is satisfied under the condition that all MPCs are strictly positive. Focusing on the effect on GDP today from deficit-financed spending today, the impact fiscal multiplier, proposition 5 can be applied for a lower bound. This bound is potentially useful as it requires a fraction of the intertemporal MPCs to enumerate — an object that little is known about empirically. And it produces sufficient conditions for the multiplier to be greater than one — a benchmark of particular interest in the literature (Christiano et al., 2011).

In section B.3, I consider a canonical Bertrand oligopoly model with differentiated products in the Industrial Organization literature (Milgrom and Roberts, 1990). This field of study has attracted renewed attention due to the rising importance of market power and oligopolistic industries (De Loecker et al., 2020; Azar and Vives, 2021). My results hold particular use for emerging research permitting rich heterogeneity, often by exploiting a network-style approach (Bimpikis et al., 2019; Galeotti et al., 2024; Pellegrino, 2023). Nodes in this framework correspond to products, and interactions

between nodes are described by the Hessian of demand between products, and the mark-up. For the comparative statics on output prices, row diagonal dominance (remark 3) is implied by workhorse demand structures, such as CES, logit, linear (with products assumed substitutes), and translog (with products assumed substitutes plus an additional restriction on the parameters). I work through the case of CES, and show that the bounds on the comparative static with respect to a cost shock ("cost pass-through") are identified from the elasticity of substitution, the firm's own market share, and the firm's mark-up. An immediate implication of the bounds is that pass-through can never be greater than 100%.

In section B.5, I consider the the ARMA(p, q) model, a widely used econometric framework for time series (Brockwell and Davis, 2016). The nodes in this framework correspond to time periods, and interactions between nodes are described by the autoregressive lag coefficients. For the comparative statics of the dependent variable in the econometric model, I show assumption 1 is satisfied if the sum (in absolute terms) of all the autoregressive coefficients is less than one. This assumption guarantees existence and uniqueness of a stationary and causal solution. The bounds on the comparative static (theorem 1) with respect to the j^{th} lag of the independent variable in the econometric model depends only on the absolute sum of all the autoregressive coefficients, and the j^{th} lag coefficient.

In section B.6, I consider the spatial autoregressive model, a workhorse framework in spatial econometrics (LeSage et al., 2009). The nodes in this framework correspond to locations, and the interactions between nodes are described by the spatial weight matrix and the spatial lag parameter. For the comparative statics of the dependent variable in the econometric model, I show row diagonal dominance (remark 3) is satisfied if the (absolute) spatial lag parameter is less than one. This condition is assumed almost without exception in the literature. The bounds on the comparative static with respect to the independent variable in any location is identified from the spatial lag parameter and the coefficient on the independent variable alone; notably, the spatial weight matrix is not needed to calculate the bounds.

5 Conclusion

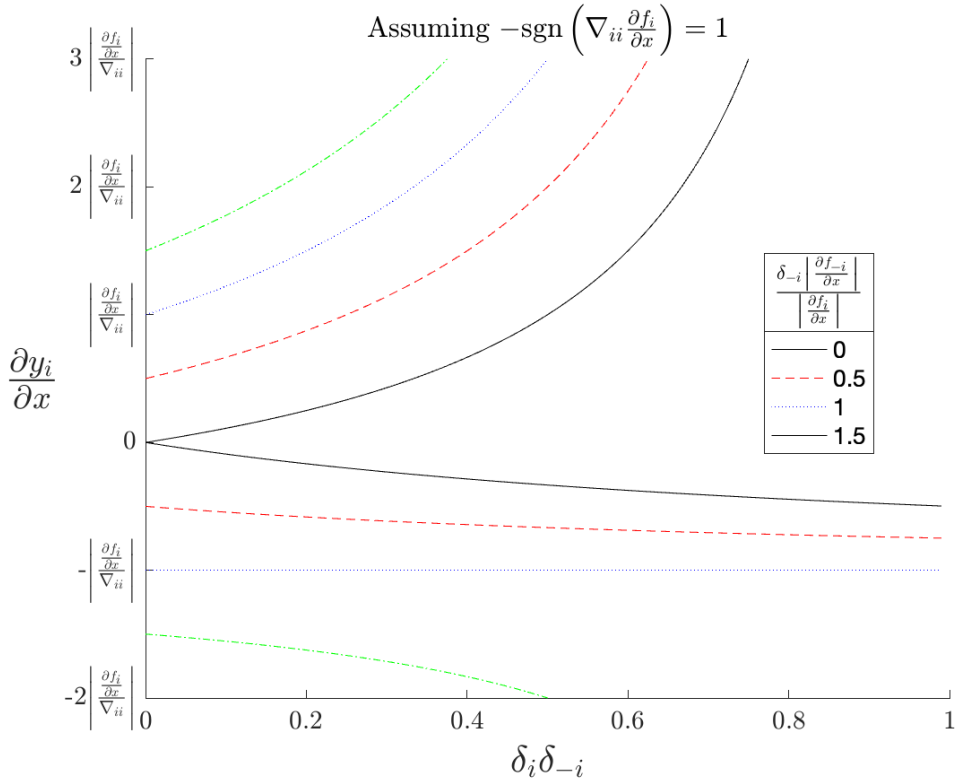
In this paper I revisit an old inquiry in economics: what can we deduce about comparative statics while making as few assumptions as possible? (Bassett et al., 1967;

Hale et al., 1999) I offer new results advancing the frontier in this subject. I exploit the widely-used assumption of diagonal dominance of the Jacobian, and derive new implications from this. Specifically, I show that this allows one to derive novel bounds on the comparative statics.

The value of this result is twofold: the bounds are analytically simpler than the exact relationship, and they are identified using low dimensional sufficient statistics. The analytic simplicity allows for easier characterization of the comparative statics in terms of the underlying model parameters, therefore possibly offering new insight into the underlying economic mechanisms. This provides an alternative to resorting to restrictive, stylized, low-dimensional versions of the model of interest.

The sufficient statistics permit one to still learn about the comparative static magnitudes in cases where full knowledge of the Jacobian is too costly or infeasible. Moreover, because the bounds are sharp, if the sufficient statistics are all that one knows about the model, then the bounds are the most that one can say about the comparative static magnitude.

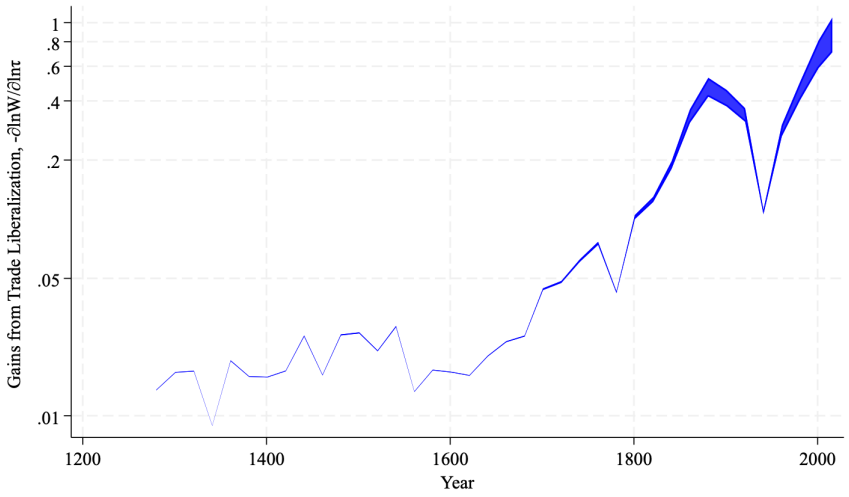
Figure 1: Visualization of Theorem 1



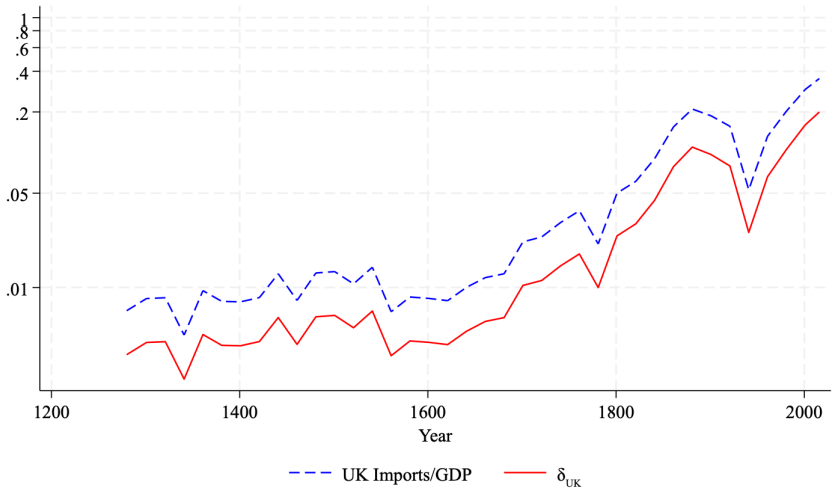
Notes. The bounds on the comparative static from theorem 1 are displayed in the case of $-\text{sgn}\left(\nabla_{ii}\right)\text{sgn}\left(\frac{\partial f_i}{\partial x}\right) = 1$. Each series shows the upper and lower bounds (inclusive of the sign) on the comparative static under different values of $\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right| / \frac{\partial f_i}{\partial x}$.

Figure 2: The Gains from Trade Liberalization in the UK over 800 Years

(a) Gains from Trade Liberalization



(b) Import share of GDP and the iDD degree, δ_i



Notes. In subfigure (a), the bounds are presented for the elasticity of UK welfare with respect to a proportional reduction in trade costs with the rest of the world. In subfigure (b), the import share of GDP is presented (data from the Bank of England).

Table 1: Diagonal Dominance in Economic Models

	Field/Model	Reference	Sufficient Condition for Diagonal Dominance	Relevance of Sufficient Condition to Model
Macroeconomics	Production Networks	Carvalho and Tahbaz-Salehi (2019)	Always	—
	New Keynesian	Auclert et al. (2024)	$\forall s,t : \text{MPC}_{st} \geq 0$ MPC: marginal propensity to consume s,t: time periods	No Inferior Income Effects
	International Trade / Economic Geography	Allen et al. (2020)	$\phi \gtrless -\frac{1}{2},\ \psi \gtrless -\frac{1}{2},\ \tau$ quasi-symmetric ϕ: demand elasticity, ψ: supply elasticity τ: trade cost matrix	Unique Interior Equilibrium
Microeconomics	Industrial Organization: Oligopoly	Milgrom and Roberts (1990)	Demand $\in$ {CES, logit, linear substitutes; restricted translog}	Unique Nash Equilibrium
	Game Theory: Network Games	Bramoullé et al. (2016)	$\lvert\phi\rvert\mu < 1,\ G$ irreducible ϕ: payoff impact parameter G: adjacency matrix μ: spectral radius of $\lvert G\rvert$	Unique Stable Nash Equilibrium
Econometrics	Time Series: ARMA(p,q)	Brockwell and Davis (2016)	$\sum_{s=1}^{p} \lvert\beta_s\rvert < 1$ β_s: s^{th} autoregressive coefficient	Unique Causal Stationary Solution
	Spatial Econometrics	LeSage et al. (2009)	$\lvert\rho\rvert < 1$ ρ: spatial lag parameter	Stable Solution

Notes. Summary of the models that I apply the theoretical results of this paper to (section 4). Cited is the main reference I use for the model. A sufficient condition for diagonal dominance is presented, either for column (assumption 1), row (remark 3), or generalized (remark 4) diagonal dominance. The last column shows a different implication of the sufficient condition that is relevant to the model.

Table 2: Bounding Peer Effects without Bilateral Network Data

	(1)	(2)	(3)
Peers GPA	0.778***		
	(0.101)		
Own Age	-0.080***	-0.061	-0.125***
	(0.022)	(0.071)	(0.022)
Observations	2156	293	2355
Sample	All	Isolated	Connected
Estimator	IV	OLS	OLS
First-stage F	42.6		

Robust standard errors in parentheses.

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Lower bound: 0.713

Notes. The dependent variable is student GPA. The first row is an average of the student's friends' GPAs, which is instrumented by an average of the student's friends' age, sex, and school grade. The second row the student's own age. Each specification also controls for the student's own school grade, whether white, whether born in USA, whether lives with mother, mother's education, and whether father is present. Column (1) uses the full sample of students; column (2) the students that report having no friends; column (3) the students that report having at least one friend. Data is from Add Health.

References

Acemoglu, Daron, Ufuk Akcigit, and William Kerr, "Networks and the Macroeconomy: An Empirical Exploration," *NBER Macroeconomics Annual*, January 2016, *30* (1), 273–335.

_ , Vasco M. Carvalho, Asuman Ozdaglar, and Alireza Tahbaz-Salehi, "The Network Origins of Aggregate Fluctuations," *Econometrica*, September 2012, *80* (5), 1977–2016.

Adão, Rodrigo, Costas Arkolakis, and Federico Esposito, "General Equilibrium Effects in Space: Theory and Measurement," Technical Report w25544, National Bureau of Economic Research, Cambridge, MA 2023.

Afrouzi, Hassan and Saroj Bhattarai, "Inflation and GDP Dynamics in Production Networks: A Sufficient Statistics Approach," *NBER*, 2023.

Alidaee, Hossein, Eric Auerbach, and Michael P. Leung, "Recovering Network Structure from Aggregated Relational Data Using Penalized Regression," *arXiv:2001.06052 [econ, q-fin, stat]*, January 2020.

Allen, Treb, Costas Arkolakis, and Xiangliang Li, "On the Equilibrium Properties of Spatial Models," *American Economic Review: Insights*, December 2024, *6* (4), 472–489.

_ , _ , and Yuta Takahashi, "Universal Gravity," *Journal of Political Economy*, 2020, *128* (2), 393–433.

Anderson, Simon P., André de Palma, and Brent Kreider, "Tax Incidence in Differentiated Product Oligopoly," *Journal of Public Economics*, August 2001, *81* (2), 173–192.

Arkolakis, Costas, Arnaud Costinot, and Andrés Rodríguez-Clare, "New Trade Models, Same Old Gains?," *The American Economic Review*, 2012, *102* (1), 94–130.

Armstrong, Mark and John Vickers, "Multiproduct Cost Pass-Through: Edgeworth's Paradox Revisited," *Journal of Political Economy*, October 2023, *131* (10), 2645–2665.

Arrow, Kenneth Joseph and Frank Hahn, "General Competitive Analysis," 1971.

Auclert, Adrien, Matthew Rognlie, and Ludwig Straub, "The Intertemporal Keynesian Cross," *Journal of Political Economy*, December 2024, *132* (12), 4068–4121.

Auerbach, Eric, "Identification and Estimation of a Partially Linear Regression Model Using Network Data," *Econometrica*, 2022, *90* (1), 347–365.

Azar, José and Xavier Vives, "General Equilibrium Oligopoly and Ownership Structure," *Econometrica*, 2021, *89* (3), 999–1048.

Bagnoli, Mark and Ted Bergstrom, "Log-Concave Probability and Its Applications," *Economic Theory*, August 2005, *26* (2), 445–469.

Ballester, Coralio, Antoni Calvó-Armengol, and Yves Zenou, "Who's Who in Networks. Wanted: The Key Player," *Econometrica*, 2006, *74* (5), 1403–1417.

Banerjee, Abhijit, Arun G Chandrasekhar, Esther Duflo, and Matthew O Jackson, "Using Gossips to Spread Information: Theory and Evidence from Two Randomized Controlled Trials," *The Review of Economic Studies*, November 2019, *86* (6), 2453–2490.

Baqaee, David and Emmanuel Farhi, "Keynesian Production Networks and the COVID-19 Crisis: A Simple Benchmark," *AEA Papers and Proceedings*, May 2021, *111*, 272–276.

Baqaee, David Rezza, "Cascading Failures in Production Networks," *Econometrica*, 2018, *86* (5), 1819–1838.

_ **and Emmanuel Farhi**, "Networks, Barriers, and Trade," *Econometrica*, 2024, *92* (2), 505–541.

Barthel, Anne-Christine and Tarun Sabarwal, "Directional Monotone Comparative Statics," *Economic Theory*, October 2018, *66* (3), 557–591.

Bassett, Lowell, Hamid Habibagahi, and James Quirk, "Qualitative Economics and Morishima Matrices," *Econometrica*, 1967, *35* (2), 221–233.

Berry, Steven, Amit Gandhi, and Philip Haile, "Connected Substitutes and Invertibility of Demand," *Econometrica*, 2013, *81* (5), 2087–2111.

Bimpikis, Kostas, Shayan Ehsani, and Rahmi İlkılıç, "Cournot Competition in Networked Markets," *Management Science*, June 2019, *65* (6), 2467–2481.

Bonadio, Barthélémy, Zhen Huo, Andrei A. Levchenko, and Nitya Pandalai-Nayar, "Global Supply Chains in the Pandemic," *Journal of International Economics*, November 2021, *133*, 103534.

Bramoullé, Yann, Andrea Galeotti, and Brian W. Rogers, *The Oxford Handbook of the Economics of Networks*, Oxford University Press, April 2016.

_ , Habiba Djebbari, and Bernard Fortin, "Identification of Peer Effects through Social Networks," *Journal of Econometrics*, May 2009, *150* (1), 41–55.

_ , Rachel Kranton, and Martin D'Amours, "Strategic Interaction and Networks," *The American Economic Review*, 2014, *104* (3), 898–930.

Breza, Emily, Arun G. Chandrasekhar, Tyler H. McCormick, and Mengjie Pan, "Using Aggregated Relational Data to Feasibly Identify Network Structure without Network Data," *American Economic Review*, August 2020, *110* (8), 2454–2484.

Brockwell, Peter J. and Richard A. Davis, *Introduction to Time Series and Forecasting*, springer publication, 2016.

Bulow, Jeremy I. and Paul Pfleiderer, "A Note on the Effect of Cost Changes on Prices," *Journal of Political Economy*, February 1983, *91* (1), 182–185.

Calvó-Armengol, Antoni, Eleonora Patacchini, and Yves Zenou, "Peer Effects and Social Networks in Education," *The review of economic studies*, 2009, *76* (4), 1239–1267.

Carvalho, Vasco M. and Alireza Tahbaz-Salehi, "Production Networks: A Primer," *Annual Review of Economics*, 2019, *11* (1), 635–663.

Cheng, Leonard, "Inverting Systems of Demand Functions," *Journal of Economic Theory*, October 1985, *37* (1), 202–210.

Chetty, Raj, "Sufficient Statistics for Welfare Analysis: A Bridge Between Structural and Reduced-Form Methods," *Annual Review of Economics*, September 2009, *1* (1), 451–488.

Christensen, Finn, "Comparative Statics and Heterogeneity," *Economic Theory*, April 2019, *67* (3), 665–702.

Christiano, Lawrence, Martin Eichenbaum, and Sergio Rebelo, "When Is the Government Spending Multiplier Large?," *Journal of Political Economy*, February 2011, *119* (1), 78–121.

Conley, Timothy G., "Spatial Econometrics," in "The New Palgrave Dictionary of Economics," London: Palgrave Macmillan UK, 2016, pp. 1–9.

de Paula, Áureo, Imran Rasul, and Pedro C L Souza, "Identifying Network Ties from Panel Data: Theory and an Application to Tax Competition," *The Review of Economic Studies*, September 2024, p. rdae088.

Dekel, Eddie, John K.-H. Quah, and Ludvig Sinander, "Comparative Statics with Adjustment Costs and the Le Chatelier Principle," February 2023.

Dekle, Robert, Jonathan Eaton, and Samuel Kortum, "Global Rebalancing with Gravity: Measuring the Burden of Adjustment," *IMF Staff Papers*, 2008, *55* (3), 511–540.

Dixit, Avinash, "Comparative Statics for Oligopoly," *International economic review*, 1986, pp. 107–122.

Fagereng, Andreas, Martin B. Holm, and Gisle J. Natvik, "MPC Heterogeneity and Household Balance Sheets," *American Economic Journal: Macroeconomics*, October 2021, *13* (4), 1–54.

Federico, Giovanni and Antonio Tena, "On the Accuracy of Foreign Trade Statistics (1909–1935): Morgenstern Revisited," *Explorations in Economic History*, 1991, *28* (3), 259–273.

_ **and Antonio Tena-Junguito**, "A Tale of Two Globalizations: Gains from Trade and Openness 1800–2010," *Review of World Economics*, August 2017, *153* (3), 601–626.

Findlay, Ronald and Kevin H. O'Rourke, *Power and Plenty: Trade, War, and the World Economy in the Second Millennium*, Princeton University Press, December 2007.

Fingleton, Bernard, "Spurious Spatial Regression: Some Monte Carlo Results with a Spatial Unit Root and Spatial Cointegration," *Journal of Regional Science*, 1999, *39* (1), 1–19.

Flynn, Joel P., Christina Patterson, and John Sturm, "Fiscal Policy in a Networked Economy," Working Paper 29619, National Bureau of Economic Research 2023.

Gale, David and Hukukane Nikaido, "The Jacobian Matrix and Global Univalence of Mappings," *Mathematische Annalen*, April 1965, *159* (2), 81–93.

Galeotti, Andrea, Benjamin Golub, and Sanjeev Goyal, "Targeting Interventions in Networks," *Econometrica*, 2020, *88* (6), 2445–2471.

_ , _ , _ , Eduard Talamàs, and Omer Tamuz, "Robust Market Interventions," 2024.

_ , Sanjeev Goyal, Matthew O. Jackson, Fernando Vega-Redondo, and Leeat Yariv, "Network Games," *The review of economic studies*, 2010, *77* (1), 218–244.

Gao, Yi-Ming and Xiao-Hui Wang, "Criteria for Generalized Diagonally Dominant Matrices and M-matrices," *Linear Algebra and its Applications*, May 1992, *169*, 257–268.

Ghassibe, Mishel, "Monetary Policy and Production Networks: An Empirical Investigation," *Journal of Monetary Economics*, April 2021, *119*, 21–39.

Gibbons, Stephen and Henry G. Overman, "Mostly Pointless Spatial Econometrics?*," *Journal of Regional Science*, May 2012, *52* (2), 172–191.

Hadar, Josef, "A Note on Dominant Diagonals in Stability Analysis," *Econometrica*, 1965, *33* (2), 442–444.

Hahn, Frank, "Chapter 16 Stability," in "Handbook of Mathematical Economics," Vol. 2, Elsevier, January 1982, pp. 745–793.

Hale, Douglas, George Lady, John Maybee, and James Quirk, *Nonparametric Comparative Statics and Stability*, Princeton University Press, 1999.

Horn, Roger A. and Charles R. Johnson, *Topics in Matrix Analysis*, Cambridge university press, 1991.

_ **and** _ , *Matrix Analysis*, Cambridge university press, 2012.

Johnson, Charles R., "Inverse M-matrices," *Linear Algebra and its Applications*, October 1982, *47*, 195–216.

Jones, Ronald W., Sugata Marjit, and Tapan Mitra, "Thes Stopler-Samuelson Theorem: Links to Dominant Diagonals," in Robert Becker, Michele Boldrin, Ronald Jones, and William Thomson, eds., *General Equilibrium, Growth, and Trade*, Academic Press, January 1993, pp. 429–441.

Kelejian, Harry H. and Dennis P. Robinson, "Spatial Correlation: A Suggested Alternative to the Autoregressive Model," in Luc Anselin and Raymond J. G. M. Florax, eds., *New Directions in Spatial Econometrics*, Berlin, Heidelberg: Springer Berlin Heidelberg, 1995, pp. 75–95.

Kolstad, Charles D. and Lars Mathiesen, "Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium," *The Review of Economic Studies*, October 1987, *54* (4), 681.

LeSage, James, Robert Kelley Pace, and Robert Kelley Pace, *Introduction to Spatial Econometrics*, Chapman and Hall/CRC, January 2009.

Lewbel, Arthur, Xi Qu, and Xun Tang, "Social Networks with Unobserved Links," *Journal of Political Economy*, April 2023, *131* (4), 898–946.

Liu, Jianzhou, Zejun Huang, and Juan Zhang, "The Dominant Degree and Disc Theorem for the Schur Complement of Matrix," *Applied Mathematics and Computation*, February 2010, *215* (12), 4055–4066.

Loecker, Jan De, Jan Eeckhout, and Gabriel Unger, "The Rise of Market Power and the Macroeconomic Implications," *The Quarterly Journal of Economics*, 2020, *135* (2), 561–644.

Long, John B. and Charles I. Plosser, "Real Business Cycles," *Journal of Political Economy*, February 1983, *91* (1), 39–69.

Manski, Charles F., *Identification for Prediction and Decision*, Harvard University Press, 2009.

Marshall, Alfred, "The Principles of Economics," Technical Report, McMaster University Archive for the History of Economic Thought 1890.

Mas-Colell, Andreu, Michael Dennis Whinston, and Jerry R. Green, *Microeconomic Theory*, Vol. 1, Oxford university press New York, 1995.

McKenzie, Lionel, "Matrices with Dominant Diagonals and Economic Theory," 1960.

Milgrom, Paul, "Multipliers and the LeChatelier Principle," in Michael Szenberg, Lall Ramrattan, and Aron A. Gottesman, eds., *Samuelsonian Economics and the Twenty-First Century*, Oxford University Press, August 2006, p. 0.

_ **and Chris Shannon**, "Monotone Comparative Statics," *Econometrica*, 1994, *62* (1), 157–180.

_ **and John Roberts**, "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica: Journal of the Econometric Society*, 1990, pp. 1255–1277.

Ord, Keith, "Estimation Methods for Models of Spatial Interaction," *Journal of the American Statistical Association*, March 1975, *70* (349), 120–126.

Ostrowski, A. M., "Note on Bounds for Determinants with Dominant Principal Diagonal," *Proceedings of the American Mathematical Society*, 1952, *3* (1), 26–30.

Pellegrino, Bruno, "Product Differentiation and Oligopoly: A Network Approach," 2023.

Quah, John K.-H, "The Comparative Statics of Constrained Optimization Problems," *Econometrica*, 2007, *75* (2), 401–431.

Ramey, Valerie A., "Ten Years after the Financial Crisis: What Have We Learned from the Renaissance in Fiscal Research?," *Journal of Economic Perspectives*, May 2019, *33* (2), 89–114.

_ **and Sarah Zubairy**, "Government Spending Multipliers in Good Times and in Bad: Evidence from US Historical Data," *Journal of Political Economy*, April 2018, *126* (2), 850–901.

Rubbo, Elisa, "Networks, Phillips Curves, and Monetary Policy," *Econometrica*, 2023.

Seade, Jesus, "Profitable Cost Increases and the Shifting of Taxation: Equilibrium Responses of Markets in Oligopoly," 1985.

Stern, Nicholas, "The Effects of Taxation, Price Control and Government Contracts in Oligopoly and Monopolistic Competition," *Journal of Public Economics*, March 1987, *32* (2), 133–158.

Tamer, Elie, "Partial Identification in Econometrics," *Annual Review of Economics*, September 2010, *2* (1), 167–195.

Thisse, Jacques-François, "Toward a Unified Theory of Economic Geography and Urban Economics," *Journal of Regional Science*, February 2010, *50* (1), 281–296.

Villas-Boas, J. Miguel, "Comparative Statics of Fixed Points," *Journal of Economic Theory*, March 1997, *73* (1), 183–198.

Weyl, E. Glen and Michal Fabinger, "Pass-Through as an Economic Tool: Principles of Incidence under Imperfect Competition," *Journal of Political Economy*, June 2013, *121* (3), 528–583.

Zhao, Jianxing, Feng Wang, and Yaotang Li, "The Diagonally Dominant Degree and Disc Separation for the Schur Complement of Ostrowski Matrix," *Journal of Applied Mathematics*, 2013, *2013.*

A Proofs and More Theoretical Results

A.1 Proof of the bounds being sharp in Lemma 1 and Theorem 1

Proof. The bounds are shown to be sharp if, conditional on assumption 1 and the information available in the bounds ($\nabla_{ii}, \delta_i, \delta_j, \delta_{-i}$ for lemma 1 and $\nabla_{ii}, \delta_i, \delta_{-i}, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$ for theorem 1), for each bound there exists a ∇ and $\frac{\partial \boldsymbol{f}}{\partial x}$ that satisfies the bound with equality. I prove this using the following Jacobian

$$\nabla = \begin{pmatrix} \nabla_{11} & \nabla_{12} & \nabla_{13} & \cdots & \nabla_{1N} \\ \nabla_{21} & \nabla_{22} & 0 & \cdots & 0 \\ 0 & 0 & \nabla_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \nabla_{NN} \end{pmatrix} \tag{29}$$

i.e. non-zero elements in the second leading principal sub-matrix, non-zero elements in the rest of the diagonal, and non-zero elements in the rest of the first row; all other elements are zero. Each non-zero element can be of arbitrary magnitude, conditional on satisfying assumption 1 (i.e. $|\nabla_{21}| \leq |\nabla_{11}|$ and $\forall i \neq 1 : |\nabla_{1i}| \leq |\nabla_{ii}|$), and be of arbitrary sign. The iDD degrees of this ∇ are

$$\delta_i = \begin{cases} i = 1: & \left|\frac{\nabla_{21}}{\nabla_{11}}\right| \\ i > 1: & \left|\frac{\nabla_{1i}}{\nabla_{ii}}\right| \end{cases} \tag{30}$$

For the direct effects, I utilize the case where

$$\forall i \notin \{1, 2\} : \quad \frac{\partial f_i}{\partial x} = 0 \tag{31}$$

which implyies the direct effect on all nodes other than $i = 1$ to be

$$\left|\frac{\partial f_{-1}}{\partial x}\right| = \left|\frac{\partial f_2}{\partial x}\right| \tag{32}$$

I will show that the Jacobian of equation (29) achieves the bounds of both lemma 1 and, along with the direct effects of equation (31), theorem 1 with equality for $i = 1$. For $i \neq 1$, one can relabel the rows in the matrix of equations (29) and (31)

accordingly and the proof follows analogously. Moreover, it is sufficient to show the proof only in the case where $i = 2$ has the maximal degree across all other nodes, i.e. $\delta_{-1} = \delta_2$. In the case where some other node $j \neq 1, 2$ has the maximal degree, one can consider a new Jacobian and direct effects which is the same as that in equation (29) equations (29) and (31) but with the labels of rows 2 and j switched.

Proving sharpness will come down to showing that there exists a particular sign pattern of $\nabla_{22}, \nabla_{12}, \nabla_{21}$, and also of $\frac{\partial f_2}{\partial x}$ in the case of Theorem 1 (all of which are not conditioned on, i.e. these signs are not part of the sufficient statistics), such that the bounds hold with equality.

Proof for Lemma 1. The inverse of the matrix of equation (29) is

$$\nabla^{-1} = \begin{pmatrix} \frac{\nabla_{22}}{\phi} & -\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{22}}{\phi} & -\frac{\nabla_{13}}{\nabla_{33}}\frac{\nabla_{22}}{\phi} & \cdots & -\frac{\nabla_{1N}}{\nabla_{NN}}\frac{\nabla_{22}}{\phi} \\ -\frac{\nabla_{21}}{\phi} & \frac{\nabla_{11}}{\phi} & \frac{\nabla_{13}}{\nabla_{33}}\frac{\nabla_{21}}{\phi} & \cdots & \frac{\nabla_{1N}}{\nabla_{NN}}\frac{\nabla_{21}}{\phi} \\ 0 & 0 & \frac{1}{\nabla_{33}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\nabla_{NN}} \end{pmatrix} \tag{33}$$

where $\phi \equiv \nabla_{11}\nabla_{22} - \nabla_{12}\nabla_{21}$. Manipulating the expression for the $(1, 1)$ element

$$\begin{aligned} \left\{\nabla^{-1}\right\}_{11} &= \frac{\nabla_{22}}{\phi} \\ &= \frac{1}{\nabla_{11}} \frac{1}{1 - \frac{\nabla_{21}}{\nabla_{11}}\frac{\nabla_{12}}{\nabla_{22}}} \\ &= \frac{1}{\nabla_{11}} \frac{1}{1 - \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}} \end{aligned} \tag{34}$$

$$\implies \left|\left\{\nabla^{-1}\right\}_{11}\right| = \frac{1}{|\nabla_{11}|} \frac{1}{1 - \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}} \tag{35}$$

where the second line used the definition of ϕ; the third line used $\frac{\nabla_{21}}{\nabla_{11}}\frac{\nabla_{12}}{\nabla_{22}} = \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}$ from equation (30), and $\delta_2 = \delta_{-1}$; the last line applied the modulus operator, and used $\delta_1\delta_{-1} < 1$. By equation (35), we see that the matrix from equation (29) achieves the upper bound in equation (7) for $i = 1$ when $\operatorname{sgn}(\nabla_{12}\nabla_{21}\nabla_{22}) = \operatorname{sgn}(\nabla_{11})$, and achieves the lower bound when $\operatorname{sgn}(\nabla_{12}\nabla_{21}\nabla_{22}) = -\operatorname{sgn}(\nabla_{11})$

Manipulating the expression for the $(i = 1, j \neq 1)$ elements of equation (33)

$$\begin{aligned} j \neq 1: \quad \left\{\nabla^{-1}\right\}_{1j} &= -\frac{\nabla_{1j}}{\nabla_{jj}}\frac{\nabla_{22}}{\phi} \\ \implies \left|\left\{\nabla^{-1}\right\}_{1j}\right| &= \delta_j \left|\left\{\nabla^{-1}\right\}_{11}\right| \end{aligned} \tag{36}$$

where the second line applied the modulus operator and used $\delta_j = \left|\frac{\nabla_{1j}}{\nabla_{jj}}\right|$ from equation (30) and $\frac{\nabla_{22}}{\phi} = \left\{\nabla^{-1}\right\}_{11}$ from equation (33). By equation (36), we see that the matrix from equation (29) achieves the bound in equation (8) for $i = 1$ with equality for $\forall j \neq i$.

Proof for Theorem 1. Calculating the comparative static, equation (3), using the matrix from equation (30)

$$\begin{aligned} \frac{\partial y_1}{\partial x} &= -\sum_{k\in\mathcal{N}} \left\{\nabla^{-1}\right\}_{1k}\frac{\partial f_k}{\partial x} \\ &= -\left\{\nabla^{-1}\right\}_{11}\frac{\partial f_1}{\partial x} - \left\{\nabla^{-1}\right\}_{12}\frac{\partial f_2}{\partial x} \\ &= \left|\left\{\nabla^{-1}\right\}_{11}\right|\left[-\text{sgn}\left(\left\{\nabla^{-1}\right\}_{11}\right)\frac{\partial f_1}{\partial x} - \text{sgn}\left(\left\{\nabla^{-1}\right\}_{12}\frac{\partial f_2}{\partial x}\right)\left|\frac{\left\{\nabla^{-1}\right\}_{12}}{\left\{\nabla^{-1}\right\}_{11}}\frac{\partial f_2}{\partial x}\right|\right] \\ &= \left|\left\{\nabla^{-1}\right\}_{11}\right|\left[-\text{sgn}\left(\left\{\nabla^{-1}\right\}_{11}\right)\frac{\partial f_1}{\partial x} + \text{sgn}\left(\left\{\nabla^{-1}\right\}_{11}\frac{\nabla_{12}}{\nabla_{22}}\frac{\partial f_2}{\partial x}\right)\delta_2\left|\frac{\partial f_2}{\partial x}\right|\right] \\ &= \left|\left\{\nabla^{-1}\right\}_{11}\right|\left[-\text{sgn}\left(\nabla_{11}\right)\frac{\partial f_1}{\partial x} + \text{sgn}\left(\nabla_{11}\frac{\nabla_{12}}{\nabla_{22}}\frac{\partial f_2}{\partial x}\right)\delta_{-1}\left|\frac{\partial f_{-1}}{\partial x}\right|\right] \\ &= \frac{1}{|\nabla_{11}|}\frac{-\text{sgn}\left(\nabla_{11}\right)\frac{\partial f_1}{\partial x} + \text{sgn}\left(\nabla_{11}\frac{\nabla_{12}}{\nabla_{22}}\frac{\partial f_2}{\partial x}\right)\left|\frac{\partial f_{-1}}{\partial x}\right|\delta_{-1}}{1 - \text{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}} \end{aligned} \tag{37}$$

where the first line is equation (3); the second line used equation (31); the third line factored out $|\{\nabla^{-1}\}_{11}|$ and wrote in terms of the sign operator; the fourth line used $k \neq 1 : \text{sgn}\left(\{\nabla^{-1}\}_{1k}\right) = \text{sgn}\left(-\frac{\nabla_{1k}}{\nabla_{kk}}\frac{\nabla_{22}}{\phi}\right) = \text{sgn}\left(-\frac{\nabla_{1k}}{\nabla_{kk}}\left\{\nabla^{-1}\right\}_{11}\right)$ from equation (33), and $\left|\frac{\left\{\nabla^{-1}\right\}_{1k}}{\left\{\nabla^{-1}\right\}_{11}}\right| = \delta_k$ from equation (36); the fifth line used that $\delta_2 = \delta_{-1}$, $\left|\frac{\partial f_2}{\partial x}\right| = \left|\frac{\partial f_{-1}}{\partial x}\right|$ from equation (32), and $\text{sgn}\left(\{\nabla^{-1}\}_{11}\right) = \text{sgn}\left(\nabla_{11}\right)$ from equation (9); the sixth line used equation (35).

Suppose equation (11) holds. Then, the modulus of equation (37) is given by

$$\left|\frac{\partial y_1}{\partial x}\right| = \frac{1}{|\nabla_{11}|}\frac{\left|\frac{\partial f_1}{\partial x}\right| - \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\partial f_2}{\partial x}\frac{\partial f_1}{\partial x}\right)\left|\frac{\partial f_{-1}}{\partial x}\right|\delta_{-1}}{1 - \operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\nabla_{21}}{\nabla_{11}}\right)\delta_1\delta_{-1}} \tag{38}$$

By equation (38), we see that the Jacobian and direct effects of equations (29) and (31) achieve the upper bound of equation (12) for $i = 1$ when $\operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\partial f_2}{\partial x}\right) = -\operatorname{sgn}\left(\frac{\partial f_1}{\partial x}\right)$ and $\operatorname{sgn}(\nabla_{21}) = \operatorname{sgn}\left(\nabla_{11}\frac{\nabla_{12}}{\nabla_{22}}\right)$, and the lower bound for $i = 1$ when $\operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\partial f_2}{\partial x}\right) = \operatorname{sgn}\left(\frac{\partial f_1}{\partial x}\right)$ and $\operatorname{sgn}(\nabla_{21}) = -\operatorname{sgn}\left(\nabla_{11}\frac{\nabla_{12}}{\nabla_{22}}\right)$.

Suppose equation (11) does not hold. Then, by equation (37), we see that the Jacobian and direct effects of equations (29) and (31) achieve the the upper bound of equation (14) for $i = 1$ when $\operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\partial f_2}{\partial x}\right) = \operatorname{sgn}(\nabla_{11})$ and $\operatorname{sgn}(\nabla_{21}) = \operatorname{sgn}\left(\nabla_{11}\frac{\nabla_{12}}{\nabla_{22}}\right)$, and the lower bound for $i = 1$ when $\operatorname{sgn}\left(\frac{\nabla_{12}}{\nabla_{22}}\frac{\partial f_2}{\partial x}\right) = -\operatorname{sgn}(\nabla_{11})$ and $\operatorname{sgn}(\nabla_{21}) = \operatorname{sgn}\left(\nabla_{11}\frac{\nabla_{12}}{\nabla_{22}}\right)$. □

A.2 Proof of theorem 1

Proof. Starting with equation (3),

$$\begin{aligned}
\frac{\partial y_i}{\partial x} &= -\sum_{j\in\mathcal{N}}\left\{\nabla^{-1}\right\}_{ij}\frac{\partial f_j}{\partial x} \\
\frac{\partial y_i}{\partial x} + \left\{\nabla^{-1}\right\}_{ii}\frac{\partial f_i}{\partial x} &= -\sum_{j\in\mathcal{N}\setminus i}\left\{\nabla^{-1}\right\}_{ij}\frac{\partial f_j}{\partial x} \\
\left|\frac{\partial y_i}{\partial x} + \left\{\nabla^{-1}\right\}_{ii}\frac{\partial f_i}{\partial x}\right| &= \left|\sum_{j\in\mathcal{N}\setminus i}\left\{\nabla^{-1}\right\}_{ij}\frac{\partial f_j}{\partial x}\right| \\
&\leq \sum_{j\in\mathcal{N}\setminus i}\left|\left\{\nabla^{-1}\right\}_{ij}\right|\left|\frac{\partial f_j}{\partial x}\right| \\
&\leq \left|\left\{\nabla^{-1}\right\}_{ii}\right|\sum_{j\in\mathcal{N}\setminus i}\delta_j\left|\frac{\partial f_j}{\partial x}\right| \\
&\leq \left|\left\{\nabla^{-1}\right\}_{ii}\right|\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right|
\end{aligned}$$

where the first line is equation (3); the second line is a simple rearrangement; the third line applied the absolute operator; the fourth line used the triangle inequality; the fifth line applied equation (8) from Lemma 1. The last line used equation (10). The last line can be rewritten as follows

$$\begin{aligned}\frac{\partial y_i}{\partial x} &\in \left[-\left\{\nabla^{-1}\right\}_{ii}\frac{\partial f_i}{\partial x} - \left|\left\{\nabla^{-1}\right\}_{ii}\right|\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right|, -\left\{\nabla^{-1}\right\}_{ii}\frac{\partial f_i}{\partial x} + \left|\left\{\nabla^{-1}\right\}_{ii}\right|\delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right|\right] \\ &\in \left|\left\{\nabla^{-1}\right\}_{ii}\right|\left[-\text{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right)\frac{\partial f_i}{\partial x} - \delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right|, -\text{sgn}\left(\left\{\nabla^{-1}\right\}_{ii}\right)\frac{\partial f_i}{\partial x} + \delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right|\right]\end{aligned} \tag{39}$$

The first line used the identity: $|a+b| < c$ is equivalent to $a \in [-b-c, -b+c]$ for any real a, b, c. The second line factored out $|\{\nabla^{-1}\}_{ii}|$.

Suppose equation (11) holds, then the sign of both upper and lower bounds is determined by $-\text{sgn}\left(\{\nabla^{-1}\}_{ii}\frac{\partial f_i}{\partial x}\right)$. Using equation (9), this is equivalent to $-\text{sgn}\left(\nabla_{ii}\frac{\partial f_i}{\partial x}\right)$, hence proving equation (13). The upper and lower bounds having the same sign implies the following bounds on the magnitude

$$\left|\frac{\partial y_i}{\partial x}\right| \in \left|\left\{\nabla^{-1}\right\}_{ii}\right|\left[\left|\frac{\partial f_i}{\partial x}\right| - \delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right|, \left|\frac{\partial f_i}{\partial x}\right| + \delta_{-i}\left|\frac{\partial f_{-i}}{\partial x}\right|\right] \tag{40}$$

Applying the lower (upper) bound of $|\{\nabla^{-1}\}_{ii}|$ from equation (7) of Lemma 1 to the lower (upper) bound of equation (40) proves equation (12).

Suppose equation (11) does not hold, then the lower (upper) bound of equation (39) is always negative (positive), and hence the sign of the comparative static is not determined in this case. Thus, for both the lower and upper bound in equation (39), we must use the upper bound on $|\{\nabla^{-1}\}_{ii}|$ from equation (7) of Lemma 1, which proves equation (14). □

A.3 Variants: Details

A.3.1 Conditioning on More Information

The following proposition presents bounds conditional on knowledge of all the direct effects, $\left\{\frac{\partial f_k}{\partial x}\right\}_{k\in\mathcal{N}}$, as opposed to only the direct effect for some i, $\frac{\partial f_i}{\partial x}$, and the summary on all other nodes $\left|\frac{\partial f_{-i}}{\partial x}\right|$.

Proposition 1. (Comparative Static Bounds using all Direct Effects). *Suppose* ∇

satisfies assumption 1. If

$$\left|\frac{\partial f_i}{\partial x}\right| > \sum_{k\in\mathcal{N}\setminus i} \delta_k \left|\frac{\partial f_k}{\partial x}\right| \tag{41}$$

then, the magnitude of the comparative static satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left[\frac{\left|\frac{\partial f_i}{\partial x}\right| - \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1+\delta_i\delta_{-i}}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1-\delta_i\delta_i}\right] \tag{42}$$

and its sign

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right) \tag{43}$$

Otherwise, the comparative static satisfies

$$\frac{\partial y_i}{\partial x} \in \frac{1}{|\nabla_{ii}|}\left[\frac{-\operatorname{sgn}\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} - \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1-\delta_i\delta_{-i}}, \frac{-\operatorname{sgn}\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} + \sum_{k\in\mathcal{N}\setminus i}\delta_k\left|\frac{\partial f_k}{\partial x}\right|}{1-\delta_i\delta_{-i}}\right] \tag{44}$$

with both ∇ and $\frac{\partial f}{\partial x}$ evaluated at $\boldsymbol{y} = \boldsymbol{y}^*(x)$. Conditional on $\left\{\nabla_{ii}, \delta_i, \frac{\partial f_i}{\partial x}\right\}_{i\in\mathcal{N}}$, the bounds in equations (42) and (44) are sharp.

Proof. □

A.3.2 Conditioning on Less Information

The following proposition presents bounds conditional on knowledge of all the direct effects, $\left\{\frac{\partial f_k}{\partial x}\right\}_{k\in\mathcal{N}}$, as opposed to only the direct effect for some i, $\frac{\partial f_i}{\partial x}$, and the summary on all other nodes $\left|\frac{\partial f_{-i}}{\partial x}\right|$.

Proposition 2. (Comparative Static Bounds without δ_{-i}). *Suppose ∇ satisfies assumption 1. If*

$$\left|\frac{\partial f_i}{\partial x}\right| \geq \left|\frac{\partial f_{-i}}{\partial x}\right| \tag{45}$$

then, the magnitude of the comparative static satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left(\frac{\left|\frac{\partial f_i}{\partial x}\right| - \left|\frac{\partial f_{-i}}{\partial x}\right|}{1+\delta_i}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \left|\frac{\partial f_{-i}}{\partial x}\right|}{1-\delta_i}\right) \tag{46}$$

and its sign

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right)\operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right) \tag{47}$$

Otherwise, the comparative static satisfies

$$\frac{\partial y_i}{\partial x} \in \frac{1}{|\nabla_{ii}|}\left(\frac{-sgn\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} - \left|\frac{\partial f_{-i}}{\partial x}\right|}{1-\delta_i}, \frac{-sgn\left(\nabla_{ii}\right)\frac{\partial f_i}{\partial x} + \left|\frac{\partial f_{-i}}{\partial x}\right|}{1-\delta_i}\right) \tag{48}$$

with $\nabla, \frac{\partial \boldsymbol{f}}{\partial x}$ *evaluated at* $\mathbf{y} = \mathbf{y}^*(x)$*. Conditional on* $\nabla_{ii}, \delta_i, \frac{\partial f_i}{\partial x}, \left|\frac{\partial f_{-i}}{\partial x}\right|$*, the bounds are sharp.*

A.3.3 Row Diagonal Dominance

Proposition 3. (Comparative Static Bounds under Row Diagonal Dominance) *Suppose* ∇ *satisfies*

$$\forall i \in \mathcal{N}: \quad |\nabla_{ii}| > \sum_{j \in \mathcal{N}\setminus i} |\nabla_{ij}| \tag{49}$$

and $\left|\frac{\partial f_{-i}}{\partial x}\right| \delta_{-i} = 0$*. Then,*

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left[\frac{\left|\frac{\partial f_i}{\partial x}\right|}{1+\delta_i\delta_{-i}}, \frac{\left|\frac{\partial f_i}{\partial x}\right|}{1-\delta_i\delta_{-i}}\right]$$

$$sgn\left(\frac{\partial y_i}{\partial x}\right) = -sgn\left(\nabla_{ii}\right) sgn\left(\frac{\partial f_i}{\partial x}\right)$$

where

$$\delta_i \equiv \frac{\sum_{j \in \mathcal{N}\setminus i} |\nabla_{ij}|}{|\nabla_{ii}|}$$

A.3.4 Sub-Unity Spectral Radius

This section presents a weakening of assumption 1 by using remark 4: although the Jacobian ∇_{ij} might not satisfy assumption 1, suppose there exists a vector g_i' such that $g_i'\nabla_{ij}$ does satisfy assumption 1. In this case, the Jacobian ∇_{ij} is generalized diagonally dominant. I provide an example of such g_i' in this section.

Define the following matrix

$$A_{ij} \equiv \begin{cases} i = j & 0 \\ i \neq j & \left|\frac{\nabla_{ij}}{\nabla_{jj}}\right| \end{cases} \tag{50}$$

Denote by ρ the spectral radius of A, and by v_i a left-eigenvector of A with eigenvalue equal in magnitude to ρ. Set $g_i' = v_i$. If A is irreducible and $\rho < 1$, then $g_i'\nabla_{ij}$ satisfies assumption 1. Proposition 4 proves this and derives the implied comparative static bounds, in terms of ρ and v_i. Note that this result conditions on more information, specifically all of the direct effects, as in proposition 1.

Proposition 4. (Comparative Static Bounds under Sub-Unity Spectral Radius). *Suppose A is irreducible and $\rho < 1$. If*

$$\left|\frac{\partial f_i}{\partial x}\right| > \rho \sum_{k \neq i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|$$

then, the magnitude of the comparative static satisfies

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|} \left[\frac{\left|\frac{\partial f_i}{\partial x}\right| - \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|}{1 + \rho^2}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|}{1 - \rho^2} \right] \tag{51}$$

and its sign

$$\operatorname{sgn}\left(\frac{\partial y_i}{\partial x}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right) \operatorname{sgn}\left(\frac{\partial f_i}{\partial x}\right)$$

Otherwise, the comparative static satisfies

$$\frac{\partial y_i}{\partial x} \in \frac{1}{|\nabla_{ii}|} \left[\frac{-\operatorname{sgn}\left(\nabla_{ii}\right) \frac{\partial f_i}{\partial x} - \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|}{1 - \rho^2}, \frac{-\operatorname{sgn}\left(\nabla_{ii}\right) \frac{\partial f_i}{\partial x} + \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|}{1 - \rho^2} \right]$$

Where $\rho > 0$, and $\{v_i\}_{i \in \mathcal{N}}$ is unique and strictly positive, and both ∇ and $\frac{\partial f}{\partial x}$ are evaluated at $\boldsymbol{y} = \boldsymbol{y}^*(x)$.

Proof. Irreducibility of $|A|$ and $\rho < 1$ is not sufficient for ∇_{ij} to satisfy assumption 1, thus we cannot directly apply theorem 1 (or proposition 1). However, it is sufficient for $\tilde{\nabla}_{ij} \equiv v_i \nabla_{ij}$ to satisfy assumption 1 (implying ∇ is generalized diagonally dominant, see remark 4). Thus, we can rewrite the comparative statics in terms of $\tilde{\nabla}$ and apply proposition 1.

To prove that $\tilde{\nabla}$ satisfies assumption 1, I rely on the Perron-Frobenius theorem (Horn and Johnson, 2012 theorem 8.4.4), which applies to A because A is nonnegative and irreducible. The theorem implies that ρ is strictly positive, that it is an eigenvalue of A, and its associated left-eigenvector $\{v_i\}_{i\in\mathcal{N}}$ is unique and has all elements strictly positive, $\forall i : v_i > 0$. Utilizing this,

$$\begin{aligned}
\left|\tilde{\nabla}_{ii}\right| - \sum_{j\neq i}\left|\tilde{\nabla}_{ji}\right| &= |v_i\nabla_{ii}| - \sum_{j\neq i}|v_j\nabla_{ji}| \\
&= v_i\,|\nabla_{ii}| - \sum_{j\neq i} v_j\,|\nabla_{ji}| \\
&= |\nabla_{ii}|\left(v_i - \sum_{j\neq i} v_j\frac{|\nabla_{ji}|}{|\nabla_{ii}|}\right) \\
&= |\nabla_{ii}|\left(v_i - \sum_{j} v_j A_{ji}\right) \\
&= |\nabla_{ii}|\,v_i\,(1-\rho) \\
&> 0
\end{aligned}$$

The first line used the definition of $\tilde{\nabla}_{ij} \equiv v_i\nabla_{ij}$; the second line used that $\forall i : v_i > 0$; the fourth line used the definition of A, equation (50); the fifth line used that ρ is eigenvalue of A with eigenvector $\{v_i\}_{i\in\mathcal{N}}$; and the last line used that $\rho < 1$. Thus, $\tilde{\nabla}$ is strictly column diagonally dominant and therefore satisfies assumption 1. The iDD degree of $\tilde{\nabla}$, which I denote by $\tilde{\delta}$, is

$$\begin{aligned}
\tilde{\delta}_i &\equiv \frac{\sum_{j\neq i}\left|\tilde{\nabla}_{ji}\right|}{\left|\tilde{\nabla}_{ii}\right|} \\
&= \frac{\sum_{j\neq i}|v_j\nabla_{ji}|}{|v_i\nabla_{ii}|} \\
&= \frac{\sum_{j\neq i} v_j\frac{|\nabla_{ji}|}{|\nabla_{ii}|}}{v_i} \\
&= \frac{\sum_{j} v_j A_{ji}}{v_i} \\
&= \frac{v_i\rho}{v_i} \\
&= \rho
\end{aligned}$$

The first line used the definition of the iDD degree, definition 1; the second line used the definition of $\tilde{\nabla}_{ij} \equiv v_i \nabla_{ij}$; the third line used that $\forall i : v_i > 0$; the forth line the definition of A, equation (50); the fifth line used that ρ is eigenvalue of A with eigenvector $\{v_i\}_{i \in \mathcal{N}}$.

Next, I rewrite the comparative statics in terms of $\tilde{\nabla}$

$$\begin{aligned}\frac{\partial y_i}{\partial x} &= -\sum_{k \in \mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \frac{\partial f_k}{\partial x} \\ &= -\sum_{k \in \mathcal{N}} \left\{\tilde{\nabla}^{-1}\right\}_{ik} \frac{\partial \tilde{f}_k}{\partial x}\end{aligned}$$

where I used $\frac{\partial \tilde{f}_k}{\partial x} \equiv v_k \frac{\partial f_k}{\partial x}$. Now, we apply proposition 1 with $\tilde{\nabla}_{ij}$ and $\frac{\partial \tilde{f}_k}{\partial x}$ in place of ∇_{ij} and $\frac{\partial f_k}{\partial x}$. First, equation (41) becomes

$$\begin{aligned} & \left|\frac{\partial \tilde{f}_i}{\partial x}\right| > \sum_{k \in \mathcal{N} \setminus i} \tilde{\delta}_k \left|\frac{\partial \tilde{f}_k}{\partial x}\right| \\ \iff \quad & \left|\frac{\partial f_i}{\partial x}\right| > \rho \sum_{k \neq i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right| \end{aligned} \tag{52}$$

Equation (42) becomes

$$\begin{aligned}\left|\frac{\partial y_i}{\partial x}\right| &\in \frac{1}{\left|\tilde{\nabla}_{ii}\right|} \left[\frac{\left|\frac{\partial \tilde{f}_i}{\partial x}\right| - \sum_{k \in \mathcal{N} \setminus i} \tilde{\delta}_k \left|\frac{\partial \tilde{f}_k}{\partial x}\right|}{1 + \tilde{\delta}_i \tilde{\delta}_{-i}}, \frac{\left|\frac{\partial \tilde{f}_i}{\partial x}\right| + \sum_{k \in \mathcal{N} \setminus i} \tilde{\delta}_k \left|\frac{\partial \tilde{f}_k}{\partial x}\right|}{1 - \tilde{\delta}_i \tilde{\delta}_{-i}}\right] \\ &= \frac{1}{|\nabla_{ii}|} \left[\frac{\left|\frac{\partial f_i}{\partial x}\right| - \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|}{1 + \rho^2}, \frac{\left|\frac{\partial f_i}{\partial x}\right| + \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left|\frac{\partial f_k}{\partial x}\right|}{1 - \rho^2}\right]\end{aligned}$$

where in the second line I used $\tilde{\nabla}_{ij} \equiv v_i \nabla_{ij}$ and $\frac{\partial \tilde{f}_k}{\partial x} \equiv v_k \frac{\partial f_k}{\partial x}$. The sign of the

comparative static, equation (43) remains the same, because

$$\begin{aligned} sgn\left(\frac{\partial y_i}{\partial x}\right) &= -sgn\left(\tilde{\nabla}_{ii}\right) sgn\left(\frac{\partial \tilde{f}_i}{\partial x}\right) \\ &= -\text{sgn}\left(v_i \nabla_{ii}\right) \text{sgn}\left(v_i \frac{\partial f_i}{\partial x}\right) \\ &= -\text{sgn}\left(\nabla_{ii}\right) \text{sgn}\left(\frac{\partial f_i}{\partial x}\right) \end{aligned}$$

given $\forall i : v_i > 0$. Equation (44) becomes

$$\begin{aligned} \frac{\partial y_i}{\partial x} &\in \frac{1}{\left|\tilde{\nabla}_{ii}\right|} \left[\frac{-sgn\left(\tilde{\nabla}_{ii}\right) \frac{\partial \tilde{f}_i}{\partial x} - \sum_{k \in \mathcal{N} \setminus i} \tilde{\delta}_k \left| \frac{\partial \tilde{f}_k}{\partial x} \right|}{1 - \tilde{\delta}_i \tilde{\delta}_{-i}}, \frac{-sgn\left(\tilde{\nabla}_{ii}\right) \frac{\partial \tilde{f}_i}{\partial x} + \sum_{k \in \mathcal{N} \setminus i} \tilde{\delta}_k \left| \frac{\partial \tilde{f}_k}{\partial x} \right|}{1 - \tilde{\delta}_i \tilde{\delta}_{-i}} \right] \\ &= \frac{1}{|\nabla_{ii}|} \left[\frac{-sgn\left(\nabla_{ii}\right) \frac{\partial f_i}{\partial x} - \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left| \frac{\partial f_k}{\partial x} \right|}{1 - \rho^2}, \frac{-sgn\left(\nabla_{ii}\right) \frac{\partial f_i}{\partial x} + \rho \sum_{k \in \mathcal{N} \setminus i} \frac{v_k}{v_i} \left| \frac{\partial f_k}{\partial x} \right|}{1 - \rho^2} \right] \end{aligned}$$

□

A.3.5 Signed Diagonal Dominance

The bounds on the comparative static for the state in i in theorem 1 can be strengthened under a restriction on the sign of the elements of $\nabla, \frac{\partial f}{\partial x}$. For ∇, all diagonal elements have the same sign, and each of the off-diagonal elements have a sign that is not equal to that of the diagonal elements (so either the opposite sign, or zero). For $\frac{\partial f_j}{\partial x}$, all $j \neq i$ have the same sign, which is different to $j = i$. These are equations (53) and (54) in the following proposition. Note that proposition 5 always requires the additional assumption on $\frac{\partial f}{\partial x}$, equation (11).

Proposition 5. (Comparative Static Bounds under Signed Diagonal Dominance). *Suppose assumption 1 holds, and equation (41) holds for i. Also suppose*

$$\forall k, j \neq k : \quad s_1 = sgn\left(\nabla_{kk}\right) \neq sgn\left(\nabla_{kj}\right) \tag{53}$$

$$\forall j \neq i : \quad s_2 = sgn\left(\frac{\partial f_i}{\partial x}\right) \neq sgn\left(\frac{\partial f_j}{\partial x}\right) \tag{54}$$

holds for some $s_1 \in \{-1, 1\}, s_2 \in \{-1, 1\}$. *Then*

$$\left|\frac{\partial y_i}{\partial x}\right| \in \frac{1}{|\nabla_{ii}|}\left[\left|\frac{\partial f_i}{\partial x}\right| - \sum_{k \in \mathcal{N}\setminus i} \delta_k \left|\frac{\partial f_k}{\partial x}\right|, \frac{\left|\frac{\partial f_i}{\partial x}\right|}{1 - \delta_i \delta_{-i}}\right] \tag{55}$$

and

$$sgn\left(\frac{\partial y_i}{\partial x}\right) = -sgn\left(\nabla_{ii}\right) sgn\left(\frac{\partial f_i}{\partial x}\right)$$

with both ∇ *and* $\frac{\partial f}{\partial x}$ *evaluated at* $\boldsymbol{y} = \boldsymbol{y}^*(x)$.

Proof. Equation (53), combined with ∇ being diagonally dominant (assumption 1), implies two properties of $s_1\nabla$[25]

$$\forall i : \left\{s_1 \nabla^{-1}\right\}_{ii} \geq \frac{1}{\{s_1 \nabla\}_{ii}} \tag{56}$$

$$\forall i, j \neq i : \left\{s_1 \nabla^{-1}\right\}_{ij} \geq 0 \tag{57}$$

To prove these, rewrite the Jacobian as

$$\nabla_{ij} \equiv \left(I_{ij} - A_{ij}\right) \nabla_{jj} \tag{58}$$

where $A_{ij} \equiv \left|\frac{\nabla_{ij}}{\nabla_{jj}}\right| (1 - I_{ij}) \geq 0$. Equation (58) follows by noting that $\forall i, j \neq i : \frac{\nabla_{ij}}{\nabla_{jj}} \leq 0$ due to equation (53). Diagonal dominance of ∇ implies the spectral radius of A, denoted ρ, is less than 1: $\rho \leq \max_j \sum_i |A_{ij}| = \max_j \sum_{i \neq j} \left|\frac{\nabla_{ij}}{\nabla_{jj}}\right| < 1$, where the first inequality follows using the Gerschgorin Circle theorem, and the last inequality follows from diagonal dominance of ∇. Thus, one can apply the Neumann expansion $(I - A)^{-1} = I + A + A^2 + \cdots$, which is valid under $\rho \leq 1$ (Johnson, 1982), implying

$$\begin{aligned} \forall i : &\quad \left\{(I - A)^{-1}\right\}_{ii} \geq 1 \\ \forall i, j \neq i : &\quad \left\{(I - A)^{-1}\right\}_{ji} \geq 0 \end{aligned}$$

which follows by noting that $\forall i, j : A_{ij} \geq 0$. Using these facts in the matrix inverse of equation (58) implies equations (56) because

$$\forall i : \left\{s_1 \nabla^{-1}\right\}_{ii} = \frac{1}{\{s_1 \nabla\}_{ii}} \left\{(I - A)^{-1}\right\}_{ii} \geq \frac{1}{\{s_1 \nabla\}_{ii}}$$

[25] $s_1\nabla$ is an M-Matrix. See Horn and Johnson (1991), chapter 2.5.

and (57) because

$$\forall i, j \neq i : \left\{s_1 \nabla^{-1}\right\}_{ij} = \frac{1}{\{s_1 \nabla\}_{ii}} \left\{(I - A)^{-1}\right\}_{ij} \geq 0$$

noting, for both cases, $\{s_1 \nabla\}_{ii} > 0$.

Now, turning to the comparative static. Using equation (3),

$$\begin{aligned} \frac{\partial y_i}{\partial x} &= -\sum_{k \in \mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \frac{\partial f_k}{\partial x} \\ &= -\left\{\nabla^{-1}\right\}_{ii} \frac{\partial f_i}{\partial x} \underbrace{\left(1 + \sum_{k \in \mathcal{N} \setminus i} \frac{\left\{\nabla^{-1}\right\}_{ik}}{\left\{\nabla^{-1}\right\}_{ii}} \frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}}\right)}_{(*)} \end{aligned} \tag{59}$$

The term $(*)$ can be bounded as follows

$$1 + \sum_{k \in \mathcal{N} \setminus i} \underbrace{\frac{\left\{\nabla^{-1}\right\}_{ik}}{\left\{\nabla^{-1}\right\}_{ii}}}_{\geq 0} \underbrace{\frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}}}_{\leq 0} \in \left[\underbrace{1 - \sum_{k \in \mathcal{N} \setminus i} \delta_k \left|\frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}}\right|}_{\geq 0}, 1\right] \tag{60}$$

where I used $\forall k \in \mathcal{N} \setminus i : \frac{\left\{\nabla^{-1}\right\}_{ik}}{\left\{\nabla^{-1}\right\}_{ii}} \geq 0$ by equation (57), and $\forall k \in \mathcal{N} \setminus i : \frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}} \leq 0$ by equation (54), which together implies $\sum_{k \in \mathcal{N} \setminus i} \frac{\left\{\nabla^{-1}\right\}_{ik}}{\left\{\nabla^{-1}\right\}_{ii}} \frac{\frac{\partial f_k}{\partial x}}{\frac{\partial f_i}{\partial x}} \leq 0$. Note that the lower bound in equation (60) is positive by equation (11).

Taking the absolute value of equation (59) and using equation (60)

$$\left|\frac{\partial y_i}{\partial x_i}\right| \in \left|\left\{\nabla^{-1}\right\}_{ii}\right| \left|\frac{\partial f_i}{\partial x}\right| \left[1 - \sum_{k \in \mathcal{N} \setminus i} \delta_k \left|\frac{\partial f_k}{\partial x} / \frac{\partial f_i}{\partial x}\right|, 1\right] \tag{61}$$

$\left|\left\{\nabla^{-1}\right\}_{ii}\right|$ can be bounded below using equation (56) and above using equation (7)

$$\forall i : \quad \left|\left\{\nabla^{-1}\right\}_{ii}\right| \in \left[\frac{1}{|\nabla_{ii}|}, \frac{1}{|\nabla_{ii}|} \frac{1}{1 - \delta_i \delta_{-i}}\right] \tag{62}$$

Inserting this into equation (61) gives the final bounds on the comparative static

magnitude

$$\left|\frac{\partial y_i}{\partial x_i}\right| \in \frac{\left|\frac{\partial f_i}{\partial x}\right|}{|\nabla_{ii}|}\left[1 - \sum_{k \in \mathcal{N}\setminus i} \delta_k \left|\frac{\partial f_k}{\partial x} / \frac{\partial f_i}{\partial x}\right|, \frac{1}{1 - \delta_i \delta_{-i}}\right]$$

The comparative static sign is unchanged from equation (13), given assumption 1 and equation (11) hold. □

B More Applications (Online Appendix)

B.1 New Keynesian (Auclert et al., 2024)

A classic question in macroeconomics is, what is the size of the fiscal multiplier (Ramey, 2019): the change in GDP given a an increase in government spending. Particular attention has been given to when is this greater than one, as this corresponds to the case where private expenditure is not crowded out by the public expenditure (Christiano et al., 2011). Utilizing the recently influential intertemporal Keynesian cross framework of Auclert et al. (2024), I show how the results of this paper can provide insight on this question.

The equations of state in this framework derive from the market clearing equations at each time period $i \in \mathcal{N}$

$$Y_i = C_i\left(\boldsymbol{Y} - \boldsymbol{T}\right) + G_i \tag{63}$$

where Y_i is GDP (the endogenous state), G_i is government expenditure (the exogenous shock), and T_i is taxation. $C_i\left(\cdot\right)$ is private consumption in period i, which is a function of after-tax income in all time periods $\boldsymbol{Y} - \boldsymbol{T}$, as agents are able to save or borrow. The lefthand-side of equation (63) is supply, while the righthand-side is total demand (private consumption C_i plus public consumption G_i). In Auclert et al. (2024), the model underlying equation (63) is infinite horizon and therefore the domain of i is $\{0, 1, ..., \infty\}$. However the results of the current paper require a finite domain, therefore I consider a truncated domain, $i \in \mathcal{N} = \{0, ..., N\}$, for some large N. This is mostly innocuous in practical terms, as Auclert et al. (2024) also truncate when numerically solving the model.

A key object in characterizing the effect of fiscal policy is the intertemporal marginal propensity to consume, given by $M_{ij} \equiv \frac{\partial C_i}{\partial Y_j}$. This describes the change in consumption in period i due to an income increase in period j. The government can save and borrow and therefore its fiscal policy is subject to a lifetime budget constraint

$$\sum_{i \in \mathcal{N}} \frac{G_i}{\left(1+r\right)^i} = \sum_{i \in \mathcal{N}} \frac{T_i}{\left(1+r\right)^i} \tag{64}$$

where $r \geq 0$ is the interest. I will focus on the comparative static corresponding to the impact fiscal multiplier, $\frac{\partial Y_0}{\partial G_0}$: the change in GDP today, $\mathrm{d}Y_0$, due to the government spending today, $\mathrm{d}G_0 > 0$ (with $\mathrm{d}G_i = 0$ for $i > 0$), that is deficit-financed through an

arbitrary tax schedule in the future, $\{\mathrm{d}T_i\}_{i>0}$.

Then, if one assumes: 1) $\forall i, j : M_{ij} > 0$, 2) there exists an n such that $\mathrm{d}Y_n \geq \mathrm{d}T_n$, the results in this paper can be used to give a lower bound on the impact fiscal multiplier given by

$$\frac{\partial Y_0}{\partial G_0} \geq \left(\frac{1 - \sum_{j \in \mathcal{N} \setminus \{0,n\}} \delta_j s_j^T}{1 - M_{00}} \right) \tag{65}$$

where

$$s_j^T \equiv \frac{(1+r)^{-j} \, \mathrm{d}T_j}{\sum_{j \in \mathcal{N} \setminus \{0,n\}} (1+r)^{-i} \, \mathrm{d}T_i}$$

is the taxation share coming from period j (in a present value sense) and

$$\delta_j = 1 - \frac{1}{(1+r)^{n-j}} \frac{M_{nj}}{1 - M_{jj}}$$

is the iDD degree corresponding to this model. I provide the proof of equation (65) at the end of this section.

Assumption 1) implies that all intertemporal MPCs are strictly positive. This rules out inferior income effects, which isn't very restrictive given this is the MPC for aggregate consumption, and assumes there is some transmission of income between all periods. Assumption 2) implies there is a time period where the GDP response is greater than the taxation levied in this period. This would likely by true if we set $n = 1$ while having most of the taxation during periods $j > 1$. Or if we only tax up to period $n - 1$ for some $n > 1$, as then the restriction reduces to $\mathrm{d}Y_n \geq 0$, and we know GDP impulse responses from fiscal spending shocks are using nonnegative empirically (see e.g. Ramey and Zubairy 2018).

The lower bound of the impact multiplier given by equation (65) can be intuitively understood as follows. The $\frac{1}{1-M_{00}}$ component is the "static" Keynesian cross analogue: spending in period zero implies output will increase by one over one minus the static MPC, M_{00}. However, the static Keynesian cross ignores the impact of the future taxation on today's output. The term $-\sum_{j \in \mathcal{N} \setminus \{0,n\}} \delta_j s_j^T$ is a sufficient statistic for a bound on this, reflecting the reduction in output today due to the negative income effect of taxation in the future. The iDD degree δ_j is bounding the effect from each period j, and this is weighted by the taxation share in each period, s_j^T.

The iDD degree δ_j is smaller, and so the lower bound on the impact multiplier higher, if M_{jj} or M_{nj} are greater, as this corresponds to more of the negative income

effect of the tax in period j being offset by consumption reduction in period j or n, respectively, as opposed to reducing consumption and therefore output, Y_0, now.

A key feature of the lower bound of the impact multiplier in equation (65) is that it requires substantially less information to enumerate than the point value of the impact multiplier. To illustrate the potential value of this, suppose the government spending is financed entirely by taxation in period 1, $\mathrm{d}T_1 = (1+r)\,\mathrm{d}G_0$, and choose $n = 2$. Then, the bound becomes

$$\frac{\partial Y_0}{\partial G_0} \geq \frac{1-\delta_1}{1-M_{00}} = \frac{M_{21}}{(1+r)\,(1-M_{00})\,(1-M_{11})} \tag{66}$$

Only three elements of the MPC matrix are required to enumerate the bound, M_{00}, M_{11}, M_{21}, while the entire matrix is required for the point value. As Auclert et al. (2024) explain, we have very little empirical evidence on most of the elements of MPC matrix, and those authors resort to using a structural model to fill in the gaps. The bound dramatically reduce this burden.

To demonstrate, suppose one assumes temporal symmetry such that $M_{11} = M_{00}$ and $M_{21} = M_{10}$, which facilitates calibration to empirical estimates as these elements are known: $M_{00} = 0.51, M_{10} = 0.18$ (Fagereng et al., 2021). Finally, set $r = 0.05$. The multiplier lower bound implied by equation (66) is then 0.71. This magnitude is non-trivial, and implies that the point value of the impact multiplier is fairly large regardless of, for instance, the structural model one might use to set the remaining elements of the MPC matrix.

The requirement of less information also allows one to derive sufficient conditions on the static MPC, M_{00}, such that the multiplier is greater one, $\frac{\partial Y_0}{\partial G_0} \geq 1$. Equation (66) immediately reveals a sufficient condition for this: $M_{00} \geq \delta_1$. Intuitively, the positive income effect on GDP today from the static MPC, M_{00}, is sufficiently greater than the negative income effect on GDP today from taxation tomorrow, δ_1 (recalling that the iDD degree appropriately bounds this). Moreover, if one continues with assuming the aforementioned temporal symmetry, and sets $M_{10} = 0.18, r = 0.05$, then the lower bound on the static MPC becomes $M_{00} \geq 1 - \sqrt{\frac{M_{10}}{1+r}} = 0.59$. This is very close to the empirical value of 0.51, implying that above-unity impact fiscal multipliers would be guaranteed in this framework if static MPCs are only slightly higher than those estimated.

Proof. Of Equation (65). Equation (63) forms the basis of the equations of state

corresponding to equation (1). However, a simplification in the comparative static bounds can be achieved by not directly using this, instead by considering it first in differential form. Totally differentiating equation (63) gives

$$\begin{aligned} i \in \mathcal{N}: \quad 0 &= \mathrm{d}Y_i - \sum_{j \in \mathcal{N}} M_{ij} \left(\mathrm{d}Y_j - \mathrm{d}T_j \right) - \mathrm{d}G_i \\ &= \left(\mathrm{d}Y_i - \mathrm{d}T_i \right) - \sum_{j \in \mathcal{N}} M_{ij} \left(\mathrm{d}Y_j - \mathrm{d}T_j \right) - \left(\mathrm{d}G_i - \mathrm{d}T_i \right) \qquad (67) \\ &= \sum_{j \in \mathcal{N}} \left(I - M \right)_{ij} \left(\mathrm{d}Y_j - \mathrm{d}T_j \right) - \left(\mathrm{d}G_i - \mathrm{d}T_i \right) \qquad (68) \end{aligned}$$

Where simple rearrangements were done to get to the last line. Solving for $\mathrm{d}Y_j$ cannot proceed directly because $I - M$ is singular; equivalently, the solutions $\mathrm{d}Y_j$ to equation (68) are underdetermined. This is because M obeys the following in order to satisfy lifetime budget constraints

$$j \in \mathcal{N}: \quad \sum_{i \in \mathcal{N}} \frac{M_{ij}}{(1+r)^i} = \frac{1}{(1+r)^j} \qquad (69)$$

Because $I - M$ is singular, there exists an eigenvector $\boldsymbol{v}$ of $I - M$ with eigenvalue zero. This means we can rewrite equation (68) without loss of generality as follows

$$i \in \mathcal{N}: \quad 0 = \sum_{j \in \mathcal{N}} \left(I - M \right)_{ij} \left(\mathrm{d}Y_j - \mathrm{d}T_j - a v_j \right) - \left(\mathrm{d}G_i - \mathrm{d}T_i \right) \qquad (70)$$

for any constant $a \in \mathbb{R}$, which reflects the indeterminacy. Essentially, the level of the solutions $\mathrm{d}Y_j$ is not determined. The model only determines the relative values $\mathrm{d}Y_j$, and these are solved for by setting $\mathrm{d}Y_j - \mathrm{d}T_j - a v_j = 0$ for one j, and using only $N-1$ of the i equations from equation (70). I set $\mathrm{d}Y_n - \mathrm{d}T_n - a v_n = 0$ and drop the $i = n$ equation, giving

$$i \in \mathcal{N} \backslash n: \quad 0 = \sum_{j \in \mathcal{N} \backslash n} \left(I - M \right)_{ij} \left(\mathrm{d}Y_j - \mathrm{d}T_j - \left(\mathrm{d}Y_n - \mathrm{d}T_n \right) \frac{v_j}{v_n} \right) - \left(\mathrm{d}G_i - \mathrm{d}T_i \right) \qquad (71)$$

noting that $v_n > 0$ ($\boldsymbol{v}$ is also an eigenvector of M with eigenvalue 1, and because $M > 0$, then $\boldsymbol{v} > 0$ by the Perron Theorem). The Jacobian of equation (71) is $\{I - M\}_{i,j \in \mathcal{N} \backslash n}$. This matrix does not satisfy assumption 1 in general, however an

appropriate transformation of the equations of state does (see remark 4):

$$i \in \mathcal{N}\backslash n: \quad 0 = \sum_{j \in \mathcal{N}\backslash n} \frac{(I-M)_{ij}}{(1+r)^i} \left(\mathrm{d}Y_j - \mathrm{d}T_j - (\mathrm{d}Y_n - \mathrm{d}T_n) \frac{v_j}{v_n} \right) - \frac{\mathrm{d}G_i - \mathrm{d}T_i}{(1+r)^i} \tag{72}$$

That is, multiplying each equation by $(1+r)^{-i}$. The Jacobian of this system is given by

$$i,j \in \mathcal{N}\backslash n: \quad \nabla_{ij} = (1+r)^{-i} (I_{ij} - M_{ij}) \tag{73}$$

which satisfies assumption 1 because

$$\begin{aligned} i \in \mathcal{N}\backslash n: \quad \sum_{j \in \mathcal{N}\backslash\{i,n\}} |\nabla_{ji}| &= \sum_{j \in \mathcal{N}\backslash\{i,n\}} (1+r)^{-j} M_{ji} \\ &= \sum_{j \in \mathcal{N}} (1+r)^{-j} M_{ji} - (1+r)^{-i} M_{ii} - (1+r)^{-n} M_{ni} \\ &= (1+r)^{-i} - (1+r)^{-i} M_{ii} - (1+r)^{-n} M_{ni} \\ &= |\nabla_{ii}| - (1+r)^{-n} M_{ni} \\ &< |\nabla_{ii}| \end{aligned} \tag{74}$$

where line three used equation (69), and the last line used the assumption $M > 0$. The iDD degrees are given by

$$\begin{aligned} i \in \mathcal{N}\backslash n: \quad \delta_i &= \frac{\sum_{j \in \mathcal{N}\backslash\{i,n\}} |\nabla_{ji}|}{|\nabla_{ii}|} \\ &= \frac{\sum_{j \in \mathcal{N}\backslash\{i,n\}} (1+r)^{-j} M_{ji}}{(1+r)^{-i} (1 - M_{ii})} \\ &= \frac{\sum_{j \in \mathcal{N}} (1+r)^{-j} M_{ji} - (1+r)^{-i} M_{ii} - (1+r)^{-n} M_{ni}}{(1+r)^{-i} (1 - M_{ii})} \\ &= \frac{(1+r)^{-i} - (1+r)^{-i} M_{ii} - (1+r)^{-n} M_{ni}}{(1+r)^{-i} (1 - M_{ii})} \\ &= 1 - \frac{(1+r)^{-n} M_{ni}}{(1+r)^{-i} (1 - M_{ii})} \end{aligned}$$

where the second line used that equation (69) and assumption $M > 0$ imply $M_{ii} \leq 1$;

the fourth line used equation (69). The direct effects are given by

$$\begin{aligned}\frac{\partial f_i}{\partial G_0} &= -\frac{1}{\mathrm{d}G_0}\frac{\mathrm{d}G_i - \mathrm{d}T_i}{(1+r)^i} \\ &= -\frac{I_{i0} - \frac{\mathrm{d}T_i}{\mathrm{d}G_0}}{(1+r)^i} \\ &= -(1+r)^i I_{i0} + s_i^T \end{aligned} \tag{75}$$

The last line follows because $\frac{\mathrm{d}T_i}{\mathrm{d}G_0} = \frac{s_i^T \sum_{i\in\mathcal{N}}(1+r)^{-i}\mathrm{d}T_i}{\mathrm{d}G_0} = s_i^T$, which used the definition of s_i^T in the first equality, and the budget balance equation (64) in the second equality. Thus, the lower bound on the impact multiplier is derived by

$$\begin{aligned}\frac{\mathrm{d}Y_0}{\mathrm{d}G_0} &= \frac{\mathrm{d}\left(Y_0 - T_0\right)}{\mathrm{d}G_0} \\ &\geq \frac{\mathrm{d}\left(Y_0 - T_0\right) - \mathrm{d}\left(Y_n - T_n\right)\frac{v_j}{v_n}}{\mathrm{d}G_0} \\ &\geq \frac{1}{|\nabla_{00}|}\left(\frac{\partial f_0}{\partial G_0} - \sum_{j\in\mathcal{N}\setminus 0}\delta_j\left|\frac{\partial f_j}{\partial G_0}\right|\right) \qquad (76) \\ &\geq \frac{1}{1-M_{00}}\left(1 - \sum_{j=1}^{n-1}\delta_j s_i^T\right) \qquad (77)\end{aligned}$$

where the first line used $T_0 = 0$ by assumption; the second line used $\mathrm{d}Y_n \geq \mathrm{d}T_n$ by assumption. The third line applied proposition 5 to equation (72) for $i = 0$, noting that both sign conditions, equations (53) and (54), are satisfied with $s_1 = 1, s_2 = -1$. The sign of the comparative static is determined because equation (41) is satisfied, $\delta_{-0}\sum_{i\in\mathcal{N}\setminus 0}\left|\frac{\partial f_i}{\partial G_0}\right| = \delta_{-0}\sum_{i\in\mathcal{N}\setminus 0} s_i^T = \delta_{-0} < 1 = |\nabla_{00}|$, implying a positive comparative static $-\mathrm{sgn}\left(\nabla_{ii}\right)\mathrm{sgn}\left(\frac{\partial f_0}{\partial G_0}\right) = -\mathrm{sgn}\left(1 - M_{00}\right)\mathrm{sgn}\left(-1\right) = 1$. The lower bound from equation (55) then implies equation (76). The last line, equation (77) follows using equations (73) and (75). □

B.2 International Trade and Economic Geography

Consider the framework in Allen et al. (2020) that encapsulates many of the workhorse models in the international trade and economic geography literature. Notably it is a strict generalization of Arkolakis et al. (2012) under their CES demand assumption

R3', which corresponds to the case where the scale elasticity (introduced below) is $\psi = 0$ (see footnote 12 in Arkolakis et al. 2012).

The model is one of $i \in \mathcal{N}, j \in \mathcal{N}$ locations, in which an aggregate good is traded across locations subject to iceberg trade frictions, $\tau_{ij} > 0$. Consumers have CES preferences, implying demand from consumers in location j for products produced in location i is $X_{ij} = \left(\frac{p_i \tau_{ij}}{P_j}\right)^{-\phi} E_j$, where p_i is the production price for products produced in location i, $P_j = \sum_{i \in \mathcal{N}} \tau_{ij}^{-\phi} p_i^{-\phi}$ the consumer price index in j, E_j is total expenditure by consumers in j, and $\phi \in \mathbb{R}$ is the demand elasticity. Quantity supplied by location i is given by $Q_i = \kappa \bar{c}_i \left(\frac{p_i}{P_i}\right)^{\psi}$, where $\psi \in \mathbb{R}$ is the supply elasticity, $\bar{c}_i$ is productivity, and $\kappa > 0$ is a (possibly endogenous) scalar. The output market clears and trade is balanced up to an exogenous deficit.

The equations of state are derived by combining these conditions. A notable case considered by the authors is the one of balanced trade and quasi-symmetric trade costs,

$$\forall i \in \mathcal{N}, j \in \mathcal{N}: \quad \tau_{ij} = \tau_i^A \tau_j^B \tilde{\tau}_{ij} \tag{78}$$

where $\tau_i^A > 0, \tau_i^B > 0$ and $\tilde{\tau}_{ij} = \tilde{\tau}_{ji} > 0$. Under this assumption, the production price solves the following set of equations (see section A2.3 of Allen et al., 2020) for all $i \in \mathcal{N}$

$$0 = f_i\left(\ln \mathbf{p}, \ln \tilde{\boldsymbol{\tau}}\right) \equiv \underbrace{\kappa p_i^{1+\psi+\psi\frac{1+\psi+\phi}{\phi-\psi}} \left(\frac{\tau_i^A}{\tau_i^B}\right)^{\frac{\psi\phi}{\phi-\psi}} \bar{c}_i^{\frac{\phi}{\psi-\phi}}}_{=Y_i} - \sum_{j \in \mathcal{N}} \underbrace{\kappa \tilde{\tau}_{ij}^{-\phi} p_i^{-\phi} \left(\tau_i^A \tau_j^A\right)^{-\phi} p_j^{-\phi}}_{=X_{ij}} \tag{79}$$

These form the equations of state of the current paper, equation (1), with the (natural logarithm of) production price, $\ln p_i$, as the endogenous state and the (symmetric component of the) trade costs, $\ln \tilde{\tau}_{ij}$, as the exogenous shock. This is an example where I explicitly include multiple shocks in the equations of state. In what follows, using theorem 1, I derive the bounds for the comparative statics

$$\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{lj}} = -\sum_{k \in \mathcal{N}} \left\{\nabla^{-1}\right\}_{ik} \frac{\partial f_k}{\partial \ln \tilde{\tau}_{lj}} \tag{80}$$

for any $i \in \mathcal{N}, l \in \mathcal{N}, j \in \mathcal{N}$. The Jacobian is

$$\nabla_{ij} \equiv \frac{\partial f_i\left(\mathbf{p}, \tilde{\boldsymbol{\tau}}\right)}{\partial \ln p_j} = \phi \frac{1+\psi+\phi}{\phi-\psi} \left(Y_i I_{ij} + \frac{\phi-\psi}{1+\psi+\phi} X_{ij}\right) \tag{81}$$

and the direct effects matrix

$$\frac{\partial f_i}{\partial \ln \tilde{\tau}_{kj}} \equiv \frac{\partial f_i\left(\mathbf{p}, \tilde{\boldsymbol{\tau}}\right)}{\partial \ln \tilde{\tau}_{kj}} = 2\phi X_{ij} I_{ki}, \quad \left|\frac{\partial f_{-i}}{\partial \ln \tilde{\tau}_{kj}}\right| = 2\left|\phi\right| \sum_{l \in \mathcal{N} \setminus i} X_{lj} I_{ki} \tag{82}$$

Hat algebra has been used to write both expressions in terms of observable objects (Y, X) rather than unobservable objects $(\tau, \bar{c})$ — see Allen et al. (2020) for details, and see Dekle et al. (2008) for the seminal paper on this. A sufficient condition for the Jacobian to satisfy assumption 1 is

$$\{\phi > -0.5,\ \psi > -0.5\} \text{ or } \{\phi < -0.5,\ \psi < -0.5\} \tag{83}$$

Allen et al. (2020) theorem 1.iii shows that equation (83) with quasi-symmetric trade costs is sufficient for the model to have a unique interior equilibrium. To see that equation (83) implies column diagonal dominance of the Jacobian, assumption 1,[26]

$$\begin{aligned} \left|\nabla_{ii}\right| - \sum_{i \neq j} \left|\nabla_{ij}\right| &= \left|\phi \frac{1+\psi+\phi}{\phi-\psi}\right| \left[\left(1 - \left|\frac{\phi-\psi}{1+\psi+\phi}\right|\right) Y_i + \left(\frac{\phi-\psi}{1+\psi+\phi} + \left|\frac{\phi-\psi}{1+\psi+\phi}\right|\right) X_{ii}\right] \\ &> 0 \end{aligned}$$

where I used that $\left|\frac{\phi-\psi}{1+\psi+\phi}\right| < 1$ under equation (83). This last fact can be confirmed by showing $|1+\phi+\psi| - |\phi-\psi| > 0$. If $\phi > -0.5, \psi > -0.5$, then

$$|1+\phi+\psi| - |\phi-\psi| = \begin{cases} \phi \geq \psi & 1+\phi+\psi-\phi+\psi = 1+2\psi > 0 \\ \phi < \psi & 1+\phi+\psi+\phi-\psi = 1+2\phi > 0 \end{cases}$$

The iDD degree in this model is given by

$$\delta_i = \left|\frac{\phi-\psi}{1+\psi+\phi}\right| \frac{1 - \frac{X_{ii}}{Y_i}}{1 + \frac{\phi-\psi}{1+\psi+\phi}\frac{X_{ii}}{Y_i}}$$

[26] Allen et al., 2020 prove the Jacobian is row diagonally dominant under $\phi > 0$ and $\psi > 0$ in lemma 5.

Thus, assuming the condition in equation (83), we can apply theorem 2

$$l = i: \quad \frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}} \in \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi}\right) \cdot \frac{2\delta_i \frac{X_{ij}}{Y_i}}{1-\frac{X_{ii}}{Y_i}} \left[\frac{1}{1+\delta_i}, \frac{1}{1-\delta_i}\right] \tag{84}$$

$$l \neq i: \quad \left|\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{lj}}\right| \leq \frac{2\delta_i}{1-\frac{X_{ii}}{Y_i}} \frac{\delta_l \frac{X_{lj}}{Y_i}}{1-\delta_i} \tag{85}$$

where I used that equation (11) is satisfied only for the comparatives statics, $\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{lj}}$, with $l = i$ because

$$\left|\frac{\partial f_i}{\partial \ln \tilde{\tau}_{lj}}\right| - \left|\frac{\partial f_i}{\partial \ln \tilde{\tau}_{lj}}\right| = 2\left|\phi\right| \left(X_{ij} I_{li} - \sum_{k \in \mathcal{N} \setminus i} X_{kj} I_{lk}\right)$$

$$= \begin{cases} l = i & 2\left|\phi\right| X_{ij} \geq 0 \\ l \neq i & -2\left|\phi\right| X_{lj} \leq 0 \end{cases}$$

Implying comparative sign for $l = i$, using (13),

$$\operatorname{sgn}\left(\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}}\right) = -\operatorname{sgn}\left(\nabla_{ii}\right) \cdot \operatorname{sgn}\left(\frac{\partial f_i}{\partial \ln \tilde{\tau}_{ij}}\right) = \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi}\right)$$

The comparative static bounds in equations (84) and (85) are enumerated with knowledge of only the demand elasticity ϕ, the supply elasticity ψ, the own-trade share for i $\frac{X_{ii}}{Y_i}$, and the level of trade between the two countries directly affected by the trade cost, l, j, relative to output in i, $\frac{X_{lj}}{Y_i}$.

I can use the above bounds on prices to also bound welfare. For a range of canonical international trade models isomorphic to the Universal Gravity framework, the change in welfare of a worker in location i is $W_i = B_i \left(\frac{p_i}{P_i}\right)^{1+\psi}$, with B_i an exogenous scalar (Allen et al. 2020 table 1). The change in welfare with respect to a

trade cost shock, $\tilde{\tau}_{ij}$, is then

$$\begin{aligned}\frac{\partial \ln W_i}{\partial \ln \tilde{\tau}_{ij}} &= (1+\psi)\frac{\partial \ln \frac{p_i}{P_i}}{\partial \ln \tilde{\tau}_{ij}} \\ &= (1+\psi)\frac{2\phi+1}{\phi-\psi}\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}} \qquad (86) \\ &\in \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi}\right)\frac{(1+\psi)(2\phi+1)}{\phi-\psi}\frac{2\delta_i\frac{X_{ij}}{Y_i}}{1-\frac{X_{ii}}{Y_i}}\left[\frac{1}{1+\delta_i},\frac{1}{1-\delta_i}\right] \qquad (87)\end{aligned}$$

where in the second line I used $\frac{\partial \ln \frac{p_i}{P_i}}{\partial \ln \tilde{\tau}_{ij}} = \frac{2\phi+1}{\phi-\psi}\frac{\partial \ln p_i}{\partial \ln \tau_{ij}}$ from Allen et al. (2020) pg 430, and in the third line I inserted equation (84).

In section 4.3, I can consider a proportional increase in trade costs with the rest of the world $\forall j \neq i : \mathrm{d}\ln \tilde{\tau}_{ij} = \mathrm{d}\ln \overline{\tau}$. The change in welfare in i is

$$\begin{aligned}\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} &= \sum_{j\in\mathcal{N}\setminus i}\frac{\partial \ln W_i}{\partial \ln \tilde{\tau}_{ij}} \\ &= (1+\psi)\frac{2\phi+1}{\phi-\psi}\sum_{j\in\mathcal{N}\setminus i}\frac{\partial \ln p_i}{\partial \ln \tilde{\tau}_{ij}} \\ &= -(1+\psi)\frac{2\phi+1}{\phi-\psi}\sum_{j\in\mathcal{N}\setminus i,k\in\mathcal{N}}\left\{\nabla^{-1}\right\}_{ik}\frac{\partial f_k}{\partial \ln \tilde{\tau}_{ij}} \\ &= -(1+\psi)\frac{2\phi+1}{\phi-\psi}\sum_{j\in\mathcal{N}\setminus i,k\in\mathcal{N}}\left\{\nabla^{-1}\right\}_{ik}2\phi X_{kj}I_{ik} \\ &= -(1+\psi)\frac{2\phi+1}{\phi-\psi}\sum_{j\in\mathcal{N}\setminus i}\left\{\nabla^{-1}\right\}_{ii}2\phi X_{ij} \\ &= -(1+\psi)\frac{2\phi+1}{\phi-\psi}\left\{\nabla^{-1}\right\}_{ii}2\phi(1-X_{ii})\end{aligned}$$

The second line used equation (4.3); the third line used equation (4.3); and the fourth line used equation (4.3). In deriving equation (26) in the main text, I further set $\psi = 0$, which corresponds to the case of Arkolakis et al. (2012),

$$\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} = -2(2\phi+1)\left\{\nabla^{-1}\right\}_{ii}(1-X_{ii}), \quad \nabla_{ij} = (1+\phi)\left(Y_iI_{ij} + \frac{\phi}{1+\phi}X_{ij}\right)$$

and use a transformed version of $\nabla_{ij} \to \frac{1}{(1+\phi)Y_i}\nabla_{ij}$ for notational convenience, giving

$$\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} = -2\frac{1+2\phi}{1+\phi}\left\{\nabla^{-1}\right\}_{ii}\left(1-\frac{X_{ii}}{Y_i}\right), \quad \nabla_{ij} = I_{ij} + \frac{\phi}{1+\phi}\frac{X_{ij}}{Y_i}$$

The bounds in equation (27) are derived by

$$\begin{aligned}
\frac{\partial \ln W_i}{\partial \ln \overline{\tau}} &= \sum_{j\in\mathcal{N}\setminus i}\frac{\partial \ln W_i}{\partial \ln \tilde{\tau}_{ij}} \\
&= \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi}\right)\frac{(1+\psi)(2\phi+1)}{\phi-\psi}2\delta_i \sum_{j\in\mathcal{N}\setminus i}\frac{\frac{X_{ij}}{Y_i}}{1-\frac{X_{ii}}{Y_i}}\left[\frac{1}{1+\delta_i},\frac{1}{1-\delta_i}\right] \\
&= \operatorname{sgn}\left(\frac{1+\psi+\phi}{\psi-\phi}\right)\frac{(1+\psi)(2\phi+1)}{\phi-\psi}2\delta_i \left[\frac{1}{1+\delta_i},\frac{1}{1-\delta_i}\right] \\
&= -\frac{2\phi+1}{|\phi|}2\delta_i\left[\frac{1}{1+\delta_i},\frac{1}{1-\delta_i}\right], \quad \delta_i = \left|\frac{\phi}{1+\phi}\right|\frac{1-\frac{X_{ii}}{Y_i}}{1+\frac{\phi}{1+\phi}\frac{X_{ii}}{Y_i}}
\end{aligned}$$

where the second line used equation (87); and the last line set $\psi = 0$ (corresponding to Arkolakis et al. 2012).

B.3 Industrial Organization: Oligopoly

Consider a model of a Bertrand oligopoly with differentiated products. I use the canonical framework from Milgrom and Roberts (1990), which nests many standard demand structures. The paper demonstrates that these demand structures satisfy diagonal dominance and uses this to establish uniqueness of the equilibrium.[27]

There are $i \in \mathcal{N}$ firms each producing one good with constant unit costs, c_i, and with demand $D_i(\mathbf{p})$, where $\mathbf{p} = \{p_i\}_{i\in\mathcal{N}}$ are the endogenous product prices. $D_i(\mathbf{p})$ is assumed to be twice-continuously differentiable. The log profits of firm i are

$$\pi_i(\mathbf{p}, c_i) = \ln\left[(p_i - c_i)D_i(\mathbf{p})\right]$$

and the firm chooses p_i to maximize $\pi_i(\mathbf{p}, c_i)$, taking all other firms' prices as given.

[27] Diagonal dominance has been shown to be relevant in other market structures too. The framework in Dixit (1986) nests Bertrand, Cournot, market-share and competitive markets structures, and they appeal to diagonal dominance for stability of the equilibrium. Pellegrino (2023) consider Bertrand with a generalized hedonic-linear demand system; one can easily verify that their (inverse) demand satisfies diagonal dominance.

The optimal price solves the first-order condition

$$\underbrace{\frac{\partial \pi_i}{\partial \ln p_i} = \frac{1}{1-\frac{c_i}{p_i}} + \frac{\partial \ln D_i(p)}{\partial \ln p_i}}_{\equiv f_i(\ln \mathbf{p}, \ln \mathbf{c})} = 0 \tag{88}$$

Equation (88) corresponds to the equations of state in my framework, equation (1), with $\ln \mathbf{p}$ the endogenous state variable, and $\ln \mathbf{c}$ the exogenous shock. The Jacobian, accordingly, is,

$$\nabla_{ij} \equiv \frac{\partial f_i(\ln \mathbf{p}, \ln \mathbf{c})}{\partial \ln p_j} = -\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} I_{ij} + \frac{\partial^2 \ln D_i(p)}{\partial \ln p_i \partial \ln p_j} \tag{89}$$

The second order condition is

$$0 > \frac{\partial^2 \pi_i}{\partial \ln p_i^2} = \nabla_{ii} \tag{90}$$

which implies the diagonal of the Jacobian is negative. The direct effect matrix is

$$\frac{\partial f_i}{\partial \ln c_j} \equiv \frac{\partial f_i(\ln \mathbf{p}, \ln \mathbf{c})}{\partial \ln c_j} = \frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} I_{ij}$$

The pass-through to p_i of a change in c_i is given by

$$\frac{\partial p_i}{\partial c_i} = \left\{\nabla^{-1}\right\}_{ii} \frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}$$

In the main for equation (28), I use a transformed version of $\nabla_{ij} \to \frac{\left(1-\frac{c_i}{p_i}\right)^2}{\frac{c_i}{p_i}} \nabla_{ij}$ for notational convenience. In deriving the bounds on this comparative static, one can use proposition 3 following remark 3 because the direct effects are only non-zero for the node shocked, $\forall i, j \neq i : \frac{\partial f_i}{\partial \ln c_j} = 0$. Remark 3 requires ∇ to be row diagonally dominant, as opposed to column diagonally dominant (assumption 1). This is useful because, as Milgrom and Roberts (1990) (pg 1271) explain, many standard demand structures for $D_i(\mathbf{p})$ imply row diagonal dominance. Specifically: CES and logit; linear under substitutes; translog under substitutes and an additional parameter

restriction.[28]

The bounds for $j = i$ from proposition 3 are

$$\frac{\partial p_i}{\partial c_i} \in \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{|\nabla_{ii}|}\left[\frac{1}{1+\delta_i\delta_{-i}}, \frac{1}{1-\delta_i\delta_{-i}}\right], \quad \delta_i = \frac{\sum_{j\in\mathcal{N}\setminus i}|\nabla_{ij}|}{|\nabla_{ii}|} \tag{91}$$

noting that the iDD degree is constructed using the row sum as we are using proposition 3. In the main text, I am interested in the lower bound, which can be manipulated further using $\delta_{-i} < 1$

$$\begin{aligned}
\frac{\partial p_i}{\partial c_i} &> \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{|\nabla_{ii}|}\frac{1}{1+\delta_i} \\
&= \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\sum_{j\in\mathcal{N}}|\nabla_{ij}|} \\
&= \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\left|-\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}+\frac{\partial^2\ln D_i(p)}{\partial\ln p_i\partial\ln p_i}\right|+\sum_{j\in\mathcal{N}\setminus i}\left|\frac{\partial^2\ln D_i(p)}{\partial\ln p_i\partial\ln p_j}\right|} \\
&= \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}-\frac{\partial^2\ln D_i(p)}{\partial\ln p_i\partial\ln p_i}+\sum_{j\in\mathcal{N}\setminus i}\left|\frac{\partial^2\ln D_i(p)}{\partial\ln p_i\partial\ln p_j}\right|}
\end{aligned} \tag{92}$$

where the second line used the definition of δ_i from equation (91); the third line used the definition of the Jacobian from equation (89); and the fourth line used the second-order condition equation (90).

In the main text I assume the Hessian of log demand is row diagonally dominant,

$$\forall i \in \mathcal{N}: \quad \left|\frac{\partial^2\ln D_i(p)}{\partial\ln p_i\partial\ln p_i}\right| > \sum_{j\in\mathcal{N}\setminus i}\left|\frac{\partial^2\ln D_i(p)}{\partial\ln p_i\partial\ln p_j}\right| \tag{93}$$

[28] Logit requires taking the equations of state to be $0 = \frac{\partial\pi_i}{\partial p_i}$, as opposed to $0 = \frac{\partial\pi_i}{\partial\ln p_i}$. This is an application of remark 4, with $g_i' = \frac{1}{p_i}$.

and the diagonal of the Hessian is positive

$$\forall i \in \mathcal{N}: \quad \frac{\partial^2 \ln D_i(p)}{\partial \ln p_i \partial \ln p_i} > 0 \tag{94}$$

Inserting these into equation (92)

$$\begin{aligned}
\frac{\partial p_i}{\partial c_i} &> \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} - \frac{\partial^2 \ln D_i(p)}{\partial \ln p_i \partial \ln p_i} + \left|\frac{\partial^2 \ln D_i(p)}{\partial \ln p_i \partial \ln p_i}\right|} \\
&> \frac{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2}}{\frac{\frac{c_i}{p_i}}{\left(1-\frac{c_i}{p_i}\right)^2} - \frac{\partial^2 \ln D_i(p)}{\partial \ln p_i \partial \ln p_i} + \frac{\partial^2 \ln D_i(p)}{\partial \ln p_i \partial \ln p_i}} \\
&> 1
\end{aligned}$$

where the first line used equation (93) and the second line used equation (94). This proves that a sufficient condition for pass-through to be greater than unity is the Hessian of log demand to be row diagonally dominant with positive diagonal, equations (93) and (94). This condition is also sufficient for demand to be log-convex. Log-convexity of demand is equivalent to the Hessian of log demand being positive semidefinite. Equations (93) and (94) imply the Hessian of log demand is positive definite, hence, implying demand is log convex.

B.4 Network Games

Consider the workhorse network game with simultaneous moves, unconstrained actions, and linear best replies; the interactions are possibly weighted and have arbitrary sign (see Bramoullé et al., 2016 chapter 5 for a unified treatment of this).

Player $i \in \mathcal{N}$ choses effort $y_i \in \mathbb{R}$ to maximize a linear-quadratic utility function

$$u_i = \beta a_i y_i - \frac{1}{2} y_i^2 + \phi \sum_{j \in \mathcal{N}} G_{ij} y_i y_j$$

where βa_i is the private benefit received by exerting effort, composed of an exogenous characteristic of the individual, $a_i \in \mathbb{R}$, with coefficient $\beta \in \mathbb{R}$; a will serve as the

exogenous shock I'll take the comparative static with respect to. The quadratic term in effort of player i and every other player j implies there are strategic interactions between players when maximizing utility. $G_{ij} \in \mathbb{R}$ is the weighted adjacency matrix, whose magnitude indicates the strength of the interaction between agents i and j. I assume $G_{ii} = 0$ as standard (there is no interaction with oneself). $\phi \in \mathbb{R}$ scales the magnitude of the strategic interactions, and is often referred to as the *payoff impact parameter*. i and j's actions are strategic complements when $\phi G_{ij} \leq 0$, and strategic substitutes when $\phi G_{ij} \geq 0$.

The first order condition of $\max_{y_i} u_i$ yields player i's best reply

$$y_i = \beta a_i + \phi \sum_{j \in \mathcal{N}} G_{ij} y_j \tag{95}$$

The Nash equilibrium is the $\mathbf{y}$ such that all players' best replies are satisfied. The best replies correspond to the equations of state, equation (1)

$$f_i(\mathbf{y}, \mathbf{a}) \equiv y_i - \beta a_i - \phi \sum_{j \in \mathcal{N}} G_{ij} y_j = 0$$

The Jacobian in this model is

$$\nabla_{ij} \equiv \frac{\partial f_i}{\partial y_j} = I_{ij} - \phi G_{ij}$$

The direct effect matrix in this model is $\frac{\partial f_i}{\partial a_j} = -\beta I_{ij}$. Using equation (3), the comparative static exact expression is given by

$$\frac{\partial y_i}{\partial a_i} = \left\{ (I - \phi G)^{-1} \right\}_{ii} \beta \tag{96}$$

As outlined in Bramoullé et al. (2016) section 5.4.1 (see also Bramoullé et al. 2014), a Nash equilibrium exists and is unique if $I - \phi G$ is invertible, and is asymptotically stable if the spectral radius of $-\phi G$ is less than one. A sufficient condition for both of these is $|\phi| \mu < 1$, where μ is the spectral radius of $|G|$.[29] If one assumes G is irreducible,[30] in addition to $|\phi| \mu < 1$, then the conditions required for proposition

[29] The spectral radius of $|\phi G|$ is always weakly greater than of $-\phi G$ by Horn and Johnson (2012) theorem 8.1.18. Under pure strategic complements, $\forall i, j \neq i : \phi G_{ij} \leq 0$, the two spectral radii are equal because $-\phi G = |\phi G|$ (recall that $\forall i : G_{ii} = 0$).

[30] Irreducibility of an adjacency matrix is equivalent to the network being *strongly connected*: any

4 are satisfied (see remark 4).[31] The proposition gives the following bounds for the comparative statics with $i = j$

$$\forall i \in \mathcal{N}: \quad \frac{\partial y_i}{\partial a_i} \in \left[\frac{\beta}{1+\phi^2\mu^2}, \frac{\beta}{1-\phi^2\mu^2}\right] \tag{97}$$

$$\forall i \in \mathcal{N}, j \in \mathcal{N}\backslash i: \quad \left|\frac{\partial y_i}{\partial a_j}\right| \leq \frac{|\phi|\,\mu\beta\frac{v_j}{v_i}}{1-\phi^2\mu^2} \tag{98}$$

and for $i \neq j$ we have

$$\forall i \in \mathcal{N}, j \in \mathcal{N}\backslash i: \quad \left|\frac{\partial y_i}{\partial a_j}\right| \leq \frac{|\phi|\,\mu\beta\frac{v_j}{v_i}}{1-\phi^2\mu^2}$$

where $\{v_i\}$ is the eigenvector of $|G|$ with eigenvalue μ. Note that the sign for $j = i$ is given because equation (11) holds for $j = i$. The expression for $j = i$ is particularly appealing as it only requires knowledge of the payoff impact parameter ϕ, and the spectral radius of the (absolute) adjacency matrix μ. Both of these parameters are of high interest in the literature on network games (e.g. Ballester et al., 2006; Bramoullé et al., 2014).

An important special case arises if one further assumes that $G_{ij} \geq 0$ and is row-normalized for non-isolated players, $\sum_{j\in\mathcal{N}} G_{ij} = 1$. This is the linear-in-means model. Here, the spectral radius equals one, $\mu = 1$ and the egienvector v_i equals the vector of ones. In this case, the comparative static from equation (97) reduces to equation (23) in the main text.

$$\forall i \in \mathcal{N}: \quad \frac{\partial y_i}{\partial a_i} \in \left[\frac{\beta}{1+\phi^2}, \frac{\beta}{1-\phi^2}\right] \tag{99}$$

$$\forall i \in \mathcal{N}, j \in \mathcal{N}\backslash i: \quad \left|\frac{\partial y_i}{\partial a_j}\right| \leq \frac{|\phi|\,\beta}{1-\phi^2} \tag{100}$$

Taking the modulus of the upper bound in equation (99), and rearranging gives $\phi^2 \geq 1 - \left|\beta/\frac{\partial y_i}{\partial a_i}\right|$, while the lower bound gives $\phi^2 \geq \left|\beta/\frac{\partial y_i}{\partial a_i}\right| - 1$. Thus, $\phi^2 \geq \left|1 - \left|\beta/\frac{\partial y_i}{\partial a_i}\right|\right|$ and hence equation (24) in the main text.

node in the network has a directed path to any other node (Bramoullé et al., 2016, page 523). This assumption is very common in the network literature, especially when considering learning dynamics, existence of a consensus, and eigenvector centralities (Bramoullé et al., 2016, section 19.3.2).

[31] Proposition 4 uses the matrix A from equation (50), which in the present model is $A_{ij} = |\phi G_{ij}|$, with $\rho = |\phi|\,\mu$. Note that $|G|$ is irreducible iff G is irreducible.

Alternatively, one could instead make the stronger restriction that $\nabla = I - \phi G$ satisfies assumption 1, which translates to

$$1 > |\phi| \sum_{j \in \mathcal{N} \setminus i} |G_{ji}| \tag{101}$$

Note that equation (101) implies $|\phi|\,\mu < 1$, and thus also implies a unique and stable Nash equilibrium.[32] Under the condition given in equation (101), therefore, we can apply theorem 1. The iDD degree of player i is

$$\delta_i = |\phi| \sum_{j \in \mathcal{N} \setminus i} |G_{ji}| \tag{102}$$

That is, δ_i is proportional to $\sum_{j \in \mathcal{N} \setminus i} |G_{ji}|$, where the latter is referred to as the (weighted) *degree centrality* in the network science literature (see e.g. Bramoullé et al. (2016) chapter 11). Equation (101), and therefore assumption 1, is satisfied if the degree centrality is sufficiently small. The bounds implied by theorem 1 are given by

$$\frac{\partial y_i}{\partial a_i} \in \left[\frac{1}{1 + \phi^2 \sum_{j \in \mathcal{N} \setminus i} |G_{ji}| \sum_{j \in \mathcal{N} \setminus i^*} |G_{ji^*}|}, \frac{1}{1 - \phi^2 \sum_{j \in \mathcal{N} \setminus i} |G_{ji}| \sum_{j \in \mathcal{N} \setminus i^*} |G_{ji^*}|} \right]$$

$$i \neq j : \quad \left| \frac{\partial y_i}{\partial a_j} \right| \leq \frac{|\phi| \sum_{k \neq j} |G_{kj}|}{1 - \phi^2 \sum_{j \in \mathcal{N} \setminus i} |G_{ji}| \sum_{j \in \mathcal{N} \setminus i^*} |G_{ji^*}|}$$

where i^* is the player with maximal degree excluding i, $i^* = \arg\max_{j \neq i} \sum_{k \neq j} |G_{jk}|$. Alternatively still, if we assume diagonal dominance, equation (101), and pure strategic complements, $\forall i, j \neq i : \phi G_{ij} \leq 0$, then we can apply proposition 5.

In any case, whichever of the three theorems/propositions one applies, the result is an expression that does not require the inversion of the Jacobian, and that requires only a few sufficient statistics, such as the spectral radius (and associated eigenvector), or the degree centrality of each node and ϕ.

The greater tractability of the bounds allow one to derive a range of properties of the comparative statics very simply. For example, if one wanted to deduce a value of the spectral radius μ that guarantees $\frac{\partial y_i}{\partial a_i} \geq x$ for some $x \in (0, 1)$, equation (97) provides an answer to this: $\frac{1}{|\phi|} \sqrt{\frac{1}{x} - 1} \geq \mu$.

[32] The Gershgorin Circle Theorem applied to the transpose of $|\phi G|$ implies $|\phi|\,\mu \leq |\phi| \sum_{j \in \mathcal{N} \setminus i} |G_{ji}|$. This combined with equation (101) therefore implies $|\phi|\,\mu < 1$.

Another fact we can quickly discern is by using that $|\phi|\,\mu < 1$, we see that $\frac{\partial y_i}{\partial a_i} \geq \frac{1}{2}$. That is, for any adjacency matrix subject to $\mu < \frac{1}{|\phi|}$, the effect of an increase in the private benefit on an agent's on action, $\frac{\partial y_i}{\partial a_i}$, is always bounded below be $\frac{1}{2}$. The intuition for this bound is as follows. With no feedback between nodes $\forall i, j \neq i : \nabla_{ij} = 0$, which corresponds to the adjacency matrix being zero between all nodes, $\forall i, j \neq i : G_{ij} = 0$, the comparative static is $\frac{\partial y_i}{\partial a_i} = 1$. With feedback between nodes, this bound can be attenuated if there is negative feedback (corresponding to strategic substitutes in this model). $\mu < \frac{1}{|\phi|}$ places an upper bound on the amount of feedback generally, and therefore also on the amount of attenuation possible. It implies that negative feedback can only at most reduce the comparative static by half relative to the case of no feedback between nodes.

B.5 Time Series

Consider the following ARMA(p, q) process $\{y_i\}_{i \in \mathcal{N}}$, where $i \in \mathcal{N}$ represents time over N periods, defined by the model

$$i \in \mathcal{N}: \quad y_i - \sum_{s=1}^{p} \beta_s y_{i-s} = x_i + \sum_{s=0}^{q} \theta_s x_{i-s} \tag{103}$$

where $\{x_i\}_{i \in \mathcal{N}}$ is a white noise process (definition 3.1.1 in Brockwell and Davis (2016)). β_s is the effect of y with a lag of s periods, and θ_s the effect of x with a lag of s periods. The equations of state in this model corresponds to

$$i \in \mathcal{N}: \quad f_i(\boldsymbol{y}, \boldsymbol{x}) = y_i - \sum_{s=1}^{p} \beta_s y_{i-s} - x_i - \sum_{s=1}^{q} \theta_s x_{i-s}$$

The Jacobian is

$$\nabla_{ij} = I_{ij} - \beta_{i-j} \cdot 1\left[i - p \leq j < i\right] \tag{104}$$

The direct effects matrix is $\frac{\partial f_i}{\partial x_j} = -I_{ij} - \theta_{i-j} \cdot 1\left[i - q \leq j < i\right]$, with

$$\left|\frac{\partial f_{-i}}{\partial x_j}\right| = \begin{cases} \sum_{s=1}^{q} |\theta_s| & j = i \\ 1 + \sum_{s \in \{1,\ldots,q\} \setminus i} |\theta_s| & j \neq i \end{cases}$$

The Jacobian ∇ satisfies assumption 1 iff

$$\sum_{s=1}^{p} |\beta_s| < 1 \tag{105}$$

Proof. Assumption 1 is satisfied iif

$$|\nabla_{jj}| - \sum_{i \neq j} |\nabla_{ij}| > 0 \tag{106}$$

Under the Jacobian of this model,

$$\begin{aligned} |\nabla_{jj}| - \sum_{i \neq j} |\nabla_{ij}| &= 1 - \sum_{i \neq j} |I_{ij} - \beta_{i-j} \cdot 1 \left[i - p \leq j < i \right]| \\ &= 1 - \sum_{s=1}^{p} |\beta_s| \end{aligned}$$

where the first line used equation (104) and the second line a manipulation of the indices. Thus, equation (106) holds iff $\sum_{s=1}^{p} |\beta_s| < 1$. □

The condition in equation (105) is sufficient for existence and uniqueness of a causal, stationary solution $\{y_i\}_{i \in \mathcal{N}}$ for this model as $N \to \infty$.

Proof. By Brockwell and Davis (2016) pg 75, a stationary solution exists and is unique iff

$$1 - \sum_{s=1}^{p} \beta_s z^s \neq 0 \tag{107}$$

for all $|z| = 1$ where $z \in \mathbb{C}$. The stationary solution is causal (y_i only depends on x_j for $j \leq i$) if equation (107) holds for all $|z| \leq 1$ where $z \in \mathbb{C}$ (Brockwell and Davis (2016) pg 75). A sufficient condition for equation (107) is

$$Re \left\{ \sum_{s=1}^{p} \beta_s z^s \right\} \neq 1$$

This holds under equation (105) because

$$\begin{aligned} Re\left\{\sum_{s=1}^{p}\beta_s z^s\right\} &\leq \left|\sum_{s=1}^{p}\beta_s z^s\right| \\ &\leq \sum_{s=1}^{p}|\beta_s|\,|z^s| \\ &\leq \sum_{s=1}^{p}|\beta_s| \\ &< 1 \end{aligned}$$

where the second line used the triangle inequality, the third used $|z| \leq 1$ and the last line used equation (105) . Thus (105) is sufficient for the existence and uniqueness of a causal stationary solution. □

In what follows I will consider only nodes $i < N - \max\{p, q\}$ to avoid having to complicate the notation to include boundary cases. However, the results naturally apply to these nodes too.

The iDD degree (definition 1) is

$$\delta_i = \sum_{s=1}^{p}|\beta_s|$$

Under equation (105), theorem 1 can be used to imply bounds on the comparative static of y_i with respect to x_j. Without any further assumptions on $\{\theta_s\}_{s=0}^{q}$, then equation (14) can be used to bound the comparative statics for all i

$$\frac{\partial y_i}{\partial x_j} \in \begin{cases} \left[\frac{1-\sum_{s=1}^{q}|\theta_s|\sum_{s=1}^{p}|\beta_s|}{1-\left(\sum_{s=1}^{p}|\beta_s|\right)^2}, \frac{1+\sum_{s=1}^{q}|\theta_s|\sum_{s=1}^{p}|\beta_s|}{1-\left(\sum_{s=1}^{p}|\beta_s|\right)^2}\right] & j = i \\ \left[\frac{\theta_{i-j}-\left(1+\sum_{s\in\{1,\ldots,q\}\setminus i}|\theta_s|\right)\sum_{s=1}^{p}|\beta_s|}{1-\left(\sum_{s=1}^{p}|\beta_s|\right)^2}, \frac{\theta_{i-j}+\left(1+\sum_{s\in\{1,\ldots,q\}\setminus i}|\theta_s|\right)\sum_{s=1}^{p}|\beta_s|}{1-\left(\sum_{s=1}^{p}|\beta_s|\right)^2}\right] & j \neq i \end{cases}$$

where $j \in [i-q, i]$. The bounds are identified from knowledge of only the sums of all the coefficients on y, $\sum_{s=1}^{p}|\beta_s|$, and the x coefficient corresponding to the shock θ_{i-j}, and the sum of the remainder, $\sum_{s\in\{1,\ldots,q\}\setminus i}|\theta_s|$. Knowledge of all the coefficients individually is not required. Note that the sign of $\frac{\partial y_i}{\partial x_i}$ is determined (it is positive) if we also assume $\sum_{s=1}^{q}|\theta_s| < 1$.

B.6 Spatial Econometrics

Consider the standard spatial autogressive model (LeSage et al., 2009, section 3.1.1)

$$y_i = \rho \sum_{j \in \mathcal{N}} W_{ij} y_j + \beta x_i + \varepsilon_i$$

where $i \in \mathcal{N}$ represents locations, y and x are some endogenous and exogenous location characteristics, respectively. $W_{ij} \geq 0$ is the spatial weight matrix, representing how strongly connected locations i and j are, often parameterized to be inversely related to distance between the two locations. $\rho \in \mathbb{R}$ is the spatial lag (or spatial autoregressive) parameter, modulating the overall magnitude and sign of the spatial interdependence. β is the direct effect of x_i on y_i, holding the indirect effect through y_j fixed. ε_i is the residual.

Typically in these models, the spatial weight matrix is constructed to have no interaction within a location, $\forall i \in \mathcal{N} : W_{ii} = 0$, and all row sums equal to one, $\forall i \in \mathcal{N} : \sum_{j \in \mathcal{N}} W_{ij} = 1$. The term $\sum_{j \in \mathcal{N}} W_{ij} y_j$ then represents a spatially weighted average of y neighboring i. Under this restriction, the parameter ρ is almost always assumed to have modulus less than one (Ord, 1975; Kelejian and Robinson, 1995; LeSage et al., 2009; Gibbons and Overman, 2012)

$$|\rho| < 1 \tag{108}$$

This assumption implies the model has a "stable" solution (Fingleton, 1999). That is, the variance of the system is finite even if the matrix has cycles or if $N \to \infty$. It also ensures the model is well-defined as $I - \rho W$ is invertible. The prevalence of this assumption, however, is perhaps more rooted in computational convenience, facilitating maximum likelihood estimation. The equations of state in this model are

$$0 = f_i(\boldsymbol{y}, \boldsymbol{x}) \equiv y_i - \rho \sum_{j \in \mathcal{N}} W_{ij} y_j - \beta x_i - \varepsilon_i$$

The Jacobian is

$$\nabla_{ij} \equiv I_{ij} - \rho W_{ij}$$

and the direct effects matrix is $\frac{\partial f_i}{\partial x_j} \equiv -\beta I_{ij}, \left| \frac{\partial f_{-i}}{\partial x_j} \right| = \beta (1 - I_{ij})$. Equation (108), along with the fact that all row sums of (nonnegative) W equal one, implies that the

Jacobian is row diagonally dominant,

$$\forall i \in \mathcal{N}: \quad \sum_{j \notin \mathcal{N}} |\nabla_{ij}| = \sum_{j \notin \mathcal{N}} |\rho| \, W_{ij} < \sum_{j \notin \mathcal{N}} W_{ij} = 1 = |\nabla_{ii}|$$

Because the direct effects are local (i.e. zero off-diagonals), we can apply remark 3 to bound the comparative static. The iDD degree is

$$\delta_i = \frac{\sum_{j \notin \mathcal{N}} |\nabla_{ij}|}{|\nabla_{ii}|} = \frac{\sum_{j \notin \mathcal{N}} |\rho| \, W_{ij}}{1} = |\rho|$$

Note that this has been calculated using definition 1 for the transpose of ∇, because we are applying remark 3. The resulting bounds are

$$\frac{\partial y_i}{\partial x_i} \in \left[\frac{\beta}{1+\rho^2}, \frac{\beta}{1-\rho^2} \right]$$

and

$$j \neq i: \quad \left| \frac{\partial y_i}{\partial x_j} \right| \leq \frac{|\rho| \, \beta}{1-\rho^2}$$

Where the sign is determined for the $j = i$ comparative static because equation (11) is satisfied.

The bounds are enumerated from knowledge of β, ρ only; no knowledge of the spatial weight matrix, W, is required. This is of potentially useful insight, as a key difficulty in spatial autoregressive models is characterizing the implications of different ρ for the distribution of Y, which is generally challenging due its interaction with the spatial weight matrix, W (Conley, 2016 pg 7).