News, Noise, and Uncertainty*

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Abstract

We investigate how uncertainty fluctuations affect the transmission of beliefs to the broader economy. In a model with noisy signals, changes in uncertainty shift the signal-to-noise ratio and hence the sensitivity of agents' expectations to new information. Accordingly, our framework captures time variation in the strength of the information friction. We embed this mechanism into a canonical macroeconomic model with financial frictions where agents receive imperfect signals about future cross-sectional risk. We solve the model nonlinearly and find that countercyclical volatility in innovations to risk fundamentals amplify belief shocks during high-volatility periods. When volatility shifts from median to high levels, the impact of noise shocks on investment more than doubles. In an estimation on actual corporate spreads and their survey-based expectations, we find that countercyclical fundamental volatility is empirically supported by the data.

KEYWORDS: imperfect information, noisy signals, beliefs, volatility, perturbation
JEL CLASSIFICATION: C61, C63, D83, E32

1 Introduction

What is the role of belief-driven fluctuations for the macroeconomy? The revival of interest in this question has generated a large literature.[1] An important ingredient in these studies is the degree of information rigidity that affects how well agents can learn about the current or future state. Most of the time, this measure of the information friction is held constant. Recent growing empirical evidence suggests that this assumption may be counterfactual. Indeed, the expectations formation literature

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[1]See, among many others, Beaudry and Portier (2004); Jaimovich and Rebelo (2009); Lorenzoni (2009); Barsky and Sims (2011, 2012); Schmitt-Grohé and Uribe (2012); Blanchard et al. (2013); Chahrour and Jurado (2018).

has documented that as volatility increases, the degree of information frictions implied by survey expectations declines (Coibion and Gorodnichenko, 2015). At the same time, numerous papers from the literature on uncertainty find that macroeconomic and financial volatility fluctuate over time substantially, be it at a business cycle frequency or due to the Great Moderation (Kim and Nelson, 1999; McConnell and Perez-Quiros, 2000; Jurado et al., 2015).[2] This change in volatility can affect the relative usefulness of signals and hence the degree of the information rigidity faced by agents, changing the speed of learning and ultimately the propagation of belief shocks.

In this paper, we investigate the macroeconomic implications of belief shocks in a model with time-varying information rigidities stemming from fluctuations in volatility. As a first step, we provide empirical evidence that a signal extraction problem with countercyclical fundamental volatility, which is the volatility of the forecasted variable, fits survey expectations and actual data on corporate spreads well. This signal extraction problem generates countercyclical signal-to-noise ratios, which makes agents' expectations more sensitive to fundamental and noise shocks during periods of high volatility via an increased sensitivity to signal innovations. Next, we embed this mechanism into a business cycle model with a costly state verification financial friction (Bernanke et al., 1999; Christiano et al., 2014) where agents learn from noisy signals about future entrepreneurial risk. We find that the effects of noise shocks on investment more than double when we move from median to high fundamental volatility. The model also generates time-varying importance of noise versus fundamental shocks with noise shocks becoming less important during highly volatile times despite their amplified effects. In contrast, linearized models with constant speed of learning cannot generate this state dependence, so they would predict that fundamental and noise shocks have the same effects throughout the business cycle.

We begin by studying the implications of a signal extraction problem with time-varying volatility on expectations dynamics. Agents need to form beliefs about the next period's fundamental after observing the current fundamental and a noisy signal about the future fundamental. We consider the consequences of time-varying volatility on the fundamental and the signal noise. Two key mechanisms are active in determining the effects of different volatility states. First, the time-varying volatility of the fundamental or noise shock is different, leading to different magnitudes of shocks. Second, as the volatility of the

[2]See also the review papers by Bloom (2014) and Fernandez-Villaverde and Guerron-Quintana (2020).

fundamental or noise shock changes, the Kalman gain on the signal innovation and therefore the speed of learning changes. Changes in the Kalman gain alters the sensitivity of expectations to changes in the signal. We find that high fundamental or low noise volatility amplifies the latter effect.

We also test these mechanisms empirically by estimating a model with time-varying fundamental volatility, time-varying noise volatility, and constant volatility on BAA-10 year spreads and their Blue Chip one quarter ahead consensus expectations. The model with time-varying fundamental volatility is best supported by the data and the model estimates a *countercyclical* signal-to-noise ratio due to countercyclical fundamental volatility.

In order to quantitatively assess the nonlinear propagation of belief shocks to the broader macroeconomy, we need to embed these mechanisms into a general equilibrium model and solve it accordingly. Our interest in investigating the nonlinear effects of beliefs necessitates technical innovations, which our companion paper Afanasyeva et al. (2025) discusses in detail. First, we analyze a class of conditionally Gaussian signal extraction problems that allow for nonlinearities and non-Gaussianities while maintaining a tractable optimal filtering problem using the conditionally Gaussian Kalman filter (Liptser and Shiryayev, 1978; Chen et al., 1989).[3] Second, we show how to recast our model with signals into a form amenable to the usual perturbation around the deterministic steady state solution approach (as in, for example, Schmitt-Grohé and Uribe (2012); Fernández-Villaverde et al. (2016)). Finally, we provide a formal proof that higher-order perturbation is applicable in this environment.

With this methodology in hand, we embed the conditionally Gaussian Kalman filter into our macro model, calibrate it based on our empirical estimates, and study its implications. We have four main results. First, our signal process with time-varying volatility generates economically important shifts in information frictions. As we move from normal conditions to high volatility, the effect of the same-sized noise shock doubles in importance for investment. This effect is entirely due to agents becoming more sensitive to the signal in a high volatility state. Second, due to the changing signal-to-noise ratios, our model produces time-varying underreaction in expectations to fundamental shocks and overreaction to noise shocks. The combination of these two effects means that agents' expectations update more rapidly

[3]Several recent papers, including Saijo (2017); Ilut and Saijo (2021); Tanaka (2025), consider signal extraction problems in this class.

in high volatility times. Third, we examine the time-varying importance of fundamental and noise shocks by computing state-dependent variance decompositions. In line with the literature (Blanchard et al., 2013), noise shocks are most important at short horizons. Despite being amplified in the high volatility state, their relative importance drops because fundamental shocks are also amplified while becoming intrinsically more volatile. Finally, our model continues to generate countercyclical uncertainty, which we define as the variance of the agents' error in forecasting one-quarter ahead risk. This result stems from the countercyclicality of the fundamental volatility in our model.

This paper is related to several strands of the literature. First, *linearized* models with imperfect information have a considerable tradition in macroeconomics. One of the earlier applications of this approach goes back to Kydland and Prescott (1982) who introduce imperfect information about productivity into a growth model. Kydland and Prescott (1982) show that in a linear-quadratic setting and under Gaussianity it is admissible to solve such a model by searching for a policy rule where latent states are replaced by their conditional expectations. Subsequent contributions build on this insight and study how learning in log-linearized models alters the transmission of shocks, explains actual expectation data features, and affects optimal (monetary) policy prescriptions (see Edge et al. (2007), Gilchrist and Saito (2008), Boz et al. (2011), and references therein, for example). Somewhat more recently, a question of the relative importance of noise versus fundamental factors in driving economic fluctuations came to the forefront of this literature, as cited in the introduction. Our contribution to this line of work is twofold: in providing the methodological foundations to solving these models with perturbation to higher order and in uncovering state-dependent effects of beliefs consistent with the data.

Second, there is a literature building off of the contributions of Van Nieuwerburgh and Veldkamp (2006) and Fajgelbaum et al. (2017) that emphasizes the importance of time-varying signal-to-noise ratios. These papers include Saijo (2017); Ilut and Saijo (2021); Tanaka (2025). In contrast with these papers, we find empirical support and emphasize the importance of countercyclical fundamental volatility in generating countercyclical signal-to-noise ratios and amplification of noise shocks in times of high volatility. This difference has important consequences for state dependence of belief shocks. If noise (rather than fundamental) volatility is countercyclical, the implication in our model would be a dampening of shocks through increased information rigidities during volatile times.

Third, there is a segment of the macro literature that roots nonlinearities and state-dependent effects in the expectation formation modeling per se. To give a few examples, the framework developed by Kozlowski et al. (2020) allows for nonparametric real-time learning about the underlying shock distribution, the rational inattention approach pioneered by Sims (2003) can give rise to attention-driven business cycles, the model in Krishnamurthy and Li (2025) investigates the interaction between beliefs about a liquidity shock and financial frictions, the work by Bianchi (2013) builds on the machinery of regime-switching to introduce learning, while Bianchi et al. (2024) present a new form of irrationality – smooth diagnostic beliefs – that jointly affects the mean and variance of the agents' subjective expectations. Our approach complements this direction of work, as we examine the relationship between learning and time-varying volatility and study its interactions with nonlinear model dynamics stemming from real and financial frictions in the economy. Compared to many of the papers in this literature, the computational burden of our perturbation-based method is lower despite accounting for nonlinearities.

Finally, there has also been a nascent literature investigating the effects of expectations shocks in nonlinear models more generally. Bretscher et al. (2021) explore the asset pricing implications of news shocks using a New Keynesian model solved with a second-order approximation. Ascari et al. (2023) examine the impacts of a non-rational shock to inflation expectations on uncertainty, inflation, and output in a model solved to third order. We contribute to this literature by providing mathematical foundations for using a higher-order perturbation approach for models with imperfect information and noisy signals.

The rest of the paper is organized as follows. Section 2 provides additional empirical motivation for state-dependent effects of beliefs on the business cycle based on survey evidence. Section 3 analyzes our signal extraction problem with time-varying volatility. Section 4 discusses our baseline macro model with financial frictions and noisy signals about borrower riskiness and how we solve the model. Finally, Section 5 discusses the key results from our model simulations, followed by concluding remarks. Technical details are provided in the Appendix.

2 Motivating Empirical Evidence

Does the sensitivity of economic and financial variables to changes in agents' beliefs about the future of the economy depend on the state of financial uncertainty? In this section, we provide a motivating piece of evidence using Senior Loan Officer Opinion Survey (SLOOS) on Bank Lending Practices[4] data that suggests this to be the case. The SLOOS surveys up to 80 large domestic banks at a quarterly frequency on bank lending conditions and business and household demand for loans. Our specific interest is in the question:

"If your bank has tightened or eased its credit standards or its terms of commercial and industrial (C&I) loans or credit lines over the past three months..., how important have been the following possible reasons for the change?" – Senior Loan Officer Opinion Survey on Bank Lending Practices, Question 3

One reason presented is whether "less (more) favorable or more (less) uncertain economic outlook" played a role in the decision. Loan officers are asked to respond "not important," "somewhat important," or "very important" to this reason.

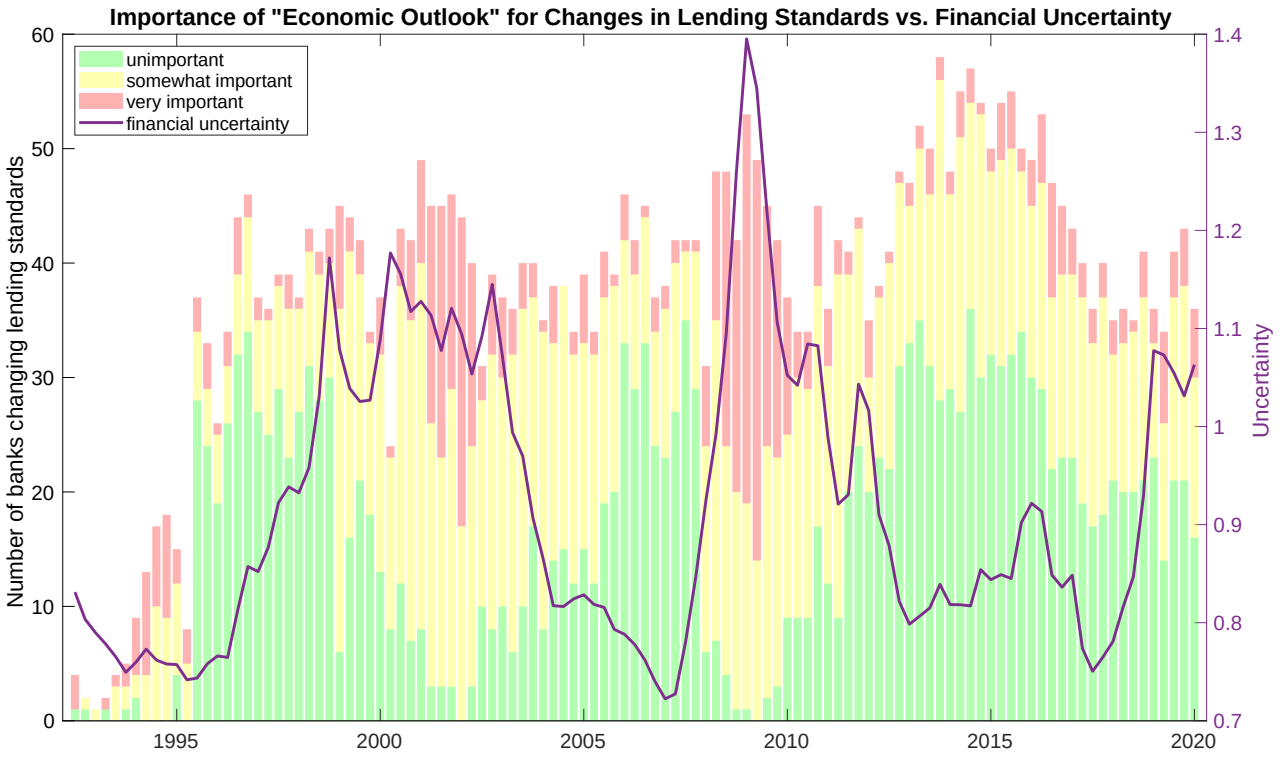
Figure 1 shows the number of banks that responded in each category to this question from 1992:Q2-2019:Q4 plotted against the 3 month ahead financial uncertainty measure of Ludvigson et al. (2021).[5] The striking fact from this figure is that when financial uncertainty is low, such as in the late-1990s, mid-2000s, or post-global financial crisis (GFC) periods, economic outlook is a largely unimportant determinant of lending standards, as seen by the large areas of green in the bar chart. By contrast, when financial uncertainty is high, such as during the early-2000s recession and GFC, economic outlook becomes a very important determinant. This fact suggests an inherent asymmetry in the importance of the economic outlook on lending standards. When financial uncertainty is low, changes in beliefs about the economic outlook do not have large effects. When it is high, however, changes in beliefs about the future become critical.[6] This fact motivates our interest in examining a nonlinear macroeconomic model that allows time-varying role for beliefs depending on financial uncertainty.

[4]Survey questions and data are available at: https://www.federalreserve.gov/data/sloos.htm.

[5]The financial uncertainty measure computes the common variation in the time-varying volatility of the unforecastable components across many financial market series.

[6]As we show in Appendix Section A, the relationship between macroeconomic uncertainty and the SLOOS survey results is lower, especially in the early-2000s recession.

Figure 1: Economic Outlook Matters More When Financial Uncertainty Is High Than When It Is Low



Note: The figure plots the number of banks that responded "unimportant" in green, "somewhat important" in yellow, and "very important" in red to whether economic outlook matters in response to Question 3 of the SLOOS. The purple line shows the 3 months ahead financial uncertainty measure from Ludvigson et al. (2021).

3 A Signal Extraction Problem with Time-Varying Volatility

We present a signal extraction framework with time-varying volatility on the fundamental process and the signal noise. This framework allows us to examine the relationship between uncertainty fluctuations, learning and expectation formation. In particular, we discuss how time-varying volatility affects the learning problem. Finally, we use survey expectations and actual data on corporate spreads to show that there is empirical support in favor of the model with learning and time-varying volatility.

3.1 Signal Extraction Problem

The signal extraction problem we consider is shown in Equation 1 below:

$$
\begin{aligned}
s_t &= x_{t+1} + \varepsilon_{s,t},\ \varepsilon_{s,t} \sim N(0, \sigma^2_{s,t}) \\
x_{t+1} &= \rho x_t + \varepsilon_{x,t+1},\ \varepsilon_{x,t+1} \sim N(0, \sigma^2_{x,t})
\end{aligned}
\tag{1}
$$

Here x_{t+1} is a univariate process that is the variable of interest to agents. We call this variable the fundamental. We assume that at time t, agents see a noisy signal of x_{t+1} that is contaminated by noise $\varepsilon_{s,t}$ as well as the time t value of the fundamental x_t. They also know the entire past history of signals and fundamentals. Note that in this formulation $\varepsilon_{s,t}$ constitutes a pure belief shock in a sense of Chahrour and Jurado (2018), as it is by construction uncorrelated with the fundamental neither contemporaneously nor at any lead or lag. Equation 2 specifies the time-varying volatility of the processes.

$$
\begin{aligned}
\log \sigma^2_{s,t} &= \omega_s + \lambda_s \log \sigma^2_{s,t-1} + \alpha_s s_{t-1} \\
\log \sigma^2_{x,t} &= \omega_x + \lambda_x \log \sigma^2_{x,t-1} + \alpha_x x_{t-1}
\end{aligned}
\tag{2}
$$

Our framework allows for time-varying volatility in both the noise and fundamental shocks. The noise volatility is a function of its lag and lags of the signal, while the fundamental volatility is a function of its lag and lags of the fundamental. This formulation of volatility has been used in the growth-at-risk literature (Adrian et al., 2019). The key parameters in this system are α_s and α_x. If $\alpha_s, \alpha_x > 0$, this means that the volatilities of the noise and fundamental shocks increase as the signal and fundamental themselves increase, respectively. If $\alpha_s, \alpha_x < 0$, then the volatilities of the noise and fundamental shocks decrease as the signal and fundamental increase, respectively. Finally, if $\alpha_s, \alpha_x = 0$, then volatility in the model is constant and the model collapses into a linear Gaussian signal extraction problem.

The model in Equations 1 and 2 encompasses two important cases. First, if noise volatility $\sigma^2_{s,t} = \infty$, then agents gain no additional information about future x_{t+1} from the signal. The model becomes one where only x_t is relevant for forecasting x_{t+1} and agents have no advance information. Second, if $\sigma^2_{s,t} = 0$, then agents perfectly see x_{t+1} at time t.

The model in state space form is shown in Equation 3:

$$
\begin{aligned}
\begin{pmatrix} s_t \\ x_t \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} + \begin{pmatrix} \sqrt{\sigma^2_{s,t}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_{s,t} \\ 0 \\ 0 \end{pmatrix}, \quad w_{s,t} \sim N(0,1) \\
\begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} &= \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{\sigma^2_{x,t}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ w_{x,t+1} \\ 0 \end{pmatrix}, \quad w_{x,t+1} \sim N(0,1) \\
\begin{pmatrix} \log \sigma^2_{s,t} \\ \log \sigma^2_{x,t} \end{pmatrix} &= \begin{pmatrix} \omega_s \\ \omega_x \end{pmatrix} + \begin{pmatrix} \rho_s & 0 \\ 0 & \rho_x \end{pmatrix} \begin{pmatrix} \log \sigma^2_{s,t-1} \\ \log \sigma^2_{x,t-1} \end{pmatrix} + \begin{pmatrix} \alpha_s & 0 \\ 0 & \alpha_x \end{pmatrix} \begin{pmatrix} s_{t-1} \\ x_{t-1} \end{pmatrix}
\end{aligned}
\tag{3}
$$

The first line specifies the observables, which we call z_t, the second the states ξ_t, and the third the dynamics of the time-varying parameters, which we call m_t. Therefore, $z_t = \{s_t, x_t\}$ and $m_t = \{\log \sigma^2_{s,t}, \log \sigma^2_{x,t}\}$. Equation 3 defines a model that falls into the class of conditionally Gauss Markov systems, as discussed in Afanasyeva et al. (2025).[7] This class of models imposes Markovianity restrictions on the conditionally Gaussian state space models of Liptser and Shiryayev (1978) and Chen et al. (1989). Note that due to the presence of time-varying volatility, the unconditional distribution of x_t is not normally distributed. Therefore, the standard time-invariant Kalman filter, which requires a linear Gaussian state space system to provide minimal mean squared estimates of the latent states, is no longer optimal (Harvey, 1989).

3.2 Conditionally Gaussian Kalman Filter

Models in the conditionally Gauss Markov class can be estimated optimally using the conditionally Gaussian Kalman filter (Liptser and Shiryayev, 1978; Chen et al., 1989). For the signal extraction model defined in Equation 3, the conditionally Gaussian Kalman filter has the following innovations

[7] Note that the third line of the system specifies the dynamics of the time-varying volatility $\log \sigma^2_{s,t}$ and $\log \sigma^2_{x,t}$, which is known to the agents with time t information. We keep this equation separate within the state space system because we think of them as defining the time-varying parameters in the conditionally Gauss Markov class.

representation:

$$\begin{aligned}
\begin{pmatrix} E_t x_{t+1} \\ x_t \end{pmatrix} &= \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_{t-1} x_t \\ x_{t-1} \end{pmatrix} + K\left(z_{t-1}, m_{t-1}, \Sigma_{t-1}\right) \Sigma_e\left(z_{t-1}, m_{t-1}, \Sigma_{t-1}\right)^{1/2} \epsilon_t, \quad \epsilon_t \sim N(0, I_2) \\
\begin{pmatrix} s_t \\ x_t \end{pmatrix} &= \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_{t-1} x_t \\ x_{t-1} \end{pmatrix} + \Sigma_e\left(z_{t-1}, m_{t-1}, \Sigma_{t-1}\right)^{1/2} \epsilon_t
\end{aligned} \tag{4}$$

where Σ_t denotes the variance covariance matrix of the error in the time t filtered estimate of the latent state. Equation 5 shows the evolution of the time-varying parameter matrices in the conditionally Gaussian Kalman filter, with $\sigma^2_{s,t}$ and $\sigma^2_{x,t}$ evolving as specified in the third line of Equation 3:

$$\begin{aligned}
K(z_t, m_t, \Sigma_t) &= \left[\begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \Sigma_t \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix}' + \begin{pmatrix} \sigma^2_{x,t+1} & 0 \\ 0 & 0 \end{pmatrix} \right] \Sigma_e\left(z_t, m_t, \Sigma_t\right)^{-1} \\
\Sigma_e\left(z_t, m_t, \Sigma_t\right) &= \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \Sigma_t \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix}' + \begin{pmatrix} \sigma^2_{s,t+1} + \sigma^2_{x,t+1} & 0 \\ 0 & 0 \end{pmatrix} \\
\Sigma_t &= \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \Sigma_{t-1} \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix}' + \begin{pmatrix} \sigma^2_{x,t} & 0 \\ 0 & 0 \end{pmatrix} - \\
&\quad K\left(z_{t-1}, m_{t-1}, \Sigma_{t-1}\right) \Sigma_e\left(z_{t-1}, m_{t-1}, \Sigma_{t-1}\right) K\left(z_{t-1}, m_{t-1}, \Sigma_{t-1}\right)'
\end{aligned} \tag{5}$$

The first line in Equation 5 specifies the dynamics of the Kalman gain, the second one does so for the conditional variance covariance of the Kalman filter innovations, and the third for the conditional variance covariance of the error in the filtered estimates of the state. These dynamics are functions of the matrices in the state space form, as well as z_t, m_t, and Σ_t. Appendix Section B derives closed-form solutions to the formulas in Equation 5. The key insight from these derivations is that:

$$\Sigma_t = \begin{pmatrix} \frac{\sigma_{s,t}^2 \sigma_{x,t}^2}{\sigma_{x,t}^2 + \sigma_{s,t}^2} & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

Equation 6 implies that Σ_t does not depend on lags of Σ_t.

There are four important points to note about the properties of the conditionally Gaussian Kalman filter. First, due to time variation in the noise and fundamental volatilities, there is also time variation in the Kalman gain, conditional variance covariance of the Kalman filter innovations, and conditional variance covariance of the error in the filtered estimate of the state. This feature differs from the standard (constant-parameter) Kalman filter, in which these objects eventually converge to constant steady state values. We examine the implications of this time variation for the agents' learning problem in Section 3.3. Second, a byproduct of the time variation in the parameters of the conditionally Gaussian Kalman filter is that when this type of learning is embedded in a macro model, m_t becomes an additional state variable for control over and above the filtered estimate of the state, which we denote by $\hat{\xi}_t$, and the observable z_t.[8] In contrast, the standard Kalman filter at steady state only has $\hat{\xi}_t$ and z_t as the relevant state variables. Third, the one-step ahead conditional distribution of latent states given time t information implied by Equation 4 continues to be normally distributed. This result obtains due to the innovations ϵ_t being normally distributed and the time-varying matrices being known. However, multi-step ahead conditional distributions are not normally distributed. Finally, in the constant-parameter case where there is no time variation in any of the matrices in the state space form of Equation 3, the conditionally Gaussian Kalman filter collapses into the Kalman filter.

3.3 Key Mechanisms of Learning with Time-Varying Volatility

We now illustrate how the signal s_t, fundamental x_t, and expectations about the one-step ahead fundamental $E_t x_{t+1}$ evolve following fundamental and noise shocks. We examine the differences between the responses in median volatility and high volatility cases. Our exercises in this subsection utilize two models. In the first model, we only consider time-varying volatility in the fundamentals while in the

[8] In the general class of conditionally Gaussian Kalman filter problems, Σ_t also appears as a state variable. In this case, however, as Equation 6 shows, Σ_t does not depend on its lag, so it is also not a state variable.

second one, we only consider time-varying volatility in the noise.

There are two key mechanisms at play when we condition on different volatility states. First, the time-varying volatility of the fundamental or noise shock is different. This means that a one standard deviation shock has different effects on the states and observables. Second, as the volatility of the fundamental or noise shock changes, the Kalman gain and therefore the speed of learning changes. Hence, a shock of the same magnitude leads to different reactions in expectations of the one-step ahead fundamental.

We begin with a calibration in which only the fundamental volatility moves over time.[9] Figure 2 shows the effects of one standard deviation fundamental and noise shocks conditional on a median fundamental volatility state in solid blue and high fundamental volatility state in dashed red. Following a fundamental shock that realizes in the first period, the signal and expectations of the fundamental increase on impact. Moving to the first period, x_{t+1} is revealed and expectations update. Afterwards, the effects of the shock revert back to baseline. In a high volatility state, both key mechanisms are at play. First, actual fundamental volatility is higher, so the increase in s_t and x_{t+1} is higher as well. Second, the reaction of expectations on impact is also larger relative to the value of the signal on impact. This is because in a high fundamental volatility state, the signal-to-noise ratio and therefore the Kalman gain on the signal innovation are higher. Intuitively, because a larger fraction of the signal comes from actual future fundamental movements, agents react more to changes in the signal.[10]

The second row of the figure shows the responses to a one standard deviation noise shock. Note here that the noise shock volatility is constant in this example. Therefore, the first two panels are identical. The third panel, which shows the reactions of expectations to the shock, displays differences. Because the sensitivity of the agents to the signal innovation is higher in the high fundamental volatility state, their expectations in turn react more to the noise shock.

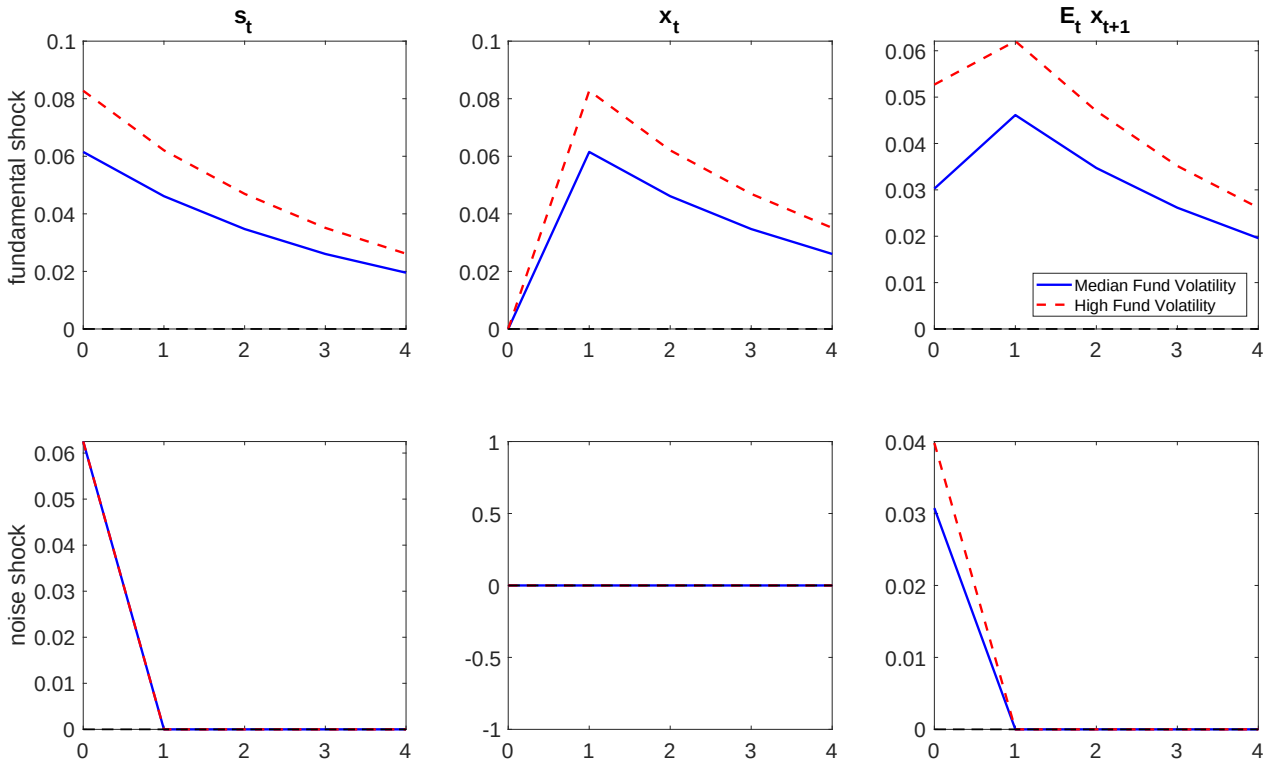
Figure 3 shows the effects of fundamental and noise shocks in an alternative model in which only the noise volatility moves over time while the fundamental volatility is constant.[11] Since the volatility of the

[9] We set $\rho = 0.75, \omega_x = \log\left(0.25^2\right), \lambda_x = 0.5, \alpha_x = 2.5, \omega_s = \log\left(0.25^2\right), \lambda_s = 0.5, \alpha_s = 0$.

[10] This effect can be seen from Figure 2 by dividing the change in expectations by the signal on impact. In the median volatility state, this ratio is around 0.5 while in the high volatility state, the ratio increases to above 0.6.

[11] We set $\rho = 0.75, \omega_x = \log\left(0.25^2\right), \lambda_x = 0.5, \alpha_x = 0, \omega_s = \log\left(0.25^2\right), \lambda_s = 0.5, \alpha_s = 2.5$.

Figure 2: Effects of Fundamental and Noise Shocks in Model with Time-Varying Volatility in Fundamentals Only

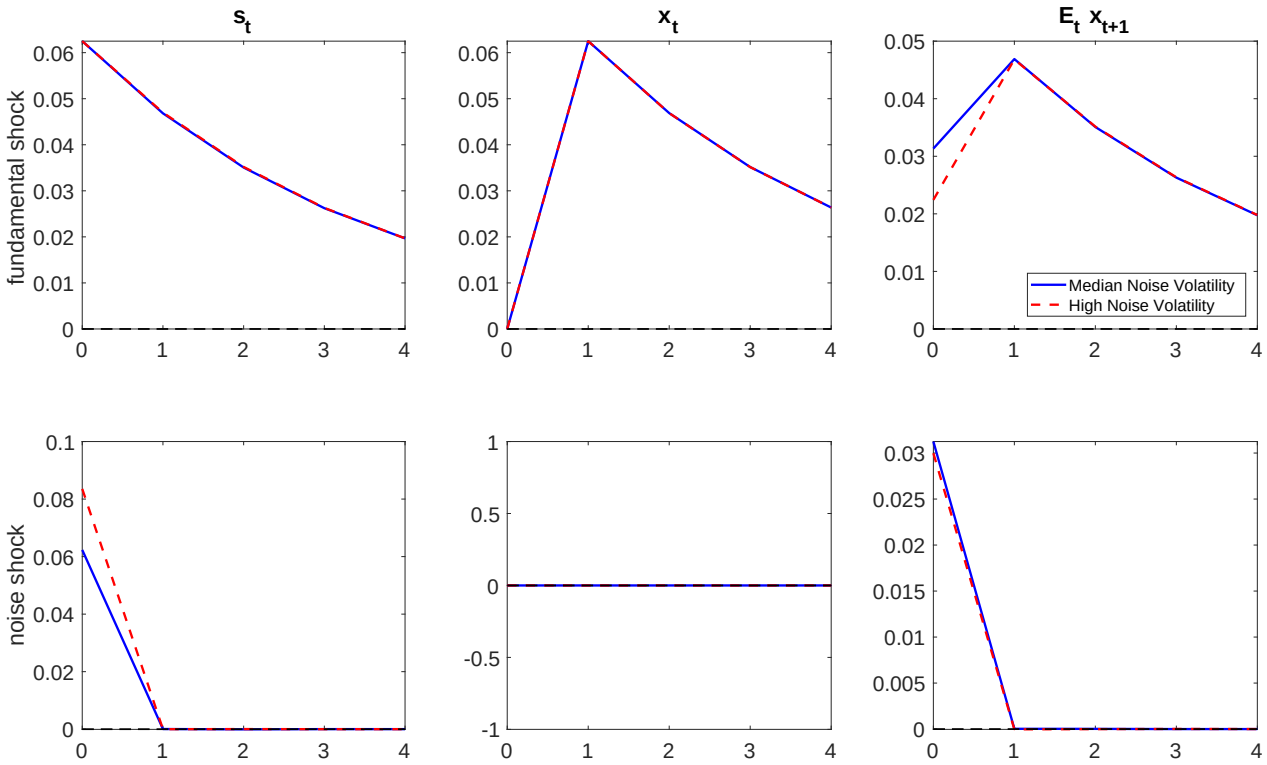


Note: The figure plots the impulse response functions of the signal, fundamental, and expected one step-ahead fundamental to a one standard deviation fundamental shock in the top row and a one standard noise shock in the bottom row. The solid blue line conditions on the median fundamental volatility state while the dashed red line conditions on the 90% fundamental volatility state.

fundamental shock is constant, the response of the signal and fundamental to a one standard deviation shock stays the same across the two noise volatility states, as can be seen by the first two panels in the top row. The conditional expectation of the one-period ahead fundamental, however, shows less sensitivity to the fundamental shock in the high noise volatility case. This is because when noise volatility is high, agents understand that the signal-to-noise ratio of the signal is lower and therefore react less to it. Turning to the second row of the figure, the noise shock now has a larger effect on the signal in the high noise volatility case. Despite this fact, its overall effects on the one-period ahead fundamental actually decrease a bit compared to the median volatility case due to the decreased sensitivity of expectations to the shock.

In summary, a high signal-to-noise ratio, whether generated by high fundamental volatility or low noise volatility, makes agents especially sensitive to the signal. If we think that uncertainty and volatility

Figure 3: Effects of Fundamental and Noise Shocks in Model with Time-Varying Volatility in Noise Only



Note: The figure plots the impulse response functions of the signal, fundamental, and expected one step-ahead fundamental to a one standard deviation fundamental shock in the top row and a one standard noise shock in the bottom row. The solid blue line conditions on the median noise volatility state while the dashed red line conditions on the 90% noise volatility state.

are heightened during recessions as is suggested by the literature (Bloom, 2014), then it would suggest that a countercyclical signal-to-noise ratio is important in amplifying the effects of belief shocks in times of crises.

3.4 Empirical Estimates of the Learning Process

We are interested in empirically testing whether agents change their speed of learning as financial volatility changes. To this end, we estimate two versions of our signal extraction problem with time-varying volatility: one with time-varying fundamental volatility and constant noise volatility and one with time-varying noise volatility and constant fundamental volatility. We also compare these two models with a constant volatility state space model that can be estimated with a Kalman filter. We use quarterly BAA-10 year spread data and one quarter ahead consensus expectations of this spread from

Blue Chip Financial Indicators for the period 1999:Q1 - 2024:Q4.[12]

In our exercises, we suppose that agents use one of the three competing models to forecast spreads. Our question is then which of the three models jointly fits the spread data and expectations the best. The two models with time-varying volatility also produce estimates of the time-varying Kalman gains, thereby shedding light on the cyclical properties of the signal-to-noise ratio.

The three models we consider are as follows:

(1) Time-Varying Fundamental Volatility Model

$$\begin{aligned} s_t &= BAA10Y_{t+1} + \varepsilon_{s,t},\ \varepsilon_{s,t} \sim N(0, \sigma_s^2) \\ BAA10Y_{t+1} &= \rho BAA10Y_t + \varepsilon_{x,t+1},\ \varepsilon_{x,t+1} \sim N(0, \sigma_{x,t}^2) \\ \log \sigma_{x,t}^2 &= \omega_x + \lambda_x \log \sigma_{x,t-1}^2 + \alpha_x \log(BAA10Y_{t-1}/\overline{BAA10Y}) \end{aligned} \tag{7}$$

(2) Time-Varying Noise Volatility Model

$$\begin{aligned} s_t &= BAA10Y_{t+1} + \varepsilon_{s,t},\ \varepsilon_{s,t} \sim N(0, \sigma_{s,t}^2) \\ BAA10Y_{t+1} &= \rho BAA10Y_t + \varepsilon_{x,t+1},\ \varepsilon_{x,t+1} \sim N(0, \sigma_x^2) \\ \log \sigma_{s,t}^2 &= \omega_s + \lambda_s \log \sigma_{s,t-1}^2 + \alpha_s \log(BAA10Y_{t-1}/\overline{BAA10Y}) \end{aligned} \tag{8}$$

(3) Constant Parameter Model

$$\begin{aligned} s_t &= BAA10Y_{t+1} + \varepsilon_{s,t},\ \varepsilon_{s,t} \sim N(0, \sigma_s^2) \\ BAA10Y_{t+1} &= \rho BAA10Y_t + \varepsilon_{x,t+1},\ \varepsilon_{x,t+1} \sim N(0, \sigma_x^2) \end{aligned} \tag{9}$$

where agents at time t observe $s_t, BAA10Y_t$, and the entire past history of signals and spreads. The term $\overline{BAA10Y}$ is the mean value of spreads over the entire sample.

[12]For Blue Chip expectations data, we use the survey conducted in the final month of the previous quarter, in line with Bordalo et al. (2020). The data are plotted in Appendix Section C.

Table 1: Parameter Estimates of the Time-Varying Fundamental Volatility Model

Parameter	Mode	5%/95%
ρ	0.83	$(0.80, 0.86)$
σ_s	1.38	$(1.12, 1.81)$
ω_x	-1.70	$(-1.91, -1.36)$
λ_x	0.02	$(-0.05, 0.16)$
α_x	1.44	$(0.92, 1.85)$

All of the models imply the conditionally Gaussian Kalman filter innovations representation of the form in Equation 10[13]:

$$\begin{pmatrix} E_t BAA10Y_{t+1} \\ BAA10Y_t \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_{t-1} BAA10Y_t \\ BAA10Y_{t-1} \end{pmatrix} + K\left(\rho, \sigma_{x,t}, \sigma_{s,t}\right) \Sigma_e \left(\rho, \sigma_{x,t}, \sigma_{s,t}\right)^{1/2} \epsilon_t \tag{10}$$

$$\epsilon_t \sim N(0, I)$$

The fundamental volatility model restricts $\sigma_{s,t}$ to be constant, the noise volatility model restricts $\sigma_{x,t}$ to be constant, and the constant parameter model restricts both to be constant. Since our observables are one-quarter-ahead expectations of corporate spreads and their current-quarter realizations, Equation 10 defines a restricted VAR that is a function of the underlying parameters in each signal extraction problem. We can then use Bayesian methods to estimate parameters of each model and compare marginal likelihoods.

We use flat priors for all of the parameters in the three models and the random walk Metropolis-Hastings algorithm with $10,000$ draws and $5,000$ burn-in. Among the three models, there is strong evidence that the fundamental volatility model fits the data the best (21.4 log points), followed by the noise volatility model (10.5), and finally by the constant parameter model (0.8), when comparing marginal likelihoods.[14]

Table 1 shows the parameter estimates from fundamental volatility model. The estimates from the other two models can be found in Appendix Section C. We estimate a persistent process for spreads

[13]The constant parameter model implies a Kalman filter innovations representation.

[14]We compute the marginal likelihoods using the modified harmonic mean of Geweke (1999) and a truncation parameter of 0.95. The results are similar when using a truncation parameter of 0.75.

along with a large amount of noise. The fundamental volatility process generally hovers around a value lower than the noise innovation, as implied by the estimate of the volatility constant ω_x. Fundamental volatility itself is not persistent, but it loads positively on the log deviation of spreads from its mean. A positive value of α_x means that when the level of spreads are high, volatility is high as well.

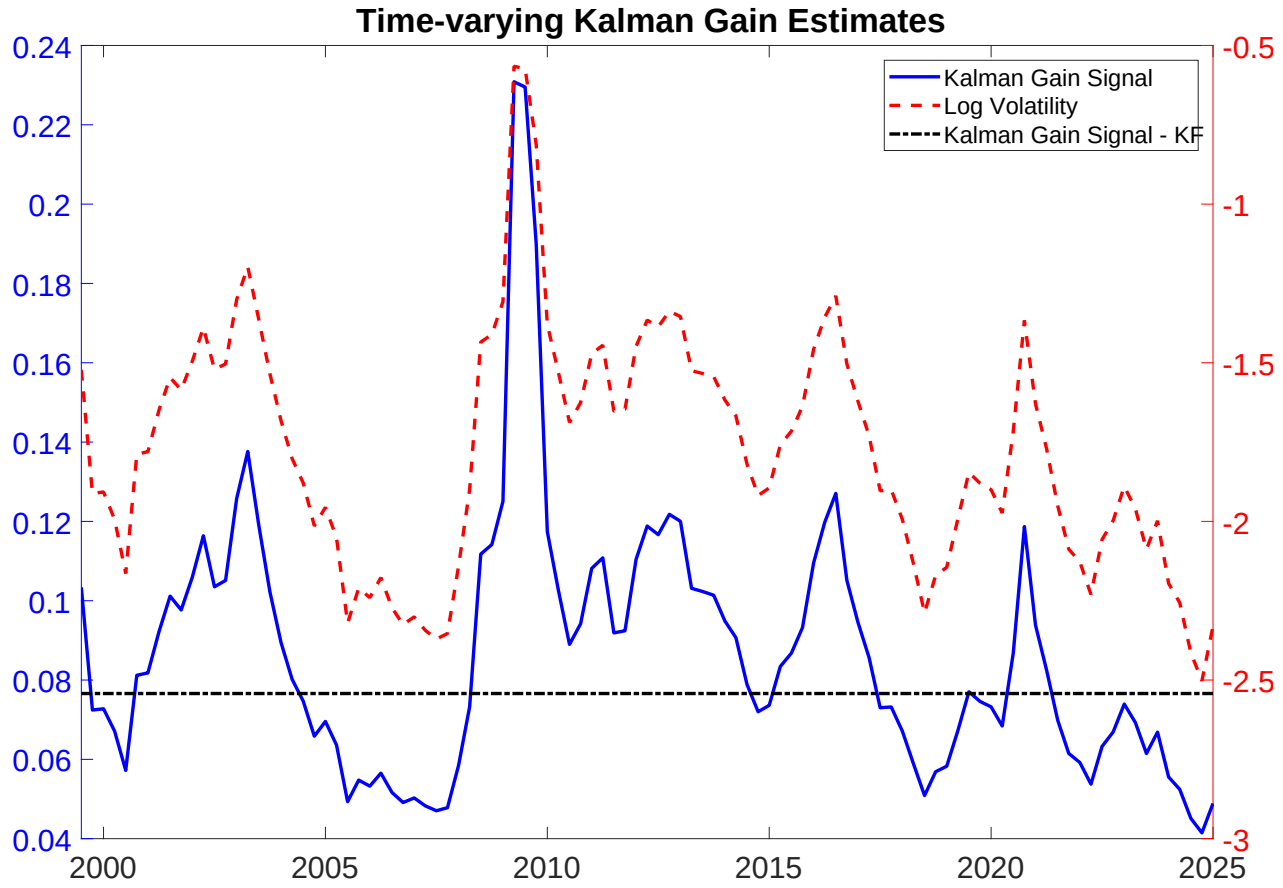
Intuitively, the fundamental volatility model fits the data the best because it can parsimoniously generate time-varying volatility in both the spread data itself and the expectations, which is what we see in the data. The noise volatility model cannot generate time-varying volatility in spreads as it can only produce a time-varying reaction of expectations to spreads. The constant parameter model cannot generate time-varying volatility in either series.

One point to note when interpreting the parameter estimates is that σ_s is far away from the two extremes $\sigma_s = 0$ and $\sigma_s = \infty$. This means that agents neither have perfect knowledge about future spreads nor do they only use lagged spreads to forecast future spreads.

Figure 4 shows the estimates of the time-varying Kalman gain on the signal in solid blue for the fundamental volatility model. The Kalman gain varies over time, reaching as low as 0.04 near the end of the sample and peaking at around 0.23 in the GFC. Since a higher Kalman gain means a larger sensitivity to the signal when forming expectations, our estimates imply a countercyclical signal-to-noise ratio. The dashed red lines are the log fundamental volatility estimates, which peak when the Kalman gain is the highest. For comparison, the Kalman gain from the constant parameter model is shown in dash-dotted black.

In Appendix Section C, we show the corresponding estimates of the time-varying Kalman gain produced by the noise volatility model. This model estimates the noise volatility to be procylical, i.e., low during the GFC. Accordingly, the signal-to-noise ratio is countercyclical in this model as well, consistent with our earlier findings.

Figure 4: Kalman Gain Estimates from Time-Varying Fundamental Volatility Model



Note: The figure plots the estimates of the Kalman gain on the signal in solid blue and the log volatility of fundamental shocks in dashed red from the posterior mode parameters. The dash-dotted black line is the Kalman gain on the signal from the constant parameter model that is estimated using the Kalman filter.

4 Incorporating Financial Uncertainty and Learning in a Macro Model

Having discussed a signal extraction framework that can generate the stylized facts from our evidence in Section 2, we are now interested in understanding its macroeconomic implications. In other words, we study how these new expectation dynamics translate into agents' decisions and affect aggregate real and financial outcomes. To this end, we employ a baseline New Keynesian model with the following key modifications: the financial accelerator from Bernanke et al. (1999) and shocks to entrepreneurial risk as in Christiano et al. (2014) augmented by the noisy signal representation that we discussed in Section 3. We briefly discuss the key aspects of the model in the next subsection and leave a full description in

Appendix Section D. Then, we discuss how we solve and simulate from the model.

4.1 Model Environment

A representative household consumes, supplies labor to labor unions, and saves in risk free deposits. The household has habit formation in consumption. A representative capital goods producer buys the non-depreciated capital stock from the entrepreneurs, makes an investment decision subject to dynamic investment adjustment costs as in Christiano et al. (2005), and sells new capital stock to the entrepreneurs. A representative intermediate goods firm rents capital from entrepreneurs, hires labor from labor unions, and sells goods to retailers. Retailers and labor unions diversify their inputs and are monopolistically competitive. They are subject to price and wage adjustment costs a la Calvo with indexation. Monetary policy follows a Taylor rule that responds to inflation and output deviations from steady state.

Entrepreneurs and the Risk Shock Following Christiano et al. (2014), a continuum of entrepreneurs combines their internal net worth with borrowing to purchase capital and rent to intermediate goods producers. They also choose capital utilization subject to utilization adjustment costs. They are subject to idiosyncratic shocks to capital productivity. The shock is iid across entrepreneurs and over time with a log normal distribution where the mean is one and the standard deviation of $\log \omega$ is σ_t. We call σ_t a risk shock, which is the only source of fundamental uncertainty in the model.

4.2 Signal Structure

We assume that the risk shock follows an autoregressive process of order 1 and has normally distributed errors with time-varying volatility:

$$\begin{aligned} \sigma_{t+1} &= \rho\sigma_t + \varepsilon^{\sigma}_{t+1}, \quad \varepsilon^{\sigma}_{t+1} \sim N\left(0, \sigma^2_{\sigma,t}\right) \\ \log \sigma^2_{\sigma,t} &= \omega_\sigma + \lambda_\sigma \log \sigma^2_{\sigma,t-1} + \alpha_\sigma \log\left(spr_{t-1}/spr_{ss}\right) \end{aligned} \tag{11}$$

We specify that fundamental volatility in the model depends on lagged log annualized spread deviations from its deterministic steady state. Spreads are defined as the interest rate paid on loans to the entrepreneurs over the risk-free interest rate. This specification choice is intended to emulate our empirical model in Section 3.4.

At time t, agents add σ_t to their information set. In addition, they receive a signal about the time $t+1$ risk s_t:

$$s_t = \sigma_{t+1} + \varepsilon_t^s, \quad \varepsilon_t^s \sim N\left(0, \sigma_s^2\right) \tag{12}$$

In the parlance of Chahrour and Jurado (2018), ϵ_t^σ is a fundamental shock while ϵ_t^s is a pure noise shock. The latter is defined as a shock that changes agents' expectations and is independent of past, current, and future fundamentals. All agents have the same information set. This information set will determine how expectations are formed within the model.

State Variables of the Model We now describe the state variables of the model, following the framework of Afanasyeva et al. (2025). Due to the information constraints of the agents, some of the variables are latent. It is over these latent variables that the agents form their beliefs.

We collect the endogenous state variables of the model in a vector called θ_t. These variables are discussed in Appendix Section D. A special endogenous state variable to highlight is spr_t. This variable is a state variable because it impacts fundamental volatility.

The (possibly) unobserved exogenous state variables are:

$$\xi_t = \{\sigma_t, \sigma_{t+1}\} \tag{13}$$

Finally, the exogenous variables observed by the agent are:

$$z_t = \{s_t, \sigma_t\} \tag{14}$$

At time t, agents can fully observe all endogenous state variables θ_t. Among the possibly unobserved exogenous state variables ξ_t, they see σ_t and do not see σ_{t+1}. Instead, they observe a partially informative

(noisy) signal s_t about σ_{t+1} and must form beliefs over it.

The information set of agents at time t is then:

$$I_t = \{z_t, \theta_t, ..., p_0, \theta_{-1}\} \tag{15}$$

where p_0 denotes the prior beliefs of the agents about σ_0. Agents use the information set I_t to evaluate $E_t[.] = E[.|I_t]$.

4.3 Solving Models with Signals

We now discuss how we use perturbation methods to solve the baseline model. Our procedure relies on first rewriting our model into a full information equivalent. Then, we show that the deterministic limit of the full information model exists and its form renders the standard perturbation methods applicable. The formal justification for this approach can be found in Afanasyeva et al. (2025). Our method extends and generalizes the previous solution techniques developed from a rich history of solving models with signals in the macro literature, which we also discuss below.

4.3.1 Full Information Equivalent and Its Deterministic Limit

The first necessary step to solving the model is to have it written in in terms of variables fully observed by the agents. This form is the *full information equivalent* and differs from the original formulation that includes latent states, ξ_t. Note that our state space representation in Equations 4 and 5 satisfies this requirement, as it is formulated in terms of agents' expectations ($\hat{\xi}_t$), their known variance covariance matrix of the error in the expectations (Σ_t), and observables (z_t, m_t). This reformulation restores Markovianity to the system, as we no longer need to condition on the entire histories of the states as in Equation 15. In particular, the conditional expectations that previously were defined as $E_t[.] = E[.|I_t]$ can now be equivalently specified as as $E_t[.] = E[.|z_t, m_t, \hat{\xi}_t, \theta_t]$, where we drop the dependence of expectations on Σ_t since it is not a state variable in this particular application. This notation applies to the Euler equations of the households and the entrepreneurs' first-order condition, respectively (see Appendix Section D for the details).

Summing up, belief evolutions specified by the innovations form of the conditionally Gaussian Kalman filter along with the remaining standard model equilibrium conditions constitute the full information equivalent. Notice that an important by-product of the conditional Gaussianity assumption is dimensionality reduction, as we are no longer dealing with an infinitely dimensional density covering the evolution of beliefs.

After the model is put into its full information equivalent form, we can proceed to the next step. To apply perturbation methods, we must compute the deterministic limit of the full information equivalent model - a reference benchmark around which perturbation can be applied. Consider again our conditional Gauss Markov system in Equation 3 and rewrite it accounting for an additional scalar parameter $\sigma > 0$, which controls the size of error terms in the system:

$$
\begin{aligned}
\begin{pmatrix} s_t^\sigma \\ x_t^\sigma \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{t+1}^\sigma \\ x_t^\sigma \end{pmatrix} + \sigma \begin{pmatrix} \sqrt{\sigma_{s,t}^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_{s,t} \\ 0 \\ 0 \end{pmatrix}, \quad w_{s,t} \sim N(0,1) \\
\begin{pmatrix} x_{t+1}^\sigma \\ x_t^\sigma \end{pmatrix} &= \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t^\sigma \\ x_{t-1}^\sigma \end{pmatrix} + \sigma \begin{pmatrix} 0 & \sqrt{\sigma_{x,t}^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ w_{x,t+1} \\ 0 \end{pmatrix}, \quad w_{x,t+1} \sim N(0,1) \\
\begin{pmatrix} \log {\sigma_{s,t}^2}^\sigma \\ \log {\sigma_{x,t}^2}^\sigma \end{pmatrix} &= \begin{pmatrix} \omega_s \\ \omega_x \end{pmatrix} + \begin{pmatrix} \rho_s & 0 \\ 0 & \rho_x \end{pmatrix} \begin{pmatrix} \log {\sigma_{s,t-1}^2}^\sigma \\ \log {\sigma_{x,t-1}^2}^\sigma \end{pmatrix} + \begin{pmatrix} \alpha_s & 0 \\ 0 & \alpha_x \end{pmatrix} \begin{pmatrix} s_{t-1}^\sigma \\ x_{t-1}^\sigma \end{pmatrix}
\end{aligned}
\tag{16}
$$

With this new notation we explicitly emphasize the dependence of the solution on the perturbation parameter σ.[15] Without loss of generality, we work with the same σ across all equations and examine the behavior of the full information equivalent model as $\sigma \to 0$.[16]

[15] In our specification, we are interested in the effects stemming from the time-varying volatility in fundamentals or noise. Hence, the state-dependence and the σ-dependence operates via the variance-covariance terms of the filtering equations only, while the matrix governing the autoregressive dynamics of the latent state and the matrix characterizing the mapping of the latent state into the observables are assumed to be constant. All our results go through when these assumptions are relaxed, i.e., when those matrices are also allowed to depend on histories of observed variables up to t (see Afanasyeva et al. (2025) for formal proofs and further discussion).

[16] This assumption simplifies notation and stresses that we consider *equal* asymptotic speed as σ approaches zero in the limit.

Assume that the starting values of the latent states are such that $\bar{\xi}_0^\sigma = \bar{\xi}_0$ and $\bar{\Sigma}_0^\sigma = \sigma^2 \bar{\Sigma}_0$, implying for the initial values of expectations $\hat{\xi}_0^\sigma \to \hat{\xi}_0^\sigma$ and $\Sigma_0^\sigma \to \sigma^2 \Sigma_0$, respectively. Using these initial values and iterating over the recursions in Equation 5, we get:

$$\Sigma_e^\sigma \to \sigma^2 \Sigma_e, \quad K^\sigma \to K, \quad \Sigma_t^\sigma \to \sigma^2 \Sigma_t.$$

This means that the Kalman gain is *invariant* to the perturbation parameter σ. The conditional expectation $\hat{\xi}_t^\sigma$ is also unaffected, while the corresponding conditional variance Σ_t^σ becomes scaled by σ^2. Accordingly, we can rewrite the innovations representation of the conditionally Gaussian Kalman filter as follows:

$$\begin{aligned}
\begin{pmatrix} E_t x_{t+1} \\ x_t \end{pmatrix} &= \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_{t-1} x_t \\ x_{t-1} \end{pmatrix} + \sigma K \left(z_{t-1}, m_{t-1}, \Sigma_{t-1} \right) \Sigma_e \left(z_{t-1}, m_{t-1}, \Sigma_{t-1} \right)^{1/2} \epsilon_t, \quad \epsilon_t \sim N(0, I_2) \\
\begin{pmatrix} s_t \\ x_t \end{pmatrix} &= \begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_{t-1} x_t \\ x_{t-1} \end{pmatrix} + \sigma \Sigma_e \left(z_{t-1}, m_{t-1}, \Sigma_{t-1} \right)^{1/2} \epsilon_t
\end{aligned} \tag{17}$$

Note that σ now features as a multiplier on stochastic terms driving the system. This representation implies that a model reformulated in this way can be directly mapped into the classic perturbation frameworks described in Judd (1996); Schmitt-Grohé and Uribe (2004); Fernández-Villaverde et al. (2016). We provide a formal proof of this result in Afanasyeva et al. (2025), Corollary 1.

4.3.2 Computing Impulse Response Functions to Structural Shocks

We need to address two issues when computing impulse response functions (IRFs) to shocks in the model. First, we are interested in computing IRFs in a nonlinear model. As pointed out in Koop et al. (1996), the definition of the IRF in this case is not straightforward. The IRFs should integrate out uncertainty regarding future structural shocks and be state-dependent. In our paper, we use the definition of the conditional nonlinear IRFs discussed in Gonçalves et al. (2021):

$$IRF_{h,\xi,I_{t-1}} = E\left[y_{t+h}(\zeta) - y_{t+h}|z_{t-1}, m_{t-1}, \hat{\xi}_{t-1}, \theta_{t-1}\right] \tag{18}$$

This definition of the IRF computes the expected value of the difference between a variable y_t in a baseline and counterfactual case given the state $\{z_{t-1}, m_{t-1}, \hat{\xi}_{t-1}, \theta_{t-1}\}$. The baseline case is generated from a sequence of random shocks. The counterfactual case is generated by adding a structural shock of size ζ to the baseline shock sequence at time t. Integrating over these paths produces the IRFs.

Second, we are ultimately interested in the impulse responses to the structural shocks ε_t^σ and ε_t^s while our full information equivalent model is written in terms of the observable forecast errors ϵ_t. Therefore, when computing the IRFs, we need to consider how the shocks of interest affect ϵ_t. Given a conditioning time $t-1$ state and a structural shock realization at time t, it is possible to compute the value of z_t by using Equations 11 and 12. We then need to search for the errors ϵ_t that force the perturbed solution of the innovations representation to equal z_t. Since this is a problem with two nonlinear equations and two unknowns, we use a nonlinear solver to find the solution.[17] Feeding these shocks into the structural model allows us to propagate the state variables forward by one period. Repeating this procedure given time $t+1$ structural shocks allows us to further propagate the system forward, until we reach the final horizon of the IRF.

4.3.3 Discussion

Our solution method and full information equivalent representation of the model are related to work beginning with Kydland and Prescott (1982) and continued more recently by Edge et al. (2007); Gilchrist and Saito (2008); Boz et al. (2011); Barsky and Sims (2012); Blanchard et al. (2013) and many others. These papers focus on the linear Gaussian signal extraction problems in the context of linear quadratic optimization. Blanchard et al. (2013), for instance, advocate for first log-linearizing the model and then using the standard Kalman Filter in innovations form to substitute for the expectations of the latent states. They also justify their solution approach with a guess-and-verify method. If we apply our approach to the standard Kalman filter and approximate the model to first order, our results are

[17] In particular, for our purposes, `fsolve` performed well.

equivalent with the approach outline by Blanchard et al. (2013). However, for the standard Kalman filter, our approach computes the full information equivalent of the fully nonlinear model, and formally shows why perturbation, including to higher order, can be applied in this case. In addition, we extend this result even further - to the models with a conditionally Gaussian Kalman filter, allowing for time variation in the speed of learning (or equivalently information friction) that can also be combined with a higher-order perturbation solution.

Proving the applicability of perturbation methods in this environment is a nontrivial task, as a priori it is not obvious that the full information equivalent model has a well-behaved deterministic limit in this environment. In general, in models with incomplete information, policy rules depend on *densities* of hidden states. Hence, such models do not have a natural deterministic counterpart, unlike full information models. First, our contribution is in showing how to define this deterministic limit. Second, for conditionally Gaussian information structures, we formally show why perturbation is a valid method for these models, thus extending and generalizing the first-order results previously known in the literature. There are a few papers in the literature, including Saijo (2017) and Tanaka (2025), that consider models within the conditionally Gaussian class and also apply higher-order perturbation to solve these models. Our contribution relative to these papers is in providing formal foundations for the applicability of this solution strategy to these and a more general class of problems.

5 Macroeconomic Implications of Financial Uncertainty and Learning

In this section, we discuss the results from our model simulations. First, we show how we calibrate the model. Then, we discuss the main result – the amplification of fundamental and noise shocks in high-volatility time periods. Next, we highlight the degree of time-varying underreaction to fundamental shocks and overreaction to noise shocks implied by our model. We then move on to examine the time-varying importance of fundamental and noise shocks for investment. Finally, we close the section by discussing the countercyclical uncertainty that our model generates.

5.1 Calibration

We calibrate most of the structural parameters to be in line with Christiano et al. (2005) and Christiano et al. (2014). These parameters can be found in Table 6 in Appendix Section D. We solve the model with a second-order perturbation with pruning (Andreasen et al., 2017).

The key set of parameters that are nonstandard are those that define the exogenous process and signal extraction problem $\{\rho, \sigma_s^2, \omega_\sigma, \lambda_\sigma, \alpha_\sigma\}$. We calibrate these parameters based off of our empirical exercise in Section 3.4. We set ρ, λ_σ, and α_σ equal to their modal estimates in the fundamental volatility model found in Table 1. To calibrate σ_s and ω_σ, we rely on the estimates in Table 1 which are then scaled down by a common factor for both σ_s and ω_σ.[18] The common factor is chosen such that at a median volatility level, a fundamental risk shock leads to a 28 bps increase in annualized spreads, which is in line with the literature (Christiano et al., 2014; Caldara et al., 2016).

Table 2: Distribution of Kalman Gains on Signal

Percentile	Kalman Gain
5%	0.04
50%	0.08
95%	0.22

Note: This table shows the distribution of Kalman gains from 20,000 simulations of our model with the benchmark calibration.

Table 2 characterizes the distribution of Kalman gains on the signal generated from a long model simulation. Comparing the simulated Kalman gains with those from Figure 4, we confirm that our model generates a distribution of Kalman gains consistent with what we see in our estimation. This check is important, as it assures us that our model is generating information frictions in line with the data.

5.2 Fundamental and Noise Shocks in Calm and Volatile Times

Our key result - that fundamental and noise shocks are amplified by the volatility state - is presented in Figure 5, which shows impulse responses to fundamental and noise shocks in median (solid blue line) and

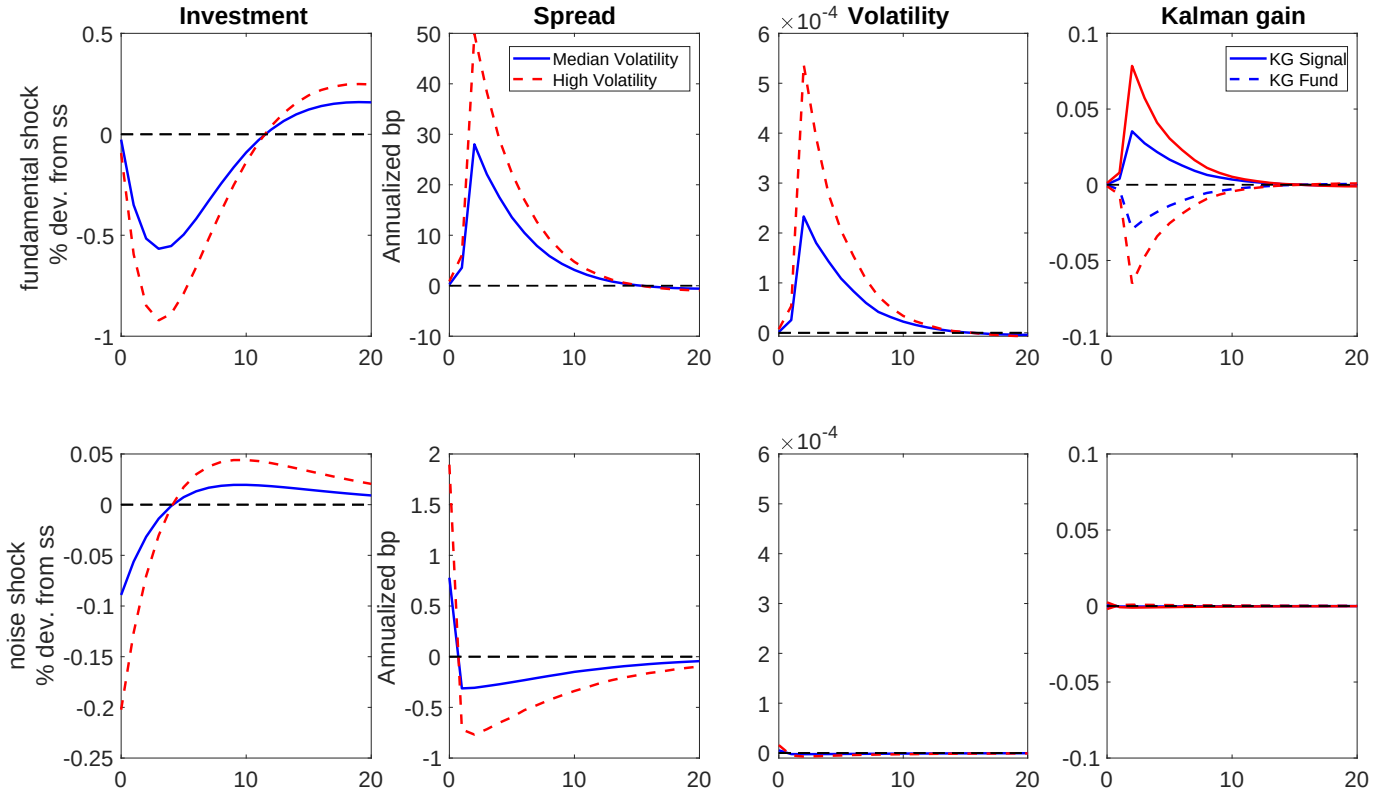
[18] This procedure entails choosing a common factor c such that $\sigma_s = \frac{\hat{\sigma}_s}{c}$ and $\omega_\sigma = \hat{\omega}_x - (1-\hat{\rho})\log(c)$. The parameters with the $\hat{}$ are the estimates from Table 1.

high (dashed red line) volatility states. An adverse fundamental risk shock that realizes in the first period has limited effects on impact, as agents must sift through the noise of the signal. When the risk shock gets revealed in the first period, however, expectations immediately update. Investment experiences a hump-shaped contraction and corporate spreads increase as entrepreneur net worth declines due to the drop in the price of capital and rise in defaults. The rise in spreads also generates a rise in volatility, which is shown in the third panel of the top row. Finally, the last panel shows the responses of the Kalman gain terms for the expectations of future risk. Agents' expectations get updated both from seeing the signal and the current risk. The responses of the Kalman gain on the signal innovation are the solid lines while those on the fundamental innovation are the dashed lines. The Kalman gain on the signal innovation increases as volatility increases, while the Kalman gain on the fundamental innovation decreases. The higher fundamental volatility makes the signal relatively more informative. As the informativeness of the signal increases, agents naturally put less weight on seeing the current fundamental.

Moving from a median- to high-volatility state amplifies the shock in two ways. First, the actual shock is larger due to the higher fundamental volatility. This difference is largely the reason behind the greater peak effects of the fundamental shock on investment, spreads, volatility, and the Kalman gains. Second, on impact, agents' expectations about future risk update more quickly due to the increased informativeness of the signal. Although the true value of risk is revealed by the first period and therefore expectations are fully updated by then in both cases, the effects of this initial expectations shift propagate throughout the entire response due to the internal mechanisms of the model.

The second row shows the responses to an adverse noise shock. The noise shock generates an expectation that future risk will be high, which leads to a drop in investment and increase in spreads in preparation for the increased risk. Moving to the first period, agents understand that this belief is mistaken, which leads to a rapid drop and overshooting in spreads. Note that the noise shock itself has a minimal effect on volatility due to the small movements in spreads. Comparing the dashed red to the solid blue lines, the high fundamental volatility state more than doubles the response of investment to the noise shock. This effect is entirely due to increased sensitivity of agents to the signal on impact, as the volatility of the noise shock itself is unchanged.

Figure 5: Fundamental and Noise Shocks Get Amplified by Fundamental Volatility



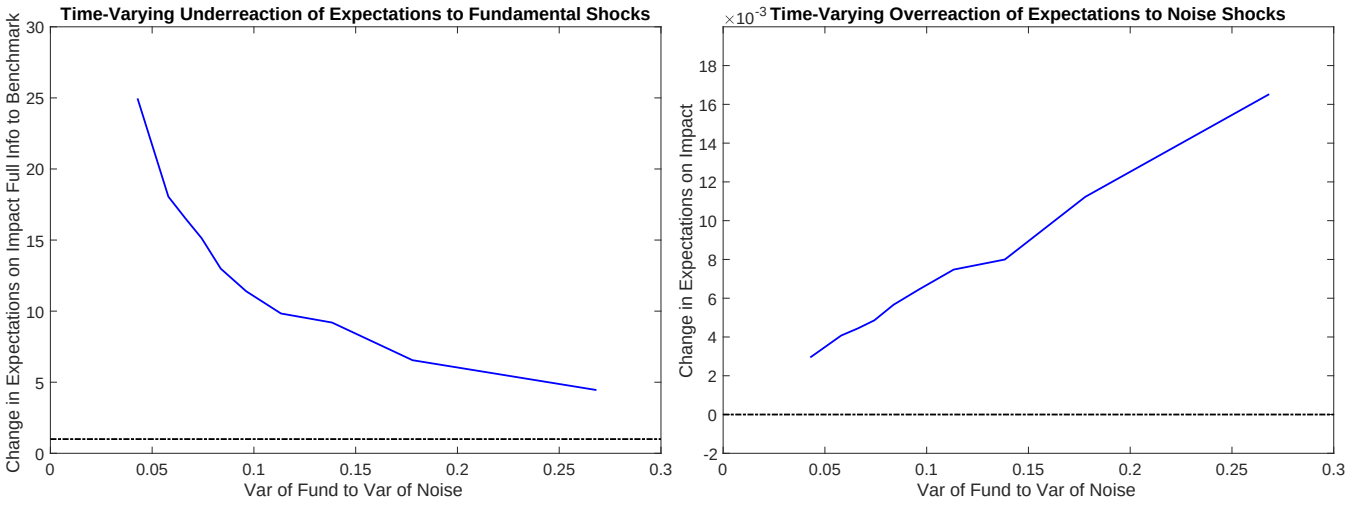
Note: The figure plots the generalized impulse response functions to a 1 standard deviation fundamental shock (top row) and noise shock (bottom row) in two cases. The solid blue line shows the responses following a shock at median volatility while the dashed red line shows the responses at high volatility (90% of its simulated distribution). For the Kalman gain panels, the solid lines show the response of the Kalman gain on the signal innovation while the dashed lines show the Kalman gain on the fundamental innovation. Blue continues to indicate median volatility while red indicates high volatility.

An interesting comparison model is one in which time-varying volatility is shut down ($\alpha_\sigma = 0$). In this case, expectation formation reverts to a constant parameter Kalman filter, but the model is still solved nonlinearly using higher-order perturbation methods. This simulation yields responses of investment and spreads to fundamental and noise shocks that are quantitatively similar to those from the benchmark model in the median-volatility case, which is shown in the blue line in Figure 5. This comparison illustrates that models with constant volatility could substantially underestimate the effects of expectations shocks during times of crisis by neglecting the increased sensitivity of agents to signals.

5.3 Time-Varying Sensitivity of Expectations to Fundamental and Noise Shocks

We now examine how the sensitivity of expectations to fundamental and noise shocks in our model changes as the informativeness of the signal changes. The left panel in Figure 6 shows the relationship between the degree of underreaction in expectations of one-quarter ahead risk to a fundamental risk shock and the relative variances of the fundamental shock to the noise shock. The blue line shows the ratio of this expectation on impact in a full information model to our benchmark one, with a higher value indicating more underreaction. This scaling is useful to control for the fact that the variance of the fundamental shock changes as we move along the x-axis. As the variance of the fundamental increases relative to the variance of the noise, agents are more confident in the signal and therefore update expectations more aggressively.

Figure 6: Time-Varying Underreaction to Fundamental Shocks (Left) and Overreaction to Noise Shocks (Right)



Note: This left panel plots the ratio of the change in expectations of one-quarter ahead risk on impact following a 1 standard deviation adverse fundamental risk shock in a full information model where the variance of the noise is 0 compared to our benchmark model as a function of the variance of the fundamental to noise shock. The dash-dotted black line is 1, which would be the ratio if both models had full information. The right panel plots the change in expectations of one-quarter ahead risk on impact following a 1 standard deviation adverse noise shock in our benchmark model. The dash-dotted black line is 0, which is the response to a noise shock in the full information model.

The right panel of the figure shows the degree of overreaction in expectations of one-quarter ahead risk to a noise shock. As the informativeness of the signal increases, expectations of risk increase more

on impact following the same-sized noise shock. In a full information model, agents do not react to noise shocks since they understand that noise shocks are uninformative about future risk. This is represented by the dash-dotted black line at 0.

Taken together, the two panels of Figure 6 show that expectations update more rapidly during highly volatile times. This result is a natural consequence of the signals becoming relatively more informative in these times.

5.4 Importance of Fundamental and Noise Shocks

We now examine the relative importance of fundamental and noise shocks for investment fluctuations in normal and more turbulent times. To do so, we compute variance decompositions conditional on median and high volatility.

As our model is nonlinear, computing variance decompositions is not straightforward. Our strategy is as follows. First, we condition on either a median- or high-volatility state. Then, we divide the variance from a model simulation in which only the fundamental shock is active by the variance from a model where both the fundamental and noise shocks are active.[19] One minus the ratio of this variance share gives a measure of the importance of the shock.[20]

Table 3: Noise Shocks Become Relatively Less Important As Volatility Increases

	Impact	1 Qtr	2 Qtr	3 Qtr	4 Qtr
Median Volatility	0.92	0.07	0.03	0.02	0.01
High Volatility	0.82	0.07	0.01	0.01	0.01

Note: This table shows the variance share of noise shocks in the median volatility state and high volatility state (90% of its simulated distribution).

Table 3 shows the variance shares of the noise shock in the median- and high-volatility states. Noise shocks in general are dominant at short horizons, but their effects quickly fade as agents realize that their beliefs were mistaken. As we move from median volatility to high volatility, noise shocks

[19] We use $20,000$ simulations to compute the variances.

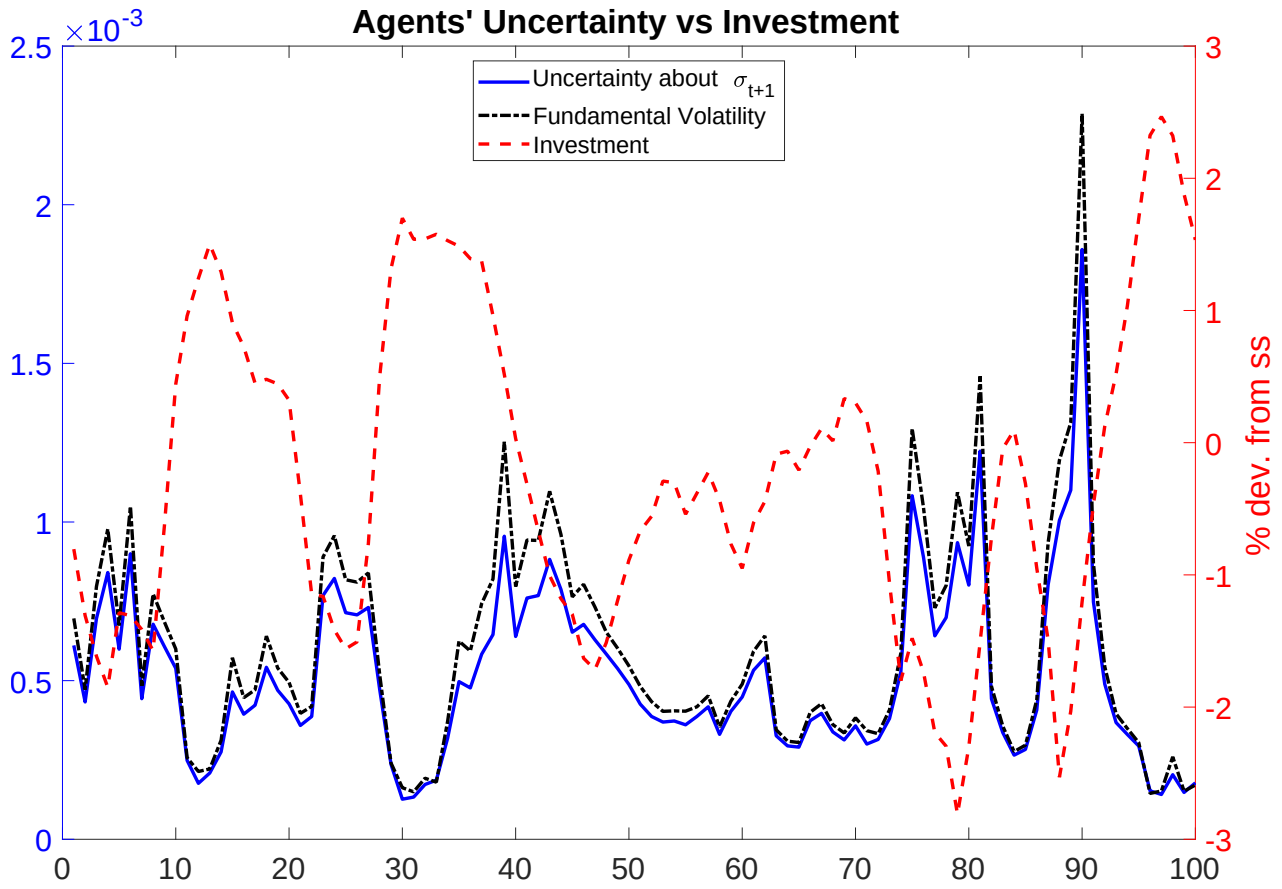
[20] Because our model is nonlinear, this variance share measure for noise is not equal to an alternative strategy where we compare the variance of simulations in which only the noise shock is active to the variance from simulations where both shocks are active. Moreover, in general it is not the case that we can use the same strategy to compute the variance share of the fundamental and have the noise and fundamental variance shares add up to 1.

become relatively less important, despite the fact that, as Figure 5 shows, their effects on investment more than double. This is because the same mechanism amplifying the effects of noise shocks in high-volatility times, namely the increased sensitivity to the signal, also amplifies the fundamental shocks. In addition, fundamental shocks become intrinsically more volatile. Hence, the combination of these two effects increases their importance.

5.5 The Countercyclicality of Uncertainty

Despite the increased informativeness of the signal in the high-volatility state, agents' uncertainty – which we define as the conditional variance of the error in the one-quarter ahead expectation of risk ($\Sigma_{11,t}$ in Equation 5) – remains countercyclical.

Figure 7: Agents' Uncertainty is Countercyclical in Model Simulations



Note: The figure plots a model simulation of agents' uncertainty about $t+1$ risk in blue and fundamental volatility in dash-dotted black (both on the left axis). The dashed red line is investment in percent deviations from steady state on the right axis.

Figure 7 illustrates this fact by plotting model simulations of agents' uncertainty about σ_{t+1} in blue against investment in dashed red. Fundamental volatility is in dash-dotted black. Investment drops due to adverse fundamental risk and noise shocks, which also raise spreads and fundamental volatility. The higher fundamental volatility generates higher uncertainty from the agents' perspective, despite the fact that the time t signal is relatively more informative. The unconditional correlation between uncertainty and investment is -0.76, while the correlation between uncertainty and fundamental volatility is near 1.[21]

The relationship between fundamental volatility and uncertainty can be understood by examining Equation 6. Higher fundamental volatility increases uncertainty as long as the standard deviation of the noise is not 0. Intuitively, agents with time t information use both s_t and σ_t to learn about σ_{t+1}. Raising fundamental volatility makes forecasting σ_{t+1} a more difficult problem overall, which reduces the informativeness of σ_t. The only case in which this effect is not at play is if the signal is perfectly informative about future risk.

6 Conclusion

Beliefs tend to shape investment and credit cycles, especially so in bad states. We analyze a signal extraction framework with time-varying fundamental volatility that generates changes in the information rigidity faced by agents which can produce this fact. We embed this into a macroeconomic model with financial frictions in which agents receive noisy signals about future entrepreneurial risk and find that this mechanism has economically important effects. The key is a countercyclical fundamental volatility which leads to a countercyclical signal-to-noise ratio. A high volatility state further amplifies the effects of fundamental and noise shocks due to an increased sensitivity of agents to their noisy signals.

[21] The correlation between uncertainty and output is -0.34 because consumption and investment move in opposite directions following fundamental and noise shocks.

References

Adrian, T., N. Boyarchenko, and D. Giannone (2019, April). Vulnerable growth. *American Economic Review 109*(4), 1263–89.

Afanasyeva, E., I. Komunjer, and M. Zhong (2025). Dynamic models with gaussian signals – theory and perturbation approximation. working paper.

Andreasen, M. M., J. Fernández-Villaverde, and J. F. Rubio-Ramírez (2017, 06). The pruned state-space system for non-linear dsge models: Theory and empirical applications. *The Review of Economic Studies 85*(1), 1–49.

Ascari, G., S. Fasani, J. Grazzini, and L. Rossi (2023). Endogenous uncertainty and the macroeconomic impact of shocks to inflation expectations. *Journal of Monetary Economics 140*(S), 48–63.

Barsky, R. B. and E. R. Sims (2011). News shocks and business cycles. *Journal of Monetary Economics 58*(3), 273–289.

Barsky, R. B. and E. R. Sims (2012). Information, Animal Spirits, and the Meaning of Innovations in Consumer Confidence. *American Economic Review 102*(4), 1343–77.

Beaudry, P. and F. Portier (2004, September). An exploration into Pigou's theory of cycles. *Journal of Monetary Economics 51*(6), 1183–1216.

Bernanke, B. S., M. Gertler, and S. Gilchrist (1999). *The Financial Accelerator in a Quantitative Business Cycle Framework*, Chapter 1C, pp. 1341–93. Handbook of Monetary Economics, edited by John B. Taylor and Michael Woodford; Amsterdam: North-Holland.

Bianchi, F. (2013). Regime switches, agents' beliefs, and post-world-war ii u.s. macroeconomic dynamics. *The Review of Economic Studies 80*(2), 463–490.

Bianchi, F., C. L. Ilut, and H. Saijo (2024). Smooth diagnostic expectations. *NBER Working Paper*.

Blanchard, O. J., J.-P. L'Huillier, and G. Lorenzoni (2013, December). News, Noise, and Fluctuations: An Empirical Exploration. *American Economic Review 103*(7), 3045–3070.

Bloom, N. (2014, May). Fluctuations in uncertainty. *Journal of Economic Perspectives 28*(2), 153–76.

Bordalo, P., N. Gennaioli, Y. Ma, and A. Shleifer (2020, September). Overreaction in Macroeconomic Expectations. *American Economic Review 110*(9), 2748–2782.

Boz, E., C. Daude, and C. Bora Durdu (2011). Emerging market business cycles: Learning about the trend. *Journal of Monetary Economics 58*(6), 616–631.

Bretscher, L., A. Malkhozov, and A. Tamoni (2021). Expectations and aggregate risk. *Journal of Monetary Economics 123*(C), 91–108.

Caldara, D., C. Fuentes-Albero, S. Gilhchrist, and E. Zakrajsek (2016). The macroeconomic impact of financial and uncertainty shocks. *European Economic Review 88*(C), 185–207.

Chahrour, R. and K. Jurado (2018, July). News or noise? the missing link. *American Economic Review 108*(7), 1702–36.

Chen, H.-F., P. R. Kumar, and J. H. van Schuppen (1989). On kalman filter for conditionally gaussian systems with random matrices. *Systems & Control Letters 5*(13), 397–404.

Christiano, L. J., M. Eichenbaum, and C. L. Evans (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of Political Economy 113*(1), 1–45.

Christiano, L. J., R. Motto, and M. Rostagno (2014, January). Risk shocks. *American Economic Review 104*(1), 27–65.

Coibion, O. and Y. Gorodnichenko (2015, August). Information Rigidity and the Expectations Formation Process: A Simple Framework and New Facts. *American Economic Review 105*(8), 2644–2678.

Edge, R. M., T. Laubach, and J. C. Williams (2007). Learning and Shifts in Long-Run Productivity Growth. *Journal of Monetary Economics 54*(8), 2421–2438.

Fajgelbaum, P. D., E. Schaal, and M. Taschereau-Dumouchel (2017). Uncertainty Traps. *The Quarterly Journal of Economics 132*(4), 1641–1692.

Fernandez-Villaverde, J. and P. Guerron-Quintana (2020, August). Uncertainty Shocks and Business Cycle Research. *Review of Economic Dynamics 37*, 118–166.

Fernández-Villaverde, J., J. Rubio-Ramírez, and F. Schorfheide (2016). Chapter 9 - solution and estimation methods for dsge models. Volume 2 of *Handbook of Macroeconomics*, pp. 527–724. Elsevier.

Geweke, J. (1999). Using simulation methods for bayesian econometric models: inference, development,and communication. *Econometric Reviews 18*(1), 1–73.

Gilchrist, S. and M. Saito (2008). Expectations, asset prices, and monetary policy: The role of learning. In *Asset Prices and Monetary Policy*, pp. 45–102. NBER Chapters.

Gonçalves, S., A. M. Herrera, L. Kilian, and E. Pesavento (2021). Impulse response analysis for structural dynamic models with nonlinear regressors. *Journal of Econometrics 225*(1), 107–130. Themed Issue: Vector Autoregressions.

Harvey, A. (1989). *Forecasting, structural time series models and the Kalman filter*. Cambridge University Press.

Ilut, C. and H. Saijo (2021). Learning, confidence, and business cycles. *Journal of Monetary Economics 117*(C), 354–376.

Jaimovich, N. and S. Rebelo (2009, September). Can News about the Future Drive the Business Cycle? *American Economic Review 99*(4), 1097–1118.

Judd, K. L. (1996). *Oxford Handbook of Innovation*, Volume I, Chapter Approximation, Perturbation, and Projection Methods in Economic Analysis. Handbook of Computational Economics.

Jurado, K., S. C. Ludvigson, and S. Ng (2015, March). Measuring Uncertainty. *American Economic Review 105*(3), 1177–1216.

Kim, C.-J. and C. R. Nelson (1999). Has the u.s. economy become more stable? a bayesian approach based on a markov-switching model of the business cycle. *The Review of Economics and Statistics 81*(4), 608–616.

Koop, G., M. H. Pesaran, and S. M. Potter (1996, September). Impulse response analysis in nonlinear multivariate models. *Journal of Econometrics 74*(1), 119–147.

Kozlowski, J., L. Veldkamp, and V. Venkatesvaran (2020). The Tail that Wags the Economy: Beliefs and Persistent Stagnation. *Journal of Political Economy 128*(8), 2839–80.

Krishnamurthy, A. and W. Li (2025). Dissecting mechanisms of financial crises: Intermediation and sentiment. *Journal of Political Economy 133*(3), 935–985.

Kydland, F. E. and E. C. Prescott (1982). Time to Build and Aggregate Fluctuations. *Econometrica 50*(6), 1345–1370.

Liptser, R. and A. Shiryayev (1978). *Statistics of Random Processes II: Applications*. Springer-Verlag.

Lorenzoni, G. (2009, December). A Theory of Demand Shocks. *American Economic Review 99*(5), 2050–2084.

Ludvigson, S. C., S. Ma, and S. Ng (2021, October). Uncertainty and business cycles: Exogenous impulse or endogenous response? *American Economic Journal: Macroeconomics 13*(4), 369–410.

McConnell, M. M. and G. Perez-Quiros (2000). Output fluctuations in the united states: What has changed since the early 1980's? *American Economic Review 90*(5), 1464 – 1476.

Saijo, H. (2017). The uncertainty multiplier and business cycles. *Journal of Economic Dynamics and Control 78*(C), 1–25.

Schmitt-Grohe, S. and M. Uribe (2006). Comparing two variants of calvo-type wage stickiness. *NBER Working Paper*.

Schmitt-Grohé, S. and M. Uribe (2004). Solving dynamic general equilibrium models using a second-order approximation to the policy function. *Journal of Economic Dynamics and Control 28*(4), 755–775.

Schmitt-Grohé, S. and M. Uribe (2012). What's news in business cycles. *Econometrica 80*(6), 2733–2764.

Sims, C. A. (2003). Implications of rational inattention. *Journal of Monetary Economics 50*, 665–690.

Tanaka, H. (2025). Equilibrium yield curves with imperfect information. *Journal of Monetary Economics 149*(C).

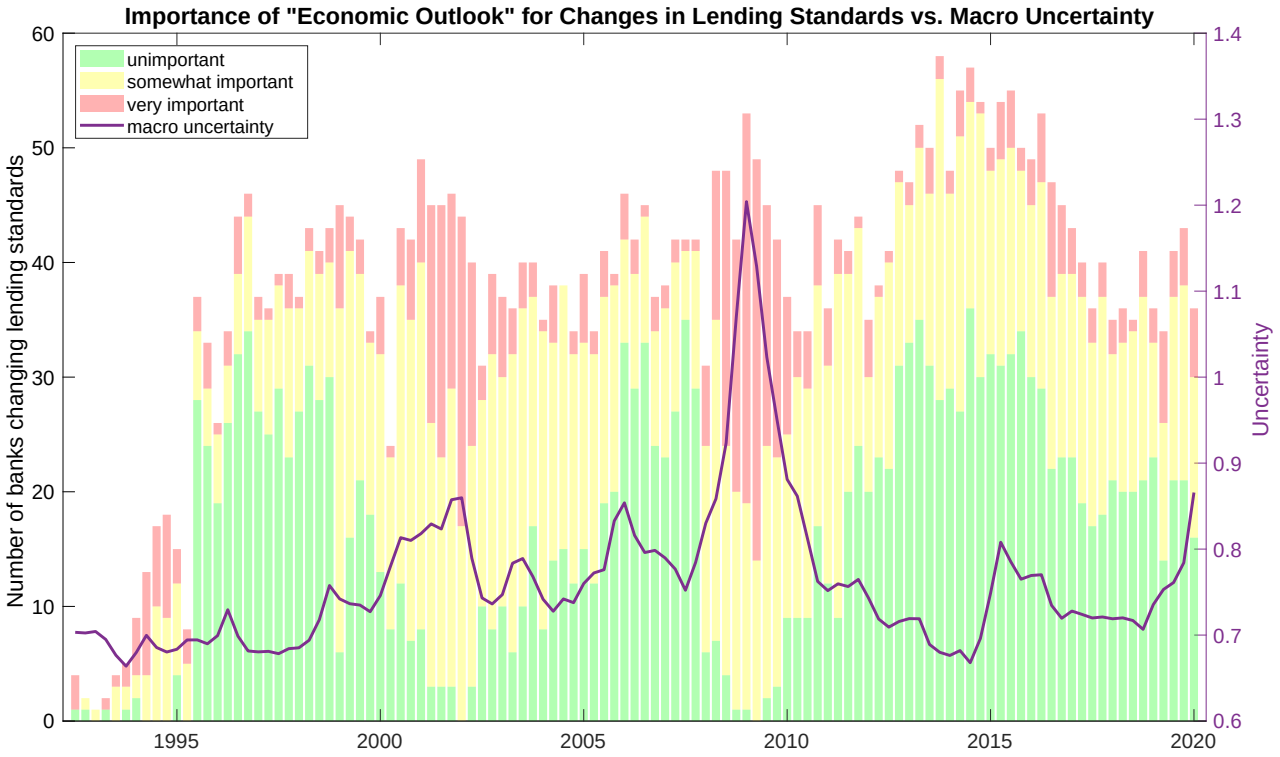
Townsend, R. M. (1979, October). Optimal contracts and competitive markets with costly state verification. *Journal of Economic Theory 21*(2), 265–293.

Van Nieuwerburgh, S. and L. Veldkamp (2006, May). Learning asymmetries in real business cycles. *Journal of Monetary Economics 53*(4), 753–772.

A SLOOS Details

A.1 Economic Outlook and Macro Uncertainty

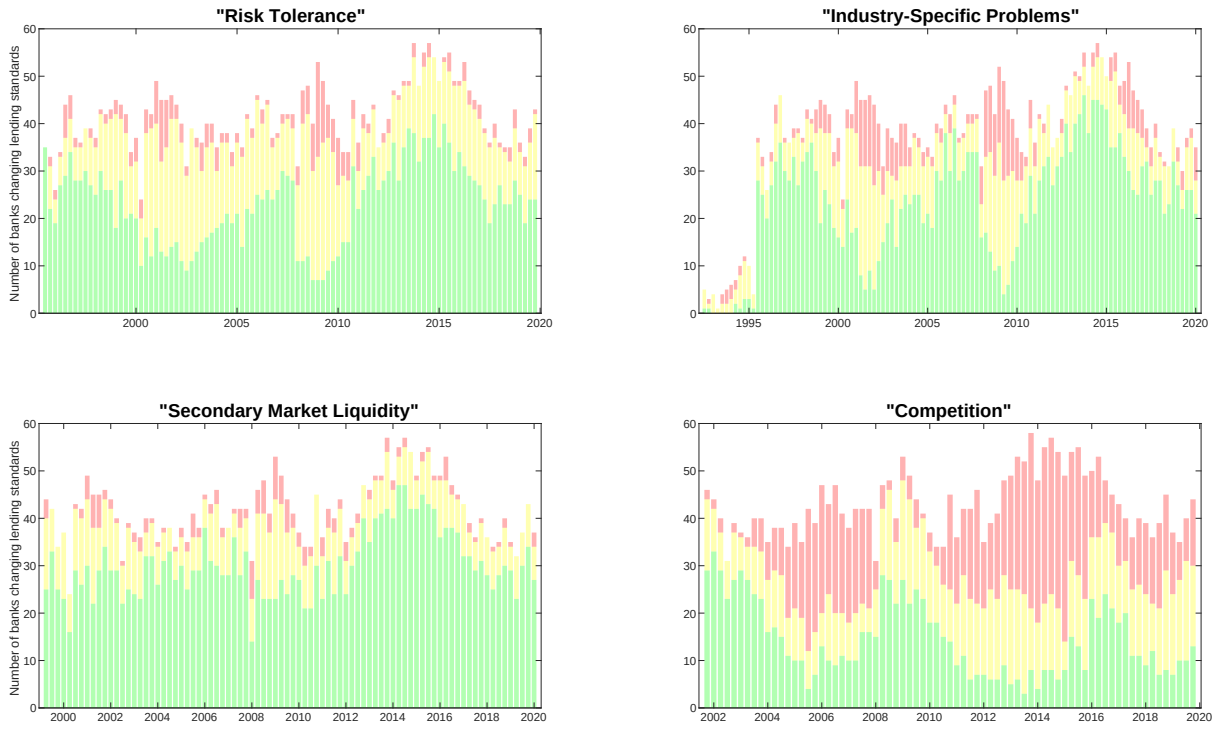
Figure 8: Relationship Between Economic Outlook and Macro Uncertainty is Weaker



Note: The figure plots the number of banks that responded "unimportant" in green, "somewhat important" in yellow, and "very important" in red to whether economic outlook matters in response to Question 3 of the SLOOS. The purple line shows the 3 months ahead macro uncertainty measure from Ludvigson et al. (2021).

A.2 Selective Other Reasons in the SLOOS

Figure 9: Other Reasons for SLOOS Question 3



Note: The figure plots the number of banks that responded "unimportant" in green, "somewhat important" in yellow, and "very important" for four other reasons in response to Question 3 of the SLOOS.

- "Risk Tolerance": Reduced (Increased) tolerance for risk
- "Industry-Specific Problems": Worsening (Improvement) of industry-specific problems
- "Secondary Market Liquidity": Decreased (Increased) liquidity in the secondary market for these loans
- "Competition": Less (More) aggressive competition from other banks or nonbank lenders (other financial intermediaries or the capital markets)

B Signal Extraction Problem Derivations

Consider the following signal extraction problem:

$$\begin{aligned} s_t &= x_{t+1} + \varepsilon_{s,t},\ \varepsilon_{s,t} \sim N(0, \sigma_{s,t}^2) \\ x_{t+1} &= \rho x_t + \varepsilon_{x,t+1},\ \varepsilon_{x,t+1} \sim N(0, \sigma_{x,t}^2) \end{aligned} \tag{19}$$

$$\begin{aligned} \log \sigma_{s,t}^2 &= \omega_s + \lambda_s \log \sigma_{s,t-1}^2 + \alpha_s s_{t-1} \\ \log \sigma_{x,t}^2 &= \omega_x + \lambda_x \log \sigma_{x,t-1}^2 + \alpha_x x_{t-1} \end{aligned} \tag{20}$$

Agents at time t see s_t, x_t and their entire history.
Let

$$w_t = \begin{pmatrix} \epsilon_{xt} \\ \epsilon_{st} \end{pmatrix}, \quad w_t \sim N(0, \mathrm{Id}_2)$$

be all the disturbances in the system (notice the stacking of shocks on top and signal errors at bottom). Then, the state transition equation is

$$\underbrace{\begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix}}_{\xi_{t+1}} = \underbrace{\begin{pmatrix} \rho & 0 \\ 1 & 0 \end{pmatrix}}_{A(z_t, m_t)} \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} + \underbrace{\begin{pmatrix} \exp[\frac{1}{2}(\omega_x + \lambda_x \log \sigma_{xt}^2 + \alpha_x x_t)] & 0 \\ 0 & 0 \end{pmatrix}}_{B(z_t, m_t)} w_{t+1} \tag{21}$$

and the observation equation is

$$\underbrace{\begin{pmatrix} s_t \\ x_t \end{pmatrix}}_{z_t} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{C(z_{t-1}, m_{t-1})} \begin{pmatrix} x_{t+1} \\ x_t \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & \exp[\frac{1}{2}(\omega_s + \lambda_s \log \sigma_{st-1}^2 + \alpha_s s_{t-1})] \\ 0 & 0 \end{pmatrix}}_{D(z_{t-1}, m_{t-1})} w_t \tag{22}$$

where we have defined

$$m_t = \begin{pmatrix} \sigma_{xt} \\ \sigma_{st} \end{pmatrix}, \quad m_0 \text{ given}$$

Note that for all $t = 0, 1, \ldots$, we have $A(z_t, m_t) = A$ and $C(z_{t-1}, m_{t-1}) = C$.

Step 2: Kalman Filtering equations The conditional distribution of ξ_t given z^t is Gaussian with mean $\hat{\xi}_t$ and covariance Σ_t. Let us start with prior mean and covariance,

$$\bar{\xi}_0 = \begin{pmatrix} \bar{x}_+ \\ \bar{x} \end{pmatrix} \quad \text{and} \quad \bar{\Sigma}_0 = \begin{pmatrix} \bar{\Sigma}_{x+} & 0 \\ 0 & \bar{\Sigma}_x \end{pmatrix}$$

At time $t = 0$, after observing $z_0 = (s_0, x_0)'$, we get from (22),

$$\begin{aligned} \hat{\xi}_0 &= \bar{\xi}_0 + \bar{\Sigma}_0 C'(C\bar{\Sigma}_0 C' + DD')^{-1}(z_0 - C\bar{\xi}_0) = \begin{pmatrix} \frac{\sigma_{s0}^2}{\bar{\Sigma}_{x+} + \sigma_{s0}^2} \bar{x}_+ + \frac{\bar{\Sigma}_{x+}}{\bar{\Sigma}_{x+} + \sigma_{s0}^2} s_0 \\ x_0 \end{pmatrix} \\ \Sigma_0 &= \bar{\Sigma}_0 - \bar{\Sigma}_0 C'(C\bar{\Sigma}_0 C' + DD')^{-1} C\bar{\Sigma}_0 = \begin{pmatrix} \frac{\sigma_{s0}^2}{\bar{\Sigma}_{x+} + \sigma_{s0}^2} \bar{\Sigma}_{x+} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{23}$$

In what follows, let

$$\hat{\xi}_{01} = \frac{\sigma_{s0}^2}{\bar{\Sigma}_{x+} + \sigma_{s0}^2}\bar{x}_+ + \frac{\bar{\Sigma}_{x+}}{\bar{\Sigma}_{x+} + \sigma_{s0}^2}s_0 \quad \text{and} \quad \Sigma_{011} = \frac{\sigma_{s0}^2}{\bar{\Sigma}_{x+} + \sigma_{s0}^2}\bar{\Sigma}_{x+}$$

so that

$$\hat{\xi}_0 = \begin{pmatrix} \hat{\xi}_{01} \\ x_0 \end{pmatrix} \quad \text{and} \quad \Sigma_0 = \begin{pmatrix} \Sigma_{011} & 0 \\ 0 & 0 \end{pmatrix}$$

Now time moves on to $t = 1$ and we have

$$\begin{aligned}
\Sigma_e(z_0, m_0, \Sigma_0) &= CA\Sigma_0 A'C' + [CB(z_0, m_0) + D(z_0, m_0)][CB(z_0, m_0) + D(z_0, m_0)]' \\
&= \begin{pmatrix} \rho^2\Sigma_{011} & \rho\Sigma_{011} \\ \rho\Sigma_{011} & \Sigma_{011} \end{pmatrix} + \begin{pmatrix} \exp(\omega_x + \lambda_x \log \sigma_{x0}^2 + \alpha_x x_0) + \exp(\omega_s + \lambda_s \log \sigma_{s0}^2 + \alpha_s s_0) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \rho^2\Sigma_{011} + \exp(\omega_x + \lambda_x \log \sigma_{x0}^2 + \alpha_x x_0) + \exp(\omega_s + \lambda_s \log \sigma_{s0}^2 + \alpha_s s_0) & \rho\Sigma_{011} \\ \rho\Sigma_{011} & \Sigma_{011} \end{pmatrix} \\
&= \begin{pmatrix} \rho^2\Sigma_{011} + \sigma_{x1}^2 + \sigma_{s1}^2 & \rho\Sigma_{011} \\ \rho\Sigma_{011} & \Sigma_{011} \end{pmatrix} \qquad (24)
\end{aligned}$$

which is invertible, so

$$\begin{aligned}
K(z_0, m_0, \Sigma_0) &= \left[A\Sigma_0 A'C' + B(z_0, m_0)B(z_0, m_0)'C' + B(z_0, m_0)D(z_0, m_0)'\right]\Sigma_e(z_0, m_0, \Sigma_0)^{-1} \\
&= \begin{pmatrix} \rho^2\Sigma_{011} + \sigma_{x1}^2 & \rho\Sigma_{011} \\ \rho\Sigma_{011} & \Sigma_{011} \end{pmatrix} \begin{pmatrix} \rho^2\Sigma_{011} + \sigma_{x1}^2 + \sigma_{s1}^2 & \rho\Sigma_{011} \\ \rho\Sigma_{011} & \Sigma_{011} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{\sigma_{x1}^2}{\sigma_{x1}^2 + \sigma_{s1}^2} & \rho\frac{\sigma_{s1}^2}{\sigma_{x1}^2 + \sigma_{s1}^2} \\ 0 & 1 \end{pmatrix} \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_1 &= A\Sigma_0 A' + B(z_0, m_0)B(z_0, m_0)' - K(z_0, m_0, \Sigma_0)\Sigma_e(z_0, m_0, \Sigma_0)K(z_0, m_0, \Sigma_0)' \\
&= \begin{pmatrix} \rho^2\Sigma_{011} + \sigma_{x1}^2 & \rho\Sigma_{011} \\ \rho\Sigma_{011} & \Sigma_{011} \end{pmatrix} - \begin{pmatrix} \rho^2\Sigma_{011} + \frac{\sigma_{x1}^2}{\sigma_{x1}^2 + \sigma_{s1}^2}\sigma_{x1}^2 & \rho\Sigma_{011} \\ \rho\Sigma_{011} & \Sigma_{011} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sigma_{s1}^2\sigma_{x1}^2}{\sigma_{x1}^2 + \sigma_{s1}^2} & 0 \\ 0 & 0 \end{pmatrix} \qquad (26)
\end{aligned}$$

Note that in this example, Σ_1 does not depend on Σ_0. This is a feature of this particular model which does not hold in general. Finally, we have

$$\begin{aligned}
\hat{\xi}_1 &= A\hat{\xi}_0 + K(z_0, m_0, \Sigma_0)\Sigma_e(z_0, m_0, \Sigma_0)^{1/2}\epsilon_1 \\
z_1 &= CA\hat{\xi}_0 + \Sigma_e(z_0, m_0, \Sigma_0)^{1/2}\epsilon_1,
\end{aligned}$$

where $\epsilon_1 \sim N(0, \mathrm{Id}_2)$ so using (24), from which it follows that

$$\Sigma_e(z_0, m_0, \Sigma_0)^{1/2} = \begin{pmatrix} \sqrt{\rho^2\Sigma_{011} + \sigma_{x1}^2 + \sigma_{s1}^2} & 0 \\ \frac{\rho\Sigma_{011}}{\sqrt{\rho^2\Sigma_{011} + \sigma_{x1}^2 + \sigma_{s1}^2}} & \sqrt{\frac{\Sigma_{011}(\sigma_{x1}^2 + \sigma_{s1}^2)}{\rho^2\Sigma_{011} + \sigma_{x1}^2 + \sigma_{s1}^2}} \end{pmatrix}$$

and (25), gives

$$\hat{\xi}_1 = \begin{pmatrix} \rho\hat{\xi}_{01} \\ \hat{\xi}_{01} \end{pmatrix} + \begin{pmatrix} \frac{\rho^2\Sigma_{011}+\sigma^2_{x1}}{\sqrt{\rho^2\Sigma_{011}+\sigma^2_{x1}+\sigma^2_{s1}}} & \rho\frac{\sigma^2_{s1}}{\sigma^2_{x1}+\sigma^2_{s1}}\sqrt{\frac{\Sigma_{011}(\sigma^2_{x1}+\sigma^2_{s1})}{\rho^2\Sigma_{011}+\sigma^2_{x1}+\sigma^2_{s1}}} \\ \frac{\rho\Sigma_{011}}{\sqrt{\rho^2\Sigma_{011}+\sigma^2_{x1}+\sigma^2_{s1}}} & \sqrt{\frac{\Sigma_{011}(\sigma^2_{x1}+\sigma^2_{s1})}{\rho^2\Sigma_{011}+\sigma^2_{x1}+\sigma^2_{s1}}} \end{pmatrix} \epsilon_1$$

$$z_1 = \begin{pmatrix} \rho\hat{\xi}_{01} \\ \hat{\xi}_{01} \end{pmatrix} + \begin{pmatrix} \sqrt{\rho^2\Sigma_{011}+\sigma^2_{x1}+\sigma^2_{s1}} & 0 \\ \frac{\rho\Sigma_{011}}{\sqrt{\rho^2\Sigma_{011}+\sigma^2_{x1}+\sigma^2_{s1}}} & \sqrt{\frac{\Sigma_{011}(\sigma^2_{x1}+\sigma^2_{s1})}{\rho^2\Sigma_{011}+\sigma^2_{x1}+\sigma^2_{s1}}} \end{pmatrix} \epsilon_1$$

Since $z_1 = (s_1, x_1)'$ it follows from the second component of both equations that $\hat{\xi}_{12} = x_1$ which is as we would expect (since $\xi_1 = (x_2, x_1)'$).

Iterating in t then gives, for every $t = 0, 1, \ldots,$

$$\begin{pmatrix} \hat{\xi}_{t+1,1} \\ s_{t+1} \\ x_{t+1} \end{pmatrix} = \begin{pmatrix} \rho\hat{\xi}_{t1} \\ \rho\hat{\xi}_{t1} \\ \hat{\xi}_{t1} \end{pmatrix} + \begin{pmatrix} \frac{\rho^2\Sigma_{t11}+\sigma^2_{x,t+1}}{\sqrt{\rho^2\Sigma_{t11}+\sigma^2_{x,t+1}+\sigma^2_{s,t+1}}} & \rho\frac{\sigma^2_{s,t+1}}{\sigma^2_{x,t+1}+\sigma^2_{s,t+1}}\sqrt{\frac{\Sigma_{t11}(\sigma^2_{x,t+1}+\sigma^2_{s,t+1})}{\rho^2\Sigma_{t11}+\sigma^2_{x,t+1}+\sigma^2_{s,t+1}}} \\ \sqrt{\rho^2\Sigma_{t11}+\sigma^2_{x,t+1}+\sigma^2_{s,t+1}} & 0 \\ \frac{\rho\Sigma_{t11}}{\sqrt{\rho^2\Sigma_{t11}+\sigma^2_{x,t+1}+\sigma^2_{s,t+1}}} & \sqrt{\frac{\Sigma_{t11}(\sigma^2_{x,t+1}+\sigma^2_{s,t+1})}{\rho^2\Sigma_{t11}+\sigma^2_{x,t+1}+\sigma^2_{s,t+1}}} \end{pmatrix} \epsilon_{t+1} \tag{27}$$

where $\{\epsilon_t\}_{t=1}^{+\infty} \sim$ iid $N(0, \mathrm{Id}_2)$, and where

$$\Sigma_{t11} = \frac{\sigma^2_{st}\sigma^2_{xt}}{\sigma^2_{xt}+\sigma^2_{st}}, \quad t = 1, \ldots, \quad \Sigma_{011} = \frac{\sigma^2_{s0}\bar{\Sigma}_{x+}}{\bar{\Sigma}_{x+}+\sigma^2_{s0}}$$

and

$$\hat{\xi}_{01} = \frac{\sigma^2_{s0}}{\bar{\Sigma}_{x+}+\sigma^2_{s0}}\bar{x}_+ + \frac{\bar{\Sigma}_{x+}}{\bar{\Sigma}_{x+}+\sigma^2_{s0}}s_0$$

The states in the full information equivalent of the model are then

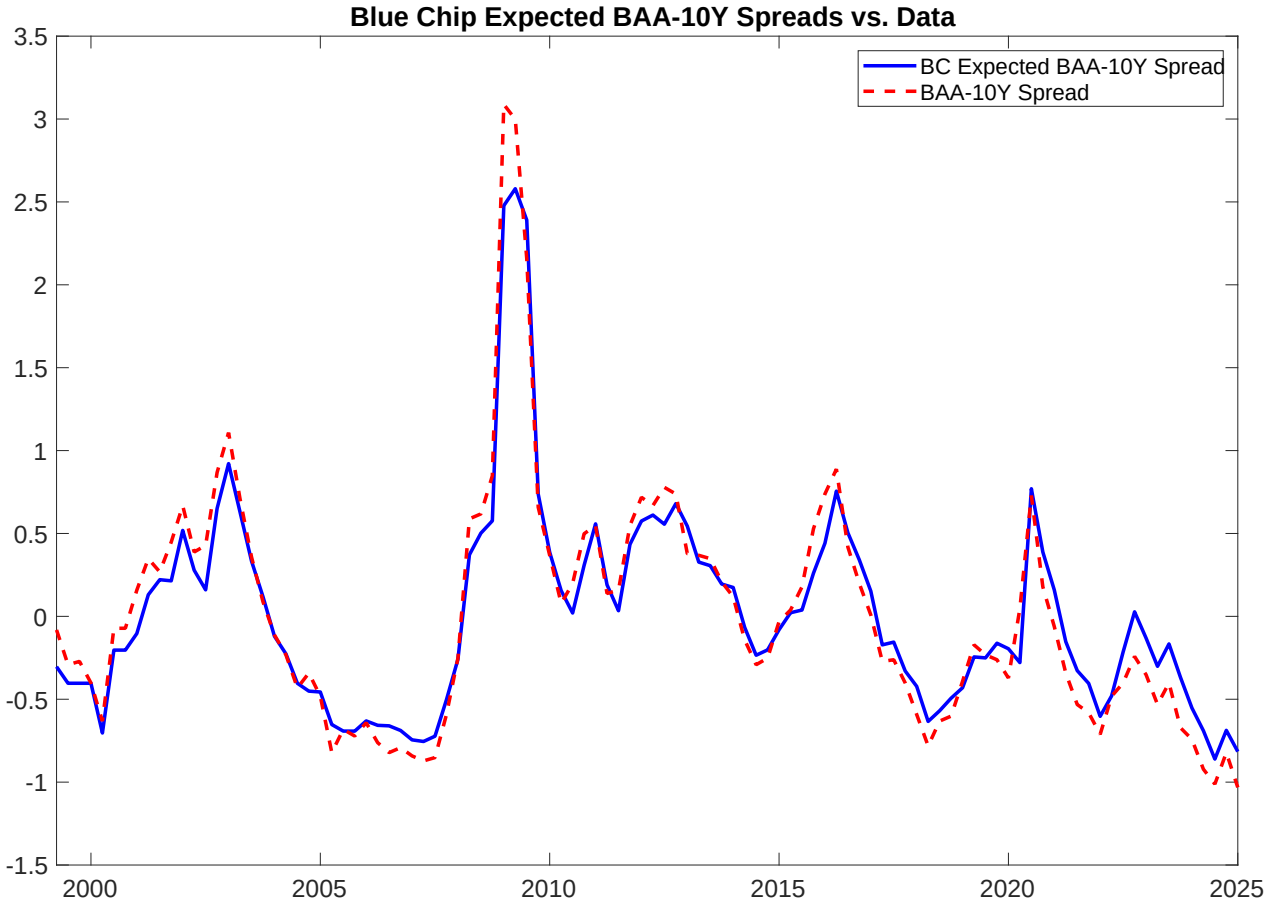
$$(\hat{\xi}_{t1}, s_t, x_t, \sigma_{xt}, \sigma_{st})$$

Note again that in this particular example, the variable Σ_{t11} does not enter since it is itself a function of the other states $(\sigma_{xt}, \sigma_{st})$.

C Further Details on Empirical Exercise

C.1 Data

Figure 10: Blue Chip Expectations of Corporate Spreads and Data



Note: The figure plots the BAA-10Y spread data in dashed red and the Blue Chip one-quarter ahead expectations data in solid blue.
Wolters Kluwer, Blue Chip Financial Forecasts. Wolters Kluwer, https://www.wolterskluwer.com/en/solutions/blue-chip.

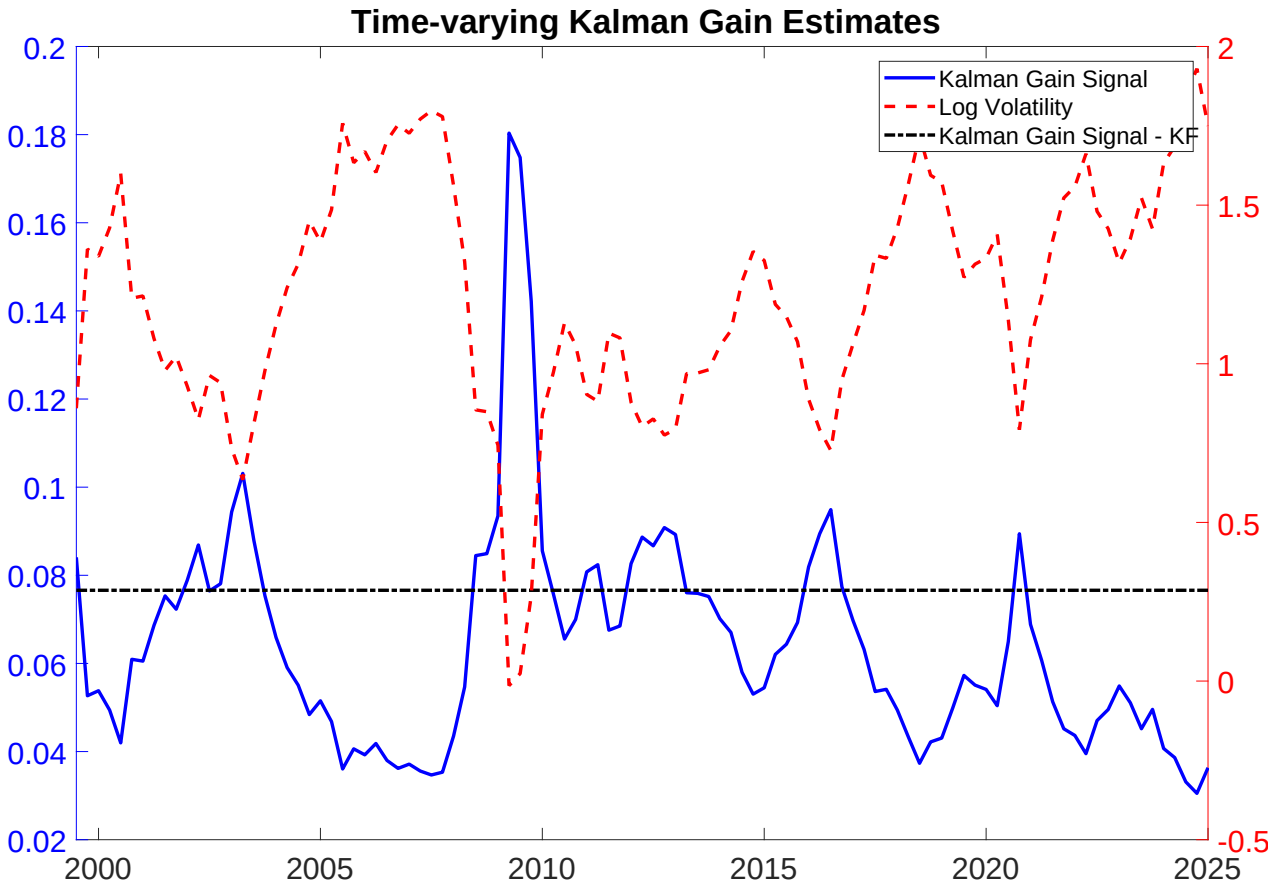
C.2 Estimates from Competing Learning Models

C.2.1 Time-Varying Noise Volatility Model

Table 4: Parameter Estimates of the CGKF with Time-Varying Noise Volatility

Parameter	Mode	5%/95%
ρ	0.83	$(0.81, 0.86)$
ω_s	1.27	$(0.75, 1.96)$
λ_s	-0.07	$(-0.49, 0.17)$
α_s	-1.69	$(-2.55, -0.78)$
σ_x	0.47	$(0.43, 0.54)$

Figure 11: Kalman Gain Estimates from Time-Varying Noise Volatility Model



Note: The figure plots the estimates of the Kalman gain on the signal in solid blue and the log volatility of noise shocks in dashed red from the posterior mode parameters. The dash-dotted black line is the Kalman gain on the signal from the constant parameter model that is estimated using the Kalman filter.

C.2.2 Constant Parameter Model

Table 5: Parameter Estimates of the Constant Parameter Model

Parameter	Mode	5%/95%
ρ	0.85	$(0.82, 0.88)$
σ_s	1.56	$(1.29, 2.03)$
σ_x	0.45	$(0.42, 0.51)$

D Details of the Extended DSGE Model

In what follows, this Section briefly summarizes the optimization problems and equations of the model outlined in Section 4. Calibration of the associated parameters is also outlined.

D.1 Frictions in the Extended Model

D.1.1 Entrepreneurs and the Financial Contract

Our model of entrepreneurs closely follows that of Christiano et al. (2014). There are a continuum of entrepreneurs indexed by $i \in [0,1]$ who are risk neutral. Entrepreneur i combines net worth N_t^i with borrowing B_t^i from a financial intermediary to purchase capital in competitive markets for total expenditure $Q_t K_t^i$.

The entrepreneur experiences an idiosyncratic shock to capital productivity ω_{t+1}^i that turns the capital into $\omega_{t+1}^i K_t^i$ efficiency units. This shock is assumed iid across entrepreneurs and over time with a log normal distribution where the mean is one and the standard deviation of $\log \omega$ is σ_t. We call σ_t a risk shock. We denote the PDF of this distribution as $f_t(\omega)$ and its CDF as $F_t(\omega)$. The entrepreneur then rents the capital efficiency units in competitive markets to the final goods firm before selling the undepreciated efficiency units of capital $\omega_{t+1}^i(1-\delta)K_t^i$ back to the capital goods producers at price Q_{t+1}. For this, the entrepreneur earns the return $\omega_{t+1}^i R_{t+1}^K$, where R_{t+1}^K is the aggregate return to capital common to entrepreneurs and equal to:

$$R_{t+1}^K = \frac{r_{t+1}^K + (1-\delta)Q_{t+1}}{Q_t} \tag{28}$$

Entrepreneurs borrow in competitive markets from a representative financial intermediary who is able to diversify over idiosyncratic risks. We assume a costly state verification framework of Townsend (1979). Therefore, the loan the entrepreneur obtains is a standard debt contract with interest rate Z_{t+1}^i and leverage $L_t^i = \frac{N_t^i + B_t^i}{N_t^i}$.

The entrepreneur's objective function is as follows:

$$E_t\left[\int_{\overline{\omega}_{t+1}^i}^{\infty} \left(\omega R_{t+1}^K Q_t K_t^i - B_t^i Z_{t+1}^i\right) dF_t(\omega)\right] = E_t\left[\left(1 - \Gamma_t\left(\overline{\omega}_{t+1}^i\right)\right) R_{t+1}^K\right] L_t^i N_t^i \tag{29}$$

where $\overline{\omega}_{t+1}^i$ is the cutoff productivity threshold below which the entrepreneur defaults and the intermediary steps in to monitor at a cost of μ per unit of asset and take the total remaining assets.[22] The term:

$$\Gamma_t\left(\overline{\omega}_{t+1}^i\right) = \left(1 - F_t(\overline{\omega}_{t+1}^i)\right)\overline{\omega}_{t+1}^i + \int_0^{\overline{\omega}_{t+1}^i} \omega dF_t(\omega) \tag{30}$$

denotes the average share of entrepreneur i revenues received by the financial intermediary.

The financial intermediary obtains its funds by issuing risk free deposits to households that pay R_t^D regardless of the realization of the aggregate shock in $t+1$. Therefore, the financial intermediary has the following constraint:

[22] The cutoff threshold is defined as $\overline{\omega}_{t+1}^i = \frac{B_t^i Z_{t+1}^i}{R_{t+1}^K Q_t K_t^i}$.

$$\left(1-F_t\left(\overline{\omega}_{t+1}^i\right)\right) Z_{t+1}^i B_t^i + (1-\mu)\int_0^{\overline{\omega}_{t+1}^i} \omega dF_t(\omega) R_{t+1}^K Q_t K_t^i \geq B_t^i R_t^D \tag{31}$$

which specifies that the average share of revenue for the loan from the financial intermediary to entrepreneur i must be greater than its funding cost to provide that loan. This constraint must hold no matter the realization of the aggregate shocks in $t+1$, so the loan interest rate Z_{t+1}^i is contingent on the $t+1$ aggregate state.

The entrepreneur chooses the cutoff threshold $\overline{\omega}_{t+1}^i$ and L_t^i to maximize Equation 29 subject to Equation 31.[23] Since N_t^i appears as a constant in the objective function and does not appear in the constraint, all entrepreneurs choose the same loan contract terms and we can drop the indexation on i.

The first-order condition from this problem is:

$$E_t\left[\left(1-\Gamma_t\left(\overline{\omega}_{t+1}\right)\right)\frac{R_{t+1}^K}{R_t} + \frac{\Gamma_t'\left(\overline{\omega}_{t+1}\right)}{\Gamma_t'\left(\overline{\omega}_{t+1}\right)-\mu G_t'\left(\overline{\omega}_{t+1}\right)}\left(\frac{R_{t+1}^K}{R_t}\left(\Gamma_t\left(\overline{\omega}_{t+1}\right)-\mu G_t\left(\overline{\omega}_{t+1}\right)\right)-1\right)\right]=0 \tag{32}$$

where $G_t\left(\omega_{t+1}\right) = \int_0^{\overline{\omega}} \omega dF_t(\omega)$. Equation 32 points to the key role played in this model of $\frac{E_t\left(R_{t+1}^K\right)}{R_t}$, which is called the external finance premium (EFP). Bernanke et al. (1999) shows that entrepreneurial leverage is an increasing function of the EFP. It is a key indicator of financial stress in the model.

Entrepreneurs are assumed to transfer a fraction $1-\gamma$ of their net worth to households after they settle all of their transactions in period $t+1$.

The entrepreneur decides on the degree of capital utilization u_t and rents part of capital services $u_t K_{t-1}$ to the intermediate goods producers (introduced below). The revenue from selling the capital services is $r_t^k u_t K_{t-1}$ and the cost (in real terms) of adjusting the capital utilization rate is $a(u_t)K_{t-1}$, where we assume $a' > 0$ and $a'' > 0$. The optimization problem of the entrepreneur is given by:

$$\max_{u_t} \quad \left[r_t^k u_t - a(u_t)\right]\omega K_{t-1}$$

In the aggregate optimum, i.e. averaged over all entrepreneurs, it must hold that

$$r_t^k = a'(u_t). \tag{33}$$

As in Christiano et al. (2014), we employ the following functional form for the adjustment cost function of the capital utilization rate:

$$a(u_t) = \frac{r_{ss}^k}{\sigma_u}\left[\exp\left\{\sigma_u(u_t-1)\right\}-1\right], \tag{34}$$

where r_{ss}^k refers to the steady-state rental rate of capital services.

The aggregate real rate of return per unit of capital in period t depends on the real rental rate of capital services, $r_t^k u_t$, and the capital gain of the non-depreciated capital stock, $(1-\delta)K_{t-1}$, between $t-1$ and t in real terms, net of capital utilization adjustment costs $a(u_t)$:

$$R_t^k = \frac{r_t^k u_t + (1-\delta)Q_t - a(u_t)}{Q_{t-1}}. \tag{35}$$

[23] This is equivalent to maximizing with respect to Z_{t+1}^i and L_t^i.

D.1.2 Household Problem

The representative household is risk-averse and derives utility from a Dixit-Stiglitz aggregate of imperfectly substitutable consumption goods,

$$C_t = \left[\int_0^1 C_t(j)^{\frac{\epsilon_p-1}{\epsilon_p}} dj\right]^{\frac{\epsilon_p}{\epsilon_p-1}}. \tag{36}$$

Households have an infinite planning horizon and discount their future expected utility with the subjective discount factor $\beta < 1$. They can transfer wealth intertemporally by saving in terms of deposits, which pay the risk-free nominal return R_t^n between t and $t+1$. We allow for habit formation in consumption. Households supply homogenous labor to the monopolistically competitive labor unions, which differentiate labor at no cost and set nominal wages. The household's constrained optimization problem can be summarized as

$$\max_{C_t, h_t, D_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left(C_t / C_{t-1}^h\right)^{1-\sigma}}{1-\sigma} - \chi \frac{h_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \right\},$$

$$C_t + D_t + r_t \tilde{a}_t = \frac{\tilde{a}_{t-1}}{\pi_t} + \frac{R_t^n}{\pi_t} D_{t-1} + h_t^d \int_0^1 w_t^m \left(\frac{w_t^m}{w_t}\right)^{-\epsilon_w} dm + \tilde{\Pi}_t, \tag{37}$$

where D_t are real deposits, $\tilde{a}_t$ is the real payoff of nominal state-contingent assets, r_t is the stochastic discount factor between period t and $t+1$, $\pi_t = P_t/P_{t-1}$ is the gross inflation rate, and $P_t \equiv \left[\int_0^1 P_t(j)^{1-\epsilon_p} dj\right]^{\frac{1}{1-\epsilon_p}}$ is the corresponding aggregate price index. $\tilde{\Pi}_t$ denotes net lump-sum transfers of profits to the household from the retailers and labor unions, whereas $h_t^d \int_0^1 w_t^m \left(\frac{w_t^m}{w_t}\right)^{-\epsilon_w} dm$ is the real wage income of the household. The parameter h determines the strength of habit formation.

The first-order conditions with respect to $\{C_t, D_t\}$, where λ_t denotes the Lagrange multiplier of the budget constraint, are

$$\begin{aligned} C_t: \quad & e_t \left(\frac{C_t}{C_{t-1}^h}\right)^{1-\sigma} C_t^{-1} - \beta h E_t \left[e_{t+1} \left(\frac{C_{t+1}}{C_t^h}\right)^{1-\sigma} C_t^{-1} \right] = \lambda_t, \\ D_t: \quad & \lambda_t = \beta E_t \left\{ \lambda_{t+1} \frac{R_t^n}{\pi_{t+1}} \right\}. \end{aligned}$$

We discuss the choice of the optimal labor supply below, jointly with the labor unions' choice of the optimal wage rate.

D.1.3 Capital Goods Producers

After production in period t has taken place, capital producers purchase the non-depreciated capital stock from entrepreneurs, invest in a Dixit-Stiglitz aggregate of imperfectly substitutable investment goods, $I_t \equiv \left[\int_0^1 I_t(j)^{\frac{\epsilon_p-1}{\epsilon_p}} dj\right]^{\frac{\epsilon_p}{\epsilon_p-1}}$, and sell the new stock of capital to entrepreneurs at the relative price Q_t. We assume that turning final output into productive capital, i.e. gross investment, is costly due

to possible disruptions of the production process, replacement of installed capital, or learning. The accumulation of physical capital can then be written as

$$K_t = (1-\delta)K_{t-1} + \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t, \tag{38}$$

where $S\left(\frac{I_t}{I_{t-1}}\right) = \frac{\phi}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2$, $S(1) = S'(1) = 0$, and $S''(1) = \phi$ (compare, e.g., Christiano et al. (2005)).

The profit-maximization problem of the representative capital goods producer, subject to the capital accumulation equation in (38), is given by

$$\max_{I_t} \quad \sum_{s=0}^{\infty} \beta^s \left\{ Q_{t+s}[K_{t+s} - (1-\delta)K_{t+s-1}] - I_{t+s} \right\},$$

while the corresponding FOC with respect to investment in period t is given by

$$Q_t \left[1 - \frac{\phi}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - \phi\left(\frac{I_t}{I_{t-1}} - 1\right)\frac{I_t}{I_{t-1}}\right] + \beta\phi E_t\left[Q_{t+1}\left(\frac{I_{t+1}}{I_t} - 1\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right] = 0. \tag{39}$$

D.1.4 Intermediate Goods Producers

Intermediate goods producers rent the productive capital stock from entrepreneurs and hire labor from households, paying a competitive rental rate on capital services and a wage rate determined in the labor market, respectively. To convert capital and labor into intermediate or wholesale goods, they use the following Cobb-Douglas production function:

$$Y_t = A\tilde{K}_t^{\alpha} h_t^{1-\alpha},$$

where A denotes constant total factor productivity (TFP).

Note that the first input argument of the production function, $\tilde{K}_t$, stands for capital *services*, defined as

$$\tilde{K}_t = u_t K_{t-1}, \tag{40}$$

where u_t is the utilization rate of capital in period t chosen by entrepreneurs.

Suppose that the price of the homogeneous wholesale good in terms of the numeraire is $1/X_t$, so that the gross flexible-price markup of retail goods over the wholesale good is X_t. The static optimization problem of the intermediate goods producer can then be summarized as

$$\max_{\tilde{K}_t, h_t} \quad \frac{1}{X_t} A\tilde{K}_t^{\alpha} h_t^{1-\alpha} - r_t^k \tilde{K}_t - w_t h_t,$$

which yields the following FOCs:

$$\begin{aligned} \tilde{K}_t: \quad & X_t r_t^k = \alpha \frac{Y_t}{\tilde{K}_t}, \\ h_t: \quad & X_t w_t = (1-\alpha)\frac{Y_t}{h_t}. \end{aligned}$$

D.1.5 Retailers

Monopolistically competitive retailers purchase homogeneous intermediate output, diversify at no cost, and resell to households and capital goods producer for consumption and investment purposes, respectively. We assume staggered price setting à la Calvo (1983), where θ_p denotes the exogenous probability of *not* being able to readjust the price.

A retailer allowed to reset its price in period t chooses the optimal price, P_t^*, in order to maximize the present value of current and expected future profits, subject to the demand function for the respective product variety in period $t+s$, $s = 0, ..., \infty$, $Y_{t+s}(j) = (P_{t,s}/P_{t+s})^{-\epsilon_p} Y_{t+s}$, where $P_{t,s}$ is the price of a retailer that was last allowed to be set in period t.[24] Hence, the profit maximization problem of a retailer in period t is

$$\max_{P_t^*} \quad E_t \left\{ \sum_{s=0}^{\infty} \theta_p^s \Lambda_{t,t+s} \Pi_{t,s} \right\},$$

where $\Lambda_{t,t+s} \equiv \beta^s E_t \left[U'(C_{t+s})/U'(C_t) \cdot P_t/P_{t+s} \right]$ denotes the stochastic discount factor and

$$\Pi_{t,s} \equiv (P_t^* - MC_{t,s}) \left[\frac{P_t^*}{P_{t+s}} \right]^{-\epsilon_p} Y_{t+s},$$

where $MC_{t,s}$ is the retailer's nominal marginal cost in period $t+s$. The corresponding optimality condition is given by

$$E_t \sum_{s=0}^{\infty} \theta_p^s \Lambda_{t,t+s} Y_{t+s} P_{t+s}^{\epsilon_p} \left[P_t^* - \frac{\epsilon_p}{\epsilon_p - 1} MC_{t,s} \right] = 0.$$

In order to arrive at the New Keynesian Phillips curve, we combine the above FOC with the definition of the aggregate price index,

$$P_t = \left\{ \theta_p P_{t-1}^{1-\epsilon_p} + (1-\theta_p) \left(P_t^*\right)^{1-\epsilon_p} \right\}^{1/(1-\epsilon_p)}.$$

D.1.6 Wage Setting

We follow Schmitt-Grohe and Uribe (2006) to introduce nominal wage stickiness. Firms hire labor from a continuum of labor markets of measure 1 indexed by $m \in [0,1]$. In each labor market m, wages are set by a monopolistically competitive labor union. The union faces labor demand $(W_t^m/W_t)^{-\epsilon_w} h_t^d$, where W_t^m is the nominal wage charged by the union in market m at time t, $W_t \equiv \left[\int_0^1 \left(W_t^m\right)^{1-\epsilon_w} dm \right]^{1/(1-\epsilon_w)}$ is an economy-wide wage index and h_t^d is aggregate labor demand by firms. In each labor market, the union takes W_t and h_t^d as exogenous. The labor supply by the union satisfies $h_t^m = \left(\frac{w_t^m}{w_t} \right)^{-\epsilon_w} h_t^d$, where $w_t^m \equiv W_t^m/P_t$ and $w_t \equiv W_t/P_t$. The resource constraint implies $h_t = h_t^d \int_0^1 h_t^m dm$, which yields

$$h_t = h_t^d \int_0^1 \left(\frac{w_t^m}{w_t} \right)^{-\epsilon_w} dm. \tag{41}$$

We assume that households have access to a complete set of nominal state-contingent assets $\tilde{A}_t$. Each period, consumers can purchase $\tilde{A}_{t+1}$ at the nominal cost $E_t r_t \tilde{A}_{t+1}$, where r_t is the stochastic

[24] The isoelastic demand schedule for the product of retailer j can be derived from the definitions of aggregate demand $Y_t = \left[\int_0^1 Y_t(j)^{\frac{\epsilon_p - 1}{\epsilon_p}} dj \right]^{\frac{\epsilon_p}{\epsilon_p - 1}}$ and the aggregate price index $P_t = \left[\int_0^1 P_t(j)^{1-\epsilon_p} dj \right]^{\frac{1}{1-\epsilon_p}}$.

discount factor between t and $t+1$. The variable $\tilde{a}_t \equiv \tilde{A}_t/P_{t-1}$ denotes the real payoff in period t of nominal state-contingent assets purchased in $t-1$.

Nominal wage stickiness is introduced by the assumption that, each period, a fraction $\theta_w \in [0,1)$ of labor unions *cannot* reoptimize the nominal wage. In these labor markets, wages are indexed to past inflation, π_{t-1}. Let $\beta^t w_t/\tilde{\mu}_t$ be the Lagrange multiplier on (41) and $\beta^t \lambda_t$ the Lagrange multiplier on the household budget constraint. Then the Lagrangian associated with the household optimization problem is given by

$$\begin{aligned}\mathcal{L} =& E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U(C_t, C_{t-1}, h_t) + \lambda_t \left[h_t^d \int_0^1 \left(\frac{w_t^m}{w_t} \right)^{-\epsilon_w} dm - C_t - D_t - r_t \tilde{a}_t + \frac{\tilde{a}_{t-1}}{\pi_t} + \frac{R_t^n}{\pi_t} D_{t-1} \right] \right. \\ & \left. + \frac{\lambda_t w_t}{\tilde{\mu}_t} \left[h_t - h_t^d \int_0^1 \left(\frac{w_t^m}{w_t} \right)^{-\epsilon_w} dm \right] \right\}.\end{aligned}$$

The FOCs with respect to h_t and w_t^m are as follows:

$$-U_h(C_t, C_{t-1}, h_t) = \frac{\lambda_t w_t}{\tilde{\mu}_t} \tag{42}$$

and

$$w_t^m = \begin{cases} \tilde{w}_t & \text{if } w_t^m \text{ is set optimally, and} \\ w_{t-1}^m \pi_{t-1}/\pi_t & \text{otherwise.} \end{cases}$$

If labor demand curves and costs of supplying labor are identical across labor markets, the optimally set wage will be the same across markets, as well. To determine $\tilde{w}_t$, we write the part of the Lagrangian relevant for the optimal wage setting,

$$\mathcal{L}^w = E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left(\frac{\prod_{k=1}^s \left(\frac{\pi_{t+k-1}}{\pi_{t+k}} \right)}{w_{t+s}} \right)^{-\epsilon_w} h_{t+s}^d \left[\tilde{w}_t^{1-\epsilon_w} \prod_{k=1}^s \left(\frac{\pi_{t+k-1}}{\pi_{t+k}} \right) - \tilde{w}_t^{-\epsilon_w} \frac{w_{t+s}}{\tilde{\mu}_{t+s}} \right].$$

Then the FOC with respect to $\tilde{w}_t$ is given by

$$E_t \sum_{s=0}^{\infty} (\theta_w \beta)^s \left(\frac{\tilde{w}_t \prod_{k=1}^s \left(\frac{\pi_{t+k-1}}{\pi_{t+k}} \right)}{w_{t+s}} \right)^{-\epsilon_w} h_{t+s}^d \left[\frac{\epsilon_w - 1}{\epsilon_w} \tilde{w}_t \prod_{k=1}^s \left(\frac{\pi_{t+k-1}}{\pi_{t+k}} \right) - \frac{-U_h(t+s)}{\lambda_{t+s}} \right] = 0. \tag{43}$$

Equation (43) implies that, when allowed to reoptimize in period t, each union sets the real wage so that its future expected marginal revenues are equal to the average marginal cost of supplying labor. The marginal revenue s periods after the most recent reoptimization equals $\frac{\epsilon_w - 1}{\epsilon_w} \tilde{w}_t \prod_{k=1}^s \left(\frac{\pi_{t+k-1}}{\pi_{t+k}} \right)$, where $\frac{\epsilon_w}{\epsilon_w - 1}$ is the markup of wages over the marginal costs of labor that would prevail without wage stickiness. The marginal cost of supplying labor equals the marginal rate of substitution between consumption and leisure, $\frac{-U_h(t+s)}{\lambda_{t+s}}$. Hence, $\tilde{\mu}_t$ denotes the wedge between the disutility of labor and the average real wage in the economy and can be interpreted as the average markup of labor unions.

In order to state (43) recursively, define

$$f_t^1 = \left(\frac{\epsilon_w - 1}{\epsilon_w} \right) \tilde{w}_t E_t \sum_{s=0}^{\infty} (\beta \theta_w)^s \lambda_{t+s} \left(\frac{w_{t+s}}{\tilde{w}_t} \right)^{\epsilon_w} h_{t+s}^d \prod_{k=1}^s \left(\frac{\pi_{t+k}}{\pi_{t+k-1}} \right)^{\epsilon_w - 1}$$

and

$$f_t^2 = -\tilde{w}_t^{-\epsilon_w} E_t \sum_{s=0}^{\infty} (\beta\theta_w)^s w_{t+s}^{\epsilon_w} h_{t+s}^d U_h(C_{t+s}, C_{t+s-1}, h_{t+s}) \prod_{k=1}^{s} \left(\frac{\pi_{t+k}}{\pi_{t+k-1}}\right)^{\epsilon_w}.$$

Then f_t^1 and f_t^2 can be written recursively as follows:

$$f_t^1 = \left(\frac{\epsilon_w - 1}{\epsilon_w}\right) \tilde{w}_t \lambda_t \left(\frac{w_t}{\tilde{w}_t}\right)^{\epsilon_w} h_t^d + \theta_w \beta E_t \left(\frac{\pi_{t+1}}{\pi_t}\right)^{\epsilon_w - 1} \left(\frac{\tilde{w}_{t+1}}{\tilde{w}_t}\right)^{\epsilon_w - 1} f_{t+1}^1; \tag{44}$$

and

$$f_t^2 = -U_h(C_t, C_{t-1}, h_t) \left(\frac{\tilde{w}_t}{w_t}\right)^{-\epsilon_w} h_t^d + \theta_w \beta E_t \left(\frac{\tilde{w}_{t+1}\pi_{t+1}}{\tilde{w}_t \pi_t}\right)^{\epsilon_w} f_{t+1}^2. \tag{45}$$

The FOC with respect to $\tilde{w}_t$ can then be written as

$$f_t^1 = f_t^2. \tag{46}$$

Aggregation across labor markets implies

$$h_t = (1-\theta_w) h_t^d \sum_{s=0}^{\infty} \theta_w^s \left(\frac{\tilde{W}_{t-s} \prod_{k=1}^{s} \left(\frac{\pi_{t+k-s-1}}{\pi_{t+k-s}}\right)}{W_t} \right)^{-\epsilon_w}. \tag{47}$$

Defining the measure of wage dispersion as

$$\tilde{s}_t \equiv (1-\theta_w) \sum_{s=0}^{\infty} \theta_w^s \left(\frac{\tilde{W}_{t-s} \prod_{k=1}^{s} \left(\frac{\pi_{t+k-s-1}}{\pi_{t+k-s}}\right)}{W_t} \right)^{-\epsilon_w},$$

we can rewrite equation (47) as

$$h_t = \tilde{s}_t h_t^d, \tag{48}$$

where the evolution of wage dispersion over time is given by

$$\tilde{s}_t = (1-\theta_w) \left(\frac{\tilde{w}_t}{w_t}\right)^{-\epsilon_w} + \theta_w \left(\frac{w_{t-1}}{w_t}\right)^{-\epsilon_w} \left(\frac{\pi_t}{\pi_{t-1}}\right)^{\epsilon_w} \tilde{s}_{t-1}. \tag{49}$$

From the definition of the wage index, $W_t \equiv \left[\int_0^1 (W_t^m)^{1-\epsilon_w} dm\right]^{1/(1-\epsilon_w)}$, it follows that the real wage rate, w_t, can be expressed as

$$w_t^{1-\epsilon_w} = (1-\theta_w)\tilde{w}_t^{1-\epsilon_w} + \theta_w w_{t-1}^{1-\epsilon_w} \left(\frac{\pi_{t-1}}{\pi_t}\right)^{1-\epsilon_w}. \tag{50}$$

Equations (44)-(50) describe the equilibrium in the labor market.

D.1.7 Monetary Policy and Market Clearing

We assume that the central bank sets the *nominal* interest rate, R_t^n, according to the following standard Taylor rule:

$$\frac{R_t^n}{R_{ss}^n} = \left(\frac{R_{t-1}^n}{R_{ss}^n}\right)^{\rho} \left[\left(\frac{\pi_t}{\pi_{ss}}\right)^{\phi_\pi} \left(\frac{Y_t}{Y_{ss}}\right)^{\phi_y}\right]^{1-\rho}. \tag{51}$$

Hence, the central bank reacts to deviations of inflation and output from their respective steady-state values and might smooth interest rates over time with a weight ρ.

The model is closed by the economy-wide resource constraint,

$$Y_t = C_t + C_t^e + C_t^b + I_t + a\left(u_t\right) K_{t-1} + \mu G(\bar{\omega}_t) R_t^k Q_{t-1} K_{t-1}, \tag{52}$$

where C_t^e and C_t^b denote the real consumption of entrepreneurial and bank net worth, respectively, $a(u_t)K_{t-1}$ denotes the real adjustment costs due to capital utilization, and $\mu G(\bar{\omega}_t) R_t^k Q_{t-1} K_{t-1}$ the aggregate monitoring costs in period t.

D.2 Calibration of the Extended Model

The calibration of model parameters and the model steady state follows the literature rather closely and is presented in Tables 6 and 7, respectively.

Table 6: Benchmark Calibration of Parameter Values.

Household and production sector	Parameter	Value
coefficient of relative risk aversion	σ	2
Frisch elasticity of labor supply	η	3
habit formation in household consumption	h	0.65
relative weight of labor in utility function	χ	5.19
quarterly discount factor of households	β	0.995
elasticity of output with respect to capital	α	0.35
quarterly depreciation rate of physical capital	δ	0.025
coefficient of quadratic investment adjustment costs	ϕ	2.5
elasticity of capital utilization adjustment costs	σ_u	0.4
elasticity of substitution between retailer varieties	ϵ_p	10
Calvo probability of quarterly price adjustments	θ_p	0.75
elasticity of substitution between labor varieties	ϵ_w	10
Calvo probability of quarterly wage adjustments	θ_w	0.75
Optimal financial contract	**Parameter**	**Value**
exogenous consumption rate of entr. net worth	$1-\gamma^e$	0.015
monitoring costs as a fraction of total return on capital	μ	0.20
variance of idiosyncratic productivity draws	σ_ω^2	0.18
steady-state default threshold of entrepreneurs	$\bar{\omega}$	0.35
Monetary policy	**Parameter**	**Value**
interest-rate persistence in monetary policy rule	ρ	0.7418
responsiveness of monetary policy to inflation	ϕ_π	1.5
responsiveness of monetary policy to output	ϕ_y	0.5

Table 7: Selected Steady-State Values for Benchmark Parameter Calibration.

Steady-State Variable or Ratio	Computation	Value
capital-output ratio	$K/(4 \cdot Y)$	1.9451
household consumption relative to output	C/Y	0.6988
entrepreneur consumption relative to output	C^e/Y	0.1010
capital investment relative to output	I/Y	0.1945
default monitoring costs relative to output	$\mu G(\bar{\omega}) R^k QK/Y$	0.0057
employment as a share of time endowment*	H	1/3
gross price markup of retailers*	$\epsilon_p/(\epsilon_p - 1)$	1.1111
gross wage markup of labor unions	$\epsilon_w/(\epsilon_w - 1)$	1.1111
leverage ratio of entrepreneurs*	QK/N	1.5441
annualized default rate of entrepreneurs*	$4 \cdot F(\bar{\omega})$	4.735%
annualized risk-free policy interest rate*	$4 \cdot (R^n - 1)$	2.010%
annualized rate of return on capital	$4 \cdot \left(R^k - 1\right)$	6.195%
annualized external finance premium	$4 \cdot \left(R^k/R^n - 1\right)$	4.164%

Note: Superscript * indicates steady-state values targeted in the benchmark calibration.