Propagation of Shocks in Networks: Identification and Applications*

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Abstract

This paper provides an estimation procedure to causally identify the impact of shocks in quite general networks, the "network GIV" (granular instrumental variables). We show that by using a panel with some large units (such as countries, industries or firms), and a parametric model on how shocks propagate between such units, we can extract idiosyncratic unit-specific shocks and use those to estimate the underlying parameters of this macroeconomic network. By measuring how idiosyncratic shocks impact other units in the network, we can estimate the underlying the key elasticities and the underlying shocks. We illustrate the network GIV in a new flexible model of macro linkages across industries, where we estimate between-sectors elasticities, and impulse responses. We delineate how the method can be used to causally estimate other quantities of interest e.g. to estimate a host of elasticities, and how productivity, demand or financial shocks propagate inside and across countries, making the network GIV a general-purpose tool to do causal identification in economics, and in particular macro-finance where instruments are traditionally scarce.

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1 Introduction

A key challenge in economics is the identification of structural (or causal) parameters. This challenge is exacerbated in the presence of network economies: (large) units such as firms, industries or countries may interact with others leading to propagation of shocks in the economy. Such propagation can often be difficult to measure, due to the lack of valid instruments to identify the shocks (and their effect on the economy) as often pointed out in the macro-finance literature (e.g., Ramey (2016), Nakamura and Steinsson (2018), Chodorow-Reich (2019), Huber (2023)).

To meet this challenge, the present paper proposes a novel method to identify parameters in economic networks, the "network granular instrumental variables", or Network GIV.

We build on the idea of granular instrumental variables (GIV) developed in Gabaix and Koijen (2024). There, the idea is to extract idiosyncratic shocks (via a factor model), and use them as primitive instruments to estimate for instance demand and supply elasticities in a given market. We generalize the idea of granular instruments in the presence of network economies, i.e., allowing shocks to propagate through the economy. We consider N units such as industries, markets or countries that interact with each other via supply and demand networks. An idiosyncratic shock to a particular unit also affects other units. Our goal is to identify and estimate a host of supply and demand cross-market elasticities. For example, we want to study the impact of a TFP shock in the steel sector on the prices and quantities in the entire economy, mediated throughout its first-order effect on customers and suppliers of steel firms, and its second-order effects on the other markets. Whereas our focus is on macroeconomic networks, our method applies generally, to a host of economic situations with networks, such as international trade, international macroeconomics, equilibrium asset prices, just to name a few.

Intuition for the procedure We observe a panel of N units observed over T periods (N is fixed and we will take the large T limit). We consider a model of the form

$$Y_t = B_0(\gamma^\star)Y_t + B_1(\gamma^\star)Y_{t-1} + \lambda\eta_t + u_t \tag{1}$$

where Y_t is the observed vector of outcomes of dimension N; $B_0(\gamma^\star)$ and $B_1(\gamma^\star)$ are $N \times N$ matrices, whose form is known from a model, up to an unknown low-dimensional parameter vector $\gamma \in \mathbb{R}^p$ that we want to identify. There is a vector of (unobserved) idiosyncratic shocks u_t, and aggregate shocks η_t, with a vector of loadings λ. The key assumption is that u_t defines idiosyncratic shocks uncorrelated between units. The lagged term B_1 and the factors $\lambda\eta_t$ are not necessary for identification: one can have an i.i.d. model with $B_1(\gamma) \equiv 0$, and also no factors $\lambda\eta_t \equiv 0$. Note that $u_t$can have heteroskedastic entries (in principle, one could also model the variance of $u_t$omitted for expositional convenience).

We identify the target parameters using the following idea. Suppose, to build intuition, that $B_1(\gamma) = 0$, $\lambda = 0$ and we observe the idiosyncratic shocks $u_{j,t}$ for simplicity. Then in the regression $Y_{it} = e_i' B_0(\gamma) Y_t + u_{i,t}$ of individual i, to instrument the endogenous regressor Y_t, we can use all shocks $u_{j,t}$ for all units j except for unit i . More precisely, we use as an instrument for Y_t that value of Y_t that would be generated by the idiosyncratic shocks of units $j \neq i$ only (i.e., the expectation of Y_t conditional on all shocks $u_{j,t}$, $j \neq i$). This allows us to recover our target parameter γ. The underlying intuition is that all shocks $u_{j,t}$ are valid instrument except for the shock of individual i.

In practice, however, we do not observe the shocks directly. Nevertheless, we show that, by using this intuition (and extending it to more complex scenarios), we can build valid moment restrictions and an iterative estimation procedure to identify and estimate both the target parameter $\gamma^\star$ and recover the idiosyncratic shocks. Given a guess of γ, we recover the idiosyncratic shock $u_{i,t}$, and given those idiosyncratic shocks, we can estimate γ. In the end, our estimator is the fixed point of this iterative procedure. The procedure is laid out in Algorithm 1.

Intuitively, our moment conditions find the value of γ that best explains how a sector i

is impacted by idiosyncratic shocks $u_{j,t}$ the other sectors j ($j \neq i$); this is the key network GIV moment, associated with the outcomes in sector i. We then aggregate each of these moment restrictions for estimation.

We study the asymptotic properties of our estimators in regimes with finite N and large T, and present a simple procedure to estimate $\gamma^\star$ and the variance of the estimator. In some relevant cases (see Example 4), the standard errors on the parameters are of order $\mathcal{O}(1/\sqrt{TN})$. Intuitively, this implies that even with a relatively short time series of length T, we can obtain precise estimates if the number of units N is large. This provides some suggestive evidence that our method can be applicable to a host of recent "big data" sets with large cross-sections.

Applications We present an application to macroeconomic fluctuations. First, we develop a new model, which is a generalized version of existing macroeconomic models (Long and Plosser (1983), Gabaix (2011), Acemoglu et al. (2012), Carvalho and Gabaix (2013), and Atalay (2017)). In particular, those models posit a uniform elasticity of demand for sectoral inputs (and indeed, one often posited to be equal to one). In contrast, here we allow for heterogeneous elasticities of demand and supply across sectors. This is useful, because it is likely that some sectors have a sector-specific elasticity of demand. Returning to our example on the effect of TFP shocks in the steel sector, it is likely that other sectors, such as the energy sectors, may have different responses to such shocks.

More generally, our procedure can be used in a host of situations. For instance, in international macroeconomics, we can ask: if country j has a supply or demand shock, what happens to each country i's demand and supply in the world economy? What about exchange rates or other relevant outcomes? Using idiosyncratic shocks to countries, and seeing their effects throughout the world economy, via the network GIV, we can answer these questions.

Our method could also be useful when studying questions in the literature on economic growth: if firm i has a positive productivity shock, how does the productivity of its competitors, its suppliers, and consumers change (as in Bloom, Schankerman, and Reenen (2013))? In principle, the network GIV allows researchers to answer that question.

Another class of applications is in finance: in the presence of a demand shock to stocks (perhaps via an idiosyncratic shock channeled through one or some sector of the economy, as in Gabaix and Koijen (2023)), what is the impact on the stock's price, as well as on bonds, exchange rates, et cetera? In a model with a demand system for several types of assets (perhaps in the spirit of Koijen and Yogo (2019)), these effect could be estimated using the network GIV.

Related literature We extend greatly (moving from one to N markets) the GIV method of Gabaix and Koijen (2024), which has been used in a number of papers (Chodorow-Reich, Ghent, and Haddad (2021), Adrian et al. (2022), Camanho, Hau, and Rey (2022), Galaasen et al. (2023), Gabaix and Koijen (2023), Qian (2024)) by using idiosyncratic shocks to large firms or sectors for identification. Baumeister and Hamilton (2023) propose a maximum likelihood (MLE) approach the GIV, without, however, allowing for network effects. Different from these references, the presence of network effects challenges the identification and estimation strategy: we show that, because individual idiosyncratic shocks of a given unit are correlated with the endogenous regressors (the other units' outcomes), moment restrictions take a new form and solve a fixed point problem. Here, we opt for transparency, using moments that are close (in a sense we will make precise in Section 2.5.2) to the (pseudo) MLE, so to preserve efficiency if certain distributional assumptions hold, without violating validity if these fail. One advantage of our method however, with respect to MLE, is its robustness to the incidental parameter problem. As we discuss in Section 2.5.2, the MLE formulation may lead to inconsistent estimates for relevant data-generating processes in the presence, for instance of time fixed effects (Neyman and Scott (1948)). As we show, in relevant data generating process, this does not occur for our estimator because it leverages specific moment conditions obtained through independence of shocks.

We more broadly connect to a large literature on vector autoregression, see for example Baumeister and Hamilton (2024) and Rubio-Ramírez, Waggoner, and Zha (2010). When these models are estimated via (pseudo) MLE these may suffer from similar issues in the presence of incidental parameter problems, different from our proposed estimator. How-

ever, as often assumed in these models, also here identification comes from restrictions on covariance and adjacency matrix.

We build on the idea that the economy is "granular" in nature— due to incompressible "grains" of volatility coming from idiosyncratic shocks to firms or industries (Gabaix (2011); Acemoglu et al. (2012), Giovanni and Levchenko (2012), Giovanni, Levchenko, and Méjean (2014), Baqaee and Farhi (2019), Carvalho and Grassi (2019), Gaubert and Itskhoki (2021), and De Graeve and Schneider (2023)). This is also confirmed in multi-sector models in the tradition of Long and Plosser (1983) (Atalay (2017)).

We link to a large literature on networks. Manski (1993)'s influential work distinguished between different types of peer effects and pointed out challenges for identification in network setups with correlated effects (see Kline and Tamer (2020) for a review). Different from standard peer effects models as in Manski (1993) (and motivated by macro-economic applications), we identify the key parameters through idiosyncratic shocks, leveraging the fact that idiosyncratic shocks are uncorrelated between units. Identification of peer effects have then been extensively studied, see Bramoullé, Djebbari, and Fortin (2009), De Giorgi, Pellizzari, and Redaelli (2010), Lee (2007), among others. These methods consider a linear in means model (without a factor structure), and leverage identification by using either sufficient variation in the rows of the adjacency matrix together with sufficient variation of auxiliary observable covariates in the outcome equation, or variation in the size of non-overlapping groups. Using related identification strategies, Blume et al. (2015) consider settings where the adjacency matrix is only partially observed and Blume et al. (2011) propose (spatial) models of the adjacency matrix. Our approach complements these existing methods in general setups by parametrizing a single network as a function of a low dimensional set of parameters and leveraging valid moment restrictions that use uncorrelated idiosyncratic shocks, and do not require variation in additional auxiliary covariates.[1]

Battaglini, Patacchini, and Rainone (2022) and Breza et al. (2020) propose Bayesian

1. Relatedly, Manresa (2013) also considers settings with unknown adjacency matrix, but assumes no endogenous peer effects (i.e., the outcome does not depend on other units' outcomes), different from our framework.

procedures to recover social networks, different from our approach here. De Paula, Rasul, and Souza (2019) present general identification results for a linear in means model leveraging variation in auxiliary covariates. Here, instead, we leverage assumptions on a low dimensional parametrization of the network structure and independence of the shocks. These assumptions allow us to identify effects through simple moment conditions that leverage exogeneity of the other units' idiosyncratic shocks. Our moment conditions can be easily extended to more complex models relevant in macroeconomic applications where a factor structure occurs. Finally, in recent work, Griffith and Peng (2023) study identification of the network assuming sparsity restrictions (e.g., only few countries may affect other countries). Although such restrictions can be plausible in micro-economics applications, these typically fail in macroeconomics where we observe dense networks. Here, we do not require such restrictions on the network, motivated by application in macroeconomics.

The rest of the paper is organized as follows. In Section 2, we present the basic econometric methodology, and in Section 2.5 a variety of extensions, e.g. to autocorrelations, heteroskedasticity, and missing data. In Section 3, we present a multi-sector macro model and illustrate how our method applies to it. In Section 4, we bring the model to the data and estimate it using our proposed methodology. Section 5 concludes.

Notations We call $e_i = (0, \dots, 0, 1, 0, \dots, 0)'$ the unit vector with 1 at position i, 0 otherwise; $\iota = (1, \dots, 1)$ a vector of ones; $\mathbb{I}$ the identity matrix — all of the appropriate dimension; when needed, we use $\mathbb{I}_N$ for the identity matrix of dimension N. For given random variables $X_{i,t}$, we denote $X_t = [X_{1,t}, \cdots, X_{N,T}]'$ the vector of such variables. We define $\mathbb{E}[\cdot], \mathbb{V}(\cdot)$ the expectation and variance operator, respectively. We call $V^y = \mathbb{E}[y_t y_t']$ the variance of a random variable y_t, which is mean 0 and vector valued, and $W^y = (V^y)^{-1}$ the inverse variance. For two vectors, $x \odot y$ denote the component-wise product.

2 Econometric Methodology: Main Ideas

Consider a panel with N units and T periods. Define $Y_{i,t} \in \mathbb{R}$ the observed outcome of interest. The key goal is to estimate a *low-dimensional* parameter $\gamma^\star$. The low dimensionality of $\gamma^\star$ embeds a large class of macro-economic applications, including our leading application in the following section. We will consider the case of a fixed N, but a large T

2.1 Some examples

We study networks of the form (1). We start by presenting some examples, and will be useful to keep in mind when interpreting the general econometric framework.

Example 1 (Influence matrix linear in factors)**.** Consider the following model:

$$Y_t = \gamma \bar{B} Y_t + u_t, \tag{2}$$

where the matrix $\bar{B}$ is known and $\gamma \in \mathbb{R}$ is an unknown parameter, so that $B(\gamma) = \gamma \bar{B}$. For instance, $\bar{B}_{ij}$ could be a known "network strength" of the flows of information from j to i (e.g., how frequently i hears j's recommendations), and we are looking to estimate the "reactivity coefficient" γ. This model is similar to Bramoullé, Djebbari, and Fortin (2009), who study a model of the form $Y_t = \gamma \bar{B} Y_t + X_t \beta + u_t$, where X_{it} are observed covariates. But while Bramoullé, Djebbari, and Fortin (2009) needs to use covariates X_{it}, we do not need any covariates. Instead, we use impact of idiosyncratic shocks u_{jt} in our instrumental variable method, and particularly of large "influencers" j, that have a large average influence $\frac{1}{N}\sum_i \bar{B}_{ij}^2$. This example generalizes to settings where

$$Y_t = \sum_{k=1}^{K} \gamma_k \bar{B}^{(k)} Y_t + u_t, \tag{3}$$

where the matrices $\bar{B}^{(k)}$ are observed by the econometrician and γ_k denotes a relative weight on the matrix $\bar{B}^{(k)}$, so that $B(\gamma) = \sum_{k=1}^{K} \gamma_k \bar{B}^k$. This would be the case if there are possibly

different channels (matrices $\bar{B}_{ij}^{(k)}$), i.e. via tweets, or emails, or Facebook messages, and each could have a different influence γ_k.

Example 2 (Supply and Demand)**.** Take a supply and demand model:

$$q_{it} = \phi_i^d p_t + \eta_{1t} + u_{it} \tag{4}$$

$$p_t = \psi q_{St} + \eta_{2t} \tag{5}$$

Firm $i \in \{1, \ldots, n\}$ has log demand q_{it} reacting to the log price p_t with demand elasticity ϕ_i^d; and the price p_t that depends on the aggregate demand $q_{St} = \sum_{i=1}^n S_i q_{it}$ (where S_i are size shares) with inverse supply elasticity ψ. We assume that u_t is uncorrelated of η_t. This fits in the framework, with $Y_t = (q_{1t}, \ldots, q_{nt}, p_t)'$, $\gamma = \left(\phi^d, \psi\right)$, $B = \begin{pmatrix} 0 & \phi^d \\ \psi S' & 0 \end{pmatrix}$, $\chi = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$. This shows how the framework generalize that of Gabaix and Koijen (2024), easily allowing for heterogeneous elasticities of demand. One could specialize it to $\phi_i^d = \varphi^d \iota$, for a uniform elasticity φ^d, or to a parametric model $\phi_i^d = \dot{\varphi} X_i$, where $\dot{\varphi}$ is a vector of parameters to estimate, and each X_i represents firm i's characteristics, which are given.

Example 3 (Macroeconomic production network)**.** A macroeconomic production network will feature n sectors with price vector p_t and output vector q_t, and a link of the type (1), where $Y_t = (\pi_t, \Delta \hat{y}_t) \in \mathbb{R}^{2n}$ records the change in price and quantities. Importantly, the matrix $B(\gamma)$ typically depends non-linearly on the parameter γ, e.g. via mixtures of elasticities and speeds of adjustment. Section 3 develops such a model, in see in particular Proposition 3.

2.2 Most basic version of the procedure

In this section we present the basic model to convey the main intuition.

Assumption 1 (Basic model)**.** *For all* $t \in \{1, \cdots, T\}$,

$$Y_t = B(\gamma^\star) Y_t + u_t, \quad \mathbb{E}[u_{i,t} | \{u_{j,t}\}_{j \neq i}] = 0, \tag{6}$$

where $\gamma^\star \in \mathbb{R}^p$ is an unknown vector of parameters in a space $U \subseteq \mathbb{R}^p$, $B : U \to \mathbb{R}^{N\times N}$ is a known, continuously differentiable function and $A(\gamma^\star) = (\mathbb{I}_N - B(\gamma^\star))^{-1}$ exists.

Assumption 1 states that the outcome of each unit depends on weighted combinations of other units' outcomes and shocks. The dependence is captured through the parameter $\gamma^\star$. We assume that we know the functional form $B(\gamma^\star)$, but do not know $\gamma^\star$. The key identification assumption is that the shocks $u_{i,t}$ are mean independent (uncorrelated) across units i. The intuition is the following: a shock $u_{i,t}$ will affect all the units j at time t. Hence, by tracing out the impact of $u_{i,t}$ to units $j \neq i$, we are able to identify $\gamma^\star$.

To derive our results for inference, we will make the simplifying assumption that shocks are independent and identically distributed over time, although we could directly relax this assumption by assuming forms of mixing and covariances stationarity (omitted for brevity).

Assumption 2. *The u_t are independent and identically distributed over time t.*

Given the model $Y_t = B(\gamma^\star)Y_t + u_t$, we cannot regress directly $Y_{i,t}$ onto the other outcomes, because $E[e_i^\top B(\gamma^\star)Y_t|u_{i,t}] \neq 0$. We therefore need to instrument the endogenous regressor $e_i^\top B(\gamma^\star)Y_t$. To do so, we use a simple idea: we instrument the endogenous regressor $e_i^\top B(\gamma^\star)Y_t$ using the unobserved shocks of all units except unit i, i.e., $\{u_{j,t}\}_{j\neq i}$. Following Chamberlain (1987), the construction of the optimal (efficient) instrument is a function of $E[e_i^\top B(\gamma^\star)Y_t|\{u_{j,t}\}_{j\neq i}]$ (as well as possible heteroskedastic components). However, without distributional assumption and arbitrary heteroskedasticity, this construction is infeasible because the optimal instrument depends on the unknown vector of parameters $\gamma^\star$. We therefore propose an intuitive approximation to the optimal instrument that retains validity of the resulting moment conditions without requiring distributional assumption on the unobserved shocks. Our idea proceed as follows.

Suppose we have a guess γ, and we imagine that the discrepancy $\Delta\gamma := \gamma^\star - \gamma$ is small. As a first step, we use a Taylor expansion to write (where $B_\gamma(\gamma^\star)$ denotes the gradient of $B(\gamma)$ evaluated at the true parameter $\gamma^\star$)

$$u_t = (I - B(\gamma^\star))\, Y_t = (I - B(\gamma + \Delta\gamma))\, Y_t = (I - B(\gamma))\, Y_t - \Delta\gamma \cdot B_\gamma(\gamma)\, Y_t + O\left(\|\Delta\gamma\|^2\right)$$

Therefore defining the residual

$$\tilde{v}_t(\gamma) := (I - B(\gamma))\, Y_t \tag{7}$$

we have

$$\tilde{v}_t(\gamma) = \Delta\gamma \cdot B_\gamma(\gamma) Y_t + u_t + O(||\Delta\gamma||^2).$$

Hence, with pre-multiplying by e_i the left and right hand side, we obtain

$$\tilde{v}_{it}(\gamma) \coloneqq e_i^\top (I - B(\gamma))\, Y_t = \Delta\gamma \cdot e_i^\top B_\gamma(\gamma)\, Y_t + u_{it} + O\left(\|\Delta\gamma\|^2\right). \tag{8}$$

The main issue of this expression is that the regressor $e_i^\top B_\gamma(\gamma)\, Y$ depends on the endogenous variable Y_t. As our second step, we use the fact that shocks are independent over units and replace $e_i^\top B_\gamma(\gamma)\, Y$ in the regression above with an instrument $z_{i,t}(\gamma)$ that approximates $E[e_i^\top B_\gamma(\gamma)\, Y | \{u_{j,t}\}_{j \neq i}]$. Such an instrument guarantees under independence of shocks across units that $E[z_{i,t}(\gamma)\tilde{v}_{it}(\gamma)\Big|_{\gamma=\gamma^\star}] = 0$. If we can construct such an instrument, it follows that we can directly estimate $\Delta\gamma$ by OLS in a simple (and intuitive) regression:

$$\tilde{v}_{it}(\gamma) = \Delta\gamma \cdot z_{it}(\gamma) + \varepsilon_{i,t}, \quad E[\varepsilon_{i,t} | z_{i,t}(\gamma^\star)] = 0. \tag{9}$$

We are left to discuss the construction of the instrument $z_{i,t}(\gamma)$. Recall that $Y_t = A(\gamma^\star)u_t$. Therefore, we have $e_i^\top B_\gamma(\gamma)\, Y_t = e_i^\top B_\gamma(\gamma)\, A(\gamma^\star)u_t$. Hence, we set

$$z_{i,t}(\gamma) = e_i^\top B_\gamma(\gamma)\, A(\gamma^\star) u_t^{-i}(\gamma), \qquad u_t^{-i}(\gamma) \coloneqq (I - e_i e_i^\top)\tilde{v}_t(\gamma) \tag{10}$$

so that $z_{i,t}$ is the instrument using only shocks to entities different from i.

We therefore obtain a simple and recursive estimation procedure where we first form a guess γ and then we update such a guess over each iteration by taking $\gamma + \Delta\gamma$ in the subsequent iteration. At convergence, the estimator provides us with a valid moment restriction for $\gamma^\star$.

This leads to Algorithm 1, which, for completeness, we present here with all its variants, and discuss each of these in the subsequent subsections.

Algorithm 1 One-step estimation of $\gamma^\star$

Require: $\hat{\gamma}$, a guess about the value of γ^*; and r, the number of factors.

1: Define $\tilde{v}_t(\hat{\gamma}) \coloneqq Y_t - B_0(\hat{\gamma})Y_t - B_1(\hat{\gamma})Y_{t-1}$. Below, we remove the explicit dependences on $\hat{\gamma}$, but everything is evaluated using $\hat{\gamma}$.

2: Run PCA with heteroskedasticity using Algorithm 2 passing as the argument $\tilde{v}_t$ and as the number of factors r. The PCA algorithm iterates to estimate the loadings $\tilde{\lambda}$. Alternatively, to control for time fixed effects, apply PCA to the cross-sectionally demeaned residuals $\check{v}_t$, as detailed in Section A.2.

3: For each i, estimate η_t^{-i} as the slope in the regression: $\tilde{v}_{jt} = \tilde{\lambda}_j \eta_t^{-i} + \tilde{u}_{jt}^{-i}$, where we only use the units $j \neq i$. We collect the residuals $\tilde{u}_{jt}^{-i}$. We also set $\tilde{u}_{it}^{-i} = 0$.

4: For each i, define $\tilde{y}_{0|-i} = 0$, and construct recursively the set of instruments $\tilde{y}_{t|-i}$, which are the values of y_t that are due to the idiosyncratic shocks from units $j \neq i$: with $A = (I - B_0)^{-1}$,

$$\tilde{y}_{t|-i} = A\left(\tilde{u}_{t|-i} + B_1 \tilde{y}_{t-1|-i}\right). \tag{11}$$

5: For each i, t, construct the instrument $z_{i,t}$, which has the same dimension as γ. It's $k-$th component is:

$$z_{i,t}^{(k)} = e_i^\top \left(\frac{\partial B_0}{\partial \gamma_k} \tilde{y}_{t|-i} + \frac{\partial B_1}{\partial \gamma_k} \tilde{y}_{t-1|-i} \right) \tag{12}$$

6: Estimate the updated $\hat{\gamma}'$ as follows:

1. For all i, regress η_t^{-i} on $z_{i,t}$, and collect the residuals $\eta_t^{-i,\perp}$: $\eta_t^{-i} = \beta z_{i,t} + \eta_t^{-i,\perp}$.
2. Run the OLS regression of $\tilde{v}_{it}$ on $z_{i,t}$, controlling for $\eta_t^{-i,\perp}$:

$$\tilde{v}_{it} = \Delta\gamma \cdot z_{i,t} + b_i \eta_t^{-i,\perp} + \varepsilon_{it} \tag{13}$$

and estimate the slope vector $\Delta\gamma$. Update $\hat{\gamma}$ as $\hat{\gamma}' \coloneqq \hat{\gamma} + \Delta\gamma$.

return $\hat{\gamma}'$

We state two propositions giving theoretical guaranties for the procedure.

Proposition 1. *Assume that u_{it} are uncorrelated with u_{jt} for $j \neq i$ (but could have heteroskedastic variance or be dependent over time periods). At the correct $\gamma^\star$, we have $\mathbb{E}\left[\tilde{v}_{it}(\gamma^\star) z_{it}(\gamma^\star)\right] = 0$, where $\tilde{v}_t(\gamma) = Y_t - B(\gamma) Y_t$ and $z_{it}(\gamma)$ is the instrument in (10).*

The advantage of the proposed procedure is its simplicity (it only requires to run a

sequence of linear regressions), and the moment conditions to identify the target parameter of interest is defined solely based on the assumption that shocks are independent across units (therefore without imposing e.g., distributional conditions on the shocks).

How many parameters can we estimate? We have the variance matrix of Y_t, which is symmetric and $N \times N$, so has $d_{VY} = \frac{1}{2}N(N+1)$ degrees of freedom. We have also one (in the homoskedastic case) free parameters for σ_u^2, or N in the heteroskedastic case. So, we can estimate up to $p^{\max} = \frac{1}{2}N(N+1) - N$ parameters, if we have a very long time series. We record:

$$p^{\max} \sim \frac{1}{2}N^2 \tag{14}$$

So in principle, we can estimate many parameters with a large time series. Even $N = 100$ units, one can in principle estimate about $5,000$ parameters, at least in the large T limit. Our empirical example will have $N \simeq 300$ units.

2.3 Enrichments

In this section we present a series of extensions of our baseline model, deferring more formal details to the Appendix.

2.3.1 The case with factors

Suppose next that $Y_t = B(\gamma)Y_t + \lambda\eta_t + u_t$, where λ indicates some (unobserved loadings) and η_t indicates unobserved factors. We define $\tilde{v}_t$ as above in (7). Whenever $\gamma = \gamma^\star$, we should have $\tilde{v}_t = u_t + \lambda\eta_t$.

As the first step, we run iteratively, for different units i, a version of heteroskedastic robust PCA to estimate a factor model for $\tilde{v}_{jt}$ for $j \neq i$: $\tilde{v}_{jt} = u_{jt} + \lambda_j\eta_t$ (see Section A.1). We collect the estimated factors η_t^{-i} and the residuals $\tilde{u}_{jt}^{-i}$, obtained from such factor models using all units except information from unit i (setting $\tilde{u}_{it}^{-i} \coloneqq 0$)

In addition, with factors, (8) becomes:

$$\tilde{v}_{it} = \Delta\gamma \cdot e_i^\top B_\gamma(\gamma) Y_t + u_{it} + \lambda_i \eta_t + O\left(\|\Delta\gamma\|^2\right) \tag{15}$$

Our idea proceeds as follows. We form the instrument $z_{it}(\gamma) \coloneqq e_i^\top B(\gamma) A(\gamma) \tilde{u}_t^{-i}(\gamma)$. Now, we also need to control for the residuals η_t^{-i}. We first residualize, i.e. estimate β in $\eta_t^{-i} = \beta z_{i,t}(\hat{\gamma}) + \eta_t^{-i,\perp}$ and collect the residuals $\eta_t^{-i,\perp}$. Then, we simply run (13), which is the basic OLS regression, but controlling for $\eta_t^{-i,\perp}$. This is done in Algorithm 1.

2.3.2 Lagged values of Y_t

With a term $B_1(\gamma^\star)Y_{t-1}$ in the process (1), we naturally define $\tilde{v}_t$ as in Step 1 of Algorithm 1, and follow verbatim our procedure now also controlling for the lagged outcomes.

2.3.3 Additional covariates

In the presence of additional known covariates X_{it}, so that the model is

$$Y_t = B_0(\gamma^\star)Y_t + B_1(\gamma^\star)Y_{t-1} + X_t b_X + \lambda\eta_t + u_t \tag{16}$$

we use a natural extension of the basic procedure. Given γ, we estimate b_X by OLS of $Y_t - B_0(\gamma)Y_t - B_1(\gamma)Y_{t-1}$ on X_t; and given b_X, we proceed as in the rest of the paper, replacing $v_t \coloneqq Y_t - B_0(\gamma)Y_t$ by $v_t \coloneqq Y_t - B_0(\gamma^\star)Y_t - X_t b_X$, in Step 1 of Algorithm 1 and its variants.[2]

2. There can be in principle another instrument, namely (imagining that we know, or have already estimated b_X, for simplicity)

$$z_{it}^X \coloneqq e_i' B_\gamma A X_t^{-i} b_X, \tag{17}$$

If we think that the value of X_{jt} a truly exogenous (i.e. X_{jt} is uncorrelated with all u_{it}, at least when when $j \neq i$), we get an extra instrument. Indeed regression (9), i.e. (13) in the Algorithm, we could have

$$\tilde{v}_{it} = \Delta\gamma \cdot z_{i,t} + \Delta\gamma^X \cdot z_{it}^X + b_i \eta_t^{-i,\perp} + \varepsilon_{it} \tag{18}$$

Hence, we have two instruments: $z_{i,t}$, using idiosyncratic shocks, and z_{it}^X, using covariates. Each of them gives a valid estimate of $\Delta\gamma$. At convergence, both should be 0.

However, it is not clear that X_t is truly exogenous. For instance, suppose that we estimate the elasticity of demand for oil, and X_{jt} is the business cycle in country j. It is a useful "control", but it is very dependent

2.4 Formal statements

We present more formal econometric results. We define

$$G^{(k)} = \frac{\partial B(\gamma)}{\partial \gamma_k}\Big|_{\gamma=\gamma^\star} A(\gamma^\star), \quad \Gamma^{(k)}_{i,j} = G^{(k)}_{i,j} 1_{i\neq j}. \tag{19}$$

so that Γ gives the network impact "removing self-impacts", as $\Gamma_{ii} = 0$, for a marginal change in γ. We also define $u_t(\gamma) \coloneqq (I - B(\gamma)) y_t$. Then, in our basic Algorithm,

$$z^{(k)}_{it} = e_i^\top \Gamma^{(k)} u_t(\gamma) \tag{20}$$

Define unit i's empirical moment restriction

$$\bar{H}_i(\gamma) = \frac{1}{T}\sum_t u_{it}(\gamma)\, z_{it}(\gamma) \tag{21}$$

the moment for $\Delta\gamma = 0$ is $\sum_i \bar{H}_i(\gamma) = 0$, i.e. more compactly:

$$\frac{1}{T}\sum_t u_t^\top(\gamma)\, \Gamma^{(k)}(\gamma)\, u_t(\gamma) = 0, \quad k \in 1, \cdots, p \tag{22}$$

We define $\mathbb{E}[u^2_{i,t}] = \sigma^2_{u_i}$. Algorithm 1 uses a regression on z_{it}. Hence, its variance should be non-degenerate. We next derive the variance of z_{it}, and assume that it is full rank.

Lemma 1. (Variance of z_{it}) *Suppose that the u_{it} are uncorrelated across periods and define $\sigma^2_u = \text{Diag}\left(\sigma^2_{u_i}\right)$. The following holds:*

$$\sum_i \mathbb{E}\left[z_{it}(\gamma^\star) z_{it}(\gamma^\star)^\top\right] = V^z, \tag{23}$$

on the price of oil, which itself will depend on the idiosyncratic demand shock by country i. So, in that case, it may be best to just use z_{it} rather than also the z^X_{it}. As always, the choice of instrument depends on careful thinking about exogeneity.

where the $p \times p$ matrix V^z is, for k, k' in $\{1, \ldots, p\}$:

$$V^z_{p_1,p_2} = \sum_{i,j} \Gamma^{(p_1)}_{ij} \Gamma^{(p_2)}_{ij} \sigma^2_{u_j} \tag{24}$$

Proof. See Appendix B.1.2. □

We will make the following assumption.

Assumption 3 (Moment restriction)**.** *Assume that for each i, $H_{i,t}(\gamma)$ is continuously differentiable almost surely, $\mathbb{E}\left[\sup_\gamma ||H_{i,t}(\gamma)||\right], \mathbb{E}\left[\sup_\gamma \left|\left|\frac{\partial H_{i,t}(\gamma)}{\partial \gamma}\right|\right|\right] < \infty, \mathbb{E}\left[\sup_\gamma \left|\left|\frac{\partial^2 H_{i,t}(\gamma)}{\partial \gamma^{(p_1)} \partial \gamma^{(p_2)}}\right|\right|\right] < \infty$, for all $p_1, p_2 \in \{1, \cdots, p\}$, $\mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^N H_{i,t}(\gamma^\star)\right)\left(\frac{1}{N}\sum_{i=1}^N H_{i,t}(\gamma^\star)\right)^\top\right]$ is positive definite, and $\frac{1}{N}\sum_{i=1}^N H_{i,t}(\gamma^\star)$ has finite third moment. Suppose in addition that $-\sum_{i=1}^N \frac{\partial \mathbb{E}[\bar{H}_i(\gamma)]}{\partial \gamma}\Big|_{g=\gamma^\star}$ is invertible and there exists a unique $\gamma^\star \in \mathcal{S}$, such that $\sum_{i=1}^N \mathbb{E}[\bar{H}_i(\gamma^\star)] = \mathbf{0}$, and $\gamma^\star$ is in the interior of $\mathcal{S}$.*

Assumption states uniqueness of the solution and invertibility of the derivative of the score function. Uniqueness is often not guaranteed for arbitrary parameter space, e.g., identification often requires signs restrictions on the parameters De Paula, Rasul, and Souza (2019). Here, we assume that the parameter is (locally) identified given the set $\mathcal{S}$. The remaining conditions guarantee that the central limit theorem applies to our setting.

Proposition 2. *Suppose that Assumptions 1, 3 hold. Then as $T \to \infty$,*

$$\sqrt{T} V(\gamma^\star)^{-1/2}\left(\hat{\gamma}^\star - \gamma^\star\right) \to^d \mathcal{N}(0, \mathbb{I}) \tag{25}$$

for some positive definite matrix $V(\gamma^\star)$. In addition, if u_{it} are homoskedastic,

$$V(\gamma^\star) = W(\gamma^\star)^{-1}, \qquad W_{p_1,p_2}(\gamma^\star) = \sum_{i,j}\left(\Gamma^{(p_1)}_{i,j}\Gamma^{(p_2)}_{i,j} + \Gamma^{(p_1)}_{i,j}\Gamma^{(p_2)}_{j,i}\right) \tag{26}$$

Proposition 2 follows directly from standard arguments for extremum estimators (Newey and McFadden 1994). The exact expression for the variance is derived in Appendix B.1.3. Note that Proposition 2 considers settings where T grows, and it holds for any (finite) N.

When N grows, the convergence rate also depends on how $V(\gamma^\star)$ converges to zero as a function of N (see e.g., Example 4, here omitted for brevity).

The expression for the variance provides us with a concrete description of the (local) identification conditions required by our model, which therefore impose restrictions (as for the OLS formulation) on the trace of the matrix obtained from $\Gamma\Gamma^\top$. We see this as a simple to verify condition as it only depends on the $B(\gamma)$ matrix and not on the variance of the shocks. [3]

The lack of dependence of the estimator's variance on σ_u^2 is because σ_u^2 determines the variance of both the endogenous regressor and the instrument.

In the following lines we illustrate that under Gaussianity and homoskedasticity of the shocks, our estimator is not only consistent but also efficient (note that similar arguments also hold with heteroskedastic shocks, by using a two-step generalized least squares estimator as opposed to simple OLS estimator proposed here). This efficiency property, together with its robustness under misspecification of the likelihood function, motivates our proposed procedure.

Corollary 1. *Suppose that the conditions in Proposition 2 hold and, in addition, u_{it} are jointly Gaussian and homoskedastic. Then Net-Giv's variance achieves the Cramer-Rao lower bound.*

Corollary 1 states that under Gaussian of the shocks, the proposed procedure is also efficient. The proof is derived in Appendix B.2.

We conclude this discussion with a comparison of the variance of our estimator with the "naive" variance computed in the last iteration of the OLS regression.

The "nominal" standard variance returned by the OLS software (applying the usual OLS formula) in the homoskedastic case is given by $V^{\gamma,OLS} = (V^z)^{-1}$. Therefore, as we

3. We only consider growing T in Proposition 2 for expositional convenience (whereas the proposition is valid for any finite N). In the presence of growing N, we should explicitly consider a positive lower bound on the matrix $\mathbb{E}\Big[\Big(\frac{1}{N}\sum_{i=1}^N H_{i,t}(\gamma^\star)\Big)\Big(\frac{1}{N}\sum_{i=1}^N H_{i,t}(\gamma^\star)\Big)^\top\Big]$ that may shrink with N and adjust the rate of convergence accordingly based on this lower bound. Here, we directly rescale by the variance $V(\gamma^\star)$ in the left-hand side of Equation (25) without being explicit about the rate of convergence in N when N grows.

may expect, the OLS regression underestimates the variance by an additional component $\text{tr}\left(\Gamma^{(p_1)}\Gamma^{(p_2)}\right)$, in the homoskedastic and Gaussian case. Interestingly, that additional is very small compared to the first term in some leading examples, see the example 4 below.

Example 4 (Sphere of influence)**.** Consider Example 1 with $\bar{B}_{i,i} = 0$ and $\bar{B}_{i,j} = c_{i,j}S_j$ for constants $c_{i,j}$, and $\sum_j S_j = 1$. That is, different countries may have different "spheres of influences", parametrized by $c_{i,j}$, and S_j (some of which may also be unknown). Suppose that $\gamma^\star = 0$, and known S, c. In this case, under homoskedasticity, we obtain that the gradient of the moment equation, and in particular $\sum_{i=1}^N \frac{\partial \mathbb{E}[\bar{H}_i(\gamma)]}{\partial \gamma}\Big|_{g=\gamma^\star} / \sigma_u^2 = \sum_i \sum_{j\neq i} c_{i,j}^2 S_j^2 + c_{i,j}c_{j,i}S_iS_j$. Suppose now that $c_{i,j}$ is iid across i, j with squared mean $\mathbb{E}[c_{i,j}^2]$ independent of (i, j). We can write, by taking expectations over c, the expression reads as $(N-1)\mathbb{E}[c_{i,j}^2]h^2 + \mathbb{E}\left[c_{ij}\right]^2$ where $h^2 = \sum_j S_j^2$ denotes the Herfindhal of country sizes. This implies that we expect the gradient to be of order $N\mathbb{E}[c_{i,j}^2]h^2$ for large N. Suppose we are in the "granular" case where there are a few large units, so that h remains of order unity even as N is large (for instance, a fixed $S_j > 0$ for a given/small number of units, even as N become large).[4] Then the variance $V(\gamma^\star)$, that is obtained through the standard Sandwich formula (and whose properties therefore depend on the gradient of the moment equation) is of order $\mathcal{O}(1/N)$, and the estimator's convergence rate is $1/\sqrt{TN}$ (more precisely, $1/\sqrt{TNh^2}$).[5] This hints at the possibility that even with a not relatively large T, but a large cross-section N, one might reach a good precision.[6]

Results with factor models The appendix (Section B.3) provides propositions for the more complex case with factor models and regressors and discuss inference on those scenarios.

4. Or, h could decay very mildly with N, as in the case of a fat-tailed distribution of sizes (Gabaix (2011)).

5. Hence, if $S_j = 1/N$ for all j, the variance $V(\gamma^\star)$ is of order $\mathcal{O}(1)$, and the estimator convergence rate much slower, as $1/\sqrt{T}$.

6. Suppose that $G_{ij} = S_j$ with $\sum_j S_j = 1$, so that S_j are influence shares. Call $h^2 = \sum_j S_j^2$ the Herfindhal. Then, $\sum_{i,j}\Gamma_{ij}^2 = (N-1)h^2$, while $\sum_{i,j}\Gamma_{ij}\Gamma_{ji} = \sum_{i\neq j} S_jS_i = 1 - h^2$. With large N and a fat-tailed distribution keep $Nh^2 \to \infty$, the first term is of larger order larger than the second one.

2.5 Discussion

In this section, we discuss how to use the algorithm in practice, and also its robustness.

2.5.1 Overidentifying restrictions

The model has many overidentifying restrictions: each individual entity idiosyncratic shock leads to a different moment. This implies that we could use such restriction to enrich the model when the model poorly approximate the data, following in spirit Gabaix and Koijen (2024), section III.C.

For example, one simple way is to decompose $z_{i,t}$ into subparts, e.g. $z_{i,t}^{(1)}$ which uses idiosyncratic shocks to the odd firms (say, heavy industry firms), and $z_{i,t}^{(2)}$ for even firms (say, light industry firms). Then, at convergence, we should have that regressing the generalization of (13):

$$\tilde{v}_{it}\left(\hat{\gamma}\right) = \Delta\gamma^{(1)} \cdot z_{i,t}^{(1)}(\hat{\gamma}) + \Delta\gamma^{(2)} \cdot z_{i,t}^{(2)}(\hat{\gamma}) + b_i \eta_t^{-i,\perp} + \varepsilon_{it}$$

should give both $\Delta\gamma^{(1)} = \Delta\gamma^{(2)} = 0$. Testing whether these coefficients are zero allows us to test whether we should consider heterogeneity in firms' characteristics in the construction of the parameters γ. Following similar approaches, one can see how to progressively enrich a model.

2.5.2 Connection to MLE

It is important to compare our procedure to MLE for two reason: (i) MLE is efficient under correct model specification; but (ii) MLE can be sensitive to misspecification of the model as discussed below. In Section B.2 of the online appendix we show that (pseudo) MLE that assumes homoskedastic and temporally uncorrelated shocks is equivalent to our procedure in the absence of unobserved factors and loadings. This result is suggestive that, despite the higher interpretability of our chosen procedure, our procedure retains efficiency when the true model does not have factors and unobservables are homoskedastic.

This comparison suggests that it is possible (and sometimes desirable) to use MLE in a

Network GIV by being explicit on the exogeneity assumptions of the model. For instance Baumeister and Hamilton (2023) propose a decomposition of the MLE with emphasis of the "granular IV" terms.

MLE has several advantages: it is a well-understood, widely-used procedure; it is easy to flexibly enrich, e.g. via measurement error; it comes with various tests.

However, we can think of three main advantages of our procedure for applications. The first one is interpretability and simplicity in the estimation, as the procedure is explicit on which exogeneity assumption (through shocks) drives identification. The second one is robustness, as our procedure requires weaker conditions on factor models compared to MLE. The third one is flexibility: as we better understand what drives identification, we can modify our procedure appropriately to incorporate information from the exogenous components only. To be more precise:

1. It is more robust when we have factors, and in particular, different from MLE, it is robust to the incidental parameter problem with e.g., time fixed effects. That is, if one interprets factors as fixed effects (which therefore do not require assumptions on the correlation of factors with covariates), in some common simple yet intuitive data generating processes, the MLE score function does not identify the parameter of interest Therefore, to achieve identification, MLE must posit assumptions on the correlation between factors and covariates (interpreting factors as correlated random effects). This is undesirable when such correlation is unknown (e.g., when unobserved economic shocks impact observed prices) whereas our method does not require assumptions on the correlation between factors and observable covariates. In particular, different from MLE, our identification strategy leverages independence of shocks to build moment restrictions that depend on the parameter of interest γ, whereas MLE builds moment restrictions that lack identification of γ. We provide formal details in Appendix B.2.

2. It is easier to interpret what drives identification: in the main case, this is idiosyncratic shocks to all (and in particular, large units). This follows in spirit to the discussion in Andrews, Kitagawa, and McCloskey (2024), who highlight the relevance of using

different sources of exogenous variation for identification of models. Our approach allows us to be explicit in this particular source of exogenous variation we leverage for identifications: independence between unit-specific unobservable shocks. But one can easily refine the identification to use only some of units. For instance, we might select "well-measured" countries to form z_{it} (just using the u_{jt} we trust), and we still have a valid instrument. For instance, in the simplest case, we form

$$z_{i,t} = e_i^\top B_\gamma A \sum_j v_{jt} 1_{j \in \mathcal{N}_{it}}$$

where $\mathcal{N}_{it}$ is a set of shocks we "trust", e.g. coming from countries in which we have very good data, or historical confirmation that we had indeed an idiosyncratic shock.[7] Likewise, we saw in Section 2.5.1 how to choose different "subsets of shocks" for the instruments.

3. Related to interpretability, one can interpret the instruments through a first stage and second stage (see the empirical application), which naturally provide with robustness checks.

In sum, our procedure is our arguably more, transparent, parsimonious and robust (for more complex models) to standard MLE procedures (while retaining efficiency guarantees in simple models).

3 A Macroeconomic Model

We illustrate the method via an empirical investigation of a multisector macroeconomic model. We present its essentials in this section, leaving the details to Appendix D. Unlike essentially all previous models, our model features potentially different elasticities of demands. For instance, the elasticity of demand for energy is likely to be lower than than for steel (which has, say, plastic as a decent substitute). We first present a static model, and

7. This is analogous to the "narratively checked" shocks in Gabaix and Koijen (2024).

then present its dynamic version, with autocorrelated shocks and slow price adjustment.

3.1 Notations

We call $X_{ij} = X_{i \leftarrow j}$ the absorption of "upstream" good j by "downstream" sector i. Upper cases are real quantities, lower cases are nominal quantities, e.g. $x_{ij} = p_j X_{ij}$ is the nominal purchase by i of good j, while X_{ij} is the real quantity. Hats denote log changes, $\hat{y}_t = \ln y_t - \ln y_{t-1}$. We call $D_\theta = \text{Diag}(\theta)$ the diagonal matrix with entries θ_i, where θ is a vector.

We define the "downstream" input-output matrix

$$A_{ij} = \frac{x_{ij}}{y_i} \tag{27}$$

which is the spending on j goods as a fraction of total gross output of downstream sector i, y_i; and the "upstream version" version

$$A^{\text{up}}_{ji} = \frac{x_{ij}}{y_j} \tag{28}$$

which is the fraction of j's output absorbed by downstream sector i. Hence, a we shall see, an expansion of downstream sector i by one percent increases the demand for an upstream sector j by A^{up}_{ji} percent, controlling for everything else. So $A^{\text{up}}_{ji} = A_{ij}\frac{y_i}{y_j}$, i.e.

$$A^{\text{up}} = D_y^{-1} A' D_y \tag{29}$$

We define the consumption basket shares: $\theta_i = \frac{c_i}{c} = \frac{p_i C_i}{\sum_j p_j C_j}$. We also define the fraction of i that goes to consumption as: $\Theta^c_i = \frac{c_i}{y_i}$,while $\Theta^y_j = 1 - \Theta^c_j = \sum_i \frac{x_{ij}}{y_j}$ is the share going to other production sectors. We also define the "share matrix": s^i_j is the fraction spent by i on intermediary goods j, out of all intermediary goods, so

$$s^i_j = \frac{x_{ij}}{\sum_k x_{ik}} = \frac{A_{ij}}{1 - b_i}, \qquad s = D^{-1}_{\iota - b} A \tag{30}$$

The shares add up to 1: $\sum_i s_i^j = 1$, i.e. $s\iota = \iota$. In particular, when $b_i = b$ for all i, $s_j^i = \frac{A_{ij}}{1-b}$, i.e. $s = \frac{1}{1-b}A$. We call $s_i^L = \frac{b_i y_i}{\sum_j b_j y_j}$ the share of total labor used by sector i.

All those were definitions and adding up constraints. We now start the economics.

3.2 Assumptions

Preferences and technology There are n goods, each produced by a different sector. We wish to have different elasticities for different goods. Hence, we adopt the heterogeneous-CES ("CRESH") of Hanoch (1971), and in the process derive some new relations in Appendix D. We start with the utility from consumption; given consumption $\boldsymbol{C} = (C_i)_{i=1...N}$, $v(\boldsymbol{C})$ is the solution of:

$$\sum_i \frac{\lambda_i \left(C_i / v\left(\boldsymbol{C}\right)\right)^{1-\frac{1}{\sigma_i}} - 1}{1-\frac{1}{\sigma_i}} + \lambda_0 = 0 \tag{31}$$

where σ_i,λ_i are parameters, positive for $i > 0$. Utility is then homogeneous of degree 1, and the Hicksian elasticity of demand for good i is then σ_i, in the case of a good with an infinitesimal share. In the general case of a finite number of goods, the relation is a more more complex, and Proposition 8) derives the Hicksian and Marshallian demand elasticities. When all σ_i are identical, and $\phi = 0$, then we have a CES utility function.

The production function is very similar: instead of a homothetic of degree 1 utility function $u(\boldsymbol{c})$, we have a CRS production function for inputs $f(\boldsymbol{X})$, where $\boldsymbol{X}$ is the vector of inputs. This $f^i(\boldsymbol{X}_i)$ is the solution of: of[8]

$$\sum_j \frac{\lambda_{ij} \left(x_{ij} / f^i\left(\boldsymbol{X}_i\right)\right)^{1-\frac{1}{e_j}} - 1}{1-\frac{1}{e_j}} + \phi_i = 0 \tag{32}$$

so that e_j is the "primitive" elasticity of demand by firms for good j. Sector i's production function is:

$$\ln Y_i = \ln z_i + b_i \ln L_i + d_i \ln \left[f^i\left(\boldsymbol{X}_i\right)\right]. \tag{33}$$

with $b_i + d_i \leq 1$. If $b_i + d_i < 1$ (respectively, $b_i + d_i = 1$), the elasticity of supply is finite

8. At a cost of parsimony, we could allow sector i's elasticity of demand e_{ij} for good j to depend on i.

(respectively infinite) as the production function is decreasing (respectively, constant) return to scale (see Section D.6).

The representative consumer has utility $u(\boldsymbol{C}, L) = v(\boldsymbol{C}) - \frac{L^{1-1/\zeta^{L,s}}}{1-1/\zeta^{L,s}}$, where $v(\boldsymbol{C})$ is the above consumption aggregator, homogeneous of degree 1, and $\zeta^{L,s}$ is the elasticity of labor supply.

Shocks and frictions We have supply shocks, $\delta_{it}^s = \ln z_{it}$. We also allow for demand shocks δ_{it}^d, which change the basket of goods demanded, for a given price and income (they can be viewed as shocks to the parameters λ_i in (31)). For the consumer, they mean that the change in demand is $\hat{c}_{it} = \hat{\delta}_{it}^d$, controlling for price and income. In addition, we also allow for demand shocks by firms. The total demand shocks for industry j is the sum of all demand shocks, by the consumers and other firms.

We also allow for a factor structure, i.e.

$$\Delta\delta_{it}^d = \lambda_i^d \eta_t + u_{it}^d \tag{34}$$

where η_t is a vector of factors (that could be autocorrelated in time), and u_{it}^d are idiosyncratic shocks. The idiosyncratic shocks can be autocorrelated through time, i.e. follow (for demand shocks; the supply shocks are analogous) $u_{it}^d = \rho_{ui}^d u_{it}^d + \varepsilon_{it}^{ud}$, where ε_{it}^{ud} is mean 0, uncorrelated across time-periods, and uncorrelated across $i \neq j$. However, we can allow for ε_{it}^{ud} and ε_{it}^{us} to be correlated, for a given i. But all the $\varepsilon_{it}^u, \varepsilon_{jt'}^u$ are uncorrelated for all t, t' and $i \neq j$.

Prices are frictional. In a given industry, only a fraction ϕ of the price adjusts per period.

3.3 Equilibrium of the model

We summarize the result of the model, from Proposition 13. We call p_{it} the price of good i, and $y_{it} = q_{it}p_{it}$ the value of good i produced, so that real growth is $\hat{y}_{it} - \hat{p}_{it}$.

Proposition 3. (Equilibrium in the frictional economy) *When prices have only a partial adjustment $\phi \in (0, 1]$ ($\phi = 1$ being the frictionless case) calling $\pi_t \coloneqq \hat{p}_t - \hat{p}_{t-1}$ the vector of*

sectoral price changes, we have:

$$\pi_t = \phi\left(A\pi_t - \Delta\delta_t^s + \left(\tilde{\zeta}^s\right)^{-1}\Delta\hat{y}_t + \tilde{\xi}^s\Delta\pi_{*t}\right) + (1-\phi)\,\pi_{t-1} \tag{35}$$

$$\Delta\hat{y}_t = A^{up}\Delta\hat{y}_t - \zeta^{d,y}\pi_t + \Delta\delta_t^d \tag{36}$$

In addition, the vector of the change in consumer expenditure satisfies:

$$\Delta\hat{c}_t = -\zeta^{d,c}\pi_t + \Delta\delta_t^c \tag{37}$$

where $\Delta\delta_t^c$ is the vector of demand shocks.

Here,

$$\zeta_{ij}^d = \frac{\partial \ln(p_i q_i)}{\partial \ln p_j} \tag{38}$$

is the matrix of Marshallian expenditure own- and cross-elasticities. The partial derivative keeps constant the total expenditure on goods (for consumers) and on intermediary inputs (for firms). We average over the demand q_i of consumers and of firms – this is detailed in Proposition 10. The proposition also details how it can be calculated from the primitive demand elasticities for good j: it depends on the consumer demand elasticity σ_j and of the firm elasticities e_j.

The matrix of supply elasticities is in (D.44), reproduced here:

$$\left(\tilde{\zeta}^s\right)^{-1} = (\zeta^s)^{-1} + \left(\zeta^L\right)^{-1} bs^{L\prime} \tag{39}$$

The first term, $(\zeta^s)^{-1}$, is the inverse elasticity of supply coming from scarce non-labor factors. We model is as a diagonal matrix; typically, for parsimony, with identical supply elasticity for all sectors, but we could relax that. The second term is the inverse elasticity coming from finite labor supply elasticity, where ζ^L is the supply elasticity of labor.

Equation (35) is the demand equation. It says that the demand for an input, $\Delta\hat{y}_{it}$, tends to increase: if there is an increase in demand of sectors that use that input, $(A^{\text{up}}\Delta\hat{y}_t)_i$; if the price of the inputs (or, complementary inputs) falls, as captured by the term $-\zeta^d\pi_t$, where

ζ^d is the matrix of the demand elasticities (and cross-elasticities); and if there is a positive demand shock $\Delta\delta^d_{it}$.

Equation (35) indicates that prices π_t change : because of price changes in the other sectors, which affect the marginal cost, as captured by $A\pi_t$ (and only a fraction ϕ is reflected on impact); because of changes in TFP, as per $\Delta\delta^s_t$; when output increases, the marginal cost $\left(\tilde{\zeta}^s\right)^{-1}\Delta\hat{y}_t$ increases due to a scarce resources, which are captured by the inverse supply elasticity $\left(\tilde{\zeta}^s\right)^{-1}$. Finally, the term $\tilde{\xi}^s\Delta\pi_{*t}$ is simply due to general inflation in the economy, and ensures long-run neutrality to inflation common to all sectors in the economy.[9]

3.4 Intuition for the sources of identification

Let us see the intuition for the identification, using Proposition 3. To build that intuition, we take the case without pricing frictions, $\phi = 1$. Then, economies are iid, and we do not use information from the time-series for identification.

Let us see how the demand elasticity by firms, $\zeta^{d,y}_{ij}$, is identified (in practice, the low-dimensionality of γ assumes that a number of elasticities will be identical, but it is useful to see, conceptually, how many could be separately identified).[10] From (36), applied to sector i $(\Delta\hat{y}_{it} - (A^{\text{up}}\Delta\hat{y}_t)_i = -\sum_j \zeta^{d,y}_{ij}\pi_{jt} + \Delta\delta^d_{it})$, we see that we need an instrument for π_{jt}. To build that instrument, we use the idiosyncratic supply shocks (and, quantitatively less powerful, the demand shocks) for industries $k \neq i$ (that is where the factor model "purifies" shocks, extracting the idiosyncratic shocks). From (35), they will change the price in industry j, for all j, via the term $(A\pi_t)_j$ —- essentially, though on top of that there are extra general equilibrium effects. As a result, they will be instruments for the term π_{jt} in (36). For instance, as steel and energy are important inputs to make cars, idiosyncratic productivity shocks in the steel and energy sectors (sector k's) are instrument for the price of cars (sector j), which allows to estimate the elasticity of demand for cars by other sectors

9. It is spelled out in (D.38), (D.40), and (D.45).

10. When the model specification assumes a common elasticity e_j to all sectors j, then the econometric procedure aggregates all those moments.

i.[11] Now, remember that we assume (for parsimony) that all industries i have the same demand elasticity e_j for sectors j. This way, we can identify e_j.

To identify the demand elasticity by consumers, $\zeta_j^{d,c}$ (which is very close to σ_j, up to corrective terms specified in 8), we use the consumer's equation (37), and we use the same instrument as a price shifter for the price of good j. The inverse elasticity of supply $\left(\tilde{\zeta}^s\right)^{-1}$ indicates (in its entry ij) how much an increase of production in a sector j affects the marginal cost in sector i (because $i = j$, or perhaps because i, j use common inputs), as (35) shows. To identify it, we likewise use idiosyncratic shocks to supply and demand in all industries $k \neq i$; they affect the total output in sector j, and we can identify the ij entry of $\left[\left(\tilde{\zeta}^s\right)^{-1}\right]$. Staying with the example of the supply elasticity of cars (sector j), a number of downstream sectors (e.g. retail distribution, and construction), use cars. Supply and demand shocks in those downstream sectors will affect the demand of cars, and their idiosyncratic components will be an instrument for the demand for cars, $\Delta\hat{y}_{jt}$. Seeing how that affects the price of cars, we get to estimate a component of $\left(\left(\tilde{\zeta}^s\right)^{-1}\right)_{jj}$; more generally, we can identify how the marginal cost of other sectors i is affected by the increased quantity of cars (instrumented as we just saw), hence identify supply cross-elasticities $\left(\left(\tilde{\zeta}^s\right)^{-1}\right)_{ij}$

The speed $1-\phi$ of price changes is easier to estimate, via (35). Idiosyncratic supply and demand shocks of sectors $k \neq i$ affect the prices j of industries, hence the marginal cost of industry i by the i-th component of $\phi A \pi_t$. As we know π_t and A, we can infer ϕ. Another sources of identification could be via the impact of lagged value $(1-\phi)\,\pi_{t-1}$. The model uses both sources of identification. As always, one could use only one source of identification, and obtain overidentification restrictions, and then refine the model, e.g. making parameters vary across industries.

11. A comparison, to identify the elasticity of demand by firm i, the one-industry GIV of Gabaix and Koijen (2024) uses the idiosyncratic demand shocks of of the other firms $k \neq i$, which affect the price, hence create a valid instrument. But the advantage here is that we can use information across many sectors.

4 Empirical Results

[Preliminary. We'll update, in particular as we update the model to introduce capital, in a next iteration]

4.1 Data

Our data set consists of an annual panel of 277 industries over the period 1964-2018. The main data source is Bureau of Economic Analysis (2023). Data on industry gross output value and price indexes, final "consumption" expenditure (which also includes investment and exports), and labor share come directly from these accounts. We deflate all price indexes using the GDP price index and demean the output and price series before the estimation.

We construct the input-output matrix A as follows. The "Use" table U reports the the dollar amount of each commodity used by each industry. For years starting in 2007, Bureau of Economic Analysis 2023 also provides an import use table U^M; for years before 2007, we provisionally fix the import share of commodity c used in industry j to its 2007 level and then scale up or down all imports of c to match aggregate imports of c as reported in U. To convert use of commodity c into use of output of industry i, we assume that the share of commodity c that j purchases from i is equal to the share of total commodity output of c produced by industry i. We normalize the make table M so that entry i, c gives the fraction of commodity c produced by i. We then define the domestic input-output matrix $A = M\left(U - U^M\right)$. We fix A at its 2007 value for our baseline estimation and report robustness for other years; in practice the input-output network is sufficiently stable that the year matters little to the results..

4.2 Parameter Identification

As a preliminary step, we estimate parameters imposing homogeneity in the elasticities across all sectors. As explained in Section A.2, we control for time fixed effects in the estimation algorithm. Since the model examines two observable outcomes per unit - price and output

Table 1: Baseline Estimated Parameters

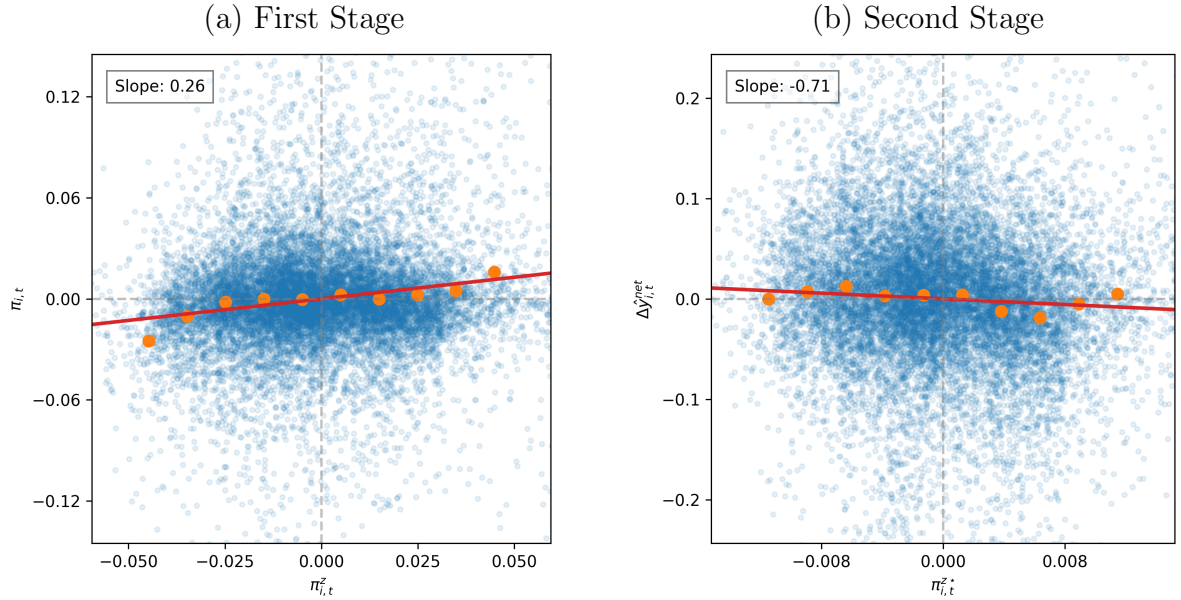
Description	Symbol	Value
Elasticity of substitution of final demand	σ	0.33
Elasticity of substitution of intermediate inputs	e	0.47
Inverse supply elasticity from fixed factors	$1/\zeta^s$	0.13
Supply elasticity from labor	$\zeta^{L,d} + \zeta^{L,s}$	0.44
Sluggish adjustment	ϕ	0.8

- two factors are required to capture time fixed effects. Additionally, we include a third unrestricted factor and allow for one autocorrelation term in the factor block. Table 1 reports the parameter values.

The elasticities of substitution are both well below one, with the intermediate goods elasticity slightly exceeding the final demand elasticity. The estimated supply elasticity coming from a fixed factor is large, suggesting that this source of decreasing returns is not key to fit the data. Instead, it generates a modest amount of upward-sloping supply from the labor market. The sluggish adjustment parameter means that about 80% of inflation comes from contemporaneous determinants and 20% from lagged inflation.

Figure 1 provides intuition for the demand elasticity following the discussion in Section 3.4. There we noted that heuristically the GIV procedure both generates excluded instruments from the inflation and output system (35) and (36) and finds parameter values given those instruments. In the case of the demand elasticity ζ^d, equation (36) can be rewritten as $(I - A^{up})\,\Delta\hat{y}_t = -\zeta^d \pi_t + \Delta\delta_t^d$. The "excluded instrument" for $\pi_{i,t}$, which we denote $\pi_{i,t}^z$, is the value of inflation given only the history of idiosyncratic supply and demand shocks to other sectors (and the estimated parameter values to optimally combine those shocks into a predictor for π_t). This instrument corresponds to expression (11) in Step 4 of Algorithm 1, $\tilde{y}_{t|-i}(\gamma^*)$. Panel (a) shows a positive, although not one-to-one, correlation between $\pi_{i,t}^z$ and actual inflation. This illustrates how the estimation procedure is able to recover the structural parameters of interest, even when noise attenuates the correlation between observed and instrumented outcomes. In this figure, each blue dot represents one sector-year observation and the large orange dots show the means of the y-axis variable for bins of the x-axis variable

Figure 1: Heuristic Identification of demand elasticity ζ^d

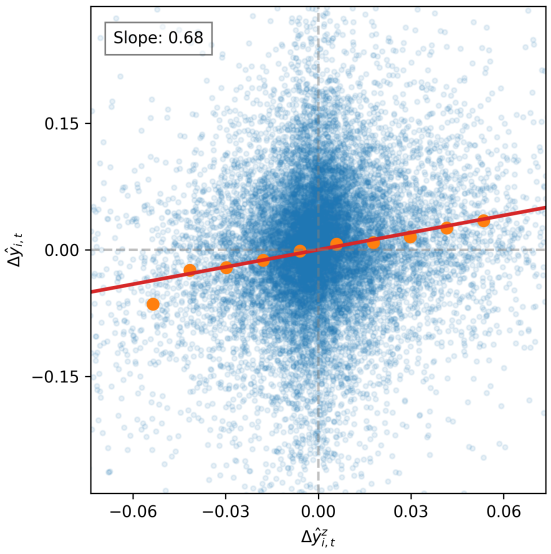


(the "bin scatter"). Panel (b) shows the relationship between $\Delta\hat{y}_{i,t}^{net} \equiv e_i' \left(I - A^{up}\right) \Delta\hat{y}_t$ and $\pi_{i,t}^{z\star}$, the fitted value for inflation from the first stage (that is, $\pi_{i,t}^{z\star} \equiv \beta^d \pi_{i,t}^{z}$, where $\beta^d$is the slope in Panel (a)). The slope of the second-stage regression equals $-\zeta^d$, the Marshallian elasticity of demand expressed in value (which is $1-\bar{\sigma}$, where $-\bar{\sigma}$ is the elasticity of demand in quantity, which is a weighted average of the elasticity from consumers, $-\sigma$, and from firms, $-e$). To further separate ζ^d into the elasticities of final demand and intermediate inputs, equation (D.34) shows that it is a weighted average of $\sigma - 1$ and $e - 1$ with weights given by the share of demand of sector i coming from final and intermediate demand. The slope in Panel (b) does not exactly equal the weighted average elasticity, because in the full GIV estimation this is one of several jointly estimated parameters.

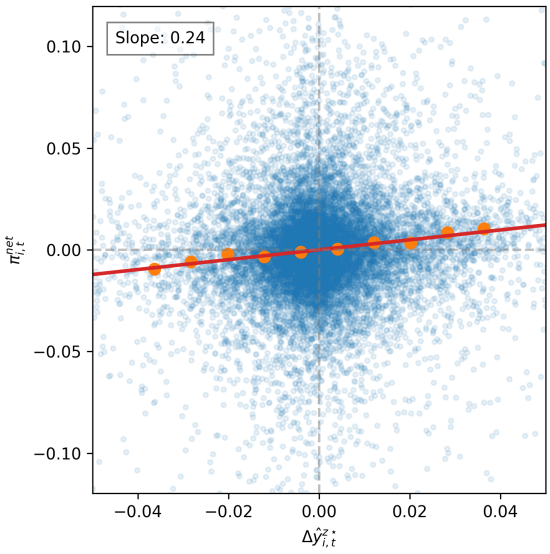
Figure 2 provides the corresponding plots for the inverse supply elasticity. Estimating this parameter requires an instrument for nominal output growth that is excluded from supply equation (35), which we again obtain by combining the idiosyncratic shocks to each other sector. Panel (a) shows that the excluded instrument $\Delta\hat{y}_{i,t}^{z}$ predicts actual output growth. From (36), the relationship between $\pi_{i,t}^{net} \equiv e_i' \left(\left(I - \phi A\right)\pi_t - \left(1-\phi\right)\pi_{t-1}\right)$ and the fitted value from Panel (a), $\Delta\hat{y}_{i,t}^{z\star}$, yields $\left(\tilde{\zeta}^s\right)^{-1}$, the composite elasticity of supply. Panel

Figure 2: Heuristic Identification of inverse supply elasticity $\left(\tilde{\zeta}^{s}\right)^{-1}$

(a) First Stage



(b) Second Stage



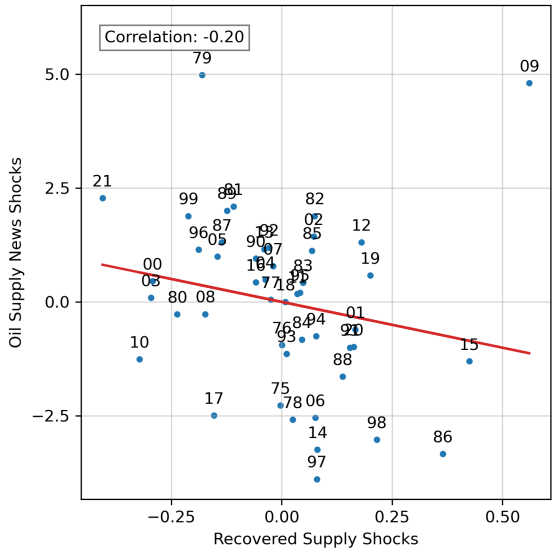
(b) shows this fit.

4.3 External Validation

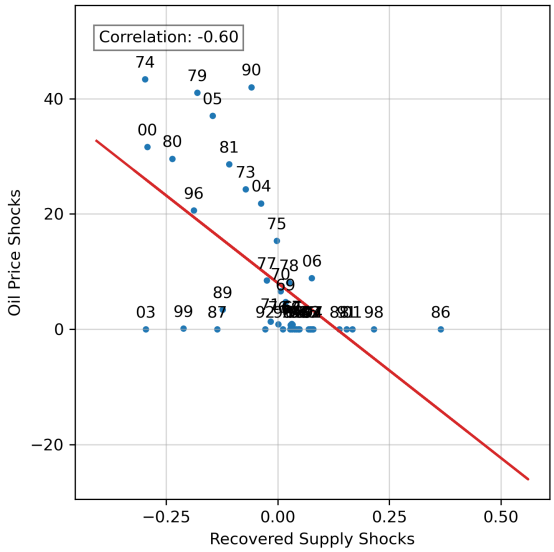
An important advantage of the GIV algorithm is recovery of supply and demand shocks for all sectors and years. We can compare these shocks to existing estimates of supply or demand shocks for individual sectors. Figure 3 compares the oil supply shocks to those from Känzig (2021) and from Hamilton (2003). The series correlate negatively, since the Känzig and Hamilton series are shocks to prices. In addition, the Hamilton methodology by construction only generates negative supply shocks. The series agree on major events such as the 1979 oil price hike and (in the case of Känzig) the positively supply shock due to the 1986 collapse of OPEC. The largest difference is 2009, which may partly reflect the divergence that year between the reduction in production in OPEC countries and the increase in production in the United States.

Figure 3: Comparison of Oil Supply Shocks

(a) Oil Supply News Shocks - Känzig (2021)



(b) Oil Price Shocks - Hamilton (2003)



5 Conclusion

We proposed a method to identify causal parameters in fairly general economies that have a network structure: the network granular instrumental variable, or network GIV. The general idea, and method, could be applied to a host of other setups, for instance for the world economy, aggregate financial markets with many assets, and more localized industries with imperfect competition. In all those cases, idiosyncratic shocks in one sector imply reaction of all the other sectors, in a way specified by the theory; if that theory is low-dimensional enough, estimation of the parameters is typically possible. In addition, there is generally a clear intuition for the way each parameter is estimated. As we saw, the procedure admits many variants that are empirically relevant, including autocorrelated shocks and lead-lag structures. We hope that the network GIV method will help economists explore a large number of causal relations in old and novel datasets.

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A Algorithmic Details for the Econometric Procedure

A.1 Heteroskedastic PCA

In this section we present the algorithm to estimate unobserved loadings in the presence of heteroskedastic residuals. We consider a model of the form

$$v_t = \lambda \eta_t + \Sigma_\varepsilon^{1/2} \varepsilon_t, \quad \mathbb{E}[\varepsilon_t^2] = \sigma_\varepsilon^2, \quad \mathbb{E}[\varepsilon_t | \eta_t] = 0,$$

where Σ_ε is a diagonal matrix capturing heteroskedastic residuals. We consider two competing algorithms for heteroskedastic PCA. The first algorithm is an alternating optimization algorithm, while the second one was first developed by Zhang, Cai, and Wu (2022), and consists of solving the equivalent of a projected gradient descent method that imputes the diagonal entries of the covariance matrix of the residuals, and then perform PCA on the remaining component.

Algorithm 2 Heteroskedastic PCA (Version 1): estimate the loadings from a vector of heteroskedastic residuals $v_t$via PCA

Require: Vector v_t of residuals, number of iterations I, number of components r

1: Initialize by setting $\Sigma_\varepsilon^0 = \mathbb{I}$, $\hat{\Sigma}_{v,0} = \frac{1}{T} \sum_t (\Sigma_{\varepsilon,0}^{-1/2} v_t)(\Sigma_{\varepsilon,0}^{-1/2} v_t)'$

2: **for** $k \in \{0, 1, \cdots, I\}$ **do**

3: Do Singular Value Decomposition (SVD) of $\hat{\Sigma}_{v,k}$ and retrive the r largest eigenvectors denoted as $\tilde{\lambda}^k$

4: Compute $\eta_t^k = \tilde{\lambda}^{k\top} \Sigma_{\varepsilon,k}^{-1/2} v_t$ (where $\tilde{\lambda}^{k\top} \tilde{\lambda}^k = \mathbb{I}$)

5: Compute $\hat{\lambda}^k = \Sigma_{\varepsilon,k}^{1/2} \tilde{\lambda}_k$

6: Compute

$$\Sigma_{\varepsilon,k,i,j} = \begin{cases} \frac{1}{T} \sum_t (v_{i,t} - \hat{\lambda}_i^k \eta_t^k)^2 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

7: **end for**

8: **return** $\hat{\lambda}^I, \Sigma_{\varepsilon,I}$

Algorithm 3 Heteroskedastic PCA (Zhang, Cai, and Wu 2022) (Version 2)

Require: Vector v_t, number of iterations I, number of components r

1: Define the covariance matrix $\hat{\Sigma}_v = \frac{1}{T}\sum_t v_t v_t'$

2: Initialize by setting $N^0 = \hat{\Sigma}_v, \text{diag}(N^0) = \mathbf{0}$

3: **for** $k \in \{0, 1, \cdots, I\}$ **do**

4: Perform SVD on N^k and denote $\tilde{N}^k$ its best best rank-r approximation

5: Update

$$N_{i,j}^{k+1} = \begin{cases} \tilde{N}_{i,j}^k & \text{if } i = j \\ \hat{\Sigma}_{i,j} & \text{otherwise} \end{cases}$$

6: **end for**

7: **return** The first r singular vectors of N^I defined as $\hat{\lambda}$ and the variances of the residuals as

$$\Sigma_\varepsilon = \text{diag}(\hat{\Sigma}_v) - \text{diag}(N^I)$$

A.2 Time fixed effects via demeaning

As mentioned in Step 2 of Algorithm 1, we can control for time fixed effects in the estimation procedure by applying PCA to the demeaned residuals $\check{v}_t$.

When there is just one outcome per entity In the simple case where, for each unit $i = 1, \ldots, N$, we observe a single outcome (that is, in our notation, $q = 1$), we do the following:

1. For each i, t, define the demeaned residuals $\check{v_{it}}(\hat{\gamma}) \coloneqq \tilde{v_{it}}(\hat{\gamma}) - \tilde{v}_{Et}(\hat{\gamma})$, where $\tilde{v}_{Et}(\hat{\gamma})$ is the mean outcome in the time-t cross-section;

2. Run PCA with heteroskedasticity using Algorithm 2 passing $\check{v}_t$ as the argument and r as the number of factors. The PCA algorithm iterates to estimate loadings $\check{\lambda}(\hat{\gamma})$ and factors $\check{\eta}_t(\hat{\gamma})$;

3. Add a N-dimensional vector of ones to the estimated loadings, $\tilde{\lambda}(\hat{\gamma}) = \left(\iota, \check{\lambda}(\hat{\gamma})\right)$ where $\iota = (1, \ldots, 1)$, and add the residuals' means to the estimated factors, $\tilde{\eta_t}(\hat{\gamma}) = (\tilde{v}_{Et}(\hat{\gamma}), \check{\eta}_t(\hat{\gamma}))$. In this way, we effectively consider $r + 1$ factors, with the first factor capturing a time fixed effect.

General case, with potentially more than one outcome per entity In general, we may have $q \geq 1$ observable outcomes per unit – for instance, in our macroeconomic application, wit have $q = 2$ observables: prices and quantities. When $q = 2$, the procedure becomes as follows (the generalization to all q is natural):

1. For each i, t, demean the two outcomes separately $\check{v}_{it}^{k}(\hat{\gamma}) \coloneqq \tilde{v_{it}^{k}}(\hat{\gamma}) - \tilde{v}_{Et}^{k}(\hat{\gamma})$ for $k = 1, 2$, where $\tilde{v}_{Et}^{k}(\hat{\gamma})$ is the mean for outcome k in the time-t cross-section;

2. Combine the two vectors of demeaned residuals $\check{v}_t(\hat{\gamma}) = (\check{v}_t{}^1(\hat{\gamma}), \check{v}_t{}^2(\hat{\gamma}))$, and apply heteroskedastic PCA to $\check{v}_t(\hat{\gamma})$. Algorithm 2 returns the estimated loadings $\check{\lambda}(\hat{\gamma})$ and the estimated factors $\check{\eta}_t(\hat{\gamma})$;

3. Add two qN-dimensional vectors of ones and zeros to the estimated loadings, $\tilde{\lambda}(\hat{\gamma}) = \left(\iota^1, \iota^2, \check{\lambda}(\hat{\gamma})\right)$, where $\iota^1 = (1, \ldots, 1, 0, \ldots 0)$ and $\iota^2 = (0, \ldots, 0, 1, \ldots 1)$, and add the two residuals' means to the estimated factors, $\tilde{\eta}_t(\hat{\gamma}) = (\tilde{v}_{Et}^{1}(\hat{\gamma}), \tilde{v}_{Et}^{2}(\hat{\gamma}), \check{\eta}_t(\hat{\gamma}))$. In this way, we effectively consider $r + 2$ factors, with the first two capturing a time fixed effect.

Online Appendix for "Propagation of shocks in networks: identification and applications"

Gabriel Chodorow-Reich, Xavier Gabaix, Ralph S.J. Koijen and Davide Viviano

March 19, 2025. Preliminary and Incomplete.

B Complements to the Econometric Procedure

B.1 Proof of the main results

B.1.1 Proof of Proposition 1

We can write $z_{it}(\gamma^\star) = e_i^\top B_\gamma(\gamma^\star)A(\gamma^\star)u_t^{-i}$, where $u_t^{-i} = u_t - e_i u_{it}$.Therefore the result follows directly under the assumption that $E[u_{it}|u_{jt}{}_{j\neq i}] = 0$.

B.1.2 Proof of Lemma 1

Recall that we can write $z_{it}^{(k)}(\gamma^\star) = e_i^\top \Gamma^{(k)} u_t$. Therefore, we can write

$$\begin{aligned} V_{k,k'}^z &= \sum_i \mathbb{E}[z_{it}^{(k)}(\gamma^\star) z_{it}^{(k')}(\gamma^\star)] = \sum_i \mathbb{E}\left[\left(\sum_j \Gamma_{ij}^{(k)} u_{jt}\right)\left(\sum_{j'} \Gamma_{ij'}^{(k')} u_{j't}\right)\right]. \\ &= \sum_i \sum_j \Gamma_{ij}^{(k)} \Gamma_{ij}^{(k')} \sigma_{u_j}^2 \end{aligned}$$

B.1.3 Proof of Proposition 2

The asymptotic normality in Proposition 2 follows directly from standard arguments for extremum estimators (Newey and McFadden 1994). Here, we are left to show the exact expression for the variance.

From standard properties of the methods of moments estimator with exact identification, we can write the variance of the estimator as $\bar{V} = \Big(N\bar{\Delta}\Big)^{-1}$ where $\Sigma = E[\Big(\sum_i H_{it}(\gamma^\star)\Big)\Big(\sum_i H_{it}(\gamma^\star)\Big)^\top]$ and $N\bar{\Delta} = -\sum_i \frac{\partial E[H_{it}(g)]}{\partial g}|_{g=\gamma^\star}$.

Denote $\sigma_i^2 = \mathbb{E}[u_i^2]$, and $H_{i,t}(\gamma, g) = \Big(Y_{i,t} - e_i^\top B(\gamma) Y_t\Big) e_i^\top \frac{\partial B(\gamma)}{\partial \gamma} y_{t|-i}(\gamma)$, where $y_{t|-i}(\gamma) = A(\gamma)[u_t - e_i u_{i,t}]$. Also, recall the definition of $G^{(k)} = \frac{\partial B(g)}{\partial g_k}\Big|_{g=\gamma^\star} A(\gamma^\star)$. First observe that because under Assumption 1, $Y_t = A(\gamma^\star)u_t$, we can write

$$\frac{\partial\Big(Y_{i,t} - e_i^\top B(\gamma) Y_t\Big)}{\partial \gamma} = -e_i^\top \frac{\partial B(\gamma)}{\partial \gamma} A(\gamma) u_t,$$

and therefore

$$\begin{aligned}\frac{\partial y_{t|-i}(\gamma)}{\partial \gamma_k} &= \frac{\partial A(\gamma)\Big[Y_t - B(\gamma)Y_t - e_i\Big(Y_t - e_i^\top B(\gamma) Y_t\Big)\Big]}{\partial \gamma_k} \\ &= -A(\gamma)(\mathbb{I} - e_i e_i^\top)\frac{\partial B(\gamma)}{\partial \gamma_k} A(\gamma) u_t + \frac{\partial A(\gamma)}{\partial \gamma_k}\Big[u_t - e_i u_{i,t}\Big].\end{aligned}$$

Consider the component $\Delta^{(p_1,p_2)}$ first. We can write

$$\begin{aligned}-\frac{\partial \mathbb{E}[H_{i,t}^{(p_1)}(\gamma,\gamma)]}{\partial \gamma_{p_2}}\Big|_{\gamma=\gamma^\star} &= \mathbb{E}\Big[e_i^\top G^{(p_2)} u_t e_i^\top G^{(p_1)}[u_t - e_i u_{i,t}]\Big] + \mathbb{E}\Big[u_{i,t} e_i^\top G^{(p_1)}(\mathbb{I} - e_i e_i^\top) G^{(p_2)} u_t\Big] \\ &= \sum_{j\neq i} G_{j,i}^{(p_1)} G_{i,j}^{(p_2)} \sigma_j^2 + \sum_{j\neq i} G_{j,i}^{(p_1)} G_{i,j}^{(p_2)} \sigma_i^2\end{aligned} \tag{B.1}$$

where we used the fact that the remaining components of the derivative are mean zero (including the component involving the second order derivative of $B(\gamma)$). Using the fact

that $\Gamma_{i,i} = 0$, and homoskedasticity, where we write $\sigma_i^2 = \sigma^2$, it follows that

$$-\sum_{i=1}^{N} \frac{\partial \mathbb{E}[H_{i,t}^{(p_1)}(\gamma, \gamma)]}{\partial \gamma_{p_2}}\Big|_{\gamma=\gamma^\star} = \sigma^2 \operatorname{tr}(\Gamma^\top \Gamma + \Gamma^2).$$

Finally, because

$$\sum_{i=1}^{N} H_{i,t}^{(k)}(\gamma^\star, \gamma^\star) = u_t^\top \Gamma^{(k)} u_t,$$

we can write the (k, k') entry as

$$E[u_t^\top \Gamma^{(k)} u_t u_t^\top \Gamma^{(k')} u_t] = E[\sum_i u_{i,t} \sum_j \Gamma_{j,i}^k u_{j,t} \sum_v u_{v,t} \sum_l \Gamma_{v,l}^{k'} u_{l,t}].$$

Now using independence across unit i and using the fact that $\Gamma_{i,i} = 0$ for all i we can simplify the expression as follows: first, we note that for for any $i \neq j$, $E[u_i u_j] = 0$, $E[u_i^2] = \sigma^2$,and $E[u_i^2 u_j^2] = \sigma^4$. Thus, $E[u_i u_j u_v u_l] \neq 0$ precisely when the indexes are split into two matching pairs. The possible matching are $i = j, v = l, i = v, j = l, i = l, j = v$. However, because $\Gamma_{i,i} = 0$ for all i any term involving $i = j$ is automatically zero in the sum and similarly for $v = l$. Consequently, the only pairings that survive are $i = v \neq j = l, i = l \neq j = v$. Let's consider the first case. Then the sum simplifies to $\sum_{i \neq j} \Gamma_{j,i}^k \Gamma_{j,i}^{k'} \sigma^4$. Let's consider the second case. Then the sum simplifies to $\sum_{i \neq j} \Gamma_{i,j}^k \Gamma_{j,i}^{k'} \sigma^4$ providing us with the desired expression:

$$E[u_t^\top \Gamma^{(k)} u_t u_t^\top \Gamma^{(k')} u_t] = \sigma^4 \left(\sum_{i,j} \Gamma_{i,j}^k \Gamma_{j,i}^{k'} + \Gamma_{j,i}^k \Gamma_{j,i}^{k'} \right)$$

B.2 Link with the MLE

We organize this subsection as follows: first we show the equivalence with MLE in the absence of a factor model. This proves our result in Corollary 1. Second, we show lack of identification of the MLE score equation, proving our claim that MLE fails to identify the parameter $\gamma^\star$ in a simple model where Net-GIV identifies $\gamma^\star$ using alternative moment conditions.

Proposition 4. (Link with MLE) *Suppose the model with no factors and no lags,* $(I - B(\gamma)) Y_t = u_t$. *Define*

$$G(\gamma) \coloneqq B_\gamma(\gamma) A(\gamma), \tag{B.2}$$

and

$$\Gamma_{ij} = G_{ij} 1_{i \neq j} \tag{B.3}$$

so that Γ *gives the network impacts "removing self-impacts", as shown by the fact that* $\Gamma_{ii} = 0$*; and the inferred* $u_t(\gamma) \coloneqq B(\gamma) y_t$. *Assume that the* σ_{u_i} *are unknown and need to be estimated. With* L *the log likelihood, the MLE estimator corresponds to the following moment condition:*

$$L_\gamma = 0, \qquad L_\gamma = \sum_t u_t'(\gamma) \Gamma(\gamma) u_t(\gamma) \tag{B.4}$$

In addition, the log likelihood satisfies:[12]

$$\mathbb{E} L_{\gamma\gamma} = -2T \operatorname{tr}\left(\Gamma\Gamma' + \Gamma^2\right) \tag{B.6}$$

which gives the efficiency of the MLE estimator :

$$\sqrt{T}\left(\gamma_T^{e,MLE} - \gamma\right) \sim N(0, V^\gamma), \qquad V^\gamma = \left(\operatorname{tr}\left(\Gamma\Gamma' + \Gamma^2\right)\right)^{-1} \tag{B.7}$$

Proof of Proposition 4. We take the case without factors and controls first, so $y_t = A(\gamma) u_t$. We write $u_{it} = \sigma_{u_i} v_{it}$, where v_{it} has variance 1. Call $V_u = Diag\left(\sigma_{u_i}^2\right)$ and $W_u = V_u^{-1}$. So, $u_t = (I - B) y_t$. The log likelihood L is given by:

$$\begin{aligned}
\frac{1}{T} L &= -\frac{1}{2}\frac{1}{T}\sum_t y_t' (I - B)' W^u (I - B) y_t + \frac{1}{2}\ln\det\left((I - B)' W^u (I - B)\right) - \frac{1}{2} N \ln 2\pi\sigma_u^2 \\
&= -\frac{1}{2}\frac{1}{T}\sum_t y_t' (I - B)' W^u (I - B) y_t + \ln\det\left((I - B)\right) - \frac{1}{2}\ln\det W^u - \frac{1}{2} N \ln 2\pi\sigma_u^2
\end{aligned}$$

12. Very explicitly, with $G^k = B_{\gamma_k} A$ a $N \times N$ matrix,

$$-\mathbb{E} L_{\gamma_k \gamma_\ell} = -2T \operatorname{tr}\left(\Gamma^k \Gamma^{\ell\prime} + \Gamma^k \Gamma^\ell\right) \tag{B.5}$$

As $W_u = \sum_i e_i e_i' \sigma_{u_i}^{-2}$, optimizing $\frac{1}{T}L$ over the variance $\sigma_{u_i}^{-2}$ gives

$$0 = \frac{2}{T}\frac{d}{d\sigma_{u_i}^{-2}}L = \frac{1}{T}\sum_t u_t' e_i e_i' u_t - \sigma_{u_i}^2$$

i.e. the MLE of variance is simply

$$\sigma_{u_i}^2 = \frac{1}{T}\sum_t u_{it}^2 \tag{B.8}$$

We observe that

$$\frac{d}{d\gamma}\left(I - B\right) y_t = -B_\gamma y_t = -B_\gamma A u_t = -G u_t$$

so

$$\frac{d}{d\gamma}\left(-\frac{1}{2} y_t' \left(I - B\right)' W^u \left(I - B\right) y_t\right) = \frac{1}{2}\left(u_t' W^u G u_t + u_t' W^u G u_t\right) = u_t' W^u G u_t$$

Also,

$$\frac{d}{d\gamma}\ln\det\left(I - B\right) = -\operatorname{tr}\left(B_\gamma \left(I - B\right)^{-1}\right) = -\operatorname{tr} G$$

so

$$\frac{1}{T}L_\gamma = \frac{1}{T}\sum_t u_t' W^u G u_t - \operatorname{tr} G \tag{B.9}$$

Now, write $u_t = u_t^{-i} + u_{it} e_i$, where $u_t^{-i} = \left(u_{jt} 1_{j \neq i}\right)_{j=1\ldots N}$. Then,

$$\begin{aligned} u_t' W^u G u_t &= u_t' \left(\sum_i e_i e_i' \sigma_{u_i}^{-2}\right) G u_t = \sum_i u_t' e_i e_i' \sigma_{u_i}^{-2} G \left(u_t^{-i} + u_{it} e_i\right) \\ &= \sum_i u_{it} \sigma_{u_i}^{-2} e_i' G u_t^{-i} + \sum_i u_{it} \sigma_{u_i}^{-2} e_i' G u_{it} e_i \\ &= \sum_i u_{it} \sigma_{u_i}^{-2} e_i' G u_t^{-i} + \sum_i G_{ii} u_{it}^2 \sigma_{u_i}^{-2} \end{aligned}$$

Hence, calling $z_{it} = \sigma_{u_i}^{-1} e_i' G u_t^{-i}$ and $\tilde{v}_{it} = \sigma_{u_i}^{-1} u_{it}$

$$\begin{aligned} -\frac{1}{T}\sum_t u_t' W^u G u_t + \operatorname{tr} G &= \frac{1}{T}\sum_t \sum_i u_{it}\sigma_{u_i}^{-2} e_i' G u_t^{-i} - \sum_i G_{ii}\left(\frac{1}{T}\sum_t u_{it}^2 \sigma_{u_i}^{-2}\right) + \operatorname{tr} G \\ &= \frac{1}{T}\sum_{t,i} \tilde{v}_{it} z_{it} - \sum_i G_{ii} + \operatorname{tr} G \\ &= \frac{1}{T}\sum_{t,i} \tilde{v}_{it} z_{it} \end{aligned}$$

So, we have

$$\frac{1}{T} L_\gamma = \frac{1}{T}\sum_{t,i} \tilde{v}_{it} z_{it} \tag{B.10}$$

which is the moment in Algorithm 1, provided we normalize by $\frac{1}{\sigma_{u_i}}$.

Now, let us study the higher order terms. Calling Γ the G matrix without diagonal terms,

$$\Gamma_{ij} = G_{ij} 1_{i \neq j} \tag{B.11}$$

we have

$$\frac{1}{T} L_\gamma = \frac{1}{T}\sum_i u_t' W_u \Gamma u_t \tag{B.12}$$

So we have

$$\mathbb{E}\frac{1}{T} L_{\gamma, W_{ii}^u} = \mathbb{E} u_{it} e_i' \Gamma u_t = 0$$

Hence, to calculate the variance of the estimator γ, we just need to consider $L_{\gamma\gamma}$. As $u_t = (I - B)\, y_t$, we have

$$\frac{d}{d\gamma} u_t = -B_\gamma y_t = -B_\gamma A y_t = -G y_t$$

hence

$$\begin{aligned} \mathbb{E}\frac{1}{T} L_{\gamma\gamma} &= \mathbb{E}\left[\frac{d}{d\gamma} u_t' W^u \Gamma u_t + u_t' W^u \Gamma \frac{d}{d\gamma} u_t + u_t' W^u \Gamma_\gamma u_t\right] \\ &= -\mathbb{E}\left[u_t' G' W^u \Gamma u_t + u_t' W^u \Gamma G u_t\right] + \operatorname{tr}\left(V^u W^u \Gamma_\gamma\right) \end{aligned}$$

As $\mathrm{tr}\left(\Gamma\left(\gamma\right)\right)=0$ for all γ, $\mathrm{tr}\,\Gamma_{\gamma}=0$. So, we have

$$-\mathbb{E}\frac{1}{T}L_{\gamma\gamma}=\mathrm{tr}\left(V^{u}G'W^{u}\Gamma+V^{u}W^{u}\Gamma G\right)=\mathrm{tr}\left(V^{u}G'W^{u}\Gamma+\Gamma G\right)$$

But, as $\Gamma_{ii}=0$, we have

$$\mathrm{tr}\left(V^{u}G'W^{u}\Gamma\right)=\sum_{i,j}V_{ii}^{u}G'_{ij}W_{jj}^{u}\Gamma_{ji}=\sum_{i,j}V_{ii}^{u}\Gamma'_{ij}W_{jj}^{u}\Gamma_{ji}=\mathrm{tr}\left(\Gamma'\Gamma\right)$$

as the term $G'_{ii}\Gamma_{ii}=0$. Likewise

$$\mathrm{tr}\left(\Gamma G\right)=\sum_{ij}\Gamma_{ij}G_{ji}=\sum_{ij}\Gamma_{ij}\Gamma_{ji}=\mathrm{tr}\left(\Gamma^{2}\right)$$

So,

$$-\mathbb{E}\frac{1}{T}L_{\gamma\gamma}=\mathrm{tr}\left(\Gamma\Gamma'+\Gamma^{2}\right) \tag{B.13}$$

which gives the efficiency of the MLE estimator, $Tvar\left(\gamma^{e}-\gamma\right)\sim\left(-\frac{1}{T}\mathbb{E}L_{\gamma\gamma}\right)^{-1}$.□

B.2.1 Example where MLE lacks identification

In the next lines we provide a simple example illustrating that, however, in simple yet more complex models with simple factor structures as for examples those formally studied below in Section B.3, our procedure identifies the parameter of interest whereas MLE might fail. The main reason of this failure is because MLE will try to recover both the factors and the parameters of interest. However, the factor may not be identified, whereas the parameter of interest $\gamma^{\star}$ might.

One simple example proceeds as follows: consider a model of the form

$$Y_{t}=\gamma^{\star}\sum_{j}S_{j}Y_{j,t}+\eta_{t}+u_{it},$$

with $\sum_{j}S_{j}=1$, and where for the purpose of this example we consider η_{t} as a fixed effect and u_{it} having unit variance (note that if η_{t} is a random effect, MLE must model its variance

covariance matrix which can be difficult in practice; we therefore consider the comparison with a fixed effect interpretation of $\eta_t$as a natural benchmark). It is easy to show that in this case the MLE estimates

$$\eta_t^e = \frac{1}{N}\sum_i Y_{it} - \gamma^\star \sum_j Y_{jt} S_j.$$

which implies that the first order conditions for $\gamma^\star$ are not data dependent, as these reads as $\sum_i (Y_{i,t} - \bar{Y}_t) \sum_j S_j Y_{jt} - \frac{\frac{1}{2}\log\det(I-B(\gamma))^\top (I-B(\gamma))}{\partial\gamma}$ where the first component equal zero by construction. That is, the MLE fails in this context to estimate $\gamma^\star$ due to the lack of identification of η_t. The key difference with our proposed approach instead here is to consider a moment condition of the form $E[(Y_{it} - B(\gamma)Y_t) z_{i,t}(\gamma)] = 0$ which preserves validity irrespective of the value of η_t, and leverage the fact that we do not need to estimate the fixed effects for estimating $\gamma^\star$.

B.3 The model with factors

In this section we extend our discussion in the presence of time-varying factors.

Assumption 4. *[Model with factors] We make Assumption 1, except that* $Y_t = B(\gamma^\star)Y_t + \lambda\eta_t + u_t$, *with* $\mathbb{E}\left[u_{it} | \{u_{jt}\}_{j\neq i}, \eta_t\right] = 0$. *Here,* $\lambda \in \mathbb{R}^N$ *is a fixed vector of loadings and* η_t *is random vector of factors. The* $\eta_t, \{u_{i,t}\}_{i=1}^N$ *are independent random variables over time* t.

Assumption 5. *[Known* λ*] Assume that* λ *is observed.*

Assumption 5 holds in settings where the vector of loadings is *proportional* to some individual-level covariates X_i as in Gabaix and Koijen (2024). Section B.3.3 discusses estimation in settings where λ is unobserved. One could directly extends to settings with estimated λ, in which case the standard errors should be adjust accordingly using, e.g., inferential theory from Bai (2003) (in practice, when λ must be estimated estimators admit a smooth representation and therefore the variance could be estimated via e.g., the bootstrap). Here, η_t can be unobserved and assumed to be independent over time for simplicity, although one could relax the inferential below results with weakly dependent factors.

The core idea is that we can orthogonalize the regression equation with respect to the presence of factors. We discuss this in the following lines.

We construct a set of instruments that take the following form

$$z_{i,t}^{\lambda}(\gamma) = B_{\gamma}(\gamma) y_{t|-i}^{\lambda}(\gamma), \quad y_{t|-i}^{\lambda}(\gamma) = A(\gamma) u_t^{-i}(\gamma), \qquad u_t^{-i}(\gamma) = \tilde{Q}_i^{\lambda}(\mathbb{I} - e_i e_i^{\top})(\mathbb{I} - B(\gamma)) Y_t. \tag{B.14}$$

The interpretation is the following. u_t^{-i} is the vector of residuals obtained by running the factor model $Y_t - B(\gamma^{\star})Y_t = \lambda \eta_t + u_t$ over the units $j \neq i$, which (at the correct γ) avoids a contamination by u_{it}, after we remove the contribution from all u_{it} from the interactive components $\eta^{-i}\lambda_t$, where η^{-i} denote the vector of loadings of all units except unit i. Then, we construct the instrument $z_{i,t}^{\lambda}(\gamma)$, similarly as in the case with no factors. The intuition behind this procedure is simple. As before, we approximate

$$Y_t - B(\gamma^{\star})Y_t = Y_t - B(\gamma)Y_t - B_{\gamma}(\gamma)Y_t + u_t + \lambda \eta_t + O(||\Delta\gamma||^2)$$

We then replace $B_{\gamma}(\gamma)Y_t$ with a prediction $E[B_{\gamma}(\gamma)Y_t | u_{jt\, j\neq i}]$. In particular, we replace $E[B_{\gamma}(\gamma)Y_t | u_{jt\, j\neq i}]$ with the simple expression for $z_{i,t}^{\lambda}(\gamma)$, which assumes that $E[\eta_t] = 0$. Note that this restriction on the factors is not necessary for validity, but it simplifies the estimation procedure. We show below validity of the procedure.

Proposition 5. *Suppose that Assumption 4, 5 hold. Then for any $i \in \{1, \cdots, N\}, t \in \{1, \cdots, T\}$,*

$$\mathbb{E}\left[\left(Y_{i,t} - e_i^{\top} B(\gamma^{\star}) Y_t\right) z_{i,t}^{\lambda}(\gamma^{\star})\right] = \mathbf{0}.$$

Proof. We can write $\tilde{Q}_i^{\lambda}(\mathbb{I} - e_i e_i^{\top})(\mathbb{I} - B(\gamma^{\star}))Y_t = \tilde{Q}_i^{\lambda}(\mathbb{I} - e_i e_i^{\top})(\lambda_{-i}\eta_t + \lambda_i \eta_t e_i + u_t) = \tilde{Q}_i^{\lambda}(u_t - u_{i,t} e_i)$. The proof is immediate since $u_{i,t}$ is mean independent of the idiosyncratic shocks of the other units. □

Proposition shows that $z_{i,t}^{\lambda}(\gamma^{\star})$ provides us with a valid instrument. The main advantage of this expression is that for estimation we do not need to know or estimate the factors η_t, since the matrix Q_i in practice remove the contribution of the factor by orthogonalizing the

expression with respect to the (known) loadings. We proceed with the iterative estimation procedure outlined in Algorithm 1. The corresponding moment conditions equals

$$\bar{H}_i^{\lambda}(\gamma) = \frac{1}{T}\sum_{t=1}^{T}\Big(Y_{i,t} - e_i^{\top}B(\gamma)Y_t\Big)e_i^{\top}\frac{\partial B(\gamma)}{\partial \gamma}y_{t|-i}^{\lambda}(\gamma), \qquad \bar{H}_i^{\lambda}(\gamma) \in \mathbb{R}^p, \qquad p = \dim \gamma. \tag{B.15}$$

B.3.1 Estimation and inference

In this section, we present our results for inference on the moment estimator that solves for $\hat{\gamma}^{\lambda} \in \mathcal{S} \subset \mathbb{R}^p$ the equation in $\mathbb{R}^p$: for given weights w_i that are not data dependent (e.g., $w_i = 1$), we set

$$\sum_{i=1}^{N} w_i \bar{H}_i^{\lambda}(\hat{\gamma}^{\lambda}) = \mathbf{0}_p \tag{B.16}$$

To express the variance of the estimator, we define:

$$\Delta_i^{\lambda} = -\mathbb{E}\Big[\frac{\partial \bar{H}_i^{\lambda}(\gamma)}{\partial \gamma}\Big|_{\gamma=\gamma^{\star}}\Big], \quad N\bar{\Delta}_w^{\lambda} = \sum_{i=1}^{N} w_i \Delta_i^{\lambda} \tag{B.17}$$

Assumption 6 (Local identification)**.** *Suppose that there exist a unique $\gamma^{\star} \in \mathcal{S}$ in the interior of $\mathcal{S}$, with $\mathcal{S}$ denoting a compact space, such that $\sum_{i=1}^{N} w_i \mathbb{E}[\bar{H}_i(\gamma^{\star})] = \mathbf{0}$ and $N\bar{\Delta}_w^{\lambda}$ is an invertible matrix.*

Assumption 6 imposes local identification of the parameter of interest (local with respect to the pre-specified set $\mathcal{S}$).

Assumption 7 (Moment restrictions)**.** *Assume that for each i, $H_{i,t}^{\lambda}(\gamma)$ is continuously differentiable almost surely, $\mathbb{E}\left[\sup_{\gamma} ||H_{i,t}^{\lambda}(\gamma)||_{\infty}\right], \mathbb{E}\left[\sup_{\gamma}\left|\left|\frac{\partial H_{i,t}^{\lambda}(\gamma)}{\partial \gamma}\right|\right|_{\infty}\right] < \infty$,*
$\mathbb{E}\left[\sup_{\gamma}\left|\left|\frac{\partial^2 H_{i,t}(\gamma)}{\partial \gamma^{(p_1)}\partial \gamma^{(p_2)}}\right|\right|_{\infty}\right] < \infty$, $\mathbb{E}\Big[\Big(\frac{1}{N}\sum_{i=1}^{N} w_i H_{i,t}^{\lambda}(\gamma^{\star})\Big)\Big(\frac{1}{N}\sum_{i=1}^{N} w_i H_{i,t}^{\lambda}(\gamma^{\star})\Big)^{\top}\Big]$ is positive definite, and $\frac{1}{N}\sum_{i=1}^{N} w_i H_{i,t}^{\lambda}(\gamma^{\star})$ has finite third moment.

Proposition 6. *Suppose that Assumptions 4, 5, 6 and 7 hold. Define*

$\Sigma_w^\lambda = \mathbb{E}\Big[\Big(\sum_{i=1}^N w_i H_{i,t}^\lambda(\gamma^\star)\Big)\Big(\sum_{i=1}^N w_i H_{i,t}^\lambda(\gamma^\star)\Big)^\top\Big]$. *Then as* $T \to \infty$,

$$\sqrt{T}\Big(V_w^\lambda(\gamma^\star)\Big)^{-1/2}\Big(\hat{\gamma}^\lambda - \gamma^\star\Big) \to_d \mathcal{N}(0, \mathbb{I}), \quad \textit{where} \quad V_w^\lambda(\gamma^\star) = \Big(N\bar{\Delta}_w^\lambda\Big)^{-1}\Sigma_w^\lambda\Big(N\bar{\Delta}_w^\lambda\Big)^{-1\top}.$$

The proof follows directly by arguments in Newey and McFadden (1994), using the fact that $u_{i,t}$ are independent over time. We can estimate Σ_w^λ using the empirical covariance of the moment restrictions, and estimate $N\bar{\Delta}_w^\lambda$ using any numerical procedure to estimate the derivative of the moment function. Consistency of the estimated variance follows directly from the law of large numbers as $T \to \infty$ under Assumption 7.

B.3.2 Variance characterization in the presence of loadings

In the following proposition, we provide an expression for the variance and of the gradient, choosing $w_i = 1$, for simplicity. Define

$$\tilde{u}_{i,t}^Q = \tilde{Q}_i[u_t - u_{i,t}e_i], \quad \tilde{\Gamma}_{i,j}^1 = \Big[\Gamma\tilde{Q}_i\Big]_{i,j,1}, \quad \tilde{\Gamma}_{i,j,2} = \Big[\tilde{Q}_i\Gamma\Big]_{j,i} \quad v_t^\lambda = \lambda\eta_t + u_t.$$

Proposition 7 (Variance)**.** *Let Assumptions 8, 9, 10, 11 hold and suppose that* $\mathbb{E}[u_{i,t}^2] = \sigma^2$. *Then*

$$N\bar{\Delta}_{w=1}^{\lambda,(p_1,p_2)} = \sigma^2\left[\mathrm{tr}\Big(\Gamma^{(p_1)\top}\tilde{\Gamma}^{1(p_2)}\Big) + \mathrm{tr}\Big(\Gamma^{(p_1)\top}\tilde{\Gamma}^{2(p_2)}\Big)\right],$$
$$\Sigma_{w=1}^{\lambda,(p_1,p_2)} = \frac{1}{T}\sum_{t=1}^T \mathbb{E}\left[\Big(v_t^{\lambda\top}G^{(p_1)}\tilde{u}_t^Q\Big)\Big(v_t^{\lambda\top}G^{(p_2)}\tilde{u}_t^Q\Big)^\top\right].$$

Proof. See Appendix B.9.1. □

The variance in Proposition 7 follows similarly to the case with no factors with some minor modifications to account for the dependence induced by the factors. Variance estimation follows similarly to typical GMM procedures using a sandwich estimator. We can estimate $\Sigma^{\lambda,(p_1,p_2)}$ using the empirical covariance of the moment restrictions, and estimate $N\bar{\Delta}^\lambda$ using any numerical procedure to estimate the derivative of the moment function. Consistency of the estimated variance follows directly from the law of large numbers as $T \to \infty$ under Assumption 11.

It is also possible that factors exhibit (weak) serial correlation, in which case practitioners can replace $\Sigma^{\lambda,(p_1,p_2)}$ with the long-run variance-covariance matrix which can be estimated using standard methods in the time-series literature (see e.g., White (2014)).

B.3.3 Complete algorithm with unobserved loadings

In this section, we turn to the description of the complete algorithm, by integrating all variants. Specifically, we consider generalized heteroskedasticity, lagged outcomes, unobserved loadings, and auto-correlated shocks. We consider a model of the form

$$Y_t = B_0(\gamma)Y_t + B_1(\gamma)Y_{t-1} + \chi(\gamma)u_t + \lambda\eta_t, \tag{B.18}$$

where $\chi(\gamma)$ is a diagonal (invertible) matrix, $B_0(\cdot), B_1(\cdot), \chi(\cdot)$ are known functions, and u_t have uncorrelated entries with arbitrary heteroskedasticity. We further let $u_t = \rho u_{t-1} + \varepsilon_t$, where ε_t are uncorrelated over time. The first step of the algorithm is described in Algorithm 1 and proceeds as follows. It takes one guess of the parameters (ρ, γ) and takes as an input the number of factors r. For simplicity, suppose that the initial guess sets $\rho = 0$. The algorithm then constructs residuals $\tilde{v}_t$ in Algorithm 1 after removing the dependence with the component $\chi(\gamma)$ (the heteroskedasticity component that depends on γ). It then estimates the loadings using an iterative principal component algorithm (PCA). The PCA algorithm allows for heteroskedasticity in the variance of the residuals and is formally described in Algorithm 2. We construct an *estimator* of the loadings and an estimated variance of the idiosyncratic shocks. We then construct an instrument $\tilde{y}_{t|-i}$ using the estimated loadings and the guessed value of γ and impose the orthogonal moment condition.

When constructing the orthogonal moment conditions, we operate with a model that divides each outcome by the standard deviation of its corresponding idiosyncratic shock to improve efficiency, and we reweight each observation by given weights w_i. We use $w_i = 1$ as the default, but other weights are possible (e.g., based on efficient GMM formula). Also, note that the algorithm can directly work if we also incorporate covariates of the form $Y_t = B_0(\gamma)Y_t + B_1(\gamma)Y_{t-1} + X_t\beta + \chi(\gamma)u_t + \lambda\eta_t$, by controlling for both the covariates in the

regression steps of the algorithm. Finally, in the last computation of Algorithm 1, we regress the residuals obtained from the previous regression (and evaluated at the $\hat{\gamma}$ from the previous iteration), the component orthogonal to the residuals from our moment condition. If the current $\hat{\gamma}$ was close to the true parameter of interest, this regression should have coefficient zero. When instead the parameter estimated in the previous iteration is different but in a neighborhood near $\gamma^\star$, we would expect that the estimated coefficient from a simple Taylor approximation.

Remark 1. (Variance with estimated loadings). In the presence of estimated factors, the variance of the estimator may be affected by the estimation error of the loadings. Whenever estimation of the factors admit a standard central limit theorem, see e.g., Bai (2003), one could use the bootstrap method to approximate the distribution of the estimated parameter. We also note that there are edge cases where estimation of the factors may not affect the asymptotic distribution.[13]

B.4 Plots to visualize the identification

We propose a way to (partially) visualize the outcomes in Algorithm 1. The following is for one given component of the γ vector, say the first one. If there are several input observables ($q > 1$), we pick one type of observable (index by $k = 1 \dots q$), e.g. price, or quantities.

First stage. We define ℓ_{it} (as in left-hand side) to be the value of $z_{i,t}$ in (12), but replacing

13. Using a first-order Taylor approximation, it is easy to show that under mild conditions, we can approximate our moment condition (choosing $w_i = 1$ for expositional convenience only)

$$0 = \frac{1}{TN}\sum_{t=1}^{T}\sum_{i=1}^{N} H_{i,t}^{\hat{\lambda}}(\hat{\gamma}) = \frac{1}{TN}\sum_{t=1}^{T}\sum_{i=1}^{N}\Big(H_{i,t}^{\lambda}(\gamma^\star) + \frac{\partial H_{i,t}^{\lambda}(\gamma)}{\partial \gamma}\Big|_{\gamma=\gamma^\star}(\hat{\gamma}-\gamma^\star) + \frac{\partial H_{i,t}^{\lambda'}(\gamma^\star)}{\partial \lambda'}\Big|_{\lambda'\in[\hat{\lambda},\lambda]}(\hat{\lambda}-\lambda) + O_p\Big(||\hat{\gamma}-\gamma^\star||_2^2\Big)\Big).$$

Observe that we can write $\frac{1}{N}\sum_{i=1}^{N}\frac{\partial H_{i,t}^{\lambda}(\gamma^\star)}{\partial \lambda} = \frac{1}{N}\sum_{i=1}^{N}\lambda_i^\top \eta_t f(\lambda_{-i}) + o_p(1)$ for some function $f(\cdot)$ which may depend on $\gamma^\star, \eta_t$. Therefore, whenever loadings are asymptotically independent and centered around zero after appropriately recentering by controlling for the loading of ones (e.g., different industries react to aggregate shocks with approximately independent industry specific effects after controlling for covariates), we may expect that $\frac{1}{N}\sum_{i=1}^{N}\frac{\partial H_{i,t}^{\lambda}(\gamma^\star)}{\partial \lambda} = O_p(N^{-v})$,for some $v > 0$ as both N, T grow. This implies that the error for estimating the loadings is asymptotically negligible for the estimated $\hat{\gamma}$,whenever the rate of convergence of $\hat{\gamma}$ is slower than $N^{-v}T^{-1/2}$. However, this does not necessarily occur if the Jacobian with respect to the loadings is non-vanishing asymptotically.

$\tilde{y}_{t|-i}(\hat{\gamma})$ by the actual value y_t: this is the "non-instrumented" value of the right-and side. We do that only for the k-th component, in this first stage, and in the second stage.

We plot z_{it} on the x axis vs ℓ_{it} on the y axis We expect that the slope is 1, and we report it.

Second stage. We pick a "default" value γ^d. For instance, if we want to visualize component 1, we take $\gamma^d = \left(\gamma_1^D, \gamma_2^e, \ldots, \gamma_K^e\right)$, where γ^e is our final estimate of γ, and γ_1^D is a default value (e.g. corresponding to 0 elasticity). We form the instrument z_{it} based on the final (obtained at convergence) value of γ^e (rather than γ^d). We define $L_t \coloneqq y_t - B_0\left(\gamma^d\right) y_t - B_1\left(\gamma^d\right) y_{t-1}$ and plot the regression:

$$L_{it} = \Delta\gamma \cdot z_{i,t} + \text{error}_{it}$$

On the left-hand side we have the values at the "default" value of γ, and on the right-hand side, we see how the instrument dictates a deviation $\gamma - \gamma^d$ from the default. The "second stage" graph plots L_t (on the y axis) vs z_{it} (on the x axis), and the slope is $\left(\gamma_1 - \gamma_1^D\right)$.

For example, in Section 4.2, in the first stage, we plot instrumented inflation $z_{it} = \pi_{i,t}^z$ vs actual inflation $R_{it} = \pi_{i,t}$. In the second stage, we plot $L_t \coloneqq (I - A^{up})\,\Delta\hat{y}_t$ vs $z_{it} = \pi_{i,t}^z$.

B.5 Model misspecification

Like most procedures, there are problems if the model is misspecified. When our $B(\gamma)$ does not span the correct matrix B^* (e.g., because it would require more parameters than allowed by our low-dimensional γ), then the chosen γ is, morally, so that $B(\gamma)$ is a "best approximation" of B^*.[14]

More optimistically, if $B(\gamma)$ does span the correct matrix B^*, then γ may be biased, but

14. For instance, in the basic case without factor, where $Y_t = (I - B)^{-1} u_t$, the predicted variance of is $V^y(\gamma) = (I - B(\gamma))^{-1} V^u (I - B'(\gamma))^{-1}$. Call $V^{y,*}$ the true correct variance of Y_t, and $D(G, H) = \text{tr}\left(G^{-1}H\right) - \ln\left|\det\left(G^{-1}H\right)\right| - N$ the Stein (1981) pseudo-distance between matrices G and H (which minimized when $G = H$). So our procedure, roughly speaking, tries to find γ and V^u (the diagonal matrix of variances of u), to minimize $D\left(V^y(\gamma), V^{y*}\right)$. Indeed, calling V^{Ye} the empirical variance-covariance matrix, and assuming Gaussian shocks, the log-likelihood is (up to a constant), $\mathcal{L} = -TD\left(V^y(\gamma), V^{Ye}\right)$.

the impulse response $B(\gamma)$ will still be correct.[15] In sum, the predictions about the impulse-responses ($B(\gamma)$) can be robust (and are completely robust in the case we mentioned), even when the details of the model of γ is partially wrong.

B.6 Missing data

We next develop how to handle missing data. Here are two examples to keep in mind. First, we could have a network model of a closed economy, but that the data on the growth of some sectors are not available. Second, the model features N prices and N quantities, and we only observe quantities, but not prices.

Here is the proposed procedure. For concreteness, divide the vector of outcomes y_t as $y_t = [y_{ot} : y_{ht}]$ where y_{ot} is $o$bserved (e.g. we have data on it), and y_{ht} is $h$idden (e.g. is from sectors with no data). Similarly, decompose $u_t = [u_{ot} : u_{ht}]$ into the shocks to observed vs hidden units. We partition the $A = (I - B)^{-1}$matrix into blocks, $A = \begin{bmatrix} A_o^o & A_h^o \\ A_0^h & A_h^h \end{bmatrix}$. We assume that, given γ, we still know the full $A(\gamma)$ matrix (e.g. we know the input-output matrix, though not price variation year to year).

B.6.1 Basic procedure for missing data: no factors, no lags

For clarity, we first take the basic model, $y_t = Au_t$, where we keep the dependence on γ implicit. It then holds

$$y_{ot} = A^o u_t = A_o^o u_{ot} + A_h^o u_{ht}. \tag{B.19}$$

15. Indeed, suppose that the true model has $B^*(\gamma^*)$, but we misspecified the model as $B(\gamma)$. But suppose that the two models have the same "expressive power", i.e. one can write $B(\gamma) = B^*(\phi(\gamma))$ for some bijective function ϕ from $\mathbb{R}^p$ to $\mathbb{R}^p$, for instance $\phi(\gamma) = \gamma + b$. Then, the model will identify a coefficient γ such that $\phi(\gamma) = \gamma^*$, so in our example $\gamma = \gamma^* - b$. So, the coefficient γ will be biased. But the impulse-response $B(\gamma)$ will be exactly equal to $B^*(\gamma^*)$: so the matrix of impulse-response will be exactly identified.

We observe that the theoretical variance matrix of y_{ot} (given γ) is $V^{y_o} \coloneqq A^o V^u A^{o\prime}$ so that we can write the model as

$$y_{ot} = \tilde{A} v_t, \qquad \tilde{A} = (V^{y_o})^{1/2} R, \tag{B.20}$$

for R a rotation matrix of dimension $\dim y_{ot}$ (so $RR' = I$), and where $V^v = I$ and where v_t are independent random variables.

Given this, we proceed as in the main procedure, with $\tilde{B} \coloneqq I - \tilde{A}$. We just enrich γ with the parameters that govern V^u. For interpretability, we pick a rotation matrix R via:

$$\min_{R:R'R=I} \left\| R - (V^{y_o})^{-1/2} A^o_o (V^{u_o})^{1/2} \right\|_2, \tag{B.21}$$

so that, if we did not have hidden units, we recover the correct A matrix.[16]

The conclusion is that we can use the general procedure, which assumes complete observability. We record, nonetheless, the influence of unobserved factors. One issue is that the "idiosyncratic shocks" are a bit harder to interpret. For instance, if we have only quantities, but not prices, the shocks are a mix of supply and demand shocks; and in the "unobserved sector" case, it's influence is still felt by influencing the recovered idiosyncratic factors v_t. Still, identification is achieved. We now show how the basic idea above extends to the other enrichments.

B.6.2 Missing data, in a model with factors

Suppose now the enriched model with factors: $y_t = Au_t + A\lambda\eta_t$. It follows that we can write $y_{ot} = A^o u_t + A^o \lambda \eta_t$. As above, we write $A^o u_t = \tilde{A} v_t$, so that,

$$y_{ot} = \tilde{A} v_t + A^o \lambda \eta_t = \left(I - \tilde{B} \right)^{-1} v_t + A^o \lambda \eta_t,$$

16. Indeed, the minimizer rotation is $R = (V^{y_o})^{-1/2} A^o_o (V^{u_0})^{1/2}$, as the latter is a rotation matrix when $V^{y_o} \coloneqq A^o_o V^{u_0} A^{o\prime}_o$. This leads to $\tilde{A} = A^o_o (V^{u_o})^{1/2}$,which is the correct matrix, up to a normalization of the standard deviation of the shocks.

it holds, with $\tilde{\lambda} = \left(I - \tilde{B}\right) A^o \lambda$,

$$y_{ot} = \tilde{B} y_{ot} + v_t + \tilde{\lambda} \eta_t. \tag{B.22}$$

Hence, we can use the general procedure, with full observability.

B.6.3 Missing data, in a model with lags

Suppose now a model with factors and lags, $y_t = Au_t + A\lambda\eta_t + Cy_{t-1}$ Allowing C to take arbitrary values would require inputing sequentially each entry of y_{t-1}. This is possible, using, e.g., methods similar in spirit to the Kalman filter or other imputation methods, with y_t the state, and y_{ot} the observed values. The high-level idea is the following: Given a guess for γ (augmented to parametrize variances such as σ_u, σ_η), we have the innovation $\tilde{y}_{ot} \coloneqq y_{ot} - \mathbb{E}_{t-1}\left[y_{ot} | (y_{o,s})_{s<t}\right]$,which has the known distribution (as a function of γ), $V^{\tilde{y}_o}$. Whereas we defer a more comprehensive treatment of this problem to future research, we suggest in practice to proceed as follows: obtain a preliminary estimate of A and η e.g., assuming no lagged effects. We start from $t = 0$, and predict $\mathbb{E}[y_0] = 0$. At this point, we can construct $\mathbb{E}_{t=1}[y_1 | y_{1o}]$, assuming e.g., joint Gaussianity with the variance-covariance matrix that depends on the corresponding correlations induced by A, λ. We then proceed recursively, each period controlling for each past observed outcome $y_{s \leq t,o}$. We can then iterate multiple times to achieve convergence also over A, λ. We can interpret this approach as similar to a Kalman filter, where here we use preliminary estimates of A, λ to capture the correlation structure.

B.7 Measurement error

A concern in practice is measurement error. Suppose that the true model satisfies our assumptions, e.g. $y_t = B(\gamma) y_t + u_t$, but we observe $y_{ot} = y_t + \varepsilon_t$, where the ε_{it} are mean-zero measurement errors — say uncorrelated with u_s, η_s at all dates, and, for simplicity, i.i.d.

with variance σ_ε^2. What to do? We observe $y_{ot} = Au_t + e_t$, with $e_t = A\varepsilon_t$, so

$$V^{y_o} = AV^u A' + V^e \tag{B.23}$$

Hence, we proceed as in Section B.6.1. We have

$$y_t = \tilde{A}v_t, \qquad \tilde{A} = \left(V^{y_o}\right)^{1/2} R$$

for a rotation matrix R, which we can choose of easy for interpretability as in (B.21). Now, the extended γ vector is made of the previous γ, and σ_u^2, σ_ε^2: $\tilde{\gamma} = (\gamma', \sigma_u^2, \sigma_\varepsilon^2)'$. One can proceed as in the main case. Generically at least, we can identify this extended γ (provided it's not of too high dimension). The interpretation is a bit less easy, but it remains "we look at the impact of idiosyncratic shocks; even though they're a bit mismeasured, the procedure accounts for that".

B.8 Summary statistics for the impulse-responses

B.8.1 The notion of a "Typical impulse response of a multidimensional shock"

Suppose that we have one outcome measure,

$$o_t = b'x_t \tag{B.24}$$

with x_t exogenous, with dimension N, and b with dimension N as well.

For a "typical" or "examplar" observation o_t, we can take the following:

$$o_t^e = b'x_t^e$$

(where the e reminds us that this is an example), where the "examplar" x_t^e might be with one of the following:

1. The simplest is: $x_t^e = (\sigma_{x_i})_{i=1\ldots N}$. This corresponds to "a one standard deviation shock

for all units"

2. We could have a "typical" shock, that respects the standard deviation of shocks: we first define $x_t^{e,0} = (\sigma_{x_i})_{i=1...N}$ as the shock before rescaling and then rescale it as $x_t^e = \frac{\sigma_o}{b' x^{e,0}} x^{e,0}$

3. A variant is: $x_t^e = \frac{\sigma_o}{b'\iota}\iota$. This is a typical shock where all components are proportional to 1. For instance, if x_{it} is a TFP shock for good i, x_t^e is a uniform TFP shock in all industries. The prefactor $\frac{\sigma_o}{b'\iota}$ ensures that $o_t^e = \sigma_{b'x}$, so reflects the typical size of the response o_t.

Now, consider an IRF, where we're interested in a series of impacts at different horizons $h \geq 0$,

$$o_{t+h} = \beta_h' x_t \tag{B.25}$$

where the β_h come from the model. We apply select the above x_t^e for $b \coloneqq \frac{1}{H}\sum_{h=0}^{H-1} \beta_h$ for some horizon H (e.g. up to 3 years).[17] This ensures the the "average" is correct up to horizon H. Then, we plot

$$o_{t+h}^e = \beta_h' x_{t^e}$$

B.8.2 Concrete application

We trace the IRF of y_{t+h} and π_{t+h} given the various shocks.

Some plots of summary quantities First, we plot some summary stats, $\delta_t^s \coloneqq \sum_i S_i^y \delta_{it}^s$, $\delta_t^{c,d} \coloneqq \sum_i S_i^c \delta_{it}^{c,d}$, aggregate firms demand shocks $\delta_t^{y,d} \coloneqq \sum_i S_i^y \delta_{it}^{y,d}$. But the estimation of IRFs is not done with those aggregate measures, it's done with the disaggregated numbers, e.g. δ_{it}^s rather than δ_t^s.

Typical IRFs We use the method in B.8.1 to plot the "typical impulse response" for the TFP shocks δ_{it}^s, firm demand shocks $\delta_{it}^{y,d}$ and consumer demand shocks $\delta_{it}^{c,d}$. They have hats in (D.25), but in this section we remove the hats.

17. There could be lots of variants, e.g. , or perhaps $b = \beta_H$ for a given horizon H.

We have all those shocks handy, except the sector-by-sector demand shocks $\delta_{ijt}^{y,d}$. Those can be extracted as a residual from the demand equation (XX Xavier or Constantin to write down the precise way to do that; but I (Xavier) is arriving in Paris with the family, and won't have the time to do that for a few days probably). But in the meantime, we can look a consumer demand and firm supply shocks.

What fraction of fluctuations comes from which shocks? We can see what's the largest fraction explaining GDP (and later, of inflation). Given

$$y_t = \sum_{h \geq 0} b_h \delta_{t-h} \tag{B.26}$$

decompose the shocks into types k, (e.g. 3 types of shocks: to consumer demand, firms demand, firm supply)[18]:

$$y_t = \sum_k y_t^{(k)}, \qquad y_t^{(k)} = \sum_{h \geq 0} b_h^{(k)} \delta_{t-h}^{(k)} \tag{B.27}$$

The fraction coming from shocks of type k is

$$s_k = \frac{cov\left(y_t, y_t^{(k)}\right)}{var\left(y_t\right)} \tag{B.28}$$

so that $\sum_k s_k = 1$.

If there's TFP shock, that increases the rational demand, so affects the measure of δ_{it}^d. Hence, we might also have a "purged" demand shocks. One way to do that it to regress:

$$\delta_{Et}^d = \beta \delta_{St}^s + \varepsilon_t \tag{B.29}$$

18. With perhaps with a special role, later, for the government; or the foreign sector. We can also decompose into idiosyncratic vs aggregate shocks.

where $S_{it} = \frac{\text{Sales}_{it}}{\text{GDP}_{it}}$ is the Domar weight of sector i, and by Hulten,

$$\delta^s_{St} = \sum_i S_{i,t-1}\delta^s_{it} \tag{B.30}$$

is the total TFP shock. We estimate β, and define

$$\delta^{d,\text{purged}}_{it} \coloneqq \delta^d_{it} - \beta\delta^s_{St} \tag{B.31}$$

which is the demand shock "purged from the effect of supply shocks". Here, there is a shortcut: ideally, we'd regress $\delta^d_{it} = \sum_j \beta_{ij} S_{j,t-1}\delta^s_{jt}$, and estimate all the β_{ij}. But there isn't enough statistical power to that, so instead as use the simpler parametrization of a constant β_{ij}. Alternatively, we could have a structural model, e.g. if TFP increase, consumers should increase their spending, which might look like a demand shock here. But the correct amount depends on the persistence of supply shocks, and the MPC of consumers. So instead, we opt for the simpler approach to purge demand shocks.

So we report the impact of TFP shocks δ^s_t and purged demand shocks, $\delta^{d,\text{purged}}_t$

B.8.3 Supply vs demand shocks in a network economy: conceptual discussion

There is a conceptual difficulty: suppose that firm i decides to use less steel, and fewer plastics (controlling for prices): this is a technological shocks — a supply shock. But, it will appear to steel producers to be a demand shock.

So, to classify as demand shocks: the demands from the consumer side (δ^c_i); and as supply shocks, the demands from the firm side, δ^y_i, and the own-TFP shock δ^s_i. We decompose the aggregate demand shock of i as (see (D.25))

$$\hat{\delta}^d_i = \Theta^c_i \hat{\delta}^{d,c}_i + \Theta^y_i \hat{\delta}^y_i \tag{B.32}$$

We directly measure $\hat{\delta}^{d,c}_i$, as a residual, and likewise measure $\hat{\delta}^y_i$.

Other version: from the flows from j to i,we get $\hat{\delta}^y_{ij}$, the change in the demand by i for

j, that is not explained by price changes. We define the total demand shock from i as:

$$\hat{\delta}^d_{i\to} = \frac{\sum_j A_{ij}\hat{\delta}_{ij}}{\sum_j A_{ij}}$$

so that the "relative" demand shocks for j is $\hat{\delta}^d_{ij} - \hat{\delta}^d_{i\to}$: this one is a TFP shock; while $\hat{\delta}^d_{i\to}$ is a demand shock.

B.9 Omitted proofs for results in the online Appendix

B.9.1 Proof of Proposition 7

Denote $\sigma^2_{i,t} = \mathbb{E}[u^2_{i,t}]$, and $H_{i,t}(\gamma) = \Big(Y_{i,t} - e_i^\top B(\gamma)Y_t\Big)e_i^\top \frac{\partial B(g)}{\partial g}A(\gamma)y^\lambda_{t|-i}(\gamma)$, where $y^\lambda_{t|-i}(\gamma) = \tilde{Q}_i(I - e_ie_i^\top)(I - B(\gamma))Y_t$. Also, recall the definition of $G^{(k)} = \frac{\partial B(g)}{\partial g_k}\Big|_{g=\gamma^\star} A(\gamma^\star)$.

First observe that because under Assumption 8, $Y_t = A(\gamma^\star)v^\lambda_t, \quad v^\lambda_t = u_t + \lambda\eta_t$, we can write

$$\begin{aligned}
\frac{\partial\Big(Y_{i,t} - e_i^\top B(\gamma)Y_t\Big)}{\partial\gamma_k} &= -e_i^\top \frac{\partial B(\gamma)}{\partial\gamma_k}A(\gamma)v^\lambda_t,\\
\frac{\partial A(\gamma)y^\lambda_{t|-i}(\gamma)}{\partial\gamma_k} &= \frac{\partial A(\gamma)}{\partial\gamma_k}y^\lambda_{t|-i}(\gamma) - A(\gamma)\tilde{Q}_i(\mathbb{I} - e_ie_i^\top)\frac{\partial B(\gamma)}{\partial\gamma_k}A(\gamma)v^\lambda_t.
\end{aligned}$$

Consider the component $\Delta^\lambda_{p_1,p_2}$ first. Because $E[\eta_t y^\lambda_{t|-i}(\gamma^\star)] = 0$ by construction, by simplifying the remaining terms (including those that depend on the second derivative of $B(\gamma)$), we can write

$$-\frac{\partial\mathbb{E}[H^{(p_1)}_{i,t}(\gamma)]}{\partial\gamma_{p_2}}\Big|_{\gamma=\gamma^\star} = \sum_{j\neq i} G^{(p_2)}_{j,i}[G^{(p_1)}\tilde{Q}_i]_{i,j}\sigma^2_{j,t} + \sum_{j\neq i} G^{(p_1)}_{i,j}[\tilde{Q}_iG^{(p_2)}]_{j,i}\sigma^2_{i,t}$$

The final expression follows from rearrangement and homoskedasticity. Finally, because

$$\sum_{i=1}^N \bar{H}_i(\gamma^\star) = v_t^\top G\tilde{u}^Q_t,$$

the proof completes.

C Data Preparation

We use the BEA IO table from from *Macro Multipliers CR-G-K/gcr/data/BEA-IO/data-BEA-detail-2007-Adomestic.dta*. Entry (i, j) in this table records how much sector j spends on sector i (i.e., how much i sells to j). Sector names are slightly different in rows and columns, so we cross-walk the columns to the rows and sort the columns to ensure they match the rows (leaving the last column, *FinalDemand*, unaffected as it has no corresponding row). We then define:

- The vector of consumptions c equal to the column *FinalDemand*
- The "values" upstream I/O matrix F^{up} equal to all rows & columns of the table except *FinalDemand*: F^{up}_{ij} records how much sector j spends on sector i
- We remove negative values in the I/O matrix F^{up}, i.e. set $F^{\text{up}}_{ij} = \max(F^{\text{up}}_{ij}, 0)$
- The "values" downstream I/O matrix $F^{\text{down}} = F^{\text{up}\prime}$: F^{down}_{ij} records how much sector i spends on sector j (this matrix is called x_{ij} in CES Propagation)
- The vector of gross outputs $\bar{y}_i = c_i + \sum_j F^{\text{down}}_{ji}$
- The "shares" downstream I/O matrix G^{down} such that $G^{\text{down}}_{ij} = \frac{F^{\text{down}}_{ij}}{\bar{y}_i}$. [19]
- We check that production equals absorption: $\bar{y} - c = G^{down\prime}\bar{y}$

We import industry nominal output (GOCP) & price (GOPI) data from *Macro Multipliers CR-G-K/gcr/data/BEA-IO/data-BEA-detail-industry.dta*, and GDP deflator data from *Macro Multipliers CR-G-K/gcr/data/A191RG3A086NBEA*.dta. We then:

- Deflate outputs & prices for each sector using the GDP deflator
- Compute log differences

19. So G^{down} is the input-output matrix A in the "macro" part of the paper (which unfortunately, as of June 2024, uses a different $A = (I - B)^{-1}$ in the econometric part, and A=input-output matrix in the macro part; so we stick to G^{down} for the input-output matrix A of the macro part).

- Drop years with missing values, resulting in data series ranging from 1964 to 2018
- Demean each output & price series by sector
- Reshape outputs to an array y of size (qN, T), stacking N sector price changes and then N sector output changes

The industry data has slightly different names for government-related sectors, so another cross-walk is required. We then check that sector names are identical and in the same order across c, y and rows & columns of G^{down}. We then export to *Macro Multipliers CR-G-K/Data/python_data/:*

- The list of N sector names to *sectors.csv*
- The $N \times N$ downstream I/O matrix G^{down} to *G_down.csv*
- The vector of size N of consumptions c to *c.csv*
- The (qN, T) array of outcomes y to *y.csv*

C.1 Network & economic inputs

We load *sectors.csv*, *G_down.csv* and *c.csv*, as described above, from *Macro Multipliers CR-G-K/Data/python_data/*. We have $N = 277$ sectors. We check that sectors match the row index of c and the rows and columns of G^{down}. We normalize consumption, i.e. replace c by $\frac{c}{\sum_j c_j}$. We then compute:

- The vector of consumption basket shares $\theta_i = \frac{c_i}{\sum_j c_j}$
- The vector of intermediate consumption shares $x_i = \sum_j G^{\text{down}}_{ij}$
- The vector of labor shares $b_i = 1 - x_i$
- The vector of "average" outputs $\bar{y} = (I - G^{\text{down}\prime})^{-1} c$
- The scalar GDP $= \sum_j c_j$

- The "upstream" IO matrix $G^{\text{up}} = D_{\bar{y}}^{-1} G^{down\prime} D_{\bar{y}}$

For good measure, we check that:

- Production equals absorption: $\bar{y} - c = G^{\text{down}\prime}\bar{y} = D_{\bar{y}} G^{\text{up}} \iota$
- Labor and intermediate consumption shares add up to 1: $b + x = \iota$
- All coefficients in G^{down}, G^{up}, θ, x and b are in $[0, 1]$

We then also compute:

- The matrix of non-labor intermediate consumption shares $s = D_{\iota - b}^{-1} G^{\text{down}}$ (i.e. $s_{ij} = \frac{G_{ij}^{\text{down}}}{1 - b_i}$). We check that $s\iota = \iota$.
- The vector of labor usage shares $s_i^L = \frac{b_i \bar{y}_i}{\sum_j b_j \bar{y}_j}$. We check that $\sum_j s_j^L = 1$.
- The vector of final consumption output shares $\Theta_i^c = \frac{c_i}{\bar{y}_i}$. We check that $\iota - \Theta^c = G^{\text{up}} \iota$.
- The vector of intermediate consumption shares $\Theta_i^y = 1 - \Theta_i^c$. We check that $\Theta^y = G^{\text{up}} \iota$.
- The matrix $M_{ij}^c = \theta_j$ which gives the mean impact of competitors prices in consumption. We check that $M^c \iota = \iota$.
- The matrix $M^p = D_{\iota - \Theta}^{-1} G^{\text{up}} s$ which gives the mean impact of competitors prices in production. We check that $M^p \iota = \iota$.
- The inflation basket weights $\theta^{\pi *} = \theta$. We check that $\sum_j \theta_j^{\pi *} = 1$.

Given the (scalar) primitive parameters $\sigma, e, \zeta^s, \zeta^{L,s} + \zeta^{L,d}, \phi$, we finally compute:

- The labor elasticity vectors $\xi^L = 1 - \frac{1}{\zeta^{L,s} + \zeta^{L,d}}$, $\xi^s = (I - G^{\text{down}})\iota - b$ and $\tilde{\xi}^s = \xi^s + b\xi^L$. Note that $\xi^s = 0$ for now.
- The demand elasticity matrix $\zeta^d = (\sigma - 1) D_{\Theta^c} (I - M^c) + (e - 1) D_{\Theta^y} (I - M^p)$. We check that $\zeta^d \iota = 0$ (demand is homogeneous of degree 0). (This works for the CES case; it's more complex in the non-homogeneous CES case).

- The (inverse) supply elasticity matrix $\left(\tilde{\zeta}^{s}\right)^{-1} = \frac{1}{\zeta^{L,s}+\zeta^{L,d}} bs^{L\prime} + \frac{1}{\zeta^{s}} I_N$.

Since the current model cannot distinguish between $\zeta^{L,s}$ and $\zeta^{L,d}$, we estimate their sum $\zeta^{L} \equiv \zeta^{L,s} + \zeta^{L,d}$ directly.

D Appendix: A macroeconomic network model with heterogeneous elasticities

We present new model of the macroeconomy. We present the main outline first. The detailed derivations are in Section D.5.

D.1 The Heterogeneous CES demand structure

We use the flexible CRESH functional form of Hanoch (1971),[20] and derives some new results about it. It (i) allows for heterogeneous elasticities of demands across goods, and (ii) keeps proportional scaling – i.e. preferences homogeneous of degree 1, or a constant return to scale production function. This is useful, as it is clear that the demand elasticity of some goods (e.g. oil) is very low, while that of other goods (e.g. leisure goods) is high. This issue is proving important e.g. to analyze the impact of energy shocks and energy embargo (Moll, Schularick, and Zachmann (2023)). The homogeneity of degree 1 is useful in macroeconomics, as we do not want ratios to change simply because the economy expands.

In macroeconomics, the typical device used is a nested CES utility functions, but they are somewhat artificial ways to generate the heterogeneous elasticities — there is a tight mapping between elasticity and expenditure share, a mapping that is unlikely to exist in the data.. In applied microeconomic, e.g. taxation, one achieves the first goal with a separable utility function of the type:

$$v\left(c\right) = \sum_{i} \frac{c_i^{1-\frac{1}{\sigma_i}} - 1}{1 - \frac{1}{\sigma_i}} \tag{D.1}$$

20. CRESH stands for for constant ratio of elasticity of substitution with homogeneity of degree 1.

But then, $v(c)$ is not homogeneous of degree 1, or any degree, so is not consistent e.g. with long term growth, hence it is not used in macro, especially not as a production function. The task of this study is to propose and analyze a utility function that behaves "similarly" to (D.1), but homogeneous of degree 1. We hope that it can be useful in macroeconomics, and perhaps related fields such as trade or taxation.

For simplicity, we mostly express things in terms of the consumption case. The production case is entirely similar.

Definition 1. (Heterogeneous CES utility function) We are given some positive parameters σ_i and a constant ϕ, we define the "Heterogeneous CES" or HCES as follows. Given a vector of consumption c, we postulate that the utility $u(c)$ is the unique solution of the equation

$$\sum_i \phi_i \frac{(c_i/u(c))^{1-\frac{1}{\sigma_i}} - 1}{1-\frac{1}{\sigma_i}} + \phi_0 = 0 \tag{D.2}$$

where the σ_i and ϕ_i are constant, strictly positive for $i > 0$.

When all σ_i are equal, this is a CES. This is in the spirit of a particular case of a class outlined in Hanoch (1971) and recently used in Matsuyama (2023), Baqaee, Farhi, and Sangani (2024)), Berlingieri et al. (2022). We shall see that despite its at first strange definition, this class is quite tractable, and we systematically analyze its properties.[21] The utility function is, transparently, homogeneous of degree 1. Lemma 2 verifies that it is also well-behaved.

We derive now the the elasticity of demand that it generates, extending Hanoch (1971). The proof is in Section 16.

Proposition 8. (Demand elasticities with heterogeneous CES: Hicksian and Marshallian demand elasticities) *To express the cross-elasticities at a given consumption c, call* $s_j =$

21. An extension to non-homothetic preferences is

$$\sum_i \frac{\left(c_i/u(c)^{1+\alpha_i}\right)^{1-\frac{1}{\sigma_i}} - 1}{1-\frac{1}{\sigma_i}} + \phi = 0$$

so that demand features Engel's curves, as in Comin, Lashkari, and Mestieri (2021).

$\frac{p_j c_j}{\sum_{j'} p_{j'} c_{j'}}$ *the expenditure share of good* j*, and define the weights*

$$m_j = \frac{\sigma_j s_j}{\sum_{j'} \sigma_{j'} s_{j'}}, \qquad M_j^i = \frac{\sigma_i m_j - s_j}{\sigma_i - 1} \tag{D.3}$$

which add up to 1 ($\sum_j m_j = \sum_j M_j^i = 1$)*. The compensated cross-elasticity of demand for good* i *is* $\sigma_i (1_{i=j} - m_j)$*, so that the own elasticity of demand is* $\sigma_i (1 - m_i)$ *and the cross-elasticity is* $-\sigma_i m_j$*. Indeed, consider the impact of a price change to the price vector,* $\hat{p}_j \coloneqq d \ln p_j$*. The compensated (Hicksian) demand changes is:*

$$\hat{c}_i = -\sigma_i (\hat{p}_i - m' \hat{p}) \tag{D.4}$$

Given also an income change $\hat{w} = d \ln w$*, the expenditure* $x_i = p_i c_i$ *spent on good* i *changes as the following Marshallian demand:*

$$\hat{x}_i \;=\; -\sigma_i (\hat{p}_i - m' \hat{p}) + (\hat{p}_i - s' \hat{p}) + \hat{w} = -(\sigma_i - 1)\left(\hat{p}_i - M^{i\prime} \hat{p}\right) + \hat{w} \tag{D.5}$$

So, the elasticity of the Marshallian consumption expenditure with respect to the price is:

$$\zeta_{ij} \coloneqq -\frac{\partial \ln (p_i c_i)}{\partial \ln p_j} = -\sigma_i (1_{i=j} - m_j) + 1_{i=j} - s_j = -(\sigma_i - 1)\left(1_{i=j} - M_j^i\right)$$

Finally, the Slutsky matrix $S_{ij} = \frac{\partial c_i(p,u)}{\partial p_j}$ *is*

$$S_{ij} = -w \frac{\sigma_i s_i}{p_i^2} 1_{i=j} + w \frac{\sigma_i \sigma_j s_i s_j}{\bar{\sigma} p_i p_j} \tag{D.6}$$

The term (D.4) shows that the own elasticity is σ_i, for a "small good" (s_i close to 0).[22] What matters is the difference in the price compared to the elasticity-times-size-weighted price change, $m' \hat{p}$. In the Marshallian demand (D.5), the first term is a substitution effect. The second and third terms are income effects.[23]The same can be used for a production

22. If the good is large, the elasticity is lower ($\sigma_i (1 - m_i)$): indeed, when the good is essentially the only good, the compensated elasticity must be 0.

23. The budget constrained is verified: for all j, $\sum_i s_i \zeta_{ij} = 0$, and the demand is homothetic of degree 1,

function.

If there is a demand shock for good i (microfounded as a small change in ϕ_i), demand shock δ_i is proportional to the change in ϕ_i, and we have:

$$\hat{c}_i = -\sigma_i \left(\hat{p}_i - m'\hat{p}\right) + \hat{w} + \delta_i - \sum_j s_j \delta_j \tag{D.7}$$

where the compensation term $-\sum_j s_j \delta_j$ ensures that the budget constraint is respected.

We next consider the aggregate price index. It turns out that we need two price indices, not one.

Proposition 9. (Price index with heterogeneous CES) *Given budget w and a vector of prices $\boldsymbol{p} = (p_i)_{i=1...N}$, there are two relevant price indices, $\mu(\boldsymbol{p})$ and $\bar{p}(\boldsymbol{p})$. The "elasticity weighted" price index $\mu(\boldsymbol{p})$ is defined as the unique solution of:*

$$\sum_i \frac{1}{1-\frac{1}{\sigma_i}} \left(\frac{p_i}{\mu}\right)^{1-\sigma_i} + \phi_0 = 0 \tag{D.8}$$

while the "share weighted" price index $\bar{p}(\boldsymbol{p})$ is:

$$\bar{p} = \mu \sum_i \left(\frac{p_i}{\mu}\right)^{1-\sigma_i} \tag{D.9}$$

Then, utility achieved by a rational consumer with budget w is $u = \frac{w}{\bar{p}}$, while the consumption of good i is:

$$c_i = \left(\frac{p_i}{\mu}\right)^{-\sigma_i} \frac{w}{\bar{p}} =: C_i\left(p_i, \bar{p}, \mu\right) w \tag{D.10}$$

The optimal consumption shares are:

$$s_i = \frac{p_i c_i}{w} \propto p_i c_i \propto \left(\frac{p_i}{\mu}\right)^{1-\sigma_i} = \left(\frac{c_i}{u}\right)^{1-\frac{1}{\sigma_i}} \tag{D.11}$$

In addition, a change in prices creates the following changes (using the weights $m_i \propto s_i \sigma_i$ in

$\sum_j \zeta_{ij} = 0$.

(D.3)):

$$\hat{\mu} = \sum_i m_i \hat{p}_i, \qquad \widehat{\bar{p}} = \sum_i s_i \hat{p}_i \tag{D.12}$$

The terminology "elasticity weighted price index" for μ comes from (D.3) and (D.12).

Usually, e.g. with CES preferences, there is just one relevant price index. Here, there are two relevant price indices, $\mu(\boldsymbol{p})$ and $\bar{p}(\boldsymbol{p})$, so that we can express consumption as $c_i = C_i(p_i, \bar{p}, \mu)$.[24] This is a bit unexpected. The loose intuition is that we need to keep track of the average price level $\bar{p}$ (to enforce the budget constraint) and also of the average elasticity (which varies with individual prices).

Nested utility or production functions Production functions are handled very similarly: instead of a homothetic of degree 1 utility function $u(\boldsymbol{c})$, we have a CRS production function $F(\boldsymbol{c})$, where $\boldsymbol{c}$ is the vector of inputs.

We next review how to handle nested utility functions. The following is well-known, but we write it down for future use. Suppose a utility $U(q_a, q_b, \dots)$ with "aggregate goods" indexed by a, and that $q_a = u^a(c_1^a, c_2^a, \dots)$, where $(c_i^a)_{i=1\dots N}$ are the subgoods in good a. For instance, a is "aggregate food intake", b is "leisure", and c_i^a are bananas, pears, etc. Then, calling ξ_a the expenditure on aggregate good a, and π_a the aggregate price of good a.

The aggregate price of good a changes as the weighted average change of its components:

$$\hat{\pi}_a = \sum_i s_i^a \hat{p}_i$$

If total spending changes by $\hat{w}$, aggregate spending on good a changes by

$$\hat{\xi}_a = \hat{w} - \zeta_b^a \hat{\pi}_b \tag{D.13}$$

where ζ_b^a are the (Marshallian) elasticities from the utility $U(q_a, q_b, \dots)$. The expenditure

24. See (D.10). Also, the same equation shows that, when at least two elasticities σ_i differ, the two indices cannot be "compressed" into one hypothetical smooth one-dimensional index $\bar{\bar{p}}(\boldsymbol{p})$, and that would ensure a functional form $c_i = wC_i(p_i, \bar{\bar{p}})$. In the CES case, μ and $\bar{p}$ are proportional, $\mu = \frac{1/\sigma - 1}{\phi}\bar{p}$, so can be made equal with $\phi = \frac{1}{\sigma} - 1$.

on each "micro" good i in aggregate good a changes as

$$\hat{x}_i^a = \hat{\xi}_a - \sum_j \zeta_j^{i|a} \hat{p}_j \tag{D.14}$$

where $\zeta_j^{i|a}$ is the elasticities from utility $u^a\left(c_1^a, c_2^a, \dots\right)$.

D.2 Setup of the competitive economy

D.2.1 General notions

We define the "running mean" or delay operator as $\mathcal{D}^\phi \coloneqq \frac{\phi}{1-(1-\phi)\mathcal{L}}$, i.e.

$$\mathcal{D}^\phi x_t \coloneqq \sum_{h\geq 0} \phi\left(1-\phi\right)^h x_{t-h} = \frac{\phi}{1-\left(1-\phi\right)\mathcal{L}} x_t \tag{D.15}$$

When $\phi = 1$, the adjustment is instantaneous, and it is slower for lower ϕ. It verifies the identities:

$$\mathcal{D}^\phi x_t = \phi x_t + \left(1-\phi\right)\mathcal{D}^\phi x_{t-1}, \qquad \Delta\mathcal{D}^\phi x_t = \phi\left(x_t - \mathcal{D}^\phi x_{t-1}\right). \tag{D.16}$$

D.2.2 Specific utility and production functions

Consumer utility is CES, with elasticity σ,or HCES, with elasticity σ_i. The production function is:

$$\ln Y_i = \ln z_i + b_i \ln L_i + \left(1-b_i\right)\frac{e}{e-1}\ln\left[\sum_j \lambda_{ij} X_{ij}^{1-\frac{1}{e}}\right] \tag{D.17}$$

and we can also replace the CES in X_{ij} by a HCES version, with elasticities e_j for the demand for good j (which could even in principle depend in the industry i).

Industries behave as price takers, for simplicity. This could be microfounded by positing a continuum of firms within each industry. Azar and Vives (2021) develop a theory with explicit oligopolistic behavior.

Labor market We call $\zeta^{L,s}$ the elasticity of labor supply. For now, the utility is

$$u\left(C_1,\ldots,C_N,L\right)=v\left(C_1,\ldots,C_N\right)-\frac{L^{1-1/\zeta^{L,s}}}{1-1/\zeta^{L,s}} \tag{D.18}$$

where $v\left(C_1,\ldots,C_N\right)$ is the consumption aggregator, homogeneous of degree 1. So, there is no income effect from labor supply, but we can add that later.

D.2.3 Demand shocks

Demand shocks from consumers. Suppose that θ exogenously changes, with Cobb-Douglas in consumption. From the production side, prices don't change, $\hat{p}=0$ (for this, the assumption of flexible labor is essential). So, as we had $c_i=\theta_i$, we have $\hat{c}=\hat{\theta}$ exogenous. As we have $\sum_i \theta_i=1$, we have

$$\theta'\hat{\theta}=0 \tag{D.19}$$

hence $\theta'\hat{c}=0$.

So we get:

$$\left(I-A^{\text{up}}\right)\hat{y}=D_\Theta\hat{\theta}. \tag{D.20}$$

which captures the intuition from demand shocks propagating upstream.

To capture a Keynesian demand shock we can allow $\theta'\hat{\theta}>0$, or perhaps write the shock as

$$\hat{c}=D_\Theta\hat{\theta}+\iota\hat{c}_t^{\text{agg}}$$

where $\hat{c}^{\text{agg}}$ is a scalar: $\hat{c}^{\text{agg}}>0$ means a positive aggregate demand shocks. For instance, in a New Keynesian model, we'd have

$$\hat{c}_t^{\text{agg}}=M\hat{c}_{t+1}^{\text{agg}}-\sigma\left(r_t-\mathbb{E}\left[\pi_{t+1}\right]-r_t^n\right)$$

The total consumption shock is

$$\hat{\delta}^c=\hat{\theta}+\iota\hat{c}^{\text{agg}} \tag{D.21}$$

We can rewrite this as:[25]

$$\hat{y} = D_y^{-1} \left(I - A'\right)^{-1} D_c \hat{\theta} \tag{D.24}$$

As prices are kept the same, the real output changes by

$$\hat{Y} = \hat{y}$$

The same works with a more general utility function. The key is that prices don't change: so, given prices, consumption is constant, hence $\hat{c}$ reflects only the taste shocks, not a change in prices.

Demand shocks from firms We might likewise have a demand shock, where firm i changes its demand by $\hat{x}_{ij} = \hat{\theta}_{ij}$, with potentially

$$\sum_j A_{ij} \hat{\theta}_{ij} = 0$$

if that's a purely redistributive shocks (i.e. firm i spends less on firm j, more on firm j', but keeps total spending constant), and $\sum_j A_{ij} \hat{\theta}_{ij} > 0$ if that's a net demand shock (e.g. going to investment).

25. Indeed, the change in income is:

$$\begin{aligned}
\widehat{wL} &= \sum_i b_i y_i \hat{y} = b' D_y \left(I - A^{\text{up}}\right)^{-1} D_\Theta \hat{c} \\
&= b' D_y \left(I - A^{\text{up}}\right)^{-1} D_y^{-1} D_c \hat{c} \text{ as } \Theta_i = \frac{c_i}{y_i} \\
&= b' \left(I - D_y A^{\text{up}} D_y^{-1}\right)^{-1} D_c \hat{c} \\
&= b' \left(I - A'\right)^{-1} D_c \hat{c} = \iota' D_c \hat{c} \qquad \text{(D.22)} \\
&= \iota' D_c \hat{c} = \hat{c}_t^{\text{agg}} \qquad \text{(D.23)}
\end{aligned}$$

which is indeed 0 if there are no aggregate demand shocks.

So it may be useful to rewrite this as:

$$\left(I - A^{\text{up}}\right) \hat{y} = D_\Theta \hat{c} = D_y^{-1} D_c \hat{c}$$

so

$$D_c \hat{c} = D_y \left(I - A^{\text{up}}\right) D_y^{-1} D_y \hat{y} = \left(I - A'\right) D_y \hat{y}$$

Total demand shock Thus the total demand shock is:

$$\hat{\delta}^d = D_{\Theta^c}\hat{\delta}^c + D_{\Theta^y}\hat{\delta}^y \tag{D.25}$$

where Θ_j^y is the share of j's output absorbed in production. It is is made of $\hat{\delta}^c$, the demand shock from the consumers, and $\hat{\delta}^y$, the total demand shock from the other firms:

$$\hat{\delta}_j^y = \sum_i A_{ji}^{\text{up}}\hat{\theta}_{ij} \tag{D.26}$$

Hence,

$$\hat{\delta}_j = \Theta_j^c\hat{\delta}_j^c + \Theta_j^y\sum_i A_{ji}^{\text{up}}\hat{\theta}_{ij} \tag{D.27}$$

D.2.4 Frictions

The good in each industry is made of a continuum of Dixit-Stiglitz firms, so that we can talk about frictional decisions. We'll consider sticky prices and partial demand pass-through.With sticky prices, only a fraction ϕ of firms adjust their price in response to cost shocks. We assume that they are myopic. So, if the price should jump by $\hat{p}_t^*$, the actual price is $\hat{p}_t = \mathcal{D}^\phi\hat{p}_t^*$ where (D.15) is the delay operator.

In the partial demand pass-through, an increase of $\hat{y}_i$ of nominal demand for industry i translates into demand for the upstream industries with a lag, so that the demand is $\hat{x}_{it} = \mathcal{D}^\chi\hat{y}_{it}$, using again the delay operator (D.15). The rest is absorbed by an increase in prices, $p_{it} = \hat{y}_{it} - \hat{x}_{it}$. For instance, on impact, if the shock is $\hat{y}_i$, the increase in nominal demand for the supplies of good i is only $\hat{x}_i = \chi\hat{y}_i$, while the rest comes about as an increase in prices by $(1-\chi)\,\hat{y}_i$. The frictionless case corresponds to $\phi = \chi = 1$.

D.2.5 Nominal equilibrium selection

The formulas are all correct, in all equilibria, but which nominal equilibrium is selected depends on another assumption about the price level, as always in economics. For concrete-

ness, on can choose a constant CPI level, so $\theta'\hat{p} = 0$. But in general, we will not make that assumption.

D.3 Results

D.3.1 Behavior of the frictionless economy

We state our main results, before deriving them (mostly in Section D.5.). We start the demand elasticity.

Proposition 10. (Aggregate demand elasticity) *The aggregate Marshallian elasticity of nominal demand ($p_i c_i$) to prices is the following matrix $\zeta^d_{ij} = \frac{\partial \ln(p_i c_i)}{\partial \ln p_j}$ with:*

$$\zeta^d = D_{\Theta^y}\zeta^y + D_{\Theta^c}\zeta^c \tag{D.28}$$

where ζ^y (ζ^c) is the elasticity from the producer (consumer) demand, and Θ^y_i (Θ^c_i) is the share of good i going into production (consumption). In general, calling $\zeta^k_{ij} = \frac{\partial \ln\left(p_i c^k_i\right)}{\partial \ln p_j}$ the elasticity of demand for i by producer k, (for a given total expenditure on intermediary goods), the aggregate demand elasticity for j by producers is:

$$\zeta^{j,y} = \frac{\sum_i A^{up}_{ji}\zeta^{j|i}}{\sum_i A^{up}_{ji}} \tag{D.29}$$

In the CES case, the consumers' matrix of expenditure elasticities is

$$\zeta^{i,c}_j = -\left(\sigma - 1\right)\left(1_{i=j} - s^c_j\right) \tag{D.30}$$

where s^c_j is the share of good j in consumption. In the HCES (heterogeneous constant elasticity of substitution), with $m^s_j = \frac{\sigma_j s^c_j}{\sum_{j'}\sigma_{j'}s^c_{j'}}$, it is:

$$\zeta^{i,c}_j = -\left(\sigma_j - 1\right)1_{i=j} + \sigma_i m^c_j - s^c_j \tag{D.31}$$

In the CES case, the producer's expenditure elasticity $\zeta_{ij}^{d,y} = \frac{\partial \ln(p_i c_i)}{\partial \ln p_j}$ *is:*

$$\zeta_j^{i,y} = -(e-1)\left(1_{i=j} - s_j^{i,y}\right) \tag{D.32}$$

where $s_j^{i,y}$ *is the share of good* j *in the producer's inputs. In the HCES case, with* $m_j^i = \frac{\sigma_j s_j^{i,y}}{\sum_{j'} \sigma_{j'} s_{j'}^{i,y}}$*, and* $M_j^i = \frac{m_j^i \sigma_i - s_j^{i,y}}{\sigma_i - 1}$*, the producer demand elasticity is:*

$$\zeta_j^{i,y} = -(e_j - 1)\, 1_{i=j} + e_i m_j^{i,y} - s_j^{i,y} \tag{D.33}$$

In the CES case, we can also write:

$$\zeta^d = (\sigma - 1)\, D_{\Theta^c}\left(I - M^c\right) + (e-1)\, D_{\Theta^y}\left(I - M^p\right) \tag{D.34}$$

where $M_{ij}^c = \theta_j^c$ in the one-region case, and in the multi-region case, $M_{ij}^c = \theta_j^{c(i)}$, the consumption share of good j in the basket of country $c(i)$ where industry i is based; and $M^p = D_{\iota - \Theta}^{-1} A^{\text{up}} s$ gives the mean impact of competitors prices in consumption (for M^c) and in production (for M^p). They satisfy $M^c \iota = M^p \iota = \iota$, which confirm that they are averaging operators.

The intuition is the following. If a good j sees its price fall, then the nominal expenditure on that good rises in a way captured by (D.34): (i) by amount $-(\sigma - 1)\, \Theta_j \hat{p_j}$ from consumers (see the term $-(\sigma - 1)\, D_\Theta \hat{p})$), who absorb a fraction Θ_j of that good; and (ii) by an amount $-(e-1)\,(1 - \Theta_j)\, \hat{p_j}$ from the other producers (see the term $-(e-1)\, D_{\iota - \Theta} \hat{p}$), who absorb the remaining fraction $1 - \Theta_j$ of that good. But the price change in good j must be compared to the average price changes faced by consumers ($\Theta \theta^{c\prime} \hat{p}$) and by the producers, weighted by the importance for good j (that's the term $A^{\text{up}} s \hat{p}$). We note

$$\zeta^d \iota = 0. \tag{D.35}$$

which comes from the demand being homogenous of degree 0: if all prices increase proportionally, and total spending is the same, then spending on each good is the same.

Elasticity of supply We refer to Section D.6. The upshot is that if there are no sector specific factor, $(\zeta^s)^{-1} = 0$. If there some, then $(\zeta^s)^{-1}$ is diagonal, with entries $\frac{\beta_{if}}{\phi_{if}}$ where β_{if} is the share of the specific factor, and ϕ_{if} is the sum or its demand and supply elasticity. In the baseline model, we take $(\zeta^s)^{-1} = 0$.

Next, we derive the aggregate equilibrium (the proof is in Section D.5). We call $\hat{\delta}^d$ is the total demand shock (D.25).

Proposition 11. (Equilibrium in the frictionless economy: main nominal quantities) *Suppose a competitive, frictionless economy. Then, the impact of productivity shocks $\hat{z}$ and consumption demand shocks $\hat{\delta}^d$ is as follows. The nominal output vector y changes according to:*

$$(I - A^{up})\,\hat{y} = -\zeta^d \hat{p} + \hat{\delta}^d \tag{D.36}$$

with the ζ^d the matrix of elasticities of nominal demand to prices in (D.28), and $\hat{\delta}^d$ the total demand shock (D.25). The price vector p changes according to:

$$(I - A)\,\hat{p} = -\hat{z} + D_b \hat{w} + (\zeta^s)^{-1}\,\hat{y} + \xi^s \pi_* \tag{D.37}$$

and the vector of wage changes is $\hat{w} = \hat{\omega}\iota$ with

$$\hat{\omega} = -\hat{z}_L + \frac{1}{\zeta^L} s^{L\prime} \hat{y} + \xi^L \pi_*, \qquad \xi^L = 1 - \frac{1}{\zeta^L} \tag{D.38}$$

where $\hat{z}_L$ is a labor supply shocks, $s_i^L = \frac{b_i y_i}{\sum_j b_j y_j}$ is the share of total labor used by sector i, ζ^L is the (scalar) some of the the elasticities of labor supply and demand,;

$$\pi_* = \theta^{\pi_*\prime} \hat{p} \tag{D.39}$$

is a measure of the aggregate price level, and the vector ξ^s ensures inflation-invariance, i.e. that the equation still hold when all prices increase by ιx for any x, so that:

$$\xi^s = (I - A)\,\iota - b - (\zeta^s)^{-1}\,\iota \tag{D.40}$$

To gain intuition for (D.36), consider first the case without a price change, so that (D.36) is:

$$\hat{y}_j = \Theta_j \hat{\theta}_j + \sum_i A^{\text{up}}_{ji} \hat{y}_i$$

This means that nominal demand rises $\hat{y}_j$ if there is a demand shock $\hat{\theta}_j$ for good j; its impact is multiplied by the share of the production of j that goes into consumption, Θ_j. In addition, if downstream sectors i's nominal production increases by $\hat{y}_i$, this ripples into the production of upstream sectors, as captured by $\sum_i A^{\text{up}}_{ji} \hat{y}_i$.

When there are no specific factors, so that supply is perfectly elastic ($(\zeta^s)^{-1} = 0$), equation (D.37) becomes $(I - A)\,\hat{p} = -\hat{z} + D_b \hat{w}$ and is well known: prices fall because of TFP increases ($-\hat{z}$), and because of the prices of suppliers fall ($A\hat{p}$), which gives $\hat{p} = -\hat{z} + A\hat{z}$. But the term with the supply elasticity appears to be new.

Suppose that TFP shocks $\hat{z}$ are uncorrelated. Then, changes in prices and gross output are generically correlated, see (D.37). However, in a model with fixed labor supply, changes in value added remain uncorrelated. To create comovement in value added, one can have either flexible labor supply ($\psi > 0$), or deviation from Cobb-Douglas (i.e., elasticities σ or e different from 1).

For the weights (D.39), we shall take the CPI weights,

$$\theta^{\pi_*} = \theta^c \tag{D.41}$$

However, later we can vary this in behaviorally interesting ways. For instance, some research finds that people use the "wrong" weights to forecast inflation, e.g. put too much weight on good purchases often, as they are more salient. That could lead to interesting effects: e.g. oil has a disproportionate impact on the inflation level if its weights $\theta^{\pi_*}_{i=\text{oil}}$ is high.

We can plug in the wage in the system of prices, and get the following simply with goods prices data:

Proposition 12. (Aggregate supply shock, aggregate supply elasticity inclusive of labor

supply) *We have*

$$(I - A)\,\hat{p} = -\delta^s + \left(\tilde{\zeta}^s\right)^{-1}\hat{y} + \tilde{\xi}^s\pi_* \tag{D.42}$$

where the total supply shock δ^s is the sum of the TFP shock and the labor supply shock:

$$\delta^s = \hat{z} + b\hat{z}_L \tag{D.43}$$

with $b = (b_i)_{i=1\ldots N}$ is the vector of the labor shares, and $\tilde{\zeta}^s$ is the supply elasticity matrix (with dimensions $N \times N$), inclusive of the effect of labor supply:

$$\left(\tilde{\zeta}^s\right)^{-1} = (\zeta^s)^{-1} + \frac{1}{\zeta^L}bs^{L\prime} \tag{D.44}$$

and vector $\tilde{\xi}^s$, is the deflator adjustment.

$$\tilde{\xi}^s = \xi^s + b\xi^L \tag{D.45}$$

D.3.2 Behavior of the frictional economy

Next, we study what happens in a frictional economy.

Proposition 13. (Equilibrium in the frictional economy) *When prices have a fractional adjustment ϕ ($\phi = 1$ being the frictionless case), in the dynamic model, we have:*

$$\hat{p} = \mathcal{D}^\phi\left(A\hat{p} - \delta^s + \left(\tilde{\zeta}^s\right)^{-1}\hat{y} + \tilde{\xi}^s\pi_*\right) \tag{D.46}$$

$$\hat{y} = A^{up}\hat{y} - \zeta^d\hat{p} + \delta^d \tag{D.47}$$

where $\mathcal{D}^\phi$ is the delay operator (D.15). Equivalently, this is:

$$\Delta\hat{p}_t = \phi\left(A\hat{p}_t - \delta^s_t + \left(\tilde{\zeta}^s\right)^{-1}\hat{y}_t + \tilde{\xi}^s\pi_{*t} - \hat{p}_{t-1}\right) \tag{D.48}$$

$$\hat{y}_t = A^{up}\hat{y}_t - \zeta^d\hat{p}_t + \delta^d_t \tag{D.49}$$

Taking the time differences, and calling $\pi_t \coloneqq \hat{p}_t - \hat{p}_{t-1}$ the vector of sectoral inflations, we

have:

$$\pi_t = \phi\left(A\pi_t - \Delta\delta_t^s + \left(\tilde{\zeta}^s\right)^{-1}\Delta\hat{y}_t + \tilde{\xi}^s\Delta\pi_{*t}\right) + (1-\phi)\,\pi_{t-1} \tag{D.50}$$

$$\Delta\hat{y}_t = A^{up}\Delta\hat{y}_t - \zeta^d\pi_t + \Delta\delta_t^d \tag{D.51}$$

In (D.46), prices react as a distributed lag of past shocks, $A\hat{p} - \delta^s$.[26] This is reflected in (D.50): the sectoral inflation is an average of "target" inflation, with a weight ϕ, and past inflation, with a weight $1-\phi$.

If different sectors have different speeds of adjustment ϕ_i, then the same expressions hold replacing the scalar ϕ by with $\phi = Diag\,(\phi_i)$. So, one can in principle estimate the speed of price adjustment in each industry.

Empirical specification

$$\pi_t = \phi\left(A_{t-1}\pi_t - \Delta\delta_t^s + \left(\tilde{\zeta}^s\right)^{-1}\Delta\hat{y}_t + \tilde{\xi}^s\Delta\pi_{*t}\right) + (1-\phi)\,\pi_{t-1} + \lambda^\pi\eta_t + c^\pi \tag{D.53}$$

$$\Delta\hat{y}_t = A_{t-1}^{\text{up}}\Delta\hat{y}_t - \zeta^d\pi_t + \Delta\delta_t^d + \lambda^y\eta_t + c^y \tag{D.54}$$

where $c_i^\pi$and c_i^y are industry-specific constants.

We deflate then demean the data. Calling $\bar{\pi}_t$ the aggregate inflation from the GDP deflator (St Louis Fred code for GDP price index: A191RG3A086NBEA). We remove the impact of $\bar{\pi}_t$, i.e. replace π_{it} by $\pi_{it} - \bar{\pi}_t$, and replace $\Delta\hat{y}_{it}$ by $\Delta\hat{y}_{it} - \bar{\pi}_t$. We also demean each variable industry-by-industry (i.e. replace π_{it} by $\pi_{it} - \frac{1}{T}\sum_{s=1}^T \pi_{is}$).

We assume $\lambda_i^\pi = \left(\lambda_i^{\pi,1}, \lambda_i^{\pi,2}, \lambda_i^{\pi,3}\right)$ (perhaps with $\lambda_i^{\pi,1} \equiv 1$; and at first we just take 1 or 2 factors). We estimate: $\phi \simeq 0.8$, $\sigma^{\text{cons}} \simeq 0.5$, $\sigma^{\text{prod}} = e \simeq 0.5$, $\zeta^{L,d} \simeq 0.5$, $\zeta^{L,s} \simeq 0.5$, $\rho_u \simeq 0.5$,

26. The delays could be source-specific, and be subject to a more specific investigation. For instance, Equation (**??**) could be generalized to

$$(I - \phi_p A)\,\hat{p} = -\phi_z\hat{z} + \phi_w D_b\hat{w} + (1-\chi)\,\hat{y} \tag{D.52}$$

where ϕ_p, ϕ_z and ϕ_w are respectively the attention to other prices, productivity, and wages, with attention between 0 and 1, and the frictionless case being an attention of 1. Then, the microfoundation for sticky prices is inattention rather than adjustment costs, which imposes that all ϕ parameters are the same.

$\sigma_{u^d} \simeq 0.08, \sigma_{u^s} \simeq 0.08$ (in annual units).

Diagnostic tools: first, we hope that the values are not too far from this calibration (i.e., within a factor of 10). We plot the time series of u^d_{St},u^s_{St}, η^1_t,η^2_t, where $S_i = \frac{Salesi}{\sum_j Salesj}$ and $u^d_{St} = \sum_i S_i u^d_{it}$. Also, we report the 10 biggest $\left|S_i u^d_{it}\right|$ and 10 biggest $|S_i u^s_{it}|$ over the whole sample (i.e. report the industry i, the t, the value of $S_i u^d_{it}$, S_i and u^d_{it}); and we hope that they make narrative sense.

D.3.3 Complementary results

We record here some further results for the frictionless economy.

We express the results of Proposition 11 as a simple supply-and-demand framework.

Proposition 14. (Total supply and demand, as a function of supply and demand shocks) *Consider a system:*

$$\hat{y} = -\bar{\zeta}^d \hat{p} + \bar{\delta}^d \tag{D.55}$$

$$\hat{p} = \left(\bar{\zeta}^s\right)^{-1} \hat{y} - \bar{\delta}^s \tag{D.56}$$

Then, its solution is

$$\hat{y} = \left(I + \bar{\zeta}^d \frac{I}{\bar{\zeta}^s}\right)^{-1} \left(\bar{\delta}^d + \bar{\zeta}^d \bar{\delta}^s\right) \tag{D.57}$$

$$\hat{p} = \left(I + \frac{I}{\bar{\zeta}^s} \bar{\zeta}^d\right)^{-1} \left(\frac{I}{\bar{\zeta}^s} \bar{\delta}^d - \bar{\delta}^s\right) \tag{D.58}$$

or, in terms of $\bar{\zeta}^s$ rather than $\frac{I}{\bar{\zeta}^s}$,

$$\hat{y} = \bar{\zeta}^s \left(\bar{\zeta}^s + \bar{\zeta}^d\right)^{-1} \left(\bar{\delta}^d + \bar{\zeta}^d \bar{\delta}^s\right) \tag{D.59}$$

$$\hat{p} = \left(\bar{\zeta}^s + \bar{\zeta}^d\right)^{-1} \left(\bar{\delta}^d - \bar{\zeta}^s \bar{\delta}^s\right) \tag{D.60}$$

We next apply this to our concrete economy.

Proposition 15. (Total supply and demand, as a function of supply and demand shocks, in the frictionless economy) *In our frictionless economy, we have the equations (D.56)-(D.56), where $\bar{\zeta}^d$ and $\bar{\zeta}^s$ are the long run demand and supply elasticities, gross of network effects, and $\bar{\delta}^d$ and $\bar{\delta}^s$ are the supply shocks, also gross of network effects:*

$$\begin{aligned}
\bar{\zeta}^d &\coloneqq (I - A^{up})^{-1} \zeta^d && \text{(D.61)}\\
\left(\bar{\zeta}^s\right)^{-1} &\coloneqq \left(I - A - \tilde{\xi}^s \theta^{\pi_*\prime}\right)^{-1} \left(\tilde{\zeta}^s\right)^{-1}, \qquad \bar{\zeta}^s = \tilde{\zeta}^s \left(I - A - \tilde{\xi}^s \theta^{\pi_*\prime}\right) && \text{(D.62)}\\
\bar{\delta}^d &\coloneqq (I - A^{up})^{-1} \hat{\delta}^d && \text{(D.63)}\\
\bar{\delta}^s &\coloneqq \left(I - A - \tilde{\xi}^s \theta^{\pi_*\prime}\right)^{-1} \hat{\delta}^s, && \text{(D.64)}
\end{aligned}$$

where $\tilde{\zeta}^s$ is in (D.44).

We now describe the rest of the economy.

Proposition 16. (Equilibrium in the frictionless economy: other real quantities) *The vectors of changes in real production $\hat{Q}$, real value added $\widehat{VA}$ and labor input $\hat{L}$ are:*

$$\hat{Q} = \hat{y} - \hat{p}, \qquad \widehat{VA} = D_{\frac{1}{b}} \hat{z} + \hat{L}, \qquad \hat{L} = \hat{y} - \hat{w} \tag{D.65}$$

With one region and one labor market, aggregate real GDP changes by

$$\hat{Y}^{tot} = \gamma' \hat{z} + \hat{L}^{tot}$$

where $\gamma_i = \frac{y_i}{gdp}$ is the Domar weight of sector i. Aggregate wage w, total labor supply L^{tot} and labor income v change by (with $\hat{w} = \hat{\omega}\iota$, where $\hat{\omega}$ is a scalar)

$$\hat{\omega} = (1 - \psi)\, s^{L\prime} \hat{y} + \psi \theta' \hat{p}, \qquad \hat{L}^{tot} = \psi \left(s^{L\prime} \hat{y} - \theta' \hat{p}\right), \qquad \hat{v} = \hat{\omega} + \hat{L}^{tot} = s^{L\prime} \hat{y} \tag{D.66}$$

With several regions and labor markets, aggregate value added in region r changes by:

$$\hat{Y}_r^{tot} = \gamma^{r\prime} \hat{z} + \hat{L}_r^{tot}$$

where $\gamma_i^r = \frac{m_{ri} y_i}{\sum_{i'} m_{ri'} b_{i'} y_{i'}}$ *is the Domar weight of industry* i *in region* r*, where* $m_{\ell i} = 1$ *iff industry* i *is in market* ℓ*, and* $\hat{L}_r^{tot} = \hat{L}_{\ell(r)}^{tot}$ *where* $\ell(r)$ *is the the labor market in region* r*. Call* $\hat{\omega} = (\hat{\omega}_\ell)_{\ell=1...N_L}$ *the vector of nominal changes in the wage in market* ℓ*, and* $\hat{L}^{tot} = \left(\hat{L}_\ell\right)_{\ell=1...N_L}$ *the vector of total labor supply in markets* ℓ*, we have:*

$$\hat{\omega} = (1-\psi) M^L \hat{y} + \psi M^{c,L} \hat{p}, \qquad \hat{L}^{tot} = \psi \left(M^L \hat{y} - M^{c,L} \hat{p}^c\right), \qquad \hat{v} = \hat{\omega} + \hat{L}^{tot} = M^L \hat{y} \tag{D.67}$$

where $M_{\ell j}^L = \frac{b_{j,\ell} y_j}{\sum_{j'} b_{j',\ell} y_{j'}}$ *is the fraction of labor of type* ℓ *going to industry* j*, and* $M_{\ell j}^{c,L} = \theta_j^\ell$ *is the the share of good* j *in the consumption basket of labor of type* ℓ*. Also,* $\hat{w}_i = \hat{\omega}_{\ell(i)}$ *where* $\ell(i)$ *is the type of labor used in industry* i*.Complements to the macroeconomic model*

D.4 Complements to the heterogeneous CES utility and production functions

D.4.1 Complements

We first verify that the preferences are well-behaved. We sometimes use the terminology "Heterogeneous CES (HCES)" for Hanoch (1971)'s "CRESH".

Lemma 2. (HCES preferences are well-behaved) *The Heterogeneous CES (HCES) test function (D.2) has a unique solution* $u(c)$*. The utility function is concave, increasing and homogeneous of degree 1.*

Proof of Lemma 2 Calling $f(u,c)$ the test function in (D.2), f is decreasing in u, and increasing in c. That guaranties the unicity of the solution, and the fact that $u(c)$ is increasing. Also, $f(u,c)$ is homogeneous of degree 0, which implies that $u(c)$ is homogeneous of degree 1: if $f(u,c) = 0$, the $f(\lambda u, \lambda c) = 0$, which implies $u(\lambda c) = \lambda u(c)$.

We next prove that function $u(c)$ is concave. First, we see that $f(u,c)$ is concave in c for any u. The heuristic argument is the following: for small random changes $d\tilde{c}$ with mean 0, by Taylor expansion the utility changes by $du = -\frac{1}{2f_u}\mathbb{E}\left[d\tilde{c} f_{cc} d\tilde{c}\right] \leq 0$, which proves that u is concave. Next, let us make this argument rigorous. Consider the price vector

$p \coloneqq \frac{u_c}{u} = \frac{-f_c}{uf_u}$, a change $\delta c = c + h\varepsilon$ for a small ε, with $p'h = 0$. We want to prove that $q \coloneqq (\delta c)' u_{cc} (\delta c) \leq 0$. Given that $u(c)$ is homogeneous of degree 1, $u_{cc}c = 0$, so $q = \varepsilon^2 h' u_{cc} h$. Next, consider $v(\varepsilon) = u(c + h\varepsilon)$, i.e. the utility impact of a change of consumption by $h\varepsilon$. We have $v_\varepsilon = \frac{-1}{f_u} f_c h = -up'h = 0$. We next calculate the second order terms. We have:

$$0 = f(v(\varepsilon), c + h\varepsilon) = f\left(v(0) + v''(0)\frac{\varepsilon^2}{2} + o\left(\varepsilon^2\right), c + h\varepsilon\right) = f_u v''(0)\frac{\varepsilon^2}{2} + f_c h\varepsilon + \frac{1}{2}h' f_{cc} h\varepsilon^2 + o\left(\varepsilon^2\right)$$

so that (as $f_c h = 0$), $v''(0) = \frac{-1}{2f_u} h' f_{cc} h$ (i.e. informally $du = -\frac{1}{f_u} dc' f_{cc} dc$, for $p'dc = 0$). This gives: $q = \varepsilon^2 h' u_{cc} h = \frac{-1}{f_u} h' f_{cc} h\varepsilon^2$, which is negative as f is concave in c. Hence, $q \leq 0$, which proves the concavity of u.

For a variety of purposes (e.g., distortions from taxation or behavioral biases), it is useful to know the second derivative of the utility function. Accordingly, we derive it in the following result.

Proposition 17. (Second order derivative of $u(c)$). *A change $\hat{c}$ creates a change in equivalent expenditure $\hat{w} \coloneqq s'\hat{c}$ to the first order; a utility gain which to the first order is $\hat{u} = \hat{w} + O\left(\|\hat{c}\|^2\right)$, and to the second order is*

$$\hat{u} = \hat{w} - \sum_i \frac{s_i}{2\sigma_i} (\hat{c}_i - \hat{w})^2 + O\left(\|\hat{c}\|^3\right) \tag{D.68}$$

The second derivative of $u(c)$ can be equivalently expressed as follows. Consider the surrogate price vector $p \coloneqq \frac{-f_c}{uf_u}$, the income $w = p'c$, and $Q = I - \frac{cp'}{w}$ the projection to the space orthogonal to c. Then:

$$u_c = up, \qquad u_{cc} = -Q' \frac{f_{cc}}{f_u} Q \tag{D.69}$$

Proof. To be added. □

So, if consumption increases proportionally for all goods ($\hat{c}_i = \hat{w}$ for all i), then $\hat{u} = \hat{w}$. Otherwise, the utility gain is less, proportionally to the squared "distortion" $(\hat{c}_i - \hat{w})^2$, weighted by by the inverse elasticity σ_i: distorting non-elastic goods ("necessities", low σ_i) is more costly in utility terms.

We state the related result.

Lemma 3. *(Deadweight losses from taxation with HCES preferences) Suppose a fractional (i.e. "percent") tax of τ_i percent on good i. Then, for small taxes, the deadweight losses from taxation, L, are given by:*

$$L = \frac{w}{2}\bar{\sigma}\left(\mathbb{E}_{s\sigma}\left[\tau_i^2\right] - \mathbb{E}_{s\sigma}\left[\tau_i\right]^2\right) = \frac{w}{2}\sum_i \sigma_i s_i \tau_i^2 - \frac{w}{2}\frac{1}{\bar{\sigma}}\left(\sum_i \sigma_i s_i \tau_i\right)^2 \tag{D.70}$$

up to third-order terms $o\left(\tau_i^3\right)$, where $\mathbb{E}_{s\sigma}\left[x_i\right] \coloneqq \frac{\sum_i s_i \sigma_i x_i}{\sum_i s_i \sigma_i}$ is the average of x_i with weights proportional to $s_i\sigma_i$.

Proof. We take the Slutsky matrix S from (D.6). So with a tax $t_i = \tau_i p_i$, the deadweight losses L from taxation t are such that:

$$\begin{aligned} 2L &= -\ t'St = -\tau' Diag\left(p\right) S Diag\left(p\right)\tau = \sum_i \sigma_i s_i \tau_i^2 w - w\sum_{i,j}\frac{\sigma_i\sigma_j}{\bar{\sigma}} s_i s_j \tau_i \tau_j \\ \frac{2L}{w\bar{\sigma}} &= \frac{1}{\bar{\sigma}}\sum_i \sigma_i s_i \tau_i^2 - \frac{1}{\bar{\sigma}^2}\left(\sum_i \sigma_i s_i \tau_i\right)^2 = \mathbb{E}_{s\sigma}\left[\tau_i^2\right] - \mathbb{E}_{s\sigma}\left[\tau_i\right]^2 \end{aligned}$$

The main virtue of this proposition is to offer, in future studies, a quantitative framework to derive well-calibrated HCES demands. Otherwise, its qualitative results are familiar. The losses are 0 with a uniform percent tax (τ_i the same for all i), otherwise, the distortions are proportional to σ_i, the elasticity of substitution of the good. For instance, if only good i is taxed, the loss is $L = \frac{w}{2}\sigma_i s_i\left(1 - \frac{s_i\sigma_i}{\bar{\sigma}}\right)\tau_i^2$, so that if its expenditure share is small, the loss is $L \sim \frac{w}{2}\sigma_i s_i \tau_i^2$. □

Behavior at extreme prices It is sometimes useful to see the behavior at extreme prices, e.g. to understand the economics of deep crises (see Dew-Becker (2023)).[27] We examine this for the heterogeneous CES preferences.

27. We thank Ian Dew-Becker for prompting us to think about this question.

Two goods For simplicity, we consider two goods, with p_2 fixed, and vary p_1. We consider what happens if $p_1 \to \infty$.[28,29]

If $\sigma_1 > 1$, then the share of good 1 goes to 0 and the prices indices μ, $\bar{p}$ go to a constant. It's as if good 1 disappeared: its consumption share is 0, and it does not influence prices and behavior.

If $\sigma_1 < 1$ and $\sigma_2 < 1$, then the "necessary" good 1 gets all the share, the the price index is proportional to p_1. It's as if good 2 disappeared. We have: $\mu \propto \bar{p} \propto p_1$, $s_1 \to 1$, so $u \to 0$.

If $\sigma_1 < 1$ and $\sigma_2 > 1$, then the situation is a bit more complex, and indeed unexpected. The price index go to infinity, but less than proportionally to p_1 — rather, as p_1^α for an $\alpha = \frac{(1-\sigma_1)\sigma_2}{\sigma_2-\sigma_1} < 1$.[30] The relatives share s_1 and s_2 both go to non-zero constants ($s_2 \to \bar{s}_2 = \frac{(\sigma_2-1)\sigma_1}{\sigma_2-\sigma_1} \in (0,1)$, $s_1 \to 1-\bar{s}_2$). This agent is in misery (her utility goes to 0), but still spends asymptotically a constant and positive share of income on "luxuries" (s_2 tends to a non-zero constant, as does s_1).

N goods More generally, suppose N goods, and that $p_i \sim e^{g_i t}$ with $g_i > 0$ (one can later refine to $g_i \geq 0$), likeDew-Becker (2023). Then $p \sim e^{gt}$ with the following value for g.

If all the σ_i are less than 1. Then,

$$g = \max_{i:\sigma_i<1} g_i \tag{D.71}$$

Indeed, the maximum term in (D.8) goes to a constant, while the other terms are $e^{(1-\sigma_i)(g_i-g)t}$ so to 0 or a constant, as $(1-\sigma_i)(g_i-g) \leq 0$. Economically, one might have thought that it is $(1-\sigma_i)g_i$ that matters, but it is only g_i. If all goods are inelastically demanded, only the most expensive good matters asymptotically, irrespective of its demand elasticity.

28. The proof works by considering first the behavior of μ from Proposition 9, which then gives the price and the shares.

29. The case $p_1 \to 0$ is symmetrical, as it is the relative price p_1/p_2 that matters.

30. Indeed, the first two terms in (D.8) go to infinity while balancing each other out: $\left(\frac{p_1}{\mu}\right)^{1-\sigma_1} \propto \left(\frac{p_2}{\mu}\right)^{1-\sigma_2}$. That implies $\mu \propto p_1^{\frac{1-\sigma_1}{\sigma_2-\sigma_1}}$ and, by (D.9), $\bar{p} \propto p_1^{\frac{(1-\sigma_1)\sigma_2}{\sigma_2-\sigma_1}}$.

If all the σ_i are greater than 1. Then, define

$$g = \min_{i:\sigma_i>1} g_i \tag{D.72}$$

Indeed, the maximum term in (D.8) goes to a constant, while the other terms are $e^{(1-\sigma_i)(g_i-g)t}$ so to 0 or a constant, as $(1-\sigma_i)(g_i-g) \leq 0$.

If some σ_i are above, some below 1. We reason as we did above with just two goods: we want the maximum of the negative and positive terms in (D.8) to balance out. Hence, $\mu \sim e^{\gamma t}$ with γ satisfying

$$G \coloneqq \max_{i:\sigma_i<1} (1-\sigma_i)(g_i-\gamma) = \max_{i:\sigma_i>1} (1-\sigma_i)(g_i-\gamma) \tag{D.73}$$

Then (D.9) gives:

$$g = \gamma + G \tag{D.74}$$

How to fit an H-CES model to real data Suppose that we have real data, including fraction s_i spent on each good. How do we fit the model to real data? We assume that we somehow estimated σ_i — there are a number of standard ways to estimate elasticities.

One way to calibrate the model is take (D.84), and to normalize $u = \mu = 1$ at the initial point. The fraction spent on good i is: $s_i = \frac{p_i c_i}{\sum_{i'} p_{i'} c_{i'}} = \frac{c_i^{1-\frac{1}{\sigma_i}}}{\sum_j c_i^{1-\frac{1}{\sigma_i}}}$. We can then fit

$$c_{i0}^{1-\frac{1}{\sigma_i}} = s_{i0} \tag{D.75}$$

i.e. $c_{i0} = s_{i0}^{\frac{\sigma_i}{\sigma_i-1}}$. Call C_i the quantity in the data, and c_i in the model. The conversion factor is $c_i = K_i C_i$, where K_i is an once-and-for-all choice of units. For instance, C_i might be the number of "quality-adjusted cars", or tons of steel, and K_i converts the units.

Then, we can then subject the model to shocks, and see how the shares evolve, following Proposition 8.

D.4.2 Proofs

Proof of Proposition 8 We consider the problem $\min_c p \cdot c$ subject to $u(c) = u$. We form

$$L = \sum_i p_i c_i - \lambda \left(\frac{\left(c_i / u(c)\right)^{1-\frac{1}{\sigma_i}} - 1}{1 - \frac{1}{\sigma_i}} + \phi \right)$$

The FOC in c_i is

$$p_i = \lambda c_i^{-\frac{1}{\sigma_i}} u^{\frac{1}{\sigma_i}-1} \tag{D.76}$$

i.e. $p_i = \mu c_i^{-\frac{1}{\sigma_i}} u^{\frac{1}{\sigma_i}}$ with $\mu \coloneqq \frac{\lambda}{u}$. Hence, after a change

$$\hat{p}_i = \hat{\mu} - \frac{1}{\sigma_i} (\hat{c}_i - \hat{u}) \tag{D.77}$$

i.e.

$$\hat{c}_i = -\sigma_i (\hat{p}_i - \hat{\mu}) + \hat{u} \tag{D.78}$$

Expenditure changes as (with $\bar{\sigma} \coloneqq \sum_i s_i \sigma_i$)

$$\hat{w} = \sum_i s_i (\hat{p}_i + \hat{c}_i) = \sum_i s_i \hat{p}_i - \sum_i s_i \sigma_i \hat{p}_i + \left(\sum_i s_i \sigma_i \right) \hat{\mu} + \hat{u} \tag{D.79}$$

or, more compactly:

$$\hat{w} = (s' - \bar{\sigma} m') \hat{p} + \bar{\sigma} \hat{\mu} + \hat{u} \tag{D.80}$$

so that

$$\hat{\mu} = \frac{\hat{w} - \hat{u} + (s' - \bar{\sigma} m') \hat{p}}{\bar{\sigma}} \tag{D.81}$$

Consider the compensated (Hicksian) demand, i.e. consumption when prices change, but not utility: $\hat{u} = 0$. But expenditure must have changed by the compensation $\hat{w} = s'\hat{p}$, so (D.81) gives $\hat{\mu} = m'\hat{p}$. Given (D.78), we obtain $\hat{c}_i = -\sigma_i (\hat{p}_i - m'\hat{p})$. If $\hat{u} \neq 0$, consumption changes by an additional $\hat{u}$ as the utility is homothetic of degree 1.

Next, consider the Marshallian demand. Then, given $\hat{w} = 0$, utility changes by $\hat{u} = -s'\hat{p}$

(by Roy's identity), so (D.81) gives also $\hat{\mu} = m'\hat{p}$. Given (D.78), we obtain

$$\hat{c}_i = -\sigma_i \left(\hat{p}_i - m'\hat{p}\right) - s'\hat{p}$$

In expenditure formulation

$$\hat{x}_i = \hat{p}_i + \hat{c}_i = -\sigma_i \left(\hat{p}_i - m'\hat{p}\right) + p_i - s'\hat{p} \tag{D.82}$$

If $\hat{w} \neq 0$, consumption changes by an additional $\hat{w}$ as the utility is homothetic of degree 1.

The Slutsky matrix can be read off (D.4):

$$S_{ij} = -\sigma_i c_i \left(1_{i=j} - m_j\right) \frac{1}{p_j}$$

When $i \neq j$, we re-express it to verify symmetry: (with $\bar{\sigma} = \sum_{j'} \sigma_{j'} s_{j'}$)

$$S_{ij} = \sigma_i c_i m_j \frac{1}{p_j} = \sigma_i \frac{p_i c_i}{w} m_j \frac{w}{p_i p_j} = \sigma_i s_i \frac{\sigma_j s_j}{\bar{\sigma}} \frac{w}{p_i p_j} = \frac{\sigma_i \sigma_j s_i s_j}{\bar{\sigma} p_i p_j} w$$

and the first term in S_{ii} is $-\sigma_i \frac{c_i}{p_i} = -\sigma_i \frac{p_i c_i}{p_i^2} = -w \frac{\sigma_i s_i}{p_i^2}$, so that for general i, j:

$$S_{ij} = -w \frac{\sigma_i s_i}{p_i^2} 1_{i=j} + w \frac{\sigma_i \sigma_j s_i s_j}{\bar{\sigma} p_i p_j}$$

□

Proof of Proposition 9 Take the f.o.c. (D.76),

$$p_i = \lambda c_i^{-\frac{1}{\sigma_i}} u^{\frac{1}{\sigma_i} - 1} = \left(\frac{c_i}{u}\right)^{-\frac{1}{\sigma_i}} \lambda u^{-1} \tag{D.83}$$

so, with $\mu = \frac{\lambda}{u}$,

$$\frac{p_i}{\mu} = \left(\frac{c_i}{u}\right)^{-\frac{1}{\sigma_i}} \tag{D.84}$$

hence

$$\left(\frac{p_i}{\mu}\right)^{1-\sigma_i} = \left(\frac{c_i}{u}\right)^{1-\frac{1}{\sigma_i}}$$

hence

$$0 = \phi_0 + \sum_i \frac{1}{1-\frac{1}{\sigma_i}} \left(\frac{c_i}{u}\right)^{1-\frac{1}{\sigma_i}} = \phi_0 + \sum_i \frac{1}{1-\frac{1}{\sigma_i}} \left(\frac{p_i}{\mu}\right)^{1-\sigma_i} \tag{D.85}$$

The budget constraint implies (using (D.84))

$$w = \sum_i p_i c_i = \sum_i p_i u \left(\frac{p_i}{\mu}\right)^{-\sigma_i} = u\mu \sum_i \left(\frac{p_i}{\mu}\right)^{1-\sigma_i}$$

and $\bar{p}$ is characterized by $u = \frac{w}{\bar{p}}$, we obtain (D.9).

In addition, (D.84) gives

$$c_i = \left(\frac{p_i}{\mu}\right)^{-\sigma_i} u = \left(\frac{p_i}{\mu}\right)^{-\sigma_i} \frac{w}{\bar{p}} =: C_i\left(p_i, \bar{p}, \mu\right) w \tag{D.86}$$

hence (D.10).

To interpret μ, consider the effect of a change in prices. Differentiating (D.8) gives:

$$0 = \sum_i \left(\frac{p_i}{\mu}\right)^{1-\sigma_i} \sigma_i \left(\hat{p}_i - \hat{\mu}\right) = \frac{\bar{p}}{\mu} \sum_i s_i \sigma_i \left(\hat{p}_i - \hat{\mu}\right)$$

Hence,

$$\hat{\mu} = \frac{\sum_i s_i \sigma_i \hat{p}_i}{\sum_i s_i \sigma_i} = \sum_i m_i \hat{p}_i$$

Also, (D.9) allows to verify $\widehat{\bar{p}} = \sum_i s_i \hat{p}_i$, which is true for all utility functions:

$$\widehat{\bar{p}} = \hat{\mu} + \sum_i \left(1 - \sigma_i\right) s_i \left(\hat{p}_i - \hat{\mu}\right) = \sum_i s_i \hat{p}_i$$

D.5 Derivations of the macroeconomic model

D.5.1 Derivation of the frictionless economy

Some accounting identities We record that:

$$A^{\text{up}}\iota = \Theta^y = \iota - \Theta^c \tag{D.87}$$

Another way to record the same thing is that the intermediate absorption of good j is:

$$y_j - c_j = \sum_i x_{ij} = \sum_i A_{ij} y_i = \left(A'y\right)_j = y_j \sum_i A^{\text{up}}_{ji}$$

so

$$y - c = A'y = D_y A^{\text{up}}\iota \tag{D.88}$$

In addition, because the production is CRS, $b_i + \sum_j A_{ij} = 1$ (recall that b_i is the labor share for producer i), i.e.

$$b + A\iota = \iota \tag{D.89}$$

Optimization by the consumer We consider first the case where total nominal income is constant, and CES demand. Because consumption is proportional to income, it is easy to add the general at the end. As nominal income is always 1, we have:

$$\hat{c}_i = (\sigma - 1)\left(\theta'\hat{p} - \hat{p}_i\right) \tag{D.90}$$

i.e.

$$\hat{c} = (\sigma - 1)\left(M^c - I\right)\hat{p} \tag{D.91}$$

with $M^c_{ij} = \theta_j$.

Cost minimization by producer i From optimization of labor by producer i gives:

$$w_i L_i = b_i y_i \tag{D.92}$$

hence

$$\hat{L}_i = \hat{y}_i - \hat{w}_i \tag{D.93}$$

The amount spending by producer i on intermediary goods, $x_i = \sum_j x_{ij}$ is

$$x_i = (1 - b_i)\, y_i$$

so

$$\hat{x_i} = \hat{y}_i \tag{D.94}$$

Now, given x_i, the change in nominal spending on by i on j is:

$$\hat{x}_{ij} = \hat{x}_i + (e-1)\left(s^i\hat{p} - \hat{p}_j\right) \tag{D.95}$$

where $s^i\hat{p}$ is the average price increase among intermediary goods used by i.

Absorption of production The change in the demand vector $\hat{c}$, rescaled by D_{Θ^c}, is (using Proposition 10)

$$D_{\Theta^c}\hat{c} = (\sigma - 1)\, D_{\Theta^c}\left(\iota\theta^{c\prime} - I\right)\hat{p} = (\sigma - 1)\left(\Theta^c\theta^{c\prime} - D_{\Theta^c}\right)\hat{p} \tag{D.96}$$

The total change in (nominal) production of j, $\hat{y}_j$, is absorbed as:

$$\begin{aligned}
\hat{y}_j &= \frac{c_j}{y_j}\hat{c}_j + \sum_i \frac{x_{ij}}{y_j}\hat{x}_{ij} = \Theta_j\hat{c}_j + \sum_i A^{\text{up}}_{ji}\hat{x}_{ij} \\
&= \Theta^c_j\hat{c}_j + \sum_i A^{\text{up}}_{ji}\left(\hat{y}_i + (e-1)\left(s^i\hat{p} - \hat{p}_j\right)\right) \\
&= \Theta^c_j\hat{c}_j + \sum_i A^{\text{up}}_{ji}\hat{y}_i + (e-1)\left(\left(\sum_i A^{\text{up}}_{ji}s^i\hat{p}\right) - \Theta^y_j\hat{p}_j\right)
\end{aligned}$$

as $\sum_i A^{\text{up}}_{ji} = \Theta^y_j = 1 - \Theta^c_j$. This also implies $A^{\text{up}} s\iota = A^{\text{up}}\iota = \Theta^y$.

The term $-(e-1)\,\Theta^y_j \hat{p}_j$ means that if good j is cheaper, and $e > 1$, then the demand for that good from downstream firms i is higher. The term $(e-1)\left(\sum_i A^{\text{up}}_{ji} s^i \hat{p}\right)$ means that this effect needs to be compared to the average price $s^i\hat{p}$ of all goods supplying to downstream firms i. In vector form:

$$\hat{y} \;=\; D_{\Theta^c}\hat{c} + A^{\text{up}}\hat{y} + (e-1)\left(A^{\text{up}} s - D_{\Theta^y}\right)\hat{p}$$

so, using (D.96),

$$(I - A^{\text{up}})\,\hat{y} = -(\sigma-1)\left(D_{\Theta^c} - \Theta^c\theta^{c\prime}\right)\hat{p} - (e-1)\left(D_{\Theta^y} - A^{\text{up}} s\right)\hat{p}. \tag{D.97}$$

If there's an increase in total nominal consumption $\hat{c}^{\text{tot}}$, we add that to the right-hand side of (D.97).

More generally, with non-homogeneous elasticities of demand. The demand for good j by sector i is

$$\hat{x}_{ij} = -\zeta^{j|i}\hat{p} + \hat{y}_i \tag{D.98}$$

where $\zeta^{j|i}$ is vector of demand elasticity vector, corresponding to the demand of sector k: its entries are $\zeta^{j|i}_l = \frac{\partial \ln\left(p_j c^i_j\right)}{\partial \ln p_l}$.It is as in as in Proposition 10,

So, the total change in (nominal) production of j, $\hat{y}_j$, is absorbed as:

$$\begin{aligned}
\hat{y}_j &= \frac{c_j}{y_j}\hat{c}_j + \sum_i \frac{x_{ij}}{y_j}\hat{x}_{ij} = \Theta^c_j\hat{c}_j + \sum_i A^{\text{up}}_{ji}\hat{x}_{ij} \\
&= \Theta^c_j\hat{c}_j + \sum_i A^{\text{up}}_{ji}\left(\hat{y}_i - \zeta^{j|i}\hat{p}\right) \\
&= \Theta^c_j\hat{c}_j + \sum_i A^{\text{up}}_{ji}\hat{y}_i - \sum_i A^{\text{up}}_{ji}\zeta^{j|i}\hat{p} \\
\hat{y}_j &= \Theta^c_j\hat{c}_j + \sum_i A^{\text{up}}_{ji}\hat{y}_i - \Theta^y_j\bar{\zeta}^{j|Y}\hat{p}
\end{aligned}$$

as

$$\sum_i A^{\text{up}}_{ji} = \Theta^y_j$$

is the fraction of good j going into production rather than consumption, and

$$\zeta^{j|Y}_l \coloneqq \frac{\sum_i A^{\text{up}}_{ji} \zeta^{j|i}_l}{\sum_i A^{\text{up}}_{ji}} \in \mathbb{R}^N \tag{D.99}$$

is the average elasticity of demand for good j w.r.t. price of good l in the production sector. With $\zeta^Y_{jl} \coloneqq \zeta^{j|Y}_l$, we have, in vector form,

$$\hat{y} = D_{\Theta^c} \hat{c} + A^{\text{up}} \hat{y} - D_{\Theta^y} \zeta^Y \hat{p} \tag{D.100}$$

We also have

$$\hat{c} = \iota s^{L\prime} \hat{y} - \zeta^C \hat{p} + \hat{\theta}$$

so

$$\left(I - A^{\text{up}} - D_{\Theta^c} \iota s^{L\prime}\right) \hat{y} = -\zeta \hat{p} + \hat{\theta}$$

where

$$\zeta = D_{\Theta^c} \zeta^c + D_{\Theta^y} \zeta^y$$

is the average elasticity of demand expenditures, averaging over demand from the consumers (ζ^c) vs producers (ζ^y).

Value added The value added to sales ratio is

$$\frac{VA_i}{p_i Q_i} = \frac{w_i L_i}{p_i Q_i} = b_i$$

The change in real value added is

$$\widehat{VA}_i = \frac{p_i dQ_i - \sum_j p_j dQ_{ij}}{VA_i} = \frac{p_i Q_i}{VA_i} \hat{z}_i + \frac{w_i L_i}{VA_i} \hat{L}_i = \frac{1}{b_i} \hat{z}_i + \hat{L}_i$$

$$\widehat{VA}_i = \frac{1}{b_i}\hat{z}_i + \hat{L}_i \tag{D.101}$$

Given that $\hat{L}_i = \hat{y}_i - \hat{w}_i$ (see (D.93)),

$$\widehat{VA}_i = \frac{1}{b_i}\hat{z}_i + \hat{y}_i - \hat{w}_i$$

Labor market: wage and labor supply *Labor demand.* Call $s_i^L = \frac{L_i}{L}$ the share of labor in sector i before the shock

$$s_i^L = \frac{L_i}{L^{tot}} = \frac{b_i y_i}{\sum_j b_j y_j} \tag{D.102}$$

As $\hat{L}_i = \hat{y}_i - \hat{w}$, total labor changes L^{tot} is

$$\hat{L}^{tot} = \sum_i s_i^L \hat{L}_i = \sum_i s_i^L \left(\hat{y}_i - \hat{w}\right) = \left(\sum_i s_i^L \hat{y}_i\right) - \hat{w}$$

$$\hat{L}^{tot} = s^{L\prime}\hat{y} - \hat{w} \tag{D.103}$$

Labor supply. The utility of the representative worker is $C - \psi L^{\frac{1}{\psi}}$ with $\psi \geq 1$, and C the utility aggregator, homogenous of degree 1. Hence, the FOC for labor supply is

$$L^{\frac{1}{\psi}-1} = \frac{w}{p^c}$$

where $p^c$is the price index of consumption. Hence, the elasticity of labor supply is

$$\zeta^{L,s} = \frac{\psi}{1-\psi} \tag{D.104}$$

and $\psi = \frac{\zeta^{L,s}}{1+\zeta^{L,s}}$. This gives the labor supply

$$\left(\frac{1}{\psi} - 1\right)\hat{L}^{tot} = \hat{w} - \hat{p}^c \tag{D.105}$$

and the elasticity of labor supply is $\frac{\psi}{1-\psi}$. Given labor demand is $\hat{L}^{tot} = s^{L\prime}\hat{y} - \hat{w}$ (cf (D.103)),

we have $\left(\frac{1}{\psi}-1\right)\hat{L}^{tot}=s^{L\prime}\hat{y}-\hat{L}^{tot}-\hat{p}^{c}$, so

$$\hat{L}^{tot}=\psi\left(s^{L\prime}\hat{y}-\hat{p}^{c}\right) \tag{D.106}$$

and

$$\hat{w}=\left(1-\psi\right)s^{L\prime}\hat{y}+\psi\hat{p}^{c} \tag{D.107}$$

and total income v changes as

$$\hat{v}=\hat{w}+\hat{L}^{tot}=s^{L\prime}\hat{y} \tag{D.108}$$

Proof of Proposition 14 Next, we solve the system.

$$\hat{y}=-\bar{\zeta}^{d}\hat{p}+\bar{\delta}^{d}=-\bar{\zeta}^{d}\left(\frac{I}{\bar{\zeta}^{s}}\hat{y}-\bar{\delta}^{s}\right)+\bar{\delta}^{d}$$

so

$$\hat{y}=\left(I+\bar{\zeta}^{d}\frac{I}{\bar{\zeta}^{s}}\right)^{-1}\left(\bar{\delta}^{d}+\bar{\zeta}^{d}\bar{\delta}^{s}\right)$$

Also,

$$\begin{aligned}\hat{p} &= \left(\bar{\zeta}^{s}\right)^{-1}\hat{y}+\bar{\delta}^{s}=\left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)^{-1}\left(\bar{\delta}^{d}+\bar{\zeta}^{d}\bar{\delta}^{s}\right)-\left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)^{-1}\left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)\bar{\delta}^{s} &\text{(D.109)}\\ &= \left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)^{-1}\left(\bar{\delta}^{d}-\bar{\zeta}^{s}\bar{\delta}^{s}\right) &\text{(D.110)}\end{aligned}$$

We also note

$$\left(I+\bar{\zeta}^{d}\frac{I}{\bar{\zeta}^{s}}\right)^{-1}=\left(\left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)\frac{I}{\bar{\zeta}^{s}}\right)^{-1}=\left(\frac{I}{\bar{\zeta}^{s}}\right)^{-1}\left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)^{-1}=\bar{\zeta}^{s}\left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)^{-1}$$

so

$$\hat{y}=\bar{\zeta}^{s}\left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)^{-1}\left(\bar{\delta}^{d}+\bar{\zeta}^{d}\bar{\delta}^{s}\right)$$

Also, we note:

$$\left(\bar{\zeta}^{s}+\bar{\zeta}^{d}\right)^{-1}=\left(\bar{\zeta}^{s}\left(I+\frac{I}{\bar{\zeta}^{s}}\bar{\zeta}^{d}\right)\right)^{-1}=\left(I+\frac{I}{\bar{\zeta}^{s}}\bar{\zeta}^{d}\right)^{-1}\frac{I}{\bar{\zeta}^{s}} \tag{D.111}$$

Proof of Proposition 15 Recall that (D.36) is:

$$(I - A^{\text{up}})\,\hat{y} = -\zeta^d \hat{p} + \hat{\delta}^d \tag{D.112}$$

which gives $\hat{y} = -\bar{\zeta}^d \hat{p} + \bar{\delta}^d$, with the announced expressions for the terms with a bar. Next, (D.42) is

$$\hat{p} = A\hat{p} - \delta^s + \left(\tilde{\zeta}^s\right)^{-1} \hat{y} + \tilde{\xi}^s \pi_* \tag{D.113}$$

so that, with $\pi_* = \theta^{\pi_*\prime} \hat{p}$, we have

$$\left(I - A - \tilde{\xi}^s \theta^{\pi_*\prime}\right) \hat{p} = -\delta^s + \left(\tilde{\zeta}^s\right)^{-1} \hat{y} \tag{D.114}$$

which again gives the announced expressions for the terms with a bar.

D.5.2 Derivation of the frictional economy

Imperfect transmission in the down input-output matrix We call $\hat{p}_t^*$ the target price, given in (D.42):

$$\hat{p}_t^* \coloneqq A\hat{p} - \delta^s + \left(\tilde{\zeta}^s\right)^{-1} \hat{y} + \tilde{\xi}^s \pi_* \tag{D.115}$$

As prices are sticky, the adjustment is

$$\hat{p}_t = \phi \hat{p}_t^* + (1 - \phi)\,\hat{p}_{t-1} \tag{D.116}$$

where $\phi \in [0, 1]$ is the fraction of firms that adjust their price. This is the route pursued by Minton and Wheaton (2023), who also add forward-looking terms. In fully dynamic terms, this is

$$\hat{p}_t = \mathcal{D}^{\phi} \hat{p}_t^*$$

This proves (13) in Proposition 13 when $\chi = 1$. Taking the time difference once, and

using the fact that for any x_t we have (D.16), $\Delta\mathcal{D}^{\phi}x_t = \phi\left(x_t - \mathcal{D}^{\phi}x_{t-1}\right)$, we get

$$\Delta\hat{p}_t = \Delta\mathcal{D}^{\phi}\hat{p}_t^* = \phi\left(\hat{p}_t^* - \mathcal{D}^{\phi}\hat{p}_{t-1}^*\right) = \phi\left(\hat{p}_t^* - \hat{p}_{t-1}\right)$$

and then, taking the time difference one more time

$$\pi_t - \pi_{t-1} = \Delta\pi_t = \Delta\Delta\hat{p}_t \quad = \quad \phi\left(\Delta\hat{p}_t^* - \Delta\hat{p}_{t-1}\right) = \phi\left(\Delta\hat{p}_t^* - \pi_{t-1}\right)$$

$$\pi_t = \phi\Delta\hat{p}_t^* + (1-\phi)\,\pi_{t-1}$$

i.e. (D.50).

Imperfect transmission in the up input-output matrix Now, we study what could create a 1-for-1 effect in the down matrix. Suppose now that, because of some behavioral rules and imperfect competition, when the nominal demand for sector i increases by $\hat{y}_i$, then it increases its price by $\hat{p}_i = (1-\chi)\,\hat{y}_i$ for some χ (that would be $\chi = 1$ in a traditional model, but it could be $\chi \in [0,1]$ more generally). Then, the real production must increase by $\hat{Q}_i = \hat{y}_i - \hat{p}_i = \chi\hat{y}_i$, and the nominal expenditure increase on its suppliers is

$$\hat{x}_i = \chi\hat{y}_i \tag{D.117}$$

rather than $\hat{x}_i = \hat{y}_i$ as in (D.94). The price adjusts to satisfy the nominal demand, so $\hat{p}_{it} = (1-\chi)\,\hat{y}_{it}$. We replace the up matrix A^{up} by χA^{up}, in all the reasoning following (D.94).

With an intertemporal adjustment, we have $\hat{x}_{it} = \mathcal{D}^{\chi}\hat{y}_{it}$, so that the impact from suppliers is $A^{\text{up}}\mathcal{D}^{\chi}y_t$. The price adjusts, so $\hat{p_{it}} = (1-\mathcal{D}^{\chi})\,\hat{y}_{it}$. Hence, the equations become:

$$\hat{p} \quad = \quad \mathcal{D}^{\phi}\left(A\hat{p} - \delta^s + \left(\tilde{\zeta}^s\right)^{-1}(I - \mathcal{D}^{\chi})\left(\hat{y} + \tilde{\xi}^s\pi_*\right)\right) \tag{D.118}$$

$$\hat{y} \quad = \quad A^{\text{up}}\mathcal{D}^{\chi}\hat{y} - \zeta^d\hat{p} + \delta^d \tag{D.119}$$

D.6 Modelling supply elasticity

The basic model has infinitely elastic supply. Here we discuss how to model inelastic supply.

An introductory example To model inelastic supply, take an example:

$$Q_i = z_i L_i^{\alpha} K_i^{\beta} \tag{D.120}$$

with $\alpha + \beta = 1$. For now, we take $z_i = 1$.

Labor L_i can be rented at rate w, and K_i are rate r_i, but K_i is a special factor that we call in "specific" factor. Perhaps factor K_i is not observed by the econometrician – it's a residual and the payment to that factor, $r_i K_i$ is sometimes called "gross operating surplus". Here we assume that it is in fixed supply.

So, labor demand follow

$$\max_{L_i} p_i L_i^{\alpha} K_i^{\beta} - w L_i$$

so that $L_i \propto p_i^{1/\beta} K_i$, and $Q_i \propto p_i^{\alpha/\beta}$. The price elasticity of quantity produced is $\psi = \frac{d \ln Q_i}{d \ln p_i}$ is $\psi = \frac{\alpha}{\beta} = \frac{1-\beta}{\beta}$, i.e.

$$\psi = \frac{1}{\beta} - 1 \tag{D.121}$$

The price elasticity of value produced is $\zeta^s = \frac{d \ln p_i Q_i}{d \ln p_i} = 1 + \psi = \frac{1}{\beta}$:

$$\zeta^s = \frac{1}{\beta} \tag{D.122}$$

If we study the impact of the price, we have

$$\hat{p}_i = -\hat{z}_i + \alpha \hat{w} + \beta \hat{r}_i$$

But as $r_i K_i = \beta y_i$ (calling $y_i = p_i Q_i$), as we (for now) assume a fixed supply of K_i, then $\hat{r}_i = \hat{y}_i$, so we have

$$\hat{p}_i = -\hat{z}_i + \alpha \hat{w} + \frac{1}{\zeta^s} \hat{y}_i \tag{D.123}$$

One can also replace $\alpha\hat{w}$ by $\sum_j \alpha_j \hat{w}_j$, summing over all the observed factor prices. We next generalize this.

The more general situation Now, suppose that K_i is partly flexible. Its supply is $\hat{K}_i = \phi^s (\hat{r}_i - \pi_*)$ for π_* some inflation index relevant for the supply of the factor (it might be just the CPI). The demand for it by unit i has demand elasticity ϕ^d, which is 1 for Cobb-Douglas, but could be different from 1 in general. The production function is indeed

$$F^i (L_i, K_i, X_{ij})$$

Then

Lemma 4. *The supply elasticity (of nominal production w.r.t price, $\zeta_i^s = \frac{\partial \ln y_i}{\partial \ln p_i}$) in sector i is:*

$$\zeta_i^s = \frac{\phi^s + \phi^d}{\beta} \tag{D.124}$$

where β is the cost share of the remaining specific factor K_i, ϕ^s and ϕ^d are its elasticities of supply and demand. The generalization of (D.123) becomes

$$\hat{p}_i = -\hat{z}_i + \sum_j \alpha_j \hat{w}_j + \frac{1}{\zeta_i^s}\hat{y}_i + \frac{1}{\zeta_i^s}\left(\phi_i^s \pi_* + \left(\phi_i^d - 1\right) M^{i\prime}\hat{p}\right) \tag{D.125}$$

The sum of of the last price coefficients is: $\frac{\phi_i^s + \phi_i^d - 1}{\zeta_i^s} = \beta_i - \frac{1}{\zeta_i^s}$. The sum of all price coefficients is 1.[31]

If there is no specific factor ($\beta = 0$), the supply elasticity is infinite, so $\frac{1}{\zeta_i^s} = 0$. We generalize the traditional equation, with extra "supply elasticity" $\frac{1}{\zeta_i^s}\hat{y}_i$, but also some price adjustments terms, $\frac{1}{\zeta_i^s}\left(\phi_i^s \pi_* + \left(\phi_i^d - 1\right) M^{i\prime}\hat{p}\right)$. Formula (D.124) generalizes our Cobb-Douglas example, which had $\phi_i^s = 0$, $\phi_i^d = 1$.

31. When $\phi^d = 1$,this is

$$\hat{p}_i = -\hat{z}_i + \sum_j \alpha_j \hat{w}_j + \frac{1}{\zeta_i^s}\hat{y}_i + \left(\beta - \frac{1}{\zeta_i^s}\right)\pi_* \tag{D.126}$$

Defining

$$\xi_i^s \coloneqq \frac{\phi_i^s}{\zeta_i^s} \tag{D.127}$$

we obtain (D.37). We also define $\zeta^s = Diag\left(\zeta_i^s\right)$. In what follows, we shall assume that $\phi_i^d = 1$, mostly for simplicity. Then, we have

$$\alpha + \frac{1}{\zeta_i^s} + \xi_i = 1 \tag{D.128}$$

so that the ξ_i term ensure that both terms increase one-for-one with inflation.

Proof. The supply of capital is

$$\hat{K}_i = \phi^s \left(\hat{r}_i - \pi_*\right)$$

and the demand is (using (D.5), replacing $\sigma_{i,K}$ by ϕ_i^d, though keeping σ_j for the elasticity of demand of the other factors)

$$\hat{K}_i = \hat{y}_i - \phi_i^d \hat{r}_i + \left(\phi_i^d - 1\right) M^{i\prime} \hat{p}$$

Hence, the rental rate equalizes supply and demand:

$$\hat{r}_i = \frac{\hat{y}_i + \phi^s \pi_* + \left(\phi_i^d - 1\right) M^{i\prime} \hat{p}}{\phi_i^s + \phi_i^d}$$

Hence the price of the good is $\hat{p}_i = A_{i\cdot}\hat{p} + \alpha\hat{w} + \beta\hat{r}_i - \hat{z}_i$ i.e.

$$\begin{aligned} \hat{p}_i &= A_{i\cdot}\hat{p} + \alpha\hat{w} - \hat{z}_i + \beta\frac{\hat{y}_i + \phi_i^s \pi_* + \left(\phi_i^d - 1\right) M^{i\prime}\hat{p}}{\phi_i^s + \phi_i^d} && \text{(D.129)} \\ &= A_{i\cdot}\hat{p} + \alpha\hat{w} - \hat{z}_i + \frac{1}{\zeta^s}\left(\hat{y}_i + \phi_i^s \pi_* + \left(\phi_i^d - 1\right) M^{i\prime}\hat{p}\right) && \text{(D.130)} \end{aligned}$$

□

D.7 A Refresher on Price deflators

Here we recall known notions of value added, and the link between price deflator of goods vs of value added. We use upper case letter for real quantities, and lower case letters for values (e.g., nominal price times a quantity).

A firm i produces a quantity Q_i, with inputs X_{ij} and labor input L_i. The net output vector is

$$\Upsilon^i = Q_i e_i - \sum_j X_{ij} e_j \tag{D.131}$$

(where e_i is the unit vector with value 1 at i and 0 at other positions): it counts outputs with a plus sign, inputs with a minus sign. Total production is the vector:

$$\Upsilon = \sum_j \Upsilon^j = \sum_j \left(Q_j - \sum_i X_{ij} \right) e_j = \sum_j C_j e_j$$

where C_j is the consumption of good j.

The nominal value added of firm i is

$$va_i = p \cdot \Upsilon^i \tag{D.132}$$

and nominal GDP is the sum of nominal value added

$$\text{gdp} = \sum_i va_i = p \cdot \Upsilon \tag{D.133}$$

If there is only one factor, labor, when $va_i = wL_i$, and $\sum_i va_i = wL$. If there are several factor inputs (e.g. different types of labor, capital), the same holds, but with multidimensional factor inputs L and factor prices w).

The real growth of value added is

$$\widehat{VA}_i = \frac{p \cdot d\Upsilon^i}{p \cdot \Upsilon^i} \tag{D.134}$$

The growth rate of real GDP is

$$\widehat{GDP} = \frac{p \cdot d\Upsilon}{p \cdot \Upsilon}$$

and we also have[32]

$$\widehat{GDP} = \sum_i \frac{va_i}{\text{gdp}} \widehat{VA}_i$$

The change in the price deflator of value added in sector i is

$$\hat{p}_i^{VA} = \frac{dp \cdot \Upsilon^i}{p \cdot \Upsilon^i} = \widehat{va}_i - \widehat{VA}_i \tag{D.135}$$

and the change in the GDP deflator is likewise

$$\hat{p}^{GDP} = \frac{dp \cdot \Upsilon}{p \cdot \Upsilon} = \widehat{\text{gdp}} - \widehat{GDP} \tag{D.136}$$

This implies that the GDP deflator is the average of sectoral deflators

$$\hat{p}^{GDP} = \sum_i \frac{va_i}{\text{gdp}} \hat{p}_i^{VA} \tag{D.137}$$

We have:

$$\hat{p}_i^{VA} = \left(\frac{p_i Q_i}{va_i} \right) \hat{p}_i - \sum_j \left(\frac{p_j X_{ij}}{va_i} \right) \hat{p}_j \tag{D.138}$$

The price of value added $\hat{p}_i^{VA}$ in sector i moves more than one for one with the price p_i of good i: the pass-through is $\frac{p_i Q_i}{va_i} \geq 1$, the share of gross to net output in the sector.

32. Indeed,

$$\widehat{GDP} = \frac{p \cdot d\Upsilon}{p \cdot \Upsilon} = \sum_i \frac{p \cdot \Upsilon^i}{gdp} \frac{p \cdot d\Upsilon^i}{p \cdot \Upsilon^i} = \sum_i \frac{va_i}{gdp} \widehat{VA}_i.$$