Mind the gap when exiting low-for-long*

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Abstract

Short-term interest rates are the primary and effective tool of monetary policy. Keeping them low for long may induce agents to believe that interest rates will remain lower for longer than intended by the central bank while stimulating the economy; however, when interest rates are lifted off, the correction of this belief gap can be perceived as surprise tightening. We articulate this potential channel of low-for-long policy by studying a New Keynesian model with learning and forward guidance. The model shows that the belief gap widens non-linearly as low rates continue while exerting expansionary effects, and its correction during the lift-off causes contraction. We study mechanisms of how the belief gap grows and provide some empirical evidence.

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Keywords: Low-for-long interest rates; learning; neutral interest rates; forward guidance; imperfect credibility.

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1 Introduction

The conduct of monetary policy generates surprises when its action or relevant information released differs from what agents have expected or believed. To promote the effectiveness of monetary policy and avoid unnecessary surprises, many central banks engage in communicating information collected and how it is transformed into policy decisions, the latter of which is called a monetary policy reaction function. Communicating a reaction function about short-term interest rates can be challenging, however. When interest rates stay constant at the effective lower bound (ELB) for a long time (low-for-long), agents would have difficulty in inferring the reaction function. A gap between the actual reaction function and what agents perceive can widen during low-for-long. And a correction of the gap, if any, can cause monetary policy surprises when the economy exits low-for-long.

After the global financial crisis of 2007–09 (GFG) many developed economies went through low-for-long, and most of them abruptly exited around 2022 in face of the post-pandemic surge in inflation. During the rapid monetary tightening, there were some disruptions including the banking sector turmoil of March 2023, although none of them triggered a recession. Lagged behind these economies, the Japanese economy is finally exiting the three-decade-long low interest rate environment as of Spring 2025.

Against this backdrop, this paper studies a mechanism of how a belief gap about a monetary policy reaction function can grow under low-for-long and its implications during an exit. To that end, we build a New Keynesian (NK) model that incorporates agents' learning about a monetary policy reaction function as well as the ELB and forward guidance with imperfect credibility. Using the model we simulate a scenario of low-for-long and study how agents' belief evolves in such a scenario and what is a consequence of a correction of a belief gap, if any. Moreover, we study them under various scenarios including an inflation surge when forward guidance keeps interest rates from rising. Furthermore, we conduct empirical analyses using Japanese data to complement the model's analyses.

Our main findings are three-fold. First, our model scenario simulations show that a gap between agents' belief about monetary policy stance implied by a monetary policy reaction function and its actual stance widens downward under low-for-long. As the period of interest rates at the ELB is prolonged, through learning, agents come to believe that monetary policy stance is more accommodative and expect lower interest rates for longer. Second, the formation of the belief gap is mainly driven by a decrease in the perceived neutral nominal

interest rate in the monetary policy reaction function. As the ELB period is prolonged, agents update the perceived neutral rate downward, giving rise to a downward widening of the gap. Third, when the economy exits low-for-long, the belief gap that has grown during low-for-long is corrected sharply, which is perceived as surprise monetary tightening and decreases output and inflation significantly. A stimulus generated from fostering expectations of lower-for-longer before the exit and the correction of the belief during the exit give rise to a boom and bust in the economy.

Our model is based on a standard three-equation NK model with the ELB constraint. It features three additional ingredients: forward guidance with imperfect credibility, additional discounting in the linearized Euler equation, and agents' learning about a monetary policy reaction function. Forward guidance is added to generate low-for-long and modeled as keeping the policy rate at the ELB for extended periods. To capture the actual conduct of forward guidance that leaves room for discretion and to address the forward guidance puzzle (Del Negro et al., 2023), we assume that agents perceive that forward guidance will be actually implemented as announced with a certain probability in the spirit similar to Bodenstein et al. (2012). The subjective probability captures the degree of imperfect credibility about forward guidance. For simplicity we abstract from unconventional monetary policy except for forward guidance. Additional discounting in the Euler equation is introduced to help the model to generate low-for-long without causing a recession that is too severe in light of the experiences of the developed economies since the GFC. It is based on households' cognitive discounting studied by Gabaix (2020) or procyclical inequality among households in the tractable heterogeneous agent new Keynesian (HANK) model studied by Bilbiie (2024).

Regarding learning, we employ the anticipated utility approach of Kreps (1998). Under this approach, agents make decisions assuming that there will be no change in a monetary policy reaction function that is characterized by a prior inherited from the previous period. After observing the macroeconomic variables including the policy rate, agents update their belief and repeat this process in the next period. We use this approach because of its simplicity and applicability to a binding ELB constraint. We closely follow Bodenstein et al. (2022), who study agents' learning under an ELB regime, and extend their learning algorithm to accommodate forward guidance with imperfect credibility.

We assume a relatively general form for a monetary policy reaction function, which is characterized by a constant term that captures the neutral nominal interest rate, a coefficient

on the current inflation rate, and a coefficient on the output gap as well as a coefficient on the lagged policy rate. We allow for agents in the model to learn about the constant term and the two coefficients on inflation and output gap.

The model is then calibrated to the Japanese economy and simulated under a baseline scenario in which a severe negative demand shock hits and drives the economy into a prolonged period of the binding ELB constraint that spans seven years including roughly two years of forward guidance. To focus on agents' learning, the scenario assumes no change in the actual monetary policy reaction function. In addition, to study learning under low-for-long, the scenario assumes the path of interest rates such that the central bank implements forward guidance as announced despite agents' imperfect credibility.

In the initial phase of the interest rate at the ELB, there is no significant change in agents' belief. This is because observing the interest rate at the ELB is too obvious and does not add meaningful information as inflation and output are far below their target or long-term levels. However, as the ELB regime is prolonged and the economy continues to recover, observing the interest rate at the ELB starts to affect their beliefs. Specifically, in updating beliefs, agents pose that the interest rate could have been positive with some probability according to their prior beliefs; but the fact that it was set at the ELB implies that the interest rate implied by a reaction function may be lower than previously thought. This reasoning leads to a downward update of the interest rate implied by the reaction function, which pushes down the neutral rate term in the reaction function in particular. The same intuition also applies to the period of forward guidance in which the interest rate is kept at the ELB. Hence, the agents' belief evolves non-linearly under the ELB regime: almost no change initially when the economic conditions clearly justify an ELB regime, but a sharp decline in the perceived neutral rate term as the ELB period is prolonged while the economy continues to recover.

The baseline scenario simulation shows that the belief gap becomes the largest downward one period before the life-off of forward guidance. After the lift-off, the interest rate is set in accordance with the reaction function and the belief gap is corrected abruptly. The correction is perceived as monetary tightening and decreases output and inflation sharply. A sensitivity analysis suggests that the belief gap one period before the lift-off becomes larger, as forward guidance becomes less credible; the period of forward guidance becomes longer; the parameters of the reaction function are perceived to be more volatile; and the perceived law of motion for the neutral nominal rate term becomes more persistent. The size of the

belief gap and the effects of its correction during the exit depends on assumptions on these exogenous parameters, but the main result continues to hold: as an ELB regime is prolonged, the belief gap widens downward, and its correction during the exit has contractionary effects on the economy.

The paper also examines alternative scenarios. An inflationary shock scenario during the period of forward guidance is considered in light of the recent inflation surge of 2021–2022. To reflect almost three-decade long experience of low interest rates in Japan, a recurrent shock scenario is added, where the interest rate is kept at the ELB for twelve years – five years longer than the baseline scenario – as a severe negative shock hits recurrently. Under these scenarios the belief gap becomes wider than in the baseline scenario. A scenario of belief accommodations after the exit is also considered because a central bank may want to keep an accommodative stance even after the exit. In this scenario, a correction of the gap is postponed and the simulation result suggests that perceived monetary tightening when belief accommodations end can be less severe than in the case of the baseline scenario.

Finally, this paper empirically examines market participants' perceptions of monetary policy using survey data from professional forecasters on short-term interest rates, real GDP growth, and CPI inflation in Japan. The estimates of the perceived neutral nominal interest rate gradually decline, especially from 2008 when the BOJ introduced various unconventional measures, including forward guidance. The gradual decline in the perceived neutral nominal interest rate under the low interest environment with forward guidance is consistent with the simulation results of our model. We further examine the relationship between monetary policy shocks and the neutral nominal interest rate. A local projection analysis reveals that the perceived neutral nominal interest rate falls a quarter later when a negative monetary policy surprise is observed. This gradual correction in the neutral rate is consistent with our model where agents are learning policy rules from the central bank actions.

Related literature This paper builds on the literature on agents' learning about parameters or variables pertaining to monetary policy, which includes learning about parameters that characterize a monetary policy reaction function (Tetlow and von zur Muehlen, 2001; Cogley et al., 2015; Bauer et al., 2024); monetary policy regimes (Gust et al., 2018; Mele et al., 2020; Bodenstein et al., 2022; Krane et al., 2023); shocks to the target rate of inflation (Erceg and Levin, 2003); trend inflation or long-run mean of inflation (Cogley and Sbordone, 2008; Carvalho et al., 2023); the real neutral interest rate (Rungcharoenkitkul and Winkler,

2023); the distribution of consumption and inflation (Christiano et al., 2024). Among them, Bodenstein et al. (2022) is closely related to ours as it studies agents' learning about a change in monetary policy strategies from an inflation-targeting rule to a price-level tarting rule under an ELB regime. We extend its learning algorithm to accommodate forward guidance with imperfect credibility.

This paper contributes to the literature on the side effects of low-for-long and unconventional monetary policy. Borio et al. (2017) and Claessens et al. (2018) empirically show that low-for-long interest rates adversely affect bank profitability. Abadi et al. (2023) theoretically demonstrate the existence of the reversal rate at which accommodative monetary policy becomes contractionary for lending in a NK model with banks. Acharya et al. (2022) and Acharya and Rajan (2024) argue that central bank balance sheet expansion and its subsequent shrinkage may increases the probability and severity of episodes of liquidity stress. Orphanides (2024) points out that forward guidance can be a trap if surprises to inflation occur when forward guidance keeps the interest rate from rising.[1] This paper adds to this literature another possible channel: low-for-long interest rates may induce agents to believe lower-for-longer interest rates, and its subsequent correction has contractionary effects on the economy.

This paper is also related to the empirical studies on monetary policy rules perceived by economic agents. The most related to ours is Bauer et al. (2024) who estimate a perceived monetary policy rule using professional forecast data. Their estimation results are consistent with a view that economic agents update their perceived monetary policy rule, especially the responsiveness to macroeconomic variables, by observing monetary policy actions. This paper focuses on the neutral nominal interest rate instead of coefficients on inflation or output in accordance with our model's results. Other papers that estimate perceived monetary policy rules include Hamilton et al. (2011) who use the United States federal funds futures; Bocola et al. (2024) who use daily bond yield data; Fendel et al. (2011) who use professional survey data for G7 countries, as we do for Japan in this paper; Schmidt and Nautz (2012) who use the Euro area financial market data; Carvalho and Nechio (2014) who use the Michigan Survey. We contribute to this literature by shedding light on the dynamics of the

[1] As a part of the European Central Bank's monetary policy review, Altavilla et al. (2021) argue that the side effects of the key monetary policy instruments employed since 2014 have been generally contained so far. In its monetary policy review, Bank of Japan (2024) concludes that although there have been certain side effects of large-scale monetary easing on financial markets and financial institutions' profits, its overall effect on the Japanese economy so far appears to have been positive.

perceived neutral nominal interest rate during the tightening cycle after the low-for-long period in Japan.

The rest of the paper is organized as follows. Section 2 presents the model including forward guidance and agents' learning. Section 3 simulates the model and presents the main results of the paper. Section 4 provides empirical support for the main results. Section 5 concludes.

2 The Model

We present a version of the NK model that incorporates forward guidance with imperfect credibility and discounting of future aggregate variables in the Euler equation. Imperfect credibility is introduced to capture uncertainty about whether forward guidance is implemented as originally announced. The discounting is introduced to address the so-called forward guidance puzzle and help the model generate a prolonged period of binding ELB.[2] The discounting can be interpreted as households' cognitive discounting as in Gabaix (2020) or procyclical inequality among households in the tractable HANK model studied by Bilbiie (2024). In addition, we introduce private agents' learning about a monetary policy rule as in Bodenstein et al. (2022). To analyze the model, we consider a linearized model and focus on a targeted equilibrium where inflation in steady state is equal to the target rate of inflation set by the central bank.[3]

Section 2.1 presents a simple three-equation linearized model with rational expectations. Section 2.2 introduces forward guidance with imperfect credibility. Section 2.3 introduces learning and describes how private agents learn about a monetary policy rule.

2.1 The linearized model

For starters a rational expectations model is presented. Let $\hat{y}_t$, $\hat{\pi}_t$, and $\hat{i}_t$ denote output, inflation, the nominal interest rate in deviation from their steady state value, respectively.[4]

[2]For the forward guidance puzzle, see McKay et al. (2016) and Del Negro et al. (2023).

[3]In the context of the low-for-long experience in Japan, for the role of non-linearity, see Iiboshi et al. (2022); for the literature that focuses on a deflationary equilibrium as opposed to a targeted equilibrium, see Aruoba et al. (2017), Hirose (2020), and Coyle et al. (2025).

[4]For the interest rate, the deviation from steady state is expressed in terms of the gross interest rate, so that $\hat{i}_t$ is given by $\hat{i}_t = (i_t - i)/(1 + i)$, where i_t is the net nominal interest rate and i is its steady state. Similarly, the ELB in deviation from steady state is given by $\hat{\underline{i}} = (\underline{i} - i)/(1 + i)$, where $\underline{i}$ is the ELB of the net nominal interest rate.

The linearized equations that describe the dynamics of inflation and output are given by

$$\hat{y}_t = M\mathbb{E}_t\hat{y}_{t+1} - \frac{1}{\sigma}\left(\hat{i}_t - \mathbb{E}_t\hat{\pi}_{t+1}\right) + z_t^d, \tag{1}$$

$$\hat{\pi}_t = \kappa\hat{y}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} + z_t^s, \tag{2}$$

where z_t^d and z_t^s are demand and supply shocks that follow an AR(1) process, respectively, $\mathbb{E}_t(\cdot)$ is an expectation operator conditional on information in period t, and $M \in [0,1]$ captures the discounting of future aggregate variables. The standard NK model corresponds to $M = 1$. The discounting of $M < 1$ can be derived from the model with households' behavioral cognitive discounting (Gabaix, 2020) or the tractable HANK model with cyclical inequality among households (Bilbiie, 2024). The other parameters are standard: $\sigma > 0$ captures relative risk aversion, β is a preference discount factor, and κ is the slope of the NK Phillips curve (2). Appendix A derives equations (1) and (2) from the original NK model.

Without forward guidance, a central bank sets the nominal interest rate following a monetary policy rule subject to the ELB constraint, given by

$$\hat{i}_t = \max\{\hat{i}_t^n, \underline{\hat{i}}\}, \tag{3}$$

$$\hat{i}_t^n = \rho_i\hat{i}_{t-1} + (1-\rho_i)\left(\hat{i}_t^* + \phi_{\pi t}\hat{\pi}_t + \phi_{yt}\hat{y}_t\right) + \epsilon_t^{mp}, \tag{4}$$

where $\hat{i}_t^n$ is the notional interest rate at which the central bank wishes to set if there were no ELB, $\hat{i}_t^*$ is the neutral nominal interest rate perceived by the central bank, and ϵ_t^{mp} is an i.i.d. monetary policy shock that follows $N(0, \sigma_{mp}^2)$.[5] The coefficients on inflation and output, $\phi_{\pi t}$ and ϕ_{yt}, are allowed to be time-varying. Private agents learning about these coefficients will be introduced in 2.3.

To focus on private agents' learning about the neutral nominal interest rate later in Section 2.3, we assume that it follows an exogenous AR(1) process:

$$\hat{i}_t^* = \rho^*\hat{i}_{t-1}^* + \epsilon_t^*, \tag{5}$$

where $0 \leq \rho^* < 1$ and ϵ_t^* is an i.i.d. shock. The assumption of $\rho^* < 1$ ensures $\hat{i}_t^* = 0$ in steady state, that is, the neutral nominal interest rate becomes equal to the steady state nominal interest rate that is consistent with inflation equal to the target rate of inflation $\hat{\pi}_t = 0$. For

[5] In steady state, the notional interest rate and the neutral nominal interest rate coincide with the actual nominal interest rate: $i^n = i^* = i$. Similar to $\hat{i}_t$, the notional and neutral rates in deviation from the steady state are given by $\hat{i}_t^n = (i_t^n - i)/(1+i)$ and $\hat{i}_t^* = (i_t^* - i)/(1+i)$.

the sake of exposition, we treat $\hat{i}_t^*$ as an exogenous time-varying policy parameter.

Without forward guidance and learning, the model is characterized by four equations (1)–(4) with four unknowns $\{\hat{y}_t, \hat{\pi}_t, \hat{i}_t, \hat{i}_t^n\}$, for given $\hat{i}_t^*$, $\phi_{\pi t}$, and ϕ_{yt}.

2.2 Forward guidance with imperfect credibility

During the period specified by forward guidance the interest rate is set according to forward guidance. Among a variety of forms of forward guidance, in light of the recent experiences of the interest rates kept at the ELB during the surge in inflation in many developed economies after the Covid-19 pandemic, we focus on a lagged exit from the ELB.[6] Specifically, consider a situation in which a severe shock hits and drives the economy into the ELB in period $t = 1$. A central bank makes a baseline projection under which the economy will move out of the ELB in period $T+1$, where T is a projected duration of the ELB without forward guidance. The central bank announces that it will keep the interest rate at the ELB for an extended period of ΔT on top of T.

In practice, the extended duration of keeping the interest rate at the ELB, from $t = 1$ to $t = T + \Delta T$, can be state-dependent. In general the duration depends on the state of the economy when it is projected and determined and thus it can be expressed as $T_t + \Delta T_t$ with time subscript t. In this analysis, for simplicity, we assume that $T + \Delta T$ is unchanged at the original value when it was announced in $t = 1$ as time evolves. This specific forward guidance can be regarded as time-dependent or calendar-based, or it can also be regarded as state-dependent where there is no change in the state of the economy as projected in $t = 1$.

We assume that the period specified by forward guidance is perfectly credible, but whether the central bank keeps the interest rate at the ELB as stipulated by the forward guidance is imperfectly credible. Specifically, agents believe that the ELB is continued with probability $q \in [0, 1]$, given that the promise made by the forward guidance has been actually implemented; with probability $1 - q$, the promise is reneged and the forward guidance is discarded thereafter. Consider a situation the economy falls in the ELB in period $t = 1$ and at the same time forward guidance to keep the interest rate at the ELB until $t = T + \Delta T$ is an-

[6] From the modeling perspective, forward guidance can be modeled as monetary policy rules (Reifschneider and Williams, 2000; Debortoli et al., 2019), news shocks (Campbell et al., 2017), and a lagged exit (Bodenstein et al., 2012; Boneva et al., 2018). Forward guidance as news shocks can deliver same outcomes as forward guidance as a lagged exit under some circumstances. For empirical applications to the Japanese economy, see Ikeda et al. (2024) and Kubota and Shintani (2024) for rule-based forward guidance, and Hayashi and Koeda (2019) and Iiboshi et al. (2022) for threshold-based forward guidance as a lagged exit.

nounced. In this situation, agents believe that the interest rate follows: for $t = 1, ..., T + \Delta T$

$$\hat{i}_t = \begin{cases} \underline{\hat{i}} & \text{if } \hat{i}_t^n \geq \underline{\hat{i}} \text{ and no renege before, with probability } q \\ \max\{\hat{i}_t^n, \underline{\hat{i}}\} & \text{otherwise} \end{cases} \tag{6}$$

Outside the period of forward guidance, agents' belief coincides with the setting of the interest rate (3).

Forward guidance is perfectly credible if $q = 1$ and not credible if $q = 0$. We assume that the degree of credibility is exogenous. In practice, it would depend on economic agents' trust in a central bank, central bank communications, future uncertainty, a central bank's discretion, time-inconsistency among others.[7]

2.3 Learning about a monetary policy rule

We now introduce private agents' learning about the monetary policy rule (4). Let $\mathbf{x}_t \stackrel{\text{def}}{=} [\hat{i}_{t-1}, \hat{\pi}_t, \hat{y}_t]'$ denote a column vector of the input variables in (4) and let $\boldsymbol{\theta}_t \stackrel{\text{def}}{=} [\hat{i}_t^*, \phi_{\pi t}, \phi_{yt}]'$ denote a column vector of time-varying parameters. A part of the notional rate that is systematically tied to inflation, output, and the lagged rate, $f(\mathbf{x}_t; \boldsymbol{\theta}_t) \stackrel{\text{def}}{=} \rho_i \hat{i}_{t-1} + (1-\rho_i)(\hat{i}_t^* + \phi_{\pi t}\hat{\pi}_t + \phi_{yt}\hat{y}_t)$, is called as the systematic part of the notional rate. With this notation, the notional rate can be written as $\hat{i}_t^n = f(\mathbf{x}_t; \boldsymbol{\theta}_t) + \epsilon_t^{mp}$.

Information structure The agents know the monetary policy rule that consists of (3) and (4), including its functional forms and parameter values, except for $\boldsymbol{\theta}_t$ and ϵ_t^{mp}. The agents believe that the neutral nominal interest rate follows (5). During the period specified by forward guidance, the agents' belief about the interest rate is given by equation (6). They observe $\hat{\pi}_t$, $\hat{y}_t$, and $\hat{i}_t$ in period t, but do not observe $\boldsymbol{\theta}_t$, ϵ_t^{mp}, or ϵ_t^*.

Beliefs and decision making In the beginning of period t, the agents have a subjective belief about the mean of unobservable monetary policy parameters, $\tilde{\boldsymbol{\theta}}_{t|t-1}$, inherited from the previous period. In line with the learning literature such as Bodenstein et al. (2022), we employ the anticipated utility approach of Kreps (1998) and assume that agents make decisions as if they expected no additional shock to $\boldsymbol{\theta}_t$ in the future.[8] Given $\tilde{\boldsymbol{\theta}}_{t|t-1}$, they form beliefs about the monetary policy shock, $\tilde{\epsilon}_t^{mp}$, and make individual decisions. Their

[7] For time-consistent forward guidance, see Batista et al. (2023).

[8] This assumption is consistent with how the model with the occasionally binding ELB constraint is solved by using the algorithm of Guerrieri and Iacoviello (2015), which we employ in this paper.

beliefs are consistent with the macroeconomic variables, $\hat{\pi}_t$, $\hat{y}_t$, and $\hat{i}_t$, which are governed by equations (1)–(6), the period specified by forward guidance, and the belief about the monetary policy shock that satisfies

$$\hat{i}_t^n = f(\mathbf{x}_t; \tilde{\boldsymbol{\theta}}_{t|t-1}) + \tilde{\epsilon}_t^{mp}, \tag{7}$$

with $\hat{i}_t^n$ given by equation (4).

How agents' belief about monetary policy shocks is formulated differs depending on whether the ELB constraint is binding and whether forward guidance is announced. Consider the period outside forward guidance. There are two cases: when the ELB constraint is slack or binding. First, when the ELB constraint is slack and thus $\hat{i}_t = \hat{i}_t^n$, agents' belief about the monetary policy shock is given by

$$\tilde{\epsilon}_t^{mp} = \hat{i}_t - f(\mathbf{x}_t; \tilde{\boldsymbol{\theta}}_{t|t-1}) = \epsilon_t^{mp} + \left[f(\mathbf{x}_t; \boldsymbol{\theta}_t) - f(\mathbf{x}_t; \tilde{\boldsymbol{\theta}}_{t|t-1}) \right]. \tag{8}$$

The belief $\tilde{\epsilon}_t^{mp}$ consists of the true monetary policy shock plus a difference between the true systematic component of the rule and agents' belief about it. Next, in the case of the binding ELB constraint, what agents learn about $\hat{i}_t^*$ is the inequality related to the ELB constraint: $\hat{i}_t^n = f(\mathbf{x}_t; \tilde{\boldsymbol{\theta}}_{t|t-1}) + \tilde{\epsilon}_t^{mp} \leq \underline{\hat{i}}$. Given $\tilde{\boldsymbol{\theta}}_{t|t-1}$, there are many values of $\tilde{\epsilon}_t^{mp}$ that satisfies this inequality. The monetary policy rule (4) suggests that any value of $\tilde{\epsilon}_t^{mp}$ that satisfies the inequality leads to the same solution to equations (1)–(4).

During the period specified by forward guidance, consider a situation that the systematic part of the notional rate is above the ELB, $f(\mathbf{x}_t; \tilde{\boldsymbol{\theta}}_{t|t-1}) > \underline{\hat{i}}$, so that agents perceive that forward guidance can take effects. If agents observe the non-binding ELB constraint, it is evident that the forward guidance has been lifted, so they form a belief about a monetary policy shock as in the case of the non-binding ELB constraint outside the period of forward guidance. If the agents observe the interest rate at the ELB, they believe that forward guidance has been implemented. In the case of $f(\mathbf{x}_t; \tilde{\boldsymbol{\theta}}_{t|t-1}) \leq \underline{\hat{i}}$, the agents perceive that the interest rate is kept at the ELB because of the low notional interest rate. Then, they form expectations about future consistent with equation (6) and make decisions in period t.

Learning and updating In the end of each period after the aggregate variables are realized, the agents update their belief about the policy parameters $\boldsymbol{\theta}_t$ in a Bayesian manner. Specifically, given the prior belief $\tilde{\boldsymbol{\theta}}_{t|t-1}$, they update it for a posterior belief $\tilde{\boldsymbol{\theta}}_{t|t}$ by solving the

filtering problem that consists of the ELB constraint (3), the actual rule for the conceptual rate (4), and the law of motion for $\boldsymbol{\theta}_t$,

$$\boldsymbol{\theta}_t = F\boldsymbol{\theta}_{t-1} + \boldsymbol{\epsilon}_t^{\theta}, \tag{9}$$

where F is a diagonal matrix given by $F = \text{diag}(\rho^*, 1, 1)$ and $\boldsymbol{\epsilon}_t^{\theta} \sim \text{i.i.d.}N(0, \Sigma_{\theta})$. Thus, monetary policy coefficients $\phi_{\pi t}$ and ϕ_{yt} are perceived as a random walk by the agents.

To simplify the filtering problem that involves an occasionally binding constraint, we make two assumptions as in Bodenstein et al. (2022). First, the agents treat endogenous variables $\mathbf{x}_t$ as independent of shocks ϵ_t^{mp} and $\boldsymbol{\epsilon}_t^{\theta}$. Second, the posterior is approximated by a normal distribution. Thanks to these assumptions, the prior belief about $\boldsymbol{\theta}_t$ is given by a normal distribution $N(\tilde{\boldsymbol{\theta}}_{t|t-1}, P_{t|t-1})$ and the posterior is given by $N(\tilde{\boldsymbol{\theta}}_{t|t}, P_{t|t})$, where $P_{t|\cdot}$ is an associated variance. From equation (9), the updated prior that is used for the agents to make decisions in $t+1$ is given by $\tilde{\boldsymbol{\theta}}_{t+1|t} = F\tilde{\boldsymbol{\theta}}_{t|t}$ and $P_{t+1|t} = FP_{t|t}F' + \Sigma_{\theta}$. Appendix B describes how to obtain the posterior from the prior.

3 Scenario Analyses

We simulate the model presented in Section 2 to study how agents' belief about the neutral nominal interest rate changes during the prolonged period of the interest rate at the ELB and its implications during the exit from the low-for-long policy. This section is organized as follows. The model is paramterized in Section 3.1. A baseline scenario is presented in Section 3.2, where a severe demand shock drives the interest rate at the ELB for a prolonged period of time under the assumption of no learning. Forward guidance and imperfect credibility are also briefly discussed. Learning is introduced in Section 3.3, which presents the evolution of agents' belief about the neutral interest rate during the low-for-long policy and how its correction during the exit affects the economy. The underlying mechanisms behind the evolution of agents' belief are studied in Section 3.4. And the sensitivity of the results to various assumptions is examined in Section 3.5. Finally, additional scenarios are examined in Section 3.6.

3.1 Parameterization

We set parameter values to reflect the Japanese economy, although the model is highly stylized. One period of the model corresponds to a quarter of a year. The parameter values

are set broadly in line with those reported in Iiboshi et al. (2022) and Hirose et al. (2024). Iiboshi et al. (2022) estimate the non-linear version of the model with the ELB, without additional discounting M, using the Japanese quarterly data from 1983 to 2016. Hirose et al. (2024) estimate a similar model but with additional discounting, using the U.S. quarterly data from 1983 to 2019. The values of parameters pertaining to equations (1)–(4) are set as $\sigma = 1.5$, $\kappa = 0.05$, $\beta = 0.998$, $\rho_i = 0.7$, $\phi_\pi = 2$, $\phi_y = 0.2$, and $M = 0.85$, where ϕ_π and ϕ_y denote benchmark monetary policy parameters. Regarding the persistence parameter of the perceived law of motion for the neutral nominal interest rate in equation (5), it is assumed to be persistent and set at $\rho^* = 0.95$. Given the target rate of inflation of 2 percent annually, the steady state nominal rate is set at $i = \pi/\beta - 1$.

The speed of learning is governed by the variance of monetary policy shocks σ^2_{mp} and the variance-covariance matrix Σ_θ that captures subjective uncertainty about the monetary policy parameters. Intuitively, according to equation (8) when the interest rate is above the ELB, the larger the ratio $\text{diag}(\Sigma_\theta)/\sigma^2_{mp}$, the more the agents attribute the perceived monetary policy shock $\tilde{\epsilon}^{mp}_t$ to changes in monetary policy parameter values. Since there is no empirical evidence regarding Σ_θ, we use the same values regarding σ_{mp} and variance attached to $\phi_{\pi t}$ that are employed by Bodenstein et al. (2022) who study learning a change in a monetary policy rule from an inflation targeting rule to a price-level targeting rule in a ELB regime. Specifically, we set $\sigma^2_{mp} = 0.01$ percent and $\Sigma_\theta/\sigma^2_{mp} = \text{diag}(0.4, 0.4, 0.4)$ percent, where we assume that variance attached to $\hat{i}^*_t$ and ϕ_{yt} is the same as that attached to $\phi_{\pi t}$. These parameter values imply the standard deviation of the monetary policy shock of 10 basis points and the perceived standard deviations of the innovations of the monetary policy parameters, $\hat{i}^*_t$, $\phi_{\pi t}$, and ϕ_{yt}, are 6 basis points.[9]

3.2 Baseline scenario with forward guidance

We construct a baseline scenario of low-for-long interest rates in view of the prolonged period of interest rates at the ELB in the Japanese economy from the late 1990s to early 2020s. We take a view that severe and persistent negative shocks hit the Japanese economy repeatedly during the period, such as deleveraging pressures from the burst of asset and land prices in the early 1990s, the domestic financial crisis in the late 1990s, the global financial crisis of

[9]According to the literature on estimating a dynamic stochastic general equilibrium model for the Japanese economy, the standard deviation of monetary policy shocks varies from about 6 basis points (Sugo and Ueda, 2008) to 10 basis points or greater (Kaihatsu and Kurozumi, 2014; Iiboshi et al., 2022).

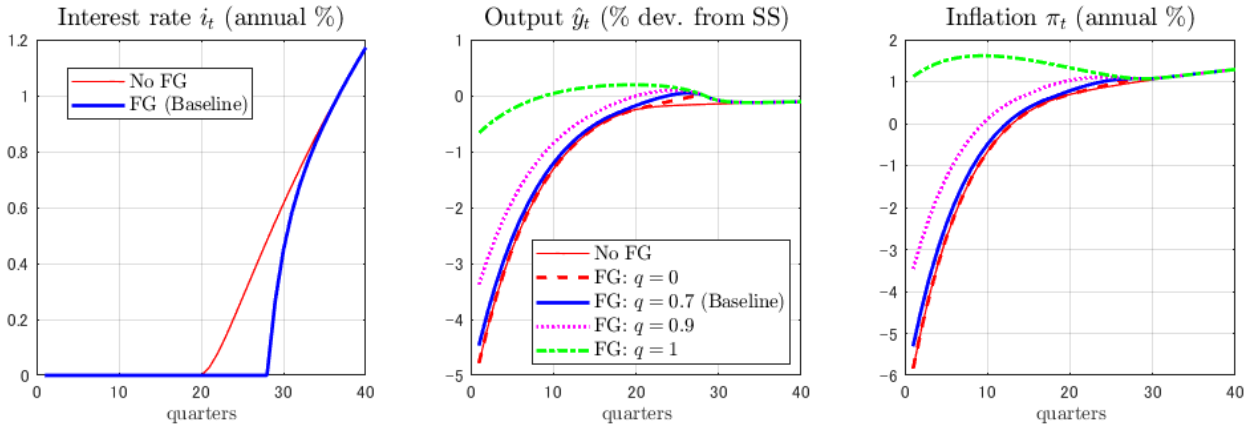
2007–09, the great east Japan earthquake of 2011, and the COVID-19 pandemic. However, it is beyond the scope of the paper to explain low-for-long in the quarter century from the late 1990s and analyze potential development of the economic agents' perception about the neutral nominal interest rate.

To shed light on mechanisms of how agents' perception about the neutral nominal interest rate changes during low-for-long interest rates, and for clarity and simplicity, we focus on a scenario in which a single negative shock hits the economy and drives it into the prolonged period of low interest rates. Specifically, we consider a severe and persistent negative demand shock that hits in period $t = 1$ and drives the interest rates at the ELB for five years without forward guidance. To generate such a long period of the binding ELB constraint with one time shock, the shock is made significantly persistent and severe: the AR(1) coefficient of z_t^d is set at 0.97 and the magnitude of the shock is set at $z_1^d = -0.55$ percent.[10] In addition, we assume that the central bank responds to this situation in period $t = 1$ by announcing forward guidance of keeping the interest rate for an extended period of two years. This, in terms of notations introduced in Section 2.2, the ELB period without forward guidance is $T = 20$ quarters and the additional period of the ELB promised by the forward guidance is $\Delta T = 8$ quarters under the baseline scenario. Importantly, to focus on agents' learning, we assume no change in the actual monetary policy rule (4) with $\hat{i}_t^* = 0$, $\phi_{\pi t} = \phi_\pi$, and $\phi_{yt} = \phi_y$. Finally, to clarify the role of learning and changes in agents' perceptions about monetary policy, we assume no learning in the baseline scenario and will introduce it as an alternative scenario subsequently. For the model's responses to shocks when the interest rate is positive, see Appendix A.3.

Figure 1 plots the baseline scenario along with the same scenario with different degree of credibility about forward guidance and the case of no forward guidance. A severe shock hits the economy in $t = 1$ and drives the interest rate at the ELB for 20 periods without forward guidance, and in the baseline scenario with forward guidance the interest rate is kept at the ELB until $t = 28$ as shown in the left panel. Output and inflation drop sharply and the recoveries are slow but smooth as shown in the middle and right panels. Admittedly, the decline of inflation is more severe than what the Japanese economy experienced from the late 1990s to the early 2020s. Thus, the baseline scenario should be considered as a hypothetical

[10]The high persistence of the shock in this simple model can be considered to reflect secular stagnation forces arising from a financial crisis and demographic aging, as studied by Ikeda and Kurozumi (2019) and Braun and Ikeda (2022) in rich quantitative models with nominal frictions, respectively.

Figure 1: Baseline scenario and imperfect credibility about forward guidance



Note: 'FG' denotes forward guidance. The interest and inflation rates are shown in annual percentage level while the output is in terms of percentage deviation from steady state (SS).

scenario that captures low inflation and low output during the prolonged period of the ELB in Japan. It is worth adding that the presence of additional discounting M in equation (1) helps the model generate deflation that is not too severe (see Appendix A.3 for details).

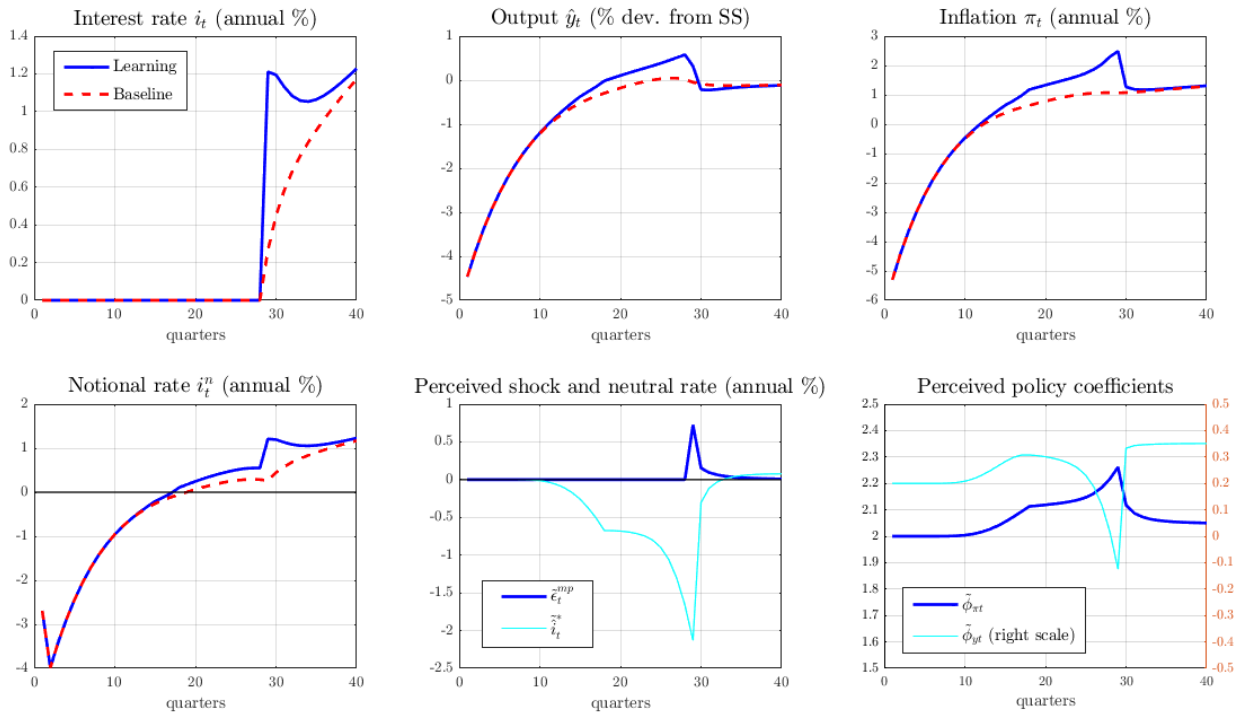
A comparison between the case of no forward guidance ('No FG') and the cases of forward guidance ('FG') in Figure 1 shows forward guidance mitigates the stagnation pressure and the degree of its effect is increasing in the credibility parameter q. When it is perfectly credible, $q = 1$, the forward guidance is extremely effective and it may be too effective to believe its effectiveness similar to in the forward guidance puzzle (Del Negro et al., 2023). In practice, no central bank would credibly commit to the path of interest rates in the next seven years and it is thus natural to assume $q < 1$. Although there is no empirical evidence about q, in light of some positive effects of monetary policy that can be interpreted as forward guidance for Japan (Kubota and Shintani, 2024), we set $q = 0.7$ as baseline.[11]

3.3 Baseline scenario when agents learn

Now we introduce agents' learning about a monetary policy rule into the model and simulate the baseline scenario. As described in Section 2.3, agents do not know key parameters in the monetary policy rule, $\boldsymbol{\theta}_t \overset{\text{def}}{=} [\hat{i}_t^*, \phi_{\pi t}, \phi_{yt}]'$. In every period, they have a belief $\tilde{\boldsymbol{\theta}}_t$ about the true parameter values $\boldsymbol{\theta}_t$, inherited from the previous period, and make a decision as if there

[11] Bodenstein et al. (2012) consider a central bank that faces the binding ELB constraint and commits to the optimal policy with probability q and reset the policy with the rest of the probability and argue that q is around 0.5 in the aftermath of the GFC in the U.S. and Sweden.

Figure 2: Baseline scenario simulation when agents learn



Note: 'Learning' and 'Baseline' indicate the baseline scenario simulation using the model with and without learning, respectively. The notional rate in 'Learning' is the perceived notional rate during the period of the binding ELB constraint. The degree of credibility is set at $q = 0.7$.

will be no additional shock to $\tilde{\boldsymbol{\theta}}_t$. We focus on a path of interest rates such that the central bank keeps the interest rate at the ELB up to period $T + \Delta T$ as originally announced.

Figure 2 shows the baseline simulation results when agents learn. Compared to the baseline results without learning, output and inflation in the model with learning show similar developments until around $t = 15$, but diverge thereafter, as shown in the upper middle and upper right panels. Specifically, output and inflation diverge upward until the liftoff of the zero interest rate policy: output increases to a level greater than its steady state level, implying a positive output gap; inflation also increases to a level greater than 2 percent – the target rate of inflation. However, when the economy exits the prolonged period of zero interest rates, output and inflation fall sharply. Hence, relative to the baseline result without learning, there are boom and bust in output and inflation around the exit from low-for-long in the model with learning.

What drives the sharp decline in output and inflation during the exit is perceived monetary tightening, as shown in the lower middle panel of Figure 2. At the time of exit the

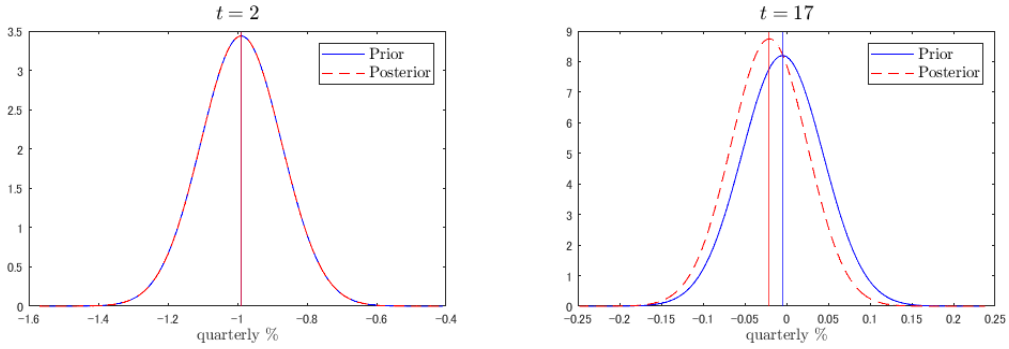
perceived monetary policy shock $\tilde{\epsilon}_t^{mp}$ is more than 70 basis points. This perceived monetary tightening lowers output and inflation by 0.8 percentage points (pp) and 1.2 pp, respectively, during the exit.

Since the structural monetary policy shock is assumed to be zero, $\epsilon_t^{mp} = 0$, equation (8) implies that the perceived monetary tightening $\tilde{\epsilon}_t^{mp}$ is caused by a difference in the actual and perceived systematic component of the monetary policy rule, $f(\mathbf{x}_t; \boldsymbol{\theta}_t) - f(\mathbf{x}_t; \tilde{\boldsymbol{\theta}}_{t|t-1})$, which in turn depends on differences in actual and perceived parameter values $\boldsymbol{\theta}_t - \tilde{\boldsymbol{\theta}}_{t|t-1}$. The decomposition of $\tilde{\epsilon}_t^{mp}$ into the three components in $\boldsymbol{\theta}_t - \tilde{\boldsymbol{\theta}}_{t|t-1}$ reveals that the perceived neutral nominal interest rate, $\tilde{\tilde{i}}_t^*$, accounts for nearly 90 percent of the monetary tightening at the time of exit. Indeed, just before the exit, the agents perceive that the neutral nominal rate is about -2 pp, while its true value is zero, and the deviations of their perceptions about monetary policy coefficients from the true values are relatively moderate, as shown in the lower middle and lower right panels of Figure 2. With these perceptions, just before the exit, the agents expect that the nominal interest rate will be low because of the low neutral nominal rate in the monetary policy rule (4). But, this expectation is violated at the time of exit, as the central bank follows the monetary policy rule (4) with $\hat{i}_t^* = 0$. The realized nominal interest rate that significantly exceeds the expectation is perceived as considerable monetary tightening.[12]

During the prolonged period of low interest rates, the agents' perception about the neutral nominal rate evolves in a nonlinear and cumulative manner. Three observations can be made from the lower middle panel of Figure 2. First, there is little change in the perception $\tilde{\tilde{i}}_t^*$ in the first 3 years. Second, as the period of low interest rates is prolonged beyond 3 years, the perceived neutral nominal rate starts decreasing and continue to do so until the central bank implements forward guidance in period $t = 18$ as originally announced. Third, although a downward revision of the perception is slow in the first half of the period of forward guidance, as the period of forward guidance is prolonged, the perception starts to be revised downward steeply in the latter half of the period. From these observations, it appears that monetary policy that keeps interest rates at the ELB for a prolonged period contributes to building agents' perception that the neutral nominal rate is lower than expected previously.

[12]The model is agnostic about the source of a change in the neutral nominal interest rate. It may be due to a change in the trend inflation rate (Cogley and Sbordone, 2008; Carvalho et al., 2023), a change in the target rate of inflation (Erceg and Levin, 2003), or a change in the real neutral rate of interest (Rungcharoenkitkul and Winkler, 2023).

Figure 3: Beliefs about the systematic part of the notional rate



3.4 Understanding the gap

What drives the nonlinear and cumulative development in the perceived nominal rate in the model with learning and what is the underlying mechanism of perceiving low neutral nominal rates during low-for-long? In this subsection, we address these questions with the aim of understanding the gap between the perceived neutral rate and its actual value.

Let $f_{t|t-1}$ and $f_{t|t}$ denote the prior and posterior beliefs about the systematic part of the notional rate, $f_t = f(\mathbf{x}_t; \boldsymbol{\theta}_t)$. Since the agents have prior $\tilde{\boldsymbol{\theta}}_{t|t-1}$ inherited from the previous period, the prior about f_t is given by $f_{t|t-1} = f(\mathbf{x}_t; \tilde{\boldsymbol{\theta}}_{t|t-1})$. Under the simplified assumptions about belief formation described in Section 2.3, when the interest rate is at the ELB because of low notional interest rates, as shown in Appendix B, the parameters of interest are updated as:

$$\tilde{\boldsymbol{\theta}}_{t|t} = \tilde{\boldsymbol{\theta}}_{t|t-1} + K_t \left(f_{t|t} - f_{t|t-1} \right), \tag{10}$$

where K_t captures a gain from updated information about f_t. Since $\hat{i}_t^n$ is unobservable when the ELB constraint (3) is binding, agents form a posterior $f_{t|t}$ about the systematic part of $\hat{i}_t^n$ given the observation of $\hat{i}_t = \underline{\hat{i}}$, and update their belief about monetary policy parameters following equation (10).

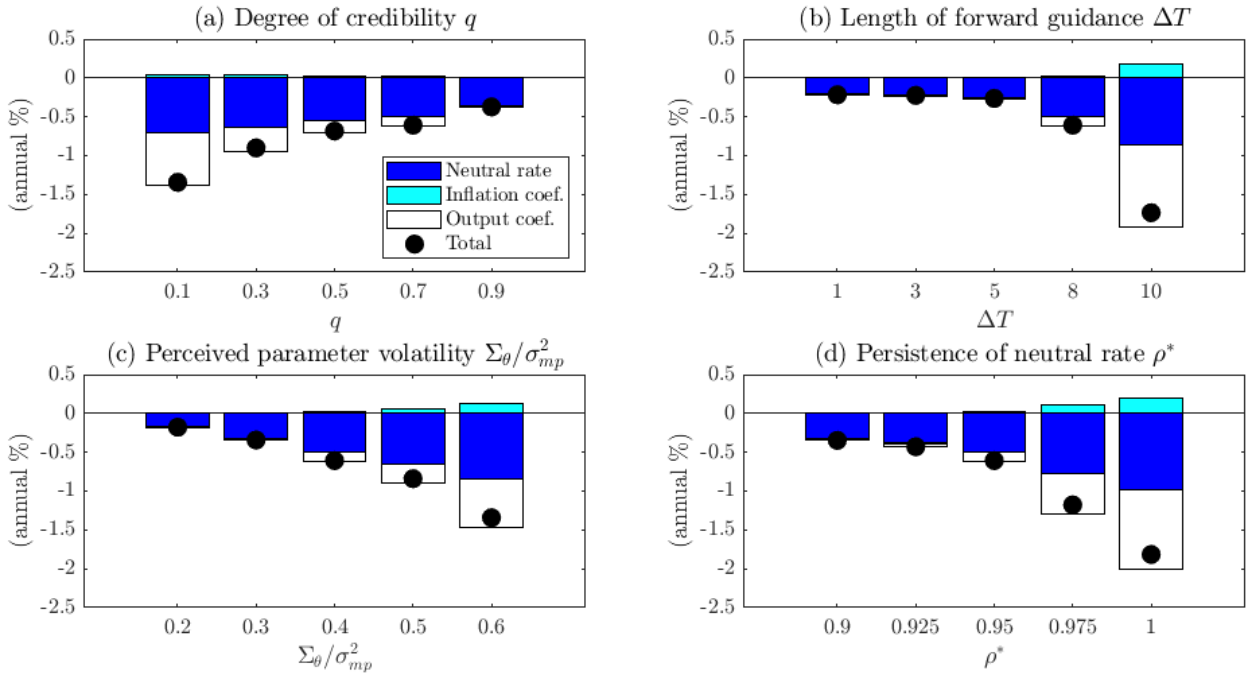
During the initial phase of the low-for-long when the notional rate is in a deep negative territory as shown in the lower left panel of Figure 2, the information gain of observing $\hat{i}_t = \underline{\hat{i}}$ is negligible, i.e., $f_{t|t} - f_{t|t-1}$ is close to zero. For example, in period $t = 2$ when the notional rate is actually the lowest below zero, the decreases in output and inflation are severe and it is obvious that the notional rate would be a deep negative territory and observing the interest rate at the ELB does not add new information. Thus, there is no difference between the posterior and prior and no difference between their means as shown in the left panel of

Figure 3. Therefore, during the downturn when output and inflation decrease substantially so that the notional rate is far below the ELB, there is essentially no new information from observing the binding ELB constraint since it is obvious, and there is essentially no change in the perceived neutral nominal rate as shown in the lower middle panel of Figure 2.

As the notional rate recovers and becomes close to the ELB, however, observing the interest rate at the ELB brings meaningful information about the notional rate. For example, in period $t = 17$, the agents' prior suggests that the perceived mean of the notional rate is still negative but close to the ELB of zero percent, as shown in the right panel of Figure 3. The prior distribution implies that the notional rate could have been positive. In this situation, upon observing the interest rate at the ELB, the agents perceive that the notional rate would be lower below the ELB and use this information to shift their belief about the systematic part of the notional rate leftward, as shown in the right panel of Figure 3. The panel shows $f_{t|t} - f_{t|t-1} < 0$ and through equation (10) this leads to the downward update of the neutral nominal rate, as shown in the lower middle panel of Figure 2.

During the period of forward guidance, the agents understand that the announced forward guidance may be implemented but they do not know whether it is actually implemented due to imperfect credibility. After observing the interest rate at the ELB, the agents form a posterior probability of forward guidance and update their belief about f_t. On the one hand, with the posterior probability of forward guidance, the agents believe that the notional rate is positive and the interest rate is kept at the ELB due to forward guidance. In this case, the information of the interest rate at the ELB shifts their posterior $f_{t|t}$ rightward, as in the opposite image of the right panel of Figure 3. On the other hand, with the posterior probability of no forward guidance, upon observing the interest rate at the ELB, the agents believe that the notional rate is negative because that's why they observe the ELB without forward guidance. Since their prior mean of the notional rate is in the positive area during the period of forward guidance, as shown in the lower left panel of Figure 2, observing the ELB shifts their posterior $f_{t|t}$ leftward. These two opposing forces cancel each other in the beginning of forward guidance, but as the perceived notional rate increases away from the ELB, the second channel of downward updating of their posterior belief $f_{t|t}$ starts to dominate. And this leads to the stronger downward revision of their perceived neutral nominal rate in the second half of the period of forward guidance, as shown in the lower middle panel of Figure 2.

Figure 4: Deviations of the perceived systematic part of the notional rate



Note: 'Neutral rate', 'Inflation coef.', and 'Output coef.' represent the contributions of the perceived neutral rate, inflation coefficient, and output coefficient, respectively, to the total deviation of the perceived systematic part of the notional rate from its true value one period before the lift-off of forward guidance.

3.5 Sensitivity

The simulation results shown above are based on the baseline scenario and agents' learning, which hinge on various assumptions. In this subsection, we study the sensitivity of the belief gaps – the deviations of the perceived systematic part of the notional nominal rate from its true value – to the degree of credibility, the length of forward guidance, and speed of learning. We focus on the belief gap one period before the lift-off of forward guidance since the gap becomes the largest and this is the culprit of a negative surprise during the exit as shown in Figure 2.

Degree of credibility Panel (a) of Figure 4 presents the belief gaps for various degrees of credibility about forward guidance, from $q = 0.1$ to 0.9, and the decomposition of the gaps into three components: the perceived neutral rate, inflation coefficient, and output coefficient, one period before the lift-off of forward guidance. The case of $q = 0.7$ corresponds to the baseline scenario simulation shown in Figure 2. Three observations can be made from Panel (a). First, the less forward guidance becomes credible, the more the deviation of

the perceived neutral rate widens downward. Although the central bank keeps the interest rate at the ELB following its forward guidance, if agents believe that forward guidance is unlikely due to low credibility, then the agents attribute the interest rate at the ELB to a low notional rate, resulting in a greater downward belief gap arising from a downward deviation of the perceived neutral rate. Second, the less forward guidance becomes credible, the more the perceived coefficient on output declines, resulting in its greater contribution to the downward belief gap between the perceived systematic part of the notional rate and its true value. When q is low, the central bank's decision to keep the interest rate at the ELB is perceived as a positive shock and stimulates output in particular. The agents attribute the interest rate at the ELB to a low notional rate in spite of positive output gaps. This leads to a lower negative value of the coefficient on output and thus a large contribution to the total deviation of the perceived systematic part of the notional rate. Third, as a result of the first two observations, the belief gap falls further in a negative territory as credibility about forward guidance declines.

Length of forward guidance Panel (b) of Figure 4 shows the belief gap and its decomposition similarly, but here in terms of the length of forward guidance from $\Delta T = 1$ to $\Delta T = 10$. The case of $\Delta T = 8$ corresponds to the baseline scenario simulation shown in Figure 2. The right panel shows that the downward deviation of the perceived neutral rate becomes greater as the length of forward guidance becomes longer. This result can be expected from the lower middle panel of Figure 2, which shows that the downward deviation of the perceived neutral rate becomes greater non-linearly as the forward guidance is continued. The contribution of the perceived coefficient on output is sensitive to and non-linear in the length of forward guidance. When it is changed from $\Delta T = 8$ to 10, the contribution of the perceived coefficient on output becomes dominant. This is due to the non-linear effect of prolonged forward guidance on output: as output is stimulated further, to reconcile it with a possibility that the notional rate may still be negative given the interest rate at the ELB, the agents update their belief such that the output coefficient is negative, leading to the downward deviation of the output coefficient component of the perceived systematic part of the notional rate.

Speed of learning Panel (c) of Figure 4 shows the sensitivity of the belief gap one period before the lift-off of forward guidance to perceived relative volatility of policy parameters,

$\Sigma_\theta/\sigma_{mp}^2$, which govern the speed of learning. The case of the identical diagonal entries of $\Sigma_\theta/\sigma_{mp}^2$ of 0.4 corresponds to the baseline scenario result, respectively. With σ_{mp} fixed, an increase in the perceived relative volatility $\Sigma_\theta/\sigma_{mp}^2$ makes a change in perceived monetary policy parameter values more volatile and thus widens the belief gap downward, as shown in Panel (c). A widening in the downward belief gap implies that agents come to perceive that the interest rate will be kept low, which stimulates output. As in the cases of Panels (a) and (b), the positive output gaps during the ELB period make agents perceive a lower negative value of the coefficient on output, leading to a large negative contribution in the case of the identical diagonal entries of $\Sigma_\theta/\sigma_{mp}^2$ of 0.6 (corresponding to the standard deviations of the innovations of parameters of interest being 8 basis points). The perceived neutral rate continues to be dominant and its downward deviation becomes greater as the relative volatility increases. With Σ_θ fixed, the result of a change in the volatility of monetary policy shocks would mirror the result shown in Panel (c). An increase in σ_{mp} makes agents perceive that structural monetary policy shocks are dominant in a change in the notional rate and thus the agents attribute a change in the notional rate less to a change in the systematic component of the notional rate.

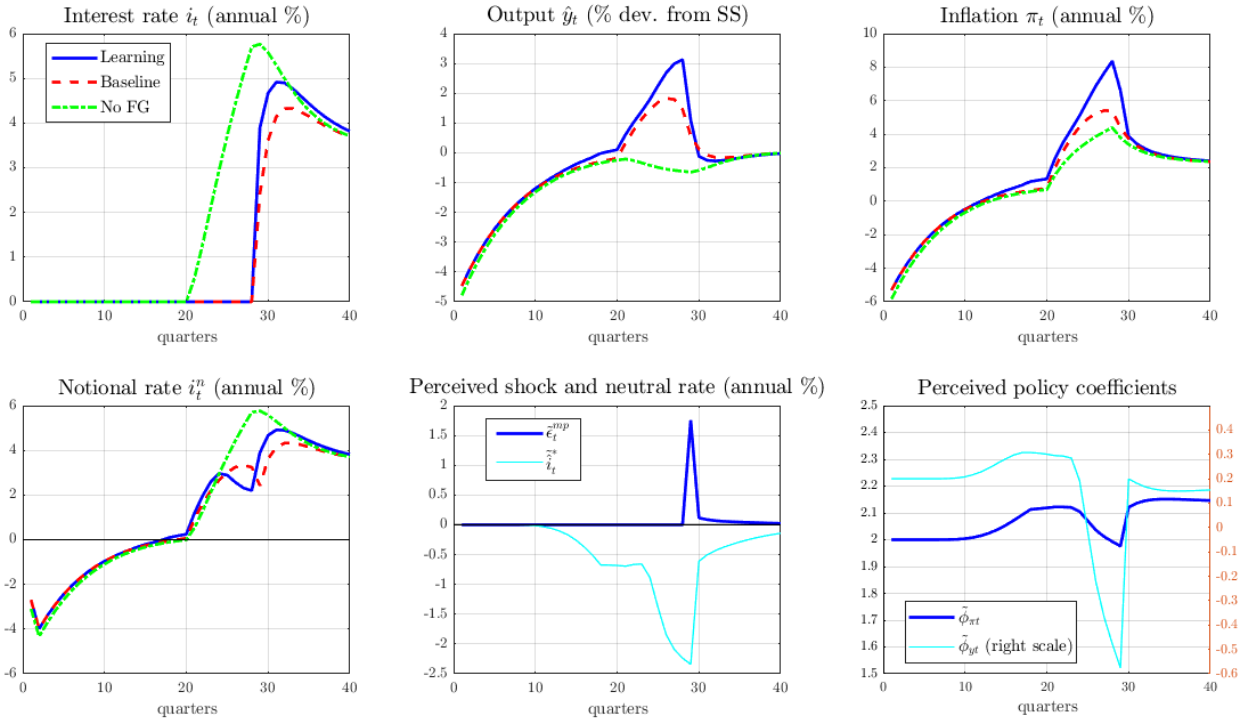
Persistence of the perceived neutral rate The persistence of the perceived neutral nominal interest rate ρ^* in equation (5) also affects the belief gap. The higher the persistence, the less quickly the perceived neutral rate returns to the steady state. Panel (d) of Figure 4 shows that the downward belief gap widens, driven by a large contribution by the downward deviation of the perceived neutral rate, as the persistence parameter increases. A lower perceived neutral rate stimulates the economy more, and for the same reason as in the case of a longer period of forward guidance presented in Panel (c), a contribution of the perceived coefficient on output becomes greater as the persistence increases.

3.6 Alternative scenarios

In addition to the baseline scenario, we consider three alternative scenarios, an inflationary shock scenario, a belief accommodation scenario, and a recurrent shock scenario.

Inflationary shocks In light of the recent inflation surge of 2021–22, especially when the interest rate was still kept at the ELB in major jurisdictions, we consider an alternative scenario that adds to the baseline scenario inflationary supply and demand shocks. These

Figure 5: With additional supply and demand shocks



Note: Unexpected supply and demand shocks are added from $t = 21$ to 28. 'Learning' and 'Baseline' use the model with and without learning, respectively, and 'No FG' uses the model without forward guidance and learning. The notional rate in 'Learning' is the perceived notional rate during the period of the binding ELB constraint.

shocks hit the economy unexpectedly during the period of forward guidance of keeping the interest rate at the ELB. For simplicity we set innovations to z_t^d and z_t^s of 0.05 percent for $t = 21, ..., 28$, where the AR(1) coefficients of the supply and demand shocks are set at 0.75. The positive demand shock increases output and inflation, while the positive supply shock decreases output but increases inflation. The combination of the two shocks is inflationary while having relatively muted effect on output.[13]

Figure 5 plots the results of the alternative scenario of additional shocks for the model with learning ('Learning'), the model without learning ('Baseline'), and the model without learning and forward guidance ('No FG'). In the case of no forward guidance, the zero interest rate is lifted in $t = 21$, and inflation surges above 4 percent in response to the series of the combination of supply and demand shocks, while the output hovers in the area just below

[13]Our model is so stylized that it has to resort to repeated supply and demand shocks to reproduce a sharp and persistent increase in inflation and a moderate effect on output. See, Erceg et al. (2024) for an extended NK model that can explain such a phenomenon with supply shocks only.

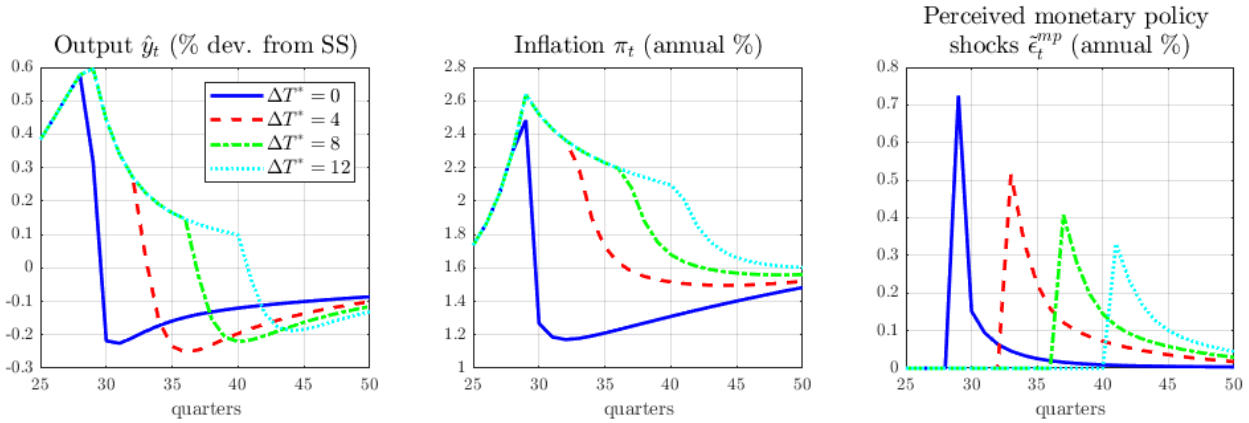
its steady state level. In the baseline case, the interest rate is kept at the ELB for two years after $t = 21$. Keeping the interest rate at the ELB in spite of inflationary supply and demand shocks boosts the economy and generates a surge in inflation and output. In particular, inflation reaches nearly 6 percent as shown in the upper right panel of Figure 5. This captures the forward guidance trap coined by Orphanides (2024).

Figure 5 shows that the presence of learning could exacerbate the forward guidance trap. Although the shocks that hits the economy are the same, the surge in inflation is more massive than in the baseline case. Observing the zero interest rate during a surge in inflation, the agents update their belief in a way that lowers the perceived nominal neutral rate as shown in the lower middle panel. With this belief, the agents expect a lower path of the nominal interest rate, which pushes up inflation further. When exiting the low-for-long policy, the agents' belief about monetary policy is not realized and the gap between their belief and the actual monetary policy is perceived as considerable monetary tightening. In sum, a boom and bust arising from a correction of a gap between agents' belief and the actual policy could become greater if inflationary supply and demand shocks hit the economy during the period of forward guidance.

Accommodating beliefs In the baseline scenario with learning, agents' belief is corrected sharply after the lift-off of forward guidance. In this situation, the central bank may want to consider accommodating the belief by offsetting perceived monetary tightening with the actual monetary easing to smooth the landing after the lift-off. To articulate this scenario, suppose that the central bank sets an unexpected monetary policy shock ϵ_t^{mp} such that the perceived monetary policy shock $\tilde{\epsilon}_t^{mp}$ becomes zero in accordance with equation (8) for ΔT^* periods after the lift-off of forward guidance. The case of $\Delta T^* = 0$ corresponds to the baseline scenario with learning presented in Figure 2.

Figure 6 plots simulation results from $t = 25$ under the alternative scenario of accommodating beliefs for $\Delta T^* = 0, 4, 8, 12$. The forward guidance is lifted in $t = 29$ after seven years since the initial shock hit in $t = 1$. In the case of no accommodation, output and inflation fall sharply as the belief gap widened under the low-for-long policy is corrected at the time of the lift-off. In the case of accommodations, a fall in output and inflation becomes more gradual and smooth as the length of belief accommodations prolongs from $\Delta T^* = 4$ to 12. Moreover, the perceived monetary tightening that comes after the end of belief accommodations becomes smaller as the central bank accommodates beliefs longer, as shown in the right

Figure 6: Accommodating beliefs after the lift-off of forward guidance



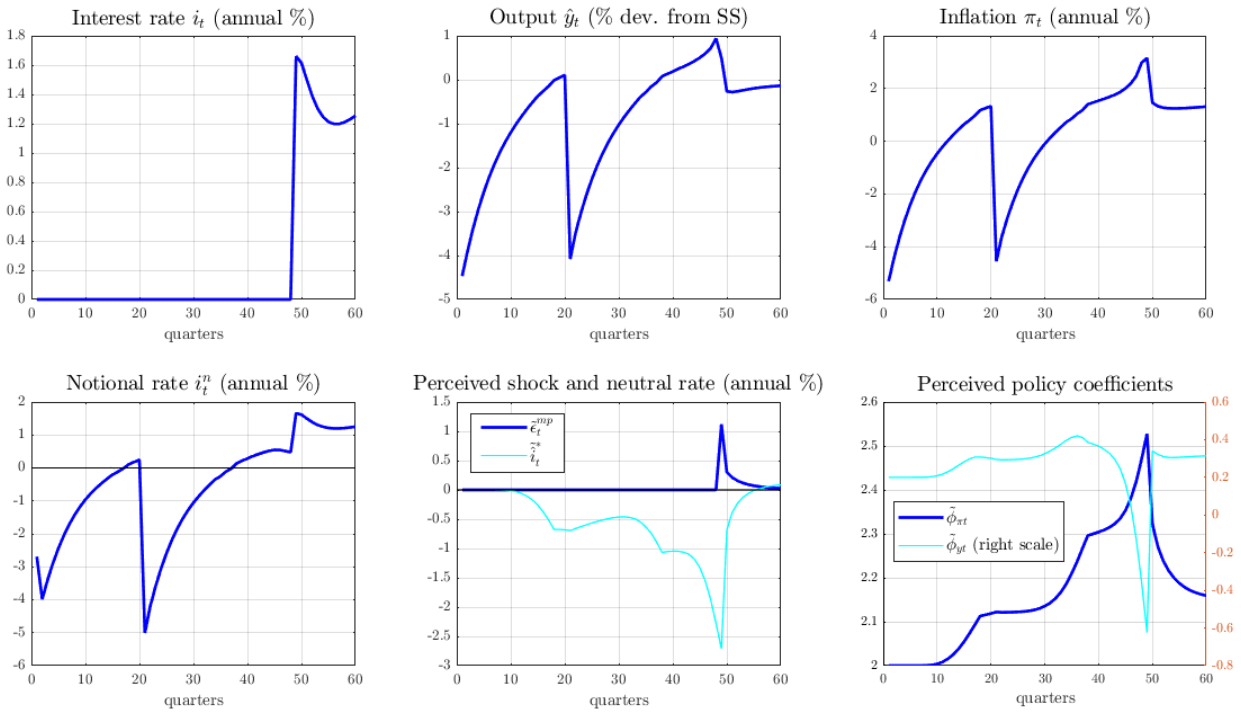
Note: ΔT^* denotes the number of periods in which agents' belief is accommodated by the central bank.

panel of Figure 6. This is driven by the assumption of $\rho^* = 0.95$, which implies that agents expect the neutral nominal rate will converge to its steady state level. For example, with $\Delta T^* = 12$, the perceived neutral nominal rate becomes about a half ($= 0.95^{12}$) of its original level in the beginning of the life-off. Instead, if the agents believe that the neutral nominal rate is more persistent, there would be less decline in the perceived monetary tightening and less smooth landing after the lift-off of forward guidance.

Recurrent shocks The Japanese economy went through a low-for-long environment from the late 1990s to the early 2020s due possibly to a series of severe negative shocks. In light of the experience, we consider a recurrent shock scenario in which a severe negative demand shock $\epsilon_t^d = -1.05$ hits in $t = 21$, which would induce additional twenty periods of the binding ELB constraint without forward guidance. In response, as in the baseline scenario, we assume that the central bank introduces forward guidance of keeping the interest rate at the ELB by additional eight periods. Except for the shock that hits in $t = 21$, the assumptions are the same as in the baseline scenario.

Figure 7 shows the simulated paths of the recurrent shock scenario. Output and inflation fall sharply and reach a second trough in $t = 21$ as the economy is hit by a second shock. This, combined with forward guidance, generates a prolonged period of the zero interest rate until $t = 48$ – twelve years since the initial shock hit. During this period, observing output, inflation, and the interest rate (at the ELB), agents continue updating their belief about monetary policy parameters. Specifically, after $t = 21$, the perceived neutral rate

Figure 7: Recurrent shock scenario



Note: A severe demand shock hits in $t = 21$ besides the shock in $t = 1$ assumed in the baseline scenario.

does not move very much around $t = 35$ as the notional rate is in a deep negative territory reflecting low levels of output and inflation. The perceived neutral rate starts to fall as the notional rate becomes close to zero. One period before the lift-off of the zero interest rate policy, the perceived neutral rate deviates downward by more than 2.5 pp, and this causes considerable monetary tightening perceived by the agents at the time of the lift-off. All in all, additional five years of the zero interest rate period to the baseline scenario exacerbates a boom and bust around the lift-off reflecting a correction of the belief gap accumulated during low-for-long, but it does not change the main features of the baseline scenario significantly.

4 Empirical Analysis

In the previous section we have demonstrated that low-for-long policy induces agents to believe lower-for-longer interest rates through learning by decreasing their perceived neutral nominal rate term in the monetary policy rule within a NK framework. In this section, we provide empirical evidence for the decline of the perceived neutral rate in Japan during the low interest rate environment and show that the perceived neutral rate responds to a

monetary policy shock consistent with our model with learning.

4.1 Estimating perceived neutral rates

Specifications Although it is difficult to directly measure perceptions, expectations can be extracted from market indicators or surveys. However, during the low interest rate environment from the late 1990s to the early 2020s in Japan, the short-term interest rate hovered around zero, and expected market rates, calculated from market indicators such as overnight index swaps, exhibited minimal variations. To address this challenge, we utilize survey data to estimate a monetary policy rule perceived by professional forecasters over a long horizon as in Bauer et al. (2024).

To be consistent with the model presented in Section 2, consider the monetary policy rule (4):

$$i_t^n = \rho_i i_{t-1} + (1-\rho_i)\left(i_t^* + \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t\right) + \epsilon_t^{mp}, \tag{11}$$

where the interest rates including the neutral nominal rate term are expressed in level. We assume that professional forecasters form expectations about the short-term interest rate based on equation (11) and the ELB constraint (3). We further assume that the the h-period ahead forecast of the interest rate is above the ELB if h is great enough. Under the assumption, $\tilde{\mathbb{E}}_{jt}(i_{t+h}^n) = \tilde{\mathbb{E}}_{jt}(i_{t+h})$ holds, where $\tilde{\mathbb{E}}_{jt}(\cdot)$ is an expectation operator by professional forecaster j conditional on information in period t. Then, the expected h-period ahead interest rate can be expressed as follows:

$$\begin{aligned}\tilde{\mathbb{E}}_{jt}(i_{t+h}) =& \tilde{\mathbb{E}}_{jt}\left[\rho_i i_{t+h-1} + (1-\rho_i)(i_{t+h}^* + \phi_\pi \hat{\pi}_{t+h} + \phi_y \hat{y}_{t+h})\right] \\ =& \rho_i \tilde{\mathbb{E}}_{jt}(i_{t+h-1}) + (1-\rho_i)\tilde{\mathbb{E}}_{jt}(i_{t+h}^*) + \tilde{\mathbb{E}}_{jt}\left[(1-\rho_i)(\phi_\pi \hat{\pi}_{t+h} + \phi_y \hat{y}_{t+h})\right].\end{aligned} \tag{12}$$

Although the perceived neutral rate term $\tilde{\mathbb{E}}_{jt}(i_{t+h}^*)$ is unobservable, it can be estimated if other variables—namely, $\tilde{\mathbb{E}}_{jt}(i_{t+h})$, $\tilde{\mathbb{E}}_{jt}(\hat{\pi}_{t+h})$, and $\tilde{\mathbb{E}}_{jt}(\hat{y}_{t+h})$—are observable at time t. However, the neutral rate term is inherently difficult to estimate due to its time-varying nature. Some empirical studies address this challenge by imposing statistical assumptions on the dynamics of i_t^*. In contrast, we estimate it taking advantage of cross-sectional variations as well as time variations in survey data from professional forecasters, specifically those reported in Consensus Forecasts (CF).

From equation (12) our baseline specification for the expected short-term interest rate

by forecaster j is given by:

$$\tilde{\mathbb{E}}_{jt}(i_{t+h}) = c + \rho_i \tilde{\mathbb{E}}_{jt}(i_{t+h-1}) + \Phi_\pi \tilde{\mathbb{E}}_{jt}(\hat{\pi}_{t+h}) + \Phi_y \tilde{\mathbb{E}}_{jt}(\hat{y}_{t+h}) + \epsilon_{jt}, \tag{13}$$

where ϵ_{jt} is an i.i.d. error term across j and over t. Then, we can infer the perceived rate term as $\widehat{\tilde{\mathbb{E}}_{jt}(i^*_{t+h})} = \widehat{c}/(1-\widehat{\rho_i})$, where a variable with $\widehat{\cdot}$ denotes its corresponding estimated value. We capture time variations in c using rolling regressions.

Data We use Consensus Forecasts (CF), which provides forecasts for GDP growth rates and CPI inflation rates on a calendar-year basis as well as the 3-month interest rates for 3 and 12 months ahead. From these data we construct $\tilde{\mathbb{E}}_{jt}(i_{t+h})$, $\tilde{\mathbb{E}}_{jt}(i_{t+h-1})$, $\tilde{\mathbb{E}}_{jt}(\hat{\pi}_{t+h})$, and $\tilde{\mathbb{E}}_{jt}(\hat{y}_{t+h})$ to estimate equation (13). The number of leads for the interest rate is set at $h = 12$ months. For $\tilde{\mathbb{E}}_{jt}(i_{t+h-1})$, we use forecasts for the interest rate after three months. The forecasting horizon for the inflation rate and GDP growth rate is set at the following year. For example, if forecasts are made at January 2024, then forecasts for 2025 are used. Regarding the target rate of inflation, it is set at 1 percent before January 2013 and at 2 percent thereafter.[14] We construct forecasts made in t for the output gap in the target year T_t as follows.

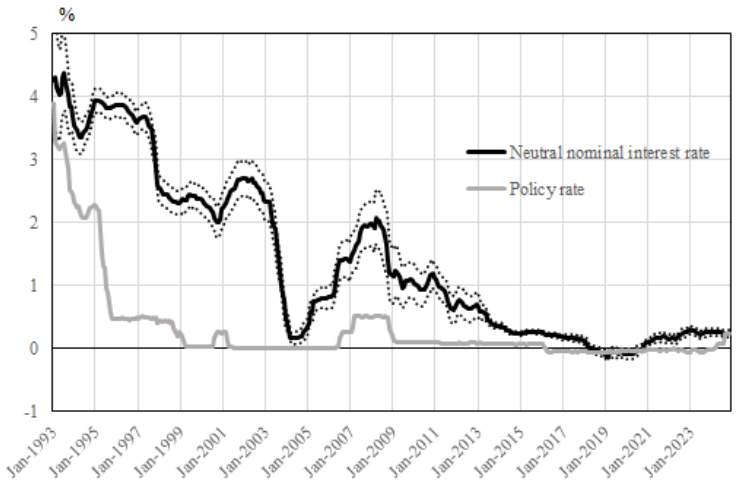
$$\tilde{\mathbb{E}}_{jt}(\hat{y}_{T_t}) = (1 + \hat{y}_{T_t-1}) \cdot \frac{1 + \tilde{\mathbb{E}}_{jt}(g_{T_t})}{1 + g^*_{T_{t-1}}} - 1, \tag{14}$$

where $\tilde{\mathbb{E}}_{jt}(g_{T_t})$ denotes forecasts for the GDP growth rate for year T_t by forecaster j. For the lagged output gap $\hat{y}_{T_{t-1}}$ and the lagged potential growth rate $g^*_{T_{t-1}}$, we use those estimated by the Bank of Japan. Underlying assumption in (14) is that forecasters expect the potential growth rate to remain unchanged from year T_{t-1} to year T_t.

Let m_t denote the month when the forecasts were made and let $\mathbb{I}(m_t = m)$ denote an indicator function that takes a value of one if the month is $m \in \{1, ..., 12\}$. With the

[14] The Bank of Japan set the price stability target at 2 percent in terms of the year-on-year rate of changes in the consumer price index in January 2013 as stated in "Joint Statement of the Government and the Bank of Japan on Overcoming Deflation and Achieving Sustainable Economic Growth" (https://www.boj.or.jp/en/mopo/mpmdeci/mpr_2013/k130122c.pdf). Prior to January 2013, there was no explicit target, but the Bank aimed at 1 percent for the time being as of February 2012 as stated in "The Price Stability Goal in the Medium o Long Term" (https://www.boj.or.jp/en/mopo/mpmdeci/mpr_2012/k120214b.pdf).

Figure 8: Estimated neutral nominal interest rates



Note: The figure shows the estimated (expected) neutral nominal interest rate calculated as $\frac{\widehat{c}}{1-\rho}$ by assuming $\rho = 0.85$, based on equation (15). The solid and dotted black lines indicate a point estimate and 90% confidence intervals, respectively. The gray line indicates the short-term policy rate.

constructed data on hand, we estimate a modified version of equation (13), given by

$$\tilde{\mathbb{E}}_{jt}(i_{t+12}) = c + \rho_i \tilde{\mathbb{E}}_{jt} i_{t+3} + \sum_{m=1}^{12} \mathbb{I}(m_t = m) \cdot \Phi_{\pi,m} \tilde{\mathbb{E}}_{jt}(\hat{\pi}_{T_t}) + \sum_{m=1}^{12} \mathbb{I}(m_t = m) \cdot \Phi_{y,m} \tilde{\mathbb{E}}_{jt}(\hat{y}_{T_t}) + \epsilon_{jt}, \tag{15}$$

This regression form implies that if the survey is conducted in January 2024 (i.e., $t =$ Jan. 2024), for example, then $m = 1$, and only $\Phi_{\pi,1}$ is active as the coefficient on the inflation rate in regressing $\tilde{\mathbb{E}}_{jt}(i_{t+12})$. The variation in the coefficients on the inflation rate and output gap across months within a year helps account for differences in forecast horizons.

Estimation results We estimate the empirical model specified in equation (15) using a pooled ordinary least squares (OLS) regression with a rolling window of 36 months, spanning from January 1990 to November 2024. The coefficient on the lagged short-term rate, ρ_i, is assumed to be 0.85 and constant over time.[15]

Figure 8 shows the estimated neutral nominal interest rate term in the perceived monetary policy rule. The estimated neutral rate term gradually declines in the late 1990s and falls sharply in the early 2000s. The timing of the decline coincides with when the BOJ

[15]The full-sample estimation yields an estimate of ρ_i equal to 0.88.

implemented a zero-interest rate policy and introduced forward guidance. The estimated neutral rate term bottoms out in 2004, however, before the BOJ lifted off the policy rate. It then rebounds to around 2%, peaking in 2008. After the GFC, the estimated neutral rate term continues to decline and reaches zero percent against the background of the prolonged period of low interest rates. This gradual decline aligns with the scenario simulation results presented in Section 3: as the low interest rate environment is prolonged, agents update their beliefs in a way to expect interest rates lower for longer, driven by a decline in the perceived neutral rate term.

Figure 8 shows that the estimated neutral rate term starts to rise but gradually after 2021. Notably, in November 2024, the estimated neutral rate term remains below 0.5 percent when the actual policy rate was 0.25 percent, whereas at the end of 2006, it is around 1.5 percent with the same policy rate of 0.25 percent. This contrast suggests that the perceived neutral rate term remains low and there may be room for corrections going forward.

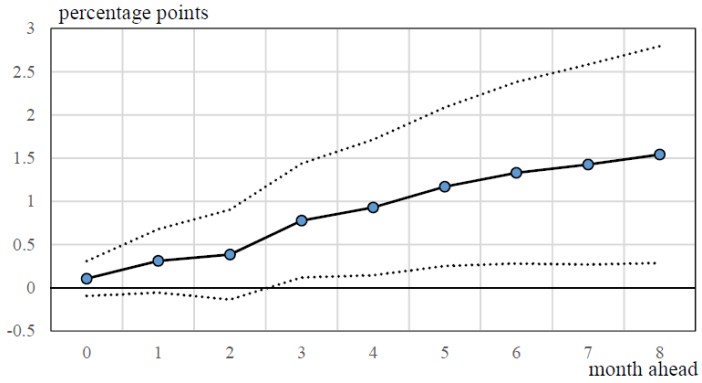
4.2 Revision of the perceived neutral rate term

Specifications The empirical results in Section 4.1 show a gradual decline in the perceived neutral rate term, but it is silent about a mechanism underlying a secular decline in the perceived neutral rate term in the past quarter century. We investigate a possible mechanism highlighted by the theoretical model presented in Section 2. As equation (8) illustrates, the observed monetary policy shock $\tilde{\epsilon}_t^{mp}$ consists of two components: a random deviation from the rule and a revision of the perceived policy rule. Therefore, if all monetary policy shocks consist solely of random deviations, these shocks do not contribute to revisions in the policy rule parameters. Conversely, if these shocks explain changes in the perceived policy rule, particularly the neutral rate term, it suggests that forecasters engage in a learning process regarding policy parameters, as postulated by our model. In other words, under learning the perceived neutral rate term is revised when the new information on the monetary policy is released.

To test this hypothesis, we investigate the relationship between monetary policy shocks and the perceived neutral rate term by extending the empirical analysis by Bauer et al. (2024). For notional simplicity, let $\tilde{i}_t^*$ denote the perceived neutral rate term in period t, estimated in Section 4.1.[16] We estimate a local projection for k-period ahead perceived

[16]If we follow the notations used in the specification (12), $\tilde{i}_t^*$ corresponds to $\tilde{\mathbb{E}}_{jt}(i_{t+12}^*)$, the perceived neutral rate term 12-month ahead conditional on information in period t.

Figure 9: Estimated impact of monetary policy shocks on the neutral nominal interest rate



Note: The figure illustrates the estimated impact of a one-percentage-point increase in the one-year Japanese government bond yield. The x-axis represents the horizon in the local projection analysis. The solid line with a dot denotes the point estimate, while the dotted line indicates the 90% confidence intervals.

neutral rate term as follows:

$$\tilde{i}^*_{t+k} = \alpha_k + \beta_k \tilde{i}^*_{t-1} + \gamma_k MP_t + e_{t+k} \tag{16}$$

where MP_t indicates a monetary policy surprise. Following Swanson (2021) and Swanson (2023), we use changes in the one-year Japanese government bond yield before and after days of monetary policy meetings and the governor's speeches and testimony in Congress as a monetary policy surprise. To focus on the low-interest rate environment, we use the sample from January 2002 to November 2024.

Estimation results Figure 9 presents the estimation results. In response to monetary policy tightening (an unexpected increase in the JGB yield), the perceived neutral rate term increases with some lags. The effect becomes statistically significant after three months. This result supports a view that agents sort out monetary policy shocks and revise their perceived monetary policy rule through learning, as in our theoretical model.

To summarize, the empirical analysis supports the idea that the neutral rate term in the perceived monetary policy rule can decline as agents learn the policy rule from new information on a central bank's policy stance. The estimated neutral rate term is still at low level in the end of the sample period of November 2024. Thus, there may be room for possible corrections of belief gaps, if any. The local projection results shown in Figure 9 suggests that belief gaps can be a source of perceived monetary policy tightening as the perceived

neutral rate term in the monetary policy rule is corrected in line with the theoretical model. Moreover, such monetary policy tightening would have negative effects on the macroeconomic variables as well as financial variables, as empirically studied by e.g. Kubota and Shintani (2022, 2024), Ikeda et al. (2024), and Nakamura et al. (2024) for Japan.

5 Conclusion

This paper studies a new channel of low-for-long interest rates by building a NK model with learning, the ELB, and forward guidance with imperfect credibility. It highlights a possible challenge facing monetary policy in normalizing from low-for-long interest rates. During a low-inflation regime, low-for-long policy supports the economy by inducing agents to believe that interest rates will be kept lower and longer. When the time comes and the interest rate is lifted up, the correction of the belief has contractionary effects on the economy. While low-for-long policy is expansionary during a low-inflation regime, unwinding the policy toward a normal regime precipitates surprise monetary tightening through the correction of the belief that has grown under low-for-long policy. We study mechanisms behind this potential channel of monetary policy and how the belief gap evolves during the ELB period and during the subsequent exit under various assumptions and scenarios. We also provide some empirical evidence that supports a practical relevance of this channel.

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Appendix

A The Model

To make the paper self-contained we derive the log-linearized equations presented in the main text from the original equilibrium conditions. We first present the standard part of the model in Appendix A.1, discuss the source of additional discounting in Appendix A.2, and study the basic properties of the model in Appendix A.3.

A.1 The standard model

Households A representative household maximizes the utility

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t d_t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} \right],$$

subject to the flow budget constraint

$$P_t c_t + B_t/(1+i_t) \leq W_t \ell_t + B_{t-1} + \Theta_t,$$

where c_t is consumption, ℓ_t is labor supply, P_t is the price level, B_t is the amount of nominal bonds in face value with the gross interest rate R_t, W_t is the nominal wage, and Θ_t consists of cash flows from firm profits. The variable d_t captures a composite shock to preferences, given by

$$d_t = \begin{cases} \exp(z_1^{*d} + z_2^{*d} + ... + z_t^{*d}) & \text{for } t = 1, 2, ... \\ 1 & \text{for } t = 0 \end{cases}$$

where z_t^{*d} is a preference (or demand) shock, which follows an AR(1) process

$$z_t^{*d} = \rho_d z_{t-1}^{*d} + \epsilon_t^{*d}.$$

Let Λ_t denote a Lagrange multiplier on the budget constraint in period t. The first-order conditions with respect to C_t, B_t, and ℓ_t are given, respectively, by

$$\begin{aligned} P_t \Lambda_t &= c_t^{-\sigma}, \\ d_t \Lambda_t &= \beta \mathbb{E}_t d_{t+1} \Lambda_{t+1} (1+i_t), \\ \psi \ell_t^{\nu} &= \Lambda_t W_t. \end{aligned}$$

Define $\pi_t = P_t/P_{t-1}$ as the gross inflation rate. By substituting out Λ_t, the equations above can be written as the Euler equation and the labor supply curve, respectively:

$$c_t^{-\sigma} = \beta \mathbb{E}_t e^{z_{t+1}^d} c_{t+1}^{-\sigma} (1+i_t)/\pi_{t+1}, \tag{A.1}$$

$$w_t = \psi c_t^{\sigma} \ell_t^{\nu}. \tag{A.2}$$

Final good firms A representative final good firm produces y_t by combining intermediate goods $\{y_t(i)\}_{i=0}^1$ according to

$$y_t = \left[\int_0^1 y_t(i)^{\frac{1}{\lambda_{p,t}}} di \right]^{\lambda_{p,t}}. \tag{A.3}$$

Let $P_t(i)$ denote the price of the i-th intermediate good. The firm maximizes profits $P_t y_t - \int_0^1 P_t(i) y_t(i) di$, taking prices as given. The solution yields demand for the i-th intermediate good, given by

$$y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\frac{\lambda_{p,t}}{\lambda_{p,t-1}}} y_t. \tag{A.4}$$

The final good is used for consumption only, so that the good market clearing condition is given by $y_t = c_t$.

Intermediate goods firms There is a continuum of intermediate goods firms, each indexed by i. Each firm uses labor to produce an intermediate good according to

$$y_t(i) = l_t(i),$$

In every period, intermediate goods firms have an opportunity to reset their prices with probability $1 - \xi_p$. Consider the i-th intermediate good firm that has an opportunity to reset its price at $\tilde{P}_t$ in period t. It maximizes the discounted sum of profits,

$$\max_{\{\tilde{P}_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \xi_p)^s \Lambda_{t,t+s} \left[P_{t+s}(i) y_{t+s}(i) - W_{t+s} y_{t+s}(i) \right],$$

subject to the demand curve (A.4), $P_{t+s}(i) = \pi P_{t+s-1}(i)$, and $P_t(i) = \tilde{P}_t$. Substituting equation (A.4) into the problem yields

$$\max_{\{\tilde{P}_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \xi_p)^s \Lambda_{t,t+s} \left(\pi^s \tilde{P}_t - W_{t+s} \right) \left(\frac{\pi^s \tilde{P}_t}{P_{t+s}} \right)^{-\frac{\lambda_{p,t+s}}{\lambda_{p,t+s-1}}} y_t,$$

The first-order condition is

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta \xi_p)^s \frac{P_{t+s} \Lambda_{t,t+s}}{\lambda_{p,t+s} - 1} \left[-\tilde{\Pi}^p_{t,t+s} \tilde{p}_t + \lambda_{p,t+s} w_{t+s} \right] \left(\tilde{\Pi}^p_{t,t+s} \tilde{p}_t \right)^{-\frac{\lambda_{p,t+s}}{\lambda_{p,t+s-1}}} y_{t+s}, \tag{A.5}$$

where $\tilde{p}_t = \tilde{P}_t / P_t$ and $\tilde{\Pi}^p_{t,t+s} = \pi^s / (P_{t+s}/P_t)$.

The price index is given by $P_t = \left[\int_0^1 P_t(i)^{1/(1-\lambda_{p,t})} di \right]^{1-\lambda_{p,t}}$ and it evolves following

$$P_t = \left\{ \xi_p \left(\pi P_{t-1} \right)^{\frac{1}{\lambda_{p,t-1}}} + (1 - \xi_p) \tilde{P}_t^{\frac{1}{\lambda_{p,t-1}}} \right\}^{1-\lambda_{p,t}}$$

This equation can be written as

$$1 = \xi_p \left(\pi / \pi_t \right)^{\frac{1}{\lambda_{p,t-1}}} + (1 - \xi_p) \tilde{p}_t^{\frac{1}{\lambda_{p,t-1}}} \tag{A.6}$$

Government The government consists of a fiscal authority and a central bank. The fiscal authority issues nominal government bonds with zero net supply. The central bank sets the nominal interest rate i_t following a monetary policy rule with the effective lower bound (ELB) constraint:

$$i_t = \max\{i_t^n, \underline{i}\}, \tag{A.7}$$

$$\log\left(\frac{1 + i_t^n}{1 + i} \right) = \rho_r \log\left(\frac{1 + i_{t-1}}{1 + i} \right) + (1 - \rho_i) \left[i_t^* + \phi_\pi \log\left(\frac{\pi_t}{\pi} \right) + \phi_y \log\left(\frac{y_t}{y} \right) \right] + \epsilon_t^{mp}, \tag{A.8}$$

where i_t^n is the notional net nominal interest rate and $\underline{i}$ is the ELB.

Log-linearization Since $c_t = y_t$, the Euler equation (A.1) can be log-linearized as

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \frac{1}{\sigma}\left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}\right) - \frac{\rho_d}{\sigma} z_t^{*d}.$$

This equation is the same as equation (1) with $M = 1$ in the main text if we set $z_t^d = -(\rho_d/\sigma) z_t^{*d}$. Log-linearizing equation (A.5) around steady state yields

$$0 = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\xi_p)^s \left(-\hat{\tilde{\Pi}}_{t,t+s}^p - \hat{\tilde{p}}_t + \hat{w}_{t+s} + \hat{\lambda}_{p,t+s}\right)$$

Solving for $\hat{\tilde{p}}_t$ yields

$$\frac{1}{1-\beta\xi_p}\hat{\tilde{p}}_t = \hat{w}_t + \hat{\lambda}_{p,t} + \frac{\beta\xi_p}{1-\beta\xi_p}\mathbb{E}_t\left(\hat{\tilde{p}}_{t+1} + \hat{\pi}_{t+1}\right)$$

Log-linearizing equation (A.6) yields

$$0 = -\xi_p \hat{\pi}_t + (1-\xi_p)\hat{\tilde{p}}_t.$$

Combining the equations above, we obtain

$$\hat{\pi}_t = \kappa_p \hat{w}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa_p \hat{\lambda}_{p,t}.$$

where $\kappa_p = (1-\xi_p)(1-\beta\xi_p)/\xi_p$. Note that $y_t = \ell_t = c_t$. Log-linearzing equation (A.2) yields

$$\hat{w}_t = (\sigma + \nu)\hat{y}_t.$$

Combining the two equations above yields equation (2) in the main text if we set $\kappa = \kappa_p(\sigma + \nu)$ and $z_t^s = \kappa_p \hat{\lambda}_{p,t}$. It is straightforward to see that monetary policy rules (3) and (4) in the main text can be derived from equations (A.7) and (A.8).

A.2 Additional discounting

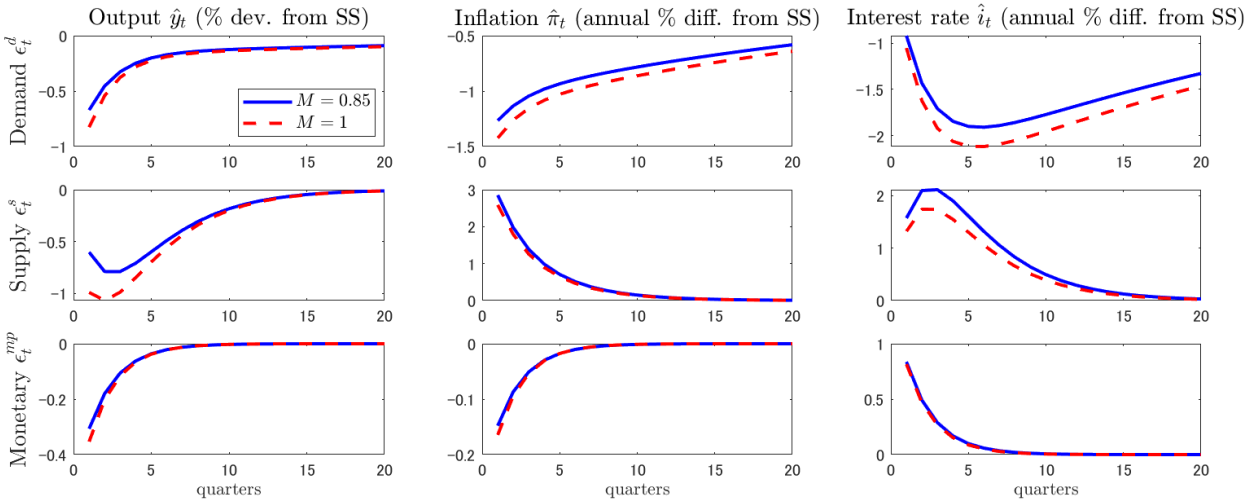
The additional discounting $M < 1$ attached to the log-linearized Euler equation (1) in the main text can be derived from two theories. First, Gabaix (2020) consider households' cognitive discounting of the future state variables and M measures attention to the future. Although $\mathbb{E}_t \hat{y}_{t+1}$ is discounted, as in Gabaix (2020) we assume that that the contemporaneous real interest rate, $\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}$, is correctly perceived, that is, the rational Fisher equation holds in the agents' perception of the world.

Second, Bilbiie (2024) studies a tractable HANK model with two types of households – savers and hand-to-mouth – whose type change stochastically and derives the aggregated Euler equation in the form of equation (1). He shows that $M < 1$ if income inequality between savers and hand-to-mouth is procylical, that is, if the income of savers relative to hand-to-mouth is procyclical. Intuitively, consider an exogenous increase in the output in the next period $\hat{y}_{t+1}$. Since income inequality is procyclical, it means lower relative income for hand-to-mouth in the next period. Households recognize the possibility of becoming hand-to-mouth in the next period and there arises demand for savings for self-insurance. But, in equilibrium there is zero saving since there is no supply of assets. The effect of an increase in $\hat{y}_{t+1}$ on $\hat{y}_t$ is muted and thus discounted to make saving zero in equilibrium.

A.3 Basic properties

Impulse responses in a non-ELB regime Figure 10 plots impulse responses to the demand policy shock ($\epsilon_t^d = -0.25$ percent), the supply shock ($\epsilon_t^s = 0.25$ percent), and the monetary policy shock ($\epsilon_t^{mp} = 0.25$ percent), in the case of with and without additional discounting, respectively. In response to the negative demand shock, output decreases more persistently in the model with additional discounting than in the standard model without it. Inflation decreases and it is less responsive in the model with additional discounting than in the standard model. Reflecting the inflation responses, the policy interest rate drops less in the model with cognitive discounting.

Figure 10: The impulse responses under a non-ELB regime



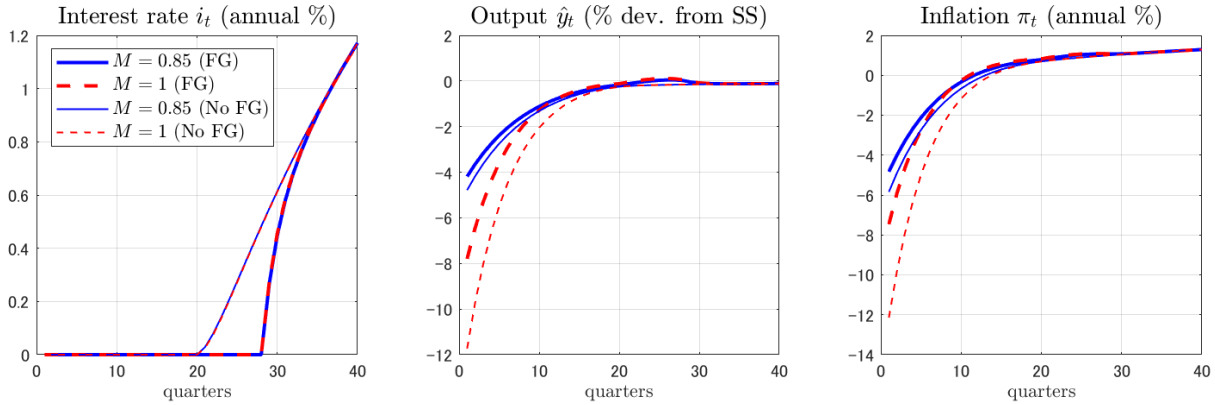
Note: '$M = 0.85$' corresponds to the model with $M = 0.85$ and '$M = 1$' corresponds to the model with $M = 1$.

In response to the inflationary supply shock, inflation rises similarly in the two models, but the decrease in output is less severe in the model with additional discounting than in the standard model. The inflationary supply shock leads to monetary tightening or a rise in the real interest rate, and equation (1) implies that the current output is given by minus the sum of discounted future real interest rates if the shock is ignored and that the future real interest rates are more heavily discounted in the model with additional discounting.

Finally, in response to the tightening shock to monetary policy, the responses of output, inflation, and the interest rate are more or less similar between the two models, although the responses of output and inflation are less responsive in the model with additional discounting.

Role of additional discounting in the ELB simulation Figure 11 plots the baseline scenarios with and without forward guidance for the model with no additional discounting $M = 1$ as well as those for the model with $M = 0.85$. For the model with $M = 1$, the magnitude of the initial shock is set to generate interest rates at the ELB for 20 periods. The paths of interest rates are virtually indifferent between the two models as shown in the left panel. However, in the model with $M = 1$, output and inflation decrease much more than those in the model with $M = 0.85$. Specifically, a decrease in inflation to -12 annual percent would be too severe to be observed in practice. Moreover, in the model with $M = 1$, forward guidance becomes extremely powerful as shown in the middle and right panels. Hence, the presence of $M < 1$ helps the model generate practical responses of output and inflation as well as moderate positive effects of forward guidance.

Figure 11: Role of additional discounting in the baseline scenario



Note: 'FG' denotes forward guidance. '$M = 0.85$' and '$M = 1$' indicate the model with $M = 0.85$ and $M = 1$, respectively. For the model with $M = 1$, the magnitude of the initial shock is set at $\epsilon_1^d = -0.5$ percent to generate interest rates at the ELB for 20 periods. In both models, $q = 0.7$ is assumed.

B Learning and updating algorithms

This appendix describes how agents update their belief about time-varying monetary policy parameters by learning from observable data. The algorithm presented here closely follows that of Bodenstein et al. (2022).

Let $\boldsymbol{\theta}_t \equiv [\hat{i}_t^*, \phi_{\pi t}, \phi_{yt}]'$ denote a column vector of time-varying unobservable monetary policy parameters. The economy starts at $t = 1$ and the agents have a prior $N(\tilde{\boldsymbol{\theta}}_{1|0}, P_{1|0})$ over the true value of $\boldsymbol{\theta}_1$. The filtering problem facing the agents outside the period of forward guidance consists of the ELB constraint (3), the actual rule for the conceptual nominal interest rate (4), and the law of motion for time-varying parameters (9), which are shown here for convenience.

$$\hat{i}_t = \max\{\hat{i}_t^n, \underline{i}\}, \tag{B.1}$$

$$\hat{i}_t^* - \rho_i \hat{i}_{t-1} = H_t \boldsymbol{\theta}_t + \epsilon_t^{mp}, \tag{B.2}$$

$$\boldsymbol{\theta}_t = F\boldsymbol{\theta}_{t-1} + \boldsymbol{\epsilon}_t^{\theta}, \tag{B.3}$$

with $\epsilon_t^{mp} \sim N(0, \sigma_{mp}^2)$, $\boldsymbol{\epsilon}_t^{\theta} \sim N(0, \Sigma_\theta)$, and $H_t \equiv [(1-\rho_i), (1-\rho_i)\hat{\pi}_t, (1-\rho_i)\hat{y}_t]$. The problem is to obtain the posterior $\tilde{\boldsymbol{\theta}}_{t|t}$ about the true parameters $\boldsymbol{\theta}_t$ given the prior $\tilde{\boldsymbol{\theta}}_{t|t-1}$.

When the ELB constraint (B.1) is slack, the problem is reduced to the standard Kalman filtering problem and the posterior is given by

$$\begin{aligned}
\tilde{\boldsymbol{\theta}}_{t|t} &= \tilde{\boldsymbol{\theta}}_{t|t-1} + K_t(\hat{i}_t - \rho_i \hat{i}_{t-1} - H_t \tilde{\boldsymbol{\theta}}_{t|t-1}), \\
P_{t|t} &= (\boldsymbol{I} - K_t H_t) P_{t|t-1}, \\
K_t &= P_{t|t-1} H_t' (H_t P_{t|t-1} H_t' + \sigma_{mp}^2)^{-1},
\end{aligned}$$

where $\boldsymbol{I}$ is an identity matrix. From equation (B.3), the updated prior in the next period is given by $N(\tilde{\boldsymbol{\theta}}_{t+1|t}, P_{t+1|t})$, where $\tilde{\boldsymbol{\theta}}_{t+1|t} = F\tilde{\boldsymbol{\theta}}_{t|t}$ and $P_{t+1|t} = FP_{t|t}F' + \Sigma_\theta$.

Now consider the case of the binding ELB constraint (B.1). First, consider the case of no forward guidance or the period outside forward guidance. In this case, the conceptual interest rate $\hat{i}_t^n$ is unobservable and what is known is the inequality $\hat{i}_t^n \leq \underline{i}$. Following Bodenstein et al. (2022), we approximate the posterior distribution of the systematic part of the notional rate, $f_t \equiv \rho_i \hat{i}_{t-1} + H_t \boldsymbol{\theta}_t$, with a normal distribution. The prior of f_t is normal with mean $f_{t|t-1} \equiv \rho_i \hat{i}_{t-1} + H_t \tilde{\boldsymbol{\theta}}_{t|t-1}$ and variances $(\sigma_{t|t-1}^f)^2 \equiv H_t P_{t|t-1} H_t'$. The

posterior mean of f_t can be calculated as

$$\begin{aligned} f_{t|t} =& \mathbb{E}(f_t|\hat{i}_t^n \leq \hat{\underline{i}}, \mathbf{x}_t) = \mathbb{E}(f_t|f_t \leq \hat{\underline{i}} - \epsilon_t^{mp}, \mathbf{x}_t) = \frac{\mathbb{E}(f_t \mathbb{1}\{f_t \leq \hat{\underline{i}} - \epsilon_t^{mp}\}|\mathbf{x}_t)}{\mathbb{P}(f_t \leq \hat{\underline{i}} - \epsilon_t^{mp}|\mathbf{x}_t)} \\ =& \frac{\mathbb{E}[f_t \mathbb{E}(\mathbb{1}\{f_t \leq \hat{\underline{i}} - \epsilon_t^{mp}\}|f_t, \mathbf{x}_t)|\mathbf{x}_t]}{\mathbb{P}(f_t \leq \hat{\underline{i}} - \epsilon_t^{mp}|\mathbf{x}_t)} = \mathbb{E}\left[f_t \frac{\mathbb{P}(\epsilon_t^{mp} \leq \hat{\underline{i}} - f_t|f_t, \mathbf{x}_t)}{\mathbb{P}(f_t + \epsilon_t^{mp} \leq \hat{\underline{i}}|\mathbf{x}_t)}|\mathbf{x}_t\right] \\ =& \int_{-\infty}^{\infty} f \frac{\Phi\left(\frac{\hat{\underline{i}}-f}{\sigma_{mp}}\right)}{\Phi\left(\frac{\hat{\underline{i}}-f_{t|t-1}}{\sqrt{(\sigma_{t|t-1}^f)^2+\sigma_{mp}^2}}\right)} \frac{1}{\sqrt{2\pi}\sigma_{t|t-1}^f} \exp\left[-\frac{(f-f_{t|t-1})^2}{2(\sigma_{t|t-1}^f)^2}\right] df \end{aligned} \tag{B.4}$$

The posterior of the second moment of f_t can be similarly obtained. By using the first and second moments, the posterior of the variance, $\sigma_{t|t}^f$, can be calculated. We assume that the posterior of f_t is normally distributed: $f_t|\hat{i}_t^n \leq \hat{\underline{i}}, \mathbf{x}_t \sim N(f_{t|t}, (\sigma_{t|t}^f)^2)$. The posterior of $\boldsymbol{\theta}_t$ can be updated by the following formula:

$$\tilde{\boldsymbol{\theta}}_{t|t} = \tilde{\boldsymbol{\theta}}_{t|t-1} + K_t(f_{t|t} - \rho_i \hat{i}_{t-1} - H_t \tilde{\boldsymbol{\theta}}_{t|t-1}), \tag{B.5}$$

$$P_{t|t} = P_{t|t-1} - K_t[H_t P_{t|t-1} H_t' - (\sigma_{t|t}^f)^2] K_t', \tag{B.6}$$

$$K_t = P_{t|t-1} H_t' (H_t P_{t|t-1} H_t')^{-1}$$

Next consider the period of forward guidance. If the agents observe the interest rate above the ELB, they learn that forward guidance has been lifted and update their belief as in the case of the non-binding ELB constraint. Instead, if the agents observe the interest rate kept at the ELB, they form the probability that the forward guidance has been implemented conditional on the observation. The probability of observing the binding ELB constraint in period t is the sum of the probability that the notional rate is above the ELB and the forward guidance is implemented, $q \times \mathbb{P}(\hat{i}_t^n \geq \hat{\underline{i}})$, and the probability that the ELB is binding due to a low conceptual rate, $\mathbb{P}(\hat{i}_t^n \leq \hat{\underline{i}})$. Then, the probability of forward guidance, perceived by the agents, conditional on observing the binding ELB constraint is given by

$$\tilde{q}_{t|ELB} = \frac{q \times \left(1 - \mathbb{P}(\hat{i}_t^n \leq \hat{\underline{i}})\right)}{q \times \left(1 - \mathbb{P}(\hat{i}_t^n \leq \hat{\underline{i}})\right) + \mathbb{P}(\hat{i}_t^n \leq \hat{\underline{i}})},$$

where

$$\mathbb{P}(\hat{i}_t^n \leq \hat{\underline{i}}) = \Phi\left(\frac{\hat{\underline{i}} - f_{t|t-1}}{\sqrt{(\sigma_{t|t-1}^f)^2 + \sigma_{mp}^2}}\right).$$

Hence, after observing the interest rate at the ELB during the period specified by forward guidance, agents believe $\hat{i}_t^n \geq \hat{\underline{i}}$ with probability $\tilde{q}_{t|ELB}$ and $\hat{i}_t^n \leq \hat{\underline{i}}$ with probability $1 - \tilde{q}_{t|ELB}$.

Now we present an updated formula for the posterior. Let $\tilde{\boldsymbol{\theta}}_{t|t}^{(+)}$ and $\tilde{P}_{t|t}^{(+)}$ denote the posterior given by equations (B.5) and (B.6), respectively, where $f_{t|t}$ and $\sigma_{t|t}^f$ are calculated conditional on information $\hat{i}_t^n > \hat{\underline{i}}$ and $\mathbf{x}_t$. Similarly, let $\tilde{\boldsymbol{\theta}}_{t|t}^{(-)}$ and $\tilde{P}_{t|t}^{(-)}$ denote the posterior given by equations (B.5) and (B.6), respectively, where $f_{t|t}$ and $\sigma_{t|t}^f$ are calculated conditional on information $\hat{i}_t^n \leq \hat{\underline{i}}$ and $\mathbf{x}_t$. The posterior mean $f_{t|t}$ and the standard deviation $\sigma_{t|t}^f$ conditional on $\hat{i}_t^n > \hat{\underline{i}}$ or $\hat{i}_t^n \leq \hat{\underline{i}}$ are calculated in a way similar to equation (B.4). Then, the posterior is given by

$$\tilde{\boldsymbol{\theta}}_{t|t} = \tilde{q}_{t|ELB} \tilde{\boldsymbol{\theta}}_{t|t}^{(+)} + (1 - \tilde{q}_{t|ELB}) \tilde{\boldsymbol{\theta}}_{t|t}^{(-)},$$

$$\tilde{P}_{t|t} = \tilde{q}_{t|ELB} \tilde{P}_{t|t}^{(+)} + (1 - \tilde{q}_{t|ELB}) \tilde{P}_{t|t}^{(-)}.$$