Economic growth with human, physical and automation capital: The role of declining population*

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Abstract

This paper examines the long-run per capita output growth when human capital complements automation and physical capital amid a declining population. The findings show that despite negative population growth, per capita output grows at the same rate as physical and human capital, while growth in automation capital lags behind. Household savings play a key role in sustaining growth. In a specific case, routine laborers' wage share vanishes in the long run, but human capital's share remains constant.

***JEL* Classification:** J11, O33, O41
Key Words: Growth, Automation, Declining population, labor share.

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1 Introduction

Extending the Solow growth model Solow (1956), we investigate the conditions of sustained per capita output growth when the population growth rate is negative, and human capital is added to the production function with automation capital. Frey and Osborne (2017) in their empirical study for developed nations shows a nearly fifty per cent decline in jobs due to automation. According to the United Nations World Population Prospects 2024 report, many countries have experienced very low fertility for several decades, making them vulnerable to negative population growth. For instance, among many other countries, Italy and Poland have experienced a population decline since 2013.

The inclusion of automation capital as a factor input can be a blessing for such countries if it can replace laborers in the production process and boost economic growth; on the contrary, countries that have positive population growth and adequate amount of human capital, like India and Pakistan, start to employ labor replacing automation capital, it will create massive unemployment and economic distress. Therefore, investigating the effect of economic growth on human and automation capital is of significance.

There is a plethora of literature like Acemoglu and Restrepo (2018), Aghion et al. (2019), and Lankisch et al. (2019), which analyze the effect of economic growth and labor share when automation capital is adopted in the production process. Jones (2022) and Sasaki and Hoshida (2017) show the impact of negative population growth on long-run economic growth. The basic framework for our analysis has been adopted from Prettner (2019), where automation capital directly substitutes labor and complements physical capital.

In this paper, we extend the earlier study by Sasaki (2023) by incorporating human capital as an additional input and adopt a different strategy to see it's effect on economic growth when population growth becomes negative in the long-run. With the inclusion of human capital as another state variable, under certain conditions, we find a decline in the long-run output growth relative to the earlier study. The long-run output growth coincides with the growth of human and physical capital, but the growth of automation capital will be lesser than that of other variables. Finally, with negative population growth in our model, we find positive output growth, as in the earlier literature. Furthermore, in the numerical section, we emphasized the role of the savings rate in boosting economic growth when the human capital share to total income is included with negative population growth.

2 Model

Following Mankiw et al. (1992), we add human capital as an additional input to production along with physical and automation capital.[1]

$$Y = F(K, H, L, P) = K^{\alpha} H^{\beta} (L + P)^{1-\alpha-\beta}. \tag{1}$$

Here Y denotes total output, K, stock of physical capital, H, stock of human capital, L, labor and P, stock of automation capital. From the total income (Y), households decide on investment in their assets, which depends on their marginal productivities. A representative household's asset profile comprises physical, human and automation capital.

Let w^h, r^k, r^p and w^l denote the return on human capital, the rental rate of physical capital, the rental rate of automation capital and the wage rate of labor, respectively. In this one good economy, price of the final good is normalized to unity and the factor returns are their respective value marginal productivities. Since labor and automation capital are perfect substitute, their returns should be equalized in an interior equilibrium.

$$w^l = r^p = (1 - \beta - \alpha)\frac{Y}{L + P}, \quad r^k = \frac{\alpha Y}{K}, \quad w^h = \frac{\beta Y}{H}. \tag{2}$$

We use the following no-arbitrage condition to guarantee that household does not have any bias in accumulating one type of asset over another. This can be written as follows.

$$r^k = r^p, \quad r^p = w^h, \quad r^k = w^h, \tag{3}$$

Using equations (2) and (3) we get the following three equations.

(i) From $r^k = r^p$, we obtain:

$$P = \left(\frac{1 - \alpha - \beta}{\alpha}\right) K - L \implies P = \max\left\{0, \left(\frac{1 - \alpha - \beta}{\alpha}\right) K - L\right\}, \tag{4}$$

(ii) From $r^p = w^h$, we obtain:

$$P = \left(\frac{1 - \alpha - \beta}{\beta}\right) H - L \implies P = \max\left\{0, \left(\frac{1 - \alpha - \beta}{\beta}\right) H - L\right\}, \tag{5}$$

[1] See Prettner (2019) and Sasaki (2023) for similar production function.

(iii) From $r^k = w^h$, we get:

$$\frac{H}{K} = \frac{\beta}{\alpha}. \tag{6}$$

For the entire analysis, we assume $r^k = w^h$ to hold so that the ratio of per capita human to physical capital will be constant. If $r^k \neq w^h$, then either $H = 0$ or $K = 0$, which in case will make $Y = 0$.

Using the three no-arbitrage relationships, we rewrite the production function as:

$$\begin{aligned} Y &= \begin{cases} H^{\alpha+\beta}\left(\frac{\alpha}{\beta}\right)^{\alpha} L^{1-\alpha-\beta} & \text{if } 0 < H \leq \bar{H} \\ \alpha^{\alpha}\beta^{\beta-1}(1-\alpha-\beta)^{1-\alpha-\beta} H & \text{if } H > \bar{H}. \end{cases} \\ \implies y &= \begin{cases} h^{\alpha+\beta}\left(\frac{\alpha}{\beta}\right)^{\alpha} & \text{if } 0 < h \leq \bar{h} \\ \alpha^{\alpha}\beta^{\beta-1}(1-\alpha-\beta)^{1-\alpha-\beta} h & \text{if } h > \bar{h}, \end{cases} \end{aligned} \tag{7}$$

where $h = H/L$ denotes per capita human capital, $\bar{H} \equiv (\frac{\beta}{1-\alpha-\beta})L$ and $\bar{h} = \bar{H}/L$. When a higher amount of human capital is accumulated (H) compared to a threshold denoted by $barH$, i.e., $H > \bar{H}$, automation capital will start accumulating making $(P > 0)$, and the production function will have an AK formulation. On the contrary, when $H \leq \bar{H}$ due to a lack of savings by the households, then the output function will have a Cobb-Douglas formulation without automation capital $(P = 0)$.

Using the no-arbitrage relationships in equation (3), we re-write the returns from all the factors as:

$$r^k = r^p = w^h = w^l = \alpha^{\alpha}\beta^{\beta}(1-\alpha-\beta)^{1-\alpha-\beta} \equiv \mathcal{R}. \tag{8}$$

Note that $\mathcal{R} > 0$. Denote the share of labour in income as Ω_l and the share of human capital in income as Ω_h. These share expressions can be solved as follows.

$$\begin{aligned} \Omega_l &= \frac{\beta}{h} = \frac{\alpha}{k}, \\ \Omega_h &= \beta. \end{aligned} \tag{9}$$

The labor share in income (Ω_l) is negatively related to per capita physical and per capita human capital. Whereas, the human capital share in income (Ω_h) is independent of any factors of production.

2.1 Asset accumulation

In this system, households use a portion of their income for asset accumulation. This simple model has no financial sector, so the total income is either consumed

or invested for asset accumulation. The asset profile of a representative household is: $A = K + H + P$, where A denotes aggregate assets of households. Therefore, the total savings profile of households is: $S = s(w^h H + r^k K + r^p P + w^l L) = sY$. Assets are accumulated dynamically following Solow (1956): $\dot{A} = sY - \delta A$, where $\delta \in [0, 1]$ is the uniform depreciation rate of all the types of capitals and $\dot{A} \equiv \frac{dA}{dt}$. Rewriting the assets accumulation equation in per capita form, we get: $\dot{a} = sy - (\delta + n)a$, where n is the population growth rate and using the no-arbitrage relationships we get $\dot{a} = \frac{\dot{k}}{\alpha} = \frac{\dot{h}}{\beta}$. Therefore, we obtain two capital accumulation equations (h & k) for our three state variables using the no-arbitrage relationships, per capita output from equation (7) and the $\dot{a}$ relationship as follows:

$$\begin{aligned} \dot{h} =& \mathcal{B}h + \beta(\delta + n), \quad h > \bar{h}, \\ \dot{k} =& \mathcal{B}k + \alpha(\delta + n), \quad k > \bar{k}, \end{aligned} \tag{10}$$

Here $\mathcal{B} \equiv s\mathcal{R} - (\delta + n)$ and we already defined $\mathcal{R}$ in equation (8).

Note that the dynamics of automation capital can be found out using equation (10) and the no-arbitrage conditions. The conditions of $h > \bar{h}$ and $k > \bar{k}$ are essential for $P > 0$ which is the case of our interest. The steady-state values of human and physical capital are obtained by letting $\dot{h} = 0$ & $\dot{k} = 0$ in equation (10). This solves for

$$\begin{aligned} h^* &= \frac{\beta(\delta + n)}{(\delta + n) - s\mathcal{R}}, \\ k^* &= \frac{\alpha(\delta + n)}{(\delta + n) - s\mathcal{R}}. \end{aligned} \tag{11}$$

We will focus on parameter values such that the above steady state values are all strictly positive as negative including zero values are meaningless in our setting. The closed-form solutions of human and physical capital are derived below, which also helps us understand the dynamics further.

$$\begin{aligned} h(t) =& h^* - e^{-\frac{\beta(\delta+n)}{h^*}t} \, (h^* - h(0)), \\ k(t) =& k^* - e^{-\frac{\alpha(\delta+n)}{k^*}t} \, (k^* - k(0)). \end{aligned} \tag{12}$$

Here the initial human and physical capital are $h(0)$ and $k(0)$ and both are assumed to be strictly greater than $\bar{h}$ and $\bar{k}$, respectively. We also assume that initials level of the values are below the steady state. Next, we derive the following dynamics depending on the various parametric combinations.

Case 1: $(\delta + n) > 0$ and $\frac{\delta+n}{\mathcal{R}} > s > \frac{(\alpha+\beta)(\delta+n)}{\mathcal{R}}$.
In this case, s is intermediate and global convergence exists when the initial levels of h and k are above $\bar{h}$ and $\bar{k}$, respectively as shown in figure 1. The per capita income growth is zero in the steady state. The share of labor and human capital in total

income is constant[2], which is a positive constant.

Case 2: $(\delta + n) < 0$ and $s > \frac{\delta+n}{\mathcal{R}}$.

In this case, s is very large $\left(s > \underbrace{\frac{(\alpha+\beta)(\delta+n)}{\mathcal{R}}}_{<0} > \underbrace{\frac{\delta+n}{\mathcal{R}}}_{<0} \right)$. There is no convergence in this case and the per capita income growth is $2\{s\mathcal{R} - (\delta + n)\} > 0$. The dynamics depends on initial value of $h(0)$ and $k(0)$. If $h(0)$ and $k(0)$ are more than h^* and k^* respectively, there is perpetual growth in per capita income where in the other case $h(t)$ and $k(t)$ will fall to $\bar{h}$ and $\bar{k}$ respectively as shown in figure 2. The share of labor in total income is zero in the long-run[3] and the share of human capital is constant.

Case 3: $(\delta + n) > 0$ and $s > \frac{\delta+n}{\mathcal{R}}$.

In this case the steady state values are negative, i.e., $h^* < 0$ and $k^* < 0$, then a higher savings rate can ensure perpetual growth in per capita output, provided the initial values are sufficiently high and above a certain threshold denoted by $\bar{h}$ & $\bar{k}$ as shown in figure 3.

Case 4: $(\delta + n) < 0$ and $s < \frac{\delta+n}{\mathcal{R}}$.

Similarly to the above case, the steady state values are negative, but there exists convergence as shown in figure 4. This is also an uninteresting case because the bound on s guarantees a negative value as $(\delta + n) < 0$, which is implausible.

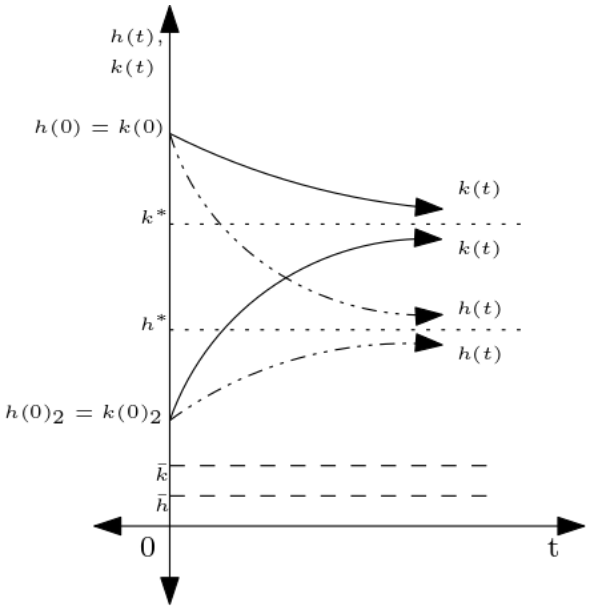


Figure 1: Transitional dynamics of $h(t)$ and $k(t)$ for case 1.

[2]Note: The share of labor and human capital in total income is given by $\Omega_l = \frac{\beta}{h_t}$, which leads to $\lim_{t\to\infty} \Omega_l = \frac{\beta}{h^*}$.

[3]Note: when $h(0) > h^*$, $\Rightarrow \lim_{t\to\infty} \Omega_l = \frac{\beta}{\infty}$, which is zero.

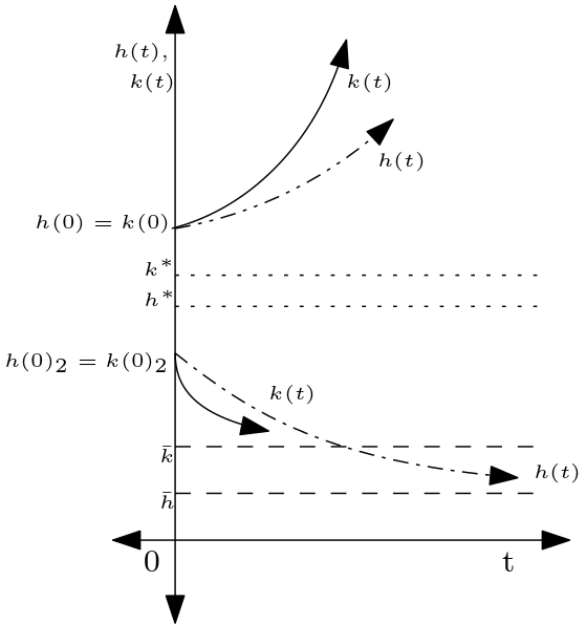


Figure 2: Transitional dynamics of $h(t)$ and $k(t)$ for case 2.

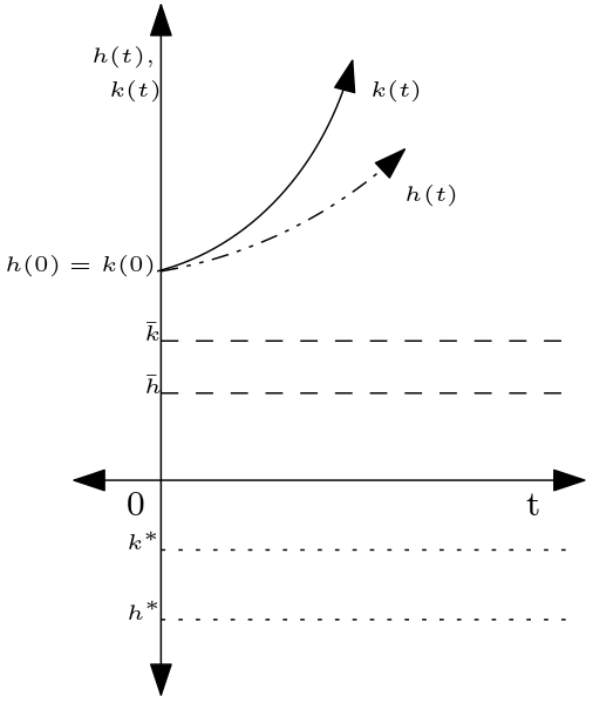


Figure 3: Transitional dynamics of $h(t)$ and $k(t)$ for case 3.

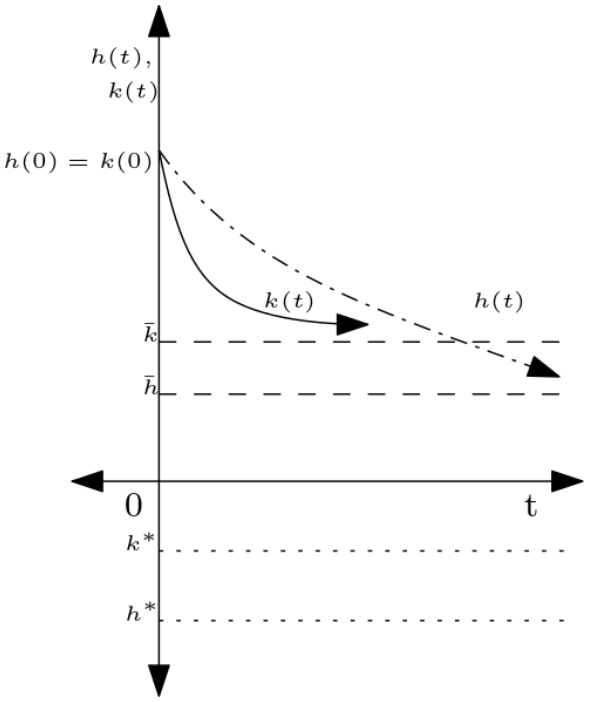


Figure 4: Transitional dynamics of $h(t)$ and $k(t)$ for case 4.

Following equations (7), (10) and the no-arbitrage relationships, the per-capita output growth rate in the long-run is equal to the growth of human and physical capital: $\frac{\dot{y}}{y} = \frac{\dot{h}}{h} = \frac{\dot{k}}{k} = 2\{s\mathcal{R} - (\delta + n)\} > 0$ (when $h > \bar{h}$). On the contrary, the per capita automation capital growth is less than the output per capita growth rate: $\frac{\dot{p}}{p} = \underbrace{\left\{\frac{(1-\alpha-\beta)(\delta+n)}{s\mathcal{R} - (\delta+n)(\alpha+\beta)}\right\}}_{\equiv\mu}\frac{\dot{h}}{h}$, where, $0 < \mu < 1$. On the other hand, when $h \leq \bar{h}$ and $P = 0$, the growth rate of per capita output is less than the growth rate of per capita human and physical capital: $\frac{\dot{y}}{y} = (\alpha + \beta)g$, where, $0 < (\alpha + \beta) < 1$ and $g \equiv \frac{\dot{h}}{h} = \frac{\dot{k}}{k}$.

Proposition 1: (i) The Addition of human capital in the production process reduces the long-run per capita output growth rate relative to the earlier studies when $\alpha < 1 - \beta$. (ii) When $(h > \bar{h})$, the per capita output growth is equivalent to the growth of per capital human and physical capital, but per capita automation capital growth is lesser than the common growth rate iff $0 < \mu < 1$. (iii) When $(h \leq \bar{h})$, the per capita output growth is lesser than the per capita human and physical capital growth iff $0 < (\alpha+\beta) < 1$. (iv) When $P > 0$, we obtain two cases to show that even if population growth becomes negative but $\delta + n > 0$, there is positive growth and convergence using the closed-form solutions of $h(t)$ and $k(t)$ iff $0 \leq s < \frac{\delta+n}{\mathcal{R}}$. (v) Finally, with $P > 0$, we obtain one relevant case of convergence when $s < 0$ and $(\delta + n) < 0$.

3 Numerical illustration

This section provides numerical examples pertaining to proposition 1. We consider two countries that have experienced negative population growth in the last decade.

We calculate various estimates for our study using the Penn World Tables v.10.01 dataset from Feenstra et al. (2015) and the World Bank dataset[4]. Italy and Poland's average annual population growth rate from 2013-2023 is $n = -0.1$ & $n = -0.3$, respectively. The share of capital in total income for both countries is $\alpha = 0.3$, and the rate of depreciation of capital is taken to be $\delta = 0.05$, which are plausible estimates used in macroeconomics literature. At last, we calculate the average share of labor in total income for Italy and Poland from 2013-2019 to be $(1-\alpha-\beta) = 0.5$ and $(1-\alpha-\beta) = 0.6$ respectively from the Penn World Tables. Therefore, the annual shares of human capital in total income for Italy and Poland from 2013-2019 are $\beta = 0.2$ & $\beta = 0.1$, respectively. We use the World Bank dataset[5] to fetch the gross savings rate of Italy and Poland as $s = 0.2$ & $s = 0.19$ respectively.

Using the above estimates and without the inclusion of human capital, the per capita output growth rate of Italy and Poland in the earlier study would have been: $\frac{\dot{y}}{y} = 0.32$ & $\frac{\dot{y}}{y} = 0.7$ respectively. Whereas with the inclusion of human capital, the per capita output growth rate of Italy and Poland become: $\frac{\dot{y}}{y} = 0.27$ & $\frac{\dot{y}}{y} = 0.67$ respectively. It shows a decline of 15.6 % and 4.3 % for Italy and Poland respectively. Now, given the current scenario, if the savings rate of Italy and Poland increase by 50 per cent from 0.2 to 0.3 and 0.19 to 0.285 respectively, portraying a scenario of case 3, where $\left(s > \frac{\delta+n}{\mathcal{R}} > \frac{(\alpha+\beta)(\delta+n)}{\mathcal{R}}\right)$, then the per capita output growth with human capital for Italy and Poland will be: $\frac{\dot{y}}{y} = 0.354$ & $\frac{\dot{y}}{y} = 0.754$ respectively. Therefore, with the inclusion of human capital as another factor of production that complements automation capital and labor, the savings rate plays an important role in navigating the economy towards positive growth, even if population growth declines and becomes negative.

To portray the effectiveness of the exogenous increase of savings rate in the earlier model (without the inclusion of human capital) compared to our model (with the inclusion of human capital), we re-calculate the per capita output growth of Italy and Poland in the earlier model using their new savings rate, i.e., $s = 0.3$ and $s = 0.285$ respectively, and obtained their new per capita output growth rate to be $\frac{\dot{y}}{y} = 0.42$ and $\frac{\dot{y}}{y} = 0.8$ respectively. Compared to the earlier estimates, the per capita output growth of Italy and Poland increased by 31.25% (from $\frac{\dot{y}}{y} = 0.32$ to 0.42) and 14.8% (from $\frac{\dot{y}}{y} = 0.8$ to 0.7), respectively, without the inclusion of human capital. On the contrary, in our model, i.e., with the inclusion of human capital we estimate the increase of per capita output of Italy and Poland with exogenous increase of

[4]To avail the population growth dataset of Italy and Poland please click here.

[5]To avail the gross savings (% of GDP) dataset of Italy and Poland please click here.

savings rate is by 30% (from $\frac{\dot{y}}{y} = 0.27$ to 0.354) and 12% (from $\frac{\dot{y}}{y} = 0.67$ to 0.754) respectively. Therefore, with the exogenous increase of savings rate and negative population growth, the earlier model will be able to increase higher per capita output growth compared to our model because in our model, with the inclusion of human capital, an additional fraction of savings is diverted to accumulate human capital as in equation (10) which hinders higher output growth. Thus, increasing the savings rate does not guarantee sustainable per capita output growth; if a more significant proportion of savings gets eroded due to additional capital accumulation, then the proposition of sustainable per capita output growth in the earlier study will cease to exist.

4 Conclusion

This study investigates the effect of human capital in explaining long-run economic growth when population growth becomes negative and automation substitutes labor in the production process. It also focuses on the transitional dynamics of the state variables in this setup.

When human capital is added as an additional automation-complementary factor of production, the per capita output growth becomes positive, but it is smaller than in the earlier study. It signifies that even if countries have negative population growth, per capita output growth need not become zero or negative in the long run if an adequate amount of savings is maintained.

The effectiveness of an increase in the savings rate to increase per capita output growth in our model is less than that of the earlier framework, which abstracts from human capital as another factor of production. Therefore, with the inclusion of another factor of production, which accumulates using a proportion of the aggregate savings rate, the output growth declines.

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A Appendix (Not for Publication)

A1.1 Derivation of total output with $P = 0$ and $P > 0$

The total output is defined as:

$$Y = K^{\alpha} H^{\beta} (L + P)^{1-\alpha-\beta}, \tag{A1.1}$$

We re-write the total output equation as:

$$\Rightarrow Y = \left(\frac{K}{L+P}\right)^{\alpha} \left(\frac{H}{L+P}\right)^{\beta} (L + P),$$

$$\Rightarrow \left(\frac{K}{L+P}\right)^{\alpha} = \frac{Y}{L+P} \left(\frac{H}{L+P}\right)^{-\beta}. \tag{A1.2}$$

Using equation (A1.1) we derive the wage rate of human capital (W^h), rental rate of physical capital (R^k), rental rate of automation capital (R^p) and wage rate of labor (W^l) as their respective marginal productivities.

$$W^h \equiv \frac{\partial Y}{\partial H} = \beta \left(\frac{K}{L+P}\right)^{\alpha} \left(\frac{L+P}{H}\right)^{1-\beta}, \tag{A1.3}$$

$$R^k \equiv \frac{\partial Y}{\partial K} = \alpha \left(\frac{L+P}{K}\right)^{1-\alpha} \left(\frac{H}{L+P}\right)^{\beta}, \tag{A1.4}$$

$$R^p \equiv \frac{\partial Y}{\partial P} = (1-\alpha-\beta) \left(\frac{K}{L+P}\right)^{\alpha} \left(\frac{H}{L+P}\right)^{\beta}, \tag{A1.5}$$

$$W^l \equiv \frac{\partial Y}{\partial L} = (1-\alpha-\beta) \left(\frac{K}{L+P}\right)^{\alpha} \left(\frac{H}{L+P}\right)^{\beta}. \tag{A1.6}$$

Using equation (A1.2) we re-write the factor returns equations in terms of total output as:

$$\begin{aligned}
W^h &= \beta \frac{Y}{H}, \\
R^k &= \alpha \frac{Y}{K}, \\
R^p &= (1-\alpha-\beta) \left(\frac{Y}{L+P}\right), \\
W^l &= (1-\alpha-\beta) \left(\frac{Y}{L+P}\right).
\end{aligned}$$

Using equations (A1.3)-(A1.6) to derive three no-arbitrage conditions:
(i) From $R^k = R^p$ we obtain:

$$P = \left(\frac{1-\alpha-\beta}{\alpha}\right) K - L, \tag{A1.7}$$

$$\Rightarrow P = \max\left\{0, \left(\frac{1-\alpha-\beta}{\alpha}\right) K - L\right\},$$

we get the condition on K by putting $P = 0$, i.e., there will be no automation capital in the production process,

$$K = \underbrace{\left(\frac{\alpha}{1-\alpha-\beta}\right)}_{\equiv \bar{K}} L,$$

and automation capital is positive ($P > 0$) iff $K > \bar{K}$.

$$\therefore \quad \text{when} \;\; 0 < K \leq \bar{K}, \;\; P = 0,$$
$$\text{and} \;\; K > \bar{K}, \;\; P > 0.$$

(ii) From $R^p = W^h$ we obtain:

$$P = \left(\frac{1-\alpha-\beta}{\beta}\right) H - L, \tag{A1.8}$$

$$\Rightarrow P = \max\left\{0, \left(\frac{1-\alpha-\beta}{\beta}\right) H - L\right\},$$

we get the condition on H by putting $P = 0$, i.e., there will be no automation capital in the production process,

$$H = \underbrace{\left(\frac{\beta}{1-\alpha-\beta}\right)}_{\equiv \bar{H}} L,$$

and automation capital is positive ($P > 0$) iff $H > \bar{H}$.

$$\begin{aligned} \therefore \quad \text{when} \;\; & 0 < H \leq \bar{H}, \;\; P = 0, \\ \text{and} \;\; & H > \bar{H}, \;\; P > 0. \end{aligned}$$

(iii) From $R^k = W^h$ we obtain:

$$\frac{H}{K} = \frac{\beta}{\alpha}, \tag{A1.9}$$

it is important to note that the above condition on H and K ensures that both the factors of production cannot be zero, if either one factor becomes zero, then the production function will cease to exist.

Now, use the no-arbitrage conditions from equations (A1.7) - (A1.9) and the total output function from equation (A1.1) we characterize the behavior of total output as per the derived conditions of H and K.
(i) Consider the case when $P > 0$, or, $H > \bar{H}$ and $K > \bar{K}$.

$$Y = K^{\alpha} H^{\beta} (L + P)^{1-\alpha-\beta},$$

from equation (A1.7) we use the optimal value of P in the above equation, we get

$$\begin{aligned} Y =& K^{\alpha} H^{\beta} \left[L + \left(\frac{1-\alpha-\beta}{\alpha} \right) K - L \right]^{1-\alpha-\beta}, \\ \Rightarrow Y =& K^{1-\beta} H^{\beta} \left(\frac{1-\alpha-\beta}{\alpha} \right)^{1-\alpha-\beta}, \end{aligned}$$

To write the output equation in one variable, substitute K from equation (A1.9), we get

$$\begin{aligned} \Rightarrow Y =& H^{\beta} \left(\frac{H\alpha}{\beta} \right)^{1-\beta} \left(\frac{1-\alpha-\beta}{\alpha} \right)^{1-\alpha-\beta}, \\ \Rightarrow Y =& H \alpha^{\alpha} \beta^{\beta-1} (1-\alpha-\beta)^{1-\alpha-\beta}. \end{aligned}$$

(ii) Consider the case when $P = 0$, or, $0 < H \leq \bar{H}$ and $0 < K \leq \bar{K}$.

$$Y = K^{\alpha} H^{\beta} L^{1-\alpha-\beta},$$

using equation (A1.9) to substitute K in the above equation, we get

$$Y = \left(\frac{H\alpha}{\beta}\right)^{\alpha} H^{\beta} L^{1-\alpha-\beta},$$

Therefore, summarizing the two cases below:

$$Y = \begin{cases} H^{\alpha+\beta}\left(\frac{\alpha}{\beta}\right)^{\alpha} L^{1-\alpha-\beta} & \text{if } 0 < H \leq \bar{H}, \\ \alpha^{\alpha}\beta^{\beta-1}(1-\alpha-\beta)^{1-\alpha-\beta} H & \text{if } H > \bar{H}. \end{cases} \tag{A1.10}$$

The per capita counterpart of the above equation can be written as:

$$\implies y = \begin{cases} h^{\alpha+\beta}\left(\frac{\alpha}{\beta}\right)^{\alpha} & \text{if } 0 < h \leq \bar{h} \\ \alpha^{\alpha}\beta^{\beta-1}(1-\alpha-\beta)^{1-\alpha-\beta} h & \text{if } h > \bar{h}, \end{cases} \tag{A1.11}$$

where $h \equiv \frac{H}{L}$ and $\bar{h} \equiv \frac{\beta}{1-\beta-\alpha}$.

A1.2 Derivation of $\mathcal{R}$

Re-writing equations (A1.3)-(A1.6) using the no-arbitrage conditions from equations (A1.7)-(A1.9), we get

$$(i) \;\; W^h = \beta\left(\frac{K}{L+P}\right)^{\alpha}\left(\frac{L+P}{H}\right)^{1-\beta},$$

using $\frac{K}{L+P} = \frac{\alpha}{1-\alpha-\beta}$ and $\frac{H}{L+P} = \frac{\beta}{1-\alpha-\beta}$ from equations (A1.7) and (A1.8) respectively, and plugging it in the above equation, we get

$$\begin{aligned} W^h =& \beta\left(\frac{\alpha}{1-\beta-\alpha}\right)^{\alpha}\left(\frac{1-\beta-\alpha}{\beta}\right)^{1-\beta}, \\ \Rightarrow W^h =& \beta^{\beta}\alpha^{\alpha}(1-\alpha-\beta)^{1-\alpha-\beta}. \end{aligned}$$

$$(ii) \;\; W^l = (1-\alpha-\beta)\left(\frac{K}{L+P}\right)^{\alpha}\left(\frac{H}{L+P}\right)^{\beta},$$

using $\frac{K}{L+P} = \frac{\alpha}{1-\alpha-\beta}$ and $\frac{H}{L+P} = \frac{\beta}{1-\alpha-\beta}$ from equations (A1.7) and (A1.8) respectively, and plugging it in the above equation, we get

$$\begin{aligned} W^l =& \beta \left(\frac{\alpha}{1-\beta-\alpha}\right)^{\alpha} \left(\frac{1-\beta-\alpha}{\beta}\right)^{\beta}, \\ \Rightarrow W^l =& \beta^{\beta}\alpha^{\alpha}(1-\alpha-\beta)^{1-\alpha-\beta}. \end{aligned}$$

$$(iii) \quad R^k = \alpha \left(\frac{L+P}{K}\right)^{1-\alpha} \left(\frac{H}{L+P}\right)^{\beta},$$

using $\frac{K}{L+P} = \frac{\alpha}{1-\alpha-\beta}$ and $\frac{H}{L+P} = \frac{\beta}{1-\alpha-\beta}$ from equations (A1.7) and (A1.8) respectively, and plugging it in the above equation, we get

$$\begin{aligned} R^k =& \beta \left(\frac{1-\beta-\alpha}{\alpha}\right)^{1-\alpha} \left(\frac{\beta}{1-\beta-\alpha}\right)^{\beta}, \\ \Rightarrow R^k =& \beta^{\beta}\alpha^{\alpha}(1-\alpha-\beta)^{1-\alpha-\beta}. \end{aligned}$$

$$(iv) \quad R^p = (1-\alpha-\beta) \left(\frac{K}{L+P}\right)^{\alpha} \left(\frac{H}{L+P}\right)^{\beta},$$

using $\frac{K}{L+P} = \frac{\alpha}{1-\alpha-\beta}$ and $\frac{H}{L+P} = \frac{\beta}{1-\alpha-\beta}$ from equations (A1.7) and (A1.8) respectively, and plugging it in the above equation, we get

$$\begin{aligned} R^p =& \beta \left(\frac{\alpha}{1-\beta-\alpha}\right)^{\alpha} \left(\frac{1-\beta-\alpha}{\beta}\right)^{\beta}, \\ \Rightarrow R^p =& \beta^{\beta}\alpha^{\alpha}(1-\alpha-\beta)^{1-\alpha-\beta}. \end{aligned}$$

Note: $W^h = W^l = R^k = R^p = \beta^{\beta}\alpha^{\alpha}(1-\alpha-\beta)^{1-\alpha-\beta} \equiv \mathcal{R}$. This solution only holds when there is an interior solution of automation capital $(P > 0)$.

A1.3 Equilibrium shares of high-skilled and low-skilled laborers

The equilibrium labor shares of high-skilled and low-skilled workers in total income are defined as:

$$\Omega_l = \frac{W^l L}{Y} \tag{A3.12}$$

Substitute W^l and Y from equations (A1.6) and (A1.1) respectively into the above equation, we get

$$\begin{aligned}\Omega_l =& \frac{(1-\alpha-\beta)\left(\frac{K}{L+P}\right)^{\alpha}\left(\frac{H}{L+P}\right)^{\beta}L}{K^{\alpha}H^{\beta}(L+P)^{1-\alpha-\beta}},\\ \Omega_l =& (1-\alpha-\beta)\frac{L}{L+P},\end{aligned}$$

re-writing the above equation in per capita form and using the no-arbitrage relationships from equations (A1.7) and (A1.9) to obtain

$$\begin{aligned}\Omega_l =& (1-\alpha-\beta)\frac{1}{1+p},\\ \Omega_l =& (1-\alpha-\beta)\frac{\alpha}{(1-\alpha-\beta)k},\\ \Rightarrow \quad \Omega_l =& \frac{\alpha}{k}=\frac{\beta}{h}.\end{aligned}$$

Similarly,

$$\Omega_h = \frac{W^h H}{Y}. \tag{A3.13}$$

Using equations (A1.3) and (A1.1) to substitute W^h and Y respectively in the aove equation, we get

$$\begin{aligned}\Omega_h =& \frac{\beta\left(\frac{K}{L+P}\right)^{\alpha}\left(\frac{L+P}{H}\right)^{1-\beta}H}{K^{\alpha}H^{\beta}(L+P)^{1-\alpha-\beta}},\\ \Omega_h =& \beta.\end{aligned}$$

It implies that Ω_l s decreasing in k and h. Conversely, Ω_h is orthogonal to k and h.

A1.4 Dynamics of Assets accumulation

A1.4.1 When $P > 0$; $h > \bar{h}$ and $k > \bar{k}$

The total asset (A) holding in this economy is:

$$A = K + P + H. \tag{A4.14}$$

Our model is a one good model, so K, P and H have the same price as the price of Y, which is normalized to unity. Therefore, the aggregate savings rate is defined as

$$\begin{aligned} S =& sY, \\ S =& s\underbrace{\left(W^l L + W^h H + R^k K + R^p P\right)}_{\equiv Y}. \end{aligned}$$

The accumulation of A is similar to that of a textbook Solow model, which is:

$$\dot{A} = S - \delta A, \quad \delta \in (0, 1), \tag{A4.15}$$

where δ is the common depreciation rate of all the assets in a portfolio.

Let the per capita assets is defined as $a \equiv \frac{A}{L}$, thus rewriting equation (A4.14) as:

$$\dot{a} = sy - (\delta + n)a. \tag{A4.16}$$

To rewrite the asset accumulation equation in terms of h and k, we use equation (A1.7) in per capita form as follows:

$$\begin{aligned} P =& \left(\frac{1-\alpha-\beta}{\alpha}\right) K - L, \\ \frac{P+K}{L} =& \frac{k}{\alpha} - \frac{\beta}{\alpha} k - 1, \end{aligned}$$

now using the total assets equation from equation (A4.14) and re-writing the above equation as:

$$\begin{aligned} \frac{A-H}{L} =& \frac{k}{\alpha} - \frac{\beta}{\alpha} k - 1, \\ a =& \frac{k}{\alpha} - \frac{\beta}{\alpha} k - 1 + h. \end{aligned}$$

Using $k = \frac{\alpha}{\beta} h$ from equation (A1.9) in per capita form and substituting it in the earlier equation, we re-write per capita assets in terms of h as:

$$a = \frac{h}{\beta} - 1. \tag{A4.17}$$

Similarly, substituting $h = \frac{\beta}{\alpha} k$ from equation (A1.9) in the earlier equation, we re-write per capita assets in terms of k as:

$$a = \frac{k}{\alpha} - 1. \tag{A4.18}$$

Now taking time derivative of equations (A4.16) and (A4.17), we get

$$\dot{a} = \frac{\dot{h}}{\beta}, \tag{A4.19}$$

$$\dot{a} = \frac{\dot{k}}{\alpha}. \tag{A4.20}$$

Using equations (A1.11) and (A4.16) to derive the accumulation equation of per capita human capital as:

$$\begin{aligned} \dot{a} =& s\frac{h}{\beta}\underbrace{\alpha^{\alpha}\beta^{\beta}(1-\alpha-\beta)^{1-\alpha-\beta}}_{\equiv\mathcal{R}} - (\delta+n)a, \\ \Rightarrow \dot{a} =& s\frac{h}{\beta}\mathcal{R} - (\delta+n)a. \end{aligned}$$

Using the $\dot{a}$ expression from equation (A4.19) and the expression of a from equation (A4.17) and put it in the above equation, we get

$$\begin{aligned} \frac{\dot{h}}{\beta} =& s\frac{h}{\beta}\mathcal{R} - (\delta+n)(\frac{h}{\beta}-1), \\ \dot{h} =& sh\mathcal{R} - (\delta+n)(h-\beta), \\ \dot{h} =& \underbrace{[s\mathcal{R} - (\delta+n)]}_{\equiv\mathcal{B}}h + (\delta+n)\beta, \end{aligned}$$

$$\dot{h} = \mathcal{B}h + (\delta+n)\beta. \tag{A4.21}$$

Similarly, to derive the accumulation equation of per capita physical capital, we use equations (A1.11), (A4.16) and (A1.9) in per capita form, we get

$$\begin{aligned} \dot{a} =& s\frac{k}{\alpha}\underbrace{\alpha^{\alpha}\beta^{\beta}(1-\alpha-\beta)^{1-\alpha-\beta}}_{\equiv\mathcal{R}} - (\delta+n)a, \\ \Rightarrow \dot{a} =& s\frac{k}{\alpha}\mathcal{R} - (\delta+n)a. \end{aligned}$$

Using the $\dot{a}$ expression from equation (A4.20) and the expression of a from equation

(A4.18) and put it in the above equation, we get

$$\begin{aligned}\frac{\dot{k}}{\alpha} =& s\frac{k}{\alpha}\mathcal{R} - (\delta+n)(\frac{1}{\alpha}-\frac{1}{k}),\\ \dot{k} =& sk\mathcal{R} - (\delta+n)(k-\alpha),\\ \dot{k} =& \underbrace{[s\mathcal{R} - (\delta+n)]}_{\equiv\mathcal{B}}k + (\delta+n)\alpha,\end{aligned}$$

$$\dot{k} = \mathcal{B}k + (\delta+n)\alpha. \tag{A4.22}$$

The steady-state values of h and k is obtained by putting equations (A4.21) and (A4.22) equal to zero and solving it for h and k respectively.

$$\begin{aligned}h^* =& -\frac{\beta(\delta+n)}{\mathcal{B}},\\ k^* =& -\frac{\alpha(\delta+n)}{\mathcal{B}}.\end{aligned}$$

Note the steady-state values are negative, but our paper considers only the cases when the steady-state values are positive, i.e., either $(\delta+n)<0$ or $\mathcal{B}<0$.

A1.4.2 When $P=0$; $0<h\leq\bar{h}$ and $0<k\leq\bar{k}$

The total asset (A) in this setup is:

$$A = K + H. \tag{A4.23}$$

Similar to the above subsection,

$$\begin{aligned}S =& sY,\\ S =& s\underbrace{(W^l L + W^h H + R^k K)}_{\equiv Y}.\end{aligned}$$

Using the same per capita asset accumulation expression as in equation (A4.16) and the no-arbitrage relationship from equation (A1.9) in per capita form, we get

$$a = h\left(\frac{\alpha+\beta}{\beta}\right). \tag{A4.24}$$

Similarly, substituting $h = \frac{\beta}{\alpha}k$ from equation (A1.9) in the earlier equation, we re-write per capita assets in terms of k as:

$$a = k\left(\frac{\alpha+\beta}{\alpha}\right). \tag{A4.25}$$

Now taking time derivative of equations (A4.24) and (A4.25), we get

$$\dot{a} = \dot{h}\left(\frac{\alpha+\beta}{\beta}\right), \tag{A4.26}$$

$$\dot{a} = \dot{k}\left(\frac{\alpha+\beta}{\alpha}\right). \tag{A4.27}$$

Using equations (A1.11) and (A4.16) to derive the accumulation equation of per capita human capital as:

$$\dot{a} = sh^{\alpha+\beta}\alpha^{\alpha}\beta^{-\alpha} - (\delta+n)a.$$

Using the $\dot{a}$ expression from equation (A4.26) and the expression of a from equation (A4.24) and put it in the above equation, we get

$$\dot{h}\left(\frac{\alpha+\beta}{\beta}\right) = sh^{\alpha+\beta}\alpha^{\alpha}\beta^{-\alpha} - (\delta+n)h\left(\frac{\alpha+\beta}{\beta}\right),$$

$$\dot{h} = \frac{sh^{\alpha+\beta}\alpha^{\alpha}\beta^{1-\alpha}}{\alpha+\beta} - (\delta+n)h.$$

Similarly, we derive the per capita capital accumulation as:

$$\dot{k} = \frac{sk^{\alpha+\beta}\beta^{\beta}\alpha^{1-\beta}}{\alpha+\beta} - (\delta+n)k.$$

The steady-state values of h and k is obtained by putting equations $\dot{h} = 0$ and $\dot{k} = 0$ respectively in the earlier equation.

$$h^* = \left[\frac{s\alpha^{\alpha}\beta^{1-\alpha}}{(\alpha+\beta)(\delta+n)}\right]^{\frac{1}{1-\alpha-\beta}},$$

$$k^* = \left[\frac{s\beta^{\beta}\alpha^{1-\beta}}{(\alpha+\beta)(\delta+n)}\right]^{\frac{1}{1-\alpha-\beta}}.$$

Note in this case the steady-state values are positive. Additionally, evaluating the steady-state values of h and k, and $\bar{h}$ and $\bar{k}$ respectively, we find when $s > \frac{(\delta+n)(\alpha+\beta)}{\mathcal{R}}$ we are in the region of $P > 0$ or, $h^* > \bar{h}$ & $k^* > \bar{k}$.

A1.5 Closed form solutions of $h(t)$ and $k(t)$

Firstly, solving for the closed form solution of $h(t)$.
Using the human capital accumulation equation from equation (A4.21), and letting $z = -\beta$ for simplicity, we re-write it as

$$\frac{\partial h(t)}{\partial t} = \mathcal{B}h(t) - z(\delta + n).$$

The integrating factor for solving the first order differential equation is:

$$\begin{aligned} \text{I.F} &= e^{\int -\mathcal{B}\, dt}, \\ &= e^{-\mathcal{B}t}. \end{aligned}$$

Multiplying the integrating factor in both hand sides in the re-written form of equation (A4.21), we get

$$\begin{aligned} e^{-\mathcal{B}t}h(t) &= -z(\delta + n) \int e^{-\mathcal{B}t} dt, \\ \text{or, } e^{-\mathcal{B}t}h(t) &= \frac{-z(\delta + n)}{-\mathcal{B}} e^{-\mathcal{B}t} + C, \\ \text{or, } e^{-\mathcal{B}t}h(t) &= \frac{z(\delta + n)}{\mathcal{B}} e^{-\mathcal{B}t} + C, \end{aligned}$$

where C is the integrating constant, therefore, we can solve it by putting $t = 0$ in the earlier equation,

$$C = h(0) - \frac{z(\delta + n)}{\mathcal{B}},$$

now, put the value of C in the earlier equation and solve for $h(t)$, we get

$$\begin{aligned} e^{-\mathcal{B}t}h(t) &= \frac{z(\delta + n)}{\mathcal{B}} e^{-\mathcal{B}t} + h(0) - \frac{z(\delta + n)}{\mathcal{B}}, \\ \text{or, } e^{-\mathcal{B}t}h(t) &= \frac{z(\delta + n)}{\mathcal{B}} [e^{-\mathcal{B}t} - 1] + h(0), \\ \text{or, } h(t) &= \frac{z(\delta + n)}{\mathcal{B}} [1 - e^{\mathcal{B}t}] + h(0)e^{\mathcal{B}t}, \\ \text{or, } h(t) &= \frac{z(\delta + n)}{\mathcal{B}} - \frac{z(\delta + n)}{\mathcal{B}} e^{\mathcal{B}t} + h(0)e^{\mathcal{B}t}, \\ \text{or, } h(t) &= \frac{z(\delta + n)}{\mathcal{B}} + e^{\mathcal{B}t} \left[h(0) - \frac{z(\delta + n)}{\mathcal{B}} \right]. \end{aligned}$$

Now, put $z = -\beta$ back in the earlier equation, we get,

$$h(t) = \frac{-\beta(\delta + n)}{\mathcal{B}} + e^{\mathcal{B}t}\left[h(0) - \left(\frac{-\beta(\delta + n)}{\mathcal{B}}\right)\right].$$

Note: finally the closed form solutions of $h(t)$ can be written as:

$$\Rightarrow h(t) = h^* + e^{\mathcal{B}t}[h(0) - h^*],$$

where $h(0)$ is the initial levels of $h(t)$.

Similarly, using equation(A4.21) we can derive the closed-form solution of $k(t)$ to be:

$$k(t) = \frac{-\alpha(\delta + n)}{\mathcal{B}} + e^{\mathcal{B}t}\left[k(0) - \left(\frac{-\alpha(\delta + n)}{\mathcal{B}}\right)\right],$$
$$\text{or, } \; k(t) = k^* + e^{\mathcal{B}t}[k(0) - k^*],$$

where $k(0)$ is the initial levels of $k(t)$.