Flight to Safety and New Keynesian Demand Recessions*

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March 18, 2025

Abstract

We incorporate flight to safety in a tractable New Keynesian model with incomplete markets and nominal safe assets. A rise in uncertainty induces a flight to safety, where investors shift portfolios from risky productive capital to nominal government bonds. Under price stickiness, the real value of nominal safe assets cannot adjust flexibly, causing capital price overshooting and aggregate demand recessions. Conventional monetary policy through the nominal rate has limited power in mitigating these recessions, as it fails to directly affect portfolio reallocation between safe and risky assets. Instead, optimal monetary policy allows temporary deviations from price stability, leveraging inflation dynamics to indirectly manage safe asset supply.

*First Version: February 2022. This paper supersedes a previous version titled "Flight-to-Safety in a New Keynesian Model." We are grateful to Joseph Abadi, Mark Aguiar, Andrey Alexandrov (discussant), Anmol Bhandari, Markus Brunnermeier, Nicolas Caramp (discussant), Sebastian Di Tella, Zhouzhou Gu, Nobu Kiyotaki, Benny Kleinman, Moritz Lenel, Ben Moll, Jonathan Payne, Krzysztof Pytka (discussant), Andreas Schaab, Dejanir Silva, Chris Sims, Alp Simsek, Gianluca Violante, Mike Woodford, Haonan Zhou, and seminar and conference participants at Princeton University, Monash University, University of Sydney, National University of Singapore, University of Porto, the CESifo Area Conference on Macro, Money, and International Finance, and the Mannheim Alumni Symposium for constructive feedback.

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1 Introduction

Following the Great Recession, a large macro literature has attributed fluctuations in real economic activity at least in part to variation in uncertainty, risk, or risk premia (e.g., Bloom, 2009; Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez and Uribe, 2011; Bloom, Floetotto, Jaimovich, Saporta-Eksten and Terry, 2018).[1] An important strand of that literature emphasizes the role played by New Keynesian price-setting frictions in transmitting uncertainty shocks to the real economy by creating aggregate demand recessions (e.g., Christiano, Motto and Rostagno, 2014; Basu and Bundick, 2017; Caballero and Simsek, 2020).[2] However, the literature has typically abstracted from the presence of safe assets in the form of government debt. This is a relevant omission for two reasons. First, outstanding stocks of government bonds in advanced economies are large and fluctuations in risk premia are typically associated with a flight to safety in financial markets, a rise in demand for these assets. Second, these assets take the form of nominal bonds so that their real value is tightly linked to nominal goods prices and possibly interacts with nominal price-setting frictions. The key point of this paper is that flight to safety is at the core of the transmission of uncertainty shocks to the real economy. Recognizing its role matters for a range of model predictions and, in particular, for conclusions about the power and optimal conduct of monetary policy.

Specifically, we provide a theoretical analysis of the link between flight to safety and aggregate demand recessions to shed light on the role played by safe asset demand for the transmission of uncertainty shocks. To this end, we build a model that augments the safe asset framework of Brunnermeier, Merkel and Sannikov (2024) with a standard New Keynesian supply side with sticky goods prices. Our model is analytically tractable, allowing us to prove all major results without resorting to numerical model analysis. The model nests, up to isomorphisms, the textbook New Keynesian

[1] The terminology in the literature is not uniform, and all three terms have been used to express similar economic ideas. For the present paper, the words uncertainty and risk are synonymous. What ultimately matters for this paper is time variation in risk premia, but the underlying exogenous variation that generates time-varying risk premia will be variation in uncertainty.

[2] One issue with flexible price models of uncertainty shocks is that they typically fail to generate business cycle comovement (Basu and Bundick, 2017). A notable exception is Di Tella and Hall (2022).

model (Galí, 2008, Chapter 3), a popular tractable extension with uncertainty shocks but without safe assets (Caballero and Simsek, 2020), and the flexible price model of Brunnermeier et al. (2024). In this sense, it represents the simplest framework to study the interaction of flight to safety and New Keynesian demand recessions.

The model features a competitive sector of household-entrepreneurs who manage productive capital subject to idiosyncratic return risk à la Angeletos (2007). They face a portfolio choice between risky capital and safe nominal government bonds. As uncertainty about the return on capital increases, households increase the fraction of wealth they hold in government bonds, a flight to safety.

Flight to safety spills over into the goods market because the real value of *nominal* government bonds is directly linked to the price level. Under flexible goods prices, a downward adjustment in nominal goods prices expands the real bond supply to accommodate the increased portfolio demand for safe assets. Nominal price adjustment frictions block this revaluation mechanism in the short run. With the real bond supply unable to expand, flight to safety instead results in a steep drop in the value of capital assets and overall household wealth, which depresses aggregate consumption demand. When short-term output is demand-determined, a decline in consumption demand initiates a general decline in economic activity.

After an uncertainty shock, the recessionary forces induced by flight to safety gradually ease over time, even if portfolios do not revert to their pre-shock positions. Specifically, the demand recession is accompanied by a period of disinflation, which gradually expands the safe asset supply to bring it closer to its "natural" level that would be obtained under flexible prices. This adjustment is analogous to the classic Pigou effect (Pigou, 1943; Patinkin, 1956) but applies to all nominal government liabilities, not just money balances. From a technical perspective, the real safe asset supply emerges as a state variable that adjusts only sluggishly to portfolio demand for safe assets and, together with the latter, is the key determinant of short-run aggregate demand.

One conclusion from our analysis concerns the relative roles of portfolio choice and intertemporal substitution in generating demand recessions in response to uncertainty shocks. The traditional literature without safe assets typically emphasizes intertemporal substitution by highlighting the effects of uncertainty on the precautionary motive

(e.g., Basu and Bundick, 2017). But a portfolio choice view is also possible (e.g., Caballero and Simsek, 2020). Both views are equivalent because intertemporal substitution and portfolio choice are not separate decision margins when the latter is only with regard to a virtual safe asset in zero net supply, while all wealth is transferred over time in the form of risky capital.[3] In contrast, in our framework, intertemporal substitution and portfolio choice correspond to separate decision margins, and we find that it is the latter that is relevant for aggregate demand. Not households' desire to save per se but their desire to increase savings in nominal safe assets generates a shortfall in aggregate demand. In Section 4, we compare our model to an environment where government bonds are in zero net supply. We show that this distinction matters, both for the baseline prediction that uncertainty shocks lead to demand recessions and for a range of comparative statics exercises.

Most importantly, the transmission mechanism emphasized in this paper suggests a diminished role of monetary policy in managing aggregate demand in response to uncertainty shocks, as it has limited effects on both the demand for and supply of safe assets. With regard to safe asset demand, manipulating nominal interest rates affects the level of asset returns, but the latter is not directly relevant to portfolio choice between safe and risky assets. It is instead the *excess return* of risky over safe assets that affects portfolio choice. Indeed, we show that, under certain assumptions on fiscal policy, portfolio choice is separated from both interest rate policy and aggregate economic activity, so that the central bank's interest rate policy has no effect on flight to safety dynamics. On the safe asset supply side, interest rate policy has limited effects via a debt revaluation and a debt growth channel. A reduction in nominal interest rates raises the market value of outstanding long-term debt (debt revaluation channel) but also reduces the nominal growth rate of government liabilities (debt growth channel). These effects give the monetary authority some influence but not perfect control over the dynamics of the safe asset supply. Indeed, we show that interest rate policy is generically unable to prevent aggregate demand recessions in response to uncertainty shocks.

The limited power of monetary policy is in stark contrast to the conventional analy-

[3]The same also applies more broadly if government bonds are in positive supply in an environment that features Ricardian equivalence under a fiscal rule that ensures debt sustainability. These are common assumptions in the wider DSGE literature.

sis in frameworks without safe assets. In these frameworks, the safe asset supply state is absent and interest rate policy affects aggregate demand directly via purely forward-looking intertemporal substitution considerations. The primary task of policy in these models is expectations management. Interest rate policy is perfectly able to accomplish this task so long as it is unconstrained. A common conclusion from the conventional analysis is therefore that uncertainty shocks are a concern for aggregate economic activity only if monetary policy is either suboptimal or constrained, e.g., by a zero lower bound (ZLB) on nominal interest rates. Our analysis, in contrast, suggests that this understates the potential for adverse effects from uncertainty shocks on economic activity. In our model, even a perfectly executed interest rate policy that is always unconstrained is generically unable to fully stabilize aggregate demand.

The ineffectiveness of interest rate policy in stabilizing aggregate demand raises two follow-up questions: (i) whether other tools, such as fiscal policy, have a role to play in aggregate demand management and (ii) how interest rate policy should be optimally deployed to at least mitigate the effects of flight to safety on aggregate demand. We study both questions in the final section of this paper.

With regard to the first question, distortionary capital taxation directly affects the portfolio choice between capital and safe assets and can mitigate flight to safety dynamics. However, this is not necessarily optimal from a welfare perspective because an increase in the fraction of wealth held in safe assets improves risk sharing, which is desirable in the face of higher uncertainty. Lump-sum taxes, in turn, represent an implicit safe asset whose real value is not linked to nominal goods prices. If lump-sum taxes are available, joint monetary-fiscal policy can implement the constrained efficient allocation, and demand recessions can be avoided.

Finally, we study the second question about optimal monetary policy when lump-sum taxes are unavailable and distortionary capital taxes are not deployed to prevent flight to safety. In this case, it is optimal for the monetary authority to under-react to shocks on impact relative to perfect demand stabilization. This allows disinflation pressures to build up and the Pigou effect to play out, which optimally smooths out the unavoidable distortions arising from flight to safety.

Related Literature. Our paper belongs to the recent literature on the transmission of uncertainty shocks in New Keynesian models.[4] Most of this work is based on the traditional Representative Agent New Keynesian (RANK) model (Christiano et al., 2014; Ilut and Schneider, 2014; Leduc and Liu, 2016; Basu and Bundick, 2017; Caballero and Simsek, 2020). Methodologically closest to our paper is Caballero and Simsek (2020), which is also based on a tractable continuous-time framework and emphasizes theoretical points. Common to these papers is that they analyze environments without government bonds or other safe assets in positive net supply.[5] While such models are also able to generate demand recessions following high uncertainty, they feature a different transmission mechanism of uncertainty shocks and lead to different policy conclusions, which we discuss in detail in Section 4.

Due to the presence of uninsurable idiosyncratic risk, our framework belongs to the class of Heterogeneous Agent New Keynesian (HANK) models that have been the primary focus of the literature in recent years. A quantitative strand of the HANK literature has revisited many classic questions in the New Keynesian literature in the context of rich numerically solved models,[6] but so far, uncertainty shocks have received relatively little attention. Notable exceptions are Bayer et al. (2019) and Schaab (2020). The former paper is most closely related to ours.[7] In that paper, a shock to the quantity of idiosyncratic income risk leads to a flight to liquidity that shares many similarities with the flight to safety in our paper. Our analysis is complementary to their quantitative work in that we provide tight theoretical results in a much simpler framework. More importantly, the focus of our work is on the aggregate implications of uncertainty shocks over and above their distributional implications. To isolate this aggregate

[4] In addition to the New Keynesian models, the literature on uncertainty shocks has investigated the "wait-and-see" effect (Bloom, 2009; Bloom et al., 2018) and various types of interactions between uncertainty and financial frictions (Gilchrist et al., 2014; Di Tella, 2017; Bianchi et al., 2018; Arellano et al., 2019; Elenev et al., 2021; Alfaro et al., 2024).

[5] Born and Pfeifer (2014) and Fernández-Villaverde et al. (2015) study the impact of policy uncertainty in environments with government bonds. However, due to Ricardian equivalence and their assumptions on lump-sum taxes, these frameworks are equivalent to a setting with bonds in zero net supply.

[6] Examples include Oh and Reis (2012), McKay, Nakamura and Steinsson (2016), Challe, Matheron, Ragot and Rubio-Ramirez (2017), Guerrieri and Lorenzoni (2017), Kaplan, Moll and Violante (2018), Bayer, Lütticke, Pham-Dao and Tjaden (2019), Auclert, Rognlie and Straub (2020), and Schaab (2020).

[7] In Schaab (2020) government bonds are real, and lump-sum taxes serve as an implicit safe asset similar to Section 6.2.

channel, our framework abstracts from heterogeneity in marginal propensities to consume (MPCs), so that the wealth distribution becomes irrelevant for macro aggregates. This difference in emphasis sets our paper apart from most of the HANK literature, including Bayer et al. (2019), which has primarily studied the interaction between the macroeconomy and the cross-sectional distribution of wealth and income.

Our paper also connects to a theoretical strand of the HANK literature that has, like us, sought to provide tight analytical characterizations in more stylized HANK models. Most of this literature has abstracted from asset holdings altogether by considering environments with vanishing liquidity (e.g., Werning, 2015; Ravn and Sterk, 2020; Bilbiie, 2020, 2024). An exception is Acharya and Dogra (2020), which also abstracts from MPC heterogeneity. But their model does not feature portfolio choice and flight to safety dynamics. We make a technical contribution to this literature by providing another tractable framework with idiosyncratic risk in returns instead of labor income.

Our paper is also related to the literature on safe assets. We refer the reader to Brunnermeier et al. (2024) for a review of that literature, which is the most closely related paper. Our paper extends their flexible price framework and studies interactions between flight to safety and demand recessions as well as optimal monetary policy. Moreover, as we discuss in Section 3.5, sticky prices generate overshooting in capital values, which corrects a major shortcoming in the asset pricing predictions of their model. Other closely related papers study safe asset shortage in New Keynesian models with a ZLB constraint. Caballero and Farhi (2018) distinguish safety and liquidity traps and emphasize the importance of the former, which arises when an excess demand for safe assets creates a shortfall in aggregate demand. In their model, a safety trap can only arise at the ZLB because changes in the risk-free interest rate directly affect demand for safe assets in zero net supply. In contrast, our analysis suggests that if safe assets are in positive net supply, safe asset demand does not depend directly on the risk-free rate, and safety traps are not limited to ZLB episodes.[8] In Acharya and Dogra (2022), safe government bonds are also in positive supply but that paper is concerned

[8]There are also several other differences regarding the details of the equilibrating mechanism in our paper and theirs. Most importantly, their paper emphasizes the evolution of the wealth distribution across risk-tolerant and risk-averse agents, whereas we isolate aggregate effects in a framework in which the wealth distribution is irrelevant.

with a different question, the normative implications of using safe asset creation to lift the economy out of a stagnant steady state.

In our model, demand for nominal government debt serves as a nominal anchor, which connects our paper to the literature on the Fiscal Theory of the Price Level (FTPL)[9]and related work in which Ricardian non-equivalence leads to a well-defined demand for nominal government bonds.[10] In line with this latter strand of the literature, Ricardian non-equivalence in our model provides a nominal anchor even under the conventional assumption that fiscal policy remains concerned with long-term debt sustainability. What distinguishes this paper is our emphasis on flight to safety dynamics, which affect the demand for nominal bonds relative to other assets.

Our welfare analysis in Section 6.1 connects to the burgeoning literature on optimal monetary policy in incomplete market models (e.g., Bhandari et al., 2021; Acharya, Challe and Dogra, 2023; Dávila and Schaab, 2023; McKay and Wolf, 2023; Yang, 2024), but differs with regard to the precise policy trade-off. The focus in the existing literature has been on environments in which achieving the traditional objective of joint output gap and price stabilization remains feasible but may be suboptimal due to distributional considerations. Here, in contrast, the traditional stabilization goals remain most desirable but may not be achievable if fiscal instruments are limited. The policy problem is how to optimally smooth unavoidable distortions from price-setting frictions over time.

2 Model

In this section, we outline our baseline model. The model is based on the safe asset model of Brunnermeier, Merkel and Sannikov (2024) in which households solve a portfolio choice problem between idiosyncratically risky capital and safe nominal gov-

[9]This literature is too vast to review here. Early contributions are Leeper (1991), Sims (1994), and Woodford (1995), a recent textbook by Cochrane (2023) summarizes the state of knowledge. This literature has also pointed out the fiscal requirements for monetary stabilization, including in situations in which government debt does not provide a nominal anchor (e.g., Woodford, 2001; Cochrane, 2017; Caramp and Silva, 2021).

[10]For example, see Bénassy (2000, 2005), Leith and von Thadden (2008), Hagedorn (2021), Angeletos, Lian and Wolf (2024). While these do not necessarily associate themselves with the FTPL, there are many similarities in the underlying mechanism, see Brunnermeier, Merkel and Sannikov (2020).

ernment bonds. The amount of idiosyncratic risk that agents face is time-varying. In times of high risk, households' demand for safe assets increases, which generates deflation pressures. We augment this model by introducing nominal rigidities in the form of price adjustment costs as in Rotemberg (1982). Nominal rigidities prevent the immediate revaluation of nominal safe assets so that deflationary pressures from flight to safety create a demand shortage.

2.1 Setup

Overview. We study a continuous-time, infinite-horizon economy. The economy is populated by a continuum of households that own (risky) physical capital and (safe) government bonds. There exists another continuum of intermediate goods firms that rent capital from households to produce differentiated intermediate goods and sell them to a competitive sector of final goods firms. Final goods firms combine intermediate goods to produce the final goods using a constant elasticity of substitution (CES) production technology. We use the final good as a numeraire.

The model features two key frictions: (1) incomplete markets limit idiosyncratic risk sharing by households, and (2) intermediate goods firms face nominal price adjustment frictions.

Households. There is a continuum of households indexed by $i \in [0,1]$. Each household i has preferences over streams of consumption $\{c_t^i\}_{t=0}^{\infty}$ and capital utilization $\{u_t^i\}_{t=0}^{\infty}$ that are described by the utility function

$$V_0^i = \mathbb{E}\left[\int_0^{\infty} e^{-\rho t}\left(\log c_t^i - \frac{(u_t^i)^{1+\varphi}}{1+\varphi}\right)\mathrm{d}t\right]. \tag{1}$$

The disutility derived from capital utilization u_t^i can be interpreted as disutility from entrepreneurial effort in production.

Each household manages physical capital that produces a flow of capital services $u_t^i k_t^i \mathrm{d}t$, which the household rents out to intermediate goods firms at rental price p_t^R (per unit of capital services). Here, k_t^i is the quantity of capital managed by household i at time t, and u_t^i is the utilization rate chosen by the household. In addition to providing

capital services, capital holdings may be subject to taxation by the government and they entitle households to profit distributions from intermediate goods firms to be specified below. Capital of household i evolves according to

$$\mathrm{d}k_t^i = \underbrace{\Big(\Phi(\iota_t^i) - \delta\Big)}_{:=g(\iota_t^i)} k_t^i \mathrm{d}t + \tilde{\sigma}_t k_t^i \mathrm{d}\tilde{Z}_t^i + \mathrm{d}\Delta_t^{K,i}, \tag{2}$$

where $\mathrm{d}\Delta_t^{K,i}$ represents agent i's market transactions in physical capital, $\iota_t^i k_t^i \mathrm{d}t$ are the agent's physical investment expenditures (in final output goods), $\Phi(\cdot)$ is an increasing and concave function that captures adjustment costs in capital accumulation, δ is the rate of capital depreciation, and $\tilde{Z}^i$ is an agent-specific Brownian motion that is i.i.d. across households i and represents idiosyncratic risk. $\tilde{\sigma}_t$ is an exogenous process that governs the magnitude of idiosyncratic risk faced by agents. We assume that $\tilde{\sigma}_t$ is driven by some (aggregate) Brownian motion Z that is independent of all the idiosyncratic $\tilde{Z}^i$ processes. To obtain simple closed-form expressions for capital prices, we choose the functional form $\Phi(\iota) = \frac{1}{\phi}\log\left(1+\phi\iota\right)$ with adjustment cost parameter $\phi \geq 0$ for the investment technology.

One of the two key frictions in the model is that households are unable to share idiosyncratic risk due to market incompleteness. While they are allowed to trade physical capital and government bonds, they cannot write financial contracts contingent on individual $\tilde{Z}^i$ histories. As a consequence, all households have to bear the idiosyncratic risk inherent in their physical capital holdings.

We formulate the decision problem of household i as a standard consumption-portfolio choice problem that does not make explicit reference to the capital trading process $\mathrm{d}\Delta_t^{K,i}$ as a choice variable and instead works with the shares of net worth invested in capital and bonds, respectively.[11] Let n_t^i be the net worth of household i and let θ_t^i be the share of net worth invested in bonds. Then household i's net worth evolves according to

$$\mathrm{d}n_t^i = n_t^i\left(\theta_t^i \mathrm{d}R_t^B + (1-\theta_t^i)\mathrm{d}R_t^{K,i}(u_t^i, \iota_t^i)\right) - c_t^i \mathrm{d}t, \tag{3}$$

where $\mathrm{d}R_t^B$ and $\mathrm{d}R_t^{K,i}(\cdot,\cdot)$ are the infinitesimal return processes for bonds and capital, respectively. We provide explicit expressions for these returns below in equations (11)

[11] The trading process $\mathrm{d}\Delta_t^{K,i}$ can be backed out ex-post from the solution to this portfolio problem.

and (12) after we have introduced the remaining model elements.

The problem of household i is to choose consumption $\{c_t\}_{t=0}^{\infty}$, capital utilization $\{u_t\}_{t=0}^{\infty}$, physical investment $\{\iota_t\}_{t=0}^{\infty}$, and the bond portfolio share $\{\theta_t\}_{t=0}^{\infty}$ in order to maximize utility (1) subject to the net worth evolution (3) and a solvency constraint $n_t^i \geq 0$ that rules out Ponzi schemes. When making decisions, the household takes initial net worth n_0^i and the return processes $\mathrm{d}R_t^B$, $\mathrm{d}R_t^{K,i}(\cdot,\cdot)$ as given.[12]

Final Goods Firms. There is a competitive sector of firms that produce the final output goods, which are used for household consumption and investment. The representative final goods firm operates a CES technology

$$Y_t = \left(\int_0^1 \left(y_t^j \right)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}j \right)^{\frac{\epsilon}{\epsilon-1}},$$

where y_t^j is the input quantity of intermediate good variety j and ϵ is the elasticity of substitution across varieties. CES cost minimization implies that final goods producers' demand functions take the standard form:

$$y_t^j = \left(\frac{\mathcal{P}_t^j}{\mathcal{P}_t} \right)^{-\epsilon} Y_t, \qquad \mathcal{P}_t = \left(\int_0^1 \left(\mathcal{P}_t^j \right)^{1-\epsilon} \mathrm{d}j \right)^{\frac{1}{1-\epsilon}}, \tag{4}$$

where $\mathcal{P}_t^j$ is the nominal price of intermediate good j and $\mathcal{P}_t$ is the nominal price index.

Intermediate Goods Firms. There is a continuum of monopolistically competitive intermediate goods firms indexed by $j \in [0,1]$. Each intermediate goods firm operates a linear production technology

$$y_t^j = a\hat{k}_t^j$$

for a differentiated good of variety j. Here, a denotes the level of productivity, and $\hat{k}_t^j$ are effective capital input services rented from households at (unit) rental price p_t^R.

Intermediate goods firms choose nominal output prices $\mathcal{P}_t^j$. The second key friction in the model is that price changes are subject to adjustment costs à la Rotemberg (1982).

[12] For the capital return process, the household takes the function $\mathrm{d}R_t^{K,i}(\cdot,\cdot)$ as given but internalizes how the choices of u_t^i and ι_t^i affect the ultimate return $\mathrm{d}R_t^{K,i}(u_t^i, \iota_t^i)$.

Specifically, the nominal price $\mathcal{P}_t^j$ of firm j evolves according to

$$\mathrm{d}\mathcal{P}_t^j = \pi_t^j \mathcal{P}_t^j \mathrm{d}t$$

and the firm incurs a flow adjustment cost $\frac{\kappa}{2}\left(\pi_t^j\right)^2 Y_t \mathrm{d}t$ when choosing the price inflation rate π_t^j, where $\psi > 0$ is a parameter and Y_t is aggregate output.

In total, the profit flow of firm j is given by

$$\omega_t^j \mathrm{d}t := \left[\frac{\mathcal{P}_t^j}{\mathcal{P}_t} a \hat{k}_t^j - p_t^R \hat{k}_t^j - \frac{\psi}{2}\left(\pi_t^j\right)^2 Y_t\right] \mathrm{d}t. \tag{5}$$

The firm chooses price inflation rates $\{\pi_t^j\}_{t=0}^{\infty}$ and capital inputs $\{\hat{k}_t^j\}_{t=0}^{\infty}$ to maximize the present value of flow profits

$$\mathbb{E}\left[\int_0^{\infty} \xi_t \omega_t^j \mathrm{d}t\right]$$

subject to equation (4), the demand for intermediate good variety j. Future profits are discounted by the stochastic discount factor $\xi_t = e^{-\rho t}\frac{1}{C_t}$ where $C_t = \int_0^1 c_t^i \mathrm{d}i$ is the aggregate household consumption.[13]

We assume that both the profit flow ω_t^j of intermediate goods firms and the price adjustment costs $\frac{\psi}{2}\left(\pi_t^j\right)^2 Y_t$ are rebated to households proportionally to their capital holdings.[14] The transfer received by households per unit of capital is

$$\omega_t = \frac{\int_0^1 \left[\omega_t^j + \frac{\psi}{2}\left(\pi_t^j\right)^2 Y_t\right] \mathrm{d}j}{K_t}, \tag{6}$$

where $K_t := \int_0^1 k_t^i \mathrm{d}i$ is the aggregate capital stock.

We also assume that the initial price distribution is degenerate, $\mathcal{P}_0^j = \mathcal{P}_0$ for all j and focus on a symmetric solution in which all firms choose identical price paths, such that $\mathcal{P}_t^j = \mathcal{P}_t$, $\pi_t^j = \pi_t$ for all j and all t.

[13] While different from any individual household's SDF, the use of this SDF maximizes the value of the profit share claims for any agent. This is the case because we distribute profits proportional to capital holdings which are idiosyncratically risky. See Brunnermeier, Merkel and Sannikov (2024) for an in-depth discussion of this weighted-average SDF.

[14] This is consistent with the redistribution strategy according to factor input shares as suggested by Kaplan, Moll and Violante (2018). Note that we include price adjustment costs in the transfers for simplicity. At the expense of additional notation, we could also make price adjustment cost a resource cost without significantly altering any of our results.

Government. There is a government that issues nominal government bonds and imposes proportional capital taxes on households. For simplicity and analytical clarity, our baseline assumption in this paper is that government bonds pay a floating nominal interest rate i_t and, therefore, have an infinitesimal duration. In Section 5, we relax this assumption and explain how our results are affected by introducing long-term bonds with arbitrary duration. We denote by $\mathcal{B}_t$ the total nominal quantity of outstanding government bonds and by τ_t the capital tax rate. The (nominal) flow budget constraint of the government is

$$\mathrm{d}\mathcal{B}_t = (i_t\mathcal{B}_t - \mathcal{P}_t\tau_t K_t)\,\mathrm{d}t, \tag{7}$$

that is, new bond issuance $\mathrm{d}\mathcal{B}_t$ must absorb the difference between total nominal interest expenditure $i_t\mathcal{B}_t\mathrm{d}t$ and nominal tax revenues $\mathcal{P}_t\tau_t K_t\mathrm{d}t$.

Note, in particular, that nominal government debt $\mathcal{B}_t$ evolves locally deterministically. That is, $\mathrm{d}\mathcal{B}_t$ only has a drift term ($\mathrm{d}t$-term) but no diffusion term ($\mathrm{d}Z_t$-term). This is the case because the cash flows from both interest payments and tax revenues do not directly load on the Brownian innovations $\mathrm{d}Z_t$.[15] We can therefore summarize the evolution of nominal bonds by the instantaneous growth rate $\mu_t^{\mathcal{B}} := \mathrm{d}\mathcal{B}_t/(\mathcal{B}_t\mathrm{d}t)$.

Henceforth, we denote by

$$\check{s}_t := \frac{\mathcal{P}_t\tau_t K_t}{\mathcal{B}_t} \tag{8}$$

the ratio of the primary surplus to outstanding government debt. With this notation, the budget constraint (7) simplifies to

$$i_t = \mu_t^{\mathcal{B}} + \check{s}_t. \tag{9}$$

In what follows, we assume that the government sets the nominal rate i_t and the surplus-debt ratio $\check{s}_t$ according to some policy rules

$$i_t = i\left(\mathbf{A}_t\right), \qquad \check{s}_t = \check{s}\left(\mathbf{A}_t\right) \tag{10}$$

and adjusts taxes τ_t and bond issuance $\mu_t^{\mathcal{B}}$ to satisfy the budget constraint (9) and the defining equation of $\check{s}_t$. Here, $\mathbf{A}_t$ is a vector of aggregate variables (e.g., $\pi_t, \tilde{\sigma}_t$) chosen

[15] For interest payments, this is dictated by the characteristics of a standard nominal debt contract. For tax revenues, a $\mathrm{d}Z_t$-loading would require a shock-contingent tax-transfer scheme that continuously alternates between positive and negative transfer payments. Such contingent tax-transfer schemes are highly counterfactual. Note that the interest and tax *rates* (i_t and τ_t) may be fully state-contingent.

by the government. In our language, we will loosely identify i_t with "monetary policy" and $\check{s}_t$ with "fiscal policy".

Asset Prices and Returns. Let q_t^K denote the market price of capital and let $q_t^B := \frac{\mathcal{B}_t}{\mathcal{P}_t}\frac{1}{K_t}$ be the real value of the total bond stock in the economy per unit of aggregate capital. We postulate that q_t^B and q_t^K follow a generic Ito evolution

$$\mathrm{d}q_t^B = \mu_t^{q,B} q_t^B \mathrm{d}t + \sigma_t^{q,B} q_t^B \mathrm{d}Z_t, \qquad \mathrm{d}q_t^K = \mu_t^{q,K} q_t^K \mathrm{d}t + \sigma_t^{q,K} q_t^K \mathrm{d}Z_t$$

with drifts $\mu_t^{q,B}$, $\mu_t^{q,K}$ and (aggregate) volatility loadings $\sigma_t^{q,B}$, $\sigma_t^{q,K}$ that are determined in equilibrium. The real return on bonds equals nominal interest net of inflation,

$$\begin{aligned}\mathrm{d}R_t^B &= i_t \mathrm{d}t + \frac{\mathrm{d}\left(1/\mathcal{P}_t\right)}{1/\mathcal{P}_t} \\ &= \left(i_t - \mu_t^{\mathcal{B}} + g_t + \mu_t^{q,B}\right)\mathrm{d}t + \sigma_t^{q,B}\mathrm{d}Z_t, \end{aligned} \tag{11}$$

where the second equality uses the identity $1/\mathcal{P}_t = q_t^B K_t/\mathcal{B}_t$ by definition of q_t^B.[16] Importantly, the return on bonds is not subject to idiosyncratic risk.

The return on capital for household i is

$$\begin{aligned}\mathrm{d}R_t^{K,i}(u_t^i, \iota_t^i) &= \overbrace{\frac{p_t^R u_t^i - \iota_t^i + \omega_t - \tau_t}{q_t^K}\mathrm{d}t}^{\text{dividend yield}} + \overbrace{\frac{\mathrm{d}\left(q_t^K\left(k_t^i - \Delta_t^{K,i}\right)\right)}{q_t^K\left(k_t^i - \Delta_t^{K,i}\right)}}^{\text{capital gain}} \\ &= \left(\frac{p_t^R u_t^i - \iota_t^i + \omega_t - \tau_t}{q_t^K} + g(\iota_t^i) + \mu_t^{q,K}\right)\mathrm{d}t + \tilde{\sigma}_t \mathrm{d}\tilde{Z}_t^i + \sigma_t^{q,K}\mathrm{d}Z_t. \end{aligned} \tag{12}$$

The dividends per unit of capital consist of four parts: revenues from renting out capital to intermediate goods firms, $p_t^R u_t^i$, physical reinvestment in the capital stock, $-\iota_t^i$, profit transfers from intermediate goods firms, ω_t, and taxes levied by the government, $-\tau_t$. The second term in the first line corresponds to the capital gains, which are the result of both changes in the capital price q_t^K and changes in the quantity of capital before market transactions ($k_t^i - \Delta_t^{K,i}$) due to investment, depreciation, and idiosyncratic shocks.

[16] We also use here $\mathrm{d}K_t = g_t K_t \mathrm{d}t$ with $g_t := \int_0^1 g(\iota_t^i) k_t^i \mathrm{d}i/K_t$, which follows readily from aggregating the individual capital evolutions (2).

Market Clearing. The market for final goods clears when total output equals the sum of consumption and investment:

$$\int_0^1 c_t^i \mathrm{d}i + \int_0^1 \iota_t^i k_t^i \mathrm{d}i = Y_t. \tag{13}$$

Bond market clearing requires that bonds held by households must equal bonds supplied by the government

$$\int_0^1 \theta_t^i n_t^i \mathrm{d}i = \frac{\mathcal{B}_t}{\mathcal{P}_t}. \tag{14}$$

Clearing of all remaining markets (for capital services, intermediate goods, and capital) is either implicit in our notation or follows from Walras' law.

Comparison with Related Models. Before moving on to the equilibrium definition, we offer a brief remark on the relation between our model and existing models in the literature. First, if we remove the price setting frictions, our model reduces to the safe asset model of Brunnermeier, Merkel and Sannikov (2024). Second, if we turn off idiosyncratic risk by setting $\tilde{\sigma}_t \equiv 0$ and capital investments by taking the limit $\phi \to \infty$, our model becomes isomorphic to a standard textbook representative agent New Keynesian model (e.g., Galí, 2008, Chapter 3).[17] In this sense, our framework represents the simplest possible combination of the two classes of models. Another notable special case is to keep idiosyncratic risk but remove safe assets by setting the bond supply to zero, $\mathcal{B}_t \equiv 0$. That version of our model is very similar to the common belief variant of the Caballero and Simsek (2020) model.[18] We discuss the case without safe assets in detail in Section 4.1.

2.2 Equilibrium

Equilibrium Definition. Natural state variables in our model are $\tilde{\sigma}_t$, $\mathcal{B}_t$, K_t, and $\mathcal{P}_t$. It is possible to reduce the state space dimension from four to two. First, because of the AK nature of the production technology, the capital stock K_t only matters for the scale of

[17] For this isomorphism, one simply re-interprets utilization u_t^i as labor supply and capital (which evolves exogenously for $\phi \to \infty$) as labor productivity.

[18] Caballero and Simsek (2020) have aggregate instead of idiosyncratic capital shocks. However, this difference is not central to their conclusions.

the economy. We can eliminate it by considering descaled quantities. Second, the states $\mathcal{B}_t$ and $\mathcal{P}_t$ only matter through their impact on the real value of government bonds, $\mathcal{B}_t/\mathcal{P}_t$. We can equivalently use the normalized value of the bond stock $q_t^B = \frac{\mathcal{B}_t}{\mathcal{P}_t K_t}$ as our second state variable. The dynamics of q_t^B follow by Ito's lemma from the dynamics of bond supply $\mathcal{B}_t$, the price level $\mathcal{P}_t$, and aggregate capital K_t:

$$\mathrm{d}q_t^B = \left(\mu_t^{\mathcal{B}} - \pi_t - g_t\right) q_t^B \mathrm{d}t. \tag{15}$$

Importantly, the value of nominal bonds is directly linked to the price level. In the presence of price stickiness, q_t^B is locally deterministic ($\sigma_t^{q,B} = 0$), i.e., it becomes a slow-moving state. In contrast, it represents an instantaneously adjusting equilibrium price under flexible nominal prices.

Definition 1 (Markov equilibrium)**.** *An equilibrium consists of laws of motion for the state variables* $\{\tilde{\sigma}_t, q_t^B\}$ *and of equilibrium mappings*

$$\left\{q^K(\tilde{\sigma}_t, q_t^B), \theta(\tilde{\sigma}_t, q_t^B), u(\tilde{\sigma}_t, q_t^B), \iota(\tilde{\sigma}_t, q_t^B), \pi(\tilde{\sigma}_t, q_t^B), i(\tilde{\sigma}_t, q_t^B), \check{s}(\tilde{\sigma}_t, q_t^B)\right\}$$

that satisfy (i) household and firm optimality, (ii) the policy rule (10),[19] *(iii) market clearing conditions (13) and (14), and (iv) the consistency requirement (15).*

Equilibrium Selection. In the context of a flexible price variant of our model, Brunnermeier, Merkel and Sannikov (2024) show that there is at most one Markov equilibrium that is "monetary" in the sense that $q_t^B > 0$ in all states. Intuitively, the demand for (nominal) safe government bonds in this economy provides a nominal anchor that overcomes price level indeterminacy.[20] A similar uniqueness result also holds in our sticky price economy (see Appendix B.2 for details). In the following, we always focus on this unique monetary Markov equilibrium.

[19] The government budget constraint (9) and the definition of $\check{s}_t$ can then be used to back out the tax rate τ_t and the bond growth rate μ_t^B.

[20] Once non-stationary equilibria are taken into account, the equilibrium may be only "locally unique." However, the off-equilibrium fiscal arrangements discussed in Brunnermeier, Merkel and Sannikov (2020) can also render it globally unique.

2.3 Model Solution

2.3.1 Solving the Household Problem

As detailed in Appendix A, our setup leads to a tight analytical characterization of the solution to the household problem.

Proposition 1 (Household Choices)**.** *The optimal choices c_t^i, ι_t^i, u_t^i, and θ_t^i of household i are characterized by*

$$c_t^i : \qquad c_t^i = \rho n_t^i \tag{16}$$

$$\iota_t^i : \qquad \iota_t^i = \frac{q_t^K - 1}{\phi} \tag{17}$$

$$u_t^i : \qquad \left(u_t^i\right)^{\varphi} = \frac{p_t^R k_t^i}{c_t^i} \tag{18}$$

$$\theta_t^i : \qquad \frac{\mathbb{E}_t\left[\mathrm{d}R_t^K - \mathrm{d}R_t^B\right]}{\mathrm{d}t} = (1-\theta_t^i)\left[(\sigma_t^{q,K})^2 + \tilde{\sigma}_t^2\right] \tag{19}$$

These conditions have intuitive economic interpretations. First, with logarithmic utility, households always consume a constant fraction ρ of their net worth. Second, the investment rate is positively associated with the price of capital, which is a standard Tobin's q relationship. Third, the optimal utilization rate equalizes the marginal disutility from utilization $(u_t^i)^{\varphi}$ with the marginal revenue product of utilization $(p_t^R k_t^i)$ weighted by marginal consumption utility $(1/c_t^i)$. In particular, there is an increasing relationship between capital utilization and the rental price of capital p_t^R. Fourth, capital earns a risk premium on both aggregate risk and idiosyncratic risk. This premium depends on the households' risk exposure $(1-\theta_t^i)$, the amount of aggregate risk $\sigma_t^{q,K}$, and the amount of idiosyncratic risk $\tilde{\sigma}_t$.

The first-order conditions (17) – (19) also imply that the household's decisions on investment ι_t, utilization u_t, and portfolio weight θ_t are independent of their own net worth or capital stock. Therefore, the households in this economy are *homogeneous up to scale*, as they all choose the same investment and utilization rates and hold the same portfolio regardless of their net worth.[21] While agents are heterogeneous with regard to their net worth n_t^i, the net worth distribution does not affect macro aggregates. It is

[21] Henceforth, we omit the i superscript since all decision variables are identical across households.

in this sense that our analysis abstracts from distributional aspects that have been the focus of much of the recent HANK literature.

2.3.2 Solving the Problem of Intermediate Goods Firms

Intermediate goods firms face a standard price-setting problem with price adjustment costs. Under our symmetry assumption, the optimal choices of firms can be summarized by a single equation for inflation dynamics:

Proposition 2 (Phillips Curve)**.** *The optimal firm choices $\pi_t^j = \pi_t$ for all intermediate goods firms j are characterized by*

$$\mathbb{E}_t\left[\mathrm{d}\left(\pi_t \frac{Y_t}{C_t}\right)\right] = \left[\rho\pi_t - \kappa\left(\frac{p_t^R - p^{R,flex}}{a}\right)\right]\frac{Y_t}{C_t}\mathrm{d}t, \tag{20}$$

where $p^{R,flex} := a\frac{\epsilon-1}{\epsilon}$ is the equilibrium capital rental price under flexible goods prices and $\kappa := \frac{\epsilon}{\psi}$ is the Phillips curve slope.

Equation (20) plays the role of the New Keynesian Phillips Curve in our model. It is a forward-looking equation[22] that encodes the standard price setting logic in New Keynesian models: firms' price setting is trading off a desire to minimize deviations of marginal unit costs p_t^R/a from their flexible price value $p^{R,flex}/a$ to achieve the statically optimal markup and a desire to smooth inflation in order to minimize price adjustment costs.

The New Keynesian Phillips Curve takes the familiar textbook form if we consider the special case without physical investment, i.e., the limit $\phi \to \infty$. In this case, $Y_t = C_t$, so that the scaling factor Y_t/C_t cancels. More generally, this scaling factor appears because firm revenues are proportional to output, but discounting in the firm SDF ξ_t is inversely proportional to consumption.

[22] By "forward-looking equation" we mean, intuitively, a condition that relates some variable, here inflation π_t, to the expected future path of other variables. Mathematically, this means the equation is a "backward stochastic differential equation": it has no initial condition but is solved backward in time from a terminal or transversality condition.

2.3.3 Equilibrium Dynamics of Relative Asset Valuations

Total wealth in the economy consists of bond wealth, $\mathcal{B}_t/\mathcal{P}_t = q_t^B K_t$, and capital wealth, $q_t^K K_t$. We denote by ϑ_t the fraction of total wealth that is due to the former component, i.e.,

$$\vartheta_t := \frac{\mathcal{B}_t/\mathcal{P}_t}{\mathcal{B}_t/\mathcal{P}_t + q_t^K K_t} = \frac{q_t^B}{q_t^B + q_t^K}. \tag{21}$$

This variable is a measure of relative asset valuations that will play a key role in the subsequent analysis. Due to homogeneity of portfolio choice, bond market clearing (14) implies $\vartheta_t = \theta_t$, that is ϑ_t also equals the common portfolio weight θ_t on bonds chosen by all households. We can characterize the equilibrium dynamics of ϑ_t by combining the household portfolio choice condition (19) with the asset return expressions (11) and (12), the government budget constraint (9), and market clearing conditions (details are provided in Appendix A):

Proposition 3 (Bond Valuation Equation)**.** *The law of motion of ϑ_t satisfies*

$$\mathbb{E}_t[\mathrm{d}\vartheta_t] = \left[\rho - (1-\vartheta_t)^2\,\tilde{\sigma}_t^2 - \check{s}_t\right]\vartheta_t \mathrm{d}t. \tag{22}$$

Like the Phillips curve, the bond valuation equation (22) is a forward-looking equation that determines the expected rate of change in ϑ_t. We defer the discussion of the factors that affect portfolio choice and relative asset valuations to Section 3.2.1.

2.3.4 Other Equilibrium Conditions

Using the optimal consumption condition (16) and aggregating over all households yields

$$\frac{c_t}{k_t} = \frac{C_t}{K_t} = \frac{\rho \int_0^1 n_t^i \mathrm{d}i}{K_t} = \rho(q_t^B + q_t^K), \tag{23}$$

where the final equality follows from the fact that total net worth in the economy consists of bond wealth, $q_t^B K_t$, and capital wealth, $q_t^K K_t$.

Using the aggregate consumption-capital ratio C_t/K_t from equation (23), the fact that all agents choose identical investment rates $\iota_t^i = \iota_t$, and that, in equilibrium, aggregate final goods production is $Y_t = a u_t K_t$, we can rewrite the goods market clearing

condition (13) as follows:

$$\rho(q_t^B + q_t^K) + \iota_t = au_t. \tag{24}$$

Substituting the individual consumption-capital ratio c_t/k_t from equation (23) into the optimal utilization choice condition (18) allows us to express the equilibrium capital rental price p_t^R in terms of asset prices and capital utilization:

$$p_t^R = \rho u_t^{\varphi}(q_t^B + q_t^K). \tag{25}$$

Together with the law of motion for the endogenous state variable q_t^B, equation (15), a (yet-to-be-specified) law of motion for the exogenous state variable $\tilde{\sigma}_t$, and policy rules for interest rate i_t and primary surplus $\check{s}_t$, the previous equations represent a complete characterization of equilibrium.

3 Flight to Safety and Sticky Prices

In this section, we analyze how uncertainty shocks are transmitted through our model economy and discuss several economic implications arising from that analysis. We start with a numerical illustration in Section 3.1 and defer an in-depth discussion of the mechanism to Section 3.2.

3.1 The Effects of Uncertainty Shocks: A Numerical Illustration

We illustrate the dynamics of our model economy by plotting impulse response functions (IRFs) to an uncertainty shock for a numerical example. Our choice of model parameters closely follows the prior literature. In particular, we follow the strategy in Brunnermeier et al. (2024) to parameterize the household side and the idiosyncratic volatility process and Kaplan et al. (2018) to parameterize the New Keynesian supply side. The exogenous process for the idiosyncratic volatility process is

$$\mathrm{d}\tilde{\sigma}_t^2 = b(\bar{\sigma}^2 - \tilde{\sigma}_t^2)\mathrm{d}t - \nu\tilde{\sigma}_t \mathrm{d}Z_t.$$

Additional details are provided in Appendix G. As we explain in Section 3.2, the qualitative dynamics discussed here are a general feature of our model and do not depend on the parameter choices.

Figure 1 depicts IRFs following a positive uncertainty shock, both under flexible (red) and sticky prices (blue). Specifically, we consider an economy that is initially (before $t = 0$) in its stochastic steady state where $\tilde{\sigma}_0^2 = \bar{\sigma}^2$ and, at $t = 0$, is hit by a single positive uncertainty shock.[23] The figure plots the transition paths back to the stochastic steady state when no further shocks are realized thereafter. To clearly isolate the effects of the shock itself, as opposed to the effects of any government policy reaction to it, we assume that the policy variables i_t and $\check{s}_t$ are kept constant throughout. The following subsection will clarify how the transmission mechanism is affected by policy reactions.

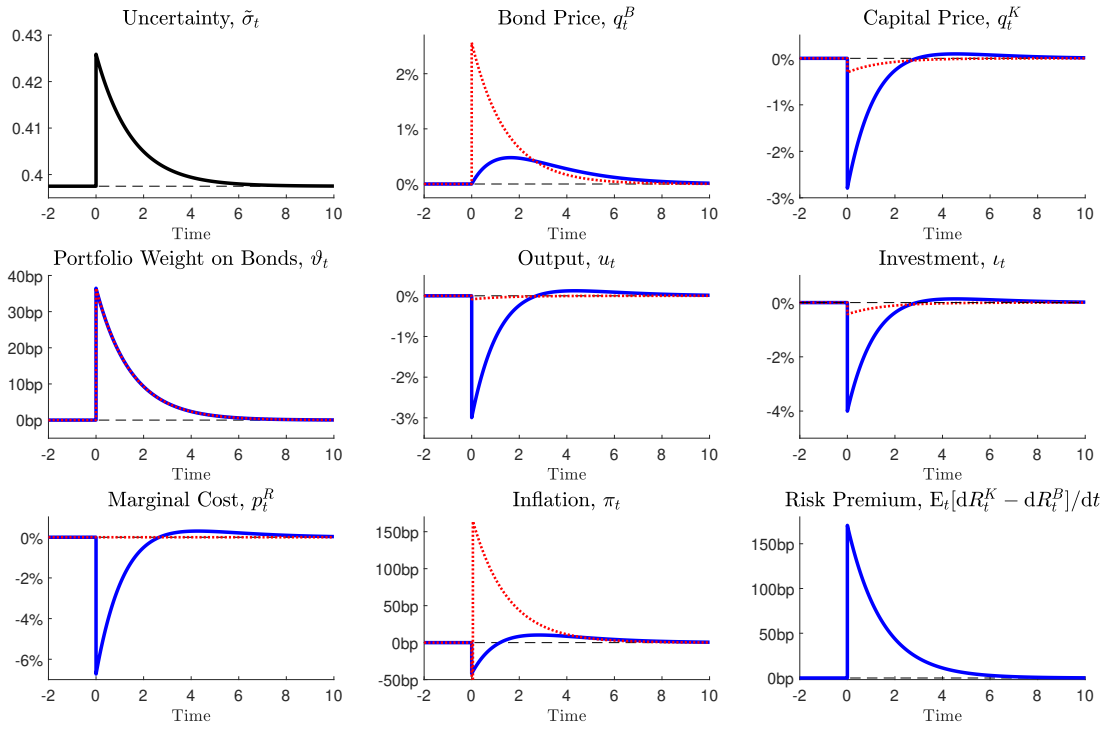


Figure 1: Impulse response functions to an uncertainty shock at $t = 0$ under sticky (blue solid lines) and flexible (red dotted lines) prices. Before $t < 0$, the economy is in its stochastic steady state without realized shocks and, for $t \geq 0$, the exogenous path for $\tilde{\sigma}_t$ depicted at the top left panel is fed into the model. Variables q_t^B, q_t^K, u_t, ι_t, p_t^R are in percentage deviations and variables ϑ_t, π_t, and $\mathrm{E}_t[\mathrm{d}R_t^K - \mathrm{d}R_t^B]/\mathrm{d}t$ in absolute deviations from the stochastic steady state.

A rise in present and future uncertainty triggers flight to safety, a reallocation in household portfolios from capital to bonds. The resulting increases in the portfolio

[23] We calibrate the size of the shock to generate a 3% recession in output.

weight ϑ_t and the required risk premium on capital are identical under flexible and sticky prices.

Under flexible prices, the portfolio reallocation results in an upward jump in bond valuations q_t^B and a downward jump in capital valuations q_t^K. The latter leads to a drop in both investment and utilization.[24] The drop in bond valuations is brought about by a downward jump in the price level $\mathcal{P}_t$ (shown as $\pi_t = -\infty$ on impact) that is followed by gradual inflation thereafter as the economy returns back to the steady state.

When prices are sticky, the value of nominal bonds can no longer adjust to clear asset markets. Instead, the heightened demand for bonds leads to overshooting in the capital price adjustment relative to the flexible price case. q_t^K drops sharply when the shock hits and then slowly converges back to the steady state as uncertainty subsides. In contrast, bond valuations q_t^B react only gradually over time in a hump-shaped fashion.

The sharp drop in the price of capital depresses aggregate demand under sticky prices. Both investment and aggregate output fall. The recession in demand reduces the marginal cost of intermediate goods producers. Since the current price level is too high, the firms optimally lower their prices, resulting in a period of disinflation.

In summary, our sticky price model implies that uncertainty leads to portfolio rebalancing towards safe bonds, disinflation, and a recession in consumption, investment, and output.

3.2 Transmission Mechanism

3.2.1 Understanding Household Portfolio Decisions

The portfolio choice between risky capital and safe nominal bonds is key for the dynamics just described. To understand how household portfolios respond to fluctuations in economic uncertainty, we analyze the bond valuation equation (22) for ϑ_t.

To gain intuition, it is helpful to write this equation in integral form,

$$\vartheta_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho(t'-t)} \vartheta_{t'} \left((1-\vartheta_{t'})^2 \tilde{\sigma}_{t'}^2 + \check{s}_{t'} \right) \mathrm{d}t' \right]. \tag{26}$$

[24] Utilization decreases (slightly) even under flexible prices because lower investment raises the consumption-capital ratio. For a given rental price, equation (18) requires a reduction in u_t.

Households' portfolio choice depends on two processes: the path of future uncertainty $\{\tilde{\sigma}_{t'}\}_{t' \geq t}$ and the path of future surplus-debt ratios $\{\check{s}_{t'}\}_{t' \geq t}$. Households demand more government bonds when they are valuable as insurance (higher $\tilde{\sigma}_t$) or when they are subsidized by government policy (higher $\check{s}_t$).[25]

Importantly, household portfolio choice does *not* directly depend on output or capital utilization (u_t), inflation (π_t), the nominal interest rate (i_t), and any parameters governing the price-setting behavior of firms (ϵ, ψ). These variables and parameters only affect household portfolio choice between capital and bonds if (fiscal) policy adjusts $\check{s}$ in response to them. This leads to the following separation result.

Proposition 4 (Separation of Portfolio Choice)**.** *Suppose that the fiscal policy rule chooses $\check{s}$ as a function of $(\tilde{\sigma}_t, \vartheta_t)$ only. Then, under technical conditions stated in Appendix B.2,*

$$\vartheta_t = \vartheta(\tilde{\sigma}_t)$$

is a function of the exogenous uncertainty state $\tilde{\sigma}_t$ only, and this function does not depend on the intermediate goods demand elasticity ϵ and the price adjustment cost ψ.

In particular, portfolio choice is unaffected by inflation π_t, capital utilization u_t, nominal interest rates i_t, and the degree of price stickiness.

This proposition has two important corollaries. First, monetary policy is *portfolio-neutral*: the nominal interest rate path i_t set by the monetary authority does not distort the portfolio choice between bonds and capital unless it induces changes in the fiscal policy instrument $\check{s}_t$.[26] Second, regardless of how sluggish nominal prices adjust, portfolio choice between capital and bonds is always "fast" in the sense that portfolios adjust precisely the same as under flexible prices. This second corollary plays an important role in the overshooting of capital values discussed below.

Having established that portfolio choice is driven by fluctuations in uncertainty $\tilde{\sigma}_t$ and the fiscal policy reaction to it, an immediate question is how uncertainty affects

[25] A higher surplus-debt ratio effectively subsidizes bonds relative to capital holdings because it implies that the government taxes capital at a larger rate to make payouts to bondholders.

[26] Note that even if $\check{s}_t$ is unresponsive to changes in the nominal interest rate, a positive $\check{s}_t$ still implies that primary surpluses rise in response to the larger debt issuance triggered by an increase in nominal rates. However, it is not surpluses but the surplus-debt ratio $\check{s}_t$ that matters for the return on bonds because this is the "dividend yield" paid by the government.

portfolios. Equation (26) suggests that an increase in $\tilde{\sigma}_t$ leads to an increase in the bond wealth share ϑ_t, which we interpret as *flight to safety*. However, equation (26) is not a closed-form solution because future portfolio decisions $\vartheta_{t'}$ appear on the right-hand side. The next proposition establishes that we nevertheless reach correct qualitative conclusions if we read the equation as if $\vartheta_{t'}$ was exogenous:

Proposition 5 (Flight to Safety)**.** *Suppose the conditions of Proposition 4 are satisfied, bonds have a positive value in equilibrium, and for any given $\vartheta \in (0,1)$, the function $\tilde{\sigma} \mapsto (1-\vartheta)^2\tilde{\sigma}^2 + \check{s}(\tilde{\sigma},\vartheta)$ is strictly increasing. Then $\vartheta'(\tilde{\sigma}) > 0$.*

In other words, unless $\check{s}$-policy is sufficiently accommodative, an increase in $\tilde{\sigma}$ leads to flight to safety.

In the following, we assume that the assumptions of Proposition 4 and 5 are satisfied, so that the bond wealth share $\vartheta_t = \vartheta(\tilde{\sigma}_t)$ is essentially exogenous for the behavior of macro aggregates and an increase in uncertainty leads to flight to safety. We make two remarks in defense of this choice of presentation. First, the requirements on $\check{s}_t$ that lead to separation of portfolio choice are not overly restrictive relative to common specifications of fiscal policy rules in the literature. Specifically, they permit policy rules that incorporate a concern for debt sustainability in the form of feedback from debt to taxes. For example, a constant and positive $\check{s}$ corresponds to a rule that adjusts primary surpluses proportionally to the value of outstanding liabilities. Second, even if the conditions for separation are not satisfied, the following discussion of the dynamics of macro aggregates and asset prices is still informative. It then needs to be interpreted as a characterization of those dynamics conditional on a given equilibrium path for ϑ_t.

3.2.2 The Transmission of Uncertainty Shocks

To understand how uncertainty affects output, we can rewrite the market clearing condition (24) as

$$u_t = \frac{1}{a}\Bigg[\underbrace{\rho(q_t^K + q_t^B)}_{\substack{\text{consumption}\\ \text{demand}}} + \underbrace{\iota_t}_{\substack{\text{investment}\\ \text{demand}}}\Bigg]. \tag{27}$$

The demand for final goods can be decomposed as demand for consumption and investment. The consumption demand is tied to the households' net worth through

wealth effects. Net worth depends on the value of both bonds and capital. The investment demand ι_t depends on q_t^K through the Tobin's q relationship (17). To determine the short-run output u_t, it is thus important to understand the behavior of the *levels of asset valuations*, q_t^K and q_t^B.

The solution of the household portfolio problem (equation (26)) determines *relative asset valuations*. Households invest more in nominal bonds as future uncertainty increases. Formally, the nominal wealth share ϑ_t rises. Recall that $\vartheta_t = (1 + q_t^K/q_t^B)^{-1}$. An increase in ϑ_t mechanically implies a drop in the price of capital relative to the value of bonds. However, the absolute movements in q_t^K and q_t^B hinge on the presence of price stickiness.

Equilibrium under Flexible Prices. When prices are flexible, intermediate goods firms' problem collapses to a static profit-maximization problem at each time, and firms engage in constant markup pricing. Price setting determines the capital rental price, $p_t^R = p^{R,flex}$, while the nominal price level $\mathcal{P}_t$ remains free to adjust in order to clear the goods market, i.e. make equation (27) hold. In that equation, $\mathcal{P}_t$ enters through the scaled bond value $q_t^B = \mathcal{B}_t/(\mathcal{P}_t K_t)$, which it is inversely proportional to.

We can solve for the flexible price asset values $q_t^{B,flex}$ and $q_t^{K,flex}$ as a function of ϑ_t by combining equation (25) for the rental price with $p_t^R = p^{R,flex}$ and the goods market clearing condition (27). In some special cases, the resulting price expressions are available in closed form, e.g., for $\varphi = 0$:

$$q_t^{B,flex} = \vartheta_t \frac{\epsilon - 1}{\epsilon} \frac{a}{\rho}, \qquad q_t^{K,flex} = (1 - \vartheta_t) \frac{\epsilon - 1}{\epsilon} \frac{a}{\rho}.$$

From these formulas, we observe the following qualitative conclusions, which also apply more generally for all $\varphi \geq 0$:

Proposition 6 (Flexible Price Equilibrium)**.** *Under flexible prices, an increase in $\tilde{\sigma}_t$ leads to an increase in the government bond value q_t^B and a drop in both the capital price q_t^K and the nominal price level $\mathcal{P}_t$.*

Economically, an increase in ϑ_t induced by heightened uncertainty means that agents desire more bond savings and fewer capital savings. As a result, both the capital price

q_t^K and the physical investment rate ι_t fall, implying lower consumption demand due to capital wealth and investment demand on the right-hand side of equation (27). For fixed utilization u_t and bond values q_t^B, aggregate demand thus falls short of supply.

However, the individual desire to save additional wealth in bonds does not translate into aggregate saving because bonds cannot be used to transfer real resources over time.[27] As households collectively demand more nominal bonds and fewer goods, the price level falls, and the real value of bonds q_t^B appreciates. A wealth effect from increased bond wealth generates additional consumption demand that eliminates the shortfall in aggregate demand without any reduction in supply.[28]

Equilibrium under Sticky Prices: Impact Effect. When the price level $\mathcal{P}_t$ is sticky, it can no longer drop immediately in response to an uncertainty shock. There is no (immediate) deflationary revaluation of nominal wealth that raises consumption demand as under flexible prices. Instead, supply must adjust to the shortfall in demand to clear the goods market.

In fact, in our continuous-time model, the state variable q_t^B for bond wealth is perfectly rigid over short time periods and thus cannot adjust at all on shock impact.[29] As a consequence, when higher uncertainty raises ϑ_t, both total wealth $q_t^K + q_t^B = q_t^B/\vartheta_t$ and capital valuations $q_t^K = (1-\vartheta_t)/\vartheta_t \cdot q_t^B$ fall. The former implies a reduction in consumption demand, the latter a reduction in investment demand through the Tobin's q relationship (17). Importantly, both components of aggregate demand in equation (27) are fully predetermined by the (separated) portfolio choice ϑ_t and the (rigid) state variable q_t^B. The only free variable to adjust on impact is u_t. A recession is inevitable.

Proposition 7 (Demand Recessions under Sticky Prices)**.** *Under sticky prices, an increase in $\tilde{\sigma}_t$ leads to an aggregate demand recession on impact. Formally, utilization $u_t = u(\tilde{\sigma}_t, q_t^B)$ is decreasing in $\tilde{\sigma}_t$ for any fixed q_t^B. For $q_t^B \approx q_t^{B,flex}$, the utilization gap $u_t/u_t^{flex} = u(\tilde{\sigma}_t, q_t^B)/u^{flex}(\tilde{\sigma}_t)$*

[27] In contrast, capital can serve as a device for aggregate saving to the extent that demand for capital leads to additional real investment into the capital stock.

[28] To be fully precise, supply and utilization only remain unaffected when the capital supply is perfectly inelastic ($\phi \to \infty$). Otherwise, supply still falls due to a reduced willingness to put utilization effort when the consumption-capital ratio rises, compare footnote 24. This is a supply-side effect that has nothing to do with a shortfall in aggregate demand.

[29] Formally, this is apparent in the lack of a dZ_t-term in equation (15).

is also decreasing in $\tilde{\sigma}_t$.[30]

We remark that the previous conclusion does not necessarily hold in models without safe assets. We discuss this in more detail in Section 4.1.

Furthermore, the capital price and investment demand always fall by more than under flexible prices. The reason is that when q_t^B is unable to adjust, a higher portfolio weight ϑ_t can only be achieved through a more substantial drop in the capital price q_t^K. That is, since the value of government bonds cannot adjust quickly to match fluctuations in the demand for bonds, uncertainty shocks depress the demand for capital even further.

The cleanest way to see this formally is to consider the effect of an unanticipated upward jump in uncertainty $\tilde{\sigma}_t$ at time t. Using $q_t^K = (1/\vartheta_t - 1)q_t^B$, we can write the jump in the capital price in response to the shock as

$$\frac{q_t^K}{q_{t-}^K} = \underbrace{\frac{1/\vartheta_{t-} - 1}{1/\vartheta_t - 1}}_{\substack{\text{portfolio} \\ \text{adjustment}}} \underbrace{\frac{q_t^B}{q_{t-}^B}}_{\substack{\text{bond value} \\ \text{adjustment}}},$$

where a subscript "$t-$" denotes the value of a variable just before the jump. In a flight to safety, the portfolio adjustment term is smaller than 1, generating downward pressure on the capital price. Under flexible prices, this is partially alleviated by an upward adjustment in bond values, $q_t^B/q_{t-}^B > 1$ (Proposition 6). Under sticky prices, the value of bonds remains stuck at the pre-shock level, $q_t^B/q_{t-}^B = 1$. Thus, the price of capital *overshoots* relative to the flexible price response following uncertainty shocks when nominal prices are sticky.

The same logic applies to our model, which features Brownian shocks instead of jump shocks. But to make it precise, we have to reformulate it in terms of the shock sensitivity $\sigma_t^{q,K}$ of q_t^K:

Proposition 8 (Capital Price Overshooting)**.** *The capital price* q_t^K *is more sensitive to uncertainty shocks* $\mathrm{d}Z_t$ *under sticky prices than under flexible prices. Formally, for all* t *and after all*

[30]Here, $u_t^{flex} = u^{flex}(\tilde{\sigma}_t)$ is the equilibrium utilization rate in the flexible price economy (which depends only on the state variable $\tilde{\sigma}_t$). The utilization gap can be interpreted as a form of output gap. In our nonlinear model, it cannot be excluded that the denominator falls faster than the numerator for q_t^B far away from $q_t^{B,flex}$, so that the restriction $q_t^B \approx q_t^{B,flex}$ cannot be dispensed with.

shock histories, $\operatorname{sgn}\sigma_t^{q,K} = \operatorname{sgn}\sigma_t^{q,K,flex}$ *and* $|\sigma_t^{q,K}| > |\sigma_t^{q,K,flex}|$.

This result is reminiscent of the classic overshooting model by Dornbusch (1976). There, stickiness in domestic prices results in overshooting in exchange rates. Here, the effective stickiness in bond valuations leads to overshooting in the capital price.

Equilibrium under Sticky Prices: Adjustment Dynamics. We have established so far that an uncertainty shock and the ensuing flight to safety create a shortfall in demand on impact. This demand shortfall generates disinflation pressures that are gradually realized after the shock: the price level $\mathcal{P}_t$ decreases over time to bring it closer to its natural (flexible price) level, leading to a period of disinflation. This adjustment follows the classic logic of the Pigou effect (e.g., Pigou, 1943; Patinkin, 1956). Because the demand shortfall is driven by the value of nominal safe assets q_t^B being too low and disinflation raises q_t^B, disinflation pressures ease over time as the safe asset value expands.

Formally, adjustment dynamics are guided by the Phillips curve (20) and by the state equation (15), restated here for the reader's convenience:

$$\mathbb{E}_t\left[\mathrm{d}\left(\pi_t \tfrac{Y}{C}(q_t^B,\vartheta_t)\right)\right] = \left[\rho\pi_t - \kappa\left(\frac{p^R(q_t^B,\vartheta_t) - p^{R,flex}}{a}\right)\right]\tfrac{Y}{C}(q_t^B,\vartheta_t)\mathrm{d}t, \tag{28}$$

$$\mathrm{d}q_t^B = \left[i_t - \pi_t - \check{s}_t - g(q_t^B,\vartheta_t)\right] q_t^B \mathrm{d}t, \tag{29}$$

where we have made explicit that p_t^R, Y_t/C_t, and g_t are functions of q_t^B and ϑ_t in equilibrium. These functions are all increasing in their first argument and decreasing in their second and in the case of p_t^R strictly so.

After an increase in uncertainty, a lack of demand for capital inputs drives the capital rental price p_t^R below its flexible price value $p^{R,flex}$. According to equation (28), inflation must gradually increase over time. As this is a (forward-looking) backward equation, the gradual increase in inflation is achieved by an immediate drop of π_t, potentially into deflationary territory.

Low and potentially negative π_t increases the rate of appreciation of the state variable q_t^B in equation (29), so that q_t^B gradually approaches its natural level $q_t^{B,flex}$ over

time. As q_t^B becomes closer to $q_t^{B,flex}$, the marginal cost gap $p_t^R - p^{R,flex}$ in the Phillips curve (28) starts to shrink, thereby dampening further adjustments in inflation.[31]

3.3 The Roles of Intertemporal Substitution and Portfolio Choice for Aggregate Demand

Our presentation in the previous subsection clarifies that the portfolio demand for nominal safe assets is the key force behind the aggregate demand effects of uncertainty shocks. Uncertainty shocks lead to a portfolio reallocation from capital assets to safe assets because they raise the risk premium that agents require to hold capital. In contrast, any economic force that affects the level of interest rates on all assets symmetrically but leaves required risk premia unaffected has no effect on the equilibrium portfolio weight ϑ_t. As we have explained, such a force is irrelevant to the impact effect of an uncertainty shock on aggregate demand.

This description contrasts with the emphasis on intertemporal substitution as the main driver of aggregate demand in the traditional textbook treatment of New Keynesian models. The key equation in that traditional analysis is the IS equation for consumption demand, a combination of the consumption Euler equation and the Fisher equation. In our setting with a growth trend in capital, it is most transparent to express the IS equation in terms of the consumption-capital ratio, C/K:

$$\frac{\mathbb{E}_t[\mathrm{d}(C_t/K_t)]}{C_t/K_t} = \left[i_t - \pi_t - \left(\rho + g_t - (\sigma_t^\vartheta)^2 - (1-\vartheta_t)^2\tilde{\sigma}_t^2\right)\right]\mathrm{d}t. \tag{30}$$

According to this equation, the expected growth rate in the consumption-capital ratio is the difference between the actual real interest rate $i_t - \pi_t$ and a "neutral" rate (here, $\rho + g_t - (\sigma_t^\vartheta)^2 - (1-\vartheta_t)^2\tilde{\sigma}_t^2$), at which the expected growth rate in the consumption-capital ratio desired by households is zero. This is a forward-looking equation that encodes intertemporal substitution considerations.

Because of its forward-looking nature, equation (30) is typically interpreted as follows: an expectation C_T/K_T far in the future is fixed ("anchored") as a terminal condition and therefore any positive deviation of actual interest rates from neutral rates

[31] Whether q_t^B asymptotically reaches $q_t^{B,flex}$ depends on the long-run behavior of nominal rates as the Phillips curve is only consistent with a zero long-run output gap if inflation settles down at zero.

between $t = 0$ and $t = T$ must depress consumption demand at date $t = 0$. This reasoning suggests that it is the level of interest rates, not the required risk premium between capital and bonds, that should matter for aggregate demand. However, while equation (30) of course holds in any equilibrium, this reasoning can be misleading: both the "terminal condition" at any finite date $t = T$ and the path of future inflation rates are not truly fixed but endogenously adjust to economic conditions.

In our model, the IS equation plays no particular role which is why we have chosen to keep it in the background in the previous discussion. In fact, once portfolio choice has determined the path for ϑ_t, the IS equation (30) is more appropriately interpreted as a backward-looking equation that tells us the (expected) future evolution of C_t/K_t given interest rates and an initial condition C_0/K_0 that is determined by the impact effect of the shock. Indeed, in equilibrium $C_t/K_t = \rho \frac{q_t^B}{\vartheta_t}$ is the ratio of a backward-looking object, the state variable q_t^B, and a forward-looking object, the portfolio weight ϑ_t. Once we have determined the latter by the portfolio choice considerations discussed in Section 3.2.1, the evolution of C_t/K_t follows "mechanically" by combining ϑ_t with the backward-looking law of motion for the state variable, equation (29). Only through the latter does the level of asset returns enter the equation.

3.4 Interest Rate Policy Ineffectiveness

A corollary from the previous discussion concerns the effectiveness of monetary policy. If forces that affect the level of interest rates leave the impact effect of uncertainty shocks unchanged, interest rate policy is unable to undo the effects of price stickiness on aggregate demand, unlike in standard models without safe assets.

Indeed, the nominal interest rate i_t enters only in the state equation (29). This means that interest rate policy only acts on the state evolution but not on forward-looking choice conditions, so it can affect the recovery dynamics after the shock but cannot prevent the initial reduction in aggregate demand on shock impact. Specifically, by raising the nominal rate, the government expands nominal bonds at a faster rate $\mu_t^{\mathcal{B}}$ (for given $\check{s}_t$), so that the price level has to adjust by less in order to expand the real quantity of safe assets q_t^B.

Proposition 9 (Interest Rate Ineffectiveness)**.** *Suppose the assumptions of Propositions 4 and 5 hold so that uncertainty shocks lead to flight to safety. Then, interest rate policy cannot eliminate output gap fluctuations. Formally, there is no policy rule $i(\mathbf{A}_t)$ such that, in equilibrium, the utilization gap u_t/u_t^{flex} is constant. In particular, no interest rate policy can implement the flexible price allocation.*

The significance of the previous result lies primarily in its stark contrast to the conventional wisdom derived from New Keynesian models that do not feature safe assets. In the conventional analysis, it is never in question that interest rate policy, unless constrained, has the power to stabilize economic activity. We discuss this issue further in Section 4.1, where we compare our model to a variant without safe assets.

Our argument leading to Proposition 9 has used the fact that interest rate policy cannot affect the safe asset value q_t^B on the impact of a shock. A reader may be worried that this argument relies on our assumption of floating-rate government bonds so that the nominal price of bonds, which is not sticky per se, must always be one. However, as we will show in Section 5, where we discuss a model extension with long-duration government bonds, the essence of Proposition 9 carries over to that case as well.

3.5 Capital Price Overshooting and Flight-to-Safety Volatility

Our model also has asset pricing implications. As emphasized by Brunnermeier et al. (2024), flight to safety from risky capital to safe bonds generates considerable variation in relative valuations between capital and bonds compared to a standard framework with bonds in zero net supply. They call this "flight-to-safety volatility" and argue that it can explain a substantial part of the large empirically observed equity price volatility. However, the flexible price model of Brunnermeier et al. (2024) has a major defect: it counterfactually implies that flight-to-safety volatility leads symmetrically also to large (countercyclical) fluctuations in safe asset valuations.

The overshooting mechanism in our sticky price model corrects this feature. By separated portfolio choice (Proposition 4), relative price volatility between capital and bonds is as large under sticky as under flexible prices. But importantly, sticky prices break the symmetry between capital and bond valuations. By making bond valuations

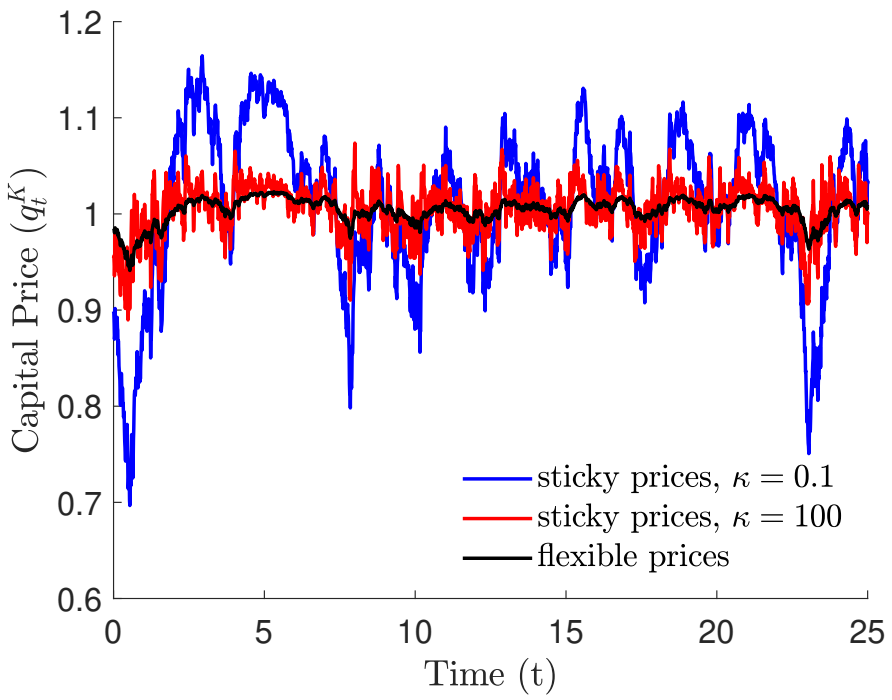


Figure 2: Simulated capital price paths under sticky and flexible prices. All lines depict simulated equilibrium paths for q_t^K based on the same sequence of exogenous shocks. The blue and red lines are from the sticky-price economy with Phillips curve slopes $\kappa = 0.1$ and $\kappa = 100$, respectively, and the black solid line is from the flexible price economy. Values on the vertical axis are relative to the stochastic steady-state value for q^K under flexible prices.

q^B a slow-moving state, sticky prices move all relative price volatility between capital and bonds into capital price fluctuations, more in line with empirical observations.

We emphasize that this point holds for any, arbitrarily small, degree of price stickiness. Figure 2 illustrates this by plotting simulated equilibrium paths for the capital price q_t^K from our model. The black line depicts dynamics under flexible prices and the blue and red lines show simulations for the same exogenous shocks under different degrees of price stickiness. While the blue line uses our baseline parameters, the red line has been generated with a very steep Phillips curve, $\kappa = 100$. The striking takeaway is that both sticky price versions generate substantially elevated capital price volatility relative to the flexible price equilibrium. As the proof of Proposition 8 reveals, this is no coincidence but a general feature: capital price volatility $\sigma_t^{q,K}$ does not converge to its flexible price counterpart in the limit $\kappa \to \infty$.

4 Comparison to Other Models

4.1 No Safe Assets

The previous literature has studied the impact of uncertainty shocks in New Keynesian models where safe bonds are in zero net supply (e.g., Basu and Bundick, 2017; Caballero and Farhi, 2018; Caballero and Simsek, 2020). In this section, we contrast models without safe bonds to our baseline model and further highlight the importance of portfolio adjustment in shaping macroeconomic dynamics following uncertainty shocks in the presence of safe asset demand.

The model without a safe asset corresponds to a special case of our model where the supply of bonds is zero ($\mathcal{B}_t \equiv 0$). In this case, households choose a portfolio entirely in capital ($\vartheta_t = 0$).[32]

Characterizing Aggregate Demand. When $\mathcal{B}_t \equiv 0$, there is no sluggishly adjusting safe asset state q_t^B that serves as a nominal anchor. Instead, aggregate demand follows the conventional IS equation logic. Specifically, short-term output is given by

$$u_t = \frac{1}{a}\Bigg[\underbrace{\rho q_t^K}_{\substack{\text{consumption}\\ \text{demand}}} + \underbrace{\iota_t}_{\substack{\text{investment}\\ \text{demand}}}\Bigg], \tag{31}$$

which is strictly increasing in the capital price q_t^K. To understand the determinants of the latter, it is useful to revisit the household portfolio choice condition (19), which now becomes

$$\underbrace{\rho + g_t + \mu_t^{q,K}}_{\mathbb{E}_t\left[\mathrm{d}R_t^K\right]/\mathrm{d}t} - \underbrace{(i_t - \pi_t)}_{\mathbb{E}_t\left[\mathrm{d}R_t^B\right]/\mathrm{d}t} = (\sigma_t^{q,K})^2 + \tilde{\sigma}_t^2. \tag{32}$$

We observe that this is a purely forward-looking equation that does not contain any endogenous state variables. Furthermore, because bonds are here in zero net supply, only the single asset price drift $\mu_t^{q,K}$ enters the excess return on the left-hand side, whereas a difference of two asset price drifts, $\mu_t^{q,K} - \mu_t^{q,B}$, enters in the safe asset model. Consequently, equation (32) determines the level of the capital price, q_t^K. This is unlike

[32] Specifically, in line with conventional New Keynesian models, we assume that agents can still trade nominal bonds and so still face a portfolio choice, but these bonds are in zero net supply.

the valuation equation (22) in the safe asset model, which determines the relative asset price (the nominal wealth share ϑ_t).

In sum, equations (31) and (32) form a purely forward-looking system of equations that directly determine the level of aggregate demand. The structure of (32) is akin to an IS equation: it relates the change in q_t^K, a sufficient statistic for aggregate demand, to the difference between the actual real interest rate, $i_t - \pi_t$, and a neutral rate $\rho + g_t - (\sigma_t^{q,K})^2 - \tilde{\sigma}_t^2$ that would obtain in an economy in which consumption grows at the capital growth rate g_t, i.e. for constant u_t.

The Effects of an Uncertainty Shock. When uncertainty increases, households demand a higher risk premium on capital; the right-hand side of equation (32) increases. For households to remain willing to operate the economy's capital stock, the expected excess return on capital over the risk-free rate must increase, either via a reduction in the capital price q_t^K, which is accompanied by a demand recession, or by a reduction in the risk-free rate $i_t - \pi_t$. Which outcome materializes depends on the monetary policy reaction and on equilibrium selection.

In Figure 3, we illustrate the effects of an uncertainty shock on resource utilization u_t under assumptions that generate the conventionally presented dynamics in the literature. For clarity of exposition, we consider a simple example without aggregate shocks with the following features. Before date $t = 0$, $\tilde{\sigma} = \bar{\sigma}$ and the economy is at its steady state, which coincides with the flexible price allocation. At $t = 0$, there is an unanticipated shock that alters the future path of $\tilde{\sigma}_t$ as follows:

$$\tilde{\sigma}_t = \begin{cases} \bar{\sigma} + \Delta, & t \in [0, T) \\ \bar{\sigma}, & t \geq T \end{cases},$$

i.e., uncertainty unexpectedly rises for T time units and then returns back to its steady-state level. We consider the equilibrium of the model with $\mathcal{B} = 0$ for which, when uncertainty recedes at $t = T$, resource utilization returns to its steady-state level immediately.[33]

[33]This equilibrium selection aligns with the standard one. It would obtain, for example, as the unique bounded equilibrium if, from some future date $\bar{T} \geq T$ onward, monetary policy followed a standard Taylor rule consistent with zero inflation at the steady state.

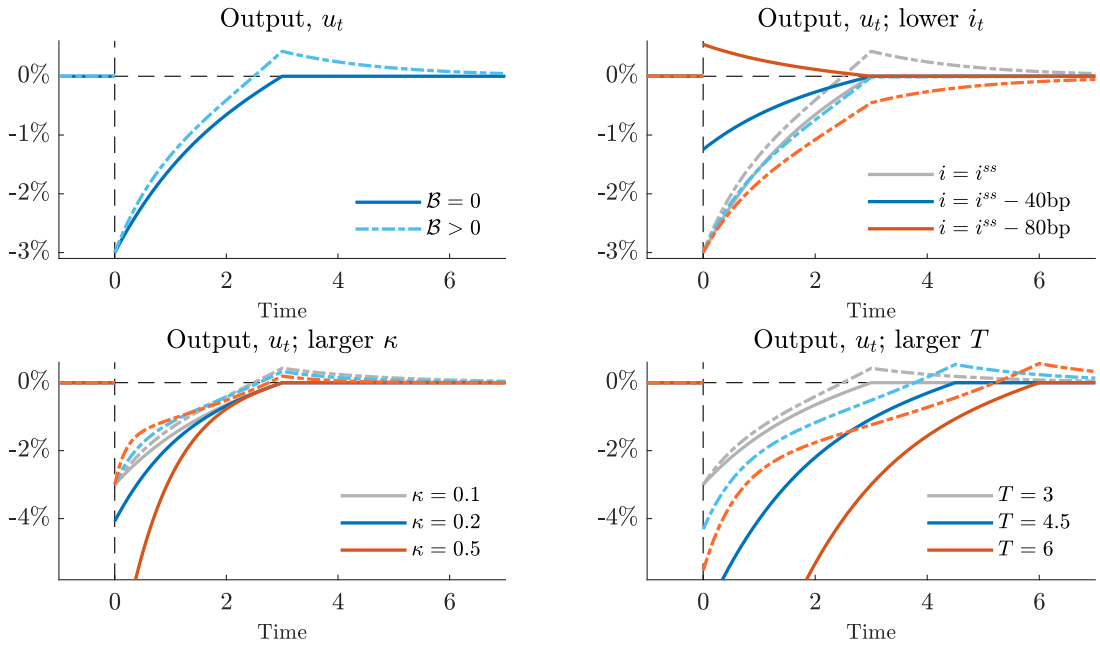


Figure 3: Effects of an uncertainty shock without (solid lines) and with (dashed-dotted lines) safe assets. The top left panel plots the impulse response function of utilization u_t in an economy without aggregate risk in response to the news at $t = 0$ that uncertainty assumes temporarily a larger value $\bar{\sigma} + \Delta$ for $T = 3$ periods and thereafter reverts back to its steady state value $\bar{\sigma}$. The remaining panels depict comparative statics of the same experiment with respect to the Phillips curve slope κ (bottom left), the length of the period of heightened uncertainty, T (bottom right), and the nominal interest rate response i before time T (top right).

The solid line in the top-left panel of Figure 3 depicts the effects of the aforementioned shock on utilization u_t under our baseline assumption that nominal interest rates are kept constant throughout. The dashed-dotted line plots the response in our safe asset model for comparison. To facilitate the comparison, we have adjusted the shock size Δ for the latter model so as to generate the same impact response to the shock. We observe that, in this specific example, the predictions of the two models closely align. In particular, a demand recession also occurs in the absence of flight to safety.

In the remaining panels of Figure 3, we illustrate that the two models generate diverging predictions with regard to several parameter changes. The bottom-left panel depicts comparative statics with respect to the slope of the Phillips curve, κ. More flexible prices (higher κ) lead to a deeper recession but do not affect its length in the model without safe assets. In contrast, they have no impact on the initial size of the recession

and generate faster recoveries in the safe asset model.[34] The bottom-right panel depicts comparative statics with respect to the length T of the period of elevated uncertainty. In both models, a higher T leads to deeper recessions, but the effects are much more pronounced without safe assets. Finally, the top-right panel illustrates the effects of a reduction in the nominal interest rate before time T. Without safe assets, interest rate changes have a direct effect on the size of the recession and, as the red line in the figure shows, can even prevent the recession altogether. In contrast, as we have previously observed, the initial recession in the safe asset model is unaffected by interest rates, while lower interest rates slow down government debt growth and thereby the speed of the recovery.

The aforementioned differences are explained by the differences in how shocks are transmitted in the two model economies. The model without safe assets follows a purely forward-looking IS equation logic with a fixed terminal condition at the time of recovery T. Increasing T mechanically increases the length of the recession and, because output must rise throughout the period, generates a deeper recession on impact. Similarly, more flexible prices speed up dynamics so that, for the same T, the initial jump in u_t must be larger to reach the same terminal condition at $t = T$. Finally, nominal interest rates directly affect aggregate demand, as evident in equation (32).

Dynamics in the safe asset model, in contrast, are guided by the interaction of two components. The first is forward-looking portfolio choice, which governs flight to safety and aggregate demand on shock impact. The second is the backward-looking evolution of the safe asset state, which is primarily driven by the Pigou effect. The former only depends on the expected future risk dynamics. A longer period of elevated risk (higher T) leads to a more pronounced portfolio adjustment, whereas changes in κ or i have no effect on portfolio choice at all. The latter depends on inflation dynamics and nominal government debt growth. Increasing either inflation or debt growth speeds up the recovery, which explains the comparative statics with respect to κ and i.

In addition to these differences, we remark that even the basic conclusion that uncertainty shocks lead to demand recessions is not robust in models without safe assets.

[34] This difference becomes starkest towards the flexible price limit, $\kappa \to \infty$. While output dynamics in the safe asset model continuously approach the flexible price dynamics, resource utilization in the model without safe assets approaches zero, a well-known pathology of standard New Keynesian analysis.

For one, the recession can be escaped with a sufficient monetary policy reaction, as illustrated by the red line in the top-right panel of Figure 3. But more importantly, changes in the assumptions about the mean-reversion dynamics of $\tilde{\sigma}_t$ may affect even the baseline prediction that a demand recession must occur in the absence of a policy reaction. In Appendix D.1, we discuss two such examples. In the first, we replace the deterministic return to steady state at T with a stochastic return that happens at a Poisson event. This is akin to the setting studied by Caballero and Simsek (2020). In the second, we study gradual deterministic mean reversion in a linearized model, similar to the analysis in Basu and Bundick (2017). In both cases, we show that uncertainty shocks only lead to demand recessions under standard equilibrium selection assumptions if the Phillips curve is sufficiently flat, i.e., if prices are sufficiently sticky. This finding contrasts with our finding in Proposition 7 that uncertainty shocks are robustly associated with aggregate demand recessions in the presence of flight to safety.

Interest Rate Policy and the Zero Lower Bound. As just observed, how uncertainty shocks affect aggregate economic activity in models without safe assets hinges on the assumptions about the monetary policy reaction. Indeed, in typical model environments in the literature such as Basu and Bundick (2017) and Caballero and Simsek (2020), the flexible price allocation can be implemented by a monetary policy that sets the nominal interest rate in all states to the natural rate, that is the equilibrium real interest rate in the flexible price model. An analogous result also holds in our model environment in the absence of safe assets:

Proposition 10 (Interest Rate Effectiveness without Safe Assets)**.** *If bonds are in zero net supply, $\mathcal{B}_t \equiv 0$, the flexible price equilibrium with zero inflation can be implemented by setting the nominal interest rate to the natural rate in all periods, i.e., with the policy*

$$i_t = r_t^n := \rho + g_t^{flex} - \tilde{\sigma}_t^2,$$

where g_t^{flex} denotes the capital growth rate process under the flexible price allocation.[35]

[35]More precisely, there are many equilibria consistent with this interest rate process, and the assertion is that the flexible price allocation is one of them. It is well known that then this equilibrium can also be uniquely implemented (among bounded equilibria) with a feedback rule that satisfies the Taylor principle, e.g., $i_t = r_t^n + \phi_\pi \pi_t$ with arbitrary $\phi_\pi > 1$.

To see why the previous proposition is correct, note that equation (32) is satisfied with zero inflation, $\pi_t = 0$, and a constant capital price at the flexible-price level, $q_t^K = q^{K,flex}$,[36] if and only if $i_t = r_t^n$. If this is the case, then also the Phillips curve equation (20) is satisfied, so we have found a valid equilibrium in the sticky price economy.

Proposition 10, which is in stark contrast to Proposition 9 for the safe asset model, has important implications for the power of monetary policy and the relevance of uncertainty shocks for aggregate fluctuations. In a world without safe assets in positive net supply, interest rate policy alone can fully stabilize the economy in response to uncertainty shocks. As a consequence, such shocks should only affect aggregate fluctuations if monetary policy is unable (or unwilling) to react appropriately, e.g., due to a binding lower bound on nominal interest rates such as a zero lower bound (ZLB) constraint. For this reason, the predominant conclusion from the previous literature is that uncertainty shocks are a concern only if they lead to a binding ZLB. Our safe asset framework suggests a different conclusion (compare Proposition 9). If uncertainty shocks induce a flight to safety into nominal safe assets in positive net supply, then they are always a concern for aggregate economic activity, regardless of the constraints faced by monetary policymakers.

Why is a natural rate policy successful in standard models but not in ours? As we have observed, equilibrium dynamics in models without safe assets are guided by purely forward-looking conditions — the IS equation and the Phillips curve.[37] When nominal rates equal natural rates, zero inflation and output gaps constitute one possible solution to these equations. While this is typically not the only possibility, the forward-looking nature of the system means that whether this solution is selected only depends on expectations. The only task of policy is, therefore, to manage expectations. By adjusting nominal interest rates, a flow quantity that affects future returns, monetary policy is perfectly able to accomplish this task.

The key difference in our model is that merely managing expectations is insufficient to stabilize aggregate demand due to the presence of the safe asset value q_t^B, which

[36] Unlike in the model with safe assets, the flexible-price capital price is constant in the model without safe assets because $\vartheta_t \equiv 0$ is constant.

[37] In richer models, there may be additional equations, including ones for endogenous state variables. However, these two typically determine how interest rate policy and sticky prices affect equilibrium dynamics.

follows a backward-looking evolution and serves as a nominal anchor in our model economy. The monetary authority could still follow a natural rate policy, and then zero inflation and output gaps would continue to constitute a valid solution to the IS equation and Phillips curve in isolation. But these are just two equilibrium conditions that have many different solutions. For a solution to be a valid equilibrium, it must also be consistent with the remaining equilibrium conditions, including the portfolio choice between capital and safe assets. In a flight to safety, the specific solution with zero inflation and output gaps could be made consistent with portfolio choice only if the safe asset state q_t^B was to react on shock impact. But, as we have discussed in Section 3.2.2, this is not the case. Instead, changes in safe asset demand generate adjustment pressures in q_t^B that result in transition periods of inflation or disinflation. Interest rate policy can manage these transitions but not eliminate them.

4.2 Safe Asset Denominated in Non-sticky Unit of Account

The model in the previous subsection without safe assets is a natural benchmark. However, removing safe assets from the model altogether alters the model's properties even in the absence of sticky prices. What really matters for the difference between our model and standard New Keynesian models is that the nominal unit of account plays a dual role: (i) as a unit of measurement for sticky goods prices and (ii) as a unit of value for the nominal safe asset.

To make this point as clearly as possible, we present in Appendix D.2 a thought experiment that separates these two roles into two distinct units of account: a "goods dollar" in which intermediate goods firms quote their prices and a "bond dollar" in which government bonds are denominated. In the model of Section 2, both units of account coincide, whereas here, there is a (flexible) exchange rate between them.

The main takeaway is that once the "bond dollar" is no longer linked to sticky goods prices and, hence, free to adjust in response to portfolio shifts, model dynamics largely resemble those observed in the model without safe assets. In response to an uncertainty shock, there is still flight to safety, but it does not necessarily generate an aggregate demand recession. Instead, for suitable interest rate policy such as $i_t = r_t^n$ (and suitable equilibrium selection), flight to safety merely leads to an appreciation of the "bond

dollar" relative to the "goods dollar" without aggregate demand effects. It is therefore not the existence of safe assets per se but the fact that they are denominated in the sticky unit of account that generates demand recessions in flight-to-safety episodes.

4.3 Models with Monetary Frictions

As the previous subsections highlight, the mechanism emphasized in this paper critically depends on two features of government bonds: (i) households' portfolio demand for bonds varies systematically with the exogenous state ($\tilde{\sigma}_t$) and (ii) bonds are nominal claims denominated in the same unit of account as the sticky goods price. In more traditional macro models that combine sticky prices and monetary frictions, money can also have both features. In Appendix D.3, we revisit a model of this type and show that its analysis can lead to similar conclusions as ours. In particular, a money demand shock causes a "flight to money," which may generate a demand recession. However, unlike in our model, monetary policy can easily avert the recession by managing the quantity of the money stock with open market operations. In contrast, an open market exchange of one government liability (bonds) for another (money/reserves) leaves the quantity of safe assets unaffected and can therefore not cure demand recessions in response to flight to safety episodes.

5 Long-term Bonds

So far, we have considered an arrangement with nominal government bonds of infinitesimally short duration so that the nominal price of these bonds must always be one. Consequently, when the price level is sticky, then so must be the real value of government bonds – and thus safe assets – in the economy. This raises the question of whether our results critically depend on this assumption. In this section, we introduce a generalized setup with long-duration government bonds and conclude that they do not. Long-duration government debt breaks the tight link between the price level and the market value of safe assets, yet our conclusions are not radically altered: the determinants of portfolio choice and the nominal wealth share ϑ_t are unaltered, almost all of our conclusions about the transmission mechanism of uncertainty shocks remain

valid, and interest rate policy is still not sufficiently powerful to eliminate sticky price distortions. In fact, even most of the model equations remain the same.

The setup of our generalized model is as before, except that we replace the assumptions on government bonds: instead of floating-rate bonds, the government issues bonds with a geometric maturity structure with duration $1/\lambda$ where $\lambda > 0$ is a model parameter. Specifically, each bond has a face value $\mathcal{b}_t$ that decays over time at rate λ, and the bond makes continuous payments $1/\mathcal{P}_t \cdot \lambda \mathcal{b}_t \mathrm{d}t$ of final goods to its holder. We denote by $\mathcal{B}_t$ the aggregate face value of the total bond stock and by $\mathcal{P}_t^B$ the nominal bond price per unit of face value. As before, we denote by $q_t^B := (\mathcal{P}_t^B \mathcal{B}_t)/(\mathcal{P}_t K_t)$ the real value of all outstanding bonds per unit of capital in the economy and by $\check{s}_t := \tau_t / q_t^B$ the surplus-to-debt ratio. Furthermore, let $i_t := \lambda(\frac{1}{\mathcal{P}_t^B} - 1)$ be the nominal yield to maturity of a long-term bond purchased at time t.[38] We distinguish the safe asset value q_t^B from the portion that depends on sluggishly adjusting state variables, $q_t^{B,0} := \mathcal{B}_t/(\mathcal{P}_t K_t)$, and the long-term nominal interest rate i_t from the nominal short rate on a hypothetical zero-maturity bond, denoted by i_t^0.

With these definitions, almost all equations from our baseline model continue to hold in the long-term debt model. The following proposition summarizes the similarities and differences between the two models. We relegate detailed derivations to Appendix C.

Proposition 11 (Long-term Bond Model)**.** *In the long-term bond model with inverse duration parameter λ, the following hold true:*

(i) All model equations characterizing equilibrium except for the state evolution equation (15) continue to hold (specifically, all numbered equations and all propositions stated in Section 2.3 remain valid). The safe asset value is given by

$$q_t^B = \frac{\lambda}{\lambda + i_t} q_t^{B,0}, \tag{33}$$

and the state variable $q_t^{B,0}$ follows an evolution akin to equation (15),

$$\mathrm{d}q_t^{B,0} = \left(\mu_t^{\mathcal{B}} - \pi_t - g_t\right) q_t^{B,0} \mathrm{d}t.$$

[38] A single face value unit of the bond purchased at time t costs $\mathcal{P}_t^B$ nominal units and generates a cash flow stream of $\lambda e^{-\lambda(t'-t)}\mathrm{d}t$ nominal units at each date $t' \geq t$. It is straightforward to determine that i_t is indeed the internal rate of return for this investment.

(ii) Conclusions from the short-term debt model on separation of portfolio choice (Proposition 4) and flight to safety (Proposition 5) continue to hold. Propositions 6, 7, and 8 on the transmission of uncertainty shocks and capital price overshooting also continue to hold, if i_t does not load on the shock $\mathrm{d}Z_t$ (or, more generally, if it does not react "too strongly" in a sense made precise in Appendix C.3).

(iii) Controlling the nominal short rate i_t^0 is sufficient for controlling the long-term bond interest rate i_t in the following sense: for any given Ito process $\{\hat{i}_t\}_{t\geq 0}$ with bounded (arithmetic) drift and volatility for which $\hat{i}_t + \lambda$ is positive, bounded, and bounded away from zero, there is a bounded stochastic process $\{i_t^0\}_{t\geq 0}$ for the nominal short rate such that, in equilibrium, $i_t = \hat{i}_t$ for all t.[39]

(iv) In the limit $\lambda \to \infty$, the model collapses to the model with floating-rate debt presented in Section 2.

Part (i) establishes a link between the long-term bond model and our baseline model, allowing us to transfer and possibly adapt most of the insights derived previously. Part (ii) formalizes some of this transfer by establishing that most of our formal propositions continue to hold, possibly up to some qualification. Part (iii) provides a justification for considering the long-term rate i_t as the policy instrument for interest rate policy in place of the more conventional nominal short rate i_t^0. This result follows from no-arbitrage relationships that hold across the nominally risk-free term structure. Controlling the path of the nominal rate for a single given duration is sufficient to determine the full term structure dynamics. Finally, part (iv) establishes that the generalized model considered here indeed nests our baseline model with infinitesimal duration.

The key difference in the long-term bond model is therefore in equation (33), which says that the safe asset value is the product of a slow-moving state variable ($q_t^{B,0}$) that evolves as before and a "jump variable" $\mathcal{P}_t^B = \frac{\lambda}{\lambda + i_t}$ that is effectively controlled by interest rate policy. All our conclusions from the baseline model therefore remain true if i_t does not load on the shock $\mathrm{d}Z_t$. More generally, however, our previous conclusion that interest rate policy has no effect on the economy on the impact of an uncertainty

[39] The additional technical conditions on $\hat{i}_t + \lambda$ are required to ensure that the nominal bond price $\mathcal{P}_t^B$ remains economically meaningful, i.e., positive and bounded.

shock needs to be qualified. Specifically, lowering the nominal interest rate i_t raises the nominal bond price, $\mathcal{P}_t^B$, which expands the real quantity of safe assets immediately.

In sum, the policy instrument i_t affects the economy with long-term bonds in two ways. First, as just explained, it affects the real quantity of safe assets through changes in the nominal bond price $\mathcal{P}_t^B$. A higher i_t lowers the nominal bond price and creates disinflationary pressures immediately. Second, like in our baseline model, it affects the growth rate of nominal government liabilities. A higher i_t raises $\mu_t^{\mathcal{B}}$, which raises the real quantity of safe assets gradually over time via its effect on the state variable $q_t^{B,0}$. This tends to reduce disinflationary pressures in the future and shorten the recession.

These two effects are associated with two policy objectives that can be in conflict with each other: (i) to stabilize aggregate demand in the short run in response to uncertainty shocks and (ii) to make the growth rate of nominal government liabilities consistent with a given inflation target in the long run, e.g., $\pi_t = 0$. Generically, the single instrument i_t is insufficient to achieve both policy objectives simultaneously. The following static example serves to illustrate this point: suppose we start the economy in a steady state with constant $\tilde{\sigma}$ and consider an unanticipated shock that permanently raises $\tilde{\sigma}$ at $t = 0$, which leads to a (one-time) permanent increase in ϑ and the flexible-price safe asset value $q^{B,flex}$. We are interested in a one-time adjustment in the nominal rate i to a new permanent level. Policy objective (i) requires setting i to the level required to produce an immediate jump in q_0^B to $q^{B,flex}$, i.e., $i = \lambda(q^{B,0}/q^{B,flex} - 1)$, which depends on λ. In contrast, policy objective (ii) requires setting i to the natural rate so that q_t^B converges to $q^{B,flex}$ in the long run, i.e., $i = i^n$, which does not depend on λ. Clearly, the same i achieves both objectives at most for a single value of λ. Generically, it is therefore impossible to achieve them both. The lessons from this static example apply more generally and lead to the following conclusion:

Corollary 1. *Proposition 9 ("Interest Rate Ineffectiveness") holds generically also in the long-term bond model.*

Beyond our model and the scope of this paper, we remark that the previous result may be overturned in a setting in which the central bank has a richer set of instruments at its disposal. Particularly, suppose the central bank can control both the nominal interest rate and the maturity structure of government bonds (a time-varying λ_t), e.g.,

by open market purchases or sales of long-term bonds for reserves. Then, it is possible to adjust λ_t dynamically to always stay in the non-generic case, in which interest rate policy can undo sticky price distortions for the next shock. This suggests that altering the maturity structure of government debt through asset purchases primarily serves the purpose of recalibrating the impact of future interest rate changes on bond prices.[40]

6 Welfare and Optimal Monetary Policy

In previous sections, we have demonstrated how flight to safety in the presence of price stickiness leads to aggregate demand recessions and how interest rate policy alone is inadequate for perfect price stabilization. In this section, we examine normative implications. We first discuss the determinants of welfare in our model and derive a constrained efficient benchmark allocation. We conclude that a certain degree of flight to safety is socially desirable and, consequently, it is generally undesirable to use fiscal policy to completely eliminate it. One implication is that the constrained efficient allocation is generically not within reach under the set of policy instruments available in our baseline model. We next discuss that enlarging the set of policy tools with lump-sum taxes and transfers can overcome the limitation on policy and put the constrained efficient allocation within reach. This is precisely because the present value of lump-sum transfers represents an implicit safe asset to the private sector whose value can be flexibly adjusted in response to shocks. While of theoretical interest, it is not very plausible that a lump-sum instrument can be utilized in practice to stabilize safe asset demand. We therefore conclude this section with some brief remarks on optimal monetary policy in the absence of lump-sum taxes. The key takeaway is that policy under-reacts relative to perfect demand stabilization on shock impact in order to let the Pigou effect play out and smooth out the unavoidable distortions that arise from our policy ineffectiveness propositions. Throughout this section, we consider the extended model with long-term government bonds and inverse duration parameter $\lambda \in (0, \infty)$.

[40] In follow-up work that builds on our paper, Alexandrov and Brunnermeier (2024) explore this idea further in the context of a richer model.

6.1 Welfare and Constrained Efficiency

We start by deriving a useful expansion for welfare in our model economy. Note that the consumption of each individual household is given by

$$c_t^i = \underbrace{(au_t - \iota_t)K_t}_{\text{aggregate consumption}} \cdot \underbrace{\frac{n_t^i}{\int n_t^j \mathrm{d}j}}_{\text{wealth share}}. \tag{34}$$

Since the households are homogeneous up to scale, the wealth share of each household follows a martingale that is only subject to idiosyncratic shocks with homogeneous geometric volatility loading $(1-\vartheta_t)\tilde{\sigma}_t$. The value function of the household can be obtained by taking logarithms on both sides of equation (34), integrating over time, and using equilibrium relationships from Section 2.3 to simplify the resulting expressions. In addition, due to logarithmic preferences and homogeneity up to scale, individual value functions differ across households only in an additively separated term that captures the effects of the initial cross-sectional wealth distribution. Therefore, for any policy that affects only aggregates but not the initial wealth distribution, all households agree on the optimal allocation:

Proposition 12 (Welfare Representation and Constrained Efficient Allocation)**.** *For any social welfare function that is strictly increasing in individual expected utilities* $\{V_0^i\}_{i\in[0,1]}$*, maximizing the welfare function is equivalent to maximizing the objective*

$$\mathbb{E}\left[\int_0^\infty \left(\mathcal{W}_{\vartheta,t}(\vartheta_t) + \mathcal{W}_u(u_t)\right)\mathrm{d}t\right]$$

with respect to $\{u_t, \vartheta_t\}_{t=0}^\infty$*, where*

$$\mathcal{W}_{\vartheta,t}(\vartheta) = \frac{1+\phi\rho}{\phi\rho}\log\left(\frac{1+\phi\rho}{1-\vartheta_t+\phi\rho}\right) + \frac{1}{\phi\rho}\log(1-\vartheta_t) - \frac{(1-\vartheta_t)^2\tilde{\sigma}_t^2}{2\rho},$$
$$\mathcal{W}_u(u) = \frac{1+\phi\rho}{\phi\rho}\log\left(\frac{1+\phi a u}{1+\phi\rho}\right) - \frac{u^{1+\varphi}}{1+\varphi}.$$

A constrained planner that can control portfolios and capital utilization directly but is unable to provide direct idiosyncratic risk insurance optimally chooses a $\tilde{\sigma}_t$*-dependent nominal wealth share* $\vartheta_t = \vartheta^*(\tilde{\sigma}_t)$ *that satisfies* $\vartheta^{*\prime} > 0$ *and a time-invariant utilization rate* $u_t = u^*$*.*

The previous proposition implies a separation of normative considerations concerning the composition of wealth into safe assets and capital assets, captured by ϑ_t, and the utilization of capital resources, captured by u_t. The term $\mathcal{W}_{\vartheta,t}(\vartheta)$ appears in a similar form also in Brunnermeier et al. (2020), where it is extensively discussed, including the pecuniary externalities that may lead to inefficient private sector behavior. We refer the reader to that paper for additional details and merely discuss the meaning of the three terms in $\mathcal{W}_{\vartheta,t}(\vartheta)$. The first and second terms capture a standard consumption-investment trade-off. For given aggregate output (given u_t), increasing the share of wealth held in safe assets raises current consumption, represented by the first term, and lowers investment and thereby consumption growth, represented by the second term. The third term captures the utility costs of idiosyncratic risk exposures. A greater allocation to safe assets improves risk sharing and lowers this cost. Importantly, this last term is sensitive to the amount of idiosyncratic risk currently present in the economy, so that, if ϑ_t can be controlled separately from u_t, a constrained planner finds it optimal to improve idiosyncratic risk sharing when $\tilde{\sigma}_t$ rises by expanding the share of wealth allocated to safe assets. In other words, *flight to safety is a feature of the constrained optimal allocation*.

The term $\mathcal{W}_u(u)$ concerns the short-term utilization of available resources, which captures a purely intertemporal trade-off between raising output and reducing disutility from utilization effort. This trade-off is independent of $\tilde{\sigma}_t$ because utilization does not affect idiosyncratic risk exposures in our model. The constrained efficient allocation therefore calls for a constant rate of capital utilization.

6.2 Implementation of the Constrained Efficient Allocation

Generic Impossibility of Implementation. Now suppose, instead of being able to control portfolios and utilization decisions directly, a policy maker is constrained to choose the paths for the policy instruments $\check{s}_t$ and i_t by adjusting the capital tax rate τ_t and the nominal short rate i_t^0. In analogy to our interest rate ineffectiveness propositions, Proposition 9 and Corollary 1, it is generically impossible to implement the constrained efficient allocation. To gain intuition, note that the constrained optimal safe asset wealth share $\vartheta_t = \vartheta^*(\tilde{\sigma}_t)$ has the properties that we have discussed in the context

of separation of portfolio choice and flight to safety in Section 3.2.1: the portfolio allocation only depends on the exogenous risk state and features flight to safety. Achieving this portfolio allocation fully constrains the fiscal instrument $\check{s}_t$ and precludes its use for aggregate demand stabilization.

This leaves the instrument i_t in charge of implementing $u_t = u^*$. However, we have already seen that interest rate policy can, at best, imperfectly control aggregate demand and, therefore, resource utilization when there is a flight to safety in equilibrium. While, in general, u^* differs from the capital utilization under flexible prices, precisely the same arguments that led to Proposition 9 and Corollary 1 also imply that achieving $u_t = u^*$ is generically out of reach for interest rate policy.

Corollary 2. *The policy instruments $\check{s}_t$ and i_t are generically insufficient to implement the constrained optimal allocation characterized in Proposition 12.*

Implementation with Lump-sum Taxes. The previous conclusion raises the question of which additional instruments would be required to bring the constrained efficient allocation within reach. From Proposition 4, we know that the fiscal instrument $\check{s}_t$ is sufficient to perfectly control portfolios ϑ_t. The issue is that it interferes with aggregate demand management because flight to safety drives the desired real value of total safe assets away from the currently outstanding stock of safe assets, whose real value cannot flexibly adjust.

On a conceptual level, a simple idea to resolve this issue is to introduce "helicopter drops" of bonds to allow the government to flexibly adjust the outstanding nominal stock of safe assets, $\mathcal{B}_t$, instead. Such a policy instrument gives the policymaker the power to adjust the state variable $q_t^{B,0}$ directly in response to uncertainty shocks. The instrument i_t then remains free to track whichever nominal rate is required to achieve the optimal resource utilization in the long run.[41] To implement such helicopter drops (and reverse drops), a state-contingent tax-transfer scheme of lump-sum taxes would be required that transfers new bonds to agents whenever $\tilde{\sigma}_t$ rises and requires agents to give up some of their bonds whenever $\tilde{\sigma}_t$ falls. Such a scheme of alternating taxes and

[41] This is generally different from tracking the natural rate because the optimal allocation differs from the flexible price allocation.

transfers on a high frequency is certainly unlike anything observed in reality (see also Footnote 15).

A more plausible implementation of the helicopter drop idea can be achieved by recognizing that Ricardian equivalence holds in our model with regard to lump-sum taxes, provided that cash flows from individual tax liabilities are in the asset span so that agents can effectively trade claims to their tax liabilities. This is the case, for example, if we assume that, in addition to bonds and capital, agents can trade state-contingent claims on aggregate shocks $\mathrm{d}Z_t$ in zero net supply.[42] Due to Ricardian equivalence with respect to this tax instrument, a government does not need to engage in an oscillating pattern of taxes and transfers to implement something like helicopter drops of bonds. Instead, it can smooth out the actual changes to taxes or transfers by merely adjusting the present value of future lump-sum taxes in response to uncertainty shocks.

Intuitively, the present value of future lump-sum taxes represents an implicit negative safe asset that reduces the overall supply of safe assets available to households. Importantly, this safe asset can be flexibly adjusted in response to uncertainty shocks without requiring a change in the level of nominal prices. Adjusting the amount of safe assets implicitly provided through lump-sum taxes and transfers can therefore insulate the aggregate goods demand from variation in safe asset demand. We provide a formal analysis of this idea in Appendix E and summarize here the conclusion:

Proposition 13 (Optimal Policy with Lump-sum Taxes)**.** *If lump-sum taxes are available and aggregate risk is tradeable, the constrained optimal allocation characterized in Proposition 12 can be implemented. Specifically, optimal policy sets capital taxes τ_t to implement the safe asset wealth share ϑ_t^* and jointly uses nominal interest rates and adjustments in future lump-sum taxes in response to $\mathrm{d}Z_t$-shocks to implement the optimal safe asset value $q_t^{B*} = \vartheta_t^* \frac{1+\phi a u^*}{1-\vartheta_t^*+\phi\rho}$. The nominal short rate is set to match the drift dynamics of q_t^{B*}, which requires*

$$i_t^0 = r_t^* + \pi_t^*,$$

where r_t^* is real interest rate in the constrained optimal allocation and π_t^* is an inflation target that makes $u_t = u^*$ consistent with the Phillips curve. Adjustments in future lump-sum taxes

[42] Allowing for these additional assets does not affect our model economy in the absence of lump-sum taxes so the previous analysis remains unaffected by this change.

in response to shocks are required to match the volatility dynamics of q_t^{B*}. The initial level and time drift of lump-sum taxes τ_t^{LS} may take arbitrary values.

6.3 Optimal Monetary Policy without Lump-sum Taxes

We conclude this section by briefly discussing aspects of optimal monetary policy when lump-sum taxes are not available. Because this question is technically more involved, we abstract here from aggregate risk and restrict attention to managing the deterministic transition back to the steady state after a single unanticipated uncertainty shock. We also abstract from physical capital investment by taking the limit $\phi \to \infty$. Specifically, suppose that the initial state $q_0^{B,0}$ is given and that, at time 0, a new deterministic path for idiosyncratic volatility $\{\tilde{\sigma}_t\}_{t=0}^{\infty}$ announced. We assume that tax policy reacts by following some time path $\{\check{s}_t\}_{t=0}^{\infty}$, so that, by Proposition 4, also the nominal wealth share $\{\vartheta_t\}_{t=0}^{\infty}$ follows a, from the perspective of the monetary authority, exogenously given time path. We further assume that the paths for $\check{s}_t$ and ϑ_t are bounded and ϑ_t remains bounded away from zero.

We study the Ramsey problem of a monetary authority that chooses the time path for interest rates, $\{i_t\}_{t=0}^{\infty}$, satisfying the technical requirements stated in Proposition 11, Part (iii), to maximize the welfare objective stated in Proposition 12 subject to the constraints imposed by competitive equilibrium. The solution to this problem represents the optimal interest rate policy under commitment. We also study a variant of this problem under discretion. We provide a formal statement of both problems in Appendix F.

In this policy problem, the monetary authority is exclusively concerned with managing resource utilization u_t to maximize the $\mathcal{W}_u$-portion of the welfare objective because portfolios are separately determined. As we have explained in Section 5, a higher interest rate lowers aggregate demand on impact by revaluing long-term bonds, but it raises the growth rate of the nominal bond supply, which tends to increase aggregate demand in the future. The optimal policy problem is therefore to distribute the shortfall or excess of aggregate demand optimally over time, recognizing that it can, generically, not be avoided entirely. The following lemma, which slightly sharpens our earlier result from Corollary 2, formalizes what this means.

Lemma 1. *In the context of the policy problem described above, there is, for given model parameters and time paths* $\{\tilde{\sigma}_t, \check{s}_t, \vartheta_t\}_{t=0}^{\infty}$ *precisely one initial condition* $q_0^{B,0} = q^{B,0*}$ *for which* $u_t = u^*$ *can be implemented with some interest rate policy. If* $q_0^{B,0} < q^{B,0*}$*, then, for any admissible interest rate policy, there is a* t *such that* $u_t < u^*$*. Symmetrically, if* $q_0^{B,0} > q^{B,0*}$*, then, for any admissible interest rate policy, there is a* t *such that* $u_t > u^*$*.*

Let us focus, for concreteness, on the case that $q_0^{B,0} < q^{B,0*}$ so that the quantity of safe assets is too small relative to what would be required to implement the optimal utilization rate at all times. This is the situation that arises, for example, if we start the economy in the constrained efficient steady state and consider an unanticipated rise in uncertainty and a fiscal policy that does not fully offset the ensuing flight to safety. In this case, the previous lemma tells us that a shortfall in aggregate demand is unavoidable at some point in the future. Intuitively, the real safe asset supply is too low, and, at least eventually, disinflation is required to raise it. By the Phillips curve (20), disinflation requires some pain in the form of under-utilization of resources in some period.

Our key result is that it is optimal to spread this pain gradually over time. To do so, optimal monetary policy under-reacts on shock impact relative to a policy that attempts full output stabilization and returns the economy back to optimal utilization at most in the long run. More specifically, returning economic activity back to its optimal level in the long run is optimal only under discretion. With commitment, it is preferable to under-utilize resources even in the long run to generate additional disinflation in the short run, so as to expand the real value of safe assets more rapidly.

Proposition 14 (Optimal Monetary Policy)**.** *Suppose that* $q_0^{B,0} < q^{B,0*}$ *in the context of Lemma 1. Under the optimal interest rate policy, both under discretion and commitment,* $u_t < u^*$ *for all* $t \geq 0$*. In addition, if* λ *is sufficiently large,* u_t *converges in the long run and*

$$\lim_{t\to\infty} u_t \begin{cases} = u^*, & \textit{discretion} \\ < u^*, & \textit{commitment} \end{cases}.$$

The proof of this proposition analyzes the first-order conditions of the Ramsey planner. It is technical and not very insightful. However, the result itself makes intuitive sense and is in the tradition of optimal smoothing results in economics, such as classic tax smoothing in simple models of distortionary taxation.

Beyond Proposition 14, the features of the optimal policy depend on parameter values. In Appendix F.2, we provide numerical examples to discuss certain aspects of the optimal policy further. Most importantly, the optimal behavior of interest rates depends crucially on the duration of government bonds $1/\lambda$. Specifically, for long bond durations, monetary policy optimally reduces interest rates to raise nominal bond prices and engineer a shallow but drawn-out recession. In contrast, for sufficiently short bond duration, it is optimal to hike interest rates to raise nominal bond growth, which generates a deeper recession followed by a fast recovery.

7 Conclusions

In this paper, we present a New Keynesian model with idiosyncratic risk and nominal government bonds to highlight a tight link between flight to safety and aggregate demand recessions in response to uncertainty shocks. Conclusions from our model differ from existing studies that do not account for the presence of nominal safe assets in positive net supply. Most importantly, our paper suggests that monetary policy plays a diminished role in managing aggregate demand in response to flight-to-safety episodes and that uncertainty shocks potentially play an important role even away from the ZLB constraint on nominal rates.

Our analysis has been purely theoretical. Potential directions for future research include a quantitative exploration of the channel we have emphasized, both with regard to business cycles and asset pricing predictions. On the methodological side, this paper develops a new tractable HANK framework, which may prove fruitful in answering other questions in the burgeoning HANK literature.

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Appendix

A Model Solution [Online Appendix]

Household Decision Problem. We derive optimal choice conditions for the household problem in a general form that includes all model variants in this paper, the sticky price and flexible price versions of our baseline model, the model without safe assets, and the model extension with long-term bonds. In all cases, households face a decision problem as outlined in Section 2.1 with a bond return $\mathrm{d}R_t^B$ as in equation (11), whose volatility loading $\sigma_t^{q,B}$ may be different from zero.

We solve dynamic decision problems using the stochastic maximum principle. The Hamiltonian of household i is

$$H_t^i = e^{-\rho t}\left(\log c_t^i - \frac{(u_t^i)^{1+\varphi}}{1+\varphi}\right) + \xi_t^i\left(-c_t^i + n_t^i\left(\theta_t^i\frac{\mathbb{E}_t[\mathrm{d}R_t^B]}{\mathrm{d}t} + (1-\theta_t^i)\frac{\mathbb{E}_t[\mathrm{d}R_t^{K,i}(\iota_t^i,u_t^i)]}{\mathrm{d}t}\right)\right)$$
$$- \varsigma_t^i\xi_t^i n_t\left(\theta_t^i\sigma_t^{q,B} + (1-\theta_t^i)\sigma_t^{q,K}\right) - \tilde{\varsigma}_t^i\xi_t^i n_t\left(1-\theta_t^i\right)\tilde{\sigma}_t,$$

where ξ_t^i denotes the costate associated with household net worth n_t^i and $-\varsigma_t^i$, $-\tilde{\varsigma}_t^i$ are the geometric volatility loadings of ξ_t^i on the aggregate and idiosyncratic shock, respectively. We remind the reader that, by equation (12),

$$\frac{\mathbb{E}_t[\mathrm{d}R_t^{K,i}(u_t^i, \iota_t^i)]}{\mathrm{d}t} = \frac{p_t^R u_t^i - \iota_t^i + \omega_t - \tau_t}{q_t^K} + \Phi(\iota_t^i) - \delta + \mu_t^{q,K}.$$

According to the stochastic maximum principle, the optimal choices must maximize the Hamiltonian H_t^i. The first-order conditions (FOCs) with respect to the four choice variables c_t^i, ι_t^i, u_t^i, and θ_t^i are

$$\begin{aligned}
c_t^i: \quad & \xi_t^i = e^{-\rho t}\frac{1}{c_t^i}, \\
\iota_t^i: \quad & \Phi'(\iota_t^i) = \frac{1}{q_t^K}, \\
u_t^i: \quad & e^{-\rho t}(u_t^i)^\varphi = \xi_t^i \frac{n_t^i(1-\theta_t^i)}{q_t^K} p_t^R, \\
\theta_t^i: \quad & \frac{\mathbb{E}_t[\mathrm{d}R_t^{K,i}(\iota_t^i, u_t^i)]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}R_t^B]}{\mathrm{d}t} = \varsigma_t^i \left(\sigma_t^{q,K} - \sigma_t^{q,B}\right) + \tilde{\varsigma}_t^i \tilde{\sigma}_t.
\end{aligned}$$

Using the functional form $\Phi(\iota) = \log(1+\phi\iota)/\phi$ and rearranging the FOC for ι_t^i, we obtain a linear equation that links q_t^K and ι_t^i. Solving for ι_t^i yields the optimal investment condition

$$\iota_t^i = \frac{q_t^K - 1}{\phi}. \tag{35}$$

Next, combining the FOCs for c_t^i and u_t^i and using that $\frac{n_t^i(1-\theta_t^i)}{q_t^K} = k_t^i$ are the capital holdings of household i, we obtain the optimal utilization condition

$$(u_t^i)^\varphi = \frac{p_t^R k_t^i}{c_t^i}. \tag{36}$$

Following Brunnermeier et al. (2024), we guess that $\xi_t^i = e^{-\rho t}/(\rho n_t^i)$. This guess can be verified using a standard verification argument based on the costate equation, which we omit here. Substituting this guess into the FOC for c_t^i and rearranging yields the optimal consumption condition

$$c_t^i = \rho n_t^i. \tag{37}$$

Finally, this guess for ξ_t^i also implies (by Ito's lemma and the net worth equation (3))

$$\varsigma_t^i = \theta_t^i \sigma_t^{q,B} + (1-\theta_t^i)\sigma_t^{q,K}, \qquad \tilde{\varsigma}_t^i = \left(1-\theta_t^i\right)\tilde{\sigma}_t,$$

and substituting these expressions into the FOC for θ_t^i yields the optimal portfolio choice condition

$$\frac{\mathbb{E}_t[\mathrm{d}R_t^{K,i}(\iota_t^i, u_t^i)]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}R_t^B]}{\mathrm{d}t} = \left(\theta_t^i \sigma_t^{q,B} + (1-\theta_t^i)\sigma_t^{q,K}\right)\left(\sigma_t^{q,K} - \sigma_t^{q,B}\right) + \left(1-\theta_t^i\right)\tilde{\sigma}_t^2. \tag{38}$$

Decision Problem of Intermediate Goods Firms. Under sticky prices, the Hamiltonian of intermediate goods firm j is

$$H_t^j = \xi_t \left(\frac{\mathcal{P}_t^j}{\mathcal{P}_t} a \hat{k}_t^j - p_t^R \hat{k}_t^j - \frac{\psi}{2}\left(\pi_t^j\right)^2 Y_t \right) + \lambda_t^j \pi_t^j \mathcal{P}_t^j,$$

where λ_t^j is the costate on the price evolution for $\mathcal{P}_t^j$. By the stochastic maximum principle, the firm maximiyes H_t^j subject to the demand equation

$$a\hat{k}_t^j = \left(\frac{\mathcal{P}_t^j}{\mathcal{P}_t}\right)^{-\epsilon} Y_t.$$

Using this equation to substitute out the choice $\hat{k}_t^j$, we can write the Hamiltonian as follows:

$$H_t^j = \underbrace{\xi_t Y_t}_{=:X_t} \left(\left(\frac{\mathcal{P}_t^j}{\mathcal{P}_t} - \frac{p_t^R}{a}\right)\left(\frac{\mathcal{P}_t^j}{\mathcal{P}_t}\right)^{-\epsilon} - \frac{\psi}{2}\left(\pi_t^j\right)^2 \right) + \lambda_t^j \pi_t^j \mathcal{P}_t^j.$$

The optimal choice condition for π_t^j is

$$\psi X_t \pi_t^j = \lambda_t^j \mathcal{P}_t^j.$$

In addition, λ_t^j must satisfy the co-state equation

$$\mathbb{E}_t[\mathrm{d}\lambda_t^j] = -\frac{\partial H_t^j}{\partial \mathcal{P}_t^j}\mathrm{d}t = -\left(\frac{X_t}{\mathcal{P}_t}\left(\frac{\mathcal{P}_t^j}{\mathcal{P}_t}\right)^{-\epsilon} \frac{\epsilon}{a}\left(\frac{1-\epsilon}{\epsilon}a + p_t^R \frac{\mathcal{P}_t}{\mathcal{P}_t^j}\right) + \lambda_t^j \pi_t^j \right) \mathrm{d}t.$$

Under our assumption that $\mathcal{P}_0^j = \mathcal{P}_0$, there is a solution for which $\mathcal{P}_t^j$, π_t^j, and λ_t^j are independent of j, so that, in particular, $\mathcal{P}_t^j = \mathcal{P}_t$ for all t by the definition of the price index. Substituting this into the costate equation yields:

$$\mathbb{E}_t[\mathrm{d}\lambda_t] = -\left(\frac{X_t}{\mathcal{P}_t}\frac{\epsilon}{a}\left(p_t^R - \frac{\epsilon-1}{\epsilon}a\right) + \lambda_t\pi_t\right)\mathrm{d}t.$$

By the optimal choice condition, $X_t\pi_t = \frac{1}{\psi}\lambda_t P_t$, so by Ito's product rule and the fact that $\mathcal{P}_t$ is locally deterministic,

$$\begin{aligned}
\mathbb{E}_t\left[\mathrm{d}\left(X_t\pi_t\right)\right] &= \frac{1}{\psi}\lambda_t\mathrm{d}\mathcal{P}_t + \frac{1}{\psi}\mathcal{P}_t\mathbb{E}_t[\mathrm{d}\lambda_t] \\
&= \frac{1}{\psi}\lambda_t\mathcal{P}_t\pi_t\mathrm{d}t - \frac{1}{\psi}\mathcal{P}_t\left(\frac{X_t}{\mathcal{P}_t}\frac{\epsilon}{a}\left(p_t^R - \frac{\epsilon-1}{\epsilon}a\right) + \lambda_t\pi_t\right)\mathrm{d}t \\
&= \frac{1}{\psi}\lambda_t\mathcal{P}_t\pi_t\mathrm{d}t - X_t\frac{\epsilon}{\psi}\left(\frac{p_t^R}{a} - \frac{\epsilon-1}{\epsilon}\right)\mathrm{d}t - \frac{1}{\psi}\lambda_t P_t\pi_t\mathrm{d}t \\
&= -X_t\frac{\epsilon}{\psi}\left(\frac{p_t^R}{a} - \frac{\epsilon-1}{\epsilon}\right)\mathrm{d}t.
\end{aligned}$$

Next, use that by our assumption on the firm stochastic discount factor,

$$X_t = \xi_t Y_t = e^{-\rho t}\frac{Y_t}{C_t},$$

and rewrite the previous equation as follows:

$$\begin{aligned}
\mathbb{E}_t\left[\mathrm{d}\left(\pi_t\frac{Y_t}{C_t}\right)\right] = \mathbb{E}_t\left[\mathrm{d}\left(e^{\rho t}X_t\pi_t\right)\right] &= \rho e^{\rho t}X_t\pi_t\mathrm{d}t + e^{\rho t}\mathbb{E}_t\left[\mathrm{d}\left(X_t\pi_t\right)\right] \\
&= \left(\rho\pi_t - \frac{\epsilon}{\psi}\left(\frac{p_t^R}{a} - \frac{\epsilon-1}{\epsilon}\right)\right)\frac{Y_t}{C_t}\mathrm{d}t. \qquad (39)
\end{aligned}$$

Under flexible prices, two aspects of the firm decision problems change. First, there are no price adjustment costs, $\psi = 0$. Second, $\mathcal{P}_t^j$ is not necessarily locally deterministic but may directly load on shocks, so that we can treat it as a choice variable instead of a state. The resulting optimal choice condition is formally identical to taking the limit $\psi \to 0$ in equation (39) and given by

$$p_t^{R,flex} = \frac{\epsilon-1}{\epsilon}a. \qquad (40)$$

We also note that, under the optimal price setting choice, regardless of price setting frictions, all firms face identical demands and therefore produce the same amount of output $y_t^j = a\hat{k}_t^j = Y_t$. Imposing market clearing in the market for capital services, we obtain $Y_t = au_tK_t$. Firm profits are therefore (compare equation (5))

$$\varpi_t = \left(\left(1 - \frac{\psi}{2}\pi_t^2\right) au_t - p_t^R\right) K_t.$$

Consequently, the profit transfer to households per unit of capital is (compare equation (6))

$$\omega_t = au_t - p_t^R. \tag{41}$$

Equilibrium and the Bond Valuation Equation. The expected excess return of capital over bonds is, by equations (11) and (12):

$$\begin{aligned}\frac{\mathbb{E}_t[\mathrm{d}R_t^K(u_t,\iota_t)]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}R_t^B]}{\mathrm{d}t} &= \left(\frac{p_t^R u_t - \iota_t + \omega_t - \tau_t}{q_t^K} + g_t + \mu_t^{q,K}\right) - \left(i_t - \mu_t^{\mathcal{B}} + g_t + \mu_t^{q,B}\right) \\ &= \underbrace{\frac{p_t^R u_t - \iota_t + \omega_t}{q_t^K}}_{A} + \underbrace{\mu_t^{\mathcal{B}} - i_t - \frac{\tau_t}{q_t^K}}_{B} + \underbrace{\mu_t^{q,K} - \mu_t^{q,B}}_{C}\end{aligned}$$

We discuss the terms A, B, and C separately:

A Using equation (41), the definition of ϑ_t (equation (21)), and the goods market clearing condition (equation (24)), we obtain:

$$\frac{p_t^R u_t - \iota_t + \omega_t}{q_t^K} = \frac{au_t - \iota_t}{q_t^K} = \frac{au_t - \iota_t}{q_t^B + q_t^K}\frac{1}{1-\vartheta_t} = \frac{\rho}{1-\vartheta_t}.$$

B Using the definition of $\check{s}_t$ (equation (8)), the government budget constraint (equation (9)), and the definition of ϑ_t (equation (21)), we obtain:

$$\mu_t^{\mathcal{B}} - i_t - \frac{\tau_t}{q_t^K} = -\check{s}_t - \frac{\tau_t}{q_t^B}\frac{q_t^B}{q_t^K} = -\check{s}_t - \check{s}_t\frac{\vartheta_t}{1-\vartheta_t} = -\frac{\check{s}_t}{1-\vartheta_t}$$

C Applying Ito's lemma to the definition of ϑ_t (equation (21)), we obtain

$$\mu_t^{\vartheta} = (1-\vartheta_t)(\mu_t^{q,B} - \mu_t^{q,K}) - \sigma_t^q\sigma_t^{\vartheta},$$

where σ_t^q is the volatility loading of $q_t := q_t^B + q_t^K$. Solving for $\mu_t^{q,K} - \mu_t^{q,B}$ yields

$$\mu_t^{q,K} - \mu_t^{q,B} = -\frac{\mu_t^\vartheta + \sigma_t^q \sigma_t^\vartheta}{1 - \vartheta_t}.$$

Putting the terms together, we obtain

$$\frac{\mathbb{E}_t[\mathrm{d}R_t^K(u_t, \iota_t)]}{\mathrm{d}t} - \frac{\mathbb{E}_t[\mathrm{d}R_t^B]}{\mathrm{d}t} = \frac{\rho - \check{s}_t - \mu_t^\vartheta - \sigma_t^q \sigma_t^\vartheta}{1 - \vartheta_t}. \tag{42}$$

Asset market clearing and homogeneity of individual portfolio choices implies $\theta_t^i = \theta_t = \vartheta_t$, so that the right-hand side of the portfolio choice condition (38) can be written as

$$\begin{aligned} &\left(\theta_t^i \sigma_t^{q,B} + (1 - \theta_t^i)\sigma_t^{q,K}\right)\left(\sigma_t^{q,K} - \sigma_t^{q,B}\right) + \left(1 - \theta_t^i\right)\tilde{\sigma}_t^2 \\ &= \underbrace{\left(\vartheta_t \sigma_t^{q,B} + (1 - \vartheta_t)\sigma_t^{q,K}\right)}_{=\sigma_t^q} \underbrace{\left(\sigma_t^{q,K} - \sigma_t^{q,B}\right)}_{=-\frac{\sigma_t^\vartheta}{1-\vartheta_t}} + (1 - \vartheta_t)\,\tilde{\sigma}_t^2. \end{aligned}$$

Substituting this expression and equation (42) into the portfolio choice condition (38) and multiplying by $1 - \vartheta_t$ yields

$$\rho - \check{s}_t - \mu_t^\vartheta - \sigma_t^q \sigma_t^\vartheta = -\sigma_t^q \sigma_t^\vartheta + (1 - \vartheta_t)^2 \tilde{\sigma}_t^2,$$

and, after canceling terms and solving for μ_t^ϑ,

$$\mu_t^\vartheta = \rho - \check{s}_t - (1 - \vartheta_t)^2 \tilde{\sigma}_t^2.$$

The bond valuation equation (22) stated in the main text follows by multiplying the previous equation by ϑ_t.

We make two remarks. First, while the previous derivation assumes $\vartheta_t, q_t^B \neq 0$ (otherwise, the geometric drift and volatility loadings are not well-defined), the ultimate equation (22) holds trivially also if $\vartheta_t = 0$. Therefore, this equation must hold in equilibrium even if bonds are worthless (i.e., $\mathcal{P}_t = \infty$) or in zero net supply (i.e., $\mathcal{B}_t = 0$). Second, the previous derivation is valid not just for our baseline model with sticky prices but also for the flexible price version (and even for the extended model with long-term bonds) because we have kept the volatility $\sigma_t^{q,B}$ of q_t^B flexible.

B Proofs [Online Appendix]

B.1 Propositions 1–3

The proofs of Propositions 1–3 follow almost directly from the model solution procedure outlined in Appendix A:

Proof of Proposition 1. Equations (16), (17), and (18) are identical to equations (37), (35), and (36), respectively. The remaining equation (19) follows from equation (38) by using that $\sigma_t^{q,B} = 0$ in the sticky price economy. □

Proof of Proposition 2. Equation (20) follows equation (39) derived in Appendix A if we substitute in $\kappa = \epsilon/\psi$ and $p^{R,flex} = a\frac{\epsilon-1}{\epsilon}$. The claim that the latter expression describes the equilibrium capital rental rate under flexible prices follows immediately from the optimal firm choice condition under flexible prices, equation (40). □

Proof of Proposition 3. Equation (22) has been derived at the end of Appendix A. □

B.2 Propositions 4 and 5

The proof of Propositions 4 and 5 builds on the BSDE arguments developed in Brunnermeier et al. (2024, Appendix B.1) to show that there is at most one nondegenerate stationary solution to the bond valuation equation (22). To apply their arguments, we make the following technical assumptions:

(A1) $\tilde{\sigma}_t$ follows a Markov diffusion process

$$\mathrm{d}\tilde{\sigma}_t = \mu_{\tilde{\sigma}}(\tilde{\sigma}_t)\mathrm{d}t + \sigma_{\tilde{\sigma}}(\tilde{\sigma}_t)\mathrm{d}Z_t$$

and it is ergodic with compact and positive support Σ;

(A2) $\check{s} : \Sigma \times [0,1] \to \mathbb{R}$ is continuous and the function

$$f : \Sigma \times [0,1] \to \mathbb{R}, f(\tilde{\sigma}, \vartheta) := (1-\vartheta)^2\tilde{\sigma}^2 - \rho + \check{s}(\tilde{\sigma}, \vartheta)$$

is strictly decreasing in ϑ for all $\tilde{\sigma} \in \Sigma$;

(A3) $\check{s}(\tilde{\sigma}, \vartheta) < \rho$ for all $(\tilde{\sigma}, \vartheta) \in \Sigma \times [0,1]$.

We remark that the monotonicity portion in (A2) is automatically satisfied if $\check{s}$ is a function of $\tilde{\sigma}$ only, the situation considered in Brunnermeier et al. (2024). To allow for state variables other than $\tilde{\sigma}_t$ to matter, we will also assume that $X_t = (\tilde{\sigma}_t, \hat{X}_t)$ is an ergodic Markov process with compact support $\mathbb{X}$. We refer to this modification of Assumption (A1) as Assumption (A1*).

Brunnermeier et al. (2024) first establish a number of properties of the *finite-horizon* BSDE

$$\vartheta_t = \mathbb{E}_t\left[\vartheta_T + \int_t^T \underbrace{\left((1-\vartheta_{t'})^2\tilde{\sigma}_{t'}^2 - \rho + \check{s}(\tilde{\sigma}_{t'}, \vartheta_{t'})\right)}_{=f(\tilde{\sigma}_{t'},\vartheta_{t'})} \vartheta_{t'}\mathrm{d}t'\right], \tag{43}$$

with a fixed terminal condition ϑ_T, which can be any $[0,1]$-valued random variable that is measurable with respect to the σ-field of information available at time T. Under our (and their) assumptions, this equation is sufficiently well-behaved that standard BSDE results are applicable. In particular, there is a unique solution for any terminal condition ϑ_T. Brunnermeier et al. (2024) then show that, in the limit $T \to \infty$, the solution does not depend on the terminal condition so long as the latter is strictly positive, and that the limit is the unique stationary non-degenerate solution to the bond valuation equation (22), if such a solution exists at all (see Proposition 15 below for a precise statement).

Unfortunately, the formal results in Brunnermeier et al. (2024) are not directly applicable here because they assume that $\check{s}$ only depends on $\tilde{\sigma}_t$, while we allow for feedback from the endogenous variable ϑ_t to $\check{s}_t$. However, an inspection of the arguments in that paper reveals that none of the proofs rely on the absence of feedback except in a single place in the proof of Lemma 3 of that paper. There, it is used that $f(\tilde{\sigma}, \vartheta)$ is strictly decreasing in ϑ for all $\tilde{\sigma} \in \Sigma$, which is trivially satisfied in their setting without feedback. Therefore, if we impose assumption (A2) as an additional technical requirement, then all arguments in Brunnermeier et al. (2024, Appendix B.1) go through. We can therefore conclude that the following proposition holds.

Proposition 15. *Suppose Assumptions (A1*), (A2), and (A3) are satisfied. Then, equation (22) has at most one stationary non-degenerate solution, that is a solution ϑ_t that can be represented in the form $\vartheta_t = \vartheta(X_t)$ with a continuous function $\vartheta : \mathbb{X} \to [0,1]$ that is not identically zero. If this solution exists, $\vartheta_t = \vartheta^*(X_t)$, then $\vartheta^*(X) > 0$ for all $X \in \mathbb{X}$.*

Furthermore, let $\hat{\vartheta} : \mathbb{X} \to (0,1)$ *be any other function and consider the unique solution* $\vartheta_t^{X,T}$ *of the finite-horizon problem* (43) *with initial state* $X_0 = X$ *and terminal condition* $\vartheta_T = \hat{\vartheta}(X_T)$*. Then the solution converges as the horizon is made large and*

$$\lim_{T\to\infty} \vartheta_0^{X,T} = \begin{cases} \vartheta^*(X), & \text{a non-degenerate stationary solution exists} \\ 0, & \text{otherwise} \end{cases}.$$

Proof. See Brunnermeier et al. (2024, Proposition 8).[43] □

With the help of the previous result, we can prove the two propositions stated in the main text:

Proof of Proposition 4. Our equilibrium definition restricts attention to Markov equilibria, so that we do not need to consider non-stationary solutions for ϑ_t. Now, consider any stationary solution, that is $\vartheta_t = \hat{\vartheta}(X_t)$ for some continuous function ϑ and some ergodic Markov process X_t with compact support. We show that then we can also write ϑ_t as $\vartheta_t = \vartheta(\tilde{\sigma}_t)$ as a function of $\tilde{\sigma}_t$ only.

For a proof, assume without loss of generality (w.l.o.g.) that X_t includes $\tilde{\sigma}_t$ as its first component. By Proposition 15, the solution $\vartheta_t = \hat{\vartheta}(X_t)$ is then either degenerate, i.e., identically zero, or the unique non-degenerate solution. In the former case, we can define $\vartheta(\tilde{\sigma}) = 0$ and are done. So let us suppose we are in the latter case and let $\vartheta^*(X) = \vartheta^*(\tilde{\sigma}, \hat{X})$ be the unique non-degenerate stationary solution.

By Proposition 15, $\vartheta^*(X) = \lim_{T\to\infty} \vartheta_0^{X,T}$, where $\vartheta_t^{X,T} = \vartheta_t^{\tilde{\sigma},\hat{X},T}$ denotes the unique solution to the finite-horizon BSDE (43) with initial condition $\tilde{\sigma}_0 = \tilde{\sigma}, \hat{X}_0 = \hat{X}$ and terminal condition $\vartheta_T = 1/2$. This finite-horizon solution $\vartheta_t^{\tilde{\sigma},\hat{X},T}$ must, in fact, be independent of $\hat{X}$ because the driver of the BSDE only depends on $\tilde{\sigma}_t$, the evolution of $\tilde{\sigma}_t$ is fully determined by the initial condition $\tilde{\sigma}_0 = \tilde{\sigma}$, and the terminal condition $1/2$ does not depend on the state. Therefore,

$$\vartheta^*(\tilde{\sigma}, \hat{X}) = \lim_{T\to\infty} \vartheta_0^{\tilde{\sigma},\hat{X},T}$$

[43] As remarked previously, the monotonicity condition on f does not appear in that paper and has been added, so that Lemma 3 in Brunnermeier et al. (2024) remains valid even when $\check{s}_t$ depends on ϑ_t. Instead, the first sentence of the proposition in that paper contains additional assumptions that follow immediately from our assumptions on X_t and $\check{s}(\tilde{\sigma}, \vartheta)$ made above. Beyond these differences, the previous proposition is precisely Proposition 8 in Brunnermeier et al. (2024).

must also be constant in the $\hat{X}$-dimension, so that we can define without ambiguity $\vartheta(\tilde{\sigma}) := \vartheta^*(\tilde{\sigma}, \hat{X})$.

For the remaining claims made in Proposition 4, note that the model parameters ϵ and ψ as well as the model variables π_t, u_t, and i_t do not enter the finite-horizon BSDE (43). The previous argument therefore also implies that $\vartheta(\tilde{\sigma})$ must be independent of these variables.

□

Proof of Proposition 5. We first prove an auxiliary result for the comparison of solutions to certain finite-horizon BSDEs. Specifically, suppose we have two stochastic paths $\tilde{\sigma}_t^1$, $\tilde{\sigma}_t^2$ over $[0, T]$ with the property that $\tilde{\sigma}_t^1 < \tilde{\sigma}_t^2$ for all $t \in [0, T]$ and that $0 < \vartheta_T^1 \leq \vartheta_T^2 < 1$ are two terminal conditions. Consider the following two BSDEs:

$$\begin{aligned} \mathbb{E}_t[d\vartheta_t] &= -\overbrace{f(\tilde{\sigma}_t^1, \vartheta_t)\vartheta_t}^{=:g_t^1(\vartheta_t)}\,\mathrm{d}t & \vartheta_T, &= \vartheta_T^1 \\ \mathbb{E}_t[d\vartheta_t] &= -\underbrace{f(\tilde{\sigma}_t^2, \vartheta_t)\vartheta_t}_{=:g_t^2(\vartheta_t)}\,\mathrm{d}t & \vartheta_T &= \vartheta_T^2. \end{aligned}$$

Let $\vartheta_t^1, \vartheta_t^2$ be the unique solutions to these BSDEs. By Brunnermeier et al. (2024, Lemma 2), these solutions are contained in the interval $(0, 1)$. Under the assumptions of the proposition, f is strictly increasing in $\tilde{\sigma}$ and therefore $g_t^1(\vartheta_t^1) < g_t^2(\vartheta_t^1)$ for all $t \in [0, 1]$. Because $\vartheta_T^1 \leq \vartheta_T^2$ by assumption, the comparison theorem for BSDEs implies that $\vartheta_0^1 < \vartheta_0^2$.

Let us now apply this result to the finite-horizon BSDE (43). Denote by $\vartheta_t^{\tilde{\sigma},T}$ the unique solution to that BSDE with initial condition $\tilde{\sigma}_0 = \tilde{\sigma}$ and terminal condition $\vartheta_T = 1/2$. By the previous result, if $\tilde{\sigma}^1 < \tilde{\sigma}^2$, then $\vartheta_0^{\tilde{\sigma}^1,T} < \vartheta_0^{\tilde{\sigma}^2,T}$. Using the limit result from Proposition 15, then

$$\vartheta(\tilde{\sigma}^1) = \lim_{T\to\infty} \vartheta_0^{\tilde{\sigma}^1,T} \leq \lim_{T\to\infty} \vartheta_0^{\tilde{\sigma}^2,T} = \vartheta(\tilde{\sigma}^2),$$

which proves that $\vartheta(\tilde{\sigma})$ is weakly increasing.

To see that it is, in fact, even strictly increasing, use the previous finite-horizon result once more, but now fix $T < \infty$ and consider, for each initial condition $\tilde{\sigma}_0 = \tilde{\sigma}$, the unique solution $\vartheta_t^{\tilde{\sigma}}$ to the finite-horizon BSDE with terminal condition $\vartheta_T = \vartheta(\tilde{\sigma}_T)$. We note that, by assumption of the proposition, bonds are valued so that $\vartheta_T \in (0, 1)$ (the

upper bound follows from Proposition 15). We can therefore apply our auxiliary result once again and conclude that $\vartheta_0^{\tilde{\sigma}^1} < \vartheta_0^{\tilde{\sigma}^2}$ with strict inequality for any $\tilde{\sigma}^1 < \tilde{\sigma}^2$. But, of course, $\vartheta_0^{\tilde{\sigma}} = \vartheta(\tilde{\sigma})$, which completes the proof. □

B.3 Propositions in Section 3.2.2

Proof of Proposition 6. Combining equation (25) with the condition $p_t^R = p^{R,flex}$ under flexible prices and solving for $q_t := q_t^B + q_t^K$ yields

$$q_t = \frac{p^{R,flex}}{\rho u_t^{\varphi}}.$$

From the optimal investment condition of households, equation (17), the investment rate is

$$\iota_t = \frac{q_t^K - 1}{\phi} = \frac{(1-\vartheta_t)q_t}{\phi} - \frac{1}{\phi} = \frac{1-\vartheta_t}{\phi}\frac{p^{R,flex}}{\rho u_t^{\varphi}} - \frac{1}{\phi},$$

where the last equality follows from the previous equation. Substituting both equations into goods market clearing (24) yields

$$\rho\frac{p^{R,flex}}{\rho u_t^{\varphi}} + \frac{1-\vartheta_t}{\phi}\frac{p^{R,flex}}{\rho u_t^{\varphi}} - \frac{1}{\phi} = a u_t \quad \Leftrightarrow \quad \left(1 + \frac{1-\vartheta_t}{\phi\rho}\right) p^{R,flex} = a u_t^{1+\varphi} + \frac{u_t^{\varphi}}{\phi}.$$

The left-hand side of this equation does not depend on u_t, and the right-hand side is strictly increasing in u_t, so that there is at most one solution. It is evident that there is also at least one and that the unique solution is nonincreasing in ϑ_t. It follows then immediately that $q_t = p^{R,flex}/(\rho u_t^{\varphi})$ is nondecreasing in ϑ_t and therefore $q_t^B = \vartheta_t q_t$ is strictly increasing in ϑ_t.

We next show that q_t^K is strictly decreasing in ϑ_t. To see this, note that the previous equations also imply that

$$q_t = \frac{1 + \phi a u_t}{1 - \vartheta_t + \phi\rho}.$$

Hence, $q_t^K = (1-\vartheta_t)q_t$ can be written as

$$q_t^K = \frac{1-\vartheta_t}{1-\vartheta_t+\phi\rho} \cdot (1 + \phi a u_t).$$

This is a product of two positive factors, the first is strictly decreasing in ϑ_t, and the second is nonincreasing because u_t is. Hence, q_t^K must be strictly decreasing in ϑ_t.

The claim in the proposition regarding q_t^B and q_t^K follows now from the standing assumption that the conditions of Propositions 4 and 5 are satisfied, so that ϑ_t is increasing in $\tilde{\sigma}_t$, compare Proposition 5. Finally, the claim about the price level $\mathcal{P}_t$ follows from the definition $q_t^B = \frac{\mathcal{B}_t}{\mathcal{P}_t K_t}$ and the fact that $\mathcal{B}_t$ and K_t do not load on the shock $\mathrm{d}Z_t$ that drives $\tilde{\sigma}_t$. □

Proof of Proposition 7. Use

$$q_t = \frac{1}{\vartheta_t} q_t^B, \qquad q_t^K = \frac{1-\vartheta_t}{\vartheta_t} q_t^B, \qquad \iota_t = \frac{q_t^K - 1}{\phi} = \frac{(1-\vartheta_t) q_t^B - \vartheta_t}{\phi \vartheta_t}$$

and substitute these expressions into equation (27), which holds under both sticky and flexible prices:

$$u_t = \frac{1}{a}\left(\frac{\rho}{\vartheta_t} q_t^B + \frac{(1-\vartheta_t) q_t^B - \vartheta_t}{\phi \vartheta_t}\right).$$

Clearly, the right-hand side is strictly decreasing in ϑ_t for fixed q_t^B. Applying Propositions 4 and 5 implies that then u_t is also strictly decreasing in $\tilde{\sigma}_t$ for fixed q_t^B. This proves the first part of the proposition.

For the second part, take the ratio of the previous formula under sticky and flexible prices (using that, by Proposition 4, $\vartheta_t = \vartheta_t^{flex}$):

$$\frac{u_t}{u_t^{flex}} = \frac{(1-\vartheta_t+\phi\rho) q_t^B - \vartheta_t}{(1-\vartheta_t+\phi\rho) q_t^{B,flex} - \vartheta_t}.$$

The derivative with respect to $\tilde{\sigma}_t$ is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tilde{\sigma}_t}\left(\frac{u_t}{u_t^{flex}}\right) &= -\frac{1+q_t^B}{(1-\vartheta_t+\phi\rho) q_t^{B,flex} - \vartheta_t}\left(1 - \frac{1+q_t^{B,flex}}{1+q_t^B}\frac{(1-\vartheta_t+\phi\rho) q_t^B - \vartheta_t}{(1-\vartheta_t+\phi\rho) q_t^{B,flex} - \vartheta_t}\right)\frac{\mathrm{d}\vartheta_t}{\mathrm{d}\tilde{\sigma}_t} \\
&\quad - \frac{1-\vartheta_t-\phi\rho}{(1-\vartheta_t+\phi\rho) q_t^{B,flex} - \vartheta_t}\frac{(1-\vartheta_t+\phi\rho) q_t^B - \vartheta_t}{(1-\vartheta_t-\phi\rho) q_t^{B,flex} - \vartheta_t}\frac{\mathrm{d}q_t^{B,flex}}{\mathrm{d}\tilde{\sigma}_t} \\
&= -\frac{1+q_t^B}{(1-\vartheta_t+\phi\rho) q_t^{B,flex} - \vartheta_t}\left(1 - \frac{1+q_t^{B,flex}}{1+q_t^B}\frac{u_t}{u_t^{flex}}\right)\frac{\mathrm{d}\vartheta_t}{\mathrm{d}\tilde{\sigma}_t} \\
&\quad - \frac{1-\vartheta_t+\phi\rho}{(1-\vartheta_t+\phi\rho) q_t^{B,flex} - \vartheta_t}\frac{u_t}{u_t^{flex}}\frac{\mathrm{d}q_t^{B,flex}}{\mathrm{d}\tilde{\sigma}_t}.
\end{aligned}$$

By Proposition 5, $\frac{\mathrm{d}\vartheta_t}{\mathrm{d}\tilde{\sigma}_t} > 0$, and, by Proposition 6, $\frac{\mathrm{d}q_t^{B,flex}}{\mathrm{d}\tilde{\sigma}_t} > 0$. Hence, the second term is always strictly negative, and the first term is at least nonpositive if $\frac{1+q_t^{B,flex}}{1+q_t^B}\frac{u_t}{u_t^{flex}} \leq 1$. For $q_t^B \approx q_t^{B,flex}$, the derivative is therefore strictly negative.

□

Proof of Proposition 8. By definition of ϑ_t and Ito's formula,

$$\sigma_t^\vartheta = (1-\vartheta_t)(\sigma_t^{q,B} - \sigma_t^{q,K}) \Rightarrow \sigma_t^{q,K} = \sigma_t^{q,B} - \frac{\sigma_t^\vartheta}{1-\vartheta_t}.$$

This formula holds under both flexible and sticky prices. Furthermore, by separation of portfolio choice (Proposition 4), the process $\frac{\sigma_t^\vartheta}{1-\vartheta_t}$ is independent of the degree of price stickiness. Under flexible prices, we have also established that q_t^B and ϑ_t move in the same direction after a dZ_t-shock whereas q_t^K moves in the opposite direction (Propositions 5 and 6). Hence, we must have

$$\operatorname{sgn}(\sigma_t^{q,K,flex}) = -\operatorname{sgn}(\sigma_t^\vartheta)$$

and

$$|\sigma_t^{q,K,flex}| < \frac{|\sigma_t^\vartheta|}{1-\vartheta_t}.$$

The assertion follows from the fact that, under sticky prices, $\sigma_t^{q,B} = 0$, so that $\sigma_t^{q,K} = -\frac{\sigma_t^\vartheta}{1-\vartheta_t}$. □

We remark that the previous proof also shows that the instantaneous volatility $\sigma_t^{q,K}$ is independent of the degree of price stickiness but takes on a strictly lower value under flexible prices. Therefore, $\sigma_t^{q,K}$ is discontinuously in the limit $\psi \to 0$ (or $\kappa \to \infty$).

B.4 Proposition 9 and Corollary 1

We remark that the formulation of Proposition 9 includes the implicit assumption that uncertainty shocks also exist, i.e., that there is a state $\tilde{\sigma} \in \Sigma$ such that $\sigma_{\tilde{\sigma}}(\tilde{\sigma}) \neq 0$. We will make use of this property in the proof of the following lemma:[44]

[44]The proof of the following lemma is technically involved. But note that this is only required to encompass the case $\alpha > 1$. For $\alpha \leq 1$, it is easy to show that $(q^B)'(\tilde{\sigma}_t) > 0$ by applying Proposition 6. Then, Ito's lemma and the previous remark immediately imply the assertion.

Lemma 2. *Suppose the assumptions of Proposition 9 hold. For any equilibrium that features a constant utilization gap u_t/u_t^{flex}, it is necessarily the case that $\sigma_t^{q,B} \neq 0$ with positive probability.*

Proof. Suppose that, in equilibrium, $u_t = \alpha u_t^{flex}$ with some constant $\alpha > 0$. Then,

$$q_t^B = \vartheta_t \frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho} = \vartheta_t \frac{1+\phi a \alpha u_t^{flex}}{1-\vartheta_t+\phi\rho}. \tag{44}$$

Recall from the proof of Proposition 6 that u_t^{flex} solves the equation

$$\frac{1-\vartheta_t+\phi\rho}{\rho} p^{R,flex} = \phi a (u_t^{flex})^{1+\varphi} + (u_t^{flex})^{\varphi}.$$

Consider two cases separately.

First, if $\phi \to \infty$, so that there is no physical investment in the model, then the previous equation implies that $u_t^{flex} = u^{flex}$ is constant over time. In this case, equation (44) represents a strictly increasing relationship between ϑ_t and q_t^B, so we can conclude that $q_t^B = q^B(\tilde{\sigma}_t)$ with a strictly increasing function q^B (because $\vartheta_t = \vartheta(\tilde{\sigma}_t)$ with $\vartheta' > 0$ by Propositions 4 and 5). By Ito's lemma,

$$\sigma_t^{q,B} = \frac{(q^B)'(\tilde{\sigma}_t)}{q^B(\tilde{\sigma}_t)} \sigma_{\tilde{\sigma}}(\tilde{\sigma}_t).$$

We have just established that the first factor on the right-hand side is strictly positive. In addition, by the remark made prior to this proof, there must be some state $\tilde{\sigma}_t$ in the ergodic set for which $\sigma_{\tilde{\sigma}}(\tilde{\sigma}_t) \neq 0$. So with positive probability, $\sigma_t^{q,B} \neq 0$.

Second, consider the case $\phi < \infty$. Then, the equation that characterizes u_t^{flex} represents a strictly decreasing relationship between u_t^{flex} and ϑ_t. Solving for ϑ_t yields

$$\vartheta_t = 1 + \phi\rho - \left(\phi a (u_t^{flex})^{1+\varphi} + (u_t^{flex})^{\varphi}\right) \frac{\rho}{p^{R,flex}}.$$

Substituting this into equation (44) yields

$$\begin{aligned} q_t^B &= \left(1 + \phi\rho - \left(\phi a (u_t^{flex})^{1+\varphi} + (u_t^{flex})^{\varphi}\right) \frac{\rho}{p^{R,flex}}\right) \frac{1+\phi\alpha a u_t^{flex}}{\phi a (u_t^{flex})^{1+\varphi} + (u_t^{flex})^{\varphi}} \frac{p^{R,flex}}{\rho} \\ &= \frac{1+\phi\rho}{(u_t^{flex})^{\varphi}} \frac{1+\phi\alpha a u_t^{flex}}{1+\phi a u_t^{flex}} \frac{p^{R,flex}}{\rho} - \left(1+\phi\alpha a u_t^{flex}\right) \\ &= \frac{\left(1+\phi\rho\right) p^{R,flex} - (u_t^{flex})^{\varphi}\left(1+\phi a u_t^{flex}\right)\rho}{(u_t^{flex})^{\varphi}\left(1+\phi a u_t^{flex}\right)\rho} \left(1+\phi\alpha a u_t^{flex}\right). \end{aligned}$$

Both the numerator and denominator of the last expression are non-constant linear combinations of powers of u_t^{flex}. It has therefore nonzero derivative with respect to u_t^{flex} almost everywhere. But then the first three factors on the right-hand side of the following expression are all nonzero for almost all exogenous states $\tilde{\sigma}_t$:

$$\sigma_t^{q,B} q_t^B = \frac{\partial q_t^B}{\partial u_t^{flex}} \frac{\partial u_t^{flex}}{\partial \vartheta_t} \frac{\partial \vartheta_t}{\partial \tilde{\sigma}_t} \sigma_{\tilde{\sigma}}(\tilde{\sigma}_t).$$

Because there is a state $\tilde{\sigma}_t$ for which $\sigma_{\tilde{\sigma}}(\tilde{\sigma}_t) \neq 0$ – and then this must be the case on a whole neighborhood around $\tilde{\sigma}_t$ – we can find a positive measure subset of Σ on which both the first three factors are all nonzero and $\sigma_{\tilde{\sigma}}(\tilde{\sigma}_t) \neq 0$. Therefore, $\sigma_t^B \neq 0$ with positive probability. □

With the help of the previous lemma, the proof of Proposition 9 is straightforward. We remark that it only uses that $\sigma_t^{q,B} \neq 0$ with positive probability and, therefore, any other allocation that requires this can also not be implemented with interest rate policy.

Proof of Proposition 9. Suppose otherwise and consider an equilibrium in the sticky price economy for which u_t / u_t^{flex} is constant. By Lemma 2, there must then be states that occur with positive probability in which $\sigma_t^{q,B} \neq 0$. This contradicts equation (15).

□

Proof of Corollary 1. We first remark that it is not possible to implement an equilibrium for which both u_t / u_t^{flex} is constant and $\mathcal{P}_t^B = 1$ throughout. This follows directly from Proposition 9 because, if $\mathcal{P}_t^B = 1$ throughout, the long-term bond model collapses to the baseline model with short-term bonds covered by that proposition. From now on, we can therefore assume that $\mathcal{P}_t^B \neq 1$ with positive probability.

The proof follows the intuition from the main text that using the bond price $\mathcal{P}_t^B$ to match the level of q_t^B is inconsistent with the requirements on the nominal interest rate to achieve a constant output gap, except for at most a single value of λ.

Suppose that there is an interest rate policy that implements $u_t = \alpha u_t^{flex}$ for a given α. We first establish two necessary requirements on the bond price $\mathcal{P}_t^B$ for this to be possible.

First, it must then be that

$$q_t^B = \frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho} = \frac{1+\phi a \alpha u_t^{flex}}{1-\vartheta_t+\phi\rho} =: \bar{q}^B(\tilde{\sigma}_t),$$

where the last expression only depends on $\tilde{\sigma}_t$ but not on $q_t^{B,0}$ because this is true for both u_t^{flex} and ϑ_t (for the latter by Proposition 4). Combining this condition with equation (33) and solving for $\mathcal{P}_t^B = \frac{\lambda}{\lambda+i_t}$ implies

$$\mathcal{P}^B(\tilde{\sigma}, q^{B,0}) = \frac{\bar{q}^B(\tilde{\sigma}_t)}{q^{B,0}}. \tag{45}$$

Second, if $q_t^B = \bar{q}^B(\tilde{\sigma}_t)$ and u_t are functions of $\tilde{\sigma}_t$ only, then so are $q_t^K = \frac{1-\vartheta_t}{\vartheta_t} q_t^B$, $\iota_t = \frac{q_t^K - 1}{\phi}$, $g_t = \Phi(\iota_t) - \delta$, $q_t = q_t^B + q_t^K$, $p_t^R = \rho u_t^{\varphi} q_t$, and $\frac{Y_t}{C_t} = \frac{a u_t}{\rho q_t}$. Then, by equation (11),

$$r_t^B := \frac{\mathbb{E}_t[\mathrm{d}R_t^B]}{\mathrm{d}t} = \check{s}_t + g_t + \mu_t^{q,B} = \bar{r}^B(\tilde{\sigma}_t) \tag{46}$$

is a also function of $\tilde{\sigma}_t$ only. Furthermore, integrating the Phillips curve (equation (20)) forward yields

$$\Pi_t := \pi_t \frac{Y_t}{C_t} = \mathbb{E}_t\left[\int_t^\infty e^{-\rho t'} \kappa \frac{(p_{t'}^R - p^{R,flex}) u_{t'}}{\rho q_{t'}} \mathrm{d}t'\right]$$

and the conditional expectation on the right is a function of $\tilde{\sigma}_t$ only. Therefore, also inflation must be a function of $\tilde{\sigma}_t$ only,

$$\pi_t = \bar{\pi}(\tilde{\sigma}_t). \tag{47}$$

We now show that equation (46) and (47) imply also $\mathcal{P}_t^B$ cannot depend on $q_t^{B,0}$. To see this, note that, by equation (49) (under sticky prices)

$$r_t^B = \mu_t^{\mathcal{P},B} - \pi_t + i_t.$$

Using $i_t = \lambda\left(\frac{1}{\mathcal{P}_t^B} - 1\right)$ and solving for $\mathbb{E}_t[\mathrm{d}\mathcal{P}_t^B]$ yields the BSDE

$$\mathbb{E}_t[\mathrm{d}\mathcal{P}_t^B] = \left(\left(r_t^B + \pi_t + \lambda\right)\mathcal{P}_t^B - \lambda\right)\mathrm{d}t.$$

Solving this forward in time and using equations (46) and (47) yields

$$\mathcal{P}_t^B = \lambda \mathbb{E}_t\left[\int_t^\infty \exp\left(-\int_t^{t'}\left(\bar{r}^B(\tilde{\sigma}_{t''}) + \bar{\pi}(\tilde{\sigma}_{t''}) + \lambda\right)\mathrm{d}t''\right)\mathrm{d}t'\right].$$

If the expression on the right does not converge to a finite value in some positive probability state, then it is impossible to implement the desired allocation with any interest rate policy, and there is nothing more to show. We can therefore assume that the expression does converge in all states. In this case, we can conclude that

$$\mathcal{P}_t^B = \bar{\mathcal{P}}^B(\tilde{\sigma}_t) \tag{48}$$

for some function $\bar{\mathcal{P}}^B$ because information about $q_t^{B,0}$ is redundant for forecasting the expression inside the conditional expectations operator.

The two properties (45) and (48) can only be satisfied simultaneously for all states $(\tilde{\sigma}_t, q_t^{B,0})$ that occur in equilibrium if $q_t^{B,0}$ is constant over time. By the law of motion for $q_t^{B,0}$, this requires

$$0 = i_t - \check{s}_t - \pi_t - g_t = \lambda\left(\frac{1}{\mathcal{P}_t^B} - 1\right) - \check{s}_t - \pi_t - g_t \Rightarrow \lambda\left(\frac{1}{\mathcal{P}_t^B} - 1\right) = \check{s}_t + \pi_t + g_t.$$

In any state in which $\mathcal{P}_t^B \neq 1$, we obtain the requirement

$$\lambda = \bar{\lambda}_t := \left(\frac{1}{\mathcal{P}_t^B} - 1\right)^{-1}\left(\check{s}_t + \pi_t + g_t\right).$$

Note that none of the expressions on the right-hand side depend on λ (for $\mathcal{P}_t^B$, this follows from property (45), for all other variables, this is clear). Therefore, this condition is satisfied for only a single value of λ. We conclude that, if there is a state in which $\mathcal{P}_t^B \neq 1$, interest rate policy can implement $u_t = \alpha u_t^{flex}$ at most for a single value of λ. That $\mathcal{P}_t^B = 1$ throughout is impossible has already been argued at the beginning of this proof. □

We close this section with a remark on the non-generic values for λ for which implementation is possible. Note that, at least for the case that $\mathcal{P}_t^B = 1$ never occurs, the previous argument shows that implementation can be possible for some value of λ if $\bar{\lambda}_t$ is constant in equilibrium. Typically, this will not be the case, so that, under additional parameter restrictions, we could strengthen our conclusion by striking the word "generically" in Corollary 1. For example, if $\phi \to \infty$, so that there is no investment, then one can show that π_t and g_t are constant. If we assume, in addition, that $\check{s}_t$ is constant, then it is easy to show that $\bar{\lambda}_t$ cannot be time-invariant. One can find many other

special cases in which a time-invariant $\bar{\lambda}_t$ can be ruled out. However, we have not been able to rule this out in general.

B.5 Proposition 10

The proposition asserts that for $i_t = r_t^n$ there is an equilibrium in the sticky price economy in which $\pi_t = 0$ and all real variables are at their flexible price values. It does not assert that this is the only equilibrium consistent with this policy rule, compare the clarifying footnote in the main text.

To prove the proposition, we only need to show that the Phillips curve equation (20) is satisfied and that $r_t = r_t^n$ is indeed the real interest rate in the flexible price allocations, which can be verified by checking that the capital pricing condition (32) is satisfied for $i_t = r_t^n$ and $\pi_t = 0$. Because all other equilibrium conditions are the same under flexible and sticky prices, and do not involve either π_t or i_t, these conditions are automatically satisfied if all real variables are at their flexible price values.

For the Phillips curve, substitute $p_t^R = p^{R,flex}$, $\pi_t = 0$, into the equation. Then,

$$\begin{aligned}\mathbb{E}_t\left[\mathrm{d}\left(\pi_t \frac{Y_t}{C_t}\right)\right] &= \mathbb{E}_t\left[\mathrm{d}\left(0 \cdot \frac{Y_t}{C_t}\right)\right] = 0 \\ &= \left(\rho \cdot 0 - \kappa \frac{p^{R,flex} - p^{R,flex}}{a}\right)\frac{Y_t}{C_t}\mathrm{d}t \\ &= \left(\rho\pi_t - \kappa \frac{p_t^R - p^{R,flex}}{a}\right)\frac{Y_t}{C_t}\mathrm{d}t,\end{aligned}$$

so equation (20) indeed holds.

Regarding equation (32), note first that $q_t^K = q^{K,flex}$ is constant over time because $\vartheta_t = 0$ is.[45] This fact implies $\mu_t^{q,K} = \sigma_t^{q,K} = 0$. Substituting these expressions and $g_t = g_t^{flex}$ into equation (32) and solving for $r_t = i_t - \pi_t$ yields

$$r_t = i_t - \pi_t = \rho + g_t^{flex} - \tilde{\sigma}_t^2 = r_t^n.$$

For $i_t = r_t^n$, $\pi_t = 0$, this equation is indeed satisfied.

[45] Recall from the proof of Proposition 6 that, under flexible prices, there is a functional relationship between u_t and ϑ_t and therefore constant ϑ_t implies constant u_t. The goods market clearing equation (31) then implies that also q_t^K must be constant.

B.6 Proposition 11

See Appendix C, where we provide formal details on the long-term bond model and explain how the model solution changes and how results from the baseline model can be transferred to the long-term bond model. In the context of this appendix, we provide a proof of all statements made in Proposition 11.

B.7 Formal Results in Section 6

Proof of Proposition 12. We first compute the expected utility V_0^i of an individual agent. To do so, let us first justify the following mathematical result: for any Ito process X_t satisfying $\mathrm{d}X_t/X_t = \mu_t^X \mathrm{d}t + \sigma_t^X \mathrm{d}Z_t + \tilde{\sigma}_t^X \mathrm{d}\tilde{Z}_t$,

$$\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\log X_t \mathrm{d}t\right] = \frac{1}{\rho}\log X_0 + \frac{1}{\rho}\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\left(\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2\right)\mathrm{d}t\right].$$

To show this, apply Ito's lemma to $\log X_t$, which implies

$$\mathrm{d}\log X_t = \left[\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2\right]\mathrm{d}t + \sigma_t^X \mathrm{d}Z_t + \tilde{\sigma}_t^X \mathrm{d}\tilde{Z}_t.$$

Writing this in integral form and taking expectations,

$$\mathbb{E}_0[\log X_t] = \log X_0 + \int_0^t \mathbb{E}_0\left[\mu_s^X - \frac{1}{2}(\sigma_s^X)^2 - \frac{1}{2}(\tilde{\sigma}_s^X)^2\right]\mathrm{d}s + \underbrace{\mathbb{E}_0\left[\int_0^t \sigma_s^X \mathrm{d}Z_s\right] + \mathbb{E}_0\left[\int_0^t \tilde{\sigma}_s^X \mathrm{d}\tilde{Z}_s\right]}_{=0}.$$

Hence,

$$\begin{aligned}
\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\log X_t \mathrm{d}t\right] &= \int_0^\infty e^{-\rho s}\mathbb{E}_0\left[\log X_t\right]\mathrm{d}s \\
&= \int_0^\infty e^{-\rho s}\log X_0 \mathrm{d}s + \int_0^\infty e^{-\rho s}\int_0^s \mathbb{E}_0\left[\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2\right]\mathrm{d}t\mathrm{d}s \\
&= \frac{1}{\rho}\log X_0 + \int_0^\infty \int_t^\infty e^{-\rho s}\mathrm{d}s\mathbb{E}_0\left[\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2\right]\mathrm{d}t \\
&= \frac{1}{\rho}\log X_0 + \frac{1}{\rho}\mathbb{E}_0\left[\int_0^\infty e^{-\rho t}\left(\mu_t^X - \frac{1}{2}(\sigma_t^X)^2 - \frac{1}{2}(\tilde{\sigma}_t^X)^2\right)\mathrm{d}t\right]
\end{aligned}$$

We now use $\log c_t^i = \log(au_t - \iota_t) + \log K_t + \log \eta_t^i$, where $\eta_t^i := n_t^i/(\int n_t^j \mathrm{d}j)$ is the wealth share of household i, compare equation (34) in the main text. Applying the

previous result for both $X_t = K_t$ and $X_t = \eta_t^i$ and using that K_t has zero risk terms whereas η_t^i has zero drift and aggregate risk, we obtain for individual expected utility

$$\begin{aligned} V_0^i &= \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log c_t^i - \frac{u_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t \right] \\ &= \frac{\log K_0 + \log \eta_0^i}{\rho} + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log \left(a u_t - \iota_t \right) + \frac{\mu_t^K}{\rho} - \frac{(\tilde{\sigma}_t^{\eta,i})^2}{2\rho} - \frac{u_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t \right]. \end{aligned}$$

Next, plug in $a u_t - \iota_t = \rho q_t = \rho \frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho}$, $\mu_t^K = \Phi(\iota_t) - \delta = \frac{1}{\phi} \log \left((1-\vartheta_t) \frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho} \right) - \delta$, $\tilde{\sigma}_t^{\eta,i} = (1-\vartheta_t)\tilde{\sigma}_t$ and rearrange,

$$\begin{aligned} V_0^i = & \frac{\log K_0 + \log \eta_0^i + \log \rho}{\rho} - \frac{\delta}{\rho^2} \\ & + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\log \left(\frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho} \right) + \frac{1}{\phi\rho} \log \left((1-\vartheta_t) \frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho} \right) - \frac{(1-\vartheta_t)^2 \tilde{\sigma}_t^2}{2\rho} - \frac{u_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t \right] \\ = & \frac{\log K_0 + \log \eta_0^i + \log \rho}{\rho} - \frac{\delta}{\rho^2} \\ & + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\frac{1+\phi\rho}{\phi\rho} \log \left(\frac{1+\phi\rho}{1-\vartheta_t+\phi\rho} \right) + \frac{1}{\phi\rho} \log (1-\vartheta_t) - \frac{(1-\vartheta_t)^2 \tilde{\sigma}_t^2}{2\rho} \right) \mathrm{d}t \right] \\ & + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\frac{1+\phi\rho}{\phi\rho} \log \left(\frac{1+\phi a u_t}{1+\phi\rho} \right) - \frac{u_t^{1+\varphi}}{1+\varphi} \right) \mathrm{d}t \right] \\ = & \underbrace{\frac{\log K_0 + \log \eta_0^i + \log \rho}{\rho} - \frac{\delta}{\rho^2}}_{=: V_0^{(0),i}} + \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(\mathcal{W}_{\vartheta,t}(\vartheta_t) + \mathcal{W}_u(u_t) \right) \mathrm{d}t \right]. \end{aligned}$$

The term $V_0^{(0),i}$ captures initial conditions which are independent of policy choices. This term is therefore irrelevant to welfare assessments of different policies. The remaining term, $V_0^i - V_0^{(0),i}$ is identical for all households and corresponds to the objective stated in the proposition. Hence, so long as the social welfare function is strictly increasing in individual welfare, maximizing the welfare function is equivalent to maximizing the objective stated in the proposition.

Now consider a constrained planner who can directly control u_t and ϑ_t but has to respect the competitive equilibrium relationships implicit in the functions $\mathcal{W}_{\vartheta,t}$ and $\mathcal{W}_u$.

This planner finds it optimal to choose u_t and ϑ_t each period in a way that maximizes $\mathcal{W}_u(u_t)$ and $\mathcal{W}_{\vartheta,t}(\vartheta_t)$, respectively.

Maximizing $\mathcal{W}_u(u_t)$ leads to a time-invariant optimal choice:

$$0 = \mathcal{W}_u'(u_t) = \frac{1+\phi\rho}{1+\phi a u_t}\frac{a}{\rho} - u_t^{\varphi} \Leftrightarrow \frac{a}{\rho} = u_t^{\varphi}\frac{1+\phi a u_t}{1+\phi\rho}.$$

While this has in general no closed-form solution, except in the limit $\phi \to \infty$, it is easy to see that there is a unique solution and that it is time-invariant.

Maximizing $\mathcal{W}_{\vartheta,t}(\vartheta_t)$ leads to the first-order condition

$$\begin{aligned} & 0 = \mathcal{W}_{\vartheta,t}'(\vartheta_t) = \frac{1+\phi\rho}{\phi\rho}\frac{1}{1-\vartheta_t+\phi\rho} - \frac{1}{\phi\rho}\frac{1}{1-\vartheta_t} + \frac{(1-\vartheta_t)\tilde{\sigma}_t^2}{\rho} \\ \Leftrightarrow \quad & 0 = (1-\vartheta_t)^3\tilde{\sigma}_t^2 + \phi\rho\,(1-\vartheta_t)^2\tilde{\sigma}_t^2 + \rho\,(1-\vartheta_t) - \rho \end{aligned}$$

The right-hand side of the last equation is a third-order polynomial in $1-\vartheta_t$, which means the equation has, in principle, up to three real solutions. However, because all coefficients on terms of positive order are positive and the coefficient on the zero order term is negative, well-known results on polynomial roots imply that there is a unique positive solution $1-\vartheta_t^* > 0$. It is easy to see that this solution must also satisfy $\vartheta_t^* > 0$ and that it is indeed the maximizer of $\mathcal{W}_{\vartheta,t}(\vartheta_t)$ (because $\mathcal{W}_{\vartheta,t}(\vartheta)$ is quasiconcave on the interval $[0,1]$). Finally, observe that ϑ_t^* is a function of $\tilde{\sigma}_t$ only, $\vartheta_t^* = \vartheta^*(\tilde{\sigma}_t)$, and a straightforward application of the implicit function theorem yields $\vartheta^{*\prime} > 0$. □

Proof of Proposition 13. The details of the model extension with lump-sum taxes are presented in Appendix E. The last part of that appendix provides an explicit construction of how to implement the constrained optimal allocation. That construction also implies the additional statements made in the proposition. □

Proof of Lemma 1. The lemma follows almost immediately from the analysis in Appendix F.1. Specifically, uniqueness of $q^{B,0*}$ and the conclusions about initial conditions $q_0^{B,0} \neq q^{B,0*}$ follows from the monotonicity result established in Lemma 4. Existence of $q^{B,0*}$ follows from the explicit construction in Appendix F that is presented after that lemma. □

Proof of Proposition 14. We refer the reader to the analysis in Appendix F.1. All assertions made in the proposition follow from the analysis of the planner's first-order conditions under commitment and discretion at the end of that appendix. □

C Additional Details on Model Extension with Long-term Bonds [Online Appendix]

C.1 Augmented Model Setup

The model is the same as in the one presented in Section 2 in the main text, except for a change in the payoff characteristics of long-term bond. This change can affect the equations of the model setup in at most three places, (i) the return on bonds $\mathrm{d}R_t^B$ faced by households, equation (11) in the baseline model, (ii) the government budget constraint, equation (9), as the government is short the bonds in the baseline model, and (iii) the endogenous state evolution, equation (15), as the backward-looking portion of bond values, $q_t^{B,0}$, is now different from the market value (per unit of capital), q_t^B. We discuss each of these three aspects in turn:

Bond Return. We show that equation (11) from the baseline model continues to hold, so that, with the notation introduced in the main text, the household problem is formally identical to the one solved in Appendix A.[46]

Suppose the nominal bond price $\mathcal{P}_t^B$ follows a generic Ito process of the form

$$\frac{\mathrm{d}\mathcal{P}_t^B}{\mathcal{P}_t^B} = \mu_t^{\mathcal{P},B}\mathrm{d}t + \sigma_t^{\mathcal{P},B}\mathrm{d}Z_t.$$

Then the return on bonds is (in general, permitting flexible prices)

$$\begin{aligned}\mathrm{d}R_t^B &= \frac{\mathrm{d}\left(\frac{\mathcal{P}_t^B}{\mathcal{P}_t}\right) + \lambda\left(\frac{1}{\mathcal{P}_t} - \frac{\mathcal{P}_t^B}{\mathcal{P}_t}\right)\mathrm{d}t}{\frac{\mathcal{P}_t^B}{\mathcal{P}_t}} \\ &= \left(\mu_t^{\mathcal{P},B} - \mu_t^{\mathcal{P}} - \sigma_t^{\mathcal{P}}\left(\sigma_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P}}\right)\right)\mathrm{d}t + \left(\sigma_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P}}\right)\mathrm{d}Z_t + \underbrace{\lambda\left(\frac{1}{\mathcal{P}_t^B} - 1\right)}_{=i_t}\mathrm{d}t.\end{aligned} \tag{49}$$

Recall the definition $q_t^B = \frac{\mathcal{P}_t^B \mathcal{B}_t}{\mathcal{P}_t K_t}$, so that $\frac{\mathcal{P}_t^B}{\mathcal{P}_t} = \frac{q_t^B K_t}{\mathcal{B}_t}$ and thus

$$\mu_t^{\mathcal{P},B} - \mu_t^{\mathcal{P}} - \sigma_t^{\mathcal{P}}\left(\sigma_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P}}\right) = \mu_t^{q,B} + g_t - \mu_t^{\mathcal{B}}, \qquad \sigma_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P}} = \sigma_t^{q,B}.$$

[46] To be precise, we show that the equality between the left-most and right-most expression in equation (11) still obtains. The intermediate equalities are no longer true, but these are not used in Appendix A to solve the model.

Substituting these equations into the previous return expression implies

$$\mathrm{d}R_t^B = \left(\mu_t^{q,B} + g_t + i_t - \mu_t^{\mathcal{B}}\right)\mathrm{d}t + \sigma_t^{q,B}\mathrm{d}Z_t,$$

which is the same expression as in equation (11) in the baseline model.

Government Budget Constraint. We show next that also the government budget constraint (9) continues to hold in this model. To see this, start from the government flow budget constraint written in a form analogous to equation (7) as an equation for the evolution of the face value of nominal bonds. With long-term bonds, this equation takes the form

$$\mathrm{d}\mathcal{B}_t = -\lambda\mathcal{B}_t\mathrm{d}t + \frac{\lambda}{\mathcal{P}_t^B}\mathcal{B}_t\mathrm{d}t - \frac{\mathcal{P}_t}{\mathcal{P}_t^B}\tau_t K_t\mathrm{d}t.$$

The first term captures decay from maturing bonds. The remaining two terms capture net issuance of new bonds to cover coupon payments on outstanding bonds in excess of tax receipts. Using $i_t = \lambda\left(\frac{1}{\mathcal{P}_t^B} - 1\right)$ and $\check{s}_t = \frac{\tau_t K_t}{\mathcal{P}_t^B\mathcal{B}_t/\mathcal{P}_t}$, we can rewrite this equation as

$$\mu_t^{\mathcal{B}} = i_t - \check{s}_t,$$

which is identical to equation (9) for the baseline model.

Endogenous State Evolution and Real Value of Safe Assets. By definition of the state variable $q_t^{B,0} := \mathcal{B}_t/(\mathcal{P}_t K_t)$, this follows an equation akin to equation (15) in the baseline model

$$\mathrm{d}q_t^{B,0} = \left(\mu_t^{\mathcal{B}} - \pi_t - g_t\right)q_t^{B,0}\mathrm{d}t = \left(i_t - \check{s}_t - \pi_t - g_t\right)q_t^{B,0}\mathrm{d}t,$$

where the second equality uses the government budget constraint. In contrast, the market value of bonds per unit of capital is $q_t^B = \frac{\mathcal{P}_t^B\mathcal{B}_t}{\mathcal{P}_t K_t}$, which is different from $q_t^{B,0}$ unless $\mathcal{P}_t^B = 1$ at all times. By definition of i_t, we have

$$\mathcal{P}_t^B = \frac{\lambda}{\lambda + i_t},$$

so that the real value of bonds per unit of capital, q_t^B, must be

$$q_t^B = \frac{\lambda}{\lambda + i_t}q_t^{B,0},$$

as claimed in equation (33).

Note that, with the notation introduced in the main text and assuming that a policy is set directly for i_t (not i_t^0), this is the only change in the equations of the model setup relative to the baseline model. In the limit $\lambda \to \infty$, we observe that $\mathcal{P}_t^B \to 1$ and $q_t^B = q_t^{B,0}$. Then, all equations in the model setup are identical to the ones in the baseline model. In this sense, the model extension here collapses then to the baseline model as claimed in part (iv) of Proposition 11.

C.2 Effects on the Model Solution

Now consider the general case, including $\lambda < \infty$. Then, the model setup is different from the baseline model. We now argue that still all equations from the baseline model characterizing equilibrium except for the state evolution for $q_t^{B,0}$ and the equation for q_t^B just stated remain valid.

First, consider the household problem. Because the return on bonds $\mathrm{d}R_t^B$ is unchanged, households face a formally identical decision problem. All solution steps in Appendix A relating to the household problem remain valid word by word, and so does Proposition 1.

Second, consider the problem of intermediate goods firms. These firms are not affected by the characteristics of government bonds. Therefore, also their decision problem remains unchanged, and the relevant parts of Appendix A as well as Proposition 2 remain valid.

Third, consider the determination of the bond wealth share ϑ_t, Proposition 3. Reviewing the derivation steps of this result in Appendix A, we observe that the bond state evolution, equation (15), or the fact that $q_t^B = q_t^{B,0}$ in the baseline model have not been used anywhere in the derivation. Therefore, also these derivations remain valid word by word, and Proposition 3 continues to hold.

Finally, consider the other equilibrium conditions derived in Section 2.3.4. Reviewing this section, we observe that also there the bond state evolution and the condition $q_t^B = q_t^{B,0}$ have not been used, only conditions for which we have already established that they continue to hold. Therefore, also this part of the model solution remains valid.

In sum, the previous arguments establish part (i) of Proposition 11.

C.3 Transferring Results from the Baseline Analysis

For a reader looking for the precise formal meaning of the assumption made in part (ii) of Proposition 11 that interest rate policy does not react "too strongly" to uncertainty shocks, let us state the relevant inequality upfront. The main idea is that q_t^B varies by less under sticky prices than it does under flexible prices. Because $\sigma_t^{q,B} = \sigma_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P}}$ and $\sigma_t^{\mathcal{P}} = 0$ under sticky prices, this translates into the condition

$$|\sigma_t^{\mathcal{P},B}| < |\sigma_t^{q,B,flex}| \Leftrightarrow |\sigma_{i,t}| < (\lambda + i_t)|\sigma_t^{q,B,flex}|,$$

where $\sigma_{i,t}$ denotes the arithmetic dZ_t-loading of i_t. The equivalence easily follows from $\mathcal{P}_t^B = \frac{\lambda}{\lambda + i_t}$ and Ito's lemma. This condition rules out the possibility that interest rate policy uses variation in $\mathcal{P}_t^B$ to "overstabilize" the safe asset supply. In fact, we only need to constrain the size of the response to shocks that move in the same direction as $q_t^{B,flex}$ (or, equivalently, that moves the interest rate in the opposite direction). We will therefore only require the weaker, but formally slightly less intuitive assumption that

$$\operatorname{sgn}(\sigma_t^{q,B,flex})\sigma_t^{\mathcal{P},B} < |\sigma_t^{q,B,flex}| \Leftrightarrow -\operatorname{sgn}(\sigma_t^{q,B,flex})\sigma_{i,t} < (\lambda + i_t)|\sigma_t^{q,B,flex}| \tag{50}$$

for all $t \geq 0$. We make this additional assumption in the following only if explicitly mentioned.

We now transfer results from the baseline analysis in Section 3, maintaining the assumption that policy directly controls i_t (instead of i_t^0). We discuss our implementation result that justifies this assumption in the following section.

First, consider Propositions 4 and 5 on the separation of portfolio choice and flight to safety, respectively. These propositions only rely on the bond valuation equation (22), which is the content of Proposition 3. We have already shown that this proposition still holds in the augmented model, so Propositions 4 and 5 hold as well.

Second, consider the first part of Proposition 6 concerning the properties of $q_t^{K,flex}$ and $q_t^{B,flex}$ after an uncertainty shock. Because these values under flexible prices do not depend on the bond duration, this part of the proposition remains valid.

Third, consider the remaining part of Proposition 6 about the behavior of the price level $\mathcal{P}_t$ under flexible prices. Recall $\mathcal{P}_t = \frac{\mathcal{P}_t^B \mathcal{B}_t}{q_t^B K_t}$. Under flexible prices, this implies

$$\sigma_t^{\mathcal{P}} = \sigma_t^{\mathcal{P},B} - \sigma_t^{q,B},$$

which has the opposite sign of $\sigma_t^{q,B}$ if and only if condition (50) holds. Therefore, under condition (50) (for all t) also this result can be transferred to the long-term bond model.

Fourth, consider Proposition 7, which establishes that uncertainty shocks lead to demand recessions. We assume, once again, that condition (50) holds for all t. To show that u_t decreases in $\tilde{\sigma}_t$ for fixed $q_t^{B,0}$, we show that σ_t^u has the opposite sign of σ_t^ϑ. Recall the formula

$$u_t = \frac{1}{a}\left(\frac{\rho}{\vartheta_t} q_t^B + \frac{(1-\vartheta_t)q_t^B - \vartheta_t}{\phi\vartheta_t}\right) = \frac{1}{a}\left(\frac{\rho}{\vartheta_t}\mathcal{P}_t^B q_t^{B,0} + \frac{(1-\vartheta_t)\mathcal{P}_t^B q_t^{B,0}}{\phi\vartheta_t} - \frac{1}{\phi}\right).$$

This implies for the arithmetic volatility of u_t, $\sigma_{u,t}$,

$$\begin{aligned}\sigma_{u,t} &= \frac{1}{a}\frac{\rho}{\vartheta_t}\mathcal{P}_t^B q_t^{B,0}\left(\sigma_t^{\mathcal{P},B} - \sigma_t^\vartheta\right) + \frac{1}{a}\frac{(1-\vartheta_t)\mathcal{P}_t^B q_t^{B,0}}{\phi\vartheta_t}\left(-\frac{1}{1-\vartheta_t}\sigma_t^\vartheta + \sigma_t^{\mathcal{P},B}\right)\\ &= \frac{1}{\phi a}(1-\vartheta_t+\phi\rho)\frac{q_t^B}{\vartheta_t}\sigma_t^{\mathcal{P},B} - \frac{1}{\phi a}(1+\phi\rho)\frac{q_t^B}{\vartheta_t}\sigma_t^\vartheta\end{aligned}$$

Next, applying Ito's lemma to $q_t^{B,flex} = \vartheta_t \frac{1+\phi a u_t^{flex}}{1-\vartheta_t+\phi\rho}$ yields

$$\begin{aligned}\sigma_t^{q,B,flex} &= \sigma_t^\vartheta + \frac{\phi a u_t^{flex}}{1+\phi a u_t^{flex}}\sigma_t^{u,flex} + \frac{\vartheta_t}{1-\vartheta_t+\phi\rho}\sigma_t^\vartheta\\ &= \frac{\phi a u_t^{flex}}{1+\phi a u_t^{flex}}\sigma_t^{u,flex} + \frac{1+\phi\rho}{1-\vartheta_t+\phi\rho}\sigma_t^\vartheta.\end{aligned}$$

Define $\mathfrak{s} := \operatorname{sgn}\sigma_t^{q,B,flex}$ for ease of notation. Combining the previous two equations with condition (50) leads to the inequality

$$\begin{aligned}\mathfrak{s}\sigma_t^u = \mathfrak{s}\frac{\sigma_{u,t}}{u_t} &< \mathfrak{s}\frac{1}{\phi a u_t}(1-\vartheta_t+\phi\rho)\frac{q_t^B}{\vartheta_t}\sigma_t^{q,B,flex} - \mathfrak{s}\frac{1}{\phi a u_t}(1+\phi\rho)\frac{q_t^B}{\vartheta_t}\sigma_t^\vartheta\\ &= \mathfrak{s}\frac{1}{\phi a u_t}(1-\vartheta_t+\phi\rho)\frac{q_t^B}{\vartheta_t}\frac{\phi a u_t^{flex}}{1+\phi a u_t^{flex}}\sigma_t^{u,flex} + \mathfrak{s}\frac{1}{\phi a u_t}\frac{q_t^B}{\vartheta_t}\left((1+\phi\rho)-(1+\phi\rho)\right)\sigma_t^\vartheta\\ &= \underbrace{\frac{1}{\phi a u_t}(1-\vartheta_t+\phi\rho)\frac{q_t^B}{\vartheta_t}\frac{\phi a u_t^{flex}}{1+\phi a u_t^{flex}}}_{>0}\underbrace{\mathfrak{s}\sigma_t^{u,flex}}_{\leq 0} \leq 0,\end{aligned}$$

where $\mathfrak{s}\sigma_t^{u,flex} \leq 0$ follows from the argument in the proof of Proposition 6 (which holds identically also in this model): u_t^{flex} is nonincreasing in ϑ_t and ϑ_t and $q_t^{B,flex}$ move in

the same direction in response to an uncertainty shock. This means, if $\mathfrak{s} > 0$, then $\sigma_t^{u,flex} \leq 0$ and if $\mathfrak{s} < 0$, then $\sigma_t^{u,flex} \geq 0$, so that in both cases $\mathfrak{s}\sigma_t^{u,flex} \leq 0$.

So we conclude that $\mathfrak{s}\sigma_t^u < 0$, which means u_t (strictly) falls after an uncertainty shock that raises $\tilde{\sigma}_t$ (and, hence, ϑ_t and $q_t^{B,flex}$). This is the first part of Proposition 7.

For the second part of Proposition 7, note that the volatility loading of the utilization gap, u_t/u_t^{flex}, is $\sigma_t^u - \sigma_t^{u,flex}$. Multiplying again by $\mathfrak{s}$ and using the previous inequality yields

$$\mathfrak{s}(\sigma_t^u - \sigma_t^{u,flex}) < \left(\frac{1}{\phi a u_t}(1 - \vartheta_t + \phi\rho)\frac{q_t^B}{\vartheta_t}\frac{\phi a u_t^{flex}}{1 + \phi a u_t^{flex}} - 1 \right) \mathfrak{s}\sigma_t^{u,flex}$$

If $q_t^B = q_t^{B,flex}$, then the first factor on the right-hand side vanishes (this is the case because $q_t^B = \vartheta_t \frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho}$ and $u_t = u_t^{flex}$), so that $\mathfrak{s}(\sigma_t^u - \sigma_t^{u,flex})$ must be strictly negative. Therefore, this expression remains strictly negative for $q_t^B \approx q_t^{B,flex}$. A negative $\mathfrak{s}(\sigma_t^u - \sigma_t^{u,flex})$ means that u_t/u_t^{flex} falls in response to an uncertainty shock that raises $\tilde{\sigma}_t$. This concludes the proof of Proposition 7 for the long-term debt model.

Fifth, consider Proposition 8 on capital price overshooting. Once again, let us assume that condition (50) holds for all t. The proof of the baseline model remains then valid with minor adjustments. Specifically, the formula

$$\sigma_t^{q,K} = \sigma_t^{q,B} - \frac{\sigma_t^{\vartheta}}{1 - \vartheta_t}$$

still holds under both flexible and sticky prices and, by Proposition 4, $\frac{\sigma_t^{\vartheta}}{1-\vartheta_t}$ is the same in both cases. We also know that

$$\sigma_t^{q,B} = \sigma_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P}}$$

under both flexible and sticky prices. Therefore,

$$\sigma_t^{q,K,flex} = \sigma_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P},flex} - \frac{\sigma_t^{\vartheta}}{1 - \vartheta_t} = \sigma_t^{q,K} - \sigma_t^{\mathcal{P},flex},$$

where $\sigma_t^{q,K}$ denotes the volatility of q_t^K under sticky prices. Hence,

$$\sigma_t^{q,K} = \sigma_t^{q,K,flex} + \sigma_t^{\mathcal{P},flex}.$$

By Proposition 6, which we have already established for the long-term bond model, $\sigma_t^{q,K,flex}$ and $\sigma_t^{\mathcal{P},flex}$ must have the same sign (and $\sigma_t^{\mathcal{P},flex} \neq 0$). This implies $|\sigma_t^{q,K}| < |\sigma_t^{q,K,flex}|$.

Note, in particular, that the previous discussion implies that all the claims made in part (ii) of Proposition 11 hold.

C.4 Using the Nominal Short Rate i_t^0 to Implement i_t-Policy

Suppose the monetary authority controls the nominal short rate process $\{i_t^0\}_{t=0}^{\infty}$. The following lemma, whose proof we defer to the end of this section, contains equilibrium restrictions on the process of long-term nominal rates $\{i_t\}_{t=0}^{\infty}$.

Lemma 3. *Suppose policy is specified in terms of a bounded process or policy rule for the nominal short rate i_t^0. Then, in any equilibrium in which the ergodic set for the state variables $(\tilde{\sigma}_t, q_t^{B,0})$ is compact, i_t follows an Ito process with bounded arithmetic drift and volatility, for which $\lambda + i_t$ is positive, bounded, and bounded away from zero. Furthermore, i_t satisfies the BSDE*

$$\mathbb{E}_t\left[\mathrm{d}i_t\right] = \left((\lambda + i_t)\left(i_t - i_t^0\right) - \sigma_t^{\vartheta}\sigma_{i,t} \right) \mathrm{d}t. \tag{51}$$

The previous lemma represents a mapping from short rates to long-term interest rates. We first show how to invert this mapping. Let us state precisely what we mean by this. We start by considering a specific equilibrium that results from specifying a policy rule directly for i_t. Our goal is to find a policy specification for i_t^0 along the equilibrium path that implements this specific equilibrium. We are not concerned with the more demanding goal of finding a policy rule for i_t^0 that generates the exact same set of equilibria as the original policy rule for i_t. This would require additional technical considerations with little gain because, in the context of this paper, we only study policies that generate a unique equilibrium prediction.

In any given equilibrium that has been generated with a policy rule for the long-term rate i_t, it is of course true that $i_t = i(\tilde{\sigma}_t, q_t^{B,0})$. Applying Ito's lemma to this equation implies that i_t must follow an Ito process in equilibrium, i.e.,

$$\mathrm{d}i_t = \mu_{i,t}\mathrm{d}t + \sigma_{i,t}\mathrm{d}Z_t$$

for some drift $\mu_{i,t}$ and volatility $\sigma_{i,t}$. Let us assume that both $\mu_{i,t}$ and $\sigma_{i,t}$ are bounded in line with the assumptions in Proposition 11. Also, we assume without loss of generality, that the separation assumptions of Proposition 4 are satisfied, so that ϑ_t and, in particular, σ_t^ϑ are determined independently of interest rates.[47] Under our usual compactness assumption on exogenous fundamentals, σ_t^ϑ will then also be bounded.

To obtain the required value for i_t^0 that implements i_t in equilibrium, we equate $\mu_{i,t}\mathrm{d}t$ with $\mathbb{E}_t\left[\mathrm{d}i_t\right]$ in the BSDE (51) and solve for i_t^0:

$$i_t^0 = i_t - \frac{\mu_{i,t} + \sigma_t^\vartheta \sigma_{i,t}}{\lambda + i_t}.$$

Under the assumption $\inf_{t\geq 0}\left(i_t + \lambda\right) > 0$, the denominator in the second term is bounded away from zero. Because, by assumption, also i_t, $\mu_{i,t}$, and $\sigma_{i,t}$ are bounded, so is the interest rate process i_t^0 required to implement i_t. This establishes part (iii) of Proposition 11.

Proof of Lemma 3. In a Markov equilibrium, $i_t = i(\tilde{\sigma}_t, q_t^{B,0})$. If the state variables are contained in a compact set, then i_t must be bounded. Furthermore, applying Ito's lemma immediately implies that i_t is also an Ito process whose arithmetic drift and volatility are (continuous) functions of the state variables, hence also bounded.

Recall that the long-term bond price is $\mathcal{P}_t^B = \frac{\lambda}{\lambda + i_t}$, which cannot be negative in equilibrium, as otherwise agents would be able to engage in the following arbitrage strategy: "purchase" more bonds, receive the cash flows resulting from the negative price, and hold the bonds forever, so that never any negative cash flow realizes. This implies $\lambda + i_t \geq 0$. To see that $\lambda + i_t$ is even bounded away from zero, note that, otherwise $\mathcal{P}_t^B$ would become arbitrarily large. However, by assumption, the ergodic set for $q_t^{B,0}$ is compact, so $q_t^{B,0} \geq \underline{q}^{B,0} > 0$.[48] Therefore, $q_t^B = \mathcal{P}_t^B q_t^{B,0} \geq \mathcal{P}_t^B \underline{q}^{B,0}$, so arbitrarily large $\mathcal{P}_t^B$ implies arbitrarily large q_t^B. The demand equation (27) then implies

[47] This is without loss of generality because we are only interested in implementing a specific equilibrium. In the desired equilibrium, it is always true that ϑ_t is a function of exogenous shock histories. While it may not be possible to represent this as a function of $\tilde{\sigma}_t$ only, we can expand the exogenous state space to a larger Markov state X_t that includes sufficient information to compute the policy variable $\check{s}_t$ in equilibrium. The arguments in Appendix B.2 immediately imply that a result akin to Proposition 4 applies to this larger state space as well.

[48] The state evolution implies $q_t^{B,0} > 0$, so $\underline{q}^{B,0} \geq 0$. By compactness, the lower bound must be assumed, which implies the strict inequality.

arbitrarily large values for utilization u_t, contradicting the fact that $u_t = u(\tilde{\sigma}_t, q_t^{B,0})$ must also be bounded in equilibrium.

It is left to derive the BSDE (51). To do so, consider the portfolio choice between (long-term) government bonds and a zero-net-supply nominal bond with infinitesimal duration. Call the latter M (for money), its return is

$$\mathrm{d}R_t^M = i_t \mathrm{d}t + \frac{\mathrm{d}\left(1/\mathcal{P}_t\right)}{1/\mathcal{P}_t} = \left(i_t^0 - \mu_t^{\mathcal{P}} + (\sigma_t^{\mathcal{P}})^2\right)\mathrm{d}t - \sigma_t^{\mathcal{P}}.$$

Using the return representation (49) for bonds, the portfolio choice condition between bonds and money becomes

$$i_t + \mu_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P}}\sigma_t^{\mathcal{P},B} - i_t^0 = \sigma_t^n \sigma_t^{\mathcal{P},B}.$$

Now, $\sigma_t^n = \sigma_t^q = \sigma_t^{q,B} - \sigma_t^{\vartheta} = \sigma_t^{\mathcal{P},B} - \sigma_t^{\mathcal{P}} - \sigma_t^{\vartheta}$, so the previous equation becomes, after rearranging,

$$\mu_t^{\mathcal{P},B} - \left(\sigma_t^{\mathcal{P},B}\right)^2 = i_t^0 - i_t - \sigma_t^{\vartheta}\sigma_t^{\mathcal{P},B}.$$

Recall $i_t = \lambda\left(1/\mathcal{P}_t^B - 1\right)$, so

$$\begin{aligned}\mathrm{d}i_t = \lambda \mathrm{d}\left(1/\mathcal{P}_t^B\right) &= -\lambda\frac{1}{\mathcal{P}_t^B}\left(\mu_t^{\mathcal{P},B} - \left(\sigma_t^{\mathcal{P},B}\right)^2\right)\mathrm{d}t - \lambda\frac{1}{\mathcal{P}_t^B}\sigma_t^{\mathcal{P},B}\mathrm{d}Z_t \\ &= \lambda\frac{1}{\mathcal{P}_t^B}\left(\left(i_t - i_t^0\right) + \sigma_t^{\vartheta}\sigma_t^{\mathcal{P},B}\right)\mathrm{d}t - \lambda\frac{1}{\mathcal{P}_t^B}\sigma_t^{\mathcal{P},B}\mathrm{d}Z_t\end{aligned}$$

which immediately implies the BSDE (51) stated above.

□

D Additional Details on Models for Comparisons in Section 4 [Supplemental Materials]

D.1 No Safe Assets

The model without safe assets is as described in the main text: $\mathcal{B}_t = 0$, $q_t^B = \vartheta_t = 0$, and households face in their portfolio choice a bond return process $\mathrm{d}R_t^B = (i_t - \pi_t)\mathrm{d}t$. For this model, all equilibrium conditions derived in Section 2.3 continue to hold, but some of them, like the bond valuation equation (22), become uninformative.

As described in the main text, instead, equation (32) for the capital price becomes the relevant determinant for aggregate demand, which can be written equivalently as

$$\frac{\mathbb{E}_t[\mathrm{d}\log q_t^K]}{\mathrm{d}t} = i_t - \pi_t - (\rho + g_t - \tilde{\sigma}_t^2). \tag{52}$$

In this appendix, we provide details on the claims made in the main text that uncertainty shocks do not necessarily lead to demand recessions in this model under alternative assumptions about mean-reversion dynamics of $\tilde{\sigma}_t$, even in the absence of a monetary policy reaction and if we stick to the conventional equilibrium selection. We first analyze gradual deterministic mean reversion in a linearized model, as this is based on the same model equations as presented in the main text (up to a linear perturbation). We then discuss stochastic mean reversion as in Caballero and Simsek (2020), which requires a modification to the model to allow for jump shocks. In both cases, we abstract from physical investment by taking the limit of infinite capital adjustment costs $\phi \to \infty$. Then capital grows at a constant rate $g_t = g$, which we take as a model parameter. The insights derived here apply more generally, but exogenous capital significantly simplifies the model equations because then $C_t = Y_t$, so that the additional scaling term in the Phillips curve (equation (20)) disappears.

D.1.1 Gradual Deterministic Mean Reversion

Suppose $\tilde{\sigma}_t$ satisfies the equation

$$\mathrm{d}\underbrace{\tilde{\sigma}_t^2}_{=:x_t} = b(\underbrace{\bar{\sigma}^2}_{=:\bar{x}} - \underbrace{\tilde{\sigma}_t}_{=x_t})\mathrm{d}t,$$

which implies that the idiosyncratic variance $x_t = \tilde{\sigma}_t^2$ follows a linear law of motion. Define $\hat{x}_t := x_t - \bar{x}$ as the deviation of x_t from steady state, then the above simplifies further to

$$\mathrm{d}\hat{x}_t = -b\hat{x}_t\mathrm{d}t, \tag{53}$$

which implies the closed-form solution $\hat{x}_t = \hat{x}_0 e^{-bt}$.

Let $\hat{q}_t := \log q_t^K - \log \bar{q}^K$, where $\bar{q}^K$ is the steady-state value of q_t^K. Substituting $g_t = g$ and the constant interest rate policy $i_t = i$ into the demand equation (52) yields

($\bar{\pi}$ is the steady-state value of π_t)

$$\frac{\mathbb{E}_t[\mathrm{d}\hat{q}_t]}{\mathrm{d}t} = i - \bar{\pi} - \underbrace{(\rho + g - \bar{x})}_{=:\bar{r}} - \hat{\pi}_t + \hat{x}_t.$$

As this equation has to hold in steady state as well, it must be that $\bar{\pi} = i - \bar{r}$, so that the equation simplifies to

$$\frac{\mathbb{E}_t[\mathrm{d}\hat{q}_t]}{\mathrm{d}t} = -\hat{\pi}_t + \hat{x}_t. \tag{54}$$

The equations so far are linear in $(\hat{x}_t, \hat{q}_t, \hat{\pi}_t)$. The only nonlinear equation in this model is the Phillips curve, which here takes the form (due to $C_t = Y_t$ in the absence of physical investment)

$$\begin{aligned}\frac{\mathbb{E}_t\left[\mathrm{d}\pi_t\right]}{\mathrm{d}t} &= \rho\pi_t - \kappa\frac{p_t^R - p^{R,flex}}{a} = \rho\pi_t - \kappa\left(u_t^{1+\varphi} - \frac{\epsilon-1}{\epsilon}\right) \\ &= \rho\pi_t - \kappa\left(\left(\frac{\rho q_t^K}{a}\right)^{1+\varphi} - \frac{\epsilon-1}{\epsilon}\right),\end{aligned}$$

where the first line uses $p_t^R = a u_t^{1+\varphi}$ (follows from equations (24) and (25) if $\iota = 0$) and the second line uses goods market clearing (equation (24)) to express u_t in terms of q_t. As this equation also has to hold in steady state, we can write it in terms of deviations as

$$\frac{\mathbb{E}_t\left[\mathrm{d}\hat{\pi}_t\right]}{\mathrm{d}t} = \rho\hat{\pi}_t - \kappa\left(\left(\frac{\rho q_t^K}{a}\right)^{1+\varphi} - \left(\frac{\rho \bar{q}^K}{a}\right)^{1+\varphi}\right) = \rho\hat{\pi}_t - \kappa\bar{u}^{1+\varphi}\left(e^{(1+\varphi)\hat{q}_t} - 1\right).$$

To be able to solve the model analytically, we take a linear approximation to the right-hand side of this equation, so that, up to an approximation error,

$$\frac{\mathbb{E}_t\left[\mathrm{d}\hat{\pi}_t\right]}{\mathrm{d}t} = \rho\hat{\pi}_t - \underbrace{\kappa\bar{u}^{1+\varphi}(1+\varphi)}_{=:\tilde{\kappa}}\hat{q}_t. \tag{55}$$

We solve the linear forward-backward system of equations (53), (54), and (55) with the method of undetermined coefficients. We look for an equilibrium that is Markovian and linear in the state variable $\hat{x}_t$, i.e.,

$$\hat{q}_t = A\hat{x}_t \qquad\qquad \hat{\pi}_t = B\hat{x}_t.$$

We remark that there is an implicit element of equilibrium selection in this parametric form as we restrict attention to the so-called minimum state variable equilibrium. This is the conventional equilibrium selection in the New Keynesian literature.[49] Using the state evolution (53), these guesses imply that

$$\frac{\mathrm{d}\hat{q}_t}{\mathrm{d}t} = -bA\hat{x}_t \qquad\qquad \frac{\mathrm{d}\hat{\pi}_t}{\mathrm{d}t} = -bB\hat{\pi}_t.$$

Substituting these equations and the guesses into equations (54) and (55) yields

$$-bA\hat{x}_t = -B\hat{x}_t + \hat{x}_t,$$
$$-bB\hat{x}_t = \rho B\hat{x}_t - \tilde{\kappa}A\hat{x}_t.$$

As these equations have to hold for all states $\hat{x}_t$, we obtain a linear equation system for the coefficients A and B. Solving the first equation for B yields $B = 1 + bA$ and substituting this into the second,

$$-b - b^2A = \rho + \rho bA - \tilde{\kappa}A \quad \Leftrightarrow \quad \big(\tilde{\kappa} - b(\rho + b)\big)\, A = \rho + b$$

Assuming that $\tilde{\kappa} \neq b(\rho + b)$, this equation can be solved for A. The equilibrium solution is

$$\hat{q}_t = \frac{\rho + b}{\tilde{\kappa} - b(\rho + b)}\hat{x}_t \qquad\qquad \hat{\pi}_t = \frac{\tilde{\kappa}}{\tilde{\kappa} - b(\rho + b)}\hat{x}_t.$$

We observe that the sign of the reaction of asset prices $\hat{q}_t = \log \frac{q_t^K}{\bar{q}^K}$ and inflation $\hat{\pi}_t = \pi_t - \bar{\pi}$ to an increase in uncertainty (recall $\hat{x}_t = \tilde{\sigma}_t^2 - \bar{\sigma}^2$) depends on the sign of the denominator. Because q_t^K is proportional to aggregate demand (and output) in equilibrium, the sign of the reaction of aggregate economic activity to an increase in

[49] The conventional procedure in the literature is to select an equilibrium implicitly by specifying interest rate policy as a Taylor rule with an intercept term that is a function of fundamental state variables only and a feedback term that satisfies the Taylor principle. This procedure does two things simultaneously: (i) it selects a unique (bounded) equilibrium, and (ii) it renders the nominal interest rate endogenous. It is well known that the selected equilibrium always depends only on the fundamental state variables. It is somewhat less well-known but equally true that the same equilibrium would be found if one were to specify an exogenous interest rate rule without feedback that coincides with the equilibrium process for i_t in the Taylor rule equilibrium and then select the minimum state variable equilibrium. This alternative procedure has the advantage that it achieves goal (i) without requiring (ii) an endogenous interest rate. This is important here as we wish to keep the interest rate constant in equilibrium to isolate the effects of the shock in the absence of a monetary policy reaction.

uncertainty is then the same. Specifically, if prices are sufficiently sticky relative to the mean reversion speed of the aggregate state such that $\tilde{\kappa} < b(\rho + b)$,[50] we obtain the same prediction as in the experiment discussed in the main text: a rise in uncertainty leads to a demand recession. However, if the Phillips curve is too steep (prices are fairly flexible) such that the opposite inequality holds, the sign switches. A rise in uncertainty leads then to an increase in aggregate demand and a positive output gap. We conclude that the prediction that uncertainty shocks lead to demand recessions is not a robust one in this model but requires sufficiently rigid prices.

D.1.2 Stochastic Recovery

Suppose that $\tilde{\sigma}_t$ follows a two-state Markov chain, $\tilde{\sigma}_t \in \{\tilde{\sigma}_l, \tilde{\sigma}_h\}$, $\tilde{\sigma}_l < \tilde{\sigma}_h$ with transition rates $\lambda_l, \lambda_h \geq 0$ (λ_s denotes the transition rate for exiting state s). This arrangement is precisely as in Caballero and Simsek (2020) except that, in their paper, capital shocks are aggregate instead of idiosyncratic shocks. However, this difference does not matter for the economic predictions of the model in the absence of government bonds.

In this model, the aggregate risk does not arise from a Brownian $\mathrm{d}Z_t$-shock but from state transitions. While this does not affect the economic insights from our model, it does change the portfolio choice conditions slightly. Formally, the demand equation (52) needs to be replaced with

$$\frac{\mathbb{E}_t[\mathrm{d}q_t^K]}{q_t^K \mathrm{d}t} = i_t - \pi_t - (\rho + g - \tilde{\sigma}_t^2) + \lambda_t \frac{(q_t^{K+} - q_t^K)^2}{q_t^{K+} q_t^K},$$

where q_t^{K+} denotes the jump target for q_t^K conditional on a jump happening at time t. In the interest of space, we omit the derivation of this equation but note that its derivation follows precisely the same steps as the derivation of equation (52). We remark that the last term captures a risk premium for aggregate (jump) risk in the capital price. In contrast, the Phillips curve, equation (20), remains unaffected by the presence of jump risk.

We assume, in line with both the main text and the analysis in Caballero and Simsek (2020), that $i_t = i(\tilde{\sigma}_t)$ only depends on the exogenous risk state and is set in the low-risk

[50] Note that $\tilde{\kappa} \propto \kappa$ depends primarily on the degree of price stickiness.

state $\tilde{\sigma}_t = \tilde{\sigma}_l$ in a way that implements the flexible price allocation.[51] This assumption ensures that $q_t^K = q_t^{K,flex}$ and $p_t^R = p^{R,flex}$ whenever the economy is in state $\tilde{\sigma}_l$. We denote by $i_t = i_h$ the nominal interest rate in state $\tilde{\sigma}_h$ and look for an equilibrium that is Markovian in the fundamental state $\tilde{\sigma}_t$, which means $\pi_t = \pi_h$ and $q_t^K = q_h^K$ are constant in state $\tilde{\sigma}_h$ and must satisfy the two equations[52]

$$\lambda_h \frac{q_l^K - q_h^K}{q_h^K} = i_h - \pi_h - (\rho + g - \tilde{\sigma}_h^2) + \lambda_h \frac{(q_l^K - q_h^K)^2}{q_l^K q_h^K}$$
$$\lambda_h (\pi_l - \pi_h) = \rho \pi_h - \kappa \left(\left(\frac{\rho q_h^K}{a} \right)^{1+\varphi} - \frac{\epsilon - 1}{\epsilon} \right)$$

Rearranging and simplifying the first equation yields

$$\pi_h = i_h - r_h^n - \lambda_h \left(1 - \frac{q_h^K}{q^{K,flex}} \right), \tag{56}$$

where $r_h^n := \rho + g - \tilde{\sigma}_h^2$ denotes the natural rate in state $\tilde{\sigma}_h$. This equation represents a linear and increasing relationship between the inflation rate π_h and the "asset price gap" $q_h^K / q^{K,flex}$ in the high-risk state.

For the second equation, note first that the Phillips curve in the low-risk state implies $\lambda_l \pi_h = (\rho + \lambda_l)\pi_l$, which allows us to express inflation in the low-risk state in terms of inflation in the high-risk state, $\pi_l = \frac{\lambda_l}{\rho + \lambda_l} \pi_h$.[53] In addition, $p_l^R = p^{R,flex}$ means $\left(\frac{\rho q_l^K}{a} \right)^{1+\varphi} = \frac{\epsilon - 1}{\epsilon}$. Using these expressions to simplify the second equation and solving for π_h yields

$$\pi_h = \frac{\rho + \lambda_l}{\rho + \lambda_l + \lambda_h} \frac{\kappa}{\rho} \frac{\epsilon - 1}{\epsilon} \left(\left(\frac{q_h^K}{q^{K,flex}} \right)^{1+\varphi} - 1 \right). \tag{57}$$

[51] Again, this is not just an assumption about monetary policy but also contains an element of equilibrium selection. A suitable Taylor rule, followed only in the low-risk state, can ensure that this is the unique equilibrium prediction.

[52] These equations follow from the dynamic equations stated earlier. Note that for a variable x_t that remains constant for a given exogenous state but jumps to x_t^+ upon a state transition, the expected change is $\mathbb{E}_t[\mathrm{d}x_t] = \lambda_t (x_t^+ - x_t) \mathrm{d}t$.

[53] Implementing the flexible price allocation in the low-risk state generally requires an inflation rate different from zero unless the low-risk state is absorbing, $\lambda_l = 0$. This is because price-setting firms take into account the change in inflation rates in a future state transition to the high-risk state and, in their desire to smooth inflation to economize on price adjustment costs, adjust inflation already in the low-risk state even if their current markup is statically optimal.

This equation represents an increasing and strictly convex (for $\varphi > 0$) relationship between inflation π_h and the asset price gap $q_h^K/q^{K,flex}$.

The previous observations imply that there are at most two pairs $(\pi_h, q_h^K/q^{K,flex})$ of values that satisfy both equations (56) and (57). Furthermore, for large $q_h^K/q^{K,flex}$, the superlinear growth of the right-hand side of equation (57) ensures that it is larger than the right-hand side of equation (56) – except for the limit case of a flat Phillips curve, $\kappa = 0$, in which case the $q_h^K/q^{K,flex}$-dependent term disappears. Therefore, a solution exists if and only if there is some positive value of $q_h^K/q^{K,flex}$ at which the right-hand side of equation (56) is no smaller than that of equation (57). A sufficient condition for that is $i_h \geq r_h^n$ as then this certainly happens for the special case $q_h^K/q^{K,flex} = 1$. In the literature, it is typically assumed that the nominal rate tracks the natural rate except if it is unable to adjust downward due to a ZLB constraint. Under such assumptions, the sufficient condition $i_h \geq r_h^n$ is naturally satisfied.

In any case, a solution always exists if $i_h = i_l$ is not adjusted downward in response to an uncertainty shock, the policy choice we are interested in here to isolate the effects of the shock itself. To see this, note that $i_l = r_l^n + \pi_l = r_l^n + \frac{\lambda_l}{\rho+\lambda_l}\pi_h$ and that, for $q_h^K/q^{K,flex} = 1$, equation (56), then implies $\pi_l, \pi_h \geq 0$ and hence $i_h \geq r_l^n > r_h^n$, so that the right-hand side of equation (56) is strictly larger than that of equation (57) at the point $q_h^K/q^{K,flex} = 1$. We conclude that, under this policy specification, there must always be at least one solution that satisfies $q_h^K > q^{K,flex}$. In addition, there may be another solution that satisfies $q_h^K < q^{K,flex}$ if there is a point $q_h^K/q^{K,flex} \in (0,1)$ at which the right-hand side of equation (57) exceeds the one of equation (56). Such a point exists if and only if this is the case at the boundary $q_h^K/q^{K,flex} = 0$, i.e., if

$$\begin{aligned} &\Leftrightarrow && -\frac{\rho+\lambda_l}{\rho+\lambda_l+\lambda_h}\frac{\kappa}{\rho}\frac{\epsilon-1}{\epsilon} > \frac{\rho+\lambda_l}{\rho}\left(r_l^n - r_h^n - \lambda_h\right) \\ &\Leftrightarrow && \frac{\kappa}{\rho+\lambda_l+\lambda_h} < \frac{\epsilon}{\epsilon-1}\left(\lambda_h - \left(\tilde{\sigma}_h^2 - \tilde{\sigma}_n^2\right)\right). \end{aligned}$$

In particular, this solution only exists if the expected time to recovery is sufficiently short such that $\lambda_h > \tilde{\sigma}_h^2 - \tilde{\sigma}_n^2$ (sufficiently fast mean reversion) and the Phillips curve is sufficiently flat, i.e., κ is small (sufficiently rigid prices).

Note that the economic predictions of these two possible equilibria are quite distinct. The always existing solution with the property $q_h^K > q^{K,flex}$ has output and inflation

increasing in uncertainty $\tilde{\sigma}_t$. If we were to consider $i_h \neq i_l$, this equilibrium solution would also have the property that inflation π_{i_h} is increasing in i_h. For this reason, we call this the "Fisherian" solution to the model because its economic predictions follow a Fisher equation intuition, just as in flexible price models. As we have shown, this solution always exists, except in the limit case $\kappa = 0$, and is the unique solution if prices are sufficiently flexible. The other possible solution with the property $q_h^K < q^{K,flex}$ may be called the "Keynesian" solution to the model. This solution has the conventional predictions that output and inflation are decreasing in uncertainty $\tilde{\sigma}_t$ and that, if i_h was adjusted, inflation π_{i_h} would be decreasing in i_h. If we select this Keynesian solution, we therefore arrive at the conventional conclusion that uncertainty shocks generate demand recessions, just as in our example with a deterministic recovery time in the main text. However, as we have shown, this solution only exists if the Phillips curve is sufficiently flat.

These results further corroborate our conclusions from the previous subsection that the prediction of a demand recession in response to an increase in uncertainty is not a robust feature of models without safe assets. In fact, in a certain sense the Fisherian solution appears to be the more robust one because it exists for any degree of price stickiness so long as the Phillips curve is not flat, so that prices are fully rigid forever. It is this limit case that is the focus in Caballero and Simsek (2020), so that, in their paper, this issue becomes never apparent.

D.2 Two Units of Account

Model Setup. The model is as in Section 2 with the following modification. There are two nominal price levels: a price level in terms of "goods dollars", $\mathcal{P}_t^g$, and a price level in terms of "bond dollars", $\mathcal{P}_t^b$. Goods dollars are a pure unit of account that does not exist in positive supply. Intermediate goods firms quote their prices in goods dollars subject to price adjustment costs, so that the Phillips curve, equation (20), continues to hold if inflation is defined as goods price inflation, $\pi_t := \mu_t^{\mathcal{P}^g} = \frac{\mathrm{d}\mathcal{P}_t^g}{\mathcal{P}_t^g \mathrm{d}t}$ (due to price stickiness, the paths of $\mathcal{P}_t^g$ are absolutely continuous, so that the last equality makes sense). Bond dollars, in contrast, are the unit of reference for nominal government bonds. That is, the quantity $\mathcal{B}_t$ is quoted in units of bond dollars, so that the real value of

bonds is $q_t^B = \frac{\mathcal{B}_t}{\mathcal{P}_t^b K_t}$. With this definition of q_t^B, the return on bonds faced by households is as defined in the main text.

Goods dollars and bond dollars are, in principle, entirely unrelated units of account. In particular, the exchange rate between the two, $\mathcal{E}_t := \mathcal{P}_t^g / \mathcal{P}_t^b$, can freely adjust in equilibrium. However, we assume that the nominal interest rate on bonds (in bond dollar units), i_t^b, and the shadow nominal interest rate on a hypothetical goods dollar bond, i_t^g, are the same, that is $i_t^g = i_t^b = i_t$. Introducing a goods dollar interest rate i_t^g (set by policy) is necessary to discipline the equilibrium behavior of the goods dollar at all despite their role as a pure unit of account, whose equilibrium behavior faces no restrictions unless there is a Fisher equation for goods dollar inflation in the model.[54] This is analogous to the conventional New Keynesian analysis with nominal assets in zero net supply, which also requires the presence of a Fisher equation. While we could in principle assume that the policy maker sets two separate nominal rates i_t^g and i_t^b, there is little gain from this additional level of generality for the conceptual point we seek to make here.[55]

Model Solution. With our definition of q_t^B from above, the household side of the model is as in the baseline model and also the government budget constraint can still be represented as in equation (9). Therefore, the optimal household choice conditions in Proposition 1 and the bond valuation equation in Proposition 3 remain valid here as well. As already stated, the optimal price setting of firms yields a Phillips curve as in Proposition 2, provided π_t is interpreted as goods dollar inflation. It is also easy to see that all equations stated in Section 2.3.4 remain valid in this augmented model.

Because the bond valuation equation remains valid, all conclusions derived from it remain valid as well, in particular, Propositions 4 and 5. Under the conditions of these propositions, portfolio choice is separated from the determination of aggregate

[54] The Phillips curve restricts the joint dynamics of goods dollar inflation and capital rental prices. However, in the absence of a Fisher equation for goods dollars, any (absolutely continuous) path for $\mathcal{P}_t^g$ is consistent with an equilibrium: simply solve the Phillips curve for the rental price p_t^R consistent with that path. The resulting rental price path is always consistent with market clearing in all markets.

[55] Effectively, having $i_t^g \neq i_t^b$ changes the expected drift of the exchange rate $\mathcal{E}_t$, akin to a type of uncovered interest parity condition (although such a condition does not exactly hold because there is a risk premium on bond dollars).

economic activity, and there is flight to safety in equilibrium.

The (only) difference in this model is how the bond value q_t^B is determined. Unlike in our baseline model, q_t^B is no longer a purely backward-looking object. Instead, the hypothetical object $q_t^{Bg} := \frac{\mathcal{B}_t}{\mathcal{P}_t^g K_t}$ (the value of bonds if they were denominated in goods instead of bond dollars) follows a state evolution akin to equation (15),

$$\mathrm{d}q_t^{Bg} = \left(\mu_t^{\mathcal{B}} - \pi_t - g_t\right) q_t^{Bg} \mathrm{d}t, \tag{58}$$

whereas the actual bond value satisfies

$$q_t^B = \mathcal{E}_t q_t^{Bg},$$

where $\mathcal{E}_t$ is an additional free variable that is absent in the model from the main text, which features $\mathcal{P}_t^b = \mathcal{P}_t^g$ (and hence $\mathcal{E}_t \equiv 1$).

The equilibrium evolution of $\mathcal{E}_t$ is constrained by only a single condition, a portfolio choice condition of households between actual bonds denominated in bond dollars and hypothetical (zero net supply) bonds denominated in goods dollars. In analogy to the portfolio choice condition between capital and bonds stated in Appendix A, this choice condition takes the form

$$\frac{\mathbb{E}_t[\mathrm{d}R_t^B]}{\mathrm{d}t} - (i_t - \pi_t) = \varsigma_t \sigma_t^{q,B}. \tag{59}$$

In equilibrium, $\sigma_t^{q,B} = \sigma_t^{\mathcal{E}}$ and we know from the baseline model solution that (recall $q_t := q_t^B + q_t^K$)

$$\varsigma_t = \vartheta_t \sigma_t^{q,B} + (1 - \vartheta_t)\sigma_t^{q,K} = \sigma_t^q = \sigma_t^{q,B} - \sigma_t^{\vartheta}.$$

Combining the two equations implies $\varsigma_t = \sigma_t^{\mathcal{E}} - \sigma_t^{\vartheta}$. In addition, the return on (bond dollar) bonds is

$$\begin{aligned}\frac{\mathbb{E}_t[\mathrm{d}R_t^B]}{\mathrm{d}t} &= i_t - \mu_t^{\mathcal{B}} + g_t + \mu_t^{q,B} = i_t - \mu_t^{\mathcal{B}} + g_t - \mu_t^{\mathcal{E}} + \mu^{q,Bg} \\ &= i_t - \mu_t^{\mathcal{B}} + g_t - \mu_t^{\mathcal{E}} + \mu_t^{\mathcal{B}} - g_t - \pi_t = i_t - \pi_t + \mu_t^{\mathcal{E}},\end{aligned}$$

where the second line uses the state evolution (58). Substituting both this expression and the one for ς_t into equation (59) and canceling terms yields

$$\mu_t^{\mathcal{E}} = (\sigma_t^{\mathcal{E}} - \sigma_t^{\vartheta})\sigma_t^{\mathcal{E}}, \tag{60}$$

which is a BSDE for $\mathcal{E}_t$ that only depends on σ_t^ϑ (which is determined separately from aggregate economic activity) and the evolution for $\mathcal{E}_t$ itself, but not on any variable related to the debt state q_t^{Bg}. Therefore, the latter does not directly constrain the evolution of $\mathcal{E}_t$ and, hence, the goods dollar price level.

Indeed, for any given path for ϑ_t, there are many possible solutions to equation (60) indicating equilibrium multiplicity in this augmented model, just like in the baseline New Keynesian model without safe assets. For example, $\mathcal{E}_t \equiv 1$ is always a valid solution. This solution yields the same equilibrium dynamics as in our baseline model with a single unit of account. Another simple example is the following continuum of solutions: let $\mathcal{E}_0 > 0$ be arbitrary and suppose that $\mathcal{E}_t$ follows a martingale with $\mathrm{d}\mathcal{E}_t = \sigma_t^\vartheta \mathcal{E}_t \mathrm{d}Z_t$ (due to separation of portfolio choice, ϑ_t does not depend on $\mathcal{E}_t$ and this is an explicit definition). Clearly, any process in this class solves equation (60) and is therefore a possibility in equilibrium.

Comparison to the Model without Safe Assets. The previous discussion clarifies that q_t^B and, hence, aggregate demand, is not tightly linked to relative safe asset demand ϑ_t because the exchange rate $\mathcal{E}_t$ can adjust flexibly in response to shocks. However, this discussion is insufficient to justify our claims in the main text that "model dynamics largely resemble those observed in the model without safe assets" and that a result akin to Proposition 10 holds, i.e., setting the nominal interest rate to the natural rate can implement the flexible price allocation. Here, we justify these claims by providing a tighter link between the exchange rate equation (60) in this model and the (IS-type) aggregate demand equation (52) for the capital price q_t^K that holds in the model without safe assets.

We first remark that, without safe assets, $q_t^K = q_t$ equals total private-sector wealth per unit of capital. In the model considered here, it is conceptually more appropriate to derive an equation for $q_t = q_t^B + q_t^K$, as this directly guides consumption demand (which is $\rho q_t K_t$). Investment demand still depends on q_t^K only, but the latter can be recovered easily from $q_t^K = (1 - \vartheta_t) q_t$.

To derive an equation for the evolution of q_t from equation (60), note that

$$q_t = q_t^B / \vartheta_t = \mathcal{E}_t \frac{q_t^{Bg}}{\vartheta_t}.$$

In this multiplicative decomposition, the second factor, $q_t^{B^g}/\vartheta_t$ depends only on a state variable and separately determined portfolio choice, which is therefore effectively exogenous for the determination of q_t. Therefore, $\mathcal{E}_t$ represents the forward-looking component in q_t that is independent of portfolio choice. Applying Ito's lemma to the previous equation yields

$$\begin{aligned}\frac{\mathbb{E}_t[\mathrm{d}q_t]}{q_t \mathrm{d}t} &= \mu_t^{\mathcal{E}} + \mu_t^{q,B^g} - \mu_t^{\vartheta} - (\sigma_t^{\mathcal{E}} - \sigma_t^{\vartheta})\sigma_t^{\vartheta} \\ &= (\sigma_t^{\mathcal{E}} - \sigma_t^{\vartheta})\sigma_t^{\mathcal{E}} + \left(i_t - \check{s}_t - \pi_t - g_t\right) - \left(\rho - \check{s}_t - (1-\vartheta_t)^2\tilde{\sigma}_t^2\right) - (\sigma_t^{\mathcal{E}} - \sigma_t^{\vartheta})\sigma_t^{\vartheta} \\ &= (\sigma_t^{\mathcal{E}} - \sigma_t^{\vartheta})^2 + (i_t - \pi_t) - \left(\rho + g_t - (1-\vartheta_t)^2\tilde{\sigma}_t^2\right),\end{aligned}$$

where the replacements in the second line follow from equations (22), (58), and (60).

Now note that $\sigma_t^{\mathcal{E}} - \sigma_t^{\vartheta} = \sigma_t^q$ and $\frac{\mathbb{E}_t[\mathrm{d}\log q_t]}{\mathrm{d}t} = \frac{\mathbb{E}_t[\mathrm{d}q_t]}{q_t \mathrm{d}t} - (\sigma_t^q)^2$. Therefore, the previous equation is equivalent to

$$\frac{\mathbb{E}_t[\mathrm{d}\log q_t]}{\mathrm{d}t} = i_t - \pi_t - \left(\rho + g_t - (1-\vartheta_t)^2\tilde{\sigma}_t^2\right).$$

Structurally, this equation is identical to equation (52) in the model without safe assets. The only difference is that the precautionary motive in the comparison interest rate contains an additional term $(1-\vartheta_t)^2$ reflecting that, due to self-insurance, residual idiosyncratic risk per unit of wealth is lower here than in the model without safe assets. But because portfolio choice is separately determined, this is a minor detail. If we adjust the exogenous idiosyncratic risk process across the two models, so that, in both models, the residual risk $(1-\vartheta_t)\tilde{\sigma}_t$ is identical, then we should expect equilibrium dynamics that are (almost) identical.[56]

Importantly, interest rate policy enters, once again, the key aggregate demand equation of the model. If, for example, the nominal rate is set to

$$i_t = r_t^n := \rho + g_t + \mu_t^{q,flex} - (\sigma_t^{q,flex})^2 - (1-\vartheta_t)^2\tilde{\sigma}_t^2,$$

then the previous equation becomes $\frac{\mathbb{E}_t[\mathrm{d}\log q_t]}{\mathrm{d}t} = \frac{\mathbb{E}_t[\mathrm{d}\log q_t^{flex}]}{\mathrm{d}t} - \pi_t$, which is solved by $\pi_t = 0$ and $q_t = q_t^{flex}$. It is easy to verify that this solution is also consistent with the

[56] Equilibrium dynamics are exactly identical under this adjustment if we consider the limit $\phi \to \infty$ without physical investment. In the model with physical investment, this is not fully true because also the effective capital growth rate g_t changes. To obtain an exact isomorphism between the two models more generally, we also would have to make the capital depreciation rate δ time-varying in one of the two, in order to ensure the same capital growth rates g_t in both model variants.

Phillips curve (equation (20)). Hence, once again, a variant of Proposition 10 holds in this economy.

D.3 Monetary Frictions

Model Setup. We make the following three modifications to the model from Section 2. First, we shut down safe asset demand by removing idiosyncratic risk, formally $\tilde{\sigma}_t = 0$. Then, the model collapses to a representative agent economy. Second, we relabel nominal government bonds and call them "money". We denote the nominal quantity of money outstanding by $\mathcal{M}_t$ (instead of $\mathcal{B}_t$) and the nominal interest rate paid on money (e.g., interest on reserves) by $i_t^{\mathcal{M}}$. If money is associated with cash holdings, then $i_t^{\mathcal{M}} \equiv 0$ for technological reasons. We reserve the symbol i_t for the market-clearing shadow nominal interest rate on a hypothetical nominal bond that does not provide transaction services. Third, we add monetary frictions in the form of a cash-in-advance constraint in production as in Merkel (2020). In all states in which the constraint binds, this modification effectively adds a quantity equation to the model:

$$\frac{\mathcal{M}_t}{\mathcal{P}_t} = \frac{Y_t}{\mathfrak{v}_t},$$

where velocity $\mathfrak{v}_t$ follows an exogenous process. We assume that the process $\mathfrak{v}_t$, which captures the tightness of the constraint, is such that the constraint is always binding in equilibrium.

The previous paragraph is sufficient to understand how the model works. Nevertheless, let us, for the sake of completeness, provide some additional details on the monetary frictions. Specifically, we introduce monetary frictions in a very stylized form by assuming that, in order to produce, capital owners must hold sufficient money balances m_t^i such that

$$\mathcal{P}_t a u_t^i k_t^i \leq \mathfrak{v}_t m_t^i.$$

This form of specifying money demand is analytically convenient because it does not distort the intertemporal consumption decision but only the portfolio choice between money and capital. Compared to more conventional specifications, this arrangement leads to greater tractability yet preserves the qualitative features of money demand

(see Merkel, 2020 for details). Note that a reduction in maximum velocity $\mathfrak{v}_t$ corresponds to a tightening of the constraint as more money is required for the same quantity of output goods.

Adding this additional constraint to the household problem leaves most model equations unaffected, with the exception of two household choice conditions, the optimal utilization choice (18) and the optimal portfolio choice (19). When the constraint is binding, it introduces an additional wedge into both equations that depresses u_t^i and increases the excess return of capital over money.

Linking Money Demand to Aggregate Goods Demand. In this augmented model, there is no portfolio choice condition that generates a safe asset demand as in Proposition 3. However, money demand can play a role with economically similar effects as safe asset demand. Let $q_t^M := \frac{\mathcal{M}_t}{\mathcal{P}_t K_t}$ be the real value of money per unit of capital in the economy. If the money stock $\mathcal{M}_t$ follows similar dynamics as the nominal bond stock $\mathcal{B}_t$ in our baseline model, then q_t^M is effectively a state variable of the model under sticky prices, just like q_t^B in our baseline model.

Then, a shock that increases money demand, i.e., a fall in velocity $\mathfrak{v}_t$, has aggregate demand implications that are qualitatively identical to the aggregate demand implications of flight to safety in our safe asset model. To see this, note that, in equilibrium, the quantity equation can be written as

$$q_t^M = \frac{a}{\mathfrak{v}_t} u_t. \tag{61}$$

When a fall in velocity $\mathfrak{v}_t$ raises money demand and q_t^M is not free to adjust on impact, then the resulting shortage of money generates a reduction in aggregate goods demand that must be matched by an equal reduction in utilization u_t. As in our baseline model, an output gap opens up that generates deflation through the Phillips curve (which is still represented by equation (20)). The evolution of the state variable q_t^M is[57]

$$\mathrm{d}q_t^M = \left(\mu_t^{\mathcal{M}} - \pi_t - g_t\right) \mathrm{d}t,$$

such that deflation gradually raises q_t^M over time and eases the downward pressure on

[57] $\mu_t^{\mathcal{M}}$ denotes the money growth rate. All other variables are as in our baseline model.

aggregate demand, the Pigou effect. These dynamics exactly mirror the dynamics of our safe asset model described in Section 3.

Contrasting Monetary Policy in the Safe Asset and Money Models. While this simple monetary model behaves similarly to our safe asset model, the following two considerations highlight important economic differences. These differences lead to a conclusion for the ability of monetary policy to cure demand recessions in the money model that differs radically from the one we have drawn in the main text for the safe asset model.

First, when monetary policy is defined as controlling nominal interest rates, then there are really two possible rates in the money model that could serve as the policy instrument: the money rate $i_t^{\mathcal{M}}$ and the (zero net supply) bond rate i_t. In the safe asset model, there is just a single rate. Controlling the nominal interest rate in the latter model is logically equivalent to controlling the money rate $i_t^{\mathcal{M}}$ in this model. If this was the monetary policy instrument, conclusions would be similar: raising $i_t^{\mathcal{M}}$ does not affect money demand, and hence aggregate demand, on impact but leads to a faster growth rate of nominal money balances, thereby speeding up the recovery after a negative money demand shock. A variant of Proposition 9 then also holds in this model. However, controlling $i_t^{\mathcal{M}}$ is different from the traditional analysis of interest rate policy in monetary models, which is concerned with controlling i_t, the "illiquid" bond rate, while keeping $i_t^{\mathcal{M}}$ fixed, e.g., at $i_t^{\mathcal{M}} = 0$ if money takes the form of cash. In that traditional analysis, to adjust i_t, a policy institution must vary the quantity of money $\mathcal{M}_t$. Varying i_t for fixed $i_t^{\mathcal{M}}$ is thus simply not possible without making the quantity of money endogenous. Any arrangement that allows the monetary authority to control i_t will automatically also give it the power to adjust $\mathcal{M}_t$ flexibly in response to shocks. But once the monetary authority has the power to adjust $\mathcal{M}_t$ flexibly, it can also control $q_t^M = \frac{\mathcal{M}_t}{\mathcal{P}_t K_t}$, which, by equation (61), is sufficient to control aggregate economic activity.

Second, it is in fact plausible to assume that central banks can control the money supply $\mathcal{M}_t$. Central banks can employ open market operations to exchange money for non-monetary assets such as government bonds or bonds issued by private agents. If there are sufficient assets the central bank can buy (and the central bank has a sufficient

asset stock to back the initially outstanding quantity of money), then the (base) money supply is entirely under the control of the central bank for any given path of fiscal policy.[58] In contrast, in our safe asset model, it is the net supply of safe assets that matters for equilibrium dynamics. By exchanging one asset for another, the central bank cannot affect the total safe asset supply. Instead, only a change in fiscal policy can achieve this.

Formally, this second point relates to Wallace neutrality (Wallace, 1981). Money as a medium of exchange earns a convenience yield and is thus not priced according to arbitrage-free asset pricing. An exchange of money for some asset with identical risk characteristics can therefore alter the properties of the total private sector portfolio as it relaxes or tightens the cash-in-advance constraint. In our safe asset framework, in contrast, there are no convenience yields on any assets, so that increasing the supply of government bonds in exchange for, say, privately issued safe bonds (in zero net supply) is completely neutral.[59]

E Additional Details on Model Extension with Lump-sum Taxes [Supplemental Materials]

Setup. The model is exactly as the model extension with long-term government bonds as described in Section 5 and Appendix C with two modifications: (i) households have access to a zero net supply asset with return

$$\mathrm{d}R_t^Z = (r_t + \varsigma_t)\mathrm{d}t + \mathrm{d}Z_t,$$

where r_t is the real risk-free rate and ς_t is the price of risk determined in equilibrium; (ii) the government levies lump-sum taxes $\tau_t^{LS} K_t \mathrm{d}t$ in units of final goods from each agent over the time interval $[t, t + \mathrm{d}t]$. These modifications imply that the household

[58] One caveat is that if there is private money creation, then the broad money supply relevant to aggregate demand may not be directly controlled by the central bank. However, so long as increasing the quantity of base money has some (positive) effect on broader monetary aggregates, the broad money supply can be effectively controlled as well.

[59] See Section VI in Brunnermeier et al. (2024) for a formal analysis of privately issued safe assets in zero net supply. Similar conclusions carry over to our sticky price economy.

net worth evolution now takes the form

$$\mathrm{d}n_t^i = n_t^i \left(\theta_t^{B,i} \mathrm{d}R_t^B + \theta_t^{Z,i} \mathrm{d}R_t^Z + (1 - \theta_t^{B,i} - \theta_t^{Z,i}) \mathrm{d}R_t^{K,i}(\iota_t^i, \iota_t^i) \right) - c_t^i \mathrm{d}t - \tau_t^{LS} K_t \mathrm{d}t$$

and the government budget constraint becomes

$$\mathrm{d}\mathcal{B}_t = \left(i_t \mathcal{B}_t - \mathcal{P}_t(\tau_t + \tau_t^{LS}) K_t \right) \mathrm{d}t. \tag{62}$$

Model Solution. Modification (i) by itself is inconsequential as aggregate risk in our baseline model is perfectly shared. Formally, note that adding the asset with return $\mathrm{d}R_t^Z$ to the household decision problem and following the solution procedure in Appendix A will add an additional first-order condition

$$\varsigma_t^i = \varsigma_t,$$

where, as before, $-\varsigma_t^i \xi_t^i$ denotes the $\mathrm{d}Z_t$-loading of the costate ξ_t^i. Because, in our baseline model (as well as the long-term bond extension), the ultimate expression for ς_t^i turns out to be independent of i, all households find it optimal to hold a zero position in the $\mathrm{d}R_t^Z$-asset, $\theta_t^{Z,i} = 0$, so that the equilibrium allocation is unaffected.

However, in combination with modification (ii), modification (i) allows us to keep the portfolio choice formulation of the household decision problem and introduce a claim to lump-sum taxes as an additional asset. The government is long in the tax claims, while households are collectively short.[60] Denote by $q_t^T K_t$ be the present value of current and future lump-sum taxes at time t. The return on the lump-sum tax claims is

$$\mathrm{d}R_t^T = \frac{\tau_t^{LS}}{q_t^T} \mathrm{d}t + \frac{\mathrm{d}(q_t^T K_t)}{q_t^T K_t} = \left(\frac{\tau_t^{LS}}{q_t^T} + g_t + \mu_t^{q,T} \right) \mathrm{d}t + \sigma_t^{q,T} \mathrm{d}Z_t,$$

where, in analogy to previous notation, $\mu_t^{q,T}$ and $\sigma_t^{q,T}$ denote the geometric drift and aggregate volatility loading of q_t^T.

Before discussing how q_t^T is determined, we first consider, in analogy to before, a household portfolio choice problem between capital and the portfolio of safe assets

[60] This means, relative to the net worth evolution stated above, we change the definition of net worth and include the implicit (negative) wealth due to the present value of future lump-sum taxes in the measurement of n_t. While, in principle, the lump-sum tax liability is not tradeable, the asset market is sufficiently complete so that it spans the return process on the lump-sum tax claim. We can therefore proceed as if the tax claim was tradeable directly.

available to the private sector, which consists of all government liabilities net of lump-sum taxes. Specifically, denote now by $\hat{q}_t^B := \frac{\mathcal{P}_t^B \mathcal{B}_t}{\mathcal{P}_t K_t}$ the gross value of government debt per unit of capital and by $q_t^B := \hat{q}_t^B - q_t^T$ the net value once we remove the present value of lump-sum taxes. Perfect aggregate risk sharing (due to modification (i)) dictates that all households hold government bonds and the tax claim in equal proportions in their safe asset portfolios. The return on this joint safe asset portfolio is therefore ($\mathrm{d}R_t^{\hat{B}}$ is the return on gross debt, which follows an equation analogous to equation (11))

$$\begin{aligned}\mathrm{d}R_t^B &= \frac{q_t^{\hat{B}}}{q_t^B}\mathrm{d}R_t^{\hat{B}} - \frac{q_t^T}{q_t^B}\mathrm{d}R_t^T \\ &= \left(\frac{q_t^{\hat{B}}(i_t - \mu_t^{\mathcal{B}}) - \tau_t^{LS}}{q_t^B} + g_t + \mu_t^{q,B}\right)\mathrm{d}t + \sigma_t^{q,B}\mathrm{d}Z_t\end{aligned}$$

Now, by the government budget constraint, equation (62),

$$q_t^{\hat{B}}(i_t - \mu_t^{\mathcal{B}}) - \tau_t^{LS} = \tau_t,$$

so that if we define $\check{s}_t := \frac{\tau_t}{q_t^B}$ as the ratio of capital taxes relative to the net value of government liabilities, we obtain precisely the same equation as before (compare equation (11) and note that both in the baseline model and the long-term bond extension, the government budget constraint can be written as in equation (9)):

$$\mathrm{d}R_t^B = \left(\check{s}_t + g_t + \mu_t^{q,B}\right)\mathrm{d}t + \sigma_t^{q,B}\mathrm{d}Z_t.$$

Hence, the optimal portfolio choice conditions as derived in Appendix A remain valid. In addition, if we define $\vartheta_t := \frac{q_t^B}{q_t^B + q_t^K}$ as the share of net government liabilities (or safe assets) in total household wealth, then the same valuation equation (22) continues to hold (the proof is identical).

Clearly, also the price-setting problem of intermediate goods firms and any aggregate clearing conditions remain unaffected by the modifications made here. Therefore, like for the long-term bond extension, all other model equations from the baseline model remain valid as well, with the exception of the state evolution (15). Let us, once again, define $\hat{q}_t^{B,0} := \frac{\mathcal{B}_t}{\mathcal{P}_t K_t}$ as the hypothetical (gross) value of government debt if all debt was due at its face value immediately. This is the economy's state variable, which

follows the equation

$$\mathrm{d}\hat{q}_t^{B,0} = \left(i_t - \frac{\check{s}_t(\hat{q}_t^B - q_t^T) + \tau_t^{LS}}{\hat{q}_t^B} - g_t - \pi_t \right) \hat{q}_t^{B,0} \mathrm{d}t$$

and, as before,

$$\hat{q}_t^B = \frac{\lambda}{\lambda + i_t} \hat{q}_t^{B,0}.$$

However, neither $\hat{q}_t^{B,0}$ nor $\hat{q}_t^B$ are directly relevant for aggregate demand. Instead, $q_t^B = \hat{q}_t^B - q_t^T$ is.

To fully characterize the equilibrium, we therefore need to determine the evolution of q_t^T. To do so, note that the return $\mathrm{d}R_t^T$ on the tax claim can be replicated by a portfolio of investments into a risk-free asset, whose return is $(i_t^0 - \pi_t)\mathrm{d}t$, and an investment into the $\mathrm{d}Z_t$-asset, whose return is $\mathrm{d}R_t^Z$ stated above. By no arbitrage, the portfolio weights for the replicating portfolio must add to one, which leads to the condition

$$\frac{\tau_t^{LS}}{q_t^T} + g_t + \mu_t^{q,T} = r_t + \varsigma_t \sigma_t^{q,T}.$$

This equation represents a BSDE that characterizes q_t^T in equilibrium. For our implementation result below, it makes sense to solve this for the lump-sum taxes required to generate a desired q_t^T-process. Specifically, suppose q_t^T follows some Ito evolution

$$\mathrm{d}q_t^T = \mu_{q^T,t}\mathrm{d}t + \sigma_{q^T,t}\mathrm{d}Z_t \tag{63}$$

with arithmetic drift and volatility $\mu_{q^T,t}$ and $\sigma_{q^T,t}$, respectively. Substituting this into the previous equation and solving for τ_t^{LS} yields

$$\tau_t^{LS} = (r_t - g_t)q_t^T + \varsigma_t \sigma_{q^T,t} - \mu_{q^T,t}. \tag{64}$$

While not necessary to characterize the equilibrium, let us use the previous equations to derive the equilibrium evolution of the safe asset value q_t^B. Recall that $q_t^B =$

$\hat{q}_t^B - q_t^T$. Hence,

$$\begin{aligned}
\mathrm{d}q_t^B &= \mathrm{d}\hat{q}_t^B - \mathrm{d}q_t^T = \mathcal{P}_t^B \mathrm{d}\hat{q}_t^{B,0} + \hat{q}_t^{B,0} \mathrm{d}\mathcal{P}_t^B - \mathrm{d}q_t^T \\
&= \left(i_t - g_t - \pi_t\right) \hat{q}_t^B \mathrm{d}t - \check{s}_t q_t^B \mathrm{d}t - \tau_t^{LS} \mathrm{d}t - \mu_{q^T,t} \mathrm{d}t - \sigma_{q^T,t} \mathrm{d}Z_t + \hat{q}_t^{B,0} \mathrm{d}\mathcal{P}_t^B \\
&= \left(i_t - g_t - \pi_t\right) \hat{q}_t^B \mathrm{d}t - (r_t - g_t) q_t^T \mathrm{d}t - \check{s}_t q_t^B \mathrm{d}t - \varsigma_t \sigma_{q^T,t} \mathrm{d}t - \sigma_{q^T,t} \mathrm{d}Z_t + \hat{q}_t^{B,0} \mathrm{d}\mathcal{P}_t^B \\
&= \left(\left(i_t - i_t^0 + \mu_t^{\mathcal{P},B}\right) \hat{q}_t^B + \left(i_t^0 - \pi_t - g_t - \check{s}_t\right) q_t^B - \varsigma_t \sigma_{q^T,t} \right) \mathrm{d}t + \left(\sigma_t^{\mathcal{P},B} \hat{q}_t^B - \sigma_{q^T,t}\right) \mathrm{d}Z_t \\
&= \left(\left(i_t^0 - \pi_t - g_t - \check{s}_t + \varsigma_t \sigma_t^{q,B}\right) \right) q_t^B \mathrm{d}t + \left(\sigma_t^{\mathcal{P},B} \hat{q}_t^B - \sigma_{q^T,t}\right) \mathrm{d}Z_t, \qquad (65)
\end{aligned}$$

where the last equality uses equation (49) under sticky prices ($\sigma_t^{\mathcal{P}} = 0$) and the fact that difference between the expected returns of the long-term bond and the short-term bond must be the risk premium $\varsigma_t \sigma_t^{\mathcal{P},B}$ arising from price risk of the long-term bond.

The previous equation clarifies the role of interest rates and lump-sum taxes in the evolution of q_t^B. We observe that lump-sum taxes do not directly affect the drift of q_t^B. In this sense, a form of Ricardian equivalence holds in this model with regard to changes in the expected timing of lump-sum taxes that do not load on Brownian shocks. Instead, nominal interest rates i_t^0 affect the drift of q_t^B, just like in our baseline model. However, lump-sum taxes matter for how q_t^B changes in response to aggregate shocks: the risk loading of the value of the tax claim, q_t^T, enters the volatility of q_t^B. It is in this sense that controlling lump-sum taxes allows the government to flexibly adjust the safe asset value in response to shocks.

Implementation of the Constrained Efficient Allocation. We now explain how the constrained efficient allocation can be implemented in this model.

First determine $\vartheta_t^* = \vartheta^*(\tilde{\sigma}_t) > 0$ as in Proposition 12. Due to our assumption that $\tilde{\sigma}_t$ follows an Ito process driven by Brownian shocks $\mathrm{d}Z_t$, Ito's lemma implies that

$$\frac{\mathrm{d}\vartheta_t^*}{\vartheta_t^*} = \mu_t^{\vartheta *} \mathrm{d}t + \sigma_t^{\vartheta *} \mathrm{d}Z_t$$

with processes $\mu_t^{\vartheta *}$ and $\sigma_t^{\vartheta *}$ that only depend on the fundamentals of the economy and the derivatives of the function ϑ^*. Substituting the drift $\mu_t^{\vartheta *} \vartheta_t^*$ into the safe asset valuation equation (22) yields

$$\mu_t^{\vartheta *} = \rho - \left(1 - \vartheta_t^*\right)^2 \tilde{\sigma}_t^2 - \check{s}_t \Leftrightarrow \check{s}_t = \rho - \left(1 - \vartheta_t^*\right)^2 \tilde{\sigma}_t^2 - \mu_t^{\vartheta *} =: \check{s}_t^*,$$

which fully pins down the path of $\check{s}_t$.

Next, combining goods market clearing (24) with the optimal investment choice condition (17) and the definition of ϑ_t allows us to express q_t^B, q_t^K, and ι_t in terms of ϑ_t and u_t

$$q_t^B = \vartheta_t \frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho}, \qquad q_t^B = (1-\vartheta_t)\frac{1+\phi a u_t}{1-\vartheta_t+\phi\rho}, \qquad \iota_t = \frac{(1-\vartheta_t)a u_t - \rho}{1-\vartheta_t+\phi\rho}. \tag{66}$$

Plugging in $\vartheta_t = \vartheta_t^*$ and $u_t = u^*$ yields the values for these three variables in the constrained optimal equilibrium. We argue next that then also $\check{s}_t$, π_t, and g_t are fully determined by the requirement that the constrained optimum is implemented.

For $\check{s}_t$, recall that, in this model, $\check{s}_t = \frac{\tau_t}{q_t^B}$ is the ratio of capital taxes to the net value of government liabilities, we obtain for the optimal capital tax rate

$$\tau_t^* = \check{s}_t^* q_t^{B*} = \check{s}_t^* \vartheta_t^* \frac{1+\phi a u^*}{1-\vartheta_t^*+\phi\rho},$$

where $\check{s}_t^*$ is as defined above.

For π_t, let us first determine the variables that enter the Phillips curve. By equation (25), the equilibrium capital rental price must be

$$p_t^{R*} = p_t^R = \rho(u^*)^\varphi (q_t^{B*} + q_t^{K*}) = \rho(u^*)^\varphi \frac{1+\phi a u^*}{1-\vartheta_t^*+\phi\rho}.$$

In addition, the output-consumption ratio is

$$\frac{Y_t^*}{C_t^*} = \frac{a u^*}{\rho(q_t^{B*} + q_t^{K*})} = \frac{a u^*}{\rho}\frac{1-\vartheta_t^*+\phi\rho}{1+\phi a u^*}.$$

Substituting both expressions into the Phillips curve (20) yields a BSDE for $\Pi_t^* := \pi_t^* \frac{Y_t^*}{C_t^*}$,

$$\mathbb{E}_t\left[\mathrm{d}\Pi_t^*\right] = \left(\rho\Pi_t^* - \kappa\left((u^*)^{1+\varphi} - \frac{\epsilon-1}{\epsilon}\frac{a u^*}{\rho}\frac{1-\vartheta_t^*+\phi\rho}{1+\phi a u^*}\right)\right)\mathrm{d}t,$$

which can be solved forward in time to determine Π_t^*:

$$\Pi_t^* = \kappa\mathbb{E}_t\left[\int_t^\infty e^{-\rho(t'-t)}\left((u^*)^{1+\varphi} - \frac{\epsilon-1}{\epsilon}\frac{a u^*}{\rho}\frac{1-\vartheta_{t'}^*+\phi\rho}{1+\phi a u^*}\right)\mathrm{d}t'\right].$$

The implied inflation rate is then

$$\pi_t^* = \Pi_t^* \frac{a u^*}{\rho}\frac{1-\vartheta_t^*+\phi\rho}{1+\phi a u^*}.$$

Finally, for g_t, by definition (compare equation (12)) and equation (66),

$$g_t = \Phi(\iota_t) - \delta = \Phi\left(\frac{(1-\vartheta_t)au^* - \rho}{1-\vartheta_t^* + \phi\rho}\right) - \delta =: g_t^*.$$

Let us now discuss how to implement $q_t^B = q_t^{B*}$. We start by discussing dynamics. $q_t^B = q_t^{B*}$ requires

$$\mathrm{d}q_t^B = \mu_t^{q,B*} q_t^{B*}\mathrm{d}t + \sigma_t^{q,B*} q_t^{B*}\mathrm{d}Z_t,$$

where $\mu_t^{q,B*}$ and $\sigma_t^{q,B*}$ can be obtained by applying Ito's lemma to $q_t^{B*} = q^{B*}(\tilde{\sigma}_t)$ and only depend on the fundamentals of the economy and the function ϑ^*. Combining the previous equation with equation (65) and yields the two conditions

$$\begin{aligned} i_t^0 - \pi_t - g_t - \check{s}_t + \varsigma_t \sigma_t^{q,B} &= \mu_t^{q,B*}, \\ \sigma_t^{\mathcal{P},B}\hat{q}_t^B - \sigma_{q^T,t} &= \sigma_t^{q,B*} q_t^{B*}. \end{aligned}$$

The first condition, together with $\pi_t = \pi_t^*$, $g_t = g_t^*$, $\check{s}_t = \check{s}_t^*$, $\varsigma_t = \varsigma_t^*$, and $\sigma_t^{q,B} = \sigma_t^{q,B*}$, implies a condition for the nominal short rate in terms of exogenous objects,

$$i_t^0 = i_t^{0*} := \underbrace{\mu_t^{q,B*} + g_t^* + \check{s}_t^* - \varsigma_t^* \sigma_t^{q,B*}}_{=:r_t^*} + \pi_t^*.$$

Once the nominal short rate is set in this way, this fully determines the long-term interest rate i_t and the bond price $\mathcal{P}_t^B$ in terms of exogenous variables (compare Lemma 3), denote the latter by $\mathcal{P}_t^{B*}$ and its volatility loading by $\sigma_t^{\mathcal{P},B*}$. Plugging this expression in the second condition above and solving for $\sigma_{q^T,t}$ yields

$$\sigma_{q^T,t} = \sigma_t^{\mathcal{P},B*}\hat{q}_t^B - \sigma_t^{q,B*} q_t^{B*} = \sigma_t^{\mathcal{P},B*}\mathcal{P}_t^{B*}\hat{q}_t^{B,0} - \sigma_t^{q,B*} q_t^{B*}. \tag{67}$$

This expresses the required value for $\sigma_{q^T,t}$ to implement $\sigma_t^{q,B} = \sigma_t^{q,B*}$ as a function of exogenous objects (the $*$-variables) and the endogenous state variable $\hat{q}_t^{B,0}$.

To complete the description of the implementation, we note that the initial value of the lump-sum tax claim must be set such that $q_0^T = q_0^{B*} - \mathcal{P}_0^{B*}\hat{q}_0^{B,0}$. The evolution of q_t^T then follows equation (63) with $\sigma_{q^T,t}$ as in equation (67) and free drift $\mu_{q^T,t}$. Given such a drift, the required tax rate τ_t^{LS} can be backed out from equation (64). The implied

evolution of the endogenous state variable is[61]

$$\mathrm{d}\hat{q}_t^{B,0} = \left(\frac{\mu_t^{q,B*} q_t^{B*}}{\mathcal{P}_t^{B*}} - \mu_t^{\mathcal{P},B*} \hat{q}_t^{B,0} \right) \mathrm{d}t + \frac{\mu_{q^T,t}}{\mathcal{P}_t^{B*}} \mathrm{d}t.$$

In particular, it is always possible to choose $\mu_{q^T,t}$, which is otherwise free, suitably to ensure that $\hat{q}_t^{B,0}$ remains strictly positive. While not required from a mathematical perspective, this is desirable because the link between government debt and aggregate demand ceases to be economically meaningful for a nonpositive bond supply.

F Additional Details on Optimal Policy without Lump-sum Taxes [Supplemental Materials]

F.1 Theoretical Analysis

The Ramsey Problem. Recall from the main text that we are restricting attention to the case without investment, $\phi \to \infty$, and to deterministic dynamics with given (exogenous) bounded paths for $\tilde{\sigma}_t$, $\check{s}_t$, and ϑ_t, such that ϑ_t is bounded away from zero. Restricting attention to deterministic dynamics is necessary to apply the deterministic optimal control and differential equation arguments below. While it may be possible to extend our results to a stochastic setting, this would be a significant extension that requires different methods. In contrast, the assumption $\phi \to \infty$ is made primarily to make our algebraic derivations more transparent and tractable. It should, in principle, be possible to extend the following analysis to the case with physical investment without additional conceptual difficulties.

The monetary authority seeks to maximize the objective function

$$\int_0^\infty e^{-\rho t} \mathcal{W}_u(u_t) dt, \qquad \mathcal{W}_u(u) := \log(u) - \frac{u^{1+\varphi}}{1+\varphi}, \tag{68}$$

which is justified by Proposition 12. u_t satisfies the demand equation (27). Using $\iota_t = 0$ (due to $\phi \to \infty$), we can write this equation as

$$u_t = \frac{\rho}{a} \frac{q_t^{B,0}}{\vartheta_t} \mathcal{P}_t^B. \tag{69}$$

[61] Note that $\mathrm{d}\hat{q}_t^B = \mathrm{d}q_t^B + \mathrm{d}q_t^B$, the first term has been computed in equation (65), and $\mathrm{d}\hat{q}_t^{B,0} = \hat{q}_t^{B,0}(\mu_t^{\hat{q},B} - \mu_t^{\mathcal{P},B} + \sigma_t^{\mathcal{P},B}(\sigma_t^{\mathcal{P},B} - \sigma_t^{\hat{q},B}))\mathrm{d}t$ (Ito's ratio rule).

We consider $\mathcal{P}_t^B = \frac{\lambda}{\lambda + i_t}$ as the policy instrument. Due to the one-to-one relationship between $\mathcal{P}_t^B$ and i_t, this is equivalent to making i_t the policy instrument. The monetary authority takes the path for ϑ_t as given and has to respect the state evolution

$$\frac{\mathrm{d}q_t^{B,0}}{\mathrm{d}t} = \left(\lambda\left(\frac{1}{\mathcal{P}_t^B} - 1\right) - \pi_t - \check{s}_t - g\right) q_t^{B,0}, \qquad q_0^{B,0} \text{ given} \tag{70}$$

as a constraint. Here, $\check{s}_t$ is, like ϑ_t, exogenously given, and the capital growth rate g is simply a constant parameter. The remaining variable, π_t, is endogenous and follows from the Phillips curve, which simplifies here to (this follows from equation (20) by using $C_t = Y_t$ and imposing deterministic dynamics)

$$\frac{\mathrm{d}\pi_t}{\mathrm{d}t} = \rho\pi_t - \kappa\left(u_t^{1+\varphi} - \frac{\epsilon - 1}{\epsilon}\right), \qquad \lim_{T\to\infty} e^{-\rho T}\pi_T = 0. \tag{71}$$

Here, we have explicitly added the terminal condition that discounted terminal inflation must vanish in the long run. This follows from the optimal price setting problem of intermediate goods firms analyzed in Appendix A.[62]

Under commitment, the problem of the monetary authority is to choose a path for $\mathcal{P}_t^B$ to maximize (68) subject to conditions (69), (70), and (71). In addition, we impose as a requirement on admissible policies that $\mathcal{P}_t^B$ and the implied path for $q_t^{B,0}$ are bounded and bounded away from zero. The assumption on $\mathcal{P}_t^B$ is equivalent to the requirements on long-term interest rates i_t in Proposition 11. The additional assumption on the state variable $q_t^{B,0}$ is without loss of generality because it only excludes policies that are clearly suboptimal. Specifically, by equation (69), it excludes policies that lead to unbounded $\log u_t$, but these can never be optimal because $\mathcal{W}_u(u) \to -\infty$ for $\log u \to \pm\infty$.

For the problem under discretion, we first define an auxiliary problem with imperfect commitment along the lines of Brunnermeier et al. (2020). Specifically, we consider a sequence of planners, indexed by $j = 1, 2, \ldots$, each of which seeks to maximize objective (68) subject to the same constraints as for the Ramsey planner under commitment. However, planner j only controls the path of $\mathcal{P}_t^B$ over the time interval $[(j-1)\Delta, j\Delta]$, where $\Delta > 0$ is the commitment horizon. An optimal policy under commitment with

[62] Specifically, in the notation from that appendix, we have derived the condition $\psi X_t \pi_t^j = \lambda_t^j \mathcal{P}_t^j$ and the right-hand side of this equation must converge to zero by a standard transversality condition. We show later in that appendix that $X_t = e^{-\rho t}\frac{Y_t}{C_t}$, which here simplifies to $X_t = e^{-\rho t}$.

horizon Δ is defined as an equilibrium in the game played by the sequence of planners. Following the usual assumptions in the literature on optimal monetary policy under discretion, we are interested in a specific form of equilibrium in this game in which each planner j's reaction to previous planners' choices depends only on the initial state $q^{B,0}_{(j-1)\Delta}$ faced by planner j but not on any other details of previous policy actions.[63] We define the optimal policy under discretion as the limit outcome for vanishing commitment horizon, $\Delta \to 0$.

A Monotonicity Lemma. Before analyzing the two policy problems further, we show that feasible outcomes for utilization u_t are monotonic in the initial state $q^{B,0}_0$ in the following (weak) sense:

Lemma 4. *Consider two initial conditions, $q^{B,0}_0 = q^{B,0,(1)}_0, q^{B,0,(2)}_0$ with $q^{B,0,(1)}_0 < q^{B,0,(2)}_0$ and two feasible policies $\mathcal{P}^{B,(1)}_t$, $\mathcal{P}^{B,(2)}_t$. Denote by $u^{(i)}_t$ the path for utilization under policy $\mathcal{P}^{B,(j)}_t$ for $j = 1, 2$. Then, there is a $t \geq 0$ such that $u^{(1)}_t < u^{(2)}_t$.*

Proof. Suppose otherwise. Then $u^{(1)}_t \geq u^{(2)}_t$ for all t. Integrating the Phillips curve (71) forward in time yields

$$\pi_t = \kappa \int_t^\infty e^{-\rho(t'-t)} \left(u_{t'}^{1+\varphi} - \frac{\epsilon - 1}{\epsilon} \right) \mathrm{d}t',$$

which is strictly increasing in $\{u_{t'}\}_{t \geq t'}$, so that necessarily $\pi^{(1)}_t \geq \pi^{(2)}_t$. Next, note that

$$\frac{\mathrm{d}(q^{B,0,(1)}_t / q^{B,0,(2)}_t)}{q^{B,0,(1)}_t / q^{B,0,(2)}_t \mathrm{d}t} = i^{(1)}_t - i^{(2)}_t - \pi^{(1)}_t + \pi^{(2)}_t \leq i^{(1)}_t - i^{(2)}_t,$$

where, for any variable x_t, $x^{(j)}_t$ denotes its path under policy $\mathcal{P}^{B,(j)}_t$ for initial condition $q^{B,0,(j)}_t$. Furthermore, the difference in interest rates satisfies

$$\begin{aligned} i^{(1)}_t - i^{(2)}_t &= \lambda \left(\frac{1}{\mathcal{P}^{B,(1)}_t} - \frac{1}{\mathcal{P}^{B,(2)}_t} \right) = \lambda \frac{\rho}{a\vartheta_t} \left(\frac{q^{B,0,(1)}_t}{u^{(1)}_t} - \frac{q^{B,0,(2)}_t}{u^{(2)}_t} \right) \\ &\leq \lambda \frac{\rho}{a\vartheta_t} \frac{q^{B,0,(2)}_t}{u^{(2)}_t} \left(\frac{q^{B,0,(1)}_t}{q^{B,0,(2)}_t} - 1 \right) \end{aligned}$$

[63] If the remaining fundamentals of the economy ($\tilde{\sigma}_t$ and $\check{s}_t$) follow an autonomous law of motion, this essentially means that we are looking for a Markov-perfect equilibrium of the game.

Combining this with the previous inequality yields

$$\frac{\mathrm{d}(q_t^{B,0,(1)}/q_t^{B,0,(2)})}{q_t^{B,0,(1)}/q_t^{B,0,(2)}\mathrm{d}t} \leq \lambda \frac{\rho}{a\vartheta_t} \frac{q_t^{B,0,(2)}}{u_t^{(2)}} \left(\frac{q_t^{B,0,(1)}}{q_t^{B,0,(2)}} - 1 \right).$$

Note that the right-hand side is strictly negative whenever $q_t^{B,0,(1)}/q_t^{B,0,(2)} < 1$. Because this is true, by assumption, for $t = 0$, the comparison theorem implies that it must be true for all $t \geq 0$. In particular, $q_t^{B,0,(1)}/q_t^{B,0,(2)}$ must decay over time.

Now, consider the limit $x := \lim_{t\to\infty} (q_t^{B,0,(1)}/q_t^{B,0,(2)}) \in [0,1)$. Distinguish two cases:

(i) If $x = 0$, then $q_t^{B,0,(1)}$ and $q_t^{B,0,(2)}$ cannot both be bounded and bounded away from zero. Either $\mathcal{P}_t^{B,(1)}$ or $\mathcal{P}_t^{B,(2)}$ must therefore be an inadmissible policy.

(ii) If $x > 0$, then necessarily $\lambda \frac{\rho}{a\vartheta_t} \frac{q_t^{B,0,(2)}}{u_t^{(2)}} \to 0$ as $t \to \infty$. Because ϑ_t is bounded, this is inconsistent with both $q_t^{B,0,(2)}$ and $u_t^{(2)} = \frac{\rho}{a} \frac{q_t^{B,0,(2)}}{\vartheta_t} \mathcal{P}_t^{B,(2)}$ being bounded and bounded away from zero. Therefore, $\mathcal{P}^{B,(2)}$ cannot be an admissible policy.

In either case, we have established a contradiction. Hence, admissible policies such that $u_t^{(1)} \geq u_t^{(2)}$ for all t do not exist. □

Existence of $q^{B,0*}$. We next show that there must be some initial condition $q^{B,0} = q^{B,0*}$ for which the constrained optimal utilization rate $u_t = u^*$ can be implemented. The proof is constructive and follows the same arguments as the implementation with lump-sum taxes in Appendix E.

Specifically, $u_t = u^*$ pins down inflation dynamics, so that, to match the drift dynamics of q_t^{B*} ($q_t^B = q_t^{B*}$ is a necessary condition for $u_t = u^*$), we need one, and only one, path for the nominal short rate i_t^0 that is independent of the initial condition. The argument is precisely the same as in Appendix E, so that we omit the details here. This unique path for the nominal short rate fully pins down the path for the nominal bond price, $\mathcal{P}_t^B = \mathcal{P}_t^{B*}$. Clearly, there is then precisely one initial condition $q_0^{B,0}$ for which also the initial level of q_t^B is consistent with the constrained optimum, namely

$$q_0^{B,0} = \frac{q_0^{B*}}{\mathcal{P}_0^{B*}}.$$

For this initial condition, following the policy $\mathcal{P}_t^B = \mathcal{P}_t^{B*}$ then ensures $q_t^B = q_t^{B*}$, and hence $u_t = u^*$, by construction because there are no further shocks.

Analyzing First-order Conditions of the Planner under Commitment. We solve the problem of the planner with the maximum principle. Unlike in other appendices, which have expressed Hamiltonians as present value Hamiltonians (in units of time-zero utility), it is here notationally simpler to work with the current value Hamiltonian (in units of time-t utility). We point this out here to clarify why the equations for the multiplier look seemingly different than in other parts of this paper. The current value Hamiltonian for the planner's problem is

$$H_t = \mathcal{W}_u(u_t) + \nu_t \left(\left(\lambda \left(\frac{1}{\mathcal{P}_t^B} - 1 \right) - \pi_t - \breve{s}_t - g \right) q_t^{B,0} \right) + \eta_t \left(\rho \pi_t - \kappa \left(u_t^{1+\varphi} - \frac{\epsilon - 1}{\epsilon} \right) \right),$$

where ν_t is the multiplier on the backward-looking constraint and η_t is the multiplier on the forward-looking constraint. Optimality necessarily requires the following three first-order conditions. First, the optimal choice condition with respect to the policy variable $\mathcal{P}_t^B$ is

$$0 = \frac{\partial H_t}{\partial u_t} \frac{\partial u_t}{\partial \mathcal{P}_t^B} - \nu_t \lambda \frac{q_t^{B,0}}{(\mathcal{P}_t^B)^2}. \tag{72}$$

Second, the costate equation for the multiplier on the backward-looking constraint is

$$\frac{\mathrm{d}\nu_t}{\mathrm{d}t} = \rho \nu_t - \frac{\partial H_t}{\partial u_t} \frac{\partial u_t}{\partial q_t^{B,0}} - \nu_t \left(\lambda \left(\frac{1}{\mathcal{P}_t^B} - 1 \right) - \pi_t - \breve{s}_t - g \right). \tag{73}$$

Third, the state equation for the multiplier on the forward-looking constraint is

$$\frac{\mathrm{d}\eta_t}{\mathrm{d}t} = \rho \eta_t - \frac{\partial H_t}{\partial u_t} \frac{\partial u_t}{\partial \pi_t} + \nu_t q_t^{B,0} - \eta_t \rho. \tag{74}$$

We can solve the first first-order condition for $\frac{\partial H_t}{\partial u_t}$, which appears in all three conditions,

$$\frac{\partial H_t}{\partial u_t} = \nu_t \lambda \frac{q_t^{B,0}}{\mathcal{P}_t^B} \frac{1}{\frac{\partial u_t}{\partial \mathcal{P}_t^B} \mathcal{P}_t^B} = \nu_t \lambda \frac{q_t^{B,0}}{\mathcal{P}_t^B} \frac{1}{u_t} = \nu_t \lambda \frac{1}{(\mathcal{P}_t^B)^2} \underbrace{\vartheta_t \frac{a}{\rho}}_{= q_t^{B*}}.$$

Plugging this previous expression into equation (73) yields

$$\begin{aligned}\frac{\mathrm{d}\nu_t}{\mathrm{d}t} &= -\left(-\rho + \lambda \frac{q_t^{B*}}{(\mathcal{P}_t^B)^2}\frac{\partial u_t}{\partial q_t^{B,0}} + \lambda\left(\frac{1}{\mathcal{P}_t^B} - 1\right) - \pi_t - \check{s}_t - g\right)\nu_t \\ &= -\left(\lambda\frac{1}{\mathcal{P}_t^B} - \rho + \lambda\left(\frac{1}{\mathcal{P}_t^B} - 1\right) - \pi_t - \check{s}_t - g\right)\nu_t.\end{aligned}$$

This equation is useful for two reasons. First, it tells us that ν_t must have the same sign for all t (by our assumptions, the first factor is bounded, so decay to zero cannot happen in finite time). Second, we see that ν_t must decay at a rate $\lambda\frac{1}{\mathcal{P}_t^B} - \rho + \mu_t^{q,B,0}$, so $\nu_t q_t^{B,0}$ decays at rate $\lambda\frac{1}{\mathcal{P}_t^B} - \rho = \lambda + i_t - \rho$. We will use this fact below.

Next, consider condition (74). Because $\frac{\partial u_t}{\partial \pi_t} = 0$, this simplifies to

$$\frac{\mathrm{d}\eta_t}{\mathrm{d}t} = \nu_t q_t^{B,0}$$

with the semi-closed-form solution

$$\eta_t = \eta_0 + \int_0^t \nu_{t'} q_{t'}^{B,0}\mathrm{d}t' = \int_0^t \nu_{t'} q_{t'}^{B,0}\mathrm{d}t'$$

where the second equality follows from the initial condition $\eta_0 = 0$ that must hold because this is the multiplier on a controlled backward equation. This implies that, for any $t > 0$, also η_t has the same sign as ν_t. Furthermore, $|\eta_t|$ is strictly increasing over time, so that $\eta_\infty := \lim_{t\to\infty}\eta_t$ exists (though it could be infinite). η_∞ is certainly finite if $\nu_t q_t^{B,0}$ decays asymptotically at a positive exponential rate.

Let us next compute $\frac{\partial H_t}{\partial u_t}$. We have

$$\lambda\frac{q_t^{B*}}{(\mathcal{P}_t^B)^2}\nu_t = \frac{\partial H_t}{\partial u_t} = \mathcal{W}_u'(u_t) - \rho\kappa(1+\varphi)u_t^{\varphi}\eta_t.$$

Hence,

$$\mathcal{W}_u'(u_t) = \lambda\frac{q_t^{B*}}{(\mathcal{P}_t^B)^2}\nu_t + \rho\kappa(1+\varphi)u_t^{\varphi}\eta_t$$

and the sign of the right-hand side is always the same as the sign of ν_t (which is constant over time). We observe that $\mathcal{W}_u'(u_t) > 0 \Leftrightarrow u_t < u^*$ if and only if $\nu_t > 0$, and that the sign of $u_t - u^*$ is constant over time. From Lemma 4, we can conclude that $u_t < u^*$ if

and only if $q_0^{B,0} < q_0^{B,0*}$ and that $u_t > u^*$ if and only if $q_0^{B,0} > q_0^{B,0,*}$. These conclusions establish the first part of Proposition 14.

Let us conclude this section by analyzing the convergence behavior of u_t as $t \to \infty$. Convergence of u_t requires convergence of $\mathcal{W}_u'(u_t)$. From the previous equation, we observe that

$$\mathcal{W}_u'(u_t) = (\lambda + i_t)\frac{u_t^*}{u_t}\nu_t q_t^{B,0} + \rho\kappa(1+\varphi)u_t^{\varphi}\eta_t.$$

Note that, because u_t is required to be bounded away from zero, the last term can only converge if η_∞ remains finite. This, in turn, requires necessarily $\nu_t q_t^{B,0} \to 0$. If this necessary condition is satisfied, then, because also $(\lambda + i_t)\frac{u_t^*}{u_t}$ must be bounded, we obtain in the limit

$$\mathcal{W}_u'(u_\infty) = \frac{1}{u_\infty} - u_\infty^{\varphi} = \rho\kappa(1+\varphi)u_\infty^{\varphi}\eta_\infty,$$

where $u_t := \lim_{t\to\infty} u_t$. It is easy to see that this equation has a unique solution that is positive and finite if

$$\eta_\infty \in \left(-\frac{1}{\rho\kappa(1+\varphi)}, \infty\right).$$

We are here interested in the recession case $q_0^{B,0} < q_0^{B,0*}$, in which $\eta_t \geq 0$, so all that is required is $\eta_\infty < \infty$.

As discussed previously, this is certainly satisfied if $\nu_t q_t^{B,0}$ decays asymptotically exponentially with a positive decay rate. We also know that the decay rate of $\nu_t q_t^{B,0}$ is $\lambda + i_t - \rho$. We show next that this is strictly positive for sufficiently large λ.

For a proof, note that

$$\begin{aligned}\frac{\mathrm{d}(\nu_t(q_t^{B,0})^2)}{\nu_t(q_t^{B,0})^2\mathrm{d}t} &= -(\lambda + i_t - \rho) + i_t - \pi_t - \check{s}_t - g \\ &= \rho - \lambda - \pi_t - \check{s}_t - g \leq \rho - (\lambda + g + \underline{\check{s}} + \underline{\pi}),\end{aligned}$$

where $\underline{\check{s}} := \inf_{t\geq 0}\check{s}_t > -\infty$ and $\underline{\pi} > -\infty$ is some lower bound on inflation for the optimal policy for arbitrarily large λ (which may be lower than $\inf_{t\geq 0}\pi_t$). Such a lower bound always exists. E.g., a crude one is $\underline{\pi} = -\frac{\kappa}{\rho}\frac{\epsilon-1}{\epsilon}$, which follows immediately from the Phillips curve (71) and the fact that $u_t \geq 0$. With the above,

$$\nu_t q_t^{B,0} = \frac{1}{q_t^{B,0}}\nu_t(q_t^{B,0})^2 \leq \frac{\nu_0(q_0^{B,0})^2}{q_t^{B,0}}e^{(\rho-\lambda-g-\underline{\check{s}}-\underline{\pi})t}.$$

Because $q_t^{B,0}$ is bounded and bounded away from zero, this converges to zero as $t \to \infty$ if and only if the exponential converges. This is the case for

$$\lambda > \rho - g - \underline{\check{s}} - \underline{\pi},$$

which is certainly satisfied for sufficiently large λ.[64]

Finally, we remark that, unless $q_0^{B,0} = q^{B,0*}$, $\nu_t \neq 0$, which implies $\eta_t \neq 0$ and, because $|\eta_t|$ is strictly increasing over time, also $\eta_\infty \neq 0$. Therefore, $\mathcal{W}'(u_\infty) \neq 0$, so that u_t does not converge to u^*. In the recession case, $q_0^{B,0} < q^{B,0*}$, the previous arguments imply further that $u_\infty < u^*$.

Analyzing First-order Conditions of the Planner under Discretion. For the active planner at time t in the sequence of planners (for finite commitment horizon Δ), the Hamiltonian is again H_t as defined above. Similarly, the first-order conditions (72), (71), and (70) remain valid. However, the initial and terminal conditions change. Planner j faces an additional terminal condition $\nu_{j\Delta}^{(j)} = \nu_{j\Delta}^{(j+1)}$, where $\nu_t^{(j)}$ denotes the time-t value of the costate faced by planner j for $t \in [(j-1)\Delta, j\Delta]$. Furthermore, because planner j reoptimizes and is not bound by past commitments, the initial condition for $\eta_t^{(j)}$ at $t = (j-1)\Delta$ is $\eta_{(j-1)\Delta}^{j} = 0$. If we define $\nu_t := \nu_t^{(\lceil t/\Delta \rceil)}$ and $\eta_t := \eta_t^{(\lceil t/\Delta \rceil)}$, then ν_t follows the same equation (71) as under commitment regardless of the value of Δ. The multiplier η_t follows, in principle, also the same equation (70) as under commitment, except that η_t is periodically reset to zero at dates $t \in \{0, \Delta, 2\Delta, \dots\}$. Therefore, in the limit $\Delta \to \infty$, $\eta_t \equiv 0$. Hence, the optimal solution satisfies the same conditions as the optimal policy under commitment, except that the state equation (70) for the multiplier η_t is replaced with the condition $\eta_t \equiv 0$. Many of the previous arguments therefore remain valid.

First, the argument that ν_t never changes sign does not rely on any condition for η_t and therefore remains valid. Similarly, the decay rate of ν_t is still the same, so that we obtain convergence $e^{\rho t}\nu_t \to 0$ under the same conditions as above.

[64] The bound here is very crude due to the crude lower bound on $\underline{\pi}$. We make two remarks here. First, any argument that leads to a larger lower bound for inflation improves on the set of λ-values for which we can predict long-run convergence to u^*. For example, if $q_t^{B,0} > q^{B,0*}$ (the boom case), we know $u_t > u^*$ and therefore $\underline{\pi} = \pi^* > 0$. Second, a refinement of the previous argument also works if $\underline{\pi}$ is not a lower bound for inflation but merely an eventual lower bound for all $t \geq t_0$ and some t_0.

Second, the discussion of the limit behavior of $\mathcal{W}'_u(u_t)$ remains valid as well, except that now $\eta_t = 0$, so that $e^{\rho t}\nu_t \to 0$ implies $\mathcal{W}'_u(u_t) \to 0$ with the consequence that $u_t \to u^*$.

F.2 Numerical Examples

In this section, we illustrate the optimal policy for a numerical example. Specifically, we start the economy in the constrained efficient steady state of the model without aggregate shocks using the same parameters as in Section 3, except for the debt duration parameter which we set to $\lambda = 0.2$, which corresponds to a debt duration of 5 years, approximately consistent with the average duration of U.S. government debt. We consider an unanticipated permanent increase in $\tilde{\sigma}$ at $t = 0$ to a new level without a change in $\check{s}$, so that ϑ_t also jumps upward at $t = 0$ and thereafter remains constant that higher level.[65] Importantly, this setup implies that $q_0^B < q_0^{B*}$, so that Proposition 14 implies permanent under-utilization relative to the constrained efficient level.

Figure 4 depicts the dynamics of the (long-term) nominal interest rate i_t, the bond price $\mathcal{P}_t^B$, the capital utilization rate u_t, and the inflation rate π_t under the optimal interest rate policy. In all panels, we plot the (percentage or percentage point) deviations from the constrained efficient equilibrium that would be attainable if lump-sum taxes were available. The blue lines show the optimal policy under discretion, and the red lines under commitment. For comparison, we also plot dynamics under the (non-optimal) policy of setting the nominal interest rate to its value in the constrained efficient equilibrium, $i_t = i^*$, depicted by the dashed black lines. In this example, the interest rate under optimal policy is lowered initially, both under discretion and commitment, to raise $\mathcal{P}_t^B$. This mitigates the initial depth of the recession considerably compared to the benchmark policy $i_t = i^*$. The reduction in interest rates is more aggressive and persistent than under discretion. Most importantly, the discretion and commitment solutions differ with respect to their long-run behavior. For the former, all variables return to the (new) constrained optimal steady state in the long run. In contrast, interest rates are kept permanently lower under commitment. Doing so depresses

[65] It is not important to assume that $\check{s}$ does not change. It could also change to a new constant level. So long as ϑ_t still jumps upward and remains constant thereafter, the qualitative takeaways from the numerical example would be the same.

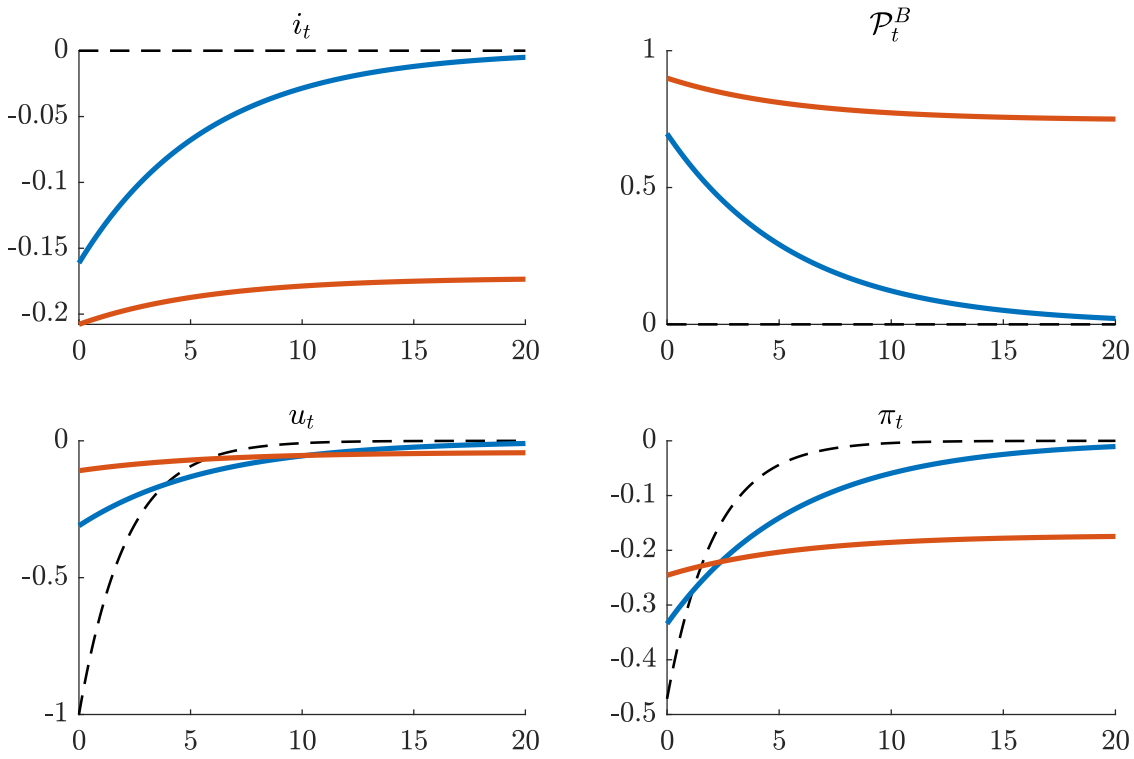


Figure 4: Optimal policy under discretion (blue) and commitment (red) in response to a permanent increase in $\tilde{\sigma}$ at $t = 0$. Panels for $\mathcal{P}_t^B$ and u_t depict percentage deviations, and panels for i_t and π_t depict percentage point differences from the values of these variables in the constrained efficient equilibrium (that would be attainable with lump-sum taxes). Black dashed lines depict dynamics under the (non-optimal) policy $i_t = i^*$ that sets the nominal interest rate to its constrained efficient value.

inflation and resource utilization in the long run, which helps the monetary authority to achieve almost the same initial inflation reduction at $t = 0$ as under discretion despite a much milder initial recession. In other words, the promise to never return economic activity back to its (constrained) optimal level allows the monetary authority to create more disinflation with less pain in the form of reduced output. Note that disinflation by itself is not costly but beneficial as it expands the real value of safe assets more rapidly.

Figures 5 and 6 show how the conclusions about optimal monetary policy depend on government debt duration $1/\lambda$. Figure 5 depicts comparative statics of the optimal interest rate policy under discretion (measured by the long-term nominal rate i_t for the respective duration $1/\lambda$) and the implied output response (measured by u_t)

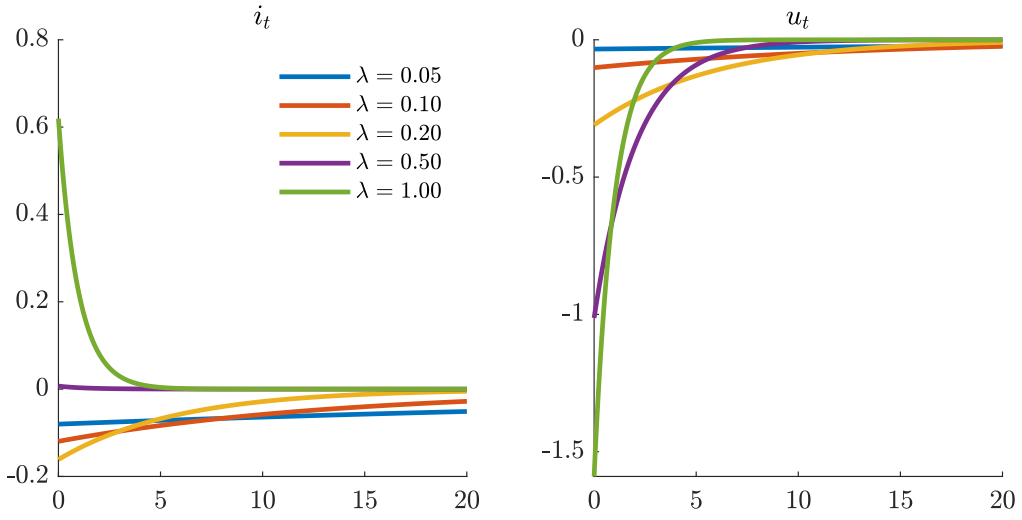


Figure 5: Optimal policy under discretion for different government debt duration ($1/\lambda$). Variables definitions and scales are as in Figure 4. The lines for $\lambda = 0.2$ correspond to the blue lines in that figure.

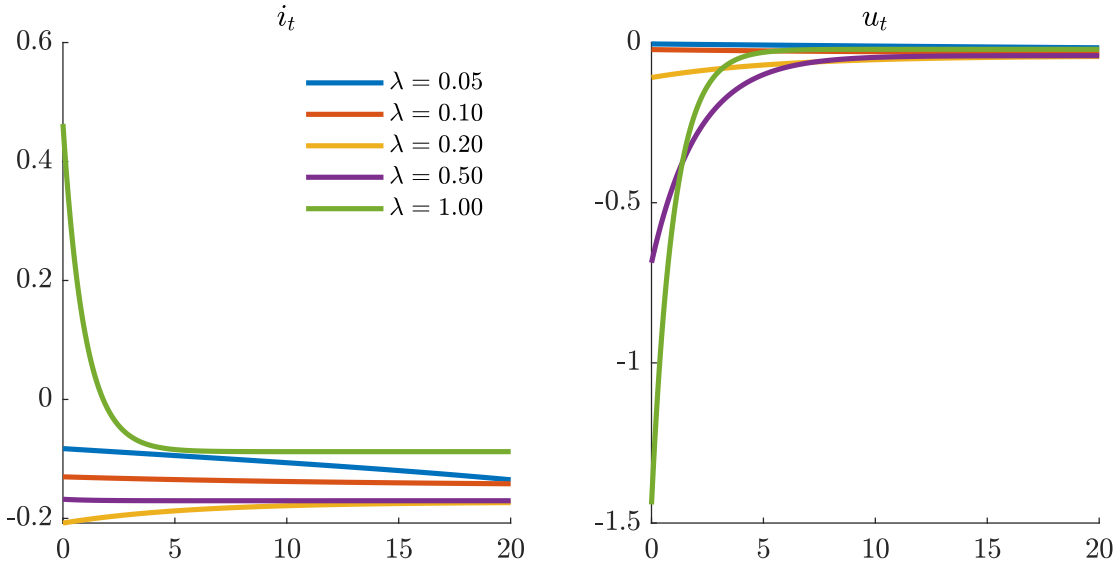


Figure 6: Optimal policy under commitment for different government debt duration ($1/\lambda$). Variables definitions and scales are as in Figure 4. The lines for $\lambda = 0.2$ corresponds to the red lines in that figure.

with respect to λ. The main takeaway is that the optimal interest rate policy depends crucially on debt duration, in contrast to the standard analysis in RANK models. For sufficiently long-term debt, i.e., low λ, interest rates are reduced and more persistently so the longer is the duration. However, if the debt duration is sufficiently short, then the optimal policy looks very different. It is then optimal to raise the interest rate because the debt growth channel is much more potent than the revaluation channel. This difference in optimal interest rate policy is reflected in the shape of the output response depicted in the right panel of Figure 5. With long-duration debt, it is optimal to engineer shallow but drawn-out recessions, whereas with short-duration debt, the optimal policy features deep recessions with fast recoveries.

Figure 6 shows the same comparative statics for the optimal policy under commitment. We observe that the same general takeaway applies: lower debt duration is associated with higher nominal interest rates in the short run and deeper recessions with faster recoveries. The main difference compared to the discretion case is that, under commitment, the economy is not returned to the constrained efficient level of economic activity, which allows the monetary authority to generate additional deflation despite a higher level of economic activity initially. The extent of the long-run under-utilization is non-monotonic in debt duration but has an inverted U-shape. It is highest for intermediate values of λ and lower for both very high and very low values. This is because, both for very high and very low debt duration, there is little conflict between the two channels through which interest rates affect economic activity: for very low-duration debt, the debt growth channel clearly dominates (because bond prices are almost insensitive to interest rates) and for very high-duration debt, the revaluation channel dominates (because then even minor changes in the interest rate have large effects on the real value of safe assets). In both cases, disinflation is less important as a mechanism to bring the safe asset supply closer to its efficient level. In the former case, it grows very fast on its own. In the latter, the nominal bond price does most of the work. Benefits from distorting long-run outcomes to generate additional disinflation in the short run are therefore small. For intermediate debt duration, in contrast, the two channels are in conflict with each other, and the Pigou effect has a greater role to play. In this case, generating additional disinflation at the cost of long-run distortions is most beneficial.

G Calibration and Numerical Solution [Supplemental Materials]

Calibration. Table 1 summarizes the model parameters for the numerical illustrations provided in the main text. We take the technology parameters a, ϕ directly from Brunnermeier et al. (2024). We also base our calibration of the parameters b, $\bar{\sigma}$, and ν on that paper but adjust the values reported there such that the implied dynamics of the exogenous component of the idiosyncratic capital risk premium are identical to their quantitative model, which features more general preferences and partial sharing of aggregate risk.[66] We also follow their strategy to calibrate the depreciation rate δ to match a steady state growth rate of 2.1% and the time discount rate ρ to match a household portfolio share in safe assets of $\vartheta^{ss} = 15\%$ in the stochastic steady state. While not directly comparable, this value for ϑ^{ss} is also inside the range of numbers for the share of liquid wealth reported by Bayer et al. (2023).[67] For the New Keynesian block, we follow Kaplan et al. (2018) in targeting a Phillips curve slope of $\kappa = 0.1$. Note that $\kappa = \epsilon/\psi$, so that the price adjustment cost parameter ψ is determined by that target for κ and a choice of the substitution elasticity ϵ of goods demand across varieties. Because markups and monopolistic competition per se play no particular role in this paper, we choose the limit $\epsilon \to \infty$, so that there is no monopoly distortion in the flexible price economy.[68] For the curvature of the disutility of utilization effort, we choose a value of $\varphi = 1.5$. While the literature does not provide any direct guidance on this parameter, this choice is in the range of common values for the inverse Frisch elasticity parameter

[66] Specifically, the idiosyncratic capital risk premium in their model is $(1-\vartheta_t)\gamma\bar{\chi}^2\tilde{\sigma}_t^2$, where γ is the coefficient of relative risk aversion and $\bar{\chi}$ is the fraction of undiversifiable idiosyncratic risk. In contrast, in our model, the idiosyncratic capital risk premium is $(1-\vartheta_t)\tilde{\sigma}_t^2$. By "exogenous component" we mean the factors $\gamma\bar{\chi}^2\tilde{\sigma}_t^2$ and $\tilde{\sigma}_t^2$, respectively, as these only depend on parameters and exogenous processes. In Brunnermeier et al. (2024), $\gamma\bar{\chi}^2 = 0.54$, so we need to scale their mean reversion level $\bar{\sigma}^2$ by 0.54 and their volatility ν by $\sqrt{0.54}$ to generate identical dynamics.

[67] Specifically, their Figure 1 shows that the ratio of (aggregate) liquid to illiquid assets based on the flow of funds fluctuates between 0.14 and 0.3 over their sample, which translates into liquid wealth shares between 12% and 23%. In Table III, they report a lower estimate of the same ratio of 0.09 based on the Survey of Consumer Finances, which translates into a liquid wealth share of 8%.

[68] Once the Phillips curve slope is fixed, the parameter ϵ only matters for the flexible price rental price $p^{R,flex} = a\frac{\epsilon-1}{\epsilon}$, which appears in the Phillips curve (20) as the target level for marginal costs. Effects on dynamics from changes in ϵ (for fixed κ) are therefore minimal and, hence, this parameter is fairly unimportant. Indeed, all our figures look very similar for the choices in Kaplan et al. (2018), $\epsilon = 10$ and $\psi = 100$.

in the disutility of labor in preference specifications that have an identical functional form.

As explained in Section 3.1, for the sake of a transparent discussion of the effects of an uncertainty shock, our baseline assumption is the admittedly unrealistic special case that government policy variables i and $\check{s}$ are kept constant throughout. This allows us to isolate the effects of the shock itself as opposed to the effects that arise from the policy reaction to the shock. We set the nominal interest rate i such that the inflation rate in the stochastic steady state is zero. For $\check{s}$, we set a value that strikes a balance between two goals: (i) to obtain a steady-state value of the ratio of primary surpluses to GDP that is empirically plausible (the postwar average of this ratio is close to zero but slightly negative in US data) and (ii) to keep $\check{s}$ positive in order to have a standard fiscal rule that stabilizes government debt in the long run and to prevent the possibility of rational bubbles in government debt (on the latter point, see Brunnermeier et al., 2020). While our analysis also works for nonpositive $\check{s}$, we require (ii) so as to emphasize that none of our results hinge on either a policy specification that makes surpluses unresponsive to outstanding government debt or public debt bubbles. To strike a balance between goals (i) and (ii), we target a steady-state primary surplus of 0.1% of GDP.

Numerical Solution Algorithm. In Section 2.3, we have characterized equilibrium up to the two dynamic equations (3) and (20) for the equilibrium portfolio weight on bonds ϑ_t and the transformed inflation rate $\Pi_t := \pi_t \frac{Y_t}{C_t}$, respectively. Mathematically, these represent backward stochastic differential equations (BSDEs). To solve them, we first use a standard procedure to transform them into partial differential equations (PDEs). Specifically, we impose a Markovian structure $\vartheta_t = \vartheta(\tilde{\sigma}_t, q_t^B)$, $\Pi_t = \Pi(\tilde{\sigma}_t, q_t^B)$ and apply Itô's lemma, which allows us to express the drifts of ϑ_t and Π_t in terms of derivatives of the unknown functions Π and ϑ. We then equate these drifts with the ones required by equations (20) and (3) to obtain PDEs for the functions ϑ and Π. We solve these equations with an iterative finite difference method as described, for example, in Brunnermeier and Sannikov (2016). This procedure determines a global (nonlinear) solution of the model.[69]

[69]Solving the model in this way can be computationally involved and potentially unstable. In special cases in which the conditions of Proposition 4 are satisfied, we improve the efficiency and stability of

Parameter	Description	Value
	Preferences	
ρ	time preference	0.1192
φ	utilization disutility	1.5
	Production	
a	productivity	0.625
δ	depreciation rate	0.126
ϕ	capital adjustment cost	8.1
	New Keynesian Block	
ϵ	elasticity of substitution	$\to \infty$
ψ	price adjustment cost	10ϵ
	Uncertainty	
b	mean reversion speed	0.67
$\bar{\sigma}^2$	mean level	0.158
ν	volatility	0.294
	Monetary-Fiscal Policy	
i	interest rate	2.3%
$\check{s}$	fiscal policy	0.13%

Table 1: Model Parameters

the algorithm by first pre-solving for the function $\vartheta_t = \vartheta(\tilde{\sigma}_t)$ that only depends on the exogenous state variable $\tilde{\sigma}_t$ and then holding this solution for ϑ fixed while applying the described algorithm to determine Π.