Belief Errors in Production Networks

William Chen*

Current version: March 18, 2025
First version: January 10, 2025

Abstract

I study the macroeconomic impact of belief errors in production networks when firms choose labor quantities before uncertainty about sectoral TFP disturbances is resolved. Belief errors influence firms' labor demand through two channels: the cyclicality of labor demand and asymmetric exposure to sectoral TFPs. The impact on aggregate output varies with the structure of the production network, the income effect on labor supply, the degree of belief disagreement between sectors, and the extent of labor reallocation. I discuss the implication of these results for ambiguity aversion in production networks.

Preliminary and incomplete. Please do not circulate.

*Department of Economics, Massachusetts Institute of Technology. Email: wyc1@mit.edu.

I am grateful for feedback from Iván Werning, Ricardo Caballero, Daron Acemoglu, and participants in the MIT Macro Lunch. I have no conflict of interest or sources of research support to disclose.

1 Introduction

Novel and complex shocks have precipitated many economic crises. The opacity of the balance sheets of shadow banks triggered the 2007-2008 financial crisis while the onset of COVID-19 entailed numerous unknowable uncertainties. When a shock is novel and complex, economic agents will likely form belief errors about the shock. These errors may further amplify the shock through two channels. First, agents may have asymmetric exposure to the shock. If agents can form belief errors, then all agents may worry that they will suffer disproportionately more than others. Even if agents initially anticipate a mild recession, they may have substantial concerns about asymmetric exposure, which amplifies the shock. Second, belief errors may induce disagreement between two sets of agents, which affects aggregate outcomes because of spillovers. The reaction of the first set of agents may be perfectly reasonable under their beliefs but a significant mistake under the beliefs of the second set of agents. If there are spillovers, then the reaction of the first set of agents can pose a significant problem to the second set.

This paper theoretically investigates these ideas within the environment of a production network in which firms potentially have errors in their beliefs about sectoral disturbances. In the model, firms choose their labor inputs and commit to pay the market-clearing wage before they learn sectoral TFP disturbances. I take firms' beliefs about TFPs as given but assume that beliefs are common knowledge, firms agree to disagree, and firms are not misspecified about the economy's structural relations. The model is an otherwise standard production network with elastic labor supply from a representative household.

Fluctuations in firms' belief errors can change GDP. The magnitude of the change depends on two channels: the cyclicality of labor demand and asymmetric exposure to sectoral TFPs. The first channel arises because belief errors about TFPs lead to belief errors about GDP. If labor demand is cyclical, then these belief errors about GDP can

cause labor allocations to fluctuate. However, the cyclicality of labor demand depends on household preferences and the structure of the production network. Even if a firm anticipates a decline in sales due to low GDP, the firm does not need to reduce hiring if the firm also anticipates wages to fall because the household's labor supply expands. The second channel arises because expenditure switching across the production network generates asymmetric exposure to sectoral TFPs, holding GDP constant. For example, sales may fluctuate because the household or downstream customers switch expenditures to other sectors. These two channel determine how belief errors propagate throughout the production network.

The model has two "stages": in the "morning," firms choose beliefs and labor, and "in the afternoon," all other decisions and prices are determined in the "afternoon." I characterize equilibrium in three steps. First, I characterize agents' optimality conditions in an arbitrary temporary equilibrium. Second, I solve GDP, sales, and prices in the afternoon as a function of firms' morning labor choices and the realized TFP disturbances. Third, I show that belief errors induce labor wedges and use restrictions on firms' beliefs to characterize the wedges as functions of sectoral belief errors. These wedges then determine morning labor choices. I apply this general solution to several example economies to study the determinants of the aggregate response to belief errors.

I establish a neutrality result and thus show that belief errors about TFP do not necessarily generate aggregate fluctuations. GDP is neutral to belief errors when the income effect on labor supply equals one and the production network is Cobb-Douglas. This neutrality result pinpoints the two channels through which belief errors may affect outcomes.

I first relax the assumption that the income effect equals one in a Cobb-Douglas network. This case shows that the impact of belief errors on GDP depends on the cyclicality of labor demand. In Cobb-Douglas economies, the relative cyclicality of sales to wages determines the cyclicality of labor demand. The cyclicality of a firm's sales is always one, so if wages match that cyclicality, then belief errors are irrelevant. Away from this benchmark,

the aggregate impact of belief errors depends on how cyclical sales are relative to wages. When the income effect is low, pessimism about TFP causes GDP to fall. However, if the income effect is high, then pessimism about TFP actually causes GDP to *rise*. In either case, belief errors about TFP only affect outcomes by causing firms to form incorrect beliefs about GDP and thus the ratio of sales to wages.

Next, I consider a horizontal economy in which the final demand aggregator is CES instead of Cobb-Douglas. Because of this change in the demand aggregator, a firm's sales are not only exposed to GDP but also asymmetrically exposed to its own sector's TFP. The reason is that, holding GDP constant, the household switches expenditures across sectors in response to TFP disturbances. If there are constraints on the total errors that firms can make about sectoral TFPs, then belief disagreement becomes necessary to generate fluctuations from belief errors.

Returning to a Cobb-Douglas demand aggregator, I show that belief errors also matter when the supply chain is CES. When the elasticities of substitution between labor and intermediate inputs are not all one, belief errors about the TFPs of upstream sectors affect firms' labor demand. The reason is that belief errors about upstream TFPs shift their beliefs about the relative distribution of sales in the afternoon between downstream and upstream sectors. If downstream sectors have complementary demand for inputs from upstream sectors, then pessimism about upstream TFP by downstream sectors causes their hiring to fall, but pessimism by upstream sectors causes their hiring to *rise*. In this case, belief disagreement directly affects GDP, even without constraints on the total errors firms can make. Without disagreement, a downturn in some sectors is offset by an expansion in others.

The sensitivity of GDP to belief errors depends on the extent of labor reallocation across sectors. The less willing the representative household is to reallocate labor, the smaller the impact that belief errors have on GDP. This result holds for arbitrary networks and income effects on labor supply. The reason is that a low willingness to reallocate labor implies a high

willingness to accept wage reductions. If firms can count on large wage changes to offset the effects of TFP disturbances, then they do not worry about their labor decision.

A novel application of these results is the study of ambiguity aversion in production networks. I model ambiguity aversion using max-min expected utility, as in Gilboa and Schmeidler (1989), and show that a spike in ambiguity causes a recession through sectoral belief errors. The magnitude of the recession depends on the structure of the production network. Expenditure switching generates asymmetric and disproportionate exposure to sectoral TFP disturbances, so firms disagree about worst-case TFPs. Disagreement aggregates to a larger recession because each sector's labor decision has spillover effects onto other sectors' decisions.

Related Literature. Information frictions are a common way to introduce errors in equilibrium beliefs into production networks. Pellet and Tahbaz-Salehi (2023) and Bui et al. (2024) focus on the noisy exogenous information case. Nikolakoudis (2024) considers learning from prices, Jamilov et al. (2024) studies endogenous attention about exogenous states, and Chahrour et al. (2021) examines the role of news media in belief formation. These papers show that specific information frictions alter the propagation of sectoral disturbances and cause noise shocks to be sources of aggregate fluctuations. In contrast to these papers, I abstract from a specific model of belief formation, which enables me to characterize what types of beliefs in general are needed to generate aggregate downturns. I do restrict attention to beliefs which satisfy a form of rationality that I clarify below.

My paper also relates to the broader production networks literature. Baqaee and Farhi (2019) and Dew-Becker (2023) show that nonlinear production networks matter for the aggregate effects of sectoral disturbances. When there are nonlinearities, sectoral disturbances change Domar weights and input-output multipliers. My paper shows that these same mechanisms have first-order effects when firms choose inputs subject to uncertainty.

Kopytov et al. (2024) has a similar result but obtains it using endogenous input-output linkages in a Cobb-Douglas network instead of an exogenous nested-CES network. Bigio and La'O (2020) and Baqaee and Farhi (2020b) study the impact of fluctuations in exogenous distortions in production networks. Labor wedges in my paper endogenously emerge from input choice under uncertainty and depend on the network structure.

2 Model

There is a representative household and n production sectors, indexed by $i \in \mathcal{N} \equiv \{1, \ldots, n\}$. A representative firm in each sector exists and sells output either to the household for consumption or for use as an intermediate input. The household owns all firms. The economy is static, but it will be convenient to divide firms' actions into two stages, the "morning" and the "afternoon." In the morning, firms choose labor in the morning. In the afternoon, the household supplies labor, and the product and intermediate goods markets clear. Wages and goods prices are both state-contingent.

2.1 Household

The representative household has preferences

$$U(Y, \{h_i^s\}_{i \in \mathcal{N}}) = \frac{Y^{1-\gamma} - 1}{1-\gamma} - \sum_{i \in \mathcal{N}} \chi_i \frac{(h_i^s)^{1+\eta_i}}{1+\eta_i},$$

where Y denotes GDP and h_i^s denotes the quantity of labor supplied to sector i by the household. This specification of labor disutility is taken from Bui et al. (2024) and implies that each sector faces a labor supply curve which is isoelastic in the real wage with sector-specific inverse Frisch elasticity η_i. These preferences keep the model tractable while limiting the degree of labor reallocation across sectors through the parameters η_i. Labor is perfectly

substitutable across sectors when $\eta_i = 0$ for all i.

GDP is a CES aggregate over sectoral goods

$$Y = \overline{Y}\left[\sum_{i\in\mathcal{N}} \beta_i \left(\frac{c_i}{\overline{c}_i}\right)^{\frac{\theta_c-1}{\theta_c}}\right]^{\frac{\theta_c}{\theta_c-1}}, \quad \sum_{i\in\mathcal{N}} \beta_i = 1.$$

where the barred variables are normalizing constants, c_i is consumption of good i, and p_i is the price of good i.[1] Let GDP be the numeraire. Then the budget constraint is

$$Y = \sum_{i\in\mathcal{N}} p_i c_i = \sum_{i\in\mathcal{N}} w_i h_i^s + \sum_{i\in\mathcal{N}} \pi_i,$$

where w_i are sector-specific wages and π_i are profits from sector i. Wages are sector-specific because labor is not perfectly substitutable across sectors unless $\eta_i = 0$ for all i.

2.2 Firms

Technology. The representative firm in sector i chooses labor h_i and intermediate inputs $\{x_{ij}\}_{j\in\mathcal{N}}$ to produce the sectoral good according to

$$y_i = \overline{y}_i z_i \left[\alpha_i \left(\frac{h_i}{\overline{h}_i}\right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} + \sum_{j\in\mathcal{N}} a_{ij}\left(\frac{x_{ij}}{\overline{x}_{ij}}\right)^{\frac{\theta_{qi}-1}{\theta_{qi}}}\right]^{\frac{\theta_{qi}}{\theta_{qi}-1}} \tag{1}$$

[1] CES aggregators are not unitless. I choose this normalization so that β_i can be interpreted as good i's steady-state budget share.

where z_i is a sectoral TFP disturbance, $\alpha_i, a_{ij} \in [0,1]$, and $\alpha_i + \sum_{j \in \mathcal{N}} a_{ij} = 1$.[2] Nominal profits are given by

$$\pi_i = p_i y_i - w_i h_i - \sum_{j \in \mathcal{N}}^{n} p_j x_{ij}.$$

Production under Uncertainty. In the morning, firms choose their labor inputs subject to uncertainty about the vector of TFP disturbances $\mathbf{z} \equiv (z_1, \ldots, z_n)^\top$. Formally, firms solve the problem

$$\max_{h_i} E_i[Y^{-\gamma} \pi_i] \tag{2}$$

where $Y^{-\gamma}$ is the household's stochastic discount factor (SDF). The *subjective* expectation operator $E_i[\cdot]$ denotes sector i's belief about a variable.

3 Equilibrium Definition and Characterization

Unless stated in the text, proofs are deferred to Appendix A. Bolded variables usually denote vectors or matrices, and the log of a vector means $\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_n)^\top$. Also, $\text{diag}(\mathbf{x})$ takes a vector as an input and returns a matrix with the vector along the diagonal.

In the remainder of the paper, $\mathbf{z}$ should be interpreted as the *realized* values of TFP in the afternoon. I denote firms' *beliefs* about these values using either hats or a subjective expectation operator.

[2] My model allows for a different elasticity of substitution between intermediate inputs from the elasticity of substitution between labor and intermediate inputs by assuming sector i purchases a composite intermediate from sector j, which uses no labor and simply aggregates intermediates from other sectors with elasticity of substitution $\theta_{qj} \neq \theta_{qi}$.

3.1 Equilibrium Definition

I formally define the equilibrium concept as follows.

Definition 1. A *sequentially rational temporary equilibrium with heterogeneous priors* is a vector of exogenous TFP disturbances $\mathbf{z}$ as well as a collection of endogenous variables $\{Y, \{w_i, c_i, p_i, h_i, y_i,\ \{x_{ij}\}_{jin\mathcal{N}},\ h_i\}_{i\in\mathcal{N}}\}$ and a collection of beliefs about those exogenous disturbances and endogenous variables for each sector such that

1. (Household Optimality) The representative household chooses consumption and labor supply in the afternoon to maximize utility subject to its budget constraint;

2. (Intermediate Inputs Optimality) Conditional on having chosen $\{h_i\}_{i\in\mathcal{N}}$ in the morning and after learning $\mathbf{z}$ in the afternoon, firms choose intermediate inputs to maximize their profits;

3. (Market Clearing) Sectoral goods prices and wages $\{p_i, w_i\}_{i\in\mathcal{N}}$ adjust in the afternoon to ensure all sectoral goods and labor markets clear.[3] Specifically,

$$y_i = c_i + \sum_{j\in\mathcal{N}} x_{ji} \quad \forall\, i, \tag{3}$$

$$h_i^s = h_i \quad \forall\, i; \tag{4}$$

4. (Sequential Rationality) Firms' beliefs about endogenous variables are consistent with conditions 1-3;

5. (Heterogeneous Priors) Sectoral labor choices are common knowledge.

6. (Labor Input Optimality) In the morning, firms choose their labor inputs to solve (2), taking as given their beliefs about exogenous disturbances.

[3] By Walras' law, the market for the aggregate consumption good will also clear.

The sequential rationality restriction can be interpreted as a subgame perfection refinement. The afternoon is a "proper subgame" of the whole model in which the economy becomes a static production network with fixed sectoral labor endowments. I require firms in the morning to understand this fact about the afternoon. Backward induction pins down firms' beliefs about endogenous variables in the afternoon as a function of the labor inputs chosen in the morning and the exogenous disturbances.

The sequential rationality condition is restrictive because it rules out some classes of beliefs. However, many popular classes of beliefs make some version of this assumption. For example, models of information frictions typically assume that all agents have the correct model of the economy but face uncertainty about exogenous disturbances and potentially each others' beliefs.

I further restrict attention to models in which higher-order beliefs are irrelevant. There is a representative firm in every sector, so higher-order uncertainty within a sector is not relevant. The heterogeneous priors restriction shuts down higher-order uncertainty across sectors because firms know each others' sectoral labor choices in the morning.

An equilibrium exists because the model is isomorphic to one in which labor wedges distort firms' labor choice. See Bigio and La'O (2020) for a proof of existence. Hereafter, whenever I write the "temporary equilibrium," I am referring to my equilibrium concept, unless stated otherwise.

3.2 Log-Linearized Solution

This section characterizes the log-linearized solution to the temporary equilibrium around the deterministic and efficient steady state in which all firms have no uncertainty about TFP disturbances. In the special case where the network is Cobb-Douglas, the log-linear solution is global. I use bars over variables to denote their steady-state values. I also choose $\overline{y}_i$ so that $\overline{\mathbf{z}} = \mathbf{1}$.

Input-Output Objects. Define the steady-state input-output matrix and Leontief inverse matrix as

$$\mathbf{\Omega}_{ij} = \frac{\overline{p}_j \overline{x}_{ij}}{\overline{p}_i \overline{y}_i}, \quad \mathbf{\Psi} = (\mathbf{I} - \mathbf{\Omega})^{-1}.$$

The Domar weights and input-output multiplier are given by

$$\boldsymbol{\lambda}_i = \sum_{i \in \mathcal{N}} \beta_i \mathbf{\Psi}_{ji}, \quad \xi = \sum_{i \in \mathcal{N}} \boldsymbol{\lambda}_i.$$

The Domar weights and labor cost shares satisfy

$$\sum_{i \in \mathcal{N}} \boldsymbol{\lambda}_i \alpha_i = 1$$

because $\boldsymbol{\lambda}_i \alpha_i$ denotes the labor share earned by sector i.

Equilibrium Conditions. I first state the log-linearized morning and afternoon equilibrium conditions derived from agents' optimality conditions and market-clearing.

Proposition 1. *To a first-order approximation of the temporary equilibrium around the deterministic steady state, changes in prices and allocations are characterized by the following system of equations*

$$d\log p_i + d\log c_i = d\log Y - (\theta_c - 1) d\log p_i, \tag{5}$$

$$d\log w_i = \gamma d\log Y + \eta_i d\log h_i, \tag{6}$$

$$d\log p_j + d\log x_{ij} = (\theta_{qi} - 1)(d\log z_i + d\log p_i - d\log p_j) + d\log s_i, \tag{7}$$

$$\begin{aligned}\left(1 + \alpha_i \frac{1 - \theta_{qi}}{\theta_{qi}}\right) d\log p_i = &\sum_{j \in \mathcal{N}} \mathbf{\Omega}_{ij} d\log p_j + \frac{\alpha_i}{\theta_{qi}}(d\log s_i - d\log h_i) \\ &- \left(1 + \alpha_i \frac{1 - \theta_{qi}}{\theta_{qi}}\right) d\log z_i,\end{aligned} \tag{8}$$

$$
\begin{aligned}
d\log h_i &= d\log s_i - \theta_{qi} d\log w_i \\
&\quad + (\theta_{qi}-1)(d\log z_i + d\log p_i) - d\log \tau_i,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
d\log \tau_i &= ((d\log s_i - d\log w_i) - (E_i[d\log s_i] - E_i[d\log w_i])) \\
&\quad + (\theta_{qi}-1)(d\log z_i + d\log p_i - d\log w_i) \\
&\quad - (\theta_{qi}-1)(E_i[d\log z_i] + E_i[d\log p_i] - E_i[d\log w_i]),
\end{aligned} \tag{10}
$$

$$
\boldsymbol{\lambda}_i d\log s_i = \beta_i (d\log p_i + d\log c_i) + \sum_{j\in\mathcal{N}} \boldsymbol{\Omega}_{ji}\boldsymbol{\lambda}_j (d\log p_i + d\log x_{ji}), \tag{11}
$$

Solution for the Afternoon.

Proposition 2. *To a first-order approximation of the temporary equilibrium around the deterministic steady state, aggregate output, sales, and prices in the afternoon fluctuate according to*

$$
d\log Y = \sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i d\log z_i + \sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i \alpha_i d\log h_i, \tag{12}
$$

$$
d\log \mathbf{s} = \mathbf{S}_z d\log \mathbf{z} + \mathbf{S}_h d\log \mathbf{h} + \mathbf{S}_Y d\log Y, \tag{13}
$$

$$
d\log \mathbf{p} = \mathbf{P}_z d\log \mathbf{z} + \mathbf{P}_h d\log \mathbf{h} + \mathbf{P}_Y d\log Y. \tag{14}
$$

where the formulas for the matrices $\mathbf{S}_z$, $\mathbf{S}_h$, $\mathbf{P}_z$, $\mathbf{P}_h$ *and vectors* $\mathbf{S}_Y$, $\mathbf{P}_Y$ *are provided in the appendix.*

Equation (12) is simply Hulten's Theorem when factor supplies are not fixed. Plugging (12) into (13) and (14) delivers expressions for sales and prices as functions of TFPs and labor demands. However, for characterizing the labor wedges, it is convenient to keep $d\log Y$. Proposition 2 holds in any temporary equilibrium, not just my equilibrium concept, since beliefs only affect the choice of $\mathbf{h}$ in the morning.

Solution for the Morning. To keep notation compact, define $\mathcal{E}_i[d\log x] \equiv E_i[d\log x] - d\log x$. The following proposition characterizes firms' morning labor demands in terms of labor wedges and provides the general formula for those wedges.

Proposition 3. *Consider a first-order approximation of the temporary equilibrium around the deterministic steady state. Equilibrium labor allocations depend on sectoral TFPs, GDP, and sectoral labor wedges according to*

$$d\log h_i = \sum_{j\in\mathcal{N}} \mathbf{H}_{z,ij} d\log z_j + \mathbf{H}_{Yi} d\log Y - \sum_{j\in\mathcal{N}} \mathbf{H}_{\tau,ij} d\log \tau_j. \tag{15}$$

If there are no errors in beliefs, then changes in GDP are given by

$$d\log Y^{(0)} = \frac{1}{1-\sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i \alpha_i \mathbf{H}_{Yi}} \sum_{i\in\mathcal{N}} \left(\boldsymbol{\lambda}_i + \sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \alpha_j \mathbf{H}_{z,ji} \right) d\log z_i. \tag{16}$$

If there are errors in beliefs, then changes in GDP are given by

$$d\log Y = d\log Y^{(0)} - \frac{1}{1-\sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i \alpha_i \mathbf{H}_{Yi}} \sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i \alpha_i \sum_{j\in\mathcal{N}} \mathbf{H}_{\tau,ij} d\log \tau_j, \tag{17}$$

where changes in sector i's labor wedge satisfy the equations

$$d\log \tau_i = -\sum_{j\in\mathcal{N}} \mathbf{T}_{z,ij} \mathcal{E}_i[d\log z_j] - \mathbf{T}_{Yi} \mathcal{E}_i[d\log Y], \tag{18}$$

$$\mathcal{E}_i[d\log Y] = \boldsymbol{\lambda}^\top \mathcal{E}_i[d\log \mathbf{z}]. \tag{19}$$

Formulas for the matrices $\mathbf{H}_z, \mathbf{H}_\tau, \mathbf{T}$ and vector $\mathbf{H}_Y$ are in the appendix.

The optimality condition for a sector's labor demand (9) suggests that a sector is affected only by its own TFP and labor wedge. However, expenditure switching throughout the production network causes a firm's labor demand (15) to potentially depends on the

TFPs and labor wedges of all sectors. The reason is that both the ratio of a sector's sales to its wage and the ratio of a sector's price to its wage depend directly on the vector of morning labor demands. By contrast, in the Cobb-Douglas case, the first ratio only depends on GDP, and the second ratio is irrelevant for a firm's labor demand.

Belief Spillovers Through the Network. In Proposition 3, while equation (15) shows that a sector's labor choice does potentially depend on other sectors' labor wedges, equations (18) and (19) show that a sector's labor wedge depends only on its own beliefs. This result simplifies the analysis, but it is a consequence of two modeling choices, not a necessary result.[4]

The first choice is the structure of the labor market. I assume that the firm pays a state-contingent wage to the household, which supplies labor in the afternoon, and that the labor supply to one sector is not affected by the labor supplied to other sectors. If a firm needed to pay a non-contingent wage, then the household's beliefs in the morning shift its labor supply to the firm, regardless of what the firm believes.[5] Similarly, if a firm's labor supply was affected by labor supplied to other sectors, then its labor supply would be shifted by other sectors' beliefs. I shut down this possibility through my specification of labor disutility. An alternative specification like

$$\frac{H^{1+\eta}}{1+\eta} \equiv \frac{1}{1+\eta}\left[\overline{H}\left(\sum_{i\in\mathcal{N}}\chi_i\left(\frac{h_i}{\overline{h}_i}\right)^{\frac{\theta_h+1}{\theta_h}}\right)^{\frac{\theta_h}{\theta_h+1}}\right]^{1+\eta}$$

would generate belief spillovers through the labor market because a sector's labor supply

[4]Removing the heterogeneous prior assumption does not necessarily cause a sector's labor wedge to depend on other sectors' beliefs. Bui et al. (2024) considers a production network with dispersed information and shows that a sector's total labor choice only depend on other sectors' beliefs when there are noise shocks that do not aggregate away (e.g. sectoral noise shocks).

[5]The wage is determined in the morning in Bui et al. (2024). However, the household's beliefs are not part of the solution because the household takes on a family structure in which workers are sent to each information island in a sector and the workers inherit the beliefs of the island.

now depends on an aggregate of sectoral labor supplies.[6]

The second choice is the structure of the market for the intermediate inputs. I assume that firms flexibly choose any intermediate inputs in the afternoon and thus all goods prices are also determined in the afternoon. There are a number of ways to change this assumption and thereby make a firm's decisions depend on other sectors' beliefs. For example, suppose a downstream sector is the only customer of an upstream supplier, and the market for this input must clear *in the morning*. The downstream sector's input choice depends on the market-clearing price, which is determined not only by the downstream sector's demand curve but also the upstream sector's supply curve. If the upstream sector has to choose some of its inputs in the morning, then its supply curve depends on the upstream sector's beliefs. Then the market-clearing price would depend on both the downstream sector's and the upstream sector's beliefs.

4 The Macroeconomic Impact of Belief Errors

While Proposition 3 gives the general solution to my model, its equations do not provide too much insight about the precise channels which determine the impact of ambiguity on aggregate output. To explain what these channels are and how they shape the response of GDP to belief errors, I study special cases of the general model in the following subsections.

4.1 A Neutrality Result

I first study an economy in which belief errors have zero impact on GDP. This neutrality result clarifies the necessary ingredients for belief errors to affect aggregate output.

Proposition 4. *If $\gamma = 1$ so that the income effect on labor supply equals one and the production network is Cobb-Douglas ($\theta_c = 1$ and $\theta_{qi} = 1$ for all $i \in \mathcal{N}$), then $d\log Y =$*

[6]This specification of labor disutility is used by Horvath (2000).

$d \log Y^{(0)}$.

This result holds not only locally but also globally. For intuition, plug the household's optimal labor supply equation into sector i's labor demand equation to obtain

$$(1+\eta_i) d \log h_i = E_i[d \log s_i] + (\theta_{qi} - 1)(E_i[d \log z_i] + E_i[d \log p_i]) - \theta_{qi} \gamma E_i[d \log Y].$$

Labor demand may fluctuate through two channels. First, labor demand can be cyclical in equilibrium.[7] Second, sales and prices may have asymmetric exposure to sectoral TFP disturbances due to expenditure switching.

A Cobb-Douglas production network shuts down the second channel because it implies that $d \log s_i = d \log Y$ and $\theta_{qi} = 1$, hence

$$(1+\eta_i) d \log h_i = (1-\gamma) E_i[d \log Y].$$

In other words, sectoral sales comove perfectly with GDP and movements in a sector's own prices or TFP become irrelevant for labor demand.[8]

Labor demand may still deviate from steady state if labor demand is cyclical. In a Cobb-Douglas network, labor demand is cyclical if the cyclicality of sales does not match the cyclicality of wages. Intuitively, an economic downturn implies not only lower sales but also lower wage costs, which insures firms against the downturn and mitigates the decline in labor demand. When $\gamma = 1$, this insurance is perfect.

[7] The parameter γ controls the income effect on the household's labor supply and thus the cyclicality of wages to GDP. Because I log-linearize around a deterministic steady state, γ does not represent risk aversion in this paper.

[8] This neutrality result may not hold if firms have belief errors about other disturbances, such as changes in monopoly distortions.

4.2 Cyclicality of Labor Demand

In this example, I show that the income effect on labor supply influences the impact of belief errors on GDP. Consider a purely horizontal economy in which all sectors hire only labor, the household's demand is Cobb-Douglas, and $\gamma \neq 1$. Also set $\eta_i = 0$ for all i to simplify expressions. By Proposition 3, sector i's belief error about GDP is

$$\mathcal{E}_i[d\log Y] = \sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \mathcal{E}_i[d\log z_j],$$

and sector i's log labor wedge is

$$d\log\tau_i = \underbrace{-(1-\gamma)\mathcal{E}_i[d\log Y]}_{\text{cyclicality of labor demand}} = -(1-\gamma)\sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \mathcal{E}_i[d\log z_j].$$

Set $d\log\mathbf{z} = \mathbf{0}$ to simplify expressions. Then the labor wedges aggregate to

$$d\log Y = \sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i \underbrace{\frac{1-\gamma}{\gamma}\overbrace{\sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \mathcal{E}_i[d\log z_j]}^{=\mathcal{E}_i[d\log Y]}}_{=d\log h_i}. \tag{20}$$

Changes in GDP result from changes in labor demand since the realized TFPs are fixed to steady state. Further, each sector's change in labor only depends on its own beliefs about $d\log Y$, holding labor allocations fixed.[9] One sector's labor choice does not affect another sector's choice at all, including through wage expectations. The reason is that $d\log\mathbf{h}$ is common knowledge. Uncertainty about future wages results only from uncertainty about GDP, holding labor allocations constant.

To build further intuition, suppose only sector i' has incorrect beliefs about TFPs.

[9] By Hulten's Theorem, the change in GDP, holding labor allocations fixed, equals the Domar-weighted sum of changes in sectoral TFPs.

Then (20) simplifies to

$$d\log Y = \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}_{i'}\sum_{j\in\mathcal{N}}\boldsymbol{\lambda}_j\mathcal{E}_{i'}[d\log z_j].$$

GDP moves only because sector i' changes its labor demand relative to steady state. The fact that other sectors are correct about sectoral TFPs does not matter for sector i'. The impact on GDP is proportional to the size of the sector and depends on which TFPs that sector i' has incorrect beliefs. It is common in the literature to assume that a sector is more informed about its own TFP than the TFPs of other sectors. A stark version is setting $\mathcal{E}_{i'}[d\log z_{i'}] = 0$. Then

$$d\log Y = \frac{1-\gamma}{\gamma}\boldsymbol{\lambda}_{i'}(1-\boldsymbol{\lambda}_{i'})\sum_{j\neq i'}\frac{\boldsymbol{\lambda}_j}{1-\boldsymbol{\lambda}_{i'}}\mathcal{E}_{i'}[d\log z_j].$$

Changes in GDP are the largest when sector i' is intermediate in size and thus $\boldsymbol{\lambda}_{i'}(1-\boldsymbol{\lambda}_{i'})$ is large.[10] With $\boldsymbol{\lambda}_{i'} \approx 1/2$, not only is sector i''s labor demand significant, but errors in i''s beliefs about the TFPs of other sectors also lead to errors about GDP changes.

The sign of $1-\gamma$ determines whether pessimisim or optimism induces a recession. Consider $\gamma < 1$ so that sales are more cyclical than wages. Then pessimistic belief errors are recessionary, and the downturn is more severe when large sectors (by Domar weight) are more pessimistic about the TFPs of large sectors. Conversely, suppose $\gamma > 1$ so that sales are less cyclical than wages. Then a firm receives so much insurance from wage declines that broad pessimism about TFP disturbances actually *increases* GDP.

Belief heterogenity matters only to the extent that it determines the magnitude of the double sum in (20). It is not required for the double sum to be nonzero. The key force is whether sectors are on average pessimistic/optimistic (depending on the sign of $1-\gamma$) about

[10] The normalized weights $\{\frac{\lambda_j}{1-\lambda_{i'}}\}_{j\neq i'}$ sum to one, so the weighted average of belief errors by sector i' about other sectors is invariant to $\lambda_{i'}$, the Domar sector of i'.

the economy. Belief errors generate larger changes in GDP when the errors are about TFPs of large sectors.

4.3 CES Final Demand

Continue to consider a purely horizontal economy, but not suppose the household's demand aggregator is CES. Continue to set $\eta = 0$ and $d \log \mathbf{z} = \mathbf{0}$ to simplify. I start with the case in which all sectors still only use labor, which implies an input-output multiplier of $\xi = 1$, and then consider when sectors use their own output as an intermediate input to assess the role of the input-output multiplier.

Purely Horizontal Economy. With a CES demand aggregator, sector i's log labor wedge becomes

$$d \log \tau_i = \underbrace{-\left(\frac{1}{\theta_c} - \gamma\right) \mathcal{E}_i[d \log Y]}_{\text{cyclicality of labor demand}} \underbrace{-\frac{\theta_c - 1}{\theta_c} \mathcal{E}_i[d \log z_i]}_{\text{asymmetric exposure}}$$

The equation for the labor wedge changes relative to the Cobb-Douglas case in two ways. First, the cyclicality term becomes $\theta_c^{-1} - \gamma$. The reason is that the price of good i changes with $d \log Y$ because demand aggregation is no longer Cobb-Douglas. The elasticity of $d \log s_i$ to $d \log Y$ changes from 1 to θ_c^{-1}.[11] Since labor is the only input, each sector's production technology is still Cobb-Douglas.

Second, sales do not perfectly comove with GDP, which gives rise to an "expenditure-switching" channel. The firm's labor demand equation remains the same because they still

[11] More flexible homothetic demand aggregators, explored in Matsuyama and Ushchev (2017), would generate sector-specific sales elasticities to GDP.

only hire labor, but the market-clearing equation for sales becomes

$$d\log s_i = d\log Y - (\theta_c - 1) d\log p_i.$$

The second term emerges because the household responds to price changes by switching expenditure shares. If the price of a good rises and the aggregator features gross complementarity ($\theta_c < 1$), then the household switches expenditures toward that good and causes $d\log s_i$ to rise, holding $d\log Y$ constant. If the price of good i does not perfectly comove with GDP, then sales of good i also does not perfectly comove with GDP. In equilibrium, a disturbance in sector i's TFP affects p_i.

Perhaps surprisingly, in the gross complementarity case, optimism by sector i about its own TFP *increases* its labor wedge and causes sector i to reduce hiring. The reason is that when sector i's TFP rises, the household's demand for sector i's good falls because the effective supply of good i has increased. This expenditure switching lowers sector i's sales.

Now consider the aggregate implications of $\theta_c \neq 1$. First set $\gamma = \theta_c^{-1}$ so that labor demand is acyclical. Then the impact of belief errors on GDP is

$$d\log Y = \sum_{i \in \mathcal{N}} \boldsymbol{\lambda}_i \underbrace{(\theta_c - 1)\mathcal{E}_i[d\log z_i]}_{= d\log h_i}. \tag{21}$$

Changes in GDP equal the Domar-weighted average of sectors' belief errors about their own TFPs. If sectors are pessimistic on average, then GDP falls if and only if the final demand aggregator exhibits gross substitutability ($\theta_c > 1$).

To build further intuition, again suppose that only sector i' has incorrect beliefs. Then the aggregate impact of beliefs errors is

$$d\log Y = \boldsymbol{\lambda}_{i'}(\theta_c - 1)\mathcal{E}_{i'}[d\log z_{i'}].$$

Since labor demand is acyclical, the only reason why GDP changes is an error by sector i' about its own TFP. If sector i' is well-informed about its own TFP, then GDP does not deviate from steady state.

Input-Output Multiplier. Now suppose each sector uses their own input as an intermediate good, and the elasticity between labor and intermediates is θ_{qi}. Sector i's input-output multiplier is α_i^{-1}, and its log labor wedge becomes

$$d\log\tau_i = \underbrace{-\theta_{qi}\left(\frac{1}{\theta_c}-\gamma\right)\mathcal{E}_i[d\log Y]}_{\text{cyclicality of labor demand}} . \underbrace{-\frac{\theta_{qi}}{\alpha_1}\frac{\theta_c-1}{\theta_c}\mathcal{E}_i[d\log z_i]}_{\text{asymmetric exposure}}$$

The input-output multiplier amplifies the asymmetric exposure channel but does not adjust the cyclicality channel. The elasticity of substitution between labor and intermediate inputs adjusts the magnitude of both channels. The more complementary that intermediate inputs and labor are, the more spending that the firm switches from intermediate inputs toward labor.

In the aggregate, the formula for GDP when $\gamma = \theta_c^{-1}$ becomes

$$d\log Y = \sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i(\theta_c - 1)\mathcal{E}_i[d\log z_i]. \tag{22}$$

This formula is qualitatively the same as (21), but the presence of an input-output multiplier amplifies belief errors because the sum of the Domar weights exceeds one. If sectors are well-informed about their own TFPs, however, then GDP changes will be small.

CES demand for inputs does not affect GDP, despite the effect on labor wedges. With $\eta_i = 0$ and $\gamma = \theta_c^{-1}$, labor demand by sector i is

$$d\log h_i = E_i[d\log s_i] + (\theta_{qi}-1)(E_i[d\log z_i] + E_i[d\log p_i]) - \frac{\theta_{qi}}{\theta_c}E_i[d\log Y].$$

By Proposition 2 and common knowledge of morning labor allocations,

$$E_i[d\log s_i] = \sum_{j\in\mathcal{N}} \mathbf{S}_{h,ij} d\log h_j + \sum_{j\in\mathcal{N}} \mathbf{S}_{z,ij} E_i[d\log z_j] + \mathbf{S}_{Yi} E_i[d\log Y],$$
$$E_i[d\log p_i] = \sum_{j\in\mathcal{N}} \mathbf{P}_{h,ij} d\log h_j + \sum_{j\in\mathcal{N}} \mathbf{P}_{z,ij} E_i[d\log z_j] + \mathbf{P}_{Yi} E_i[d\log Y].$$

Since firms do not make errors about morning labor allocations, they recognize that there is feedback between the labor they hire and the afternoon's sales and prices. This feedback offsets the impact of CES demand on labor wedges.

Adding Back the Cyclicality Channel. Now remove the restriction $\gamma \neq \theta_c^{-1}$ so that labor wedges depend on the cyclicality of labor demand. Belief errors affect GDP according to

$$d\log Y = \sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i \frac{\frac{1}{\theta_c} - \gamma}{\gamma} \sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \alpha_j \mathcal{E}_i[d\log z_j] + \sum_{i\in\mathcal{N}} \boldsymbol{\lambda}_i \frac{1}{\gamma} \frac{\theta_c - 1}{\theta_c} \mathcal{E}_i[d\log z_i]. \tag{23}$$

GDP is proportional to the weighted average log labor wedge. The first term is the cyclicality channel while the second term is the asymmetric exposure channel. This equation is essentially the sum of (20) and (22). There are some differences because the cyclicality of labor demand is different with CES production, and γ is not restricted to equal θ_c^{-1}.

It is perhaps surprising that (23) does not have interactions between the asymmetric exposure channels of each sector's labor wedge. The reason is these interactions only arise when a sector's sales, wages, or prices depend on other sectors' labor demands. With only CES final demand and no other input-output linkages between sectors, there are still no such spillovers.[12] As a result, a sector's labor demand depends only on its own labor wedge.

[12] A different specification of labor disutility could introduce interactions.

Belief Disagreement. In this example, I argue that belief disagreement between sectors matters for aggregate outcomes. First consider (22), which is the GDP equation when labor demand is acyclical. Assuming the household has complementary demand for goods ($\theta_c < 1$), optimism by each sector about their own TFP reduces GDP. Belief errors about the economy in general are not particularly relevant. If belief errors about the vector of TFPs are subject to some constraints on how large the errors might be, then relative optimism about one's own sector requires relative pessimism about other TFPs. In this sense, disagreement generates larger fluctuations in GDP. I discuss in Section 5 that ambiguity aversion can introduce such a constraint.

4.4 CES Supply Chains

Return to the case where the household's demand aggregator is Cobb-Douglas, but now suppose that each of the economy's sectors is composed of a downstream industry and an upstream industry. Figure 1 illustrates this network. The downstream industry sells to the household while the upstream industry sells output only to the downstream industry. To improve readability, I make an abuse of notation and redefine $\mathcal{N} = \{1d, 1u, \ldots, nd, nu\}$ so that id and iu denote, respectively, the downstream and upstream industries of sector i. To keep expressions interpretable, I assume that the upstream industries only use labor and the downstream industry only uses the output of its own sector's upstream industry as an intermediate. I also maintain $d \log \mathbf{z} = \boldsymbol{\eta} = \mathbf{0}$. Then the Domar weights and input-output multiplier are given by

$$\boldsymbol{\lambda} = \Big((1-\alpha_{1d})\beta_1, \quad \beta_1, \quad \cdots, \quad (1-\alpha_{nd})\beta_n, \quad \beta_n \Big)^\top, \quad \xi = 1 + \sum_{i=1}^{n} (1-\alpha_{id})\beta_i.$$

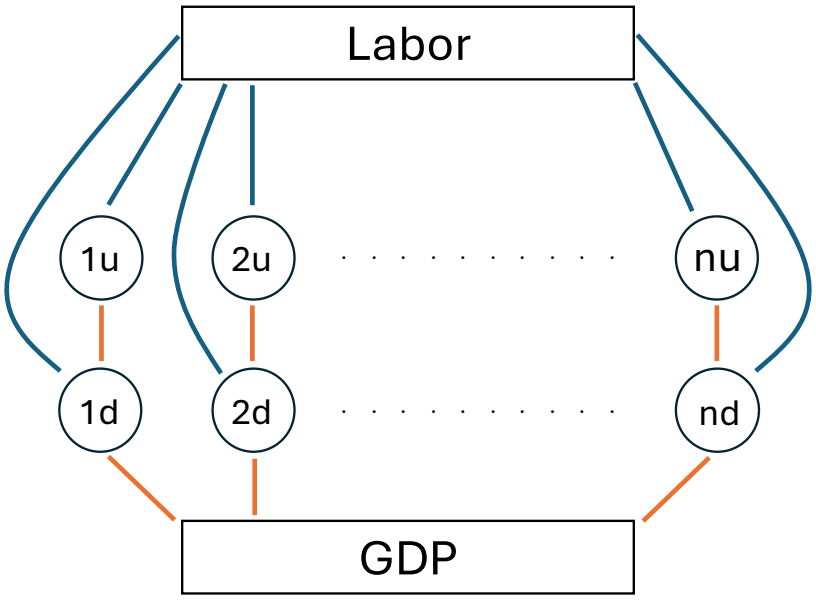


Figure 1: CES Supply Chain

The log labor wedges in the upstream and downstream industries are

$$d\log\tau_{iu} = (1-\gamma)\mathcal{E}_{iu}[d\log Y] - \alpha_{id}\frac{\theta_{q,id}-1}{\theta_{q,id}}\mathcal{E}_{iu}[d\log z_{iu}],$$

$$d\log\tau_{id} = \theta_{q,id}(1-\gamma)\mathcal{E}_{id}[d\log Y] + (1-\alpha_{id})(\theta_{q,id}-1)\mathcal{E}_{id}[d\log z_{iu}].$$

The log labor wedge in this case only depends on beliefs about upstream TFP. Downstream TFP is irrelevant for labor allocations because it does not shift sales or prices except through movements in GDP. Fixing the belief error about upstream TFP, the signs are opposite for the two labor wedges. When $\theta_{q,id} < 1$, optimism by upstream causes the labor wedge to increase while optimism by downstream causes the labor wedge to decrease. The intuition is that, when downstream has complementarity over its inputs, high upstream TFP causes the downstream industry to switch expenditures from intermediate inputs toward labor. This expenditure switching reduces sales for upstream, holding GDP constant, so upstream wants to reduce labor. In contrast, intermediate inputs are relatively cheap for downstream,

so downstream can expand hiring. Thus, if upstream and downstream have common belief errors, then the two labor wedges move in opposite directions.

Set $\gamma = 1$ to shut down the cyclicality channel. Then GDP fluctuates according to

$$d \log Y = \sum_{i=1}^{n} \boldsymbol{\lambda}_{id}(1 - \alpha_{id})\alpha_{id} \frac{\theta_{q,id} - 1}{\theta_{q,id}} (\mathcal{E}_{iu}[d \log z_{iu}] - \mathcal{E}_{id}[d \log z_{iu}]). \tag{24}$$

Changes in GDP occur if and only if downstream and upstream disagree about upstream's TFP. The intuition for this result is the same intuition for the behavior of labor wedges. When both industries agree about upstream's TFP, the labor wedges move in opposite directions, so a shift in upstream's labor demand is offset by the shift in downstream's labor demand. It is only when the industries disagree about upstream's TFP that the labor wedges move in the same direction and thereby fail to offset each other.

This result differs from the CES final demand case. In the latter case, disagreement matters for GDP fluctuations only when there is a constraint on belief errors. With a CES supply chain, disagreement always matters. The reason for this difference is that expenditure switches caused by upstream's TFP are zero-sum between the upstream and downstream industries. In contrast, while CES final demand causes asymmetric exposure to sectoral TFP disturbances, it does not induce zero-sum expenditure switches *horizontally across* sectors.

Inspecting the Result. To clarify why disagreement directly matters for GDP fluctuations, I discuss some of the calculations which lead to (24). A key property of the CES supply chain economy is that, within each sector, labor choices have spillovers between industries. Figure 2 illustrates these spillovers in sector i when downstream has complementary input demand. Suppose $d \log \tau_{id}$ initially increases. By Figure 2, downstream's labor demand $d \log h_{id}$ falls in response. Then the scale of downstream's operation falls, which decreases demand for intermediate inputs. Upstream responds by reducing its labor demand $d \log h_{iu}$, hence the negative spillover from downstream's labor demand to upstream's labor demand

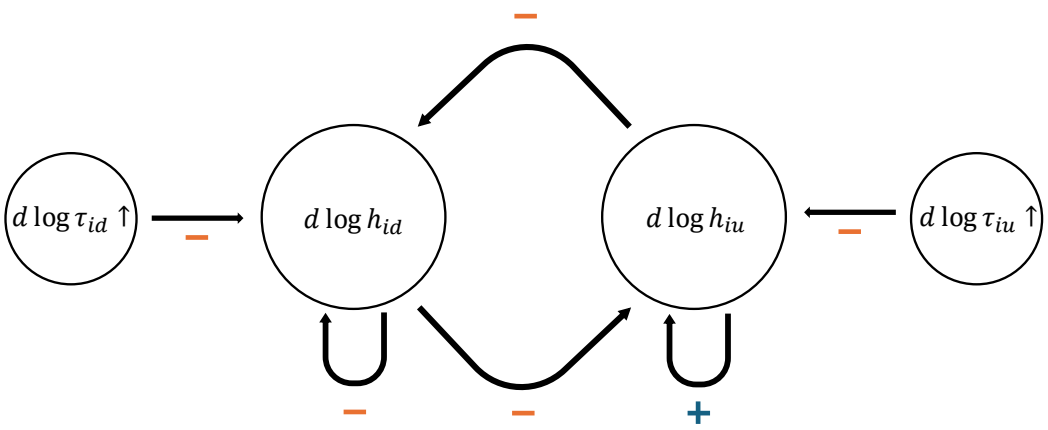


Figure 2: Labor Spillovers ($\theta_{q,id} < 1$)

in Figure 2. The price of upstream's output rises, which generates a negative spillover back to downstream. The reason is that downstream anticipates spending more on intermediate inputs and thus switches expenditures away from labor. There is also a feedback loop within each industry because changes in labor allocations directly affect goods prices.

I solve the matrix system of equations underlying Figure 2 to obtain expressions for labor demand in each industry as a function of the labor wedges.

$$\begin{pmatrix} d\log h_{iu} \\ d\log h_{id} \end{pmatrix} = \begin{pmatrix} -1-\alpha_{id}(\theta_{q,id}-1) & \alpha_{id}\frac{\theta_{q,id}-1}{\theta_{q,id}} \\ (1-\alpha_{id})(\theta_{q,id}-1) & -1+\alpha_{id}\frac{\theta_{q,id}-1}{\theta_{q,id}} \end{pmatrix} \begin{pmatrix} d\log \tau_{iu} \\ d\log \tau_{id} \end{pmatrix}.$$

With complementary input demand, both industries redduce their labor demand when upstream's labor wedge increases, although the decline in upstream's demand is partially mitigated by the change in the price of its output.

The impact on GDP equals the sum of labor changes, weighted by each industry's labor share of GDP. For keep notation compact, define disagreement in belief errors within sector i by $D_i \equiv \mathcal{E}_{iu}[d\log z_{iu}] - \mathcal{E}_{id}[d\log z_{iu}]$. Also, suppose only sector i' makes a belief error. Then

the change in GDP is

$$
\begin{aligned}
d\log Y = & \underbrace{\boldsymbol{\lambda}_{i'd}(1-\alpha_{i'd})}_{=\boldsymbol{\lambda}_{i'u}} \underbrace{\alpha_{i'u}}_{=1} d\log h_{i'u} + \boldsymbol{\lambda}_{i'd}\alpha_{i'd} d\log h_{i'd} \\
= & \quad \boldsymbol{\lambda}_{i'd} \underbrace{\alpha_{i'd}(1-\alpha_{i'd})\frac{\theta_{q,i'd}-1}{\theta_{q,i'd}}(D_{i'} + \theta_{q,i'd}\mathcal{E}_{i'd}[d\log z_{i'u}] + \alpha_{i'd}(\theta_{q,i'd}-1)D_{i'})}_{=(1-\alpha_{i'd})d\log h_{i'u}} \\
& + \boldsymbol{\lambda}_{i'd} \underbrace{\alpha_{i'd}(1-\alpha_{i'd})\frac{\theta_{q,i'd}-1}{\theta_{q,i'd}}(\quad - \theta_{q,i'd}\mathcal{E}_{i'd}[d\log z_{i'u}] - \alpha_{i'd}(\theta_{q,i'd}-1)D_{i'})}_{=\alpha_{i'd}d\log h_{i'd}}
\end{aligned}
$$

The second equality shows the intuition described previously. When the downstream industry has a belief error about upstream TFP ($\mathcal{E}_{i'd}[\log z_{i'u}] \neq 0$), both industries adjust their labor in exactly the correct amounts such that GDP is unchanged. When the two industries disagree, upstream and downstream further adjust their labor responses. GDP ultimately changes because the upstream industry reacts to disagreement more than the downstream industry.

Three Industry Supply Chain. Are changes in GDP more or less sensitive to belief errors when the supply chain is longer? It turns out that the answer is *more sensitive*. Suppose the supply chain now has three industries: downstream, midstream, and upstream. Maintaining $\gamma = 1$, the labor wedges become

$$
\begin{aligned}
d\log\tau_{iu} &= -\frac{\theta_{q,id}-1}{\theta_{q,id}}\alpha_{id}\mathcal{E}_{iu}[d\log z_{im}] - \left(\alpha_{im}\frac{\theta_{q,im}-1}{\theta_{q,im}} + \alpha_{id}(1-\alpha_{im})\frac{\theta_{q,id}-1}{\theta_{q,id}}\right)\mathcal{E}_{iu}[d\log z_{iu}] \\
d\log\tau_{im} &= -\theta_{q,im}\frac{\theta_{q,id}-1}{\theta_{q,id}}\alpha_{id}\mathcal{E}_{im}[d\log z_{im}]\theta_{q,im}(1-\alpha_{im})\left(\frac{\theta_{q,im}-1}{\theta_{q,im}} - \alpha_{id}\frac{\theta_{q,id}-1}{\theta_{q,id}}\right)\mathcal{E}_{im}[d\log z_{iu}] \\
d\log\tau_{id} &= (1-\alpha_{id})(\theta_{q,id}-1)\mathcal{E}_{id}[d\log z_{im}] + (\theta_{q,id}-1)(1-\alpha_{im}-\alpha_{id}(1-\alpha_{im}))\mathcal{E}_{id}[d\log z_{iu}]
\end{aligned}
$$

Belief errors about midstream's and upstream's TFP disturbances generate labor wedges, and there is asymmetric exposure across industries, which is determined by whether the

disturbance affects an industry as a customer or a supplier. For example, downstream's exposure to $d\log z_{im}$ is the opposite of the other industries because midstream's TFP affects downstream as a customer and the other industries as suppliers.

Changes in GDP are given by

$$\begin{aligned} d\log Y = \sum_{i=1}^{n} \boldsymbol{\lambda}_{id}(1-\alpha_{id}) \Bigg[& \alpha_{id}\frac{\theta_{q,id}-1}{\theta_{q,id}}((1-\alpha_{im})D_{im}^{ud} + \alpha_{im}D_{im}^{md}) \\ & + (1-\alpha_{im})\left(\alpha_{im}\frac{\theta_{q,im}-1}{\theta_{q,im}}D_{iu}^{um} + \alpha_{id}\frac{\theta_{q,id}-1}{\theta_{q,id}}((1-\alpha_{im})D_{iu}^{um} + D_{iu}^{md})\right)\Bigg] \end{aligned} \tag{25}$$

where $D_{ia}^{bc} \equiv \mathcal{E}_{ib}[d\log z_{ia}] - \mathcal{E}_{ic}[d\log z_{ia}]$. For example, D_{iu}^{um} is the disagreement between upstream and midstream about $d\log z_{iu}$, midstream's TFP disturbance. Suppose upstream and midstream have the same beliefs about $d\log z_{im}$ but disagree with downstream so that $D_{im}^{ud} = D_{im}^{md} \neq 0$. Then the first line of (25) coincides with (24). Thus, the addition of a third industry does not significantly change the impact of disagreement about the TFP of downstream's immediate supplier, $d\log z_{im}$. The only difference is that there are now two supplier industries exposed to $d\log z_{im}$ instead of one, and their disagreement is weighted by their labor share of sales from downstream.

The second line of (25) shows that the impact of disagreement about upstream's TFP adds to the impact from disagreement about $d\log z_{im}$. Suppose input demand is complementary throughout the network ($\theta_{q,id} < 1$ and $\theta_{q,im} < 1$). Then positive disagreement about upstream's TFP does not offset the response of GDP to positive disagreement about midstream's TFP, and vice-versa. Positive disagreement about either industry's TFP reduces GDP. Thus, when a production network features complementary demand throughout the supply chain, recessions are larger if downstream industries are more pessimistic than upstream industries.

4.5 Frictions in Labor Reallocation

I model frictions in labor reallocation across sectors using the sector specific inverse Frisch elasticities. Let $\underline{\eta} = \min_{i \in \mathcal{N}} \eta_i$. A high $\underline{\eta}$ implies a relatively inelastic labor supply to every sector. For example, consider a two-sector economy in which the household supplies h_1 and h_2. The household would accept a significant wage reduction from sector 1 and would require a significant wage increase from sector 2 before reallocating labor from sector 1 to sector 2. The impact of $\underline{\eta}$ on outcomes is summarized in the following Proposition.

Proposition 5. *Fix firms' belief errors, and assume $\boldsymbol{\chi}$ changes to ensure that the steady state remains the same as $\underline{\eta}$ changes. Then changes in belief errors have a vanishing effect on changes in GDP as $\underline{\eta} \to \infty$.*

When labor is not mobile across sectors, firms can count on wages dropping significantly if they ever have a reason to reducing hiring. Absent wage rigidities, firms know that in the limit as $\underline{\eta} \to \infty$ wages will adjust to offset whatever occurs in the afternoon, so they do not need to reduce hiring. High $\underline{\eta}$ also reduces the importance of spillovers between sectors through wages. Each sector's wage depends more and more on the sector's own demand as $\underline{\eta}$ increases and relatively less on GDP and thus labor demand in other sectors.

5 Ambiguity Aversion and Uncertainty Shocks

Fluctuations in ambiguity have been proposed as an exogenous driver of business cycles by Bidder and Smith (2012), Ilut and Schneider (2014), Bhandari et al. (2024), and (Pei, 2024) or as an endogenous propagation mechanism for other business cycle shocks by Ilut and Saijo (2021). Michelacci and Paciello (2020) and Ilut et al. (2024) also show that ambiguity aversion can change aggregate outcomes by generating belief disagreement between borrowers and savers. I make conjectures about the macroeconomic impact of shocks to ambiguity in

production networks and emphasize the role of belief disagreement.

Setup. Suppose I model ambiguity aversion using max-min expected utility, as in Gilboa and Schmeidler (1989). Each sector in the morning faces ambiguity about the sectoral TFPs and solves

$$\max_{h_i} E_i[C^{-\gamma}\pi_i] \equiv \max_{h_i} \min_{\widetilde{\mathbf{z}}\in\Delta_i} C^{-\gamma}\pi_i. \tag{26}$$

The object Δ_i is some potentially *sector*-specific set of priors and defines the "plausible" values of the TFP disturbances. When choosing their perceived disturbances, each firm takes as given the (hiring) strategies and beliefs of all other firms. I maintain the sequential rationality and heterogeneous priors restrictions.

Because I still log-linearize around the deterministic steady state, my setup ignores the role of *statistical* risk in the formation of beliefs.[13] Firms have multiple Dirac priors $\mathbf{z}$ in the sense that the prior distributions in Δ_i are all concentrated at individual vectors. This assumption isolates variation in firms' (mean) perceptions of $\mathbf{z}$ as the source of my results rather than other moments of their joint distribution. A numerical global analysis is also tractable because, without risk, the subjective expectation operator satisfies $E_i[\mathbf{xy}] = E_i[\mathbf{x}]E_i[\mathbf{y}]$ and $f(E_i[\mathbf{x}]) = E_{ik}[f(\mathbf{x})]$ for arbitrary vectors $\mathbf{x}, \mathbf{y}$ and function $f(\cdot)$.

To define the set Δ_i, I generalize the approach in Ilut and Schneider (2014) to a multivariate setting.

$$\Delta_i \equiv \{\widetilde{\mathbf{z}} \in \mathbb{R}^{|\mathcal{N}|} : (\log\widetilde{\mathbf{z}} - \log\mathbf{z})^\top \boldsymbol{\Sigma}_i^{-1}(\log\widetilde{\mathbf{z}} - \log\mathbf{z}) \leq Q_i^2\}, \tag{27}$$

where the matrix $\boldsymbol{\Sigma}_i$ encodes the relative ambiguity among the exogenous disturbances and

[13] Technically, this log-linearization is accurate in only one direction. Alternatively, one could imagine that when Q_i becomes negative, firms become ambiguity *loving* and have "max-max" expected utility.

the scalar Q_i parameterizes the overall level of ambiguity.[14] I interpret a change in the value of Q_i as a shock to sector i's ambiguity, and a change in Q_i for every sector as an aggregate ambiguity shock. I center firms' ambiguity around the realized disturbances so that firms are not biased in the limit as $Q_i \to 0$.

Implications. As I mentioned in the CES final demand example, ambiguity aversion imposes a constraint on the total errors a firm can make about the vector of sectoral TFPs. In my specific setup, a weighted sum of the squared errors cannot exceed Q_i^2. Recall that GDP changes according to

$$d \log Y = (\theta_c - 1) \sum_{i \in \mathcal{N}} \boldsymbol{\lambda}_i \alpha_i \frac{1}{\alpha_i} \mathcal{E}_i[d \log z_i],$$

assuming the labor demand cyclicality channel is shut off. Given the constraint on belief errors in Δ_i, GDP changes the most when each sector only makes an error about its own TFP and thus disagrees with every other sector about the overall vector $d \log \mathbf{z}$. Further, the optimal beliefs in this example are $\mathcal{E}_i[d \log z_i] = -\operatorname{sgn}(\theta_c - 1) dQ_i$. Intuitively, each sector is disproportionately exposed to its own TFP disturbance, so the worst-case outcome for each sector is a large change in its own TFP. Ambiguity aversion endogenously generates belief disagreement that maximally decreases GDP, given the sets $\{\Delta_i\}_{i \in \mathcal{N}}$. This result broadly applies to nested CES production networks because these networks induce expenditure switching, which creates disproportionate and asymmetric exposure of firms to sectoral TFP disturbances.

Since ambiguity shocks operate through a precautionary mechanism in my model, I conjecture that risk shocks behave similarly to ambiguity shocks in nested CES networks. However, the analysis is far more tractable in my case because ambiguity shocks have first-order effects whereas risk shocks would require at least a second-order approximation. My

[14] Remember that each vector in Δ_i represents a Dirac prior.

analysis thus complements Kopytov et al. (2024), which can study risk shocks in production networks by modeling endogenous input-output linkages in a Cobb-Douglas network. I take a different approach and change how the uncertainty shock is modeled.

6 Conclusion

How does the aggregate impact of a novel and complex shock depend on the degree of heterogeneity in the shock's effects and belief disagreement between agents when there are spillovers? This paper investigates this question by modeling such a shock as a fluctuation in ambiguity and studying the consequences for firms in a production network. A spike in ambiguity can trigger a recession, and the magnitude of the recession depends on two channels, which are shaped by the production network. The first channel is the exposure of a sector's relative Domar weight to sectoral demand reallocation. The second channel is imperfect correlation between aggregate consumption and the wages that the firm pays, holding relative Domar weights constant. The importance of these two channels varies with the income effect on labor supply, the structure of the production network, and the extent of labor reallocation.

I have focused on the case in which agents perfectly know the model but face uncertainty about exogenous states. In a sense, this approach is a minimal departure from complete information rational expectations because I alter agents' beliefs about exogenous disturbances and not the structural relations of the model. I defer analysis of the interaction between beliefs with model misspecification and production networks to future research.

References

Baqaee, D. and E. Farhi (2020a): "Nonlinear Production Networks with an Application to the Covid-19 Crisis," Working paper, National Bureau of Economic Research.

Baqaee, D. R. and E. Farhi (2019): "The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem," *Econometrica*, 87, 1155–1203.

——— (2020b): "Productivity and Misallocation in General Equilibrium," *The Quarterly Journal of Economics*, 135, 105–163.

Bhandari, A., J. Borovička, and P. Ho (2024): "Survey Data and Subjective Beliefs in Business Cycle Models," *Review of Economic Studies*, rdae054.

Bidder, R. M. and M. E. Smith (2012): "Robust Animal Spirits," *Journal of Monetary Economics*, 59, 738–750.

Bigio, S. and J. La'O (2020): "Distortions in Production Networks," *The Quarterly Journal of Economics*, 135, 2187–2253.

Bui, H., Z. Huo, A. A. Levchenko, and N. Pandalai-Nayar (2024): "Noisy Global Value Chains," Working paper.

Chahrour, R., K. Nimark, and S. Pitschner (2021): "Sectoral Media Focus and Aggregate Fluctuations," *American Economic Review*, 111, 3872–3922.

Dew-Becker, I. (2023): "Tail Risk in Production Networks," *Econometrica*, 91, 2089–2123.

Gilboa, I. and D. Schmeidler (1989): "Maxmin Expected Utility with Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141–153.

Horvath, M. (2000): "Sectoral Shocks and Aggregate Fluctuations," *Journal of Monetary Economics*, 45, 69–106.

Ilut, C. and H. Saijo (2021): "Learning, Confidence, and Business Cycles," *Journal of Monetary Economics*, 117, 354–376.

Ilut, C. L., P. Krivenko, and M. Schneider (2024): "Uncertainty or Frictions? A Quantitative Model of Scarce Safe Assets," Working paper.

Ilut, C. L. and M. Schneider (2014): "Ambiguous Business Cycles," *American Economic Review*, 104, 2368–2399.

Jamilov, R., A. Kohlhas, O. Talavera, and M. Zhang (2024): "Granular Sentiments," Working paper.

Kopytov, A., B. Mishra, K. Nimark, and M. Taschereau-Dumouchel (2024): "Endogenous Production Networks Under Supply Chain Uncertainty," *Econometrica*, 92, 1621–1659.

Matsuyama, K. and P. Ushchev (2017): "Beyond CES: Three Alternative Classes of Flexible Homothetic Demand Systems," Working paper.

Michelacci, C. and L. Paciello (2020): "Ambiguous Policy Announcements," *The Review of Economic Studies*, 87, 2356–2398.

Nikolakoudis, G. (2024): "Incomplete Information in Production Networks," Working paper.

Pei, G. (2024): "Pessimism, Disagreement, and Economic Fluctuations," *Journal of the European Economic Association*, 22, 1177–1227.

Pellet, T. and A. Tahbaz-Salehi (2023): "Rigid Production Networks," *Journal of Monetary Economics*, 137, 86–102.

Rojas-Bernal, A. and A. Karimirad (2023): "Nonlinearities in Production Network Economies with Distortions," Working paper.

Rubbo, E. (2023): "Networks, Phillips Curves, and Monetary Policy," *Econometrica*, 91, 1417–1455.

Appendices

A Proofs

A.1 Proposition 1

A.1.1 Nonlinear Optimality Conditions

Household Problem. The household's Lagrangian takes the form

$$\mathcal{L} = u\left(\overline{Y}\left[\sum_{i\in\mathcal{N}}\beta_i\left(\frac{c_i}{\overline{c}_i}\right)^{\frac{\theta_c-1}{\theta_c}}\right]^{\frac{\theta_c}{\theta_c-1}}\right) - \sum_{i\in\mathcal{N}}\chi_i\frac{(h_i^s)^{1+\eta_i}}{1+\eta} + M\left(\sum_{i\in\mathcal{N}}(w_i h_i^s + \pi_i) - \sum_{i\in\mathcal{N}}p_i c_i\right),$$

where $u(Y) = \frac{Y^{1-\gamma}}{1-\gamma}$ and M is the multiplier on the household's budget constraint. The household's controls are $\{c_i, h_i^s\}_{i\in\mathcal{N}}$.

The FOC w.r.t. c_i is

$$u'(Y)\beta_i\left(\frac{\overline{Y}}{\overline{c}_i}\right)^{\frac{\theta_c-1}{\theta_c}}\left(\frac{Y}{c_i}\right)^{\frac{1}{\theta_c}} = Mp_i.$$

Since a CES aggregator is homogeneous of degree one, and GDP is the numeraire, it can be shown that[15]

$$\beta_i\left(\frac{\overline{Y}}{\overline{c}_i}\right)^{\frac{\theta_c-1}{\theta_c}}\left(\frac{Y}{c_i}\right)^{\frac{1}{\theta_c}} = p_i,$$

which implies $M = u'(Y)$ and re-arranges to

$$p_i c_i = \beta_i^{\theta_c}\left(\frac{\overline{Y}}{\overline{c}_i}\right)^{\theta_c-1} p_i^{1-\theta_c} Y. \tag{28}$$

[15] See the proof of Lemma 1 in the supplementary appendix of Rubbo (2023).

To obtain a formula for the consumption expenditure share β_i, divide both sides by Y and evaluate at the steady state to obtain

$$\beta_i = \frac{\overline{p}_i \overline{c}_i}{\overline{Y}}.$$

The FOC w.r.t. h_i^s is

$$w_i = \chi_i (h_i^s)^{\eta_i} Y^{\gamma}. \tag{29}$$

Firm Problem in the Afternoon. In the afternoon, sector i chooses intermediate inputs to minimize its costs, given some required output y_i, and then determines its optimal scale.

The cost minimization problem for firms is

$$\min_{\{x_{ij}\}_{j \in \mathcal{N}}} \sum_{j \in \mathcal{N}} p_j x_{ij}$$

subject to the production function

$$y_i = z_i \overline{y}_i \left[a_{hi} h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}} + \sum_{j \in \mathcal{N}} a_{ij} x_{ij}^{\frac{\theta_{qi}-1}{\theta_{qi}}} \right]^{\frac{\theta_{qi}}{\theta_{qi}-1}}.$$

The FOC w.r.t. x_{ij} then is

$$0 = p_j - amc_i^{1/\theta_{qi}} (z_i \overline{y}_i)^{\frac{\theta_{qi}-1}{\theta_{qi}}} y_i^{1/\theta_{qi}} a_{ij} x_{ij}^{-1/\theta_{qi}}$$

Re-arrange to

$$x_{ij} = \left[p_j^{-1} (z_i \overline{y}_i)^{\frac{\theta_{qi}-1}{\theta_{qi}}} (amc_i y_i)^{1/\theta_{qi}} a_{ij} \right]^{\theta_{qi}}$$

Plug this expression for x_{ij} into the production function.

$$y_i = z_i \overline{y}_i \left[a_{hi} h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}} + \sum_{j \in \mathcal{N}} a_{ij} \left((z_i \overline{y}_i)^{\theta_{qi}-1} a_{ij}^{\theta_{qi}} \frac{amc_i y_i}{p_j^{\theta_{qi}}} \right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} \right]^{\frac{\theta_{qi}}{\theta_{qi}-1}}$$

To get an expression for $amc_i y_i$, re-arrange to

$$amc_i y_i = (z_i \overline{y}_i)^{-(\theta_{qi}-1)} \left(\sum_{j \in \mathcal{N}} \frac{a_{ij}^{\theta_{qi}}}{p_j^{\theta_{qi}-1}} \right)^{-\frac{\theta_{qi}}{\theta_{qi}-1}} \left[\left(\frac{y_i}{(z_i \overline{y}_i)} \right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi} h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}} \right]^{\frac{\theta_{qi}}{\theta_{qi}-1}}$$

This expression concludes the inner minimization problem.

Now consider the firm's optimal scale. Write expenditures on intermediate inputs as

$$p_j x_{ij} = \frac{a_{ij}^{\theta_{qi}}}{p_j^{\theta_{qi}-1}} \left(\sum_{k \in \mathcal{N}} \frac{a_{ik}^{\theta_{qi}}}{p_k^{\theta_{qi}-1}} \right)^{-\frac{\theta_{qi}}{\theta_{qi}-1}} \left[\left(\frac{y_i}{(z_i \overline{y}_i)} \right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi} h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}} \right]^{\frac{\theta_{qi}}{\theta_{qi}-1}}.$$

Using this expression and some algebra, profits become

$$\pi_i = p_i y_i - w_i h_i - \left(\sum_{k \in \mathcal{N}} \frac{a_{ik}^{\theta_{qi}}}{p_k^{\theta_{qi}-1}} \right)^{-\frac{1}{\theta_{qi}-1}} \left[\left(\frac{y_i}{(z_i \overline{y}_i)} \right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi} h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}} \right]^{\frac{\theta_{qi}}{\theta_{qi}-1}}$$

The FOC of profits w.r.t. y_i yields a relationship between the firm's price and its marginal cost

$$p_i = \left(\sum_{k \in \mathcal{N}} \frac{a_{ik}^{\theta_{qi}}}{p_k^{\theta_{qi}-1}} \right)^{-\frac{1}{\theta_{qi}-1}} \left[\left(\frac{y_i}{(z_i \overline{y}_i)} \right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi} h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}} \right]^{\frac{1}{\theta_{qi}-1}} (z_i \overline{y}_i)^{-\frac{\theta_{qi}-1}{\theta_{qi}}} y_i^{-\frac{1}{\theta_{qi}}}. \tag{30}$$

This expression can be re-arranged to write

$$\left(\sum_{k\in\mathcal{N}} \frac{a_{ik}^{\theta_{qi}}}{p_k^{\theta_{qi}-1}}\right)^{-\frac{\theta_{qi}}{\theta_{qi}-1}} \left[\left(\frac{y_i}{(z_i\overline{y}_i)}\right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi}h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}}\right]^{\frac{\theta_{qi}}{\theta_{qi}-1}} = (z_i\overline{y}_i)^{\theta_{qi}-1}p_i^{\theta_{qi}}y_i. \tag{31}$$

Then expenditures on intermediate goods takes the simpler form

$$p_j x_{ij} = (z_i\overline{y}_i)^{\theta_{qi}-1}\left(\frac{p_i}{p_j}\right)^{\theta_{qi}-1} a_{ij}^{\theta_{qi}} p_i y_i. \tag{32}$$

Firm Problem in the Morning. The firm solves

$$\max_{h_i} E_i\left[Y^{-\gamma}\left(p_i y_i - w_i h_i - \sum_{j\in\mathcal{N}} p_j x_{ij}\right)\right].$$

Since firms are sequentially rational, they know their optimality conditions in the afternoon, and they also know that those conditions hold for every value of $\mathbf{z}$. As a result, they know

$$\sum_{j\in\mathcal{N}} p_j x_{ij} = \left(\sum_{k\in\mathcal{N}} \frac{a_{ik}^{\theta_{qi}}}{p_k^{\theta_{qi}-1}}\right)^{-\frac{1}{\theta_{qi}-1}} \left[\left(\frac{y_i}{(z_i\overline{y}_i)}\right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi}h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}}\right]^{\frac{\theta_{qi}}{\theta_{qi}-1}},$$

hence the firm's FOC w.r.t. h_i is

$$\begin{aligned}
0 = {} & E_i\left[Y^{-\gamma}\left(p_i\frac{\partial y_i}{\partial h_i}\right)\right] - E_i[Y^{-\gamma}w_i] \\
& - E_i\left[Y^{-\gamma}\left(\sum_{k\in\mathcal{N}} \frac{a_{ik}^{\theta_{qi}}}{p_k^{\theta_{qi}-1}}\right)^{-\frac{1}{\theta_{qi}-1}} \frac{\partial}{\partial y_i}\left[\left(\frac{y_i}{(z_i\overline{y}_i)}\right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi}h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}}\right]^{\frac{\theta_{qi}}{\theta_{qi}-1}} \frac{\partial y_i}{\partial h_i}\right] \\
& - E_i\left[Y^{-\gamma}\left(\sum_{k\in\mathcal{N}} \frac{a_{ik}^{\theta_{qi}}}{p_k^{\theta_{qi}-1}}\right)^{-\frac{1}{\theta_{qi}-1}} \frac{\partial}{\partial h_i}\left[\left(\frac{y_i}{(z_i\overline{y}_i)}\right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi}h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}}\right]^{\frac{\theta_{qi}}{\theta_{qi}-1}}\right]
\end{aligned}$$

The sum of the first and third terms equals zero because they compose the firm's FOC w.r.t. y_i in the afternoon. Using (31), the remaining two terms become

$$E_i[Y^{-\gamma}w_i] = E_i\left[Y^{-\gamma}\left[\left(\sum_{k\in\mathcal{N}}\frac{a_{ik}^{\theta_{qi}}}{p_k^{\theta_{qi}-1}}\right)^{-\frac{1}{\theta_{qi}-1}}\left(\frac{y_i}{(z_i\overline{y}_i)}\right)^{\frac{\theta_{qi}-1}{\theta_{qi}}} - a_{hi}h_i^{\frac{\theta_{qi}-1}{\theta_{qi}}}\right]^{-\frac{1}{\theta_{qi}-1}} a_{hi}h_i^{-1/\theta_{qi}}\right]$$
$$= E_i\left[Y^{-\gamma}(z_i\overline{y}_i)^{\frac{\theta_{qi}-1}{\theta_{qi}}}p_i y_i^{\frac{1}{\theta_{qi}}}a_{hi}h_i^{-1/\theta_{qi}}\right].$$

Since sector i knows its own labor choice,

$$h_i^{\frac{1}{\theta_{qi}}} = a_{hi}\frac{E_i[Y^{-\gamma}(z_i p_i\overline{y}_i)^{\frac{\theta_{qi}-1}{\theta_{qi}}}(p_i y_i)^{\frac{1}{\theta_{qi}}}]}{E_i[Y^{-\gamma}w_i]}. \tag{33}$$

A.1.2 Log-Linearization

Household Problem. Households' optimality conditions are log-linear, so it is straightforward to obtain (5) and (6).

Firm Problem in the Afternoon The two optimality conditions are firms' expenditures on intermediate inputs (32) and the relationship between sectoral prices and marginal costs (30). The first condition is log-linear, so its log-linearization is straightforward. After using properties of CES aggregators and some algebra, the second condition log-linearizes to

$$d\log p_i = \frac{1}{1-\alpha_i}\sum_{j\in\mathcal{N}}\mathbf{\Omega}_{ij}d\log p_j + \frac{1}{\theta_{qi}}\frac{\alpha_i}{1-\alpha_i}(d\log y_i - d\log h_i) - \left(1+\frac{1}{\theta_{qi}}\frac{\alpha_i}{1-\alpha_i}\right)d\log z_i$$

Multiply both sides by $1-\alpha_i$ and re-arrange to get

$$\left(1+\frac{1-\theta_{qi}}{\theta_{qi}}\alpha_i\right)d\log p_i$$
$$= \sum_{j\in\mathcal{N}}\mathbf{\Omega}_{ij}d\log p_j + \frac{1}{\theta_{qi}}\alpha_i(d\log s_i - d\log h_i) - \left(1+\frac{1-\theta_{qi}}{\theta_{qi}}\alpha_i\right)d\log z_i$$

Firm Problem in the Morning. The firm's labor demand condition (33) is log-linear. Equation (9) is obtained after some algebra and using the definition of the log-lineaized labor wedge (10).

Market-Clearing for Sectoral Expenditures. Using the fact that $\overline{p}_i\overline{x}_{ji} = \mathbf{\Omega}_{ji}\overline{p}_j\overline{q}_j$ and dividing both sides by $\overline{Y}$, the market-clearing equation for sectoral expenditures log-linearizes to

$$\boldsymbol{\lambda}_i d\log s_i = \beta_i(d\log p_i + d\log c_i) + \sum_{j\in\mathcal{N}} \mathbf{\Omega}_{ji}\boldsymbol{\lambda}_j(d\log p_i + d\log x_{ji}),$$

which is (11).

A.2 Proposition 2

Hulten's Theorem for GDP holds because sectoral labor choices are determined in the morning. When the afternoon arrives, the economy is isomorphic to a static production network model in which there are multiple factors with endogenous supply. See Baqaee and Farhi (2020a) or Rojas-Bernal and Karimirad (2023) for the proof of Hulten's Theorem in such a setting.

Matrix Equation for Sectoral Sales. Take the market-clearing equation for sectoral sales (11) and plug in expenditures demand on consumption (5) and intermediate inputs to get

$$\begin{aligned}\boldsymbol{\lambda}_i d\log s_i = \beta_i(d\log Y + (1-\theta_c)d\log p_i) \\ + \sum_{j\in\mathcal{N}} \mathbf{\Omega}_{ji}\boldsymbol{\lambda}_j(d\log s_j + (\theta_{qj}-1)(d\log z_j + d\log p_j - d\log p_i)).\end{aligned}$$

In matrix form,

$$
\begin{aligned}
&(\mathbf{I} - \boldsymbol{\Omega}^\top)\,\mathrm{diag}(\boldsymbol{\lambda})d\log\mathbf{s} \\
&= \boldsymbol{\beta}d\log Y + \boldsymbol{\Omega}^\top\,\mathrm{diag}(\boldsymbol{\lambda})\,\mathrm{diag}(\boldsymbol{\theta}_q - \mathbf{1})d\log\mathbf{z} \\
&\quad + (\boldsymbol{\Omega}^\top\,\mathrm{diag}(\boldsymbol{\lambda}\odot(\boldsymbol{\theta}_q - \mathbf{1})) - \mathrm{diag}(\boldsymbol{\Omega}^\top(\boldsymbol{\lambda}\odot(\boldsymbol{\theta}_q - \mathbf{1}))) + (1-\theta_c)\,\mathrm{diag}(\boldsymbol{\beta}))d\log\mathbf{p}.
\end{aligned}
$$

nDefine the matrices

$$
\begin{aligned}
\mathbf{A}_z &= \boldsymbol{\Psi}^\top\boldsymbol{\Omega}^\top\,\mathrm{diag}(\boldsymbol{\lambda}\odot(\boldsymbol{\theta}_q - \mathbf{1})) \\
\mathbf{A}_p &= \mathbf{A}_z - \boldsymbol{\Psi}^\top\,\mathrm{diag}(\boldsymbol{\Omega}^\top(\boldsymbol{\lambda}\odot(\boldsymbol{\theta}_q - \mathbf{1}))) + \boldsymbol{\Psi}^\top(1-\theta_c)\,\mathrm{diag}(\boldsymbol{\beta}).
\end{aligned}
$$

Then the sectoral sales equation becomes

$$
\mathrm{diag}(\boldsymbol{\lambda})d\log\mathbf{s} = \boldsymbol{\Psi}^\top\boldsymbol{\beta}d\log Y + \mathbf{A}_z d\log\mathbf{z} + \mathbf{A}_p d\log\mathbf{p}. \tag{34}
$$

Matrix Equation for Prices. Now consider the relationship between prices and marginal costs (8) in matrix form:

$$
\begin{aligned}
&(\mathrm{diag}(\mathbf{1} + \boldsymbol{\alpha}\odot(\mathbf{1}-\boldsymbol{\theta}_q)\oslash\boldsymbol{\theta}_q) - \boldsymbol{\Omega})d\log\mathbf{p} \\
&= \mathrm{diag}(\boldsymbol{\alpha}\oslash\boldsymbol{\theta}_q)(d\log\mathbf{s} - d\log\mathbf{h}) - \mathrm{diag}(\mathbf{1} + \boldsymbol{\alpha}\odot(\mathbf{1}-\boldsymbol{\theta}_q)\oslash\boldsymbol{\theta}_q)d\log\mathbf{z}.
\end{aligned}
$$

Define the matrix

$$
\boldsymbol{\Psi}_\theta = (\mathrm{diag}(\mathbf{1} + \boldsymbol{\alpha}\odot(\mathbf{1}-\boldsymbol{\theta}_q)\oslash\boldsymbol{\theta}_q) - \boldsymbol{\Omega})^{-1},
$$

which resembles the Leontief inverse matrix but adjusts for the fact that labor allocations are decided before wages are paid. Then prices solve

$$\begin{aligned} d\log \mathbf{p} = \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash \boldsymbol{\theta}_q)(d\log \mathbf{s} - d\log \mathbf{h}) \\ - \boldsymbol{\Psi}_\theta \operatorname{diag}(\mathbf{1} + \boldsymbol{\alpha} \odot (\mathbf{1} - \boldsymbol{\theta}_q) \oslash \boldsymbol{\theta}_q) d\log \mathbf{z} \end{aligned} \tag{35}$$

Remainder of Proof. Plug (35) into (34).

$$\begin{aligned} \operatorname{diag}(\boldsymbol{\lambda}) d\log \mathbf{s} = \boldsymbol{\Psi}^\top \boldsymbol{\beta} d\log Y + \mathbf{A}_z d\log \mathbf{z} + \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash \boldsymbol{\theta}_q)(d\log \mathbf{s} - d\log \mathbf{h}) \\ - \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\mathbf{1} + \boldsymbol{\alpha} \odot (\mathbf{1} - \boldsymbol{\theta}_q) \oslash \boldsymbol{\theta}_q) d\log \mathbf{z}. \end{aligned}$$

Re-arrange to get

$$\begin{aligned} &(\mathbf{I} - \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash (\boldsymbol{\theta}_q \odot \boldsymbol{\lambda}))) \operatorname{diag}(\boldsymbol{\lambda}) d\log \mathbf{s} \\ &= \boldsymbol{\Psi}^\top \boldsymbol{\beta} d\log Y - \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash \boldsymbol{\theta}_q) d\log \mathbf{h} \\ &\quad + (\mathbf{A}_z - \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\mathbf{1} + \boldsymbol{\alpha} \odot (\mathbf{1} - \boldsymbol{\theta}_q) \oslash \boldsymbol{\theta}_q)) d\log \mathbf{z} \end{aligned}$$

Solving for $d\log \mathbf{s}$ delivers (13) and plugging the result into the matrix equation for prices (35) obtains (14).

For reference, the formulas for the matrices are as follows.

$$\begin{aligned} \boldsymbol{\Psi}_\theta &= (\operatorname{diag}(\mathbf{1} + \boldsymbol{\alpha} \odot (\mathbf{1} - \boldsymbol{\theta}_q) \oslash \boldsymbol{\theta}_q) - \boldsymbol{\Omega})^{-1} \\ \mathbf{A}_z &= \boldsymbol{\Psi}^\top \boldsymbol{\Omega}^\top \operatorname{diag}(\boldsymbol{\lambda} \odot (\boldsymbol{\theta}_q - \mathbf{1})) \\ \mathbf{A}_p &= \mathbf{A}_z - \boldsymbol{\Psi}^\top \operatorname{diag}(\boldsymbol{\Omega}^\top (\boldsymbol{\lambda} \odot (\boldsymbol{\theta}_q - \mathbf{1}))) + \boldsymbol{\Psi}^\top (1 - \theta_c) \operatorname{diag}(\boldsymbol{\beta}) \\ \mathbf{S}_z &= \operatorname{diag}(\boldsymbol{\lambda})^{-1} (\mathbf{I} - \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash (\boldsymbol{\theta}_q \odot \boldsymbol{\lambda}))^{-1} (\mathbf{A}_z - \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\mathbf{1} + \boldsymbol{\alpha} \odot (\mathbf{1} - \boldsymbol{\theta}_q) \oslash \boldsymbol{\theta}_q)) \\ \mathbf{S}_h &= -\operatorname{diag}(\boldsymbol{\lambda})^{-1} (\mathbf{I} - \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash (\boldsymbol{\theta}_q \odot \boldsymbol{\lambda}))^{-1} \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash \boldsymbol{\theta}_q) \end{aligned}$$

$$\mathbf{S}_Y = \operatorname{diag}(\boldsymbol{\lambda})^{-1}(\mathbf{I} - \mathbf{A}_p \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash (\boldsymbol{\theta}_q \odot \boldsymbol{\lambda}))^{-1} \boldsymbol{\Psi}^\top$$

$$\mathbf{P}_z = \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash \boldsymbol{\theta}_q)\mathbf{S}_z - \boldsymbol{\Psi}_\theta \operatorname{diag}(\mathbf{1} + \boldsymbol{\alpha} \odot (\mathbf{1} - \boldsymbol{\theta}_q) \oslash \boldsymbol{\theta}_q)$$

$$\mathbf{P}_h = \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash \boldsymbol{\theta}_q)(\mathbf{S}_h - \mathbf{I})$$

$$\mathbf{P}_Y = \boldsymbol{\Psi}_\theta \operatorname{diag}(\boldsymbol{\alpha} \oslash \boldsymbol{\theta}_q)\mathbf{S}_Y.$$

A.3 Proposition 3

Labor Wedges. Plug (6), (13), and (14) into (10) to get

$$\begin{aligned} d\log \tau_i = &-\mathbf{e}_i^\top (\mathbf{S}_z + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})(\mathbf{I} + \mathbf{P}_z))\mathcal{E}_i[d\log \mathbf{z}] \\ &- \mathbf{e}_i^\top ((\mathbf{S}_Y + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_Y)\boldsymbol{\beta} - \boldsymbol{\theta}_q \gamma)\mathcal{E}_i[d\log Y] \\ &- \mathbf{e}_i^\top (\mathbf{S}_h + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})(\mathbf{S}_h + \operatorname{diag}(\boldsymbol{\theta}_q - 1)\mathbf{P}_h - \operatorname{diag}(\boldsymbol{\theta}_q \odot \boldsymbol{\eta}))\mathcal{E}_i[d\log \mathbf{h}] \end{aligned}$$

This calculation is valid because of the sequential rationality assumption.

Define

$$\begin{aligned} \mathbf{T}_z &= \mathbf{S}_z + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})(\mathbf{I} + \mathbf{P}_z), \\ \mathbf{T}_Y &= \mathbf{S}_Y + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_Y)\boldsymbol{\beta} - \boldsymbol{\theta}_q \gamma. \end{aligned}$$

Under the heterogeneous priors assumption, $\mathcal{E}_i[d\log \mathbf{h}] = \mathbf{0}$ for all i. Then sector i's log labor wedge can be written as

$$d\log \tau_i \equiv -\mathbf{T}_{z,i:}\mathcal{E}_i[d\log \mathbf{z}] - \mathbf{T}_{Yi}\mathcal{E}_i[d\log Y],$$

where the notation $\mathbf{T}_{z,i:}$ means the ith row of $\mathbf{T}_z$.

By Hulten's theorem,

$$d\log Y = \boldsymbol{\lambda}^\top d\log \mathbf{z} + \boldsymbol{\lambda}^\top \operatorname{diag}(\boldsymbol{\alpha}) d\log \mathbf{h}.$$

Take expectations of both sides to get

$$E_i[d\log Y] = \boldsymbol{\lambda}^\top E_i[d\log \mathbf{z}] + \boldsymbol{\lambda}^\top \operatorname{diag}(\boldsymbol{\alpha}) E_i[d\log \mathbf{h}].$$

Then the expectation error is simply

$$\mathcal{E}_i[d\log Y] = \boldsymbol{\lambda}^\top \mathcal{E}_i[d\log \mathbf{z}] + \boldsymbol{\lambda}^\top \operatorname{diag}(\boldsymbol{\alpha}) \mathcal{E}_i[d\log \mathbf{h}]$$

Under the heterogeneous prior assumption, $\mathcal{E}_i[d\log \mathbf{h}] = \mathbf{0}$, hence

$$d\log \tau_i = -(\mathbf{T}_{z,i:} + \mathbf{T}_{Yi}\boldsymbol{\lambda}^\top)\mathcal{E}_i[d\log \mathbf{z}].$$

Labor Demand in Terms of Labor Wedges. Plug (13), (14), and (6) into (9) to get

$$\begin{aligned}
&(\mathbf{I} + \operatorname{diag}(\boldsymbol{\theta}_q \odot \boldsymbol{\eta}) - \mathbf{S}_h - \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_h) d\log \mathbf{h} \\
&= (\mathbf{S}_z + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_z) d\log \mathbf{z} + ((\mathbf{S}_Y + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_Y)\boldsymbol{\beta} - \gamma\boldsymbol{\theta}_q) d\log Y - d\log \boldsymbol{\tau}.
\end{aligned}$$

Define the matrices $\mathbf{H}_\tau$, $\mathbf{H}_z$, and $\mathbf{H}_Y$

$$\begin{aligned}
\mathbf{H}_\tau &= (\mathbf{I} + \operatorname{diag}(\boldsymbol{\theta}_q \odot \boldsymbol{\eta}) - \mathbf{S}_h - \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_h)^{-1} \\
\mathbf{H}_z &= \mathbf{H}_\tau(\mathbf{S}_z + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})(\mathbf{I} + \mathbf{P}_z)) \\
\mathbf{H}_Y &= \mathbf{H}_\tau((\mathbf{S}_Y + \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_Y)\boldsymbol{\beta} - \gamma\boldsymbol{\theta}_q)
\end{aligned}$$

so that

$$d\log \mathbf{h} = \mathbf{H}_z d\log \mathbf{z} + \mathbf{H}_Y d\log Y - \mathbf{H}_\tau d\log \boldsymbol{\tau}.$$

Aggregation. Plug the expression for labor demand in terms of labor wedges into $d\log Y$ from Hulten's Theorem and re-arrange to

$$(1 - \text{diag}(\boldsymbol{\alpha})\mathbf{H}_Y) d\log Y = \boldsymbol{\lambda}^\top (\mathbf{I} + \text{diag}(\boldsymbol{\alpha})\mathbf{H}_z) d\log \mathbf{z} - \boldsymbol{\lambda}^\top \text{diag}(\boldsymbol{\alpha})\mathbf{H}_\tau d\log \boldsymbol{\tau}.$$

Define

$$d\log Y^{(0)} = \frac{1}{1 - \text{diag}(\boldsymbol{\alpha})\mathbf{H}_Y} \boldsymbol{\lambda}^\top (\mathbf{I} + \text{diag}(\boldsymbol{\alpha})\mathbf{H}_z) d\log \mathbf{z}.$$

Then, writing the matrix algebra into sums,

$$\begin{aligned}
d\log Y &= d\log Y^{(0)} - \frac{1}{1 - \text{diag}(\boldsymbol{\alpha})\mathbf{H}_Y} \sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \alpha_j \sum_{i\in\mathcal{N}} \mathbf{H}_{\tau,ji} d\log \tau_i \\
&= d\log Y^{(0)} + \frac{1}{1 - \text{diag}(\boldsymbol{\alpha})\mathbf{H}_Y} \sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \alpha_j \sum_{i\in\mathcal{N}} \mathbf{H}_{\tau,ji} (\mathbf{T}_{z,i:} + \mathbf{T}_{Yi}\boldsymbol{\lambda}^\top) \mathcal{E}_i[d\log \mathbf{z}] \\
&= d\log Y^{(0)} + \frac{1}{1 - \text{diag}(\boldsymbol{\alpha})\mathbf{H}_Y} \sum_{i\in\mathcal{N}} (\mathbf{T}_{z,i:} + \mathbf{T}_{Yi}\boldsymbol{\lambda}^\top) \mathcal{E}_i[d\log \mathbf{z}] \sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \alpha_j \mathbf{H}_{\tau,ji} \\
&= d\log Y^{(0)} + \frac{1}{1 - \text{diag}(\boldsymbol{\alpha})\mathbf{H}_Y} \sum_{i\in\mathcal{N}} \underbrace{\left(\sum_{j\in\mathcal{N}} \boldsymbol{\lambda}_j \alpha_j \mathbf{H}_{\tau,ji} \right)}_{= \boldsymbol{\lambda}^\top \text{diag}(\boldsymbol{\alpha})\mathbf{H}_\tau \mathbf{e}_i} (\mathbf{T}_{z,i:} + \mathbf{T}_{Yi}\boldsymbol{\lambda}^\top) \mathcal{E}_i[d\log \mathbf{z}].
\end{aligned}$$

A.4 Proposition 4

Consider a Cobb-Douglas network with $\gamma = 1$. Plugging in intermediates expenditure demand (7) into the sales market-clearing equation (11) from Proposition 1 implies that $d\log s_i = d\log Y$. Then the sales to wage ratio is $d\log s_i - d\log w_i = -\eta_i d\log h_i$. Because be-

liefs are common knowledge, $E_i[d\log h_i] = d\log h_i$, hence $d\log\tau_i = 0$ and $d\log Y = d\log Y^{(0)}$.

A.5 Proposition 5

By Appendix A.3, which proves Proposition 3,

$$d\log\mathbf{h} = \mathbf{H}_z d\log\mathbf{z} + \mathbf{H}_Y d\log Y - \mathbf{H}_\tau d\log\tau.$$

All the matrices on the RHS depend on $\mathbf{H}_\tau$, which in thecase of a uniform inverse Frisch elasticity is given by

$$\mathbf{H}_\tau = (\mathbf{I} + \operatorname{diag}(\boldsymbol{\eta})\operatorname{diag}(\boldsymbol{\theta}_q) - \mathbf{S}_h - \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_h)^{-1}.$$

Since $\underline{\eta} = \min_{i\in\mathcal{N}}\eta_i$,

$$\mathbf{I} + \operatorname{diag}(\boldsymbol{\eta})\operatorname{diag}(\boldsymbol{\theta}_q) - \mathbf{S}_h - \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_h \geq \mathbf{I} + \operatorname{diag}(\underline{\eta}\boldsymbol{\theta}_q) - \mathbf{S}_h - \operatorname{diag}(\boldsymbol{\theta}_q - \mathbf{1})\mathbf{P}_h \to \infty$$

as $\underline{\eta} \to \infty$, hence $\mathbf{H}_\tau \to \mathbf{0}$ and thus $d\log\mathbf{h} \to \mathbf{0}$. Note that $d\log Y \neq 0$ because sectoral TFPs can still deviate from steady state.