Believe it or Not:
The Role of Investor Beliefs for Private Equity Valuation*

Aleksandr Ermakov† Benjamin Holcblat‡

January 30, 2025

Abstract

We show investors' beliefs can explain the undervaluation of buyout private equity (PE) funds. The value of PE funds, by finance principles, corresponds to expected cash flows discounted for time and risk. Therefore, the undervaluation may stem from an incorrect stochastic discount factor (SDF) or a difference between investors' beliefs and the true distribution of discounted cash flows. We estimate investors' beliefs from PE cash flows. Estimated beliefs are consistent with investors' sentiment surveys. We find that investors' pessimism about PE relative to public markets, rather than SDF misspecification, offers a potential explanation for PE funds' undervaluation.

Keywords: Subjective Beliefs; Belief Premium; Investors' Expectations; Private Equity Funds; Generalized Empirical Likelihood; Relative Entropy; Survey Data.

JEL Codes: G12, G23, G24, D84.

*We are grateful to Laurent Barras, Gustavo Freire, Will Gornall, Michael Halling, Juan Imbet, Marie Lambert, Elena Loutskina, Julien Pénasse, Patrick Roger, Denitsa Stefanova, Roberto Steri, Thomas Gruenthaler, Fabio Trojani, Tereza Tykvová, and Yuanji Wen, and seminar/conference participants at the University of Luxembourg, 9th Finance Forum of the Spanish Finance Association, 14th Annual Hedge Fund Research Conference, 2023 Financial Markets and Corporate Governance Conference, 39th Conference of the French Finance Association, 6th Dauphine Finance PhD Workshop, International Accounting & Finance Doctoral Symposium 2023, IAAE 2023 Annual Conference, 7th Entrepreneurial Finance Association Conference, EFA Doctoral Tutorial 2023, the seminar at the University of Virginia Darden School of Business, CEPR/ESSFM (Gerzensee), FinEML Conference (Lugano). The present paper is based on the first chapter of Alex's Ph.D. thesis. Both authors acknowledge financial support from the Luxembourg National Research Fund (FNR) through the PRIDE grant MATHCODA. Any errors are our own.

†University of Manchester, aleksandr.ermakov AT manchester.ac.uk .
‡University of Luxembourg, benjamin.holcblat AT uni.lu

1 Introduction

Private equity (PE) refers to capital shares in firms not publicly traded on a stock market. Many firms, at various stages of their development, require long-term capital or external managerial expertise to transform their business models. By offering both, PE funds have played an increasing role in the economy. As of 2021, their total size exceeds one-tenth of the $54 trillion U.S. asset management industry.[1] Despite PE's impressive growth, there is a persistent undervaluation of one of its main fund types, namely buyout PE funds (Higson and Stucke 2012, Harris et al. 2014, Kaplan and Sensoy 2015, Robinson and Sensoy 2016). Hereafter, we refer to buyout PE funds simply as PE funds. This undervaluation, equivalent to an outperformance of PE funds net of fees, is one of the few stylized facts about PE (Korteweg 2023).

The present paper proposes an explanation for the undervaluation of PE funds from the lenses of PE investors, also known as Limited Partners (LPs). Usual valuation methods, such as the Internal Rate of Return (IRR) and various multiples, systematically undervalue PE funds (e.g., Korteweg 2023, Table 1). One possible reason for this failure is their reliance on a *constant* discount factor, so their risk adjustment does not align with finance principles (LeRoy 1973). Specifically, these methods do not account for the higher value of payoffs in bad states of the economy: In poverty, one dollar is worth more than in abundance. Nevertheless, even the Public Market Equivalent (PME) approach (Long and Nickels 1996, Kaplan and Schoar 2005), which adjusts for risk with a *stochastic* discount factor (SDF) in line with finance principles (Sorensen and Jagannathan 2015), implies PE funds undervaluation.

According to basic finance principles, the value of a PE fund should equal its expected future cash flows discounted for time and risk. Therefore, the undervaluation of PE funds may stem from an incorrect SDF or a difference between investors' beliefs and the true distribution of discounted cash flows. The present paper investigates whether PE investors' beliefs can explain the undervaluation. In line with rational expectations (Muth 1961), standard valuation approaches have assumed perfect investors' information, so investors' subjective beliefs equate the true objective probability distribution of data (e.g., Adam and Nagel 2022, and references therein). Nevertheless, there is a long and growing list of evidence difficult to reconcile with perfect investor's information (e.g., Pesaran 1987, Manski 2004, Coibon and Gorodnichenko 2015, Greenwood and Shleifer 2014, Bordalo et al. 2019). Moreover, in the case of PE, the funds' opacity makes it particularly likely that PE investors' beliefs differ from the true distribution of discounted cash flows. The relative

[1] According to Preqin, a leading provider of data, analytics, and insights for PE funds, and the BCG Global Asset Management 2022 report (page 23).

opacity of PE funds stems at least from two sources. First, PE funds have been subject to lighter disclosure requirements than other types of funds, such as mutual funds. Thus, PE investors' have limited information about the funds they invest in. Second, by definition, PE funds are not publicly traded, which means that transaction data are nonexistent or very rare. In consequence, valuation has to rely on cash flow data or net asset values (NAVs). The latter are subject to potential manipulation by PE managers, also known as General Partners (GPs).[2]

In the present paper, we estimate beliefs reflected in PE cash flows —both inflows and outflows. The amount invested (inflows) into a PE fund reflects investors' beliefs about future outflows they will receive. Since PE investors are not forced to invest in PE funds and can instead invest more in public stock markets, our method relies on the idea that under investors' beliefs, one dollar invested in PE should yield the same expected discounted cash flow as one dollar in public markets. If this were not the case, investors would invest more in the asset with higher expected discounted cash flows, hence eliminating the difference. In a nutshell, investors' beliefs should be such that investors are indifferent between PE and the public stock markets. We propose an empirical-likelihood-type method to estimate beliefs. Following the parsimony principle, we estimate the belief $\mathbb{P}^*$ closest to the true distribution $\mathbb{P}$ so that investors are indifferent between PE and the public stock markets, i.e., the expected discounted PE cash flows in excess of public markets' discounted cash flows are zero. We estimate the belief $\mathbb{P}^*$ that diverges as little as possible from the true distribution $\mathbb{P}$, in the sense of the relative entropy, also known as the Kullback-Leibler divergence criterion.[3] The belief $\mathbb{P}^*$ is a proxy for PE investors' beliefs.

The main empirical result of the paper is that PE investors' beliefs can explain the undervaluation of PE funds. More precisely, we find a belief premium due to the difference between PE investors' beliefs and the true distribution of discounted cash flows. Figure 1 illustrates this main result. It displays the estimated investors' beliefs about the average PE fund's discounted excess cash flows and their estimated true distribution. Figure 1 shows the source of the belief premium: The excess average discounted cash flows perceived by investors are smaller than the true ones. The estimated beliefs exhibit a fatter negative tail and a thinner positive tail compared to the estimated true distribution. Figure 1 also explains the main empirical result: On average, investors appear to be overly pessimistic about PE funds' ability to generate cash flows in comparison to the public stock markets and, hence, the relative undervaluation of PE funds.

We investigate whether SDF misspecification and omitted pricing factors are likely to drive our

[2] See Phalippou and Gottschalg (2009), Brown et al. (2019, 2020), Jenkinson et al. (2020).

[3] See Hansen and Sargent (2008) and references therein for similar use of the relative entropy, and motivations for this measure of discrepancy.

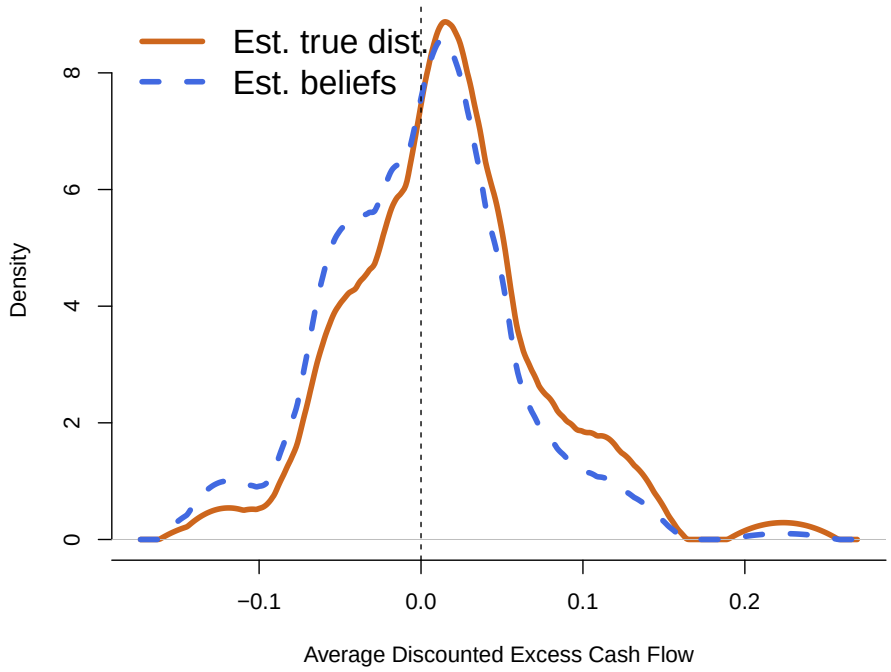


Figure 1: Estimated investors' beliefs $\hat{\mathbb{P}}^*$ vs estimated true distribution $\hat{\mathbb{P}}$ of average discounted excess cash flows. The cash flows are quarterly data from Q1 1996 to Q4 2014 in excess of the S&P500. Values are presented in millions of dollars. Cash flows are discounted with the exponential CAPM SDF used in Korteweg and Nagel (2016). The Epanechnikov kernel is used to smooth the estimated true distribution and the estimated belief.

main empirical result. First, we show that it is robust to a large range of asset pricing models, including the Consumption-CAPM (C-CAPM) by Rubinstein (1976), the External Habit model by Campbell and Cochrane (1999), and the Long-Run Risks model based on Epstein-Zin preferences by Bansal and Yaron (2004). Each of these models implies different time and risk preferences for PE investors, and thus different SDFs.

Second, we show our main empirical result is robust to factors that could have a price impact but are not necessarily accounted for in usual SDFs. For this purpose, we orthogonalize estimated beliefs against these factors. Specifically, we consider the following factors identified in the literature: the expected equity risk premium (Haddad et al. 2017), Moody's BBB to AAA credit spread (Schmid et al. 2019), the Chicago Board Options Exchange S&P 100 Volatility Index (Gompers et al. 2008), and the aggregate liquidity factor proposed by Pástor and Stambaugh (2003) (Franzoni et al. 2012). The results reveal that these orthogonalized beliefs yield qualitatively similar results: Orthogonalized estimated beliefs have a strong correlation with original estimated beliefs. We also note that the liquidity factor appears to be not only qualitatively but also quantitatively irrelevant across all asset pricing models. This result supports the idea that PE investors are

long-term investors who trade infrequently and require minimal compensation for illiquidity, as in Constantinides (1986).

Finally, we show that estimated beliefs are consistent with investor sentiment reported in survey data. The main focus is on sentiment among private equity investors. For them, we analyze two key indices: the Silicon Valley VC Confidence Index (SVVCCI) and the Central Europe PE Confidence Index (CEPECI). The SVVCCI is a quarterly survey of venture capitalists in the San Francisco Bay Area, conducted by Cannice and Goldberg (2009), while the CEPECI is a semi-annual survey of private equity professionals in Central Europe, conducted by Deloitte. The results reveal a positive correlation between these PE sentiment indices and belief-adjusted average excess cash flows, with correlations ranging from 0.57 to 0.6 for the CEPECI, and from 0.68 to 0.73 for the SVVCCI, indicating that estimated beliefs do indeed contain information about investors' expectations.

Understanding the source of PE funds undervaluation is not a mere academic curiosity. Funds GPs likely want to increase their funds' valuations, and hence raise more capital. The undervaluation of PE funds is also likely to have an impact beyond PE. Assets' valuations reflect and guide capital allocation in the economy, which is a key driver of economic growth. As previously mentioned, PE is not a small asset class anymore. While the public stock market is shrinking, PE is growing.[4]

The remainder of the paper is organized as follows. In Section 2, we present the methodology and the moment condition that we use to estimate the investors' beliefs from the observed excess cash flows. Section 3 details the results of the model-implied subjective beliefs and explains the existence of the positive excess cash flow. Additionally, we explore various risk preferences assumptions and conduct a bootstrap exercise to compare the economic implications of estimated beliefs in Section 3. Section 4 compares estimated beliefs with survey data from individual, institutional, and PE investors. Finally, the paper concludes with suggestions for future research.

Related Literature. In addition to the aforementioned articles, this paper builds on and contributes to several literatures. First, it builds on the literature about PE funds' valuation, or equivalently, the literature about PE funds performance (e.g., Ang et al. 2018, Harris et al. 2014, Gupta and Van Nieuwerburgh 2021, among others). As mentioned above, usual valuation methods are the Internal Rate of Return (IRR) and various multiples such as the Total Value to Paid-in Capital (TVPI) ratio (e.g., Gompers et al. 2016). These valuation methods suffer from several documented problems (e.g., Phalippou 2013, 2014, Korteweg 2023, Sec. 3.2.3 for a summary). As mentioned above, one of their main problems is that they do not adjust for risk according to finance principles. Building on Long and Nickels (1996), Kaplan and Schoar (2005) proposes

[4] E.g., see JPMorgan Chase & Co Annual Report 2023 on p. 36.

the PME approach, which benchmarks PE funds against the public stock market. Sorensen and Jagannathan (2015) shows the PME approach is equivalent to an SDF for investors with log-utility preferences, so it is in line with finance principles. Korteweg and Nagel (2016) develops the *generalized* PME (GPME) approach that allows for any SDFs. Gredil et al. (2019) applies Korteweg and Nagel (2016)'s GPME approach with SDF implied by non-time-separable utilities. The present paper also builds on Korteweg and Nagel (2016)'s GPME approach but introduces investors' beliefs potentially different from the true distribution of cash flows. It finds evidence that the difference between investors' beliefs and the true distribution of cash flows commands a premium. The difference could be due to behavioral biases or informational frictions (e.g., Adam and Nagel 2022, Sec. 2.1). The present paper is agnostic regarding the origin of the difference. Unlike a risk premium, a belief premium can be positive or negative depending on whether beliefs are optimistic or pessimistic with respect to the true distribution.

Second, the paper builds on a long literature about the role of beliefs and expectations in investment decisions. Following the pioneering work of Pigou (1927) and Keynes (1936), several models of belief formation were proposed: extrapolative expectations (Metzler 1941), adaptative expectations (Nerlove 1958), and rational expectations (Muth 1961). The latter, which equates investors' subjective beliefs with true probability, became dominant apparently because of its tractability, robustness to policy changes (Lucas 1976), and its intrinsic microfoundation as the best predictor in terms of mean squared error. Nevertheless, probably starting with Miller (1977), a stream of literature investigates deviations from rational expectations in asset pricing. The emergence of survey data and techniques to estimate beliefs have generated renewed interest in beliefs (e.g., Hansen 2007, Adam et al. 2021). Within the latter literature, the present paper specifically contributes to a growing stream of literature that tries to uncover investors' beliefs. From a conceptual point of view, the present paper builds on an idea that goes back at least to Ramsey (1926/1931) and de Finetti (1937), who propose to measure beliefs (subjective probabilities) from one's willingness to bet. Our estimation technique is based on PE investors' willingness to bet (invest) in PE funds with respect to public stock markets. From a technical point of view, the present paper is close to Ghosh and Roussellet (2020), who relies on Kitamura et al. (2004) non-parametric methodology to estimate conditional beliefs. In the present paper, the limited data time span renders perilous the fine-tuning of nuisance parameters involved in nonparametric estimation so we only estimate unconditional beliefs, thereby avoiding nonparametric techniques and the associated curse of dimensionality. Chen et al. (2020) and Korsaye (2022) use survey data and asset prices to estimate investors's beliefs. In the present paper, we augment this literature by extracting the unconditional beliefs of PE investors from cash flow data.

Finally, the present paper is also closely related to the empirical likelihood (EL) literature through its methodology. EL was developed by Owen (1988) and extended to moment conditions models by Qin and Lawless (1994). Smith (1997) and Newey and Smith (2004) proposed the Generalised Empirical Likelihood (GEL) class of estimators, which includes the exponential tilting (ET) estimator of Kitamura and Stutzer (1997) and Imbens et al. (1998). This paper utilises the ET approach, which goes back at least to Efron (1981), to estimate investors' beliefs. By Lagrangian duality, the ET approach yields an estimate of the probability distribution that is the closest to the estimated true probability distribution in the sense of the relative entropy. The use of EL-type methods has gained popularity in the financial literature. Regarding the estimation of beliefs, the methodology of this paper is mainly inspired by Stutzer (1996) and Julliard and Ghosh (2012). Stutzer (1996) used EL-type methods to estimate the risk-neutral probability distribution, while Julliard and Ghosh (2012) applied EL-type methods to evaluate the empirical validity of the rare disasters explanation for the equity premium puzzle. Other works with applications of EL-type methods to finance include Almeida and Garcia (2012, 2017), Ghosh et al. (2017), Almeida et al. (2020), Almeida and Freire (2022), Korsaye et al. (2019), Holcblat and Sowell (2022), Almeida et al. (2022), Almeida, Freire, Azevedo and Ardison (2023), Almeida, Ardison and Freire (2023), Korsaye et al. (2023).

2 Methodology

2.1 SDF Pricing

Traditional asset pricing theory determines the fair value of an asset (V_t) as the expected future cash flow (CF_{t+1}) plus residual value (V_{t+1}), discounted for time and risk using an SDF ($M_{t,t+1}$):

$$V_t = \mathbb{E}^{\mathbb{P}}[M_{t,t+1}(CF_{t+1} + V_{t+1})] \tag{1}$$

An asset's value is not solely determined by the risks associated with the investment but also by investors' beliefs about the distribution of future cash flows. The Euler equation (1) captures this essence by focusing on three pivotal components: (i) the stochastic future cash flows, (ii) the one-step-ahead SDF, and (iii) the true probability measure ($\mathbb{P}$). Valuation errors might arise not only from inaccuracies in reflecting risk preferences in the SDF but also from potential discrepancies in capturing the representative investor's understanding of the DGP. This latter challenge becomes particularly crucial given the RE assumption, which suggests that agents are aware of the true DGP.

This paper investigates the violation of the RE hypothesis as the main source of valuation errors. While the application of RE is widely acknowledged for analytical purposes, this paper introduces an alternative version of the Euler equation that incorporates investors' beliefs, represented by the measure $\mathbb{P}^*$, which generally might deviate from the true measure $\mathbb{P}$.

$$V_t = \mathbb{E}^{\mathbb{P}^*}[M_{t,t+1}(CF_{t+1} + V_{t+1})] \tag{2}$$

In this paper, we emphasize the discrepancy between the true market beliefs ($\mathbb{P}$) and investors' subjective beliefs ($\mathbb{P}^*$) and their significance in the valuation of PE funds. We demonstrate that the standard SDF approach cannot adequately explain the outperformance of PE funds relative to the public market. To assess the performance of PE funds, we utilize the benchmarking technique known as the Generalized Public Market Equivalent (GPME) metric introduced by Korteweg and Nagel (2016). The GPME approach computes the Net Present Value (NPV) of PE cash flows using the SDF as a discount factor. Our findings indicate an abnormal performance for PE funds of 25 cents for every invested dollar on average using the CAPM model[5].

A detailed discussion of the GPME framework and our adaptation is available in Appendix C. This paper primarily focuses on the following NPV formulation:

$$NPV_i^{PE} = \sum_{t=1}^{T} M_{1,t} CF_{i,t}^{PE} \tag{3}$$

This formulation ensures that all cash flows are consistently compared, as they are discounted to a single reference point for each fund. This consistency is crucial for our analysis since the beliefs estimation requires aggregation across funds to form the underlying time series for the moment condition. Here, $t = 1, ..., T$ represents the timeline of observed PE cash flows, while $M_{1,t}$ denotes the multi-period SDF, which is derived by compounding the single-period discount factors:

$$M_{1,1} = 1; M_{1,2} = M_{1,2}; M_{1,3} = M_{1,2}M_{1,3}; M_{1,t} = M_{1,2}M_{1,3}...M_{1,t}$$

Under the null hypothesis, the NPV of a PE fund is expected to be zero, that is, $\mathbb{E}[NPV_i^{PE}] = 0$, given that the cash flows ($CF_{i,t}^{PE}$) include both the initial investments (capital calls) and the returned cash flows (capital distributions). If this condition is met for each individual fund, then the aggregated NPV across all funds should also be zero:

[5] For estimation details, see Table D.5.

$$\mathbb{E}^{\mathbb{P}} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{PE} = 0 \tag{4}$$

Similarly, a benchmark fund, which simulates cash flows from investments in a market index like CRSP, should also have an NPV of zero:

$$\mathbb{E}^{\mathbb{P}} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{mrkt} = 0 \tag{5}$$

Combining the NPV conditions for both PE funds and benchmark funds yields the condition for excess cash flows:

$$\mathbb{E}^{\mathbb{P}} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} [CF_{i,t}^{PE} - CF_{i,t}^{mrkt}] = \mathbb{E}^{\mathbb{P}} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0 \tag{6}$$

Equation 6 implies that the NPV of the aggregate excess cash flows equals zero. If the GPME metric deviates from zero, the excess cash flows stand out as the sole source of abnormal performance. The occurrence of abnormal performance for PE funds implies that this equation is not satisfied, given the combination of risk preferences ($M_{1,t}$) and probability distribution ($\mathbb{P}$).

The initial step entails estimating the parameters of the SDF under the empirical probability distribution, $\hat{\mathbb{P}}$. This estimation relies on the moment condition formulated in Equation 6. We show that the SDF alone can not explain the observed excess cash flows with a plausible level of risk aversion. In the subsequent step, we apply an empirical likelihood-type methodology. The aim is to pinpoint the subjective probability distribution, represented by $\hat{\mathbb{P}^*}$. This alternative distribution eliminates the pricing errors in combination with the SDF constructed using risk-aversion parameters from the literature.

2.2 Empirical Likelihood Estimation

To estimate investors' subjective beliefs, we seek an alternative probability distribution $\mathbb{P}^*$ that is closest to $\mathbb{P}$ in the Kullback-Leibler Information Criterion (KLIC) sense and satisfies the moment condition for excess average cash flows, as shown in Equation (6):

$$\begin{aligned} &\min_{\mathbb{P}^*} D(\mathbb{P}^*||\mathbb{P}) \equiv \min_{\mathbb{P}^*} \int \log\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}\right) d\mathbb{P}^* \\ &\text{subject to} \quad \mathbb{E}^{\mathbb{P}^*} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0 \end{aligned} \tag{7}$$

Here, $D(\mathbb{P}^*||\mathbb{P})$ represents the closeness between the subjective probability distribution $\mathbb{P}^*$ and the true probability distribution $\mathbb{P}$. The KL divergence is non-negative and equals zero if and only if $\mathbb{P}^* = \mathbb{P}$, which holds under the RE hypothesis. The probability distribution closest to the true one represents the investors' subjective beliefs.

The Exponential Tilting (ET) estimator, introduced by Efron (1981), Kitamura and Stutzer (1997), and Imbens et al. (1998), is employed to solve this problem. Anatolyev and Gospodinov (2011) demonstrates that this solution yields identical model-implied probabilities to those formulated for the KL divergence minimisation problem. However, since $\mathbb{P}$ and $\mathbb{P}^*$ are unobservable, the estimated subjective beliefs are the solution to the following minimization problem:

$$\begin{aligned} &\min_{p_t} D(\hat{\mathbb{P}^*}||\hat{\mathbb{P}}) = \sum_{t=1}^{T} p_t \log\left(\frac{p_t}{\frac{1}{T}}\right) \\ &\text{subject to} \quad \sum_{t=1}^{T} p_t M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0, \\ &\qquad\qquad \sum_{t=1}^{T} p_t = 1. \end{aligned} \tag{8}$$

Here, $1/T$ denotes the empirical probability distribution, and $\{p_t\}_{t=1}^{T}$ is the discrete distribution of subjective beliefs that assigns probabilities to the same points in time as $CF_{i,t}^{exc},\ t = 1, \ldots, T$. Introducing Lagrange multipliers to accommodate constraints, the Lagrangian is formed as:

$$L(p_t, \lambda, \mu) = \sum_{t=1}^{T} p_t \log\left(\frac{p_t}{\frac{1}{T}}\right) + \lambda' \sum_{t=1}^{T} p_t M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} + \mu\left(\sum_{t=1}^{T} p_t - 1\right) \tag{9}$$

Here, λ and μ enforce the moment and normalization constraints, respectively. With subsequent first-order conditions and normalization requirements, the optimal probability measure assumes a specific form as per Csiszár (1975)'s I-projection solution, leading to:

$$p_t = \frac{e^{\lambda'(M_{1,t}\sum_{i=1}^{N} CF_{i,t}^{exc})}}{\sum_{t=1}^{T} e^{\lambda'(M_{1,t}\sum_{i=1}^{N} CF_{i,t}^{exc})}} \tag{10}$$

In this equation, p_t is the probability measure satisfying the moment condition while remaining as close as possible to the empirical distribution $\frac{1}{T}$ in the KLIC sense. It is important to note that investors' subjective beliefs and model-implied probabilities, while intricately connected, represent distinct statistical objects. Model-implied probabilities derive from the alternative probability measure, while subjective beliefs emerge from applying these implied probabilities to specific variables of interest, such as excess cash flows in this case.

2.3 Data

To estimate investors' subjective beliefs, we use cash flow data for PE funds employing a buyout strategy sourced from Preqin (2020), spanning the period from 1980 to 2020. Furthermore, we filtered out funds incepted between 1980 and 2015 to exclude premature funds, characterized by substantial undistributed capital, from the analysis.

The cash flow data comprises three elements: capital calls, capital distributions, and valuations. Capital calls occur when the General Partner (GP) requests capital from the Limited Partner (LP) to facilitate investments. Conversely, capital distributions take place when the LP begins receiving returns from past investments. Valuations offer insights into the remaining capital awaiting distribution to investors. While valuations are crucial, particularly for premature funds, serving as estimates of the fund's remaining capital, they are often subject to potential manipulation by GPs. For a detailed discussion and other issues related to valuations, see Phalippou and Gottschalg (2009), Brown et al. (2019, 2020), Jenkinson et al. (2020).

To circumvent issues related to valuations and to employ more reliable cash flows, we restrict the sample to closed and liquidated funds. Liquidated funds refer to those that have terminated their operations and returned all capital to investors, while closed funds denote those that have concluded their fundraising campaigns but have not yet distributed all capital.

2.3.1 Excess Cash Flows. The key quantity for performing the ET estimation is excess cash flows. We use quarterly excess cash flows from 1996 Q3 to 2014 Q4, truncating and averaging the sample due to concerns about the stationarity of the underlying time series. This is imperative to consistently estimate investors' subjective distribution of cash flows in the ET estimation procedure and to make a valid statistical inference. Appendix A provides detailed insights into the construction and stationarity checks.

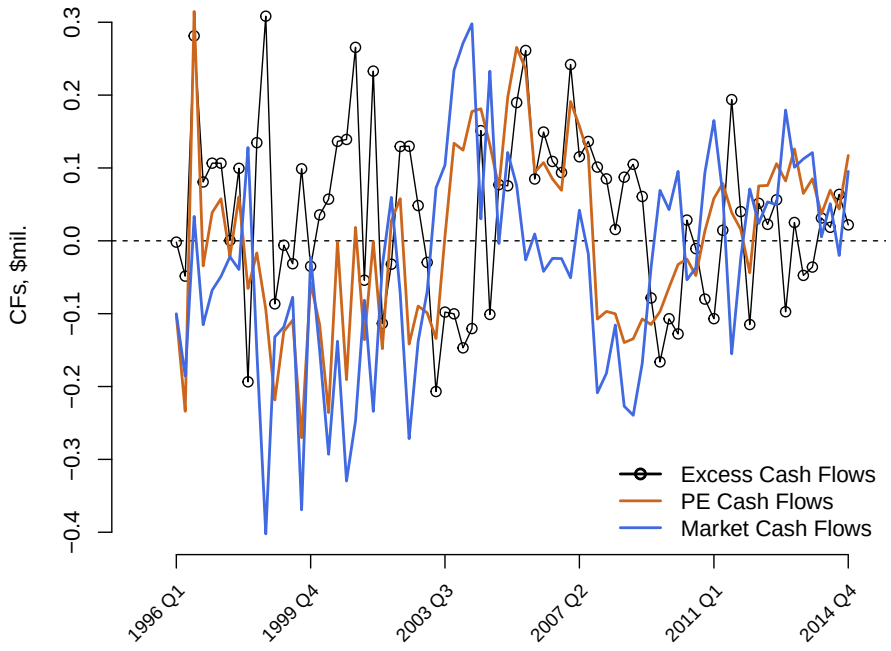


Figure 2: Average Excess Cash Flow and Its Components. This figure illustrates the value-weighted average excess cash flow of PE funds (black line) from Q1 1996 to Q4 2014. The average excess cash flow is derived from the difference between the average PE cash flows (brownish line) and the average market (CRSP) cash flows (blue line).

Aggregate excess cash flows are constructed by subtracting the aggregate benchmark public market mimicking cash flows ($\sum_{i=1}^{N} CF_{i,t}^{\text{mrkt}}$) from the aggregate PE cash flows ($\sum_{i=1}^{N} CF_{i,t}^{PE}$). The market benchmark fund is formulated following the methodology described in Korteweg and Nagel (2016). To construct mimicking cash flows, we set the date of transactions and assume that an LP investor allocates identical amounts of capital to both funds, ensuring capital calls of the two funds are identical in timing and magnitude.

While capital distributions share timing, they differ in magnitude. The size of capital distributions from benchmark funds is contingent upon public market performance. Specifically, the payout equals the accumulated returns between two consecutive dates, plus a fraction of the capital in a benchmark fund. This fraction is meticulously selected to align with the fund's lifespan.

Figure 2 presents the value-weighted average excess cash flow for PE funds spanning from 1996 Q1 to 2014 Q4. The excess cash flow fluctuates between a low of −\$0.21 million and a high of \$0.31 million, with a mean of \$0.04 million. It is essential to note that my analysis focuses on average excess cash flows due to their superior stationarity properties. Such properties are critical for ensuring robust beliefs estimation. All the cash flows considered are adjusted for inflation and are expressed in 1990 dollars. To better grasp the economic relevance of these excess cash flows, we compare them with the average payout, which is the average capital distributions across all

funds for each period. The mean of this ratio is 6.6%, representing a substantial portion of the typical PE cash flow.

2.3.2 Summary Statistics. Table 1 presents the descriptive statistics for the sample of PE funds used in the analysis. The sample encompasses 976 funds orchestrated by 454 PE firms. On average, a PE firm in the sample has raised two funds, each with an average commitment of $1.9 billion. The typical fund in the sample experiences 30 unique cash flows (inclusive of capital calls and distributions) and spans a period of 10 years between its initial and final observed cash flow.

The performance metrics are notably striking, both in absolute and relative terms. The median (average) fund posts an IRR of 10.78% (11.34%) and a total value-to-paid-in (TVPI) ratio of 1.52 (1.65). The GMPE metric is computed based on the GMM estimation of risk preferences in the SDF[6]. The abnormal performance for PE funds in the sample is affirmatively positive, equating to 25 cents and 15 cents for the GMPE and PME metrics, respectively. Notably, buyout funds have a reputation for outperforming the public market, in stark contrast to, for instance, VC funds, see Korteweg (2019). This divergence underscores the reason for centring the paper's focus on these funds: to avert interpretational ambiguity in the findings. Nevertheless, this focus is not predominantly derived from a methodological perspective.

Table 1: Summary Statistics: PE Funds Data. Data sourced from Preqin focuses on closed and liquidated funds incepted between 1980-2015, with a commitment threshold of at least $5 million (in 1990 dollars). Variables showcased include 'Fund size' (total commitment in millions), 'Fund effective years' (duration between the first and last observed cash flows), and performance measures such as IRR, TVPI and GPME. The GPME (Generalized Public Market Equivalent) captures discounted cash flows across funds using a specific SDF.

	Mean	Median	St.dev
# funds	976		
PE firms	454		
Funds/PE Firms	2.15	2.00	1.64
Fund size	1848	783	2812
Fund effective years	10.31	10.63	4.60
# of cash flows / fund	29.37	30.00	12.29
IRR, %	11.34	10.78	13.96
TVPI	1.65	1.52	0.76
PME, % – a/r per $1 com.	15.43	11.85	39.64
GPME, % – a/r per $1 com.	25.43	-0.71	125.38

[6] Results from the GMM estimation are detailed in Table D.5.

3 Empirical Results

Our methodology estimates beliefs using "off-the-shelf" SDFs. We estimate a subjective probability measure that aligns with observed PE valuations while minimizing deviation from the empirical distribution of cash flows. The organization of this section is as follows: 1) We elaborate on how we construct various mainstream SDFs in sub-section 3.1. 2) Next, utilizing the estimated SDFs, we delve into the economic implications of these beliefs, using the K-N CAPM SDF from the prior PE literature, in sub-section 3.2. 3) We then generalize the economic implications across a broader class of SDFs in sub-section 3.3. 4) Finally, by conducting a counterfactual exercise using model-implied probabilities, we demonstrate the statistical significance of the subjective beliefs in the PE context in sub-section 3.4.

3.1 Mainstream SDFs Construction and Estimation

Our methodology facilitates the extraction of beliefs under diverse assumptions regarding risk preferences. We consider a range of consumption-based asset pricing models, moving beyond the exponential CAPM (K-N CAPM) of Korteweg and Nagel (2016). This spectrum includes the C-CAPM (Consumption-CAPM) by Rubinstein (1976), the EHB (external habit formation) model by Campbell and Cochrane (1999), the LRR (long-run risks) model by Bansal and Yaron (2004), the exponential CAPM complemented with the size premium (CAPM+SMB), and Kaplan and Schoar (2005) version of the CAPM SDF (K-S CAPM).

Table 2 outlines the functional form of each SDF. For all the models, we estimate the relative risk aversion (RRA) parameter using a GMM procedure. This estimation procedure is designed to select the SDF parameters that can accurately price excess cash flows. In the CAPM and CAPM+SMB models, we introduce additional factors such as the Treasury Bill and the small growth portfolio benchmark. This approach facilitates the estimation of both the temporal component and the risk price tied to investing in a small-growth portfolio, one of the six portfolios highlighted by Fama and French (1993).

Constructing the LRR and EHB models involves significant complexity. We adhere to the methodology outlined in the original paper for the EHB model and to the approach described by Colacito and Croce (2011) for the LRR model. Investors in the EHB model of Campbell and Cochrane (1999) aim to keep their consumption above the current habit in the economy. This behaviour aligns with the objective of the PE investors to smooth their future consumption. On the other hand, the LRR model, as introduced by Bansal and Yaron (2004), posits that consumption

Table 2: SDFs Construction. Displayed in this table are the specific models and their functional forms used to derive Stochastic Discount Factors (SDFs), as shown in Panel A. Models require the estimation of parameters (like risk-preferences) distinctly from subjective beliefs, and if estimated, they are denoted with 'hats', e.g., $\hat{a}$, $\hat{b}$. For models like LRR and EHB, Panel B reveals how to build the one-period-ahead SDF, specifying the needed time series and the required calibration parameters. Notably, lowercase variables represent their log values. The consumption data is sourced from the National Income and Product Accounts of the Bureau of Economic Analysis, spanning from 1949 Q4 to 2019 Q4.

Panel A: Functional form

Model	One-period-ahead SDF (*log*)	Paper
K-N CAPM	$m_{t+1} = \hat{a} - \hat{b} r_{m,t+1}$	Korteweg & Nagel (2016)
K-S CAPM	$m_{t+1} = -r_{m,t+1}$	Kaplan & Schoar (2005)
CAPM+SMB	$m_{t+1} = \hat{a} - \hat{b} r_{m,t+1} - \hat{c} r_{smb,t+1}$	Korteweg & Nagel (2016)
C-CAPM	$m_{t+1} = log(\delta) - \hat{b}\Delta c_t$	Rubinstein (1976)
EHB	$m_{t+1} = log(\delta) - \hat{b}\Delta(c_t - h_t)$	Campbell & Cochrane (1999)
LRR	$m_{t+1} = log(\delta) - \frac{1}{\psi} x_t - \kappa_c \frac{1-\psi\hat{b}}{\psi(1-\rho_x\kappa_c)} e_{x,t+1} - \hat{b} e_{c,t+1}$	Bansal & Yaron (2004)

Panel B: Construction

Model	Consumption-based time series
EHB	$H_t = C_t - C_t S_t$ $s_t = (1-\phi_{QQ})\log(\bar{S}) + \phi_{QQ} s_{t-1} + \lambda(s_{t-1}) u_t$ $\lambda(S_t) = 1/\bar{S}\sqrt{1 - 2(s_{t-1} - \log(\bar{S}))} - 1$, $s_t \leq s_{\max}$, and 0 otherwise $S_{\max} = \exp(\log(\bar{S}) + \frac{1}{2}(1-\bar{S}^2))$ $\bar{S} = \sigma(\Delta c_t)\sqrt{\frac{\gamma}{(1-\phi_{QQ})}}$ $\phi_{QQ} = \phi_{mm}^3 = 0.89^3$, $\gamma = 2$, $\delta = 0.998$
LRR	$x_t = \Delta\hat{c}_t$, $\Delta c_t = [\Delta c_{t-1}, \Delta cy_{t-1}, pd_{t-1}, rf_{t-1}, spread_{t-1}]$ $e_{x,t+1} = x_{t+1} - \rho_x x_t$, $e_{c,t+1} = \delta\hat{c}_{t+1} - x_t$ $sd(e_{c,t+1}) = \sigma_{QQ}$, $sd(e_{x,t+1}) = \phi_e \sigma_{QQ}$ $\rho_x = \rho_{mm}^3 = 0.987^3$, $\psi = 2$, $\kappa_c = 0.9649$, $\sigma_{QQ} = \sqrt{3}\cdot\sigma_{mm} = \sqrt{3}\cdot 0.0078$, $\phi_e = 0.1085$ and $\delta = 0.998$

has a small but persistent component inducing uncertainty. Investors in the LRR model are concerned with long-term consumption uncertainty and aim to mitigate it. Although the central idea resonates with the habit model, the fundamental mechanisms are distinct, leading to different asset pricing implications.

Table 3 showcases the estimation outcomes of all models using the following moment condition:

$$\mathbb{E}\sum_{t=1}^{T} M_{1,t}(\theta) \sum_{i=1}^{N} CF_{i,t}^{exc} = 0,$$

where θ denotes the relative risk aversion (RRA) coefficient.

Table 3: GMM Estimation Results. The table reports the relative risk aversion (RRA) estimates for different SDFs using the GMM approach.

SDF	K-N CAPM	CAPM+SMB	C-CAPM	EHB	LRR
RRA	14.5	16.2	917.3	1611.9	39.9

High RRA coefficient values are required to explain the observed average excess cash flows. Particularly, those exceeding the widely accepted threshold of 10 indicate extreme risk aversion. In many economic contexts, such high values are often considered implausible, highlighting potential limitations within the traditional Rational Expectations (RE) paradigm. This evidence aligns with studies using the consumption Euler Equation for public markets, such as Julliard and Ghosh (2012), who employed the C-CAPM SDF to explain the equity premium puzzle across countries. Their findings indicate that the data requires high RRA levels to account for the stock market risk premium.

A similar challenge emerges for results in the PE market: adhering strictly to the RE framework results in improbable SDF parameter estimations. This accentuates a potential model misspecification inherent in the RE models when determining RRA. This observation is not merely a critique of existing methodologies but rather emphasizes the importance of subjective beliefs in the PE context.

Given implausible SDF parameter estimates using the moment condition for average excess cash flow, we estimate subjective beliefs using established 'off-the-shelf' SDF parameters. For the K-N CAPM and CAPM+SMB, the reference is Korteweg and Nagel (2016), particularly their GPME estimates for VC funds. Specifically, we set $a = 0.012$, on a quarterly basis, and RRA = 2.65. For the C-CAPM SDF, we use the upper bound for RRA as 10, as suggested by Julliard and Ghosh (2012). For the SDF determined by the LRR model, we rely on Bansal and Yaron (2004). The specifics of the LRR SDF construction are detailed in Table 2. In particular, we set $\sigma_{QQ} = 0.0135$, $\phi_e = 0.1085$, $\rho_x = \rho_{mm}^3 = 0.987^3$, $\psi = 2$, $\kappa_c = 0.9649$, and the RRA parameter to 10. For the EHB model, we follow Campbell and Cochrane (1999), and we set $\phi_{QQ} = \phi_{mm}^3 = 0.89^3$, $\gamma = 2$, $\delta = 0.998$, and the RRA parameter to 2. The $\sigma(\Delta c_t)$ is adjusted to the sample standard deviation of log consumption growth, following Ghosh et al. (2017).

3.2 Subjective Beliefs of PE Investor

Having constructed SDFs, we can identify subjective beliefs consistent with the observed excess cash flows, thereby eliminating the pricing error resulting from misspecification. To accomplish this, we ascertain an alternative probability distribution using Equation (10), ensuring that $\sum_{t=1}^{T} p_t M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0$, where p_t denotes model-implied probabilities. Subsequently, we apply these probabilities to excess cash flows to obtain investors' subjective beliefs about these cash flows. In this sub-section, our focus is centered on the beliefs inferred from the K-N CAPM SDF, aiming to unravel the primary economic information they contain. We later demonstrate that this analysis extends to other SDFs as well.

The results illustrating model-implied probabilities and belief-adjusted components of excess cash flows are depicted in Figure 3. The figure's upper panel showcases the profile of model-implied probabilities for the K-N CAPM SDF. These probabilities exhibit heightened sensitivity during recessionary periods, particularly in the aftermath of the dot-com bubble and the global financial crisis – periods defined by shaded grey regions. The spikes observed in 2003 Q2 and 2009 Q3 indicate that investors slowly re-adjust their expectations.

The figure's dashed line corresponds to the empirical (or historical, represented by $1/T$) probabilities associated with the occurrence of excess cash flow. Deviations of implied probabilities from this line suggest discrepancies between investor expectations and market trends. The ET methodology strives to minimize these discrepancies, aligning them with the zero excess moment condition stipulated in Equation 8. For instance, in the aftermath of pronounced recessions, the inferred probabilities diverge significantly from those implied by the RE hypothesis. This picture underscores the crucial role of crises on the divergence of investors' beliefs from market dynamics when asset pricing models based on the RE hypothesis fail to predict prices.

In the middle and bottom panels of Figure 3, we examine the belief-adjusted components of excess cash flow – the PE cash flow and the public market cash flow. We aim to identify the underlying reasons for the positive abnormal performance of PE funds. The middle panel displays the difference between the belief-adjusted and observed aggregate PE cash flows, termed as the *beliefs discrepancy*. Remarkably, for around 55% of the observed period, the belief discrepancy was negative. This suggests that investors often underestimate the performance of PE funds, especially during crises. However, at the onset of the global financial crisis, investors considerably underestimated its effect on PE cash flows. This optimism faded by 2009 Q3 when the discrepancy between expected and actual PE cash flow reached about \$17 million.

The bottom panel of Figure 3 presents an analysis of the belief-adjusted public market compo-

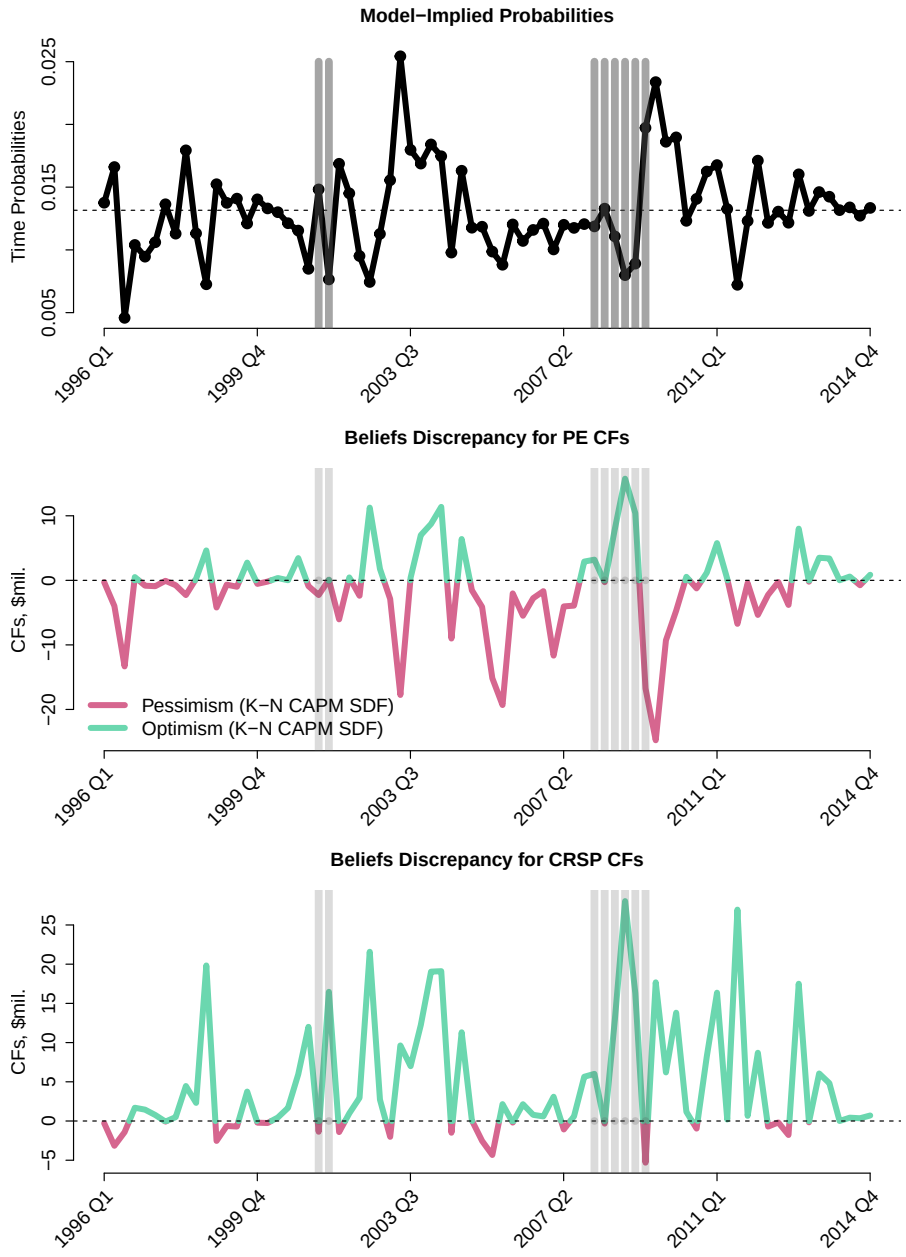


Figure 3: Model-Implied Probabilities and Subjective Beliefs. The figure is structured into three panels, capturing different dimensions of model-implied probabilities and belief-adjusted cash flows over the period 1996 Q1 to 2014 Q4. On the upper panel, the figure plots the model-implied probabilities estimated under K-N CAPM SDF. On the middle and bottom panel, the figure captures the *Beliefs Discrepancy*, which is the difference between belief-adjusted and observed aggregate cash flows, presented in $ million. The middle panel delves into the PE fund cash flows, while the bottom panel delves into the mimicking cash flow of the CRSP benchmark fund.

nent. The findings contrast sharply with the PE cash flow results. Around 69% of the observed period showed a positive beliefs discrepancy, which was also greater in magnitude. Investors typi-

cally overestimate the public market cash flow, even in crisis times. A notable discrepancy was in 2009 Q2 when they expected the CRSP cash flow to be $4 million less than the actual. However, in the crisis apex in 2008 Q4, PE investors held a strong belief in the public market's cash flow, resulting in a $21 million overestimation of public market cash flow.

The results of our analysis indicate that investors are pessimistic about PE cash flows and, concurrently, overly optimistic about the public market. Such beliefs lead to excessive compensation for the risk of holding PE fund stakes, resulting in the observed positive abnormal performance of PE funds. We do not take a stance on the potential sources of observed biases in expectations. PE investors might still be entirely rational, use all available information, and update their expectations using the Bayes rule. The bias may arise from the opacity of the asset class. An alternative explanation might be behavioral. Overconfidence about current portfolio holdings is a well-documented phenomenon in behavioral finance, as noted by Glaser and Weber (2007). This type of behavioral bias is not confined to unsophisticated investors; even experts, as shown by Elan and Goodrich (2010), can overestimate their ability to forecast the future of their investments compared to the average person.

3.3 Subjective Beliefs for Various SDFs

In this sub-section, we further explore the economic implications of subjective beliefs, as discussed in the previous sub-section, to a broader set of SDFs. Figures 4 and 5 display the *belief discrepancies* for the aggregate PE and market cash flows across various models. Notably, patterns of investor expectations for components of excess cash flows appear consistent when compared to beliefs estimated under the K-N CAPM SDF.

PE Cash Flows. Figure 4 illustrates the evolution of belief discrepancy over time with prevailing pessimism of PE investors across the majority of the observed periods. For the C-CAPM, LRR, and EHB models, skepticism regarding the cash flow-generating abilities of PE funds is strong at the start of the sample period. Pre-dot-com bubble era resulted in a positive belief discrepancy across all models. Especially, EHB and LRR investors demonstrated notable optimism during the late 1990s when the dot-com bubble was inflating. The peak was observed in 1999 Q3, several quarters before the bubble burst, leading to the subsequent recession in 2001 Q2-Q3. Remarkably, the C-CAPM model demonstrates high accuracy in pricing the PE cash flow time series, particularly for the post-2000 era, outperforming even the more sophisticated LRR and EHB models.

These findings underscore the varying perceptions across different models regarding distinct recession periods. According to the findings of Ning et al. (2015), the impact of the 2008 financial

crisis on the venture industry was less profound than that of the 2000 tech crash, potentially because venture firms were still rebounding from the steep decline post the 2000 peak. The subjective beliefs estimated for the EHB, LRR, and C-CAPM investors validate these findings for the PE sector.

In sum, the insights derived from the analysis of PE cash flows reveal a persistent pattern of pessimistic beliefs among investors, regardless of their underlying risk preferences. Episodes of pronounced optimism frequently antecede intervals of marked pessimism, while catastrophic events undeniably influence the shaping of PE investors' expectations. This analysis underscores the complexity inherent in the investors' expectation formation process.

Market Cash Flows. Figure 5 depicts a prevailing trend of overoptimism preceding the dot-com bubble across all models. The most pronounced surges of optimism, corresponding with the most significant belief discrepancies, are prominent for the EHB, LRR, and C-CAPM models around 1998 Q3.

In summary, PE investors exhibit greater optimism regarding public market cash flows and are more pessimistic about PE cash flows. This dichotomy culminates in a positive excess cash flow, which in turn translates into positive performance benchmarking metrics, such as GMPE. Consequently, we show that the primary source of positive abnormal performance for PE funds — unexplained by conventional asset pricing models — lies in the discrepancy between investors' beliefs about excess cash flows and those suggested by the RE hypothesis. The economic significance of the implications stemming from investor subjective beliefs is large. The statistical significance of these beliefs will be formally tested in the next sub-section, employing a bootstrap method.

3.4 Counterfactual Evidence

In this sub-section, we analyse the counterfactual cash flows implied by alternative probability distributions. A similar exercise was conducted by Julliard and Ghosh (2012) to evaluate the empirical validity of the rare disasters hypothesis in solving the equity premium puzzle. They used an alternative probability distribution to construct the counterfactual distribution of rare disasters (in terms of size and frequency). Similarly, we use alternative probability distributions to construct the counterfactual average risk-adjusted excess cash flow and its components based on the estimated beliefs. The results of the bootstrap analysis are presented in Table 4 in three distinct panels.

Panel A in Table 4 shows the results of the counterfactual analysis for excess cash flows. Here, the historical mean of discounted excess cash flows is juxtaposed against the bootstrapped mean.

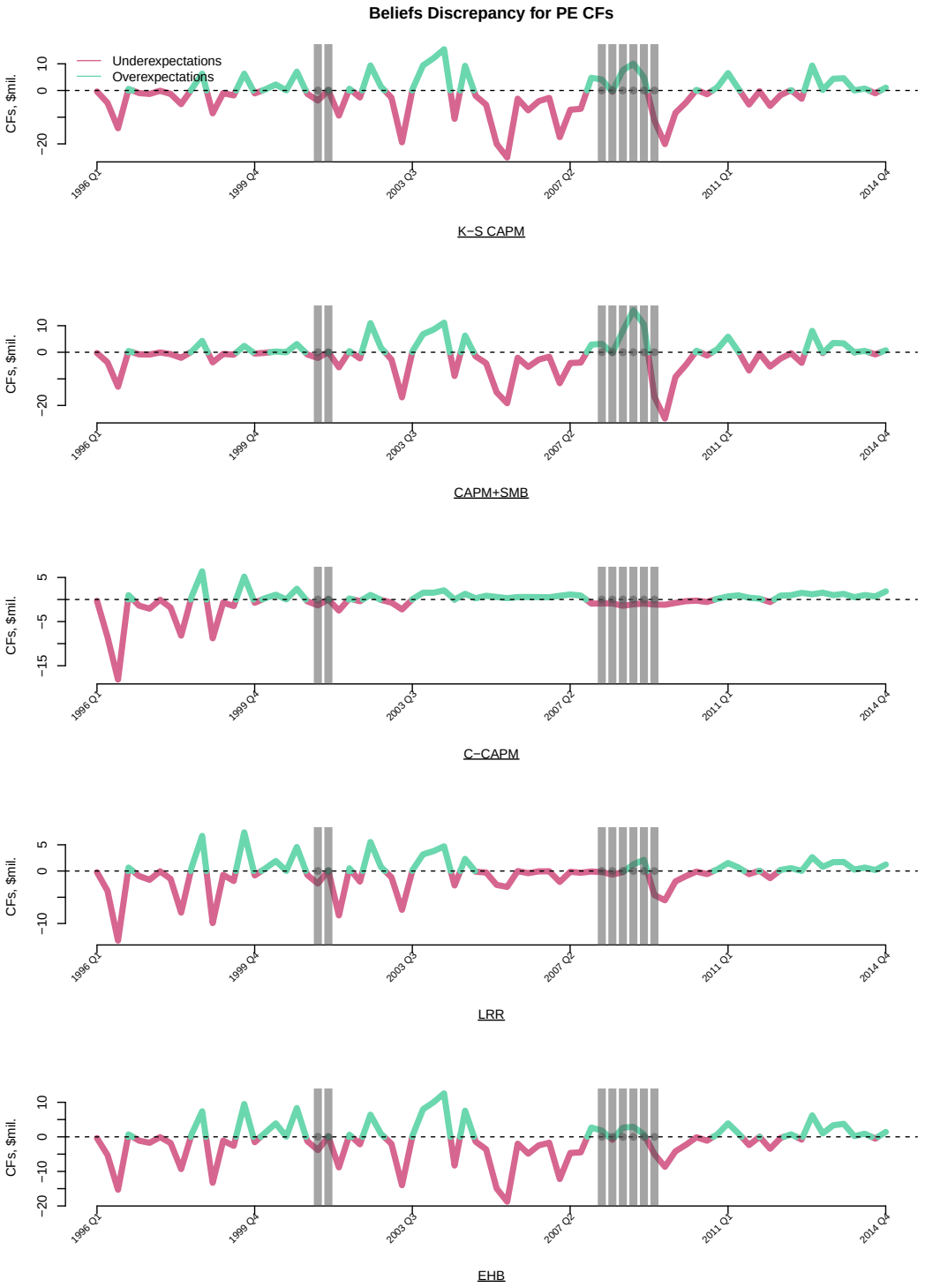


Figure 4: Aggregate PE Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate cash flows for PE funds from 1996 Q1 to 2014 Q4. Each plot title reflects the SDF under which beliefs are estimated.

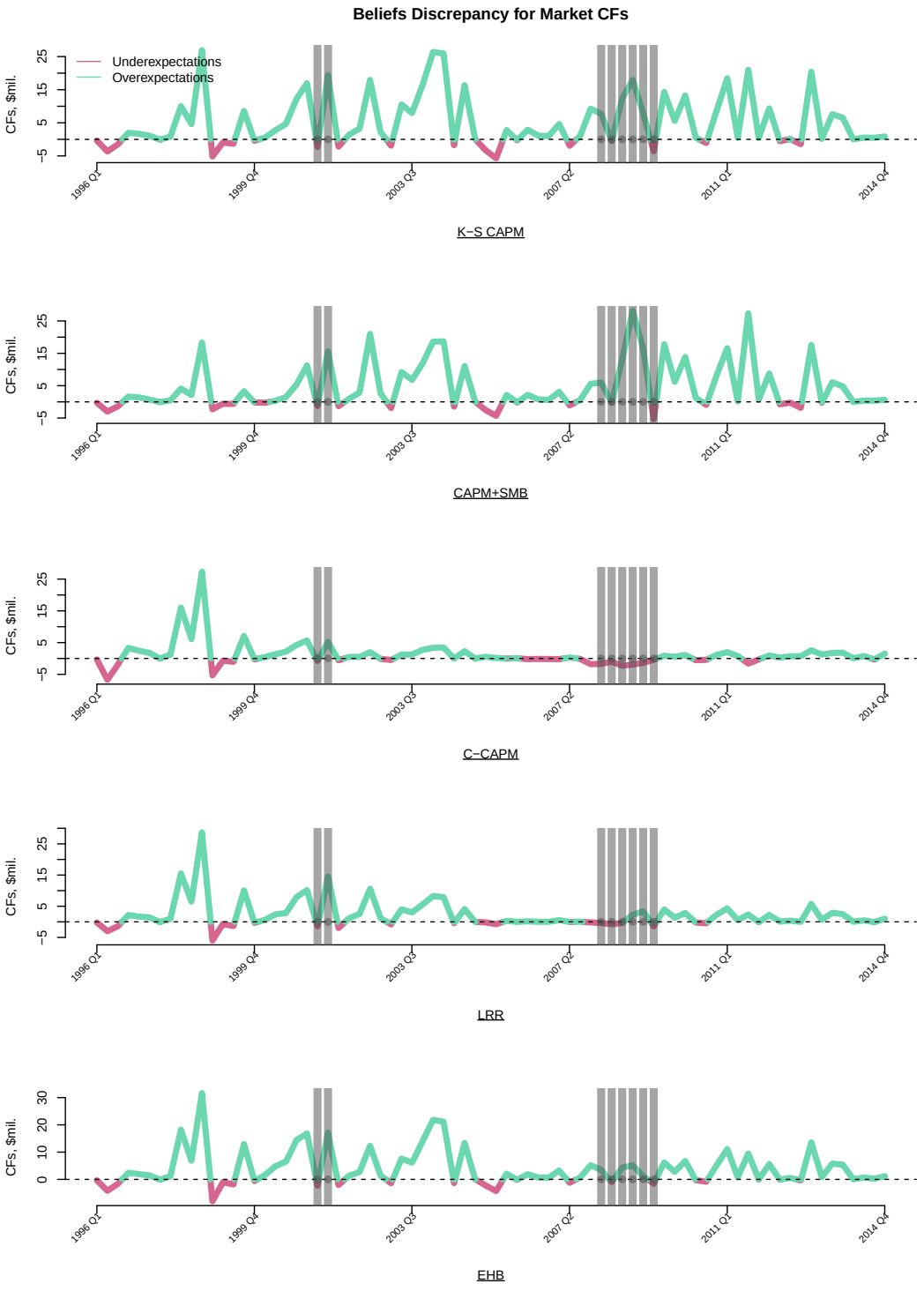


Figure 5: Aggregate Market Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate mimicking cash flows for CRSP-benchmark funds from 1996 Q1 to 2014 Q4. Each plot title reflects the SDF under which beliefs are estimated.

Table 4: Counterfactual Excess, PE, and Market Cash Flows. This table presents the findings from a counterfactual analysis using 100,000 simulations for cash flows from 1996 Q1 to 2014 Q4. Each column reflects the SDF under which beliefs are estimated. Panel A, B, and C each display results for Excess Cash Flows, PE Cash Flows, and Market Cash Flows, respectively. Statistics in each panel include the historical mean, bootstrap mean, 95% confidence interval for the bootstrap mean, and the probability of the counterfactual mean surpassing the historical mean. Values presented are in percentages to the Mean Average Payout (MAP).

	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	LRR	EHB
			Panel A: Excess Cash Flows (in % of MAP)			
$\overline{CF_T^{exc}}$	6.25	8.36	6.22	2.55	5.02	7.41
$\overline{CF_{boot}^{exc}}$	-0.01	0	0	0	-0.01	-0.01
$[\overline{CF_{boot}^{exc}}_{2.5\%}; \overline{CF_{boot}^{exc}}_{97.5\%}]$	[-5.01;4.98]	[-5.73;5.74]	[-5.14;5.14]	[-1.87;1.87]	[-4.62;4.60]	[-5.1;5.07]
$Pr(CF_{boot}^{exc} > \overline{CF_T^{exc}})$	0.73%	0.24%	0.92%	0.59%	1.67%	0.22%
			Panel B: PE Cash Flows (in % of MAP)			
$\overline{CF_T^{PE}}$	-1.78	-1.12	-1.61	-1.52	-4.72	-2.76
$\overline{CF_{boot}^{PE}}$	-3.55	-3.76	-3.34	-3.42	-6.64	-5.31
$[\overline{CF_{boot}^{PE}}_{2.5\%}; \overline{CF_{boot}^{PE}}_{97.5\%}]$	[-8.86;1.76]	[-9.88;2.37]	[-8.76;2.09]	[-6.24;-0.61]	[-11.36;-1.92]	[-11.01;0.40]
$Pr(CF_{boot}^{PE} > \overline{CF_T^{PE}})$	25.84%	19.98%	26.87%	7.87%	21.33%	19.10%
			Panel C: Market Cash Flows (in % of MAP)			
$\overline{CF_T^{mrkt}}$	-8.03	-9.48	-7.83	-4.07	-9.73	-10.17
$\overline{CF_{boot}^{mrkt}}$	-3.54	-3.76	-3.34	-3.43	-6.63	-5.3
$[\overline{CF_{boot}^{mrkt}}_{2.5\%}; \overline{CF_{boot}^{mrkt}}_{97.5\%}]$	[-9.77;2.69]	[-10.95;3.43]	[-9.74;3.06]	[-6.01;-0.85]	[-11.85;-1.41]	[-11.73;1.14]
$Pr(CF_{boot}^{mrkt} > \overline{CF_T^{mrkt}})$	91.86%	93.89%	91.34%	70.05%	87.74%	92.83%

For all subjective beliefs, the bootstrapped mean of excess cash flows is virtually zero. Meanwhile, the SDF valuation under the RE hypothesis (first row, Panel A, Table 4) results in economically significant non-zero means for excess cash flows. The historical mean of risk-adjusted excess cash flows ranges from 2.55% for the C-CAPM SDF to 7.41% for the EHB SDF. The significant difference between the historical and bootstrapped average highlights a conflict between beliefs implied by the RE hypothesis and the subjective beliefs estimated from PE cash flows.

In the third row of Panel A, the results of the counterfactual exercise are used to test the validity of using the RE hypothesis in PE valuation. The 95% confidence interval for the counterfactual excess cash flows across all models does not include the historically observed mean. This suggests that the RE hypothesis should be rejected. The discrepancy between subjective beliefs and those implied by the RE hypothesis is, therefore, the primary source of pricing error for PE funds. The SDF misspecification cannot solely offer a plausible explanation for the observed outperformance

of PE funds. The valuation of PE funds based on subjective beliefs, as captured in the simulations, outputs the zero risk-adjusted excess cash flow that resolves the emerging valuation puzzle.

The fourth row of Panel A reveals that investors tend to value PE funds, assuming a very low probability of large excess cash flows. On average, the likelihood of observing a counterfactual excess cash flow that is at least as large as the historical one is less than 1.7% across all models. In the following panels, we will delve deeper into the implications of these findings for the components of the excess cash flow.

Panels B and C of Table 4 present the counterfactual distributional properties of the components of the excess cash flow. The third row of Panel B indicates that the historical PE cash flow is within the 95% confidence interval. Similarly, the average market cash flow that was observed empirically is also plausible across all models. Thus, the counterfactual experiment reveals that both components are plausible separately, but the likelihood of their combination leads to the unrealistically high historical excess cash flow.

The results from the fourth row of each panel in Table 4 provide insight into why the RE hypothesis is rejected for excess cash flows. With a probability greater than 70%, investors expect market cash flows to be higher than what was actually observed. On the other hand, with a probability not higher than 22%, investors expect PE cash flows to be higher than observed. As shown in Figures 4 and 5, investors tend to be more optimistic about market cash flows compared to PE cash flows. Thus, the bootstrap exercise supports the idea that the PE valuation puzzle is due to a combination of pessimism regarding PE cash flows and optimism regarding stock market cash flows.

In summary, the results of the bootstrap exercise show that subjective beliefs play a crucial role in explaining the abnormal excess cash flows of PE funds. The RE hypothesis can be rejected with substantial statistical confidence, indicating that investors tend to overestimate public market performance while underestimating PE funds' performance. Such discrepancy in beliefs leads investors to demand additional compensation for the uncertainty associated with future cash flows, resulting in the observed undervaluation of funds. These findings are consistent across all mainstream asset pricing models.

However, there are several unaddressed concerns. The first concern is how to ensure that estimated beliefs represent the investors' sentiment. The following section addresses this issue. The second concern is investors' (LPs) ability to translate their beliefs into a price discount. The potential heterogeneity in fee schemes and its influence on the economic implications of estimated beliefs is elaborated upon in Appendix B. The third concern is about alternative approaches for beliefs estimation. Appendix D provides insights on this, with Appendix D.1 examining the economic

implications of beliefs estimated using a two-step approach when we first estimate the SDF parameters following Korteweg and Nagel (2016) and only then estimate subjective beliefs. Appendix D.2 investigates the feasibility of estimating subjective beliefs through the moment condition for aggregate cash flows despite their suboptimal stationarity properties compared to average cash flows. Lastly, Appendix D.3 contemplates the prospect of estimating investors' subjective beliefs on a fund-by-fund basis, thereby harnessing the available cross-sectional heterogeneity.

4 Analysis of Beliefs

In this section, we will address the question of whether extracted subjective beliefs accurately reflect the opinions of investors. To do this, we will compare estimated beliefs with the results of publicly available surveys, as described in sub-section 4.2. In sub-section 4.1, we will provide an overview of the survey data. Additionally, we will address potential concerns regarding the sensitivity of estimated beliefs to misspecification of the SDF, as well as the adequacy of PE sentiment indices in reflecting opinions of investors, in sub-sections 4.3 and 4.4, respectively.

4.1 Survey Data

We make use of survey data for three types of investors: individual, institutional, and Private Equity investors. The opinions of individual investors have received significant attention in the paper by Greenwood and Shleifer (2014). We utilize two surveys from their paper: 1) the monthly Gallup survey (*Gallup*) conducted between 1996 and 2012 with occasional gaps and 2) Robert Shiller's survey of wealthy individual investors (*Shiller (Ind)*) available sporadically between 1999 and July 2001 and conducted monthly afterwards. The opinions of institutional investors are represented by Robert Shiller's institutional investor survey (*Shiller (Inst)*) conducted at six-month intervals from July 1989 to July 2001 and then monthly afterwards.

The Silicon Valley Venture Capital Confidence Index (*SVVCCI*) and Central Europe Private Equity Confidence Index (*CEPECI*) capture the beliefs of PE investors. As there is limited data specifically for investors focused on buyout (BO) strategies, these indices serve as a proxy for the opinions of PE investors in general. Figure 6 provides a visual representation of the time series for the PE indices and highlights that aggregated sentiments are similar. Table 5 shows a high correlation of 0.82 between the moods of PE and VC investors in the San Francisco area and Central Europe, suggesting that the sentiment of PE investors is closely tied to that of VC investors.

Table 5: Sentiment Indices Correlation. The table reports the correlations between Institutional (*Shiller (inst)*), Individual (*Shiller (ind), Gallup, ESI*) and PE (*CEPECI, SVVCCI*) sentiment indices.

	Shiller (inst)	Shiller (ind)	Gallup	ESI	CEPECI	SVVCCI
Shiller (inst)	—	0.51***	-0.52***	-0.55***	0.09	-0.05
Shiller (ind)	0.51***	—	-0.10	-0.17	0.57***	0.43***
Gallup	-0.52***	-0.10	—	0.78***	0.90***	0.83***
ESI	-0.55***	-0.17	0.78***	—	0.86***	0.76***
CEPECI	0.09	0.57***	0.90***	0.86***	—	0.82***
SVVCCI	-0.05	0.43***	0.83***	0.76***	0.82***	—

Note: *p<0.1; **p<0.05; ***p<0.01

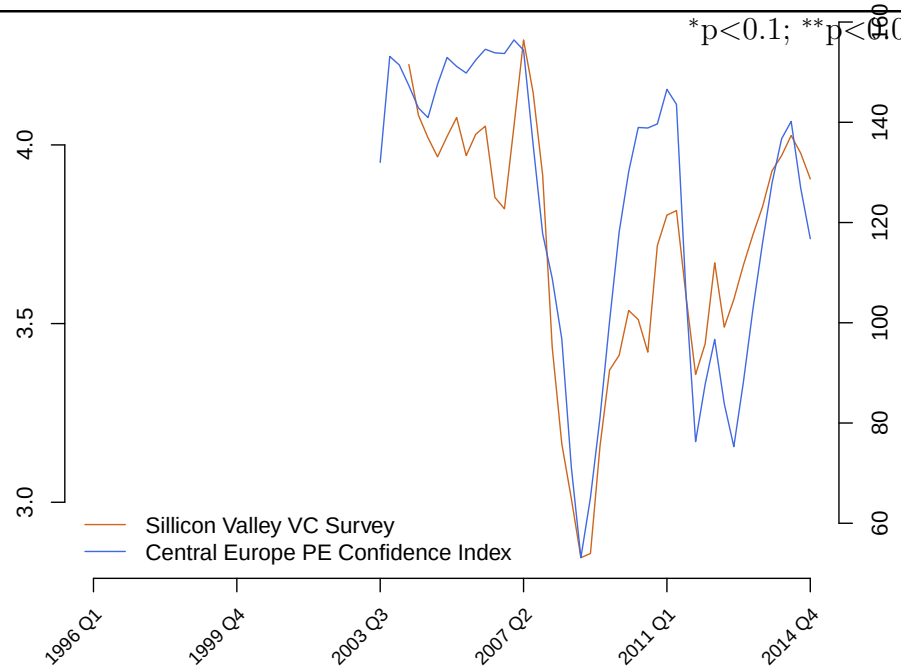


Figure 6: PE Investors Sentiments. The figure plots investors' expectations about the private equity market in Central Europe (blue line) and the San Francisco Bay area (brownish line). For Europe, the confidence index is represented by a survey of PE professionals conducted by Deloitte, while investors' moods of Silicon Valley VC capitalists are represented by a survey conducted by the University of San Francisco.

The SVVCCI is based on a recurring quarterly survey of San Francisco Bay Area venture capitalists, which has been conducted since Q1 2004. This index measures and reports the opinions of professional venture capitalists regarding their estimates of the high-growth entrepreneurial environment for the next 6 to 18 months. For more information about the quarterly survey, please see Appendix 2 of Cannice and Goldberg (2009).

The CEPECI is a semi-annual survey conducted by Deloitte of private equity professionals focused on Central Europe, see Deloitte's website. The index is based on the responses to the first

seven questions of the survey, which focus on topics such as economic climate, debt availability, investors' focus, size of transactions, market activity, investment return, and investors' activities. The positive answer ratio is calculated for each period by dividing the number of positive answers by the sum of positive and negative answers. This ratio is then compared to the base period of spring 2003 to obtain the final index value.

Further, we investigate how market expectations are related across different types of investors. Figure 7 depicts the evolution of three indices: Shiller Individual, Shiller Institutional, and CEPECI. The sentiment about PE and the public market is often misaligned. Even public institutional and individual investors do not always agree on the movement of the aggregate stock market, as shown by the correlation in the first column of Table 5. This finding is consistent with previous literature, as shown in the analysis of beliefs of heterogeneous public market investors by Ghosh et al. (2020).

Additionally, it is important to note that there can be disagreements within the same type of investor group. For example, the correlation between the Gallup survey and Shiller Individual survey is low and negative (-0.1), but it is not statistically significant. This discrepancy could be due to the fact that the two groups of investors are not as homogeneous as they may seem. As noted by Adam and Nagel (2022), the Shiller index surveys wealthy individuals, while the Gallup index surveys the general population.

4.2 Subjective Beliefs and Investors' Sentiment

The central question of this sub-section is whether beliefs extracted from models align with survey data. We must first understand how expectations influence excess cash flows to answer this question. One important point to consider is that investors are asked in questionnaires to forecast cash flows over some distant future. To reflect this aspect of sentiment indices, we include them in Table 6 with a one-year lag. Also, it is crucial to understand that subjective beliefs estimated using my approach are related to the excess market cash flow. At the same time, investors usually make predictions about other quantities. Therefore, survey data may affect excess cash flows indirectly through PE and stock market components.

First, we inspect the PE survey data, represented by CEPECI and SVVCCI indices. For example, San Francisco VC investors forecast the 'future high-growth venture entrepreneurial environment' on a scale from one to five. We interpret the level of optimism (pessimism) as their subjective probability of a positive (negative) PE cash flow, which in turn would result in a positive (negative) excess cash flow. As a result, it is expected that PE survey indices would positively correlate with the belief-adjusted average excess and PE cash flows with a one-year lag.

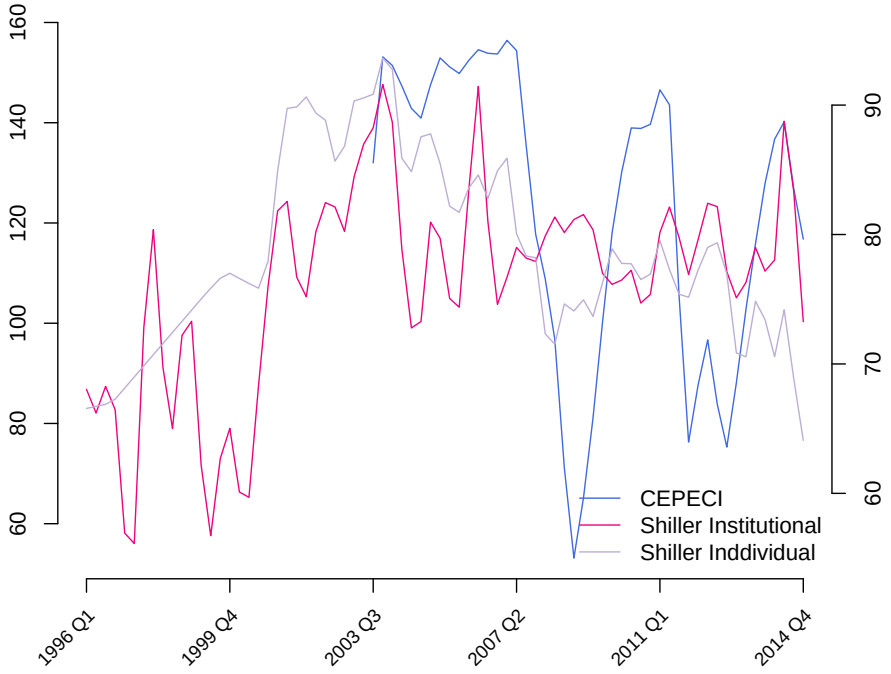


Figure 7: Sentiments for PE, Institutional, and Individual Investors. The figure plots investors' sentiments towards the stock market. It contrasts Shiller's Institutional Investors (pink line, right scale) and Individual Investors (violet line, left scale). Simultaneously, sentiments towards the private equity market in Central Europe are visualized by a blue line (right scale).

Table 6 shows that the survey data aligns well with subjective beliefs, as the correlation is high and statistically significant. These results suggest that the opinions of PE investors play a significant role in predicting the excess cash flows in the PE market.

Further, we inspect the individual and institutional sentiment indices. The results show a spuriously puzzling finding: individual investors correctly predict the belief-adjusted cash flows, while institutional investors do not. The correlation between the belief-adjusted cash flows and the institutional sentiment index, represented by *Shiller (inst)*, is negative and mostly insignificant. This suggests that institutional sentiments still may impact PE excess cash flows, but to a lesser extent than those of PE and individual investors.

The explanation for this finding is provided by Greenwood and Shleifer (2014), who discovered that individual investors' expectations in survey data are inconsistent with the rational expectations model. They find that individual investors' positive expectations generally lead to lower average stock returns. Thus, the optimism of individual investors leads to lower public market

cash flows (misprediction), which in turn results in higher excess cash flows. This is why a positive correlation is observed. On the other hand, Ghosh et al. (2020) find that institutional investors, who are countercyclical in their expectations, accurately predict the aggregate stock market. Their positive expectations result in higher public market cash flows, reducing excess cash flows. This explains the negative correlation sign.

Table 6: Correlations between Surveys and Subjective Cash Flows. The table reports the correlations between various sentiment indices, presented row-wise, and belief-adjusted average excess cash flows for PE funds. Each column reflects the SDF under which beliefs are estimated.

	Panel A: Average Excess Cash Flow adjusted for Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.23*	-0.18	-0.23**	-0.09	-0.10	-0.10
Shiller (ind)	-0.02	-0.02	-0.03	0.12	0.04	0.09
Gallup	0.42***	0.36***	0.42***	0.28**	0.28**	0.29**
ESI	0.36***	0.32***	0.37***	0.27**	0.25**	0.26**
CEPECI	0.59***	0.57***	0.59***	0.58***	0.57***	0.58***
SVVCCI	0.72***	0.70***	0.72***	0.66***	0.69***	0.68***
	Panel B: Placebo Test for Contemporaneous Correlations					
Shiller (inst)	-0.16	-0.14	-0.16	-0.02	-0.08	-0.06
Shiller (ind)	-0.15	-0.17	-0.15	0.00	-0.11	-0.07
Gallup	-0.05	-0.11	-0.05	-0.19	-0.17	-0.17
ESI	0.30***	0.23**	0.30***	0.14	0.14	0.17
CEPECI	0.11	0.06	0.11	0.12	0.05	0.11
SVVCCI	0.26*	0.21	0.26*	0.24	0.20	0.24
Note:						*p<0.1; **p<0.05; ***p<0.01

In summary, we show that estimated beliefs are in line with survey data. The opinions of PE investors have the strongest impact on predicting excess cash flows from investments in PE funds. The beliefs of individual and institutional investors affect excess cash flows through their impact on the public market component.

4.3 Beliefs Adjusted for SDF Misspecification

If risk preferences are not adequately reflected in SDF, it may result in significant pricing errors. The errors associated with model misspecification might distort beliefs. In this sub-section, we test the potential impact of misspecification on estimated beliefs and investigate the potential sources of risks not captured by SDFs used in our analysis.

Public Market Conditions. Mainly, we investigate market fundamentals as a potential source of omitted factors. In Table 7, we test for expected market premia (*expected premia*), Moody's BBB to AAA credit spread (*credit spread*), the Chicago Board Options Exchange S&P 100 Volatility Index (*vxo*), and the aggregate liquidity factor of Pástor and Stambaugh (2003) (*liquidity*). This exercise aims to 'purify' the model-implied probabilities estimated in the previous section by adjusting for the potential SDF misspecification. The residuals from all four regressions will be used to construct the 'purified' beliefs. Furthermore, Table 7 provides insights into the most plausible public market factors that might have been missed in the SDFs considered in this paper.

Table 7: How Model-Implied Probabilities Align with Market Fundamentals. The table reports the regression analysis of model-implied probabilities against market fundamentals. The market fundamentals are presented by expected market premia, credit spread, volatility index, aggregate liquidity factor, or their contemporaneous changes (Δ_t).

	Model-Implied Probabilities under:					
	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	EHB	LRR
	(1)	(2)	(3)	(4)	(5)	(6)
expected premia	0.0002	0.0003**	0.0002	−0.0001	0.0002	0.00002
	(0.0001)	(0.0002)	(0.0001)	(0.0001)	(0.0002)	(0.0001)
Δ_t(credit spread)	−0.006	−0.001	−0.006	0.007	0.005	0.006
	(0.004)	(0.005)	(0.004)	(0.004)	(0.005)	(0.004)
Δ_t(vxo)	−0.013***	−0.015***	−0.013***	−0.007*	−0.014***	−0.009**
	(0.004)	(0.004)	(0.004)	(0.004)	(0.005)	(0.004)
Δ_t(liq.agg) 0.008	0.012	0.008	0.003	0.011	0.009	
	(0.008)	(0.009)	(0.008)	(0.008)	(0.010)	(0.008)
Constant	0.012***	0.011***	0.012***	0.013***	0.012***	0.013***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
Observations	75	75	75	75	75	75
R^2	0.367	0.313	0.373	0.070	0.189	0.142
Adjusted R^2	0.331	0.273	0.337	0.017	0.142	0.093
F Statistic	10.135***	7.956***	10.395***	1.324	4.070***	2.886**
Note:						*p<0.1; **p<0.05; ***p<0.01

The results from Table 7 indicate that changes in market volatility have a statistically significant effect on all model-implied probabilities. This is in line with the research by Gompers et al. (2008), who find that VC firms increase their investments during periods of high volatility in the public market, with no effect on their performance. Limited Partners appear to respond similarly by allocating more capital into private equity, as they seek to escape the volatility of the public stock

market.[7] As market volatility increases, it signals a contraction in public market cash flows, which leads to an increase in excess cash flows, all else being equal. As a result, changes in volatility are expected to positively correlate with average excess cash flows and negatively with model-implied probabilities.[8]

The other factors do not exert any statistically significant effect on estimated beliefs. However, their potential importance was highlighted in the literature, along with potential transmission mechanisms. For example, Schmid et al. (2019) explore the potential to value PE funds using information about their loan portfolio. They develop the credit market equivalent metric and show the significance of credit market movements on PE valuation. An increase in the credit spread signifies a decrease in public market cash flow and, therefore, a corresponding increase in the excess cash flow.

The research by Haddad et al. (2017) show that changes in the aggregate risk premium can impact the PE industry. They find that a low risk premium increases the present value of performance gains and reduces the cost of holding an illiquid investment. As the expected market premia increases, it signifies a higher public market cash flow and, as a result, a lower excess cash flow.

Another factor is the aggregate market liquidity, as suggested by Pástor and Stambaugh (2003). Despite its potential importance in PE, see, for example, Franzoni et al. (2012), the regression results in Table 7 indicate that the aggregate market liquidity does not have a significant impact on model-implied probabilities. It is possible that the effect of aggregate market liquidity is overshadowed by other public market conditions considered here.

In conclusion, it appears that changes in market volatility may be the crucial factor omitted in existing asset pricing models applied to PE cash flows. The question remains: how much of an impact does this potential misspecification have on subjective beliefs? To address this question, we use the residuals from the presented regressions as ('purified') orthogonalized model-implied probabilities to construct belief-adjusted cash flows and repeat the validation process as outlined in sub-section 4.2.

Orthogonalized Beliefs. To further examine the reliability of the results, we assess the robustness of the correlations observed in Table 6 when using 'purified' beliefs. To do this, we extract orthogonalized probabilities in the form of residuals from the regressions in Table 7. They are or-

[7] June 1, 2022, Altexchange.com; May 26, 2022,Wall Street Journal

[8] The distinction between subjective beliefs and model-implied probabilities lies in their calculation. Model-implied probabilities are the alternative probability measure ($\mathbb{P}^*$), which eliminates the pricing errors, while subjective beliefs are obtained by applying a new probability measure to cash flows.

Table 8: Correlations Between Survey Data and Orthogonalized Beliefs. The table reports the correlations between distinct sentiment indices and excess cash flow that has been adjusted for beliefs. These subjective beliefs, estimated using different models, are further refined to be orthogonal to market fundamentals, as detailed in Table 7.

	Average Excess Cash Flow adjusted for Purified Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.22*	-0.18	-0.22*	-0.10	-0.10	-0.10
Shiller (ind)	-0.02	-0.01	-0.02	0.11	0.05	0.08
Gallup	0.40***	0.36***	0.41***	0.29**	0.29**	0.30**
ESI	0.37***	0.33***	0.37***	0.29**	0.27**	0.28**
CEPECI	0.60***	0.58***	0.60***	0.58***	0.58***	0.58***
SVVCCI	0.71***	0.69***	0.71***	0.66***	0.67***	0.66***

thogonalized to the market conditions potentially omitted in the risk preferences. Then, we scale the time series of the residuals to ensure that they satisfy the probability conditions of summing to one and being non-negative.

As shown in Figure 8, the 'purified' probabilities strongly correlate with the original model-implied probabilities, indicating that subjective beliefs extracted using my methodology capture investors' opinions rather than misspecification. The correlation for the probabilities estimated under different pricing kernels is as follows: C-CAPM – 0.96; LRR – 0.93; EHB – 0.9. A higher correlation is observed for beliefs estimated under the SDFs that produce a lower pricing error for excess cash flows, which suggests that a more calibrated SDF outputs more informative beliefs.

We repeat the correlation analysis for subjective beliefs obtained using orthogonalized model-implied probabilities and survey indices. The results of Table 8 are similar to those found previously. The correlation between subjective beliefs and PE survey indices remains strong, with little to no change in statistical significance. The correlation between subjective beliefs and both individual and institutional investors also remains unchanged. Therefore, we conclude that my methodology robustly captures investors' subjective beliefs, even in the presence of potential misspecification of SDFs. The misspecification does not significantly impact model-implied probabilities and associated subjective beliefs.

4.4 PE Market Conditions and Investors' Sentiment

Beyond the concerns associated with estimated beliefs, we also examine the relationship between sentiment indices and the PE market. Our main objective is to assess if there is a correlation between PE market conditions and investors' opinions and if this relationship has been supported

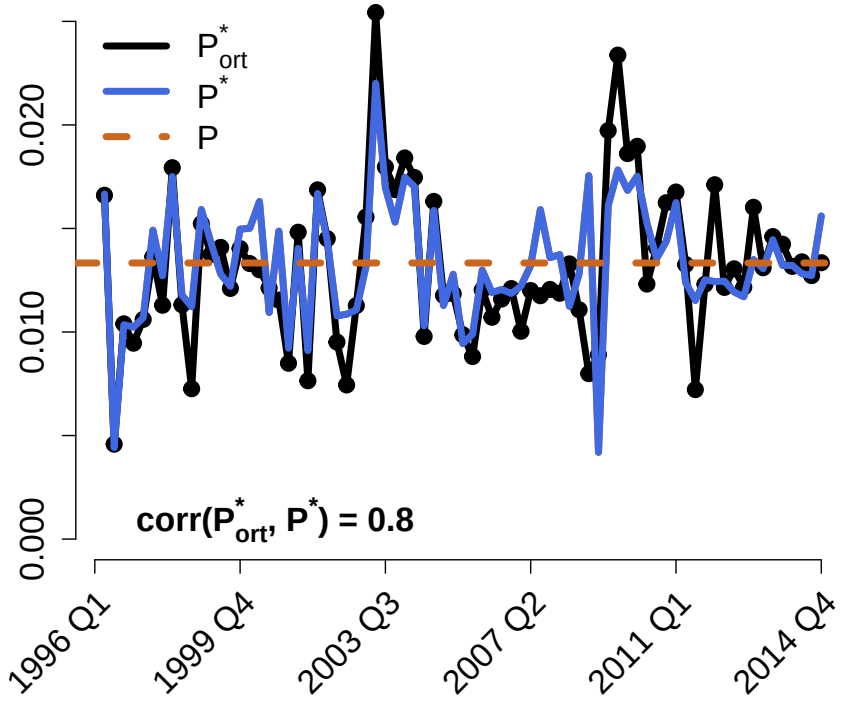


Figure 8: Original Versus Orthogonalized Model-Implied Probabilities. The figure plots the model-implied original probabilities and their orthogonalized counterparts. These orthogonalized probabilities are obtained post-projection to public market fundamentals as detailed in Table 7 for K-N CAPM SDF for 1996 Q1 – 2014 Q4.

by previous research. This examination sheds light on potential factors influencing the belief formation process. We take into account factors like entry and exit conditions and transactions in the PE secondary market. Table 9 presents the results of the correlation analysis between these factors and one-step-ahead subjective beliefs.

The correlation analysis for PE sentiment indices reveals that PE market entry conditions, represented by fundraising activities, exert a pronounced influence on investor expectations. High levels of fundraising correlate with pessimistic expectations among PE investors. This observation aligns with Brown et al. (2021) findings, which indicate that periods of increased fundraising precede periods marked by subpar performance. This implies that fundraising activities adversely affect investors' outlook on future cash flows.

An active secondary market tends to have a negative influence on PE investors' sentiment. High levels of transactions in the secondary market correspond with more pessimistic expectations for future cash flows. This sentiment might be rooted in the notion that observing a surge in investors

Table 9: PE Market Fundamentals and PE Sentiments. This table reports the correlations between PE market fundamentals and PE sentiments. Key PE market conditions encompass factors like the capital raised by PE funds (*fundraise*), the number of secondary transactions (*scnd. transactions*), the number of BO IPO deals (*#BO IPO deals*), and the mean value of exit deals (*avg. BO exit size*). Moreover, contemporaneous changes in these factors (Δ_t) are considered. IPO data is provided by Ritter's website.

	CEPECI	SVVCCI
fundraise	-0.55***	-0.34**
Δ_t(fundraise)	0.17	0.14
scnd. transactions	-0.35**	-0.14
Δ_t(scnd. transactions)	0.03	0.03
#BO IPO deals	0.17	0.37**
Δ_t(#BO IPO deals)	0.24	0.23
avg. BO exit size	-0.19	0.04
Δ_t(avg. BO exit size)	0.02	0.11

eager to exit the market dampens confidence in PE funds' future cash flow generation capabilities. This perspective gains credence from Nadauld et al. (2019), who demonstrate that buyers in the secondary market outperform sellers by offering liquidity to those eager to exit. Consequently, a spike in secondary market transactions could be perceived as indicative of a less favourable PE market environment, leading to a negative correlation with investor sentiment. Table 9 supports this perspective.

Traditionally, activity in the IPO market serves as a barometer of investor sentiment. Favorable exit conditions typically trigger positive investor expectations. As evidenced by Nanda et al. (2020), each additional IPO among a VC firm's first ten investments forecasts an elevated IPO rate—up to 8% higher—for subsequent investments. As illustrated in Table 9, there exists a positive correlation between the sentiment of VC investors, as captured by the SVVCCI index, and a dynamic IPO market. It is noteworthy that the volume of IPOs exerts a greater influence on shaping investor sentiment than the magnitude of exit transactions.

In summary, we show that the PE survey data, employed to validate estimated subjective beliefs, aligns sensibly with the prevailing conditions of the PE market. The analysis shows that robust fundraising activity and a bustling secondary market tend to dampen the optimism of PE investors. In contrast, a buoyant IPO market is linked with heightened investor optimism. These observations are consistent with prior research.

5 Conclusion

The valuation of PE funds is challenging. The existing asset pricing models fail to explain the PE valuation from the perspective of Limited Partners. In this paper, we argue that the conventional approach of discounting for time and risk using SDF is insufficient for explaining the observed abnormal performance of PE funds. The overlooked component in existing models, it appears, is the uncertainty tied to future cash flows. Investors in PE may demand additional compensation for this uncertainty beyond just the time and risk factors.

We find that investors tend to be pessimistic about the cash flows of PE funds but overly optimistic about public market cash flows. This results in general pessimism regarding excess cash flows, which is the sole source of abnormal performance when using benchmarking methods like GPME. To estimate investors' subjective beliefs from cash flow data, we introduce a novel methodology based on an empirical-likelihood-type estimator. This approach estimates investors' beliefs from the time series of excess cash flows unconditionally, requiring only that the underlying time series is stationary.

We eliminate any remaining pricing errors from the time series of excess cash flow after discounting using the 'off-the-shelf' SDF parameters. This allows us to extract an alternative probability distribution for excess cash flows. When paired with SDFs in use, this distribution can rationalize the observed PE fund valuations. Investors' subjective beliefs are reweighted excess cash flows using model-implied probabilities. Moreover, estimated subjective beliefs are the closest in the Kullback-Leibler sense to the historical excess cash flows, implied by the Rational Expectation (RE) hypothesis.

We further show that there is a significant discrepancy, both economically and statistically, between investors' subjective beliefs and the cash flow distribution under the RE hypothesis. To validate the practical significance of these estimated beliefs, we have conducted a correlation analysis using survey data from PE investors. We also consider sentiment indices from public market investors, as these could indirectly influence the observed excess cash flow through the public market component. We show that estimated beliefs have the largest statistical significance and economic magnitude for the core PE investors. Furthermore, the public market sentiment shows substantial alignment with the observed public market component of excess cash flow.

Lastly, we have executed a battery of robustness tests to determine how robust these beliefs are against potential model misspecification. We found that SDF misspecification can not critically distort estimated beliefs. Moreover, this methodology proves robust even when alternative moment conditions are considered. It is worth noting that our method could be extended to other asset

classes similar to PE, such as Venture Capital or Real Estate funds. It could also be used to discern investors' beliefs for various public market applications.

References

Adam, K., Matveev, D. and Nagel, S.: 2021, Do survey expectations of stock returns reflect risk adjustments?, *Journal of Monetary Economics* **117**, 723–740.

Adam, K. and Nagel, S.: 2022, Expectations data in asset pricing, *Technical report*, National Bureau of Economic Research.

Almeida, C., Ardison, K. and Freire, G.: 2023, Which (nonlinear) factor models? Available at SSRN 4421179.

Almeida, C., Ardison, K., Freire, G., Garcia, R. and Orłowski, P.: 2022, High-frequency tail risk premium and stock return predictability, *Journal of Financial and Quantitative Analysis* pp. 1–38.

Almeida, C., Ardison, K. and Garcia, R.: 2020, Nonparametric assessment of hedge fund performance, *Journal of Econometrics* **214**(2), 349–378.

Almeida, C. and Freire, G.: 2022, Pricing of index options in incomplete markets, *Journal of Financial Economics* **144**(1), 174–205.

Almeida, C., Freire, G., Azevedo, R. and Ardison, K.: 2023, Nonparametric option pricing with generalized entropic estimators, *Journal of Business & Economic Statistics* **41**(4), 1173–1187.

Almeida, C. and Garcia, R.: 2012, Assessing misspecified asset pricing models with empirical likelihood estimators, *Journal of Econometrics* **170**(2), 519–537.

Almeida, C. and Garcia, R.: 2017, Economic implications of nonlinear pricing kernels, *Management Science* **63**(10), 3361–3380.

Anatolyev, S. and Gospodinov, N.: 2011, *Methods for estimation and inference in modern econometrics*, CRC Press.

Ang, A., Chen, B., Goetzmann, W. N. and Phalippou, L.: 2018, Estimating private equity returns from limited partner cash flows, *The Journal of Finance* **73**(4), 1751–1783.

Bansal, R. and Yaron, A.: 2004, Risks for the long run: A potential resolution of asset pricing puzzles, *The Journal of Finance* **59**(4), 1481–1509.

Bordalo, P., Gennaioli, N., Porta, R. L. and Shleifer, A.: 2019, Diagnostic expectations and stock returns, *The Journal of Finance* **74**(6), 2839–2874.

Brown, G., Ghysels, E. and Gredil, O.: 2020, Nowcasting net asset values: The case of private equity, *The Review of Financial Studies* (20-02).

Brown, G., Gredil, O. R. and Kaplan, S. N.: 2019, Do private equity funds manipulate reported returns?, *Journal of Financial Economics* **132**(2), 267–297.

Brown, G., Harris, R., Hu, W., Jenkinson, T., Kaplan, S. N. and Robinson, D. T.: 2021, Can investors time their exposure to private equity?, *Journal of Financial Economics* **139**(2), 561–577.

Campbell, J. Y. and Cochrane, J. H.: 1999, By force of habit: A consumption-based explanation of aggregate stock market behavior, *Journal of Political Economy* **107**(2), 205–251.

Cannice, M. V. and Goldberg, C. S.: 2009, Venture capitalists' confidence, asymmetric information, and liquidity events, *Journal of Small Business & Entrepreneurship* **22**(2), 141–164.

Chen, X., Hansen, L. P. and Hansen, P. G.: 2020, Robust identification of investor beliefs, *Proceedings of the National Academy of Sciences* **117**(52), 33130–33140.

Chung, J.-W., Sensoy, B. A., Stern, L. and Weisbach, M. S.: 2012, Pay for performance from future fund flows: The case of private equity, *The Review of Financial Studies* **25**(11), 3259–3304.

Coibon, O. and Gorodnichenko, Y.: 2015, Information rigidity and the expectations formation process: A simple framework and new facts, *American Economic Review* **105**(3), 2644–2678.

Colacito, R. and Croce, M. M.: 2011, Risks for the long run and the real exchange rate, *Journal of Political Economy* **119**(1), 153–181.

Constantinides, G. M.: 1986, Capital market equilibrium with transaction costs, *Journal of Political Economy* **94**(4), 842–862.

Csiszár, I.: 1975, I-divergence geometry of probability distributions and minimization problems, *The Annals of Probability* pp. 146–158.

de Finetti, B.: 1937, La prévision: ses lois logiques, ses sources subjectives, *Annales de l'Institut Henri Poincaré* **7**(1), 1–68. Translated to English in "Studies in Subjective Probability" by Henry E. Kyrburgh and Howard E. Smokler, 1964.

Efron, B.: 1981, Nonparametric standard errors and confidence intervals, *canadian Journal of Statistics* **9**(2), 139–158.

Elan, S. L. and Goodrich, M.: 2010, Behavioral patterns and pitfalls of us investors, *Federal Research Division, Library of Congress*, p. 01.

Fama, E. F. and French, K. R.: 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* **33**(1), 3–56.

Franzoni, F., Nowak, E. and Phalippou, L.: 2012, Private equity performance and liquidity risk, *Journal of Finance* **67**(6), 2341–2373.

Ghosh, A., Julliard, C. and Taylor, A. P.: 2017, What is the consumption-capm missing? an information-theoretic framework for the analysis of asset pricing models, *The Review of Financial Studies* **30**(2), 442–504.

Ghosh, A., Korteweg, A. G. and Xu, Q.: 2020, Recovering heterogeneous beliefs and preferences from asset prices, *Available at SSRN 3650199* .

Ghosh, A. and Roussellet, G.: 2020, Identifying beliefs from asset prices, *Proceedings of Paris December 2019 Finance Meeting EUROFIDAI-ESSEC*.

Glaser, M. and Weber, M.: 2007, Overconfidence and trading volume, *The Geneva Risk and Insurance Review* **32**(1), 1–36.

Gompers, P., Kaplan, S. N. and Mukharlyamov, V.: 2016, What do private equity firms say they do?, *Journal of Financial Economics* **121**(4), 449–476.

Gompers, P., Kovner, A., Lerner, J. and Scharfstein, D.: 2008, Venture capital investment cycles: The impact of public markets, *Journal of Financial Economics* **87**(1), 1–23.

Gredil, O., Sorensen, M. and Waller, W.: 2019, Evaluating private equity performance using stochastic discount factors, *Available at SSRN 3506847* .

Greenwood, R. and Shleifer, A.: 2014, Expectations of returns and expected returns, *The Review of Financial Studies* **27**(3), 714–746.

Gupta, A. and Van Nieuwerburgh, S.: 2021, Valuing private equity investments strip by strip, *The Journal of Finance* **76**(6), 3255–3307.

Haddad, V., Loualiche, E. and Plosser, M.: 2017, Buyout activity: The impact of aggregate discount rates, *The Journal of Finance* **72**(1), 371–414.

Hansen, L. P.: 2007, Beliefs, doubts and learning: Valuing macroeconomic risk, *American Economic Review* **97**(2), 1–30.

Hansen, L. P. and Sargent, T. J.: 2008, *Robustness*, Princeton University Press.

Harris, R. S., Jenkinson, T. and Kaplan, S. N.: 2014, Private equity performance: What do we know?, *The Journal of Finance* **69**(5), 1851–1882.

Higson, C. and Stucke, R.: 2012, The performance of private equity, *SSRN Electronic Journal* .

Holcblat, B. and Sowell, F.: 2022, The empirical saddlepoint estimator, *Electronic Journal of Statistics* **16**(1), 3672–3694.

Imbens, G. W., Spady, R. H. and Johnson, P.: 1998, Information theoretic approaches to inference in moment condition models, *Econometrica* **66**(2), 333.

Jenkinson, T., Landsman, W. R., Rountree, B. R. and Soonawalla, K.: 2020, Private equity net asset values and future cash flows, *The Accounting Review* **95**(1), 191–210.

Julliard, C. and Ghosh, A.: 2012, Can rare events explain the equity premium puzzle?, *The Review of Financial Studies* **25**(10), 3037–3076.

Kaplan, S. N. and Schoar, A.: 2005, Private equity performance: Returns, persistence, and capital flows, *The Journal of Finance* **60**(4), 1791–1823.

Kaplan, S. N. and Sensoy, B. A.: 2015, Private equity performance: A survey, *Annual Review of Financial Economics* **7**(1), 597–614.

Keating, D.: 2006, G p confidential, *Private Equity International* **June**, 70–73.

Keynes, J. M.: 1936, *General Theory of Employment, Interest, and Money*, Macmillan London.

Kitamura, Y. and Stutzer, M.: 1997, An information-theoretic alternative to generalized method of moments estimation, *Econometrica* pp. 861–874.

Kitamura, Y., Tripathi, G. and Ahn, H.: 2004, Empirical likelihood-based inference in conditional moment restriction models, *Econometrica* **72**(6), 1667–1714.

Korsaye, S. A.: 2022, Investor beliefs and market frictions. Available at SSRN 4277979.

Korsaye, S. A., Quaini, A. and Trojani, F.: 2019, Smart sdfs. Available at SSRN 3878300.

Korsaye, S. A., Trojani, F. and Vedolin, A.: 2023, The global factor structure of exchange rates, *Journal of Financial Economics* **148**(1), 21–46.

Korteweg, A.: 2019, Risk adjustment in private equity returns, *Annual Review of Financial Economics* **11**(1), 131–152.

Korteweg, A.: 2023, *Handbook of the Economics of Corporate Finance, Vol 1: Private Equity and Entrepreneurial Finance*, Elsevier, chapter Risk and return in private equity, pp. 43–418.

Korteweg, A. and Nagel, S.: 2016, Risk-adjusting the returns to venture capital, *The Journal of Finance* **71**(3), 1437–1470.

Lerner, J. and Leamon, A.: 2011, Yale university investments office: February 2011, *Harvard Business School Entrepreneurial Management Case* (812-062).

LeRoy, S. F.: 1973, Risk aversion and the martingale property of stock prices, *International Economic Review* **14**(2), 436–446.

Long, A. and Nickels, C.: 1996, A private investment benchmark, *AIMR Conference on Venture Capital Investing* pp. 1–17.

Lucas, R. E.: 1976, Econometric policy evaluation: A critique, *Carnegie-Rochester conference series on public policy*, Vol. 1, pp. 19–46.

Manski, C. F.: 2004, Measuring expectations, *Econometrica* **72**(5), 1329–1376.

Metzler, L. A.: 1941, The nature and stability of inventory cycles, *The Review of Economics and Statistics* **23**(3), 113–129.

Miller, E. M.: 1977, Risk, uncertainty, and divergence of opinion, *The Journal of Finance* **32**(4), 1151–1168.

Muth, J. F.: 1961, Rational expectations and the theory of price movements, *Econometrica: Journal of the Econometric Society* pp. 315–335.

Nadauld, T. D., Sensoy, B. A., Vorkink, K. and Weisbach, M. S.: 2019, The liquidity cost of private equity investments: Evidence from secondary market transactions, *Journal of Financial Economics* **132**(3), 158–181.

Nanda, R., Samila, S. and Sorenson, O.: 2020, The persistent effect of initial success: Evidence from venture capital, *Journal of Financial Economics* **137**(1), 231–248.

Nerlove, M.: 1958, Adaptive expectations and cobweb phenomena, *The Quarterly Journal of Economics* **72**(2), 227–240.

Newey, W. K. and Smith, R. J.: 2004, Higher order properties of gmm and generalized empirical likelihood estimators, *Econometrica* **72**(1), 219–255.

Ning, Y., Wang, W. and Yu, B.: 2015, The driving forces of venture capital investments, *Small Business Economics* **44**(2), 315–344.

Owen, A. B.: 1988, Empirical likelihood ratio confidence intervals for a single functional, *Biometrika* **75**(2), 237–249.

Pástor, L. and Stambaugh, R. F.: 2003, Liquidity risk and expected stock returns, *Journal of Political Economy* **111**(3), 642–685.

Pesaran, H.: 1987, *The Limits to Rational Expectations*, New York and Oxford: Basil Blackwell Inc.

Phalippou, L.: 2013, Yale's endowment returns: Case study in gips interpretation difficulties, *The Journal of Alternative Investments* **15**(4), 97.

Phalippou, L.: 2014, The hazards of using irr to measure performance: The case of private equity, *The Journal of Performance Measurement* **12**, 55–66.

Phalippou, L. and Gottschalg, O.: 2009, The performance of private equity funds, *Review of Financial Studies* **22**(4), 1747–1776.

Pigou, A. C.: 1927, *Industrial fluctuations*, Macmillan and Co, Limited.

Qin, J. and Lawless, J.: 1994, Empirical likelihood and general estimating equations, *Annals of Statistics* **22**(1), 300–325.

Ramsey, F. P.: 1926/1931, *The Foundations of Mathematics and other Logical Essays*, London: Routledge and Kegan Paul, chapter Truth and Probability (1926, Ch. VII), pp. 156–198.

Robinson, D. T. and Sensoy, B. A.: 2016, Cyclicality, performance measurement, and cash flow liquidity in private equity, *Journal of Financial Economics* **122**(3), 521–543.

Rubinstein, M.: 1976, The valuation of uncertain income streams and the pricing of options, *The Bell Journal of Economics* pp. 407–425.

Schmid, L., Huether, N. and Steri, R.: 2019, Credit market equivalents and the valuation of private firms, *Available at SSRN 3463333* .

Sensoy, B. A., Wang, Y. and Weisbach, M. S.: 2014, Limited partner performance and the maturing of the private equity industry, *Journal of Financial Economics* **112**(3), 320–343.

Smith, R. J.: 1997, Alternative semi-parametric likelihood approaches to generalised method of moments estimation, *The Economic Journal* **107**(441), 503–519.

Sorensen, M. and Jagannathan, R.: 2015, The public market equivalent and private equity performance, *Financial Analysts Journal* **71**(4), 43–50.

Stutzer, M.: 1996, A simple nonparametric approach to derivative security valuation, *The Journal of Finance* **51**(5), 1633–1652.

Appendices

A Stationary Excess Cash Flows

The ET estimator relies on the stationarity of the underlying time series of excess cash flows for estimating unconditional subjective beliefs. However, challenges with the aggregate cash flows spanning the entire sample are apparent, as depicted in Figure A.1, top panel. One issue emerging at the sample's outset is the limited availability of cash flows, attributed to the mere existence of a handful of funds. Predominantly, these cash flows arise from capital calls, aligning directly with the benchmark market fund's cash flows and leading to limited variability. Another challenge emerges at the sample's end: cash flows from funds incepted post-2015 (the last inception year in my dataset) display cash flows heavily influenced by valuations, leading to a surge in excess cash flows.

Truncating the time series to a shorter period partially addresses the non-stationarity inherent in the aggregate excess cash flows. The enhancements in stationarity properties become evident in Figure A.1, middle panel. However, it is important to note that the excess cash flows before 2000 are less volatile than those post-2000. The rising number of funds in the industry has increased both the magnitude and fluctuations of the aggregate excess cash flows. These patterns cast doubt on their use in belief extraction due to stationarity concerns.

A potential resolution lies in averaging the time series, shifting from the moment condition of $\mathbb{E}^{P^*} \sum_{t=1}^{T} M_{1,t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0$ to $\mathbb{E}^{P^*} \sum_{t=1}^{T} M_{1,t} \frac{1}{N_t} \sum_{i=1}^{N} CF_{i,t}^{exc} = 0$. This strategy aims to 'normalize' the aggregate cash flows by accounting for the active fund count within each time interval.

Assessing the impact of averaging on the stationarity characteristics of the aggregate excess cash flows series involves dividing the original series by the count of funds exhibiting non-zero cash flows, defined N_t. The bottom panel of Figure A.1 suggests that the average excess cash flows display marginally enhanced stationarity compared to aggregate excess cash flows. The autocorrelation function (ACF) in Figure A.2 supports this observation, showcasing a swift descent to zero after one lag for the average excess cash flows. Using this averaged series appears to attenuate the data's persistent nature, proving beneficial for subjective beliefs estimation.

Subsequent formalized testing, detailed in Table A.1, employ the Augmented Dickey-Fuller (ADF), Phillips-Perron (PP), and Kwiatkowski-Phillips-Schmidt-Shin (KPSS) tests. Each test adopts its unique approach to stationarity, featuring distinct null hypotheses:

- **ADF test**: The null hypothesis asserts the presence of a unit root in the series, indicating

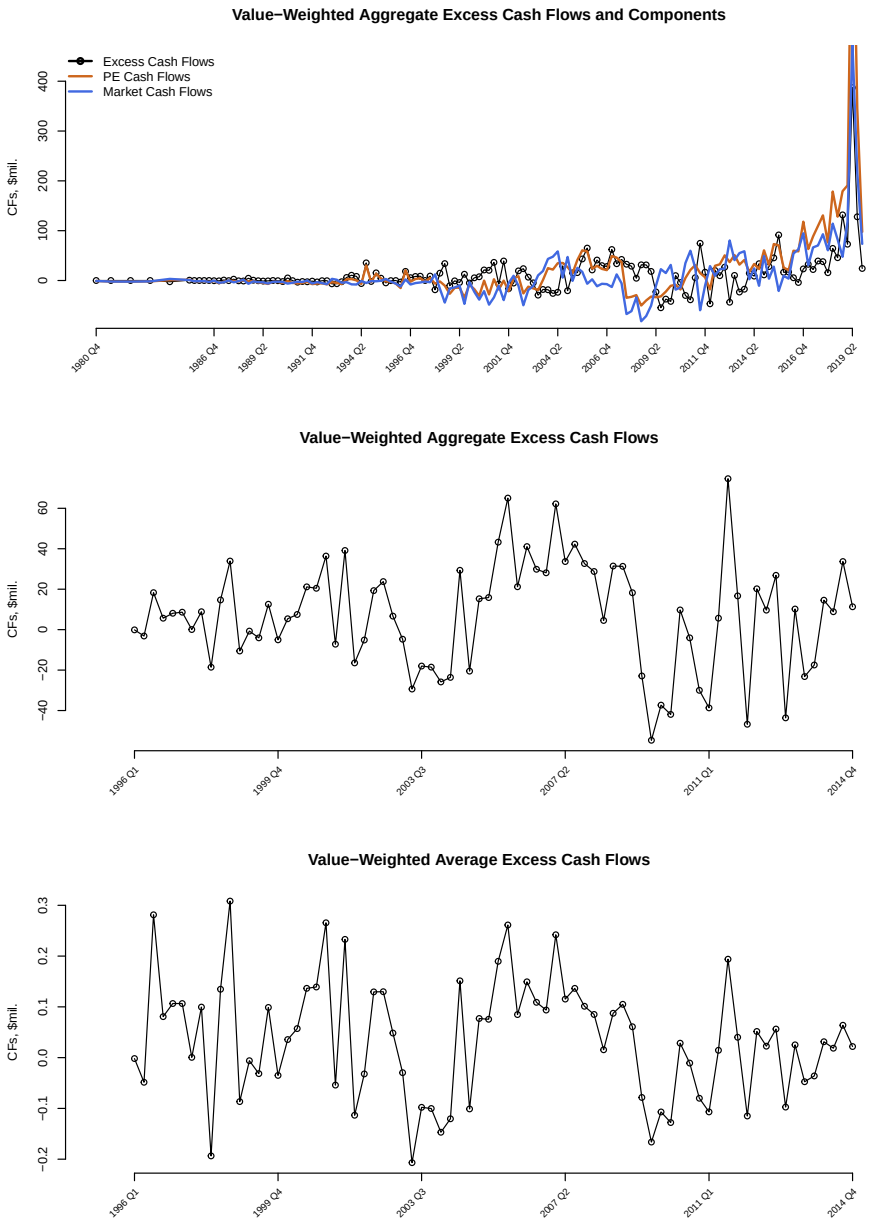


Figure A.1: Excess Cash Flow and Its Components. This figure plots the excess cash flow dynamics for PE funds. The top panel contrasts aggregate PE cash flow with aggregate CRSP cash flows from 1980 Q4 to 2019 Q4. The middle and bottom panels offer a detailed view of the aggregate and average excess cash flows for PE funds between 1996 Q1 and 2014 Q4, respectively.

non-stationarity. A significant p-value implies this hypothesis's rejection

- **PP test**: Aligning with the ADF test, the PP test's null hypothesis posits that the series contains a unit root. However, this test adjusts for any serial correlation within error terms,

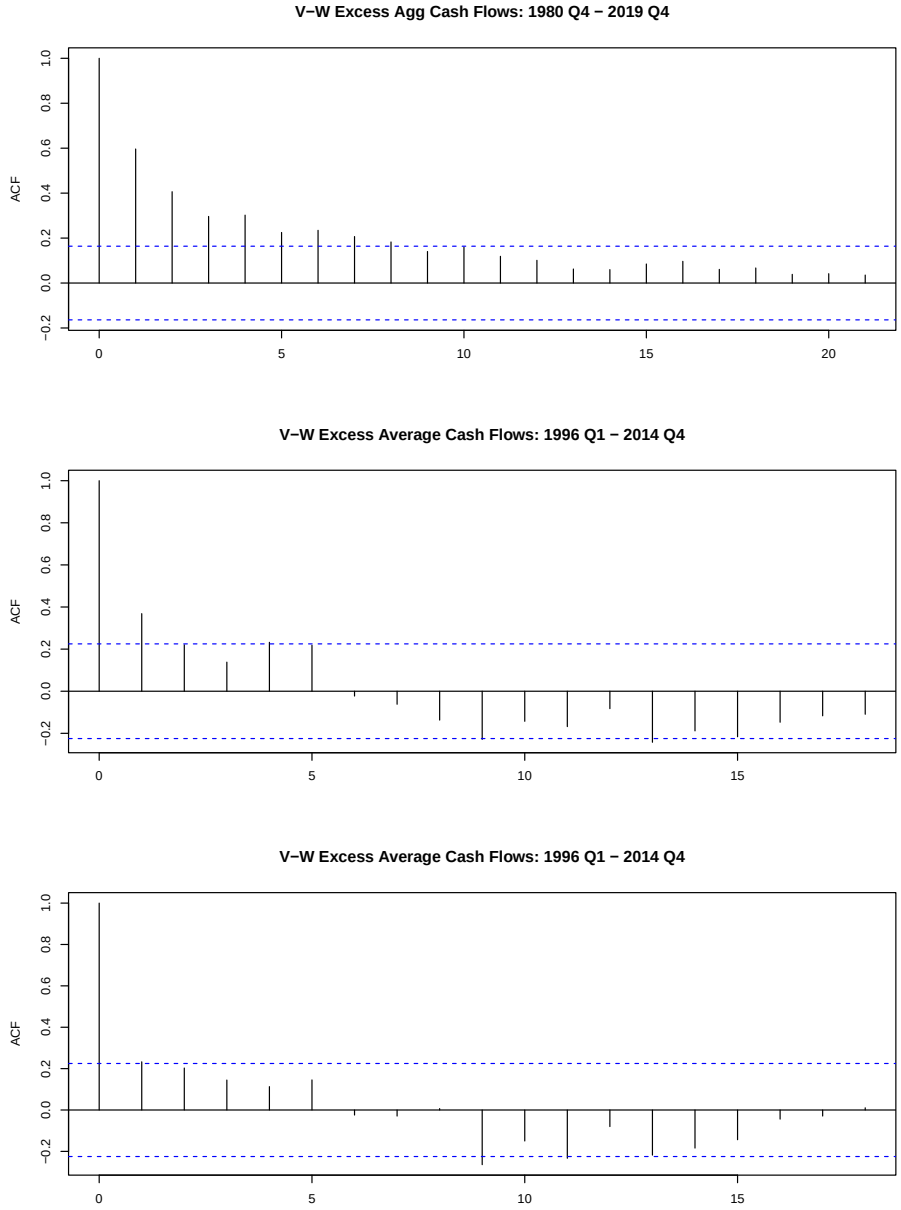


Figure A.2: Autocorrelation Function for Excess Cash Flows. The figure plots the autocorrelation function (ACF) for aggregate and average excess cash flows for PE funds. The x-axis at each plot shows the number of lags ACF is computed for. Dashed blue lines represent a zero bound.

ensuring a robust analysis.

- **KPSS test**: Its null hypothesis posits the series is stationary around a deterministic trend. A significant p-value negates this hypothesis.

Both time series produce conflicting conclusions between the ADF and PP tests. While ADF

Table A.1: Stationarity Test Results for Aggregate and Average Excess Cash Flows. The table presents the results of stationarity tests applied to both aggregate and average excess cash flows. Columns provide the test statistic and corresponding p-values for each applied test. Rows denote specific stationarity tests: Augmented Dickey-Fuller (ADF), Phillips-Perron (PP), and Kwiatkowski-Phillips-Schmidt-Shin (KPSS).

	Aggregate Excess CFs		Average Excess CFs	
Test	Test Statistic	p-value	Test Statistic	p-value
ADF	-2.30	0.45	-2.68	0.30
PP	-47.88	0.01	-63.48	0.01
KPSS	0.099	0.10	0.183	0.10

results insinuate non-stationarity, the PP test leans towards a stationary stance. The inferences drawn from the KPSS test align with those of the PP, further hinting at stationarity.

Switching to the averaged series seems to reduce the persistence and the longer memory in the data, as evidenced by the diminished and faster-decaying autocorrelations. The averaged series appears closer to stationarity, although formal statistical tests do not unequivocally favor one over the other. Hence, focusing on the average excess cash flows from 1996 Q1 to 2014 Q4 appears suitable for the EL-type methodology used to extract investors' beliefs.

However, it is important to acknowledge that, given the foundation laid by the GMPE-linked moment condition derivation for aggregate cash flows, extracting beliefs using this approach remains rational. A detailed examination of this approach is conducted in Appendix D.2.

B 'Billion Dollar Club'

In this appendix, we investigate the capability of Limited Partners (LPs) to translate their beliefs into price discounts during negotiations for investments in PE funds. While the main analysis has demonstrated the significant economic role of beliefs in driving excess cash flows, it remains uncertain whether the average investor possesses sufficient bargaining power to negotiate more favorable contract terms based on these beliefs. If this holds true, the premium tied to future cash flow uncertainties might be at least partly priced through the fee agreement. To probe this further, we focus on a subset of funds representing investments made by members of the so-called 'Billion Dollar Club', investors committing $1bn or more to PE.

According to Preqin, in 2018, there were 359 investors that met the criteria, allocating a total of $1.54tn to private equity. The majority of these investors (54%) are based in North America, while 28% are based in Europe, and 9% are based in Asia and the rest of the world. As pointed out by Christopher Elvin[9], this elite group of investors exert significant influence over the PE industry due to their ability to shape standards, negotiate fees, and gain access to oversubscribed vehicles[10]. This advantage could feasibly enable them to obtain price discounts via negotiating fee structures or accessing the best deals offered by the fund. The concentration of such bargaining power might lead to additional compensation relative to time and risk, potentially aligning subjective beliefs more closely with the RE hypothesis.

The academic literature indicates that high-performing General Partners (GPs) do not increase the size of their subsequent funds in line with the rising demand for their stakes. Consequently, these funds often become oversubscribed. In this situation, large investors such as those in the 'Billion Dollar Club' have a distinct advantage in investing in top-tier GPs, a trend underlined by the research of Keating (2006). As highlighted by Lerner and Leamon (2011), certain practitioners, like David Swensen[11], have strategically reinvested in these leading GPs to ensure greater accessibility to their future funds. It is also important to note that only the best GPs typically are able to raise the follow-on funds. This is supported by studies from Sensoy et al. (2014), Chung et al. (2012), and Kaplan and Schoar (2005), underscoring that the prospects of launching a new fund hinge heavily on the historical performance of their previous ones.

To construct the sample of investments made by the 'Billion Dollar Club', we rely on data from Pitchbook[12] detailing LPs' capital allocation to PE funds. This dataset encompasses details

[9] a former head of PE products at Preqin
[10] https://docs.preqin.com/press/PE-1bn-Club-Jun-18.pdf
[11] a former head of Yale University endowment
[12] a comprehensive provider of M&A, private equity, and venture capital data

about the funds LPs invest in, their investment sizes, and the associated dates. An LP qualifies as a club member if they have committed at least 1 billion dollars specifically to buyout funds—this is notably stricter than the typical definition which pertains to the broader category of PE. The timing of investments also plays a role in the classification. For example, if an LP achieves club membership in 2008, only the funds they committed to post-2008 are taken into account. CalPERS (California Public Employees' Retirement System) was the pioneering LP to meet these standards in 1995, and the club's roster expanded to 58 members by 2014.

Table B.2: Descriptive Statistics for the 'Billion Dollar Club': LPs Investment Data. The table reports characteristics of the LPs in the "Billion Dollar Club" for the years 2005 and 2014 from the Pitchbook dataset. Detailed statistics include the count of funds invested, the cumulative capital commitment (presented in $ millions), and the mean fund size. The LPs are divided by their membership status in the club (members and non-members) and further differentiated by their institutional types.

Type of LP	# LPs		Total Commitment		Average Fund Size	
	club	non-club	club	non-club	club	non-club
Fund of Funds	2	52	11820.5	3497.2	636.4	539.8
Government Agency	1	3	4068.2	466.0	513.8	522.9
Insurance Company	2	71	2307.6	4941.8	498.3	475.0
Public Pension Fund	21	61	58302.2	9654.3	526.9	537.9
Wealth Management Firm	1	5	1059.5	451.0	440.6	608.8
Total (2005):	27	570	77558.0	39404.2	539.7	521.1
Corporate Pension	4	79	6894.0	9837.9	564.3	538.2
Endowment	3	10	7762.0	974.8	499.0	551.5
Foundation	1	41	1024.0	3126.6	627.6	504.7
Fund of Funds	4	31	12570.7	6356.7	669.4	544.5
Government Agency	1	1	7947.8	13.4	530.9	230.5
Insurance Company	4	69	10982.2	12585.6	543.2	500.5
Public Pension Fund	39	80	233139.9	18734.2	549.1	534.7
Sovereign Wealth Fund	1	1	1550.5	301.3	623.2	710.7
Wealth Management Firm	1	4	1066.6	210.0	430.1	642.1
Total (2014):	58	470	282937.7	58282.9	567.3	541.7

Table B.2 presents the characteristics of LPs, segregating between members (*club*) and non-members (*non-club*) of the 'Billion Dollar Club' as of the midpoint year 2005 and the concluding year 2014. The table indicates that Public Pension Funds are the primary investors in buyout funds, constituting over 50% of club members. Additionally, the collective commitment from the club has seen a consistent uptrend, amplifying its investments threefold to culminate at $283 billion by 2014. In contrast, the non-members, comprising 470 LPs, had aggregated investments of merely $58 billion in 2014. Generally speaking, club members tend to commit to larger funds compared

to their non-member counterparts across nearly all LP classifications. Still, this discrepancy is not vast. For context, back in 2005, the mean fund size chosen by Public Pension Funds within the club was marginally less than that picked by non-members. However, this trend pivoted in the opposite direction by 2014.

Table B.3 showcases the characteristics of funds from the Preqin sample invested by both members (*club*) and non-members (*non-club*) of the 'Billion Dollar Club'. The median fund size is almost double for club members; thus, the difference is significantly higher than in the Pitchbook sample. However, the performance of funds invested by club members is not significantly better, as the median IRR (with a 1 bp advantage for club members) and TVPI are nearly identical. It should be noted that this performance parity could stem from the fact that many of the same funds attract investments from LPs across both categories. And, given that performance metrics are tracked at the fund level and not at the individual LP level, such nuances might not surface. Further insights from Table B.3 suggest that club LPs have a propensity towards more liquid funds. This is inferred from the fact that the funds they lean towards tend to turn over capital more swiftly, as indicated by *Fund effective years* column. Overall, there are no significant differences between member and non-member funds.

Table B.3: Fund Performance Metrics for the 'Billion Dollar Club' Investors. This table reports the performance metrics of the funds in which members of the "Billion Dollar Club" and their non-member counterparts have invested. The metrics assessed include the median fund size, represented as the overall commitment (in $ millions), the IRR (%), TVPI, and the fund's effective years, which indicate the span between the first and last observed cash flows of a particular fund.

	Fund Size	IRR,%	TVPI	Fund effective years
club	912	11	1.54	10.25
non-club	456	10	1.50	11.25

Beliefs. Figure B.3 shows belief-adjusted average excess cash flows for the subset of funds where investments were made by the 'Billion Dollar Club'. Using the K-N CAPM SDF for estimating beliefs, the belief-adjusted distribution closely mirrors the distribution implied by the RE hypothesis, especially when juxtaposed against Figure 1. Nonetheless, this distribution reveals a marginally fatter negative tail and exhibits a slight left skew. On the other hand, the EHB SDF provides a very accurate estimation for the sub-sample of 'billionaires' funds by producing very low pricing errors for excess cash flows. In fact, the belief-adjusted distribution, estimated through the EHB SDF, nearly aligns seamlessly with the one suggested by the RE hypothesis. Such observations bolster the premise that club members can translate their subjective beliefs and the associated

uncertainties about cash flows into price discounts through limited partner agreements (LPAs).

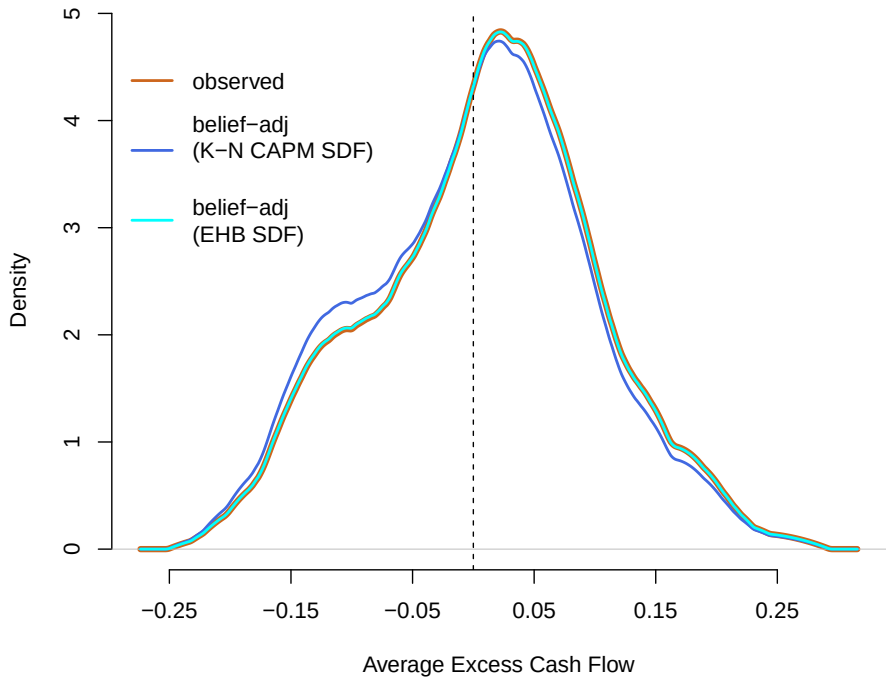


Figure B.3: Discrepancy in PE Excess Cash Flows: True vs. Belief-Adjusted. This figure plots the observed and belief-adjusted average excess cash flows for 'Billion Dollar Club' PE funds from Q1 1996 to Q4 2014. Adjustments are based on beliefs estimated under the K-N CAPM and EHB models. Values are presented in millions of dollars, with density estimations derived using the Epanechnikov kernel method.

In Figure B.4, the top panel displays the impact of beliefs estimated under the K-N CAPM SDF on the components of excess cash flow for the sub-sample of "billionaire" funds. The figure underscores that the discrepancy between subjective beliefs and those suggested by the RE hypothesis persists. However, the magnitude of this discrepancy is reduced in comparison to the full sample of funds. Despite this reduction, the magnitude remains considerable, indicating that even in the 'billionaire' funds sub-sample, the belief discrepancy carries economic significance when assessed using the K-N CAPM SDF. Thus, the narrative of subjective beliefs remains relevant, even for the 'Billion Dollar Club' sub-sample.

In Figure B.4, the bottom panel illustrates the influence of beliefs calculated under the EHB SDF on the components of excess cash flow for the sub-sample of "billionaire" funds. The panel reveals a diminished presence of the 'pessimism - optimism' pattern, and the magnitude of the errors has substantially decreased, spanning several orders of magnitude. This suggests that the combination of well-calibrated risk preferences (as in the EHB SDF) and their application to funds

backed by investors with significant bargaining clout can notably mitigate pricing errors. This further underscores a potential linkage between the alignment of subjective beliefs and the terms set forth in LPAs.

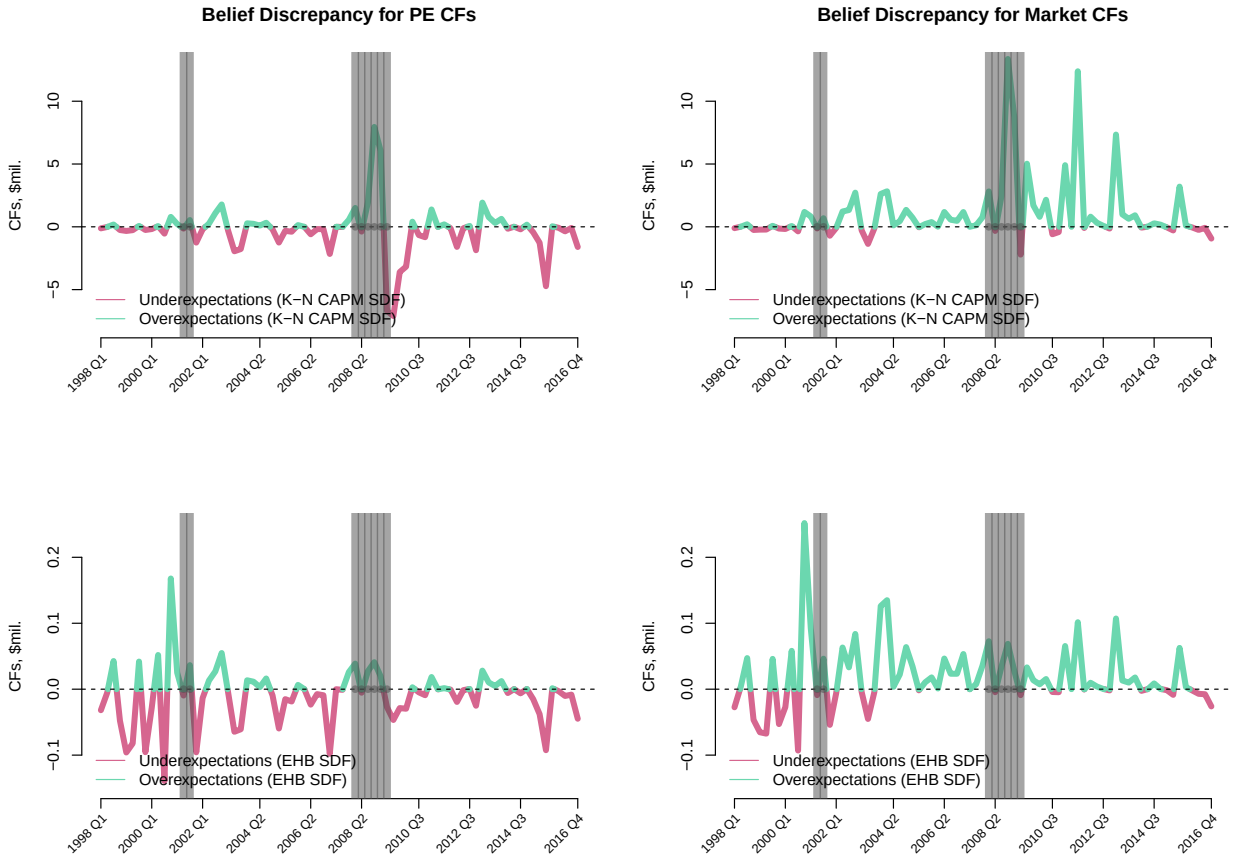


Figure B.4: Comparison of Belief-Adjusted and Observed Aggregate Cash Flows: PE vs. Market. This figure illustrates the discrepancy between belief-adjusted and observed aggregate cash flows for PE funds and market benchmark funds spanning from 1996 Q1 to 2014 Q4. Organized in two columns, the left graph details the aggregate cash flows of PE funds, while the right graph showcases those of CRSP-mimicking funds. The rows differentiate the subjective beliefs: the top row adheres to the K-N CAPM SDF, and the bottom row adheres to the EHB SDF. NBER-designated recession periods are highlighted with shaded areas.

Furthermore, in Table B.4, we validate the estimated beliefs against survey data. The results emphasize a positive and substantial correlation between subjective beliefs and survey indices. Nonetheless, the correlation's magnitude is notably lower. This could suggest that sentiment indices like CEPECI and SVVCCI encompass a broader spectrum of investors, not just those from the 'Billion Dollar Club', which is a logical assumption.

In this appendix, we examine the subjective beliefs of LPs from the 'Billion Dollar Club'. The

Table B.4: Correlations between Surveys and Excess Cash Flow. The table reports the correlations between various sentiment indices, presented row-wise, and belief-adjusted average excess cash flows for 'Billion Dollar Club' PE funds. Across the columns, the subjective beliefs are estimated using K-N CAPM SDF and EHB SDF.

	Average Excess Cash Flow adjusted for Purified Beliefs estimated for:	
	K-N CAPM	EHB
Shiller (inst)	-0.15	-0.13
Shiller (ind)	0.10	0.12
Gallup	0.38***	0.33**
ESI	0.26**	0.23*
CEPECI	0.52***	0.50***
SVVCCI	0.62***	0.56***

findings suggest that these LPs remain over-optimistic about the stock market cash flows while harboring pessimistic views on the cash flows of PE funds. Yet, it appears that members of the 'Billion Club' can use their bargaining power to secure price discounts, which, in turn, narrows the discrepancy in beliefs, especially when well-calibrated SDFs are in play. These insights bolster the main analysis, underscoring that subjective beliefs persistently influence the investment-making process, particularly when bargaining power among investors is not apparent and they 'vote with their feet', shifting the demand for PE fund stakes.

C Different NPV Methodologies

The GPME framework, as described by Korteweg and Nagel (2016), employs an NPV methodology that discounts each fund's cash flows back to its individual inception date. While theoretically robust, this approach encounters challenges when it comes to the beliefs estimation process, which requires aggregation across funds. Since each fund has a distinct inception point, it complicates the aggregation process. To elaborate, the original GPME framework computes the NPV as:

$$NPV_i^{PE,KN16} = \sum_{t=1}^{T} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{PE}$$

Here, $M_{\emptyset(i)}$ represents the multi-period SDF at the inception of fund i. $t = 1, ..., T$ denotes the range of all dates for which PE cash flows are observed. Furthermore, $M_{1,t}$ is the multi-period SDF, derived by successively compounding the single-period discount factors.

This approach captures the economic conditions at the start of the fund and aligns with the GMM estimation of SDF parameters tailored for PE investors. The GPME methodology also accounts for irregularly spaced PE cash flows by referring them back to each fund's inception. Consequently, despite potential time inconsistencies among data points, the GMM seeks to minimize pricing errors by optimally selecting the parameters of the asset pricing model.

However, in the context of the belief estimation, which aims to minimize the KLIC divergence, the temporal aggregation of consistent economic objects becomes crucial, particularly given the overlapping structure of PE funds. Directly aggregating cash flows with varied discounting points poses challenges—akin to combining 'apples and oranges'. One potential avenue is to consider segmenting funds based on inception dates or similar criteria, aggregating within cohorts, and analyzing these aggregated values. However, such segmentation is impractical here due to overlapping investment horizons and difficulty delineating consistent fund cohorts.

To circumvent this obstacle, we introduce a more straightforward strategy for beliefs extraction, which discounts all cash flows to the start of the dataset using the corresponding SDFs. This revision ensures the comparability of all aggregated cash flows—maintaining an 'apples to apples' comparison. Under this approach, the NPV for an individual PE fund is calculated as:

$$NPV_i^{PE} = \sum_{t=1}^{T} M_{1,t} CF_{i,t}^{PE}$$

This formulation guarantees uniformity in the cash flows. Notably, for the GMM estimation of SDF parameters, cash flows are normalized to one invested dollar. In contrast, for belief extraction,

we resort to the original cash flows. Thus, we maintain the economic importance of each fund proportionate to its genuine size. Merging original (size-sensitive) cash flows with a unified reference point for discounting, without leaning on $M_{\emptyset(i)}$ normalization, offers a logical and economically meaningful solution.

D Alternative Approaches to Belief Estimation

D.1 Estimating Beliefs with estimated SDFs. In the main analysis, we used the 'off-the-shelf' SDF parameters. To assess the robustness of the findings, we want to examine the economic implications of subjective beliefs using an SDF estimated following the approach of Korteweg and Nagel (2016). For this, we introduce the estimation step for a moment condition for benchmark funds that invest in the public stock market. The alternative moment condition for a benchmark fund should output an NPV of zero:

$$\mathbb{E}\sum_{t=1}^{T} M_{1,t}(\theta)\sum_{i=1}^{N} CF_{i,t}^{mrkt} = 0,$$

where θ denotes the relative risk aversion (RRA) coefficient.

Table D.5 showcases the estimation outcomes of all models. The point estimate of the RRA coefficient for the K-N CAPM model stands at 3.63, which is considerably higher than the assumption dictated by the log-utility model proposed by Kaplan and Schoar (2005). The J-test results do not negate the null hypothesis of zero pricing errors. However, the GPME value, being economically significant at 25 cents of outperformance per committed dollar, indicates potential model misspecification.

The alternate models unanimously reject the null hypothesis of zero-pricing error at the 10% significance level. The findings for the K-S CAPM model reflect a GPME estimate of 15 cents. Similarly, the C-CAPM model by Rubinstein (1976) exhibits similar estimates, 14 cents. Both the LRR model and the EHB model show relatively lower pricing errors, with abnormal performance of 10 and 13 cents, respectively. These findings highlight the important role of the EHB and LRR models in understanding PE investor behaviour. All else being equal, the beliefs implied by these models should be more accurate for the pricing kernels since their pricing error is the smallest under the RE hypothesis.

To shed light on the dynamics of pricing errors, which underpins the observed outperformance of PE funds using the GMPE metric, we applied the estimated SDFs to the aggregate excess cash flows. Figure D.5 graphically represents the pricing error in terms of millions of dollars over time, supplemented by the Mean Standard Errors (MSE) for each SDF. Both the EHB and LRR models consistently exhibit superior precision in their pricing of the excess cash flows. The differences among the pricing errors for the K-N CAPM, K-S CAPM, and C-CAPM SDFs are small. By minimizing these pricing errors, we estimate an alternative probability distribution for excess cash flows.

Table D.5: GPME Assessment of PE Funds Using Various SDFs. The table examines the abnormal performance of PE funds by utilizing the SDFs referenced from Table 2. The following models are estimated: *K-N CAPM* – Korteweg and Nagel (2016) SDF, *K-S CAPM* – Kaplan and Schoar (2005) SDF, *C-CAPM* – Rubinstein (1976) SDF, *EHB* – Campbell and Cochrane (1999) SDF, *LRR* – Bansal and Yaron (2004) SDF. These models are calibrated to price benchmark funds, resembling the inflow structure of PE funds, and investing in the CRSP value-weighted index and T-bills. Each SDF is mapped to its respective abnormal performance and parameters. Additionally, t-tests are applied to these parameters, with p-values provided in round brackets. For SDF parameters with absent p-values, values are sourced directly from existing literature. The table concludes with the J-test, analyzing if the abnormal performance is statistically equal to zero.

SDF	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	EHB	LRR
Abnorm.perf. (per $1)	0.25	0.15	0.38	0.14	0.13	0.10
			SDF parameters			
a	0.161	0	0.276			
	(0.01)		(0.00)			
b	3.63	1	4.00	2.01	3.23	8.01
	(0.00)		(0.00)	(0.00)	(0.02)	(0.09)
J–stat.	2.46	12.17	22.18	13.30	2.89	3.14
$P(\chi^2_1 > J)$	0.11	0.00	0.00	0.00	0.09	0.07

Having estimated SDFs, we extract investors' subjective beliefs using Equation 8. The objective is three-fold. First, we aim to understand if the economic mechanism underscored in Figure 3 remains consistent for beliefs estimated using two stage estimation procedure. Second, we evaluate if the statistical significance of this mechanism documented in Table 4 holds. Third, we test if beliefs retain their validity using survey data, as demonstrated in Table 6.

As evident from Figures D.6 and D.7, the dynamics of beliefs discrepancy for PE cash flows consistently exhibit the prevailing pessimism of PE investors across the majority of the observed periods. Episodes of pronounced optimism frequently antecede intervals of marked pessimism. While catastrophic events undeniably influence the shaping of PE investors' expectations, the pattern is not consistently observed across all subjective beliefs. This analysis underscores the consistent behaviour of beliefs across the two analyses.

Meanwhile, PE investors exhibit greater optimism regarding public market cash flows. This dichotomy culminates in a positive excess cash flow, which translates into positive performance benchmarking metrics like GMPE. As in the main analysis, the economic significance of the implications stemming from investors' subjective beliefs derived using the estimation step is undeniable.

Moreover, we examine whether the statistical significance of subjective beliefs persists upon the application of estimated SDFs. Table D.6 reinforces the statistical robustness of the employed

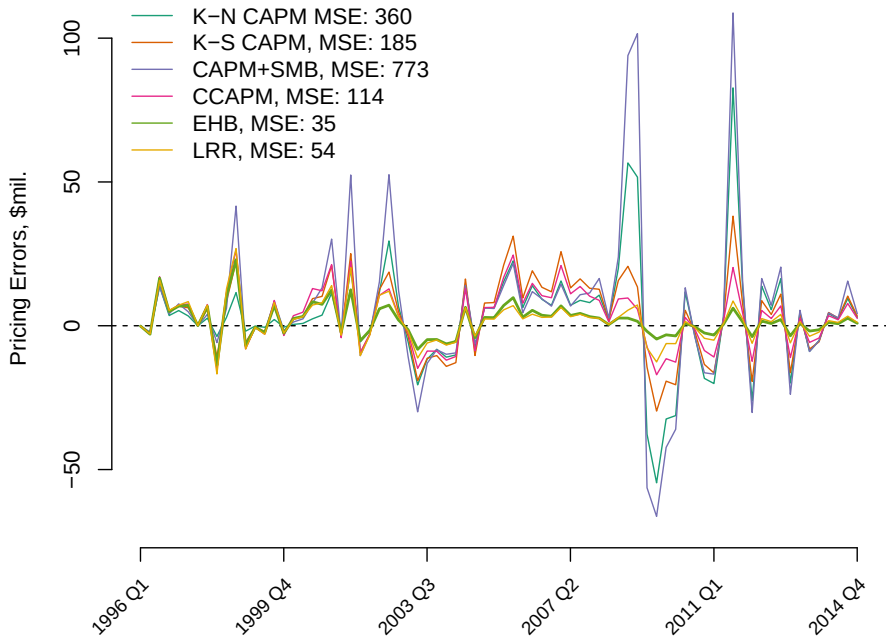


Figure D.5: Analysis of Pricing Errors Across Time and SDFs. The figure plots the period-wise errors of average excess cash flow pricing using different SDFs for 1996 Q1 – 2014 Q4. The legend of the plot presents considered SDFs and their corresponding Mean Squared Errors (MSE). All numbers are in millions of dollars.

methodology. Beliefs determined under estimated SDFs consistently eliminate the pricing error remaining after SDF discounting. For all SDFs, the historically observed misvaluation falls outside the 95% bootstrap confidence interval, which is constructed based on the extracted beliefs. The EHB model displays the narrowest confidence interval for excess cash flows. Yet, despite the small discounted mean of excess cash flows yielded by the EHB SDF, the formal test still rejects the validity of the RE hypothesis. This evidence supports that investors' subjective beliefs likely diverge from the empirical distribution of excess cash flows.

Furthermore, we check if extracted beliefs for estimated SDFs match with survey data. As presented in Table D.7 Panel A, the correlation of these subjective beliefs is even bigger than in the main analysis (Table 6). The statistical significance for core investors, as represented by the CEPECI and SVVCCI indices, remains pronounced across all models. This accentuates that the economic significance of subjective beliefs can not be rejected by the survey evidence. Moreover, for non-core individual and institutional investors in the public market, the correlation signs align

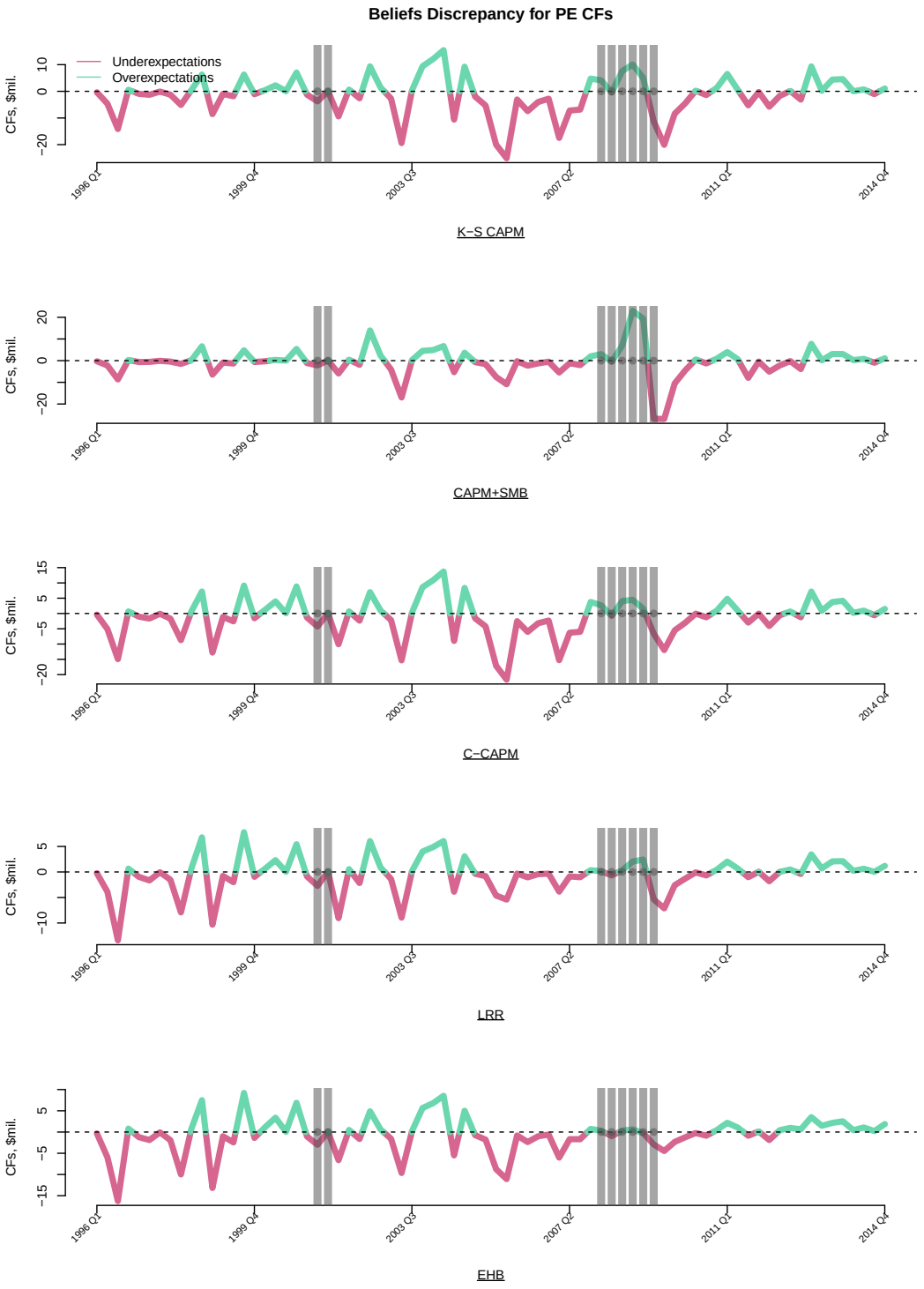


Figure D.6: Aggregate PE Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate cash flows for PE funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived.

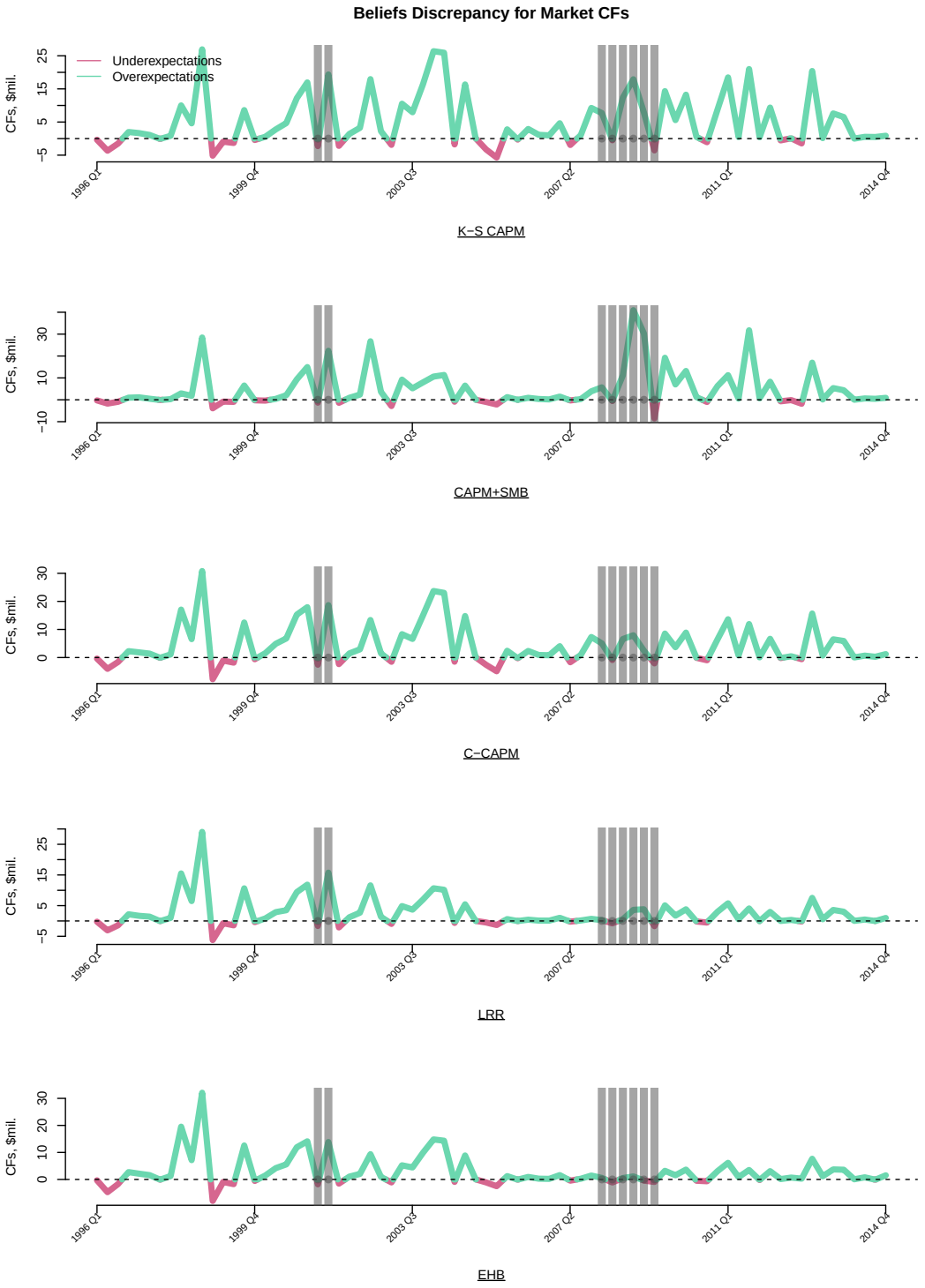


Figure D.7: Aggregate Market Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate mimicking cash flows for CRSP-benchmark funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived.

Table D.6: Counterfactual Excess, PE, and Market Cash Flows. This table presents the findings from a counterfactual analysis using 100,000 simulations, emphasizing subjective beliefs as importance weights for cash flows from 1996 Q1 to 2014 Q4. Each column represents an SDF, indicating where subjective beliefs originate. Panel A, B, and C each display results for Excess Cash Flows, PE Cash Flows, and Market Cash Flows, respectively. Statistics in each panel include the historical mean, bootstrap mean, 95% confidence interval for the bootstrap mean, and the probability of the counterfactual mean surpassing the historical mean. Values are portrayed as percentages of the Mean Average Payout (MAP).

	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	LRR	EHB
			Panel A: Excess Cash Flows (in % of *MAP*)			
$\overline{CF_T^{exc}}$	6.39	8.36	12.27	8.11	5.53	5.54
$\overline{CF_{boot}^{exc}}$	-0.02	0	0.88	0	-0.01	-0.01
$[\overline{CF_{boot}^{exc}}_{2.5\%}; \overline{CF_{boot}^{exc}}_{97.5\%}]$	[-6.18;6.15]	[-5.73;5.74]	[-8.49;10.25]	[-5.45;5.46]	[-4.86;4.84]	[-3.99;3.97]
$Pr(CF_{boot}^{exc} > \overline{CF_T^{exc}})$	2.14%	0.24%	1.09%	0.2%	1.32%	0.33%
			Panel B: PE Cash Flows (in % of *MAP*)			
$\overline{CF_T^{PE}}$	-3.29	-1.12	-12.15	-2.53	-4.68	-3.07
$\overline{CF_{boot}^{PE}}$	-4.27	-3.76	-13.99	-5.23	-6.68	-5.41
$[\overline{CF_{boot}^{PE}}_{2.5\%}; \overline{CF_{boot}^{PE}}_{97.5\%}]$	[-10.8;2.26]	[-9.88;2.37]	[-24.13;-3.85]	[-11.28;0.82]	[-11.72;-1.64]	[-9.99;-0.82]
$Pr(CF_{boot}^{PE} > \overline{CF_T^{PE}})$	39.05%	19.98%	37.47%	19.11%	21.77%	15.88%
			Panel C: Market Cash Flows (in % of *MAP*)			
$\overline{CF_T^{mrkt}}$	-9.68	-9.48	-24.43	-10.65	-10.21	-8.62
$\overline{CF_{boot}^{mrkt}}$	-4.26	-3.76	-14.87	-5.23	-6.67	-5.4
$[\overline{CF_{boot}^{mrkt}}_{2.5\%}; \overline{CF_{boot}^{mrkt}}_{97.5\%}]$	[-12.19;3.68]	[-10.95;3.43]	[-29.4;-0.35]	[-12.13;1.67]	[-12.31;-1.03]	[-10.36;-0.44]
$Pr(CF_{boot}^{mrkt} > \overline{CF_T^{mrkt}})$	90.43%	93.89%	89.54%	93.63%	88.84%	89.53%

with expectations, underscoring the economic coherence across diverse investor categories.

Panel B of Table D.7 reports also the results of a placebo test. In this test, we examine the correlation between contemporaneous levels of sentiments and belief-adjusted excess cash flows, without considering the potential lag between the time when the answers were given and the time when they might affect cash flows. The results reveal no correlation for the estimated beliefs, except for four pairs: CEPECI and K-N CAPM, ESI and K-N CAPM, ESI and K-S CAPM, and ESI and SMB+CAPM. It is worth noting that the subjective beliefs estimated under these models were the least accurate. Therefore, SDFs less contaminated by misspecification lead to a more precise estimation of subjective beliefs.

In summary, based on the examination of beliefs derived for estimated SDFs, it is evident that the adopted methodology consistently extracts beliefs that are both statistically robust and

economically compelling. This further corroborates the hypothesis that PE funds' observed positive abnormal performance can be attributed to the pronounced pessimism of LPs regarding the cash flow-generating capabilities of PE funds. Concurrently, there is a marked overestimation of the potential cash flows from their holdings in the public market.

Table D.7: Correlations between Surveys and Excess Cash Flow. The table reports the correlations between various sentiment indices, presented row-wise, and the two key metrics: 1) belief-adjusted excess cash flow (shown in Panel A), and 2) a placebo test that omits the nuanced term structure of investors' expectations (shown in Panel B). Across the columns, the subjective beliefs are estimated using distinct SDFs.

	Panel A: Average Excess Cash Flow adjusted for Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.25**	-0.18	-0.18	-0.12	-0.08	-0.11
Shiller (ind)	-0.02	-0.02	-0.02	0.03	0.08	0.08
Gallup	0.45***	0.36***	0.36***	0.29**	0.26**	0.29**
ESI	0.39***	0.32***	0.32***	0.26**	0.24**	0.26**
CEPECI	0.60***	0.57***	0.57***	0.57***	0.57***	0.58***
SVVCCI	0.73***	0.70***	0.70***	0.69***	0.68***	0.68***
	Panel B: Placebo Test for Contemporaneous Correlations					
Shiller (inst)	-0.16	-0.14	-0.14	-0.10	-0.05	-0.07
Shiller (ind)	-0.13	-0.17	-0.17	-0.13	-0.07	-0.08
Gallup	-0.03	-0.11	-0.11	-0.16	-0.19	-0.16
ESI	0.33***	0.23**	0.23**	0.16	0.13	0.17
CEPECI	0.14	0.06	0.06	0.05	0.07	0.10
SVVCCI	0.29*	0.21	0.21	0.19	0.21	0.24
Note:						*p<0.1; **p<0.05; ***p<0.01

D.2 Estimating Beliefs Using the Moment Condition for Aggregate Excess Cash Flows. In this appendix, we explore the consequences of using the moment condition from Equation 8. This comes with a particular focus on potential stationarity issues tied to aggregate excess cash flows. As mentioned in Appendix A, these cash flows do not have the best stationarity features, especially when compared to average cash flows.

The ET methodology we employ to derive beliefs is especially sensitive to any lack of stationarity in the series under consideration. Because of this, this section zooms in on how such shortcomings might influence the economic implications of beliefs estimated for aggregate cash flow. Ultimately, the aim is to see if these beliefs coincide with those pinpointed in the main analysis.

The inclination towards using aggregate excess cash flows arises from the Euler equation that is directly derived from the GPME moment condition. Opting for averaged excess cash flows may slightly deviate from the economic consistency of the original aggregate series, but it offers improved stationarity characteristics. The main goal of the current analysis is to comprehend the nuances of this trade-off.

We begin by examining the dynamics of belief discrepancies for the components of PE cash flows. As evident from Figures D.8 and D.9, the economic story appears more congruent than the analysis presented for the 'off-the-shelf' SDFs in Appendix D.1. The interpretation of the pessimism-optimism patterns remains consistent, with the only variation being a slight amplification in the magnitude of belief discrepancies for both PE and market cash flows.

Next, we turn to the statistical significance of the subjective beliefs narrative. The updated counterfactual exercise for aggregate cash flows for bootstrap based on model-implied probabilities for the aggregate excess cash flow moment condition is laid out in Table D.8. The findings are noteworthy for a couple of reasons. First, as previously, the ET methodology for extracting subjective beliefs effectively eliminates the observed pricing error, as evidenced in the first and second rows of Panel A. Second, the results from the formal test of the RE hypothesis are not as unanimous as before. Except for the K-N CAPM model, the constructed 95% confidence interval challenges the validity of the RE hypothesis.

The case of the K-N CAPM model deserves special attention. As mentioned, the occurrence of historically observed excess cash flow cannot be rejected at a 5% significance level. Moreover, the probability of the mean discounted excess cash flow being greater than its historical mean is almost 7%, which is considerably higher than for other models where this probability does not surpass 2%.

From Panel B of Table D.8, it is evident that the observed mean of PE cash flow nearly

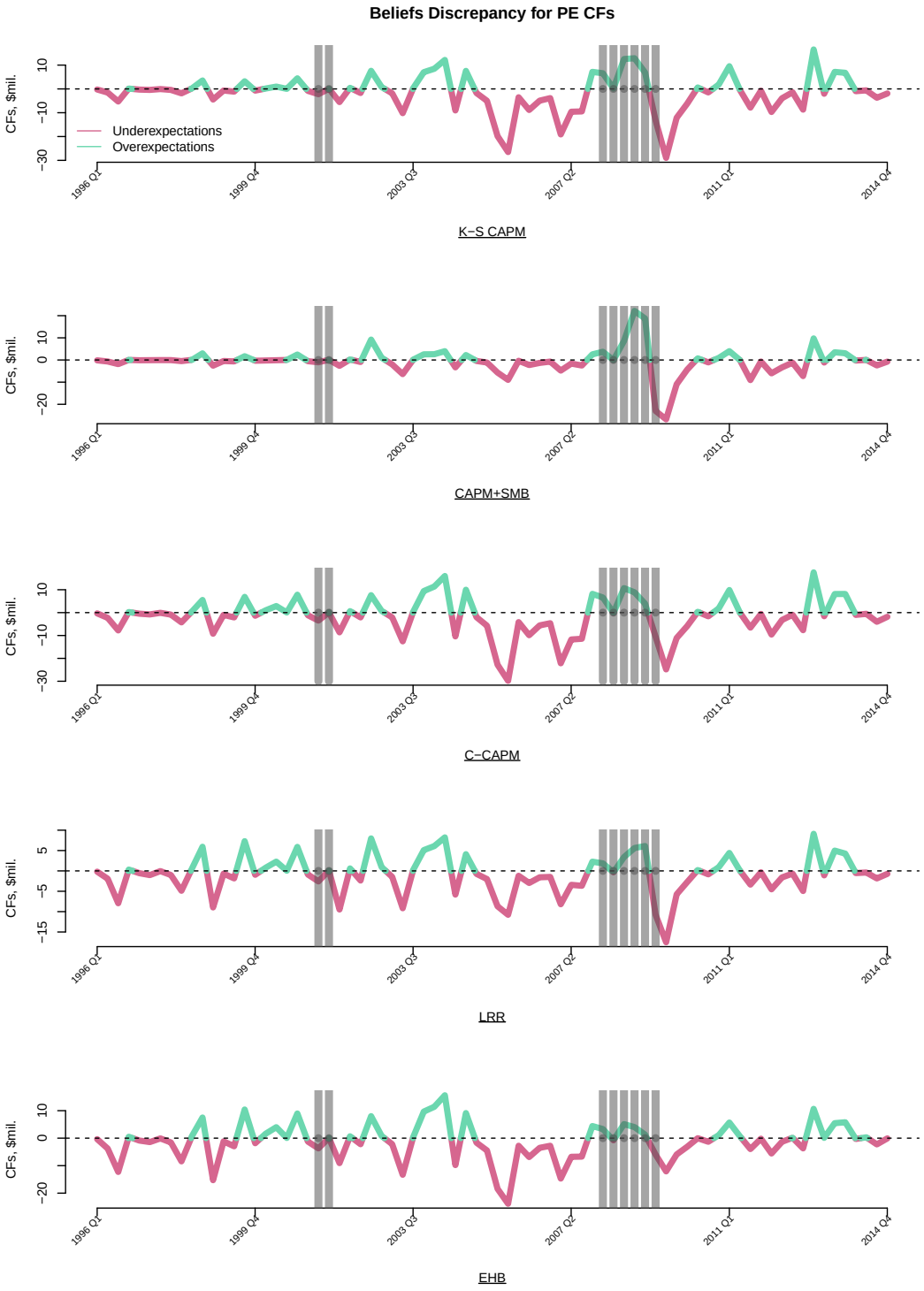


Figure D.8: Aggregate PE Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate cash flows for PE funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived. Subjective beliefs are estimated using the moment condition for aggregate cash flows.

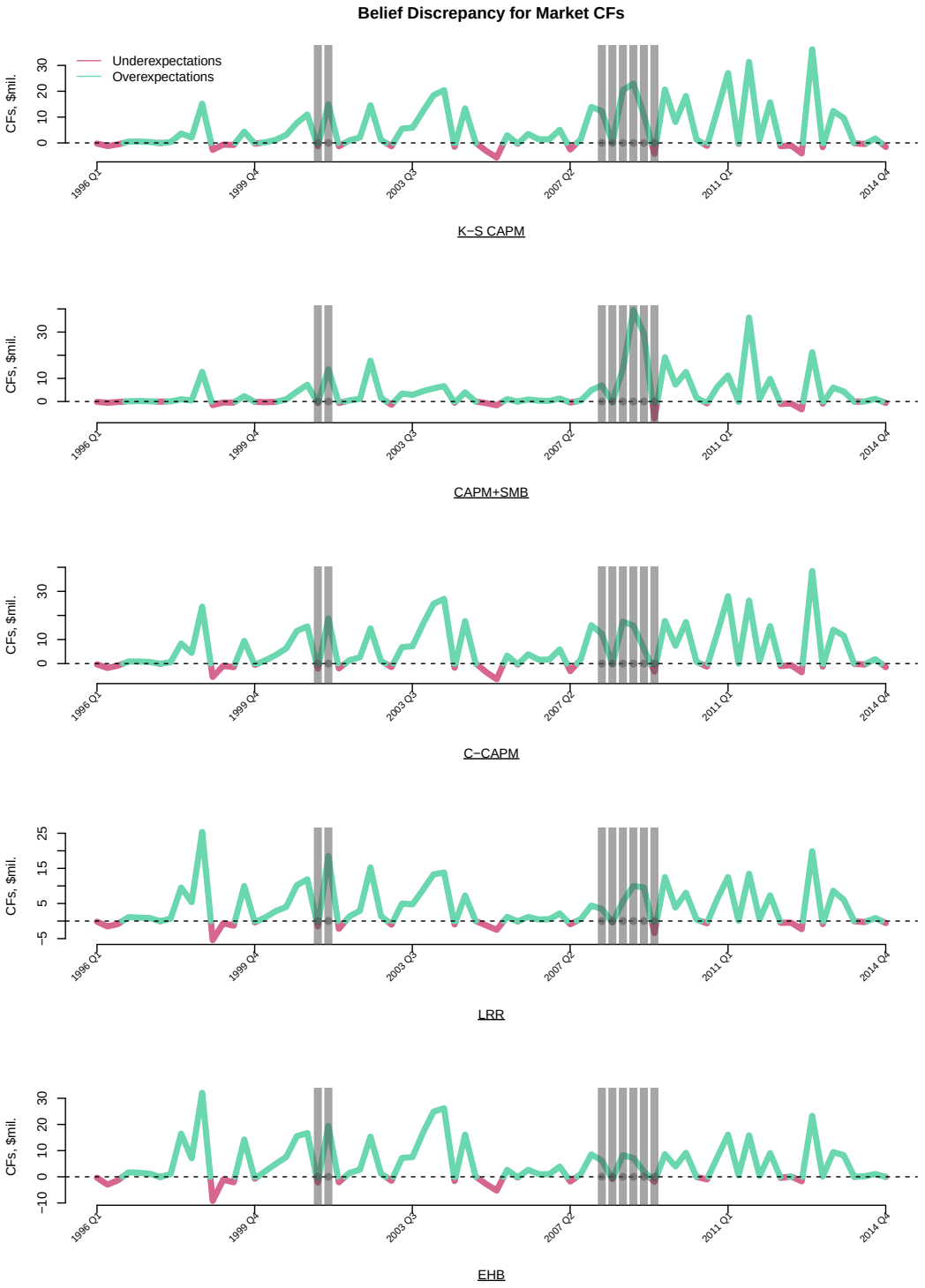


Figure D.9: Aggregate Market Cash Flows: Beliefs Discrepancy Between Belief-Adjusted and Observed. The figure plots the difference between belief-adjusted and observed aggregate mimicking cash flows for CRSP-benchmark funds from 1996 Q1 to 2014 Q4. Model-implied subjective beliefs, stemming from different risk preferences, are illustrated, with each plot's title highlighting the specific SDF from which beliefs are derived. Subjective beliefs are estimated using the moment condition for aggregate cash flows.

reaches the lower counterfactual bound, suggesting that LPs no longer exhibit excessive pessimism regarding the PE funds' cash flow generating abilities. When this result is compared with other models, it becomes clear that none yields a lower bound for PE cash flows proximate to the historical mean of the observed PE cash flow. As a consequence, the probability of observing the PE cash flow in a counterfactual world is almost 50%, reinforcing the lack of systematic pessimism toward PE funds.

However, Panel C of Table D.8 indicates that the deviation in investors' subjective beliefs is notably greater for the public market component of excess cash flow. The bootstrap mean for counterfactual market cash flow is -1.97% of the Mean Aggregate Payout, compared to -7.15% for the historically observed market mean. Consequently, 86% of the time, LPs anticipate a higher cash flow from their public market portfolio.

Figure D.10 offers insight into the economic implications of subjective beliefs estimated under the K-N CAPM SDF. It is evident that the K-N CAPM SDF successfully prices both PE and market cash flows relative to other models. Before the GFC, discrepancies in beliefs regarding both components of excess cash flows were negligible. However, the GFC had a marked impact, causing a substantial divergence between investors' subjective beliefs and those implied by the market.

The broader economic story portrayed by the K-N CAPM model indicates that LPs see the excess cash flow as having a tilt to the left with a noticeable negative tail. However, this viewpoint has two significant divergences for beliefs estimated for aggregate excess cash flow from our baseline understanding.

First, the shift in subjective beliefs about excess cash flow primarily stems from over-optimism regarding market cash flow rather than a mix of pessimism about PE cash flows combined with the said optimism. Second, the gap between these subjective beliefs and the RE hypothesis no longer holds strong statistically.

This drop in the importance of subjective beliefs can be logically connected to the results related to the GPME metric presented in Table D.5. Specifically, the abnormal performance of the K-N CAPM does not stand out statistically at the 10% level, even though there is a large abnormal performance. While this could suggest that the model is not perfectly specified, with the current analysis, it is clear that the main influence comes from active funds around the time of the Global Financial Crisis (GFC).

Further, we assess whether the extracted beliefs for aggregate excess cash flow align with survey data. As per Table D.9, the validity of the estimated beliefs holds, even for the K-N CAPM model. While the magnitude of correlations with PE sentiment indices is slightly reduced, they

Table D.8: Counterfactual Excess, PE, and Market Cash Flows. This table presents the findings from a counterfactual analysis using 100,000 simulations, emphasizing subjective beliefs as importance weights for cash flows from 1996 Q1 to 2014 Q4. Each column represents an SDF, indicating where subjective beliefs originate. Subjective beliefs are estimated using the moment condition for aggregate cash flows. Panel A, B, and C each display results for Excess Cash Flows, PE Cash Flows, and Market Cash Flows, respectively. Statistics in each panel include the historical mean, bootstrap mean, 95% confidence interval for the bootstrap mean, and the probability of the counterfactual mean surpassing the historical mean. Values are portrayed as percentages of the Mean Average Payout (MAP).

	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	LRR	EHB
			Panel A: Excess Cash Flows (in % of *MAP*)			
$\overline{CF_T^{exc}}$	5.31	6.19	10.06	5.48	2.78	3.03
$\overline{CF_{boot}^{exc}}$	-0.02	0.01	-0.27	-0.01	-0.01	0
$[\overline{CF_{boot}^{exc}}_{2.5\%}; \overline{CF_{boot}^{exc}}_{97.5\%}]$	[-7.07;7.03]	[-5.15;5.16]	[-10.53;9.99]	[-3.89;3.88]	[-2.68;2.66]	[-2.07;2.07]
$Pr(CF_{boot}^{exc} > \overline{CF_T^{exc}})$	6.88%	0.92%	2.71%	0.29%	2.08%	0.22%
			Panel B: PE Cash Flows (in % of *MAP*)			
$\overline{CF_T^{PE}}$	-1.83	2	-9.14	1	-1.77	-0.72
$\overline{CF_{boot}^{PE}}$	-2.99	0.67	-9.96	-0.21	-2.31	-1.42
$[\overline{CF_{boot}^{PE}}_{2.5\%}; \overline{CF_{boot}^{PE}}_{97.5\%}]$	[-1.99;0.67]	[-9.96;-0.21]	[-21.16;1.25]	[-4.28;3.85]	[-5.1;0.48]	[-3.73;0.9]
$Pr(CF_{boot}^{PE} > \overline{CF_T^{PE}})$	49.89%	30.26%	46.66%	28.15%	35.54%	28.09%
			Panel C: Market Cash Flows (in % of *MAP*)			
$\overline{CF_T^{mrkt}}$	-7.15	-4.19	-19.2	-4.49	-4.55	-3.75
$\overline{CF_{boot}^{mrkt}}$	-1.97	0.67	-9.69	-0.2	-2.3	-1.42
$[\overline{CF_{boot}^{mrkt}}_{2.5\%}; \overline{CF_{boot}^{mrkt}}_{97.5\%}]$	[-11.31;7.38]	[-5.68;7.01]	[-25.89;6.51]	[-5.25;4.85]	[-5.69;1.09]	[-4.16;1.33]
$Pr(CF_{boot}^{mrkt} > \overline{CF_T^{mrkt}})$	85.94%	93.2%	87.14%	95.15%	90.12%	94.94%

remain statistically significant. Additionally, the correlation analysis for non-core individual and institutional investors in the public market remains largely consistent with the baseline analysis, even after introducing the new moment condition.

To sum it up, subjective beliefs that match observed PE valuations demonstrate a consistent trend: a pessimistic view towards aggregate excess cash flows. Yet, what drives this conclusion can change based on the specific conditions used to estimate these beliefs. For instance, in the case of the K-N CAPM SDF and the moment condition for aggregate cash flow, the main factors influencing these beliefs seem to alter. This shift challenges the prevailing notion of investor pessimism about the cash flow-generating capabilities of PE funds. Instead, it suggests a tilt in importance towards over-optimism about stock market cash flows.

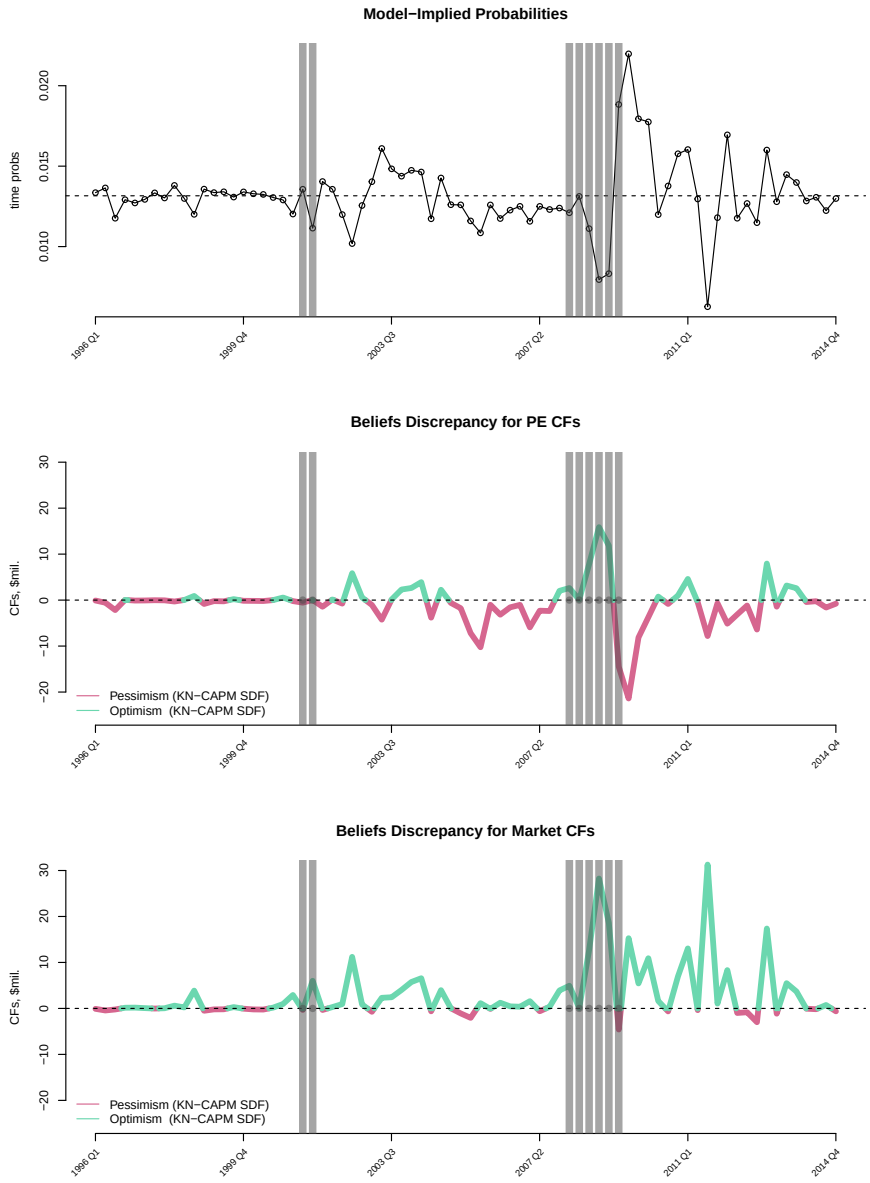


Figure D.10: Model-Implied Probabilities and Subjective Beliefs. The figure is structured into three panels, capturing different dimensions of model-implied probabilities and belief-adjusted cash flows over the period 1996 Q1 to 2014 Q4. Subjective beliefs are estimated using the moment condition for aggregate cash flows. On the upper panel, the figure plots the model-implied probabilities estimated under K-N CAPM SDF. On the middle and bottom panel, the figure captures the *Beliefs Discrepancy*, which is the difference between belief-adjusted and observed aggregate cash flows, presented in $ million. The middle panel delves into the PE fund cash flows, while the bottom panel delves into the mimicking cash flow of the CRSP benchmark fund.

More calibrated models, such as the EHB and LRR, yield smaller abnormal performances. Also, subjective beliefs estimated for them are statistically robust and match with survey data.

Therefore, potential belief distortions for the K-N CAPM could arise from non-stationarity, severe model misspecification, or a mix of both. we argue that it is prudent to avoid at least one potential source of distraction in belief estimations. This rationale underpins the choice of average excess cash flows for the baseline analysis, as detailed in Appendix A.

Table D.9: Correlations between Surveys and Excess Cash Flow. The table reports the correlations between various sentiment indices, presented row-wise, and belief-adjusted excess cash flow. Across the columns, the subjective beliefs are estimated using distinct SDFs using the moment condition for aggregate cash flows.

	Panel A: Average Excess Cash Flow adjusted for Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.17	-0.19	-0.19	-0.17	-0.12	-0.13
Shiller (ind)	0.10	0.05	0.05	0.05	0.07	0.10
Gallup	0.45***	0.45***	0.45***	0.42***	0.36***	0.38***
ESI	0.38***	0.39***	0.39***	0.37***	0.32***	0.33***
CEPECI	0.58***	0.55***	0.55***	0.53***	0.54***	0.56***
SVVCCI	0.68***	0.65***	0.65***	0.64***	0.63***	0.65***
Note:						*p<0.1; **p<0.05; ***p<0.01

D.3 Estimating Beliefs Fund by Fund. Estimating investors' subjective beliefs on a fund-by-fund basis is attractive because it leverages the cross-sectional information from funds' cash flows. From a methodological perspective, this approach can be linked to the null hypothesis underlying the GPME calculation. Specifically, the individual PE fund cash flows, including the initial investment when discounted with the SDF, should be zero. As described in Appendix C, this null hypothesis can be formulated for the individual PE funds in NPV terms as:

$$NPV_i^{PE,KN16} = \sum_{t=1}^{T} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{PE}$$

Here, $M_{1,t}$ is the multi-period SDF, derived by successively compounding the single-period discount factors. $M_{\emptyset(i)}$ represents the multi-period SDF at the inception of fund i. $t = 1, ..., T$ denotes the range of all dates for which PE cash flows are observed.

The moment condition introduced in the original SDF estimation methodology by Korteweg and Nagel (2016) employs the mimicking cash flows of the benchmark fund investing in the public stock market. A similar NPV formulation for the market benchmark fund produces:

$$NPV_i^{mrkt,KN16} = \sum_{t=1}^{T} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{mrkt}$$

The difference between the two NPV moment conditions yields the excess moment condition for the individual fund, analogous to Equation 8. This moment constraint for excess cash flows is used to estimate beliefs using the ET methodology:

$$NPV_i^{exc,KN16} = \sum_{t=1}^{T} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{exc}$$

By leveraging ideas similar to those in the main analysis, we aim to estimate investors' subjective beliefs for individual funds. This process eliminates the pricing errors at the fund level instead of the aggregate level. The optimization problem at the fund level is:

$$\min_{p_{i,t}} D(\hat{\mathbb{P}}_i^*||\hat{\mathbb{P}}_i) = \sum_{t=1}^{T_i} p_{i,t} \log\left(\frac{p_{i,t}}{\frac{1}{T_i}}\right)$$
$$\text{subject to} \quad \mathbb{E}^{\hat{\mathbb{P}}_i^*} \sum_{t=1}^{T_i} \frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{exc} = 0, \tag{11}$$
$$\sum_{t=1}^{T} p_{i,t} = 1.$$

Solving Problem 11 provides the set of individual fund model-implied probabilities: $\hat{\mathbb{P}}_1^*, \hat{\mathbb{P}}_2^*, ..., \hat{\mathbb{P}}_N^*$. Subsequently, applying these probabilities to fund-level cash flows generates the alternative series of a fund's cash flows, referred to as individual fund subjective cash flows. We aggregate these cash flows across all funds, period by period, to examine whether this methodology yields distributional results similar to those at the aggregate level. All the results presented in this section are based on beliefs estimated using the K-N CAPM SDF.

The outcomes of the fund-specific estimation are showcased in Figure D.11. The top panel illustrates both the fund's excess cash flow, $CF_{i,t}^{exc}$, and its discounted version, $\frac{M_{1,t}}{M_{\emptyset(i)}} CF_{i,t}^{exc}$. The middle panel displays the specific probability measure for the fund, $\hat{\mathbb{P}}_i^*$, that minimizes the KLIC divergence from the historical distribution (delineated by the dashed line) while eliminating the pricing error at the individual fund level. The bottom panel demonstrates the outcome of applying the model-implied probabilities (from the middle panel) to discounted excess cash flows. When adjusted for beliefs, these discounted excess cash flows ensure the NPV identity holds true for every fund by balancing the sum of discounted investments (capital calls) with that of discounted capital payouts (capital distributions).

Figure D.12 visually portrays the discrepancy between the actual (observed) and belief-adjusted cash flows. The figure illustrates the influence of investors' beliefs during the fundraising phase for each fund within the Preqin sample from the primary analysis, affecting the valuation of PE funds. For a consistent comparison, we have limited the time series to the identical sample period (from Q1 1996 to Q4 2014) employed in the primary analysis. The patterns closely mirror those in Figure 1. Once investors' subjective beliefs are aggregated across the funds, the resulting time series reveals a fatter negative tail and a more left-skewed distribution than the historical cash flows. This new methodological approach reinforces the inference that investors must factor in a premium for uncertainties tied to future cash flows to rationalise the observed PE fund valuations.

In Figure D.13, we investigate the potential sources of investors' pessimism concerning excess

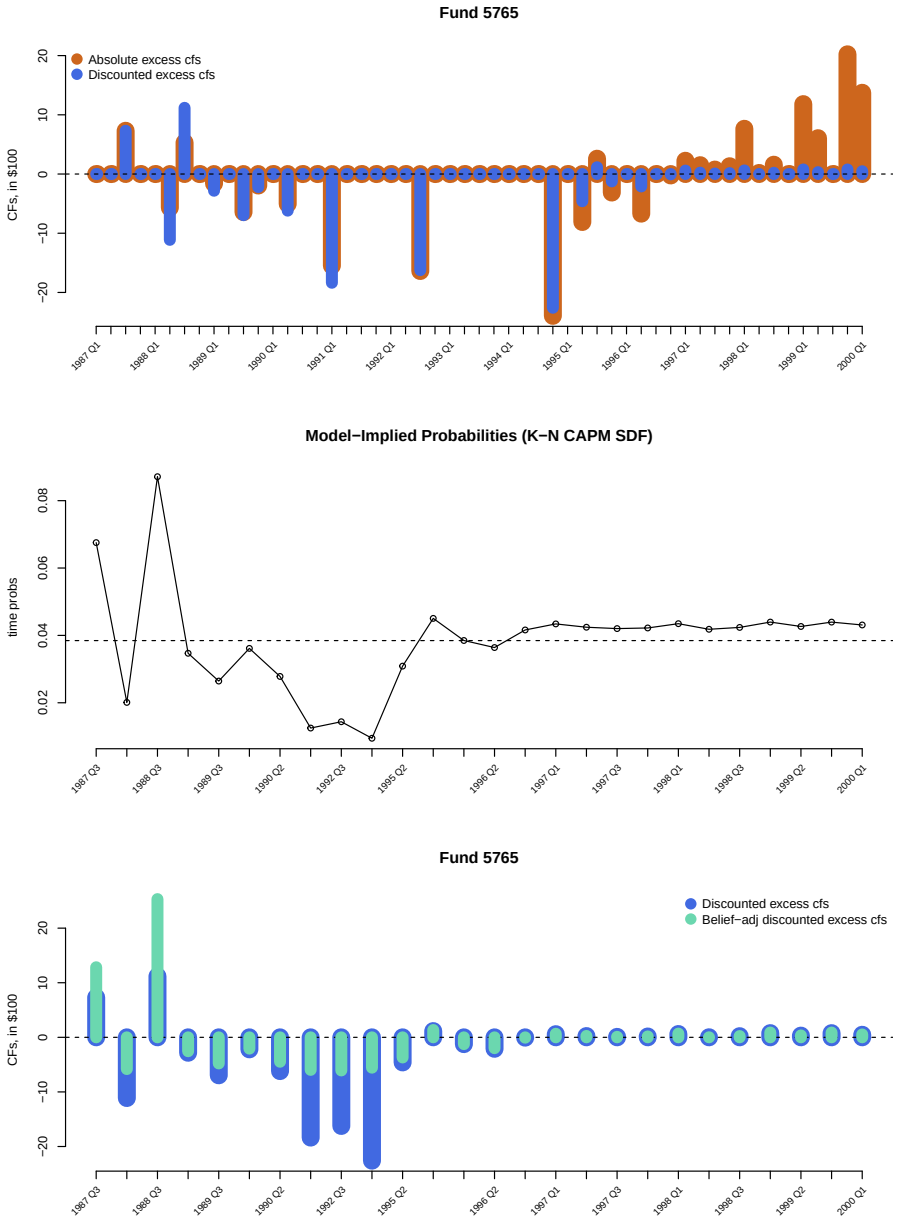


Figure D.11: Model-Implied Probabilities and Subjective Beliefs for One Fund. The figure plots the original and discounted excess cash flows for an individual fund on the upper panel. On the middle panel, the figure plots the model-implied probabilities obtained using an Empirical Likelihood–type estimator on the 'fund-by-fund' basis. On the bottom panel, the figure contrasts the risk-adjusted and belief-adjusted excess cash flows.

cash flows. As depicted in the top panel, their skepticism towards the cash flows of PE funds is more noticeable than in the primary analysis shown in Figure 3. In about 84% of the examined period,

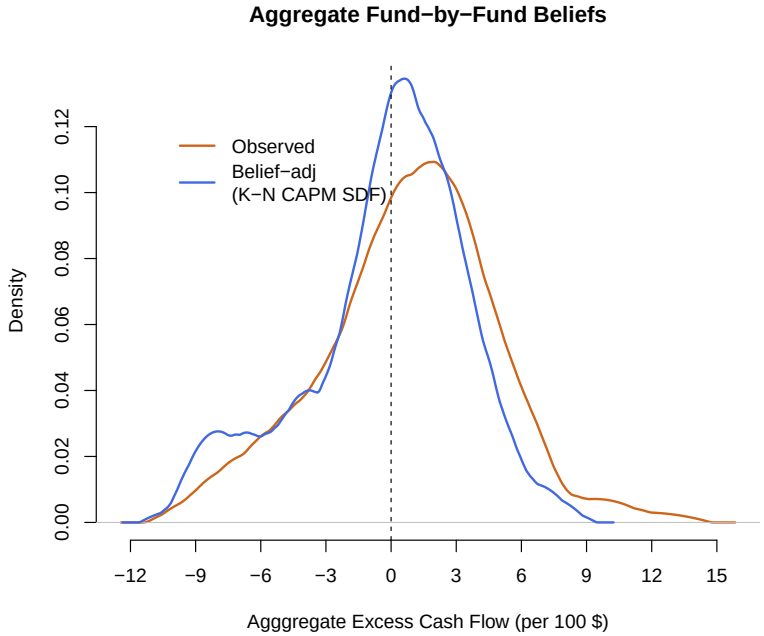


Figure D.12: Discrepancy in PE Excess Cash Flows: True vs. Belief-Adjusted. This figure plots the observed and belief-adjusted aggregate excess cash flows for PE funds from Q1 1996 to Q4 2014. Adjustments are based on beliefs estimated under the K-N CAPM SDF on a 'fund-by-fund' basis. Values are presented in 100 dollars, with density estimations derived using the Epanechnikov kernel method.

the discrepancy in beliefs is negative for fund-specific subjective beliefs. In contrast, this figure was only 63% when beliefs were estimated using average cash flows. A more detailed analysis of this discrepancy is available in Table D.10. The intensity of negative sentiments about PE cash flows significantly outweighs the positive sentiments. The $max - min$ range of discrepancy for the K-N CAPM model exceeds 5-to-1. Notably, this pessimism is not primarily clustered around significant market events such as the dot-com bubble or the Global Financial Crisis (GFC), corroborating evidence derived from subjective beliefs across different SDFs in the main analysis.

The bottom panel of Figure D.13 delivers an analysis of the belief-adjusted component linked to the public market. As discerned from the figure and further verified by Table D.10, investor sentiments towards public market cash flows are more restrained compared to the main analysis. Only about 51% of the time do they exhibit a positive discrepancy in beliefs, significantly less than the 60-70% observed in the primary analysis, and more in line with the historical distribution of cash flows, suggesting a 50% ratio. The discrepancy's magnitude also markedly diminished, plunging from the minimum 4.7-to-1 $max - min$ ratio (as per the K-S CAPM model in Table **??**) to 1.2-to-1 for the K-N CAPM model in this analysis. Intriguingly, the impact of the GFC remains evident in fund-specific belief estimates; in Q4 2009, investors had an overly optimistic outlook for

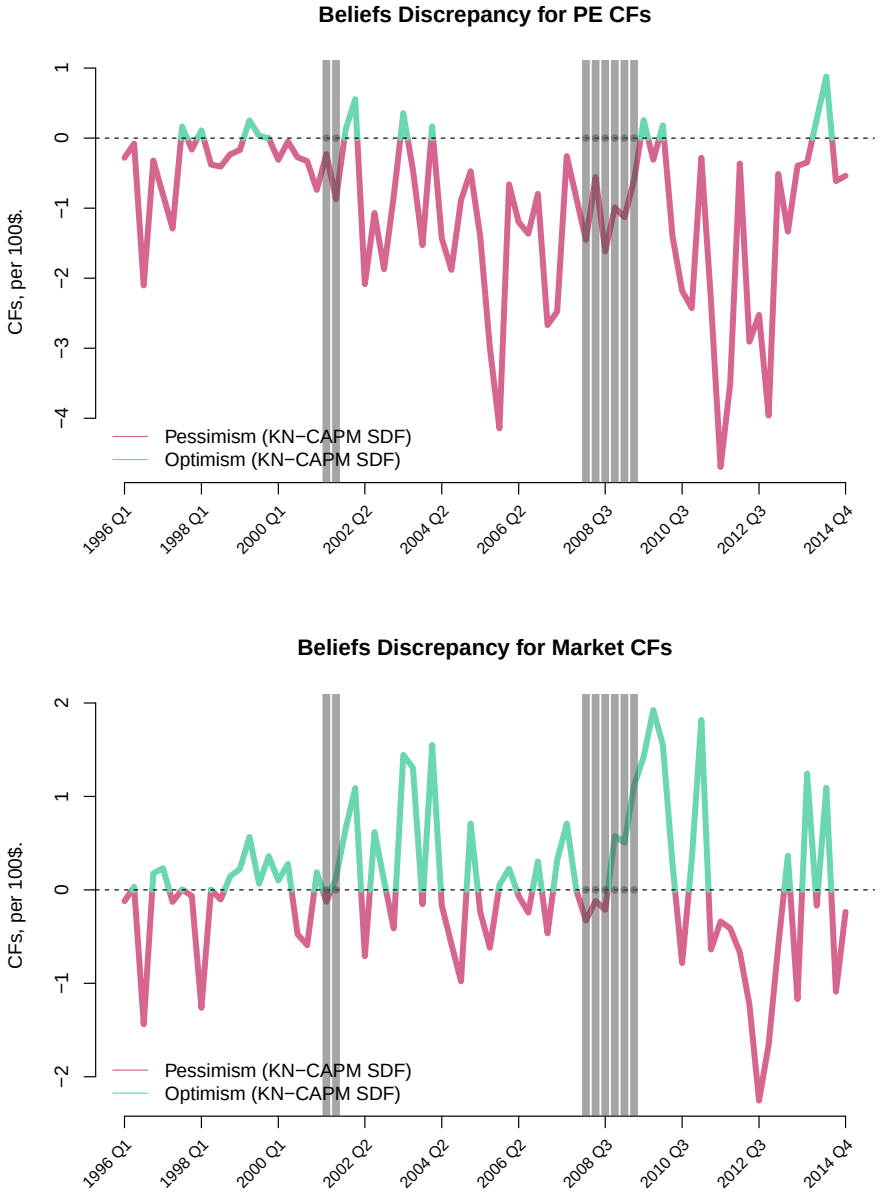


Figure D.13: Comparison of Belief-Adjusted and Observed Aggregate Cash Flows: PE vs. Market. This figure illustrates the discrepancy between belief-adjusted and observed aggregate cash flows for PE funds and market benchmark funds from 1996 Q1 to 2014 Q4. The top panel details the aggregate cash flows of PE funds, while the bottom panel showcases those of CRSP-mimicking funds. The discrepancy is calculated for beliefs estimated on a 'fund-by-fund' basis for the K-N CAPM SDF. NBER-designated recession periods are highlighted with shaded areas.

the market cash flow, expecting the CRSP cash flow to exceed the actual by \$1.92 for every \$100 invested. Predictably, this excessive optimism eventually gave way to heightened pessimism by Q3

2012, with a deviation of $2.26 per $100 invested, as detailed in Table D.10.

In summary, the fund-specific analysis of subjective beliefs suggests that investors tend to perceive excess cash flows with a negative skew. This is predominantly driven by deep-seated pessimism about PE cash flows. Interestingly, the secondary influential factor—public market cash flows—does not carry the same importance weight. This investigation implies that the observed undervaluation of PE funds primarily arises from investor skepticism regarding PE funds' cash flow-generating capacities, rather than any excessive optimism about their public market portfolios.

Table D.10: Discrepancy in Beliefs for PE and Market Cash Flows. The table provides statistics related to the *beliefs discrepancy* for PE and market cash flows, as visualized in Figure D.13.

	Beliefs Discrepancy for:	
	PE Cash Flows	Market Cash Flows
Underexpectations, % time	84.21%	48.68%
Overexpectations, % time	15.79%	51.32%
Max Discrepancy, $	0.88	1.92
Max Discrepancy - YQ	2014 Q2	2009 Q4
Min Discrepancy, $	-4.7	-2.26
Min Discrepancy - YQ	2011 Q3	2012 Q3
Avg Discrepancy, $	-0.97	0.04

The validity assessment of these fund-specific subjective beliefs, as detailed in Table D.11, confirms that the estimated beliefs align with survey data. Although the magnitude of correlations with PE sentiment indices has somewhat diminished, it remains statistically significant when juxtaposed with the main analysis results in Table D.7. It is worth noting that the correlation insights for non-core individual and institutional investors in the public market remain largely congruent with the baseline analysis. For both the Gallup and ESI surveys, the PE market beliefs narrative aligns with the public market, indicating positive correlation trends. However, the Shiller survey presents a potential anomaly: its statistically significant negative correlation may hint at the estimator's potential noisiness, which is somewhat echoed by the market cash flow findings earlier.

Table D.11: Correlations between Surveys and Excess Cash Flow. The table reports the correlations between various sentiment indices, presented column-wise, and belief-adjusted average excess cash flows. Subjective beliefs are estimated on a 'fund-by-fund' basis for the K-N CAPM SDF.

	Shiller (inst)	Shiller (ind)	Gallup	ESI	CEPECI	SVVCCI
K-N CAPM	0.02	-0.24**	0.50***	0.46***	0.57***	0.66***
Note:						*p<0.1; **p<0.05; ***p<0.01

While both methodologies (fund-by-fund versus average excess cash flow) produce similar findings, it is crucial to underscore some theoretical limitations connected with estimating subjective beliefs on an individual fund basis. The granularity afforded by the fund-by-fund method reveals cross-sectional nuances of investors' subjective beliefs. However, we spotlight three aspects that could critically distort the estimated beliefs and their economic implications when employing this novel approach: 1) the constraint of limited observations, 2) potential stationarity issues, and 3) the risk of overfitting.

First, the extraction of beliefs at the individual fund level intrinsically means that the analyses are often based on a constrained set of observations. This scarcity of data raises prominent concerns regarding the statistical reliability and robustness of the derived estimates. More succinctly, the limited data points might inhibit the estimator's ability to converge accurately to the true underlying measure.

Second, the ET methodology in use requires the stationarity of underlying processes. If the data exhibits non-stationary characteristics – such as trends, cycles, or other temporal patterns – the resultant estimates could be spurious, potentially leading to misleading inferences about the model-implied probabilities and corresponding subjective beliefs.

Third, engaging with a limited dataset invariably raises the risk of overfitting. In cases where the model excessively adapts to the specificities and noise within the data, like the fund-level cash flows, it might capture anomalous patterns rather than the true underlying dynamics. Consequently, a new approach trades the model's richer economic implications for the noisiness of beliefs' estimator.

Bearing these caveats in mind – nuances which we admit are not exhaustively explored or addressed in this paper – we proceed to examine the cross-sectional economic implications for PE funds using subjective beliefs estimated on a fund-by-fund basis.

Cross-Sectional Economic Implications.

The results of cross-sectional analysis for PE funds using subjective-beliefs primarily hinge on contrasting two time-series: the original cash-flows and their belief-adjusted counterparts. This comparison utilizes the concept of partial ordering between these two random variables via the notion of first-order stochastic dominance. In this context, both the original and belief-adjusted cash flows act as distinct random variables. The goal is to classify all funds into one of three categories based on this comparison: optimistic, pessimistic, and neutral. Such a classification provides insight into the LPs' sentiment at the time of capital commitment to the respective funds.

To fix ideas, we introduce the following notation. Let $CF^{\hat{P}}$ represent the original PE fund's cash flows as observed under the empirical probability measure, and $CF^{\hat{P}^*}$ symbolize the counterfactual cash-flows observed under model-implied probabilities, which reflect subjective beliefs about the fund's cash flows. we then define the *First-Order Stochastic Dominance* (FOSD) in terms of these cash flows:

Definition 1 (FOSD for Cash-Flows)**.** For any given cash-flows $F \equiv CF^{\hat{P}}$ and $G \equiv CF^{\hat{P}^*}$, F *first-order stochastically dominates* G if and only if

$$F_{CF}(cf) \leq G_{CF}(cf)$$

for all $cf \in CF$ with a strict inequality over some intervals. Here, $F_{CF}(cf)$ and $G_{CF}(cf)$ are the cumulative distribution functions (CDFs) of $CF^{\hat{P}}$ and $CF^{\hat{P}^*}$ respectively.

Using the above dominance criteria, we can categorize all funds into three mutually exclusive groups:

- **Optimistic Funds**: Here, $G_{CF}(cf)$ first-order stochastically dominates $F_{CF}(cf)$. All quantiles of the beliefs-adjusted cash-flows exceed those of the original series. This suggests that LPs anticipate that larger cash flows are more likely based on their subjective beliefs, indicating an optimistic outlook on future fund cash flows.
- **Pessimistic Funds**: In this case, $F_{CF}(cf)$ first-order stochastically dominates $G_{CF}(cf)$. All quantiles of the beliefs-adjusted cash-flows are less than those of the original series. This indicates that LPs, based on their beliefs, expect smaller cash flows, pointing to a pessimistic view of future fund cash flows.
- **Neutral Funds**: The *FOSD for Cash-Flows* criterion does not yield a conclusive result. Not all quantiles of the beliefs-adjusted cash-flows are consistently greater or smaller than the original series. This suggests that the LPs maintain a neutral outlook on the future cash flows of the fund.

To gain a deeper understanding of how subjective beliefs operate, consider the following simplified example. This illustration will shed light on the interaction between model-implied probabilities and subjective cash flows, as well as the mechanics of the FOSD criterion.

Assume a fund has three distinct cash flows, and we know the estimated model-implied probabilities:

1. **Original Cash Flows** (CF^P)

$$CF_i = CF_i^{\hat{P}_i} = \{5, 10, -5\}$$
$$\hat{P}_i = \{p_1, p_2, p_3\} = \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

2. **Belief-Adjusted Cash Flows** (CF^{P^*})

$$\hat{P^*} = \{p_1^*, p_2^*, p_3^*\} = \left\{\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right\} \quad \text{(estimated using ET methodology for fund } i\text{)}$$
$$T = \text{length of the cash flow series}$$
$$\begin{aligned} CF^{\hat{P^*}} &= \{p_1^* \cdot CF_1 \cdot T, p_2^* \cdot CF_2 \cdot T, p_3^* \cdot CF_3 \cdot T\} \\ &= \left\{\frac{2}{3} \cdot 5 \cdot 3, \frac{1}{6} \cdot 10 \cdot 3, \frac{1}{6} \cdot (-5) \cdot 3\right\} \\ &= \{10, 5, -2.5\} \end{aligned}$$

3. **FOSD**

$$F_{CF}(cf) = \{(Value; Prob)\} = \{(-5; \frac{1}{3}), (5; \frac{2}{3}), (10; 1)\}$$
$$G_{CF}(cf) = \{(Value; Prob)\} = \{(-2.5; \frac{2}{3}), (5; \frac{5}{6}), (10; 1)\}$$

The *FOSD for Cash-Flows* criterion does not yield a conclusive result. Therefore, Fund i is classified as **neutral**.

As demonstrated by this example, the *FOSD for Cash-Flows* is a particularly conservative measure of LPs' sentiments. This design choice was deliberate, aiming to mitigate the potentially noisy nature of the fund-by-fund estimator for investors' subjective beliefs.

We start the cross-sectional analysis by examining the distribution of fund types over time. This will offer insights into how the PE industry has evolved through the years. As depicted in Figure D.14, during the early stages of the PE industry, LPs were predominantly neutral about

fund cash flows. However, for funds with vintage years in the 1990s, LPs exhibited a markedly optimistic outlook on the PE industry. This trend shifted in the 2000s, coinciding with the aftermath of the first dot-com bubble burst. Specifically, for the 2000 vintage year, pessimistic LPs outnumbered their optimistic counterparts, registering at 17% versus 11%, respectively. Post-bubble, sentiments rebounded, and leading up to the GFC, LPs were primarily optimistic about PE funds. In contrast, post-GFC for funds launched in 2010, we observe a stark pessimism with 45% of LPs showing pessimistic sentiments as opposed to a mere 21% being optimistic. This shift indicates that following the GFC, PE investors were overwhelmingly pessimistic.

Another interesting trend is the consistent decline in neutral investors over time, suggesting a growing polarization in opinions among PE investors. This observation aligns well with the emergence of the secondary market around 2007–2009, as documented by Nadauld et al. (2019). A divergence in opinions is a critical precursor for trading—a concept that traces its roots back to Miller (1977).

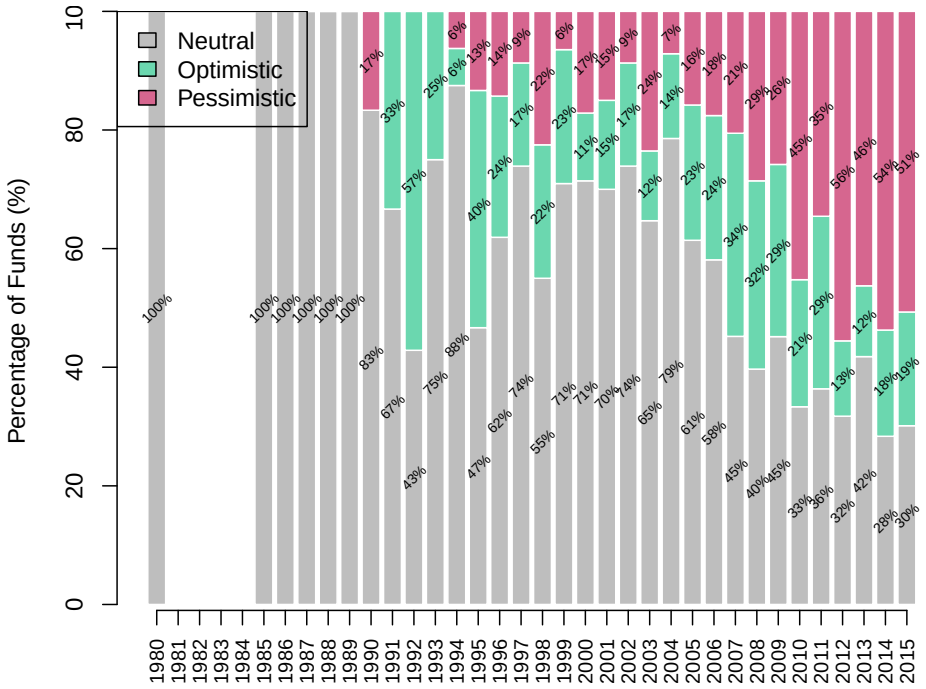


Figure D.14: Sentiment Distribution of Funds Across Vintage Years. The figure plots the distribution of LPs' sentiments (Optimism, Pessimism, or Neutral) regarding fund types, as deduced from quantile comparisons of belief-adjusted and original values. Each bar represents the distribution of sentiments for a specific vintage year, offering insights into how perceptions of different fund types have evolved over time.

We conduct a similar analysis in the context of geography. We focus on two key characteristics of funds: their primary geographical focus and the location of General Partners (GPs). For this

analysis, we aggregate all geographies into four regions: Asia, Europe, North America, and the rest of the world.

The analysis concerning geographical focus is presented in Figure D.15. As anticipated, North America dominates in terms of the number of funds, substantially outnumbering the other regions. Funds originating in this region are mostly neutral according to my *FOSD for Cash-Flows* criterion. This neutrality is observed across all geographical categories. Asia showcases the most uniform distribution, with the highest proportion of optimistic LPs: 15 neutral funds, 15 optimistic funds, and 14 pessimistic funds. Conversely, funds focused on the rest of the world—likely less developed nations—reveal a substantial proportion of pessimistic investors, accounting for 34% over all time periods. This observation aligns with common sense, as investments in less-developed countries often entail heightened legal risks, a factor of paramount significance in PE investments.

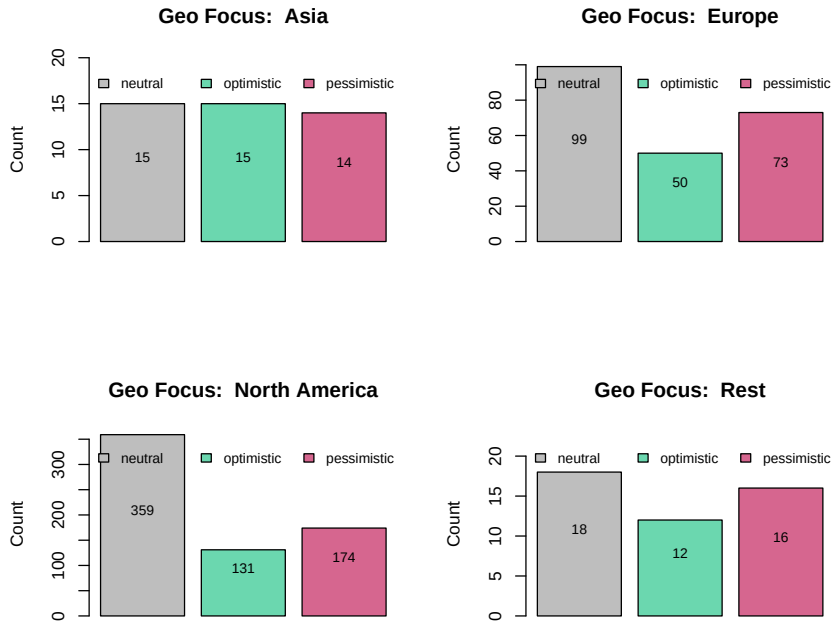


Figure D.15: Sentiment Distribution by Geographic Focus. The figure plots a breakdown of sentiments (Optimism, Pessimism, and Neutral) related to fund types, categorized by their primary geographic focus. Each subplot represents one of the enlarged regions - North America, Europe, Asia, and the Rest.

The distribution based on the locations of GPs is depicted in Figure D.16. North American GPs are highly sought after by investors. Mirroring the trend observed with geographical focus, investors maintain a predominantly neutral stance toward North American GPs. Intriguingly, both Europe and the rest of the world register the highest levels of pessimism concerning GPs' cash-flow

generating capabilities. This skepticism may be rooted in the fee structures proposed by GPs in these regions. In contrast, investors display the most optimistic views toward GPs located in Asia, with the proportion standing at 36%.

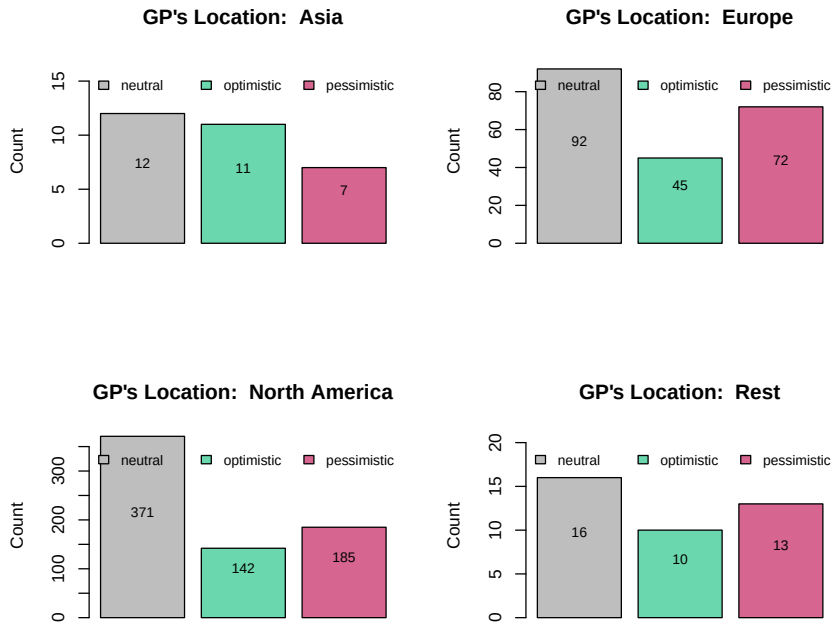


Figure D.16: Sentiment Distribution by GP's Location. The figure plots a segmentation of sentiments (Neutral, Optimistic, and Pessimistic) corresponding to fund types, arranged according to the location of their GP. Each subplot is dedicated to one of the aggregated regions - North America, Europe, Asia, and the Rest.

Finally, we analyze the industrial context of investing in PE. Before the analysis, we categorized funds into broader industrial groups. A detailed description of these categories can be found in Table D.12. The table presents the original Preqin industry classification in the second column, outlines which sub-industries each industry includes, and provides the rationale for consolidating these industries in the fourth column. As a result of this exercise, we identified six distinct consolidated industrial groups, which are listed in the first column: Tech and Media, Health and Services, Finance and Real Estate, Energy and Resources, Consumer and Industrial, and Diversified.

The sentiments of PE investors across these industries are depicted in Figure D.17. Most of the funds examined lean towards a diversified industrial focus, accounting for more than half of all funds. Investors are predominantly neutral about diversified funds. The Energy and Resources sector is viewed most pessimistically by LPs. About 41% of 17 initiated funds had investors who

Table D.12: Industry Groupings and Rationale. This table categorizes various industries into consolidated groups based on shared characteristics and functions. The rationale behind each grouping provides insight into the commonalities and dynamics that shape each category.

Consolidated Groups	Original Industries	Sub-Industries	Rationale
Tech & Media	Information Technology, Telecoms & Media	Electronics, Internet, IT Infrastructure, Software, Media, Telecoms	Both IT and Telecoms & Media companies are focused on the transmission, storage, and processing of data. Whether it's via hardware, software, or signals, these firms are central to the digital age, rendering their consolidation logical.
Health & Services	Healthcare, Business Services	Biotech, Healthcare IT, Pharmaceuticals, Business Support Services, Marketing/Advertising	While healthcare provides intangible services related to patient care, business services offer expert-driven intangible products/services. Both primarily deliver non-physical value, making them compatible.
Finance & Real Estate	Financial & Insurance Services, Real Estate	Financial Services, Insurance, Commercial Property, Real Estate Development	Both these sectors are pillars of the economy, dealing with assets, capital, and investments. Real estate, often financed through financial services, is intertwined with economic cycles, credit availability, and investor sentiment.
Energy & Resources	Energy & Utilities, Raw Materials & Natural Resources	Oil & Gas, Renewable Energy, Agribusiness, Chemicals, Mining	This consolidation is underpinned by the extraction, refinement, production, and distribution of natural resources, whether for energy generation or other purposes. These sectors form the backbone of primary industry operations, making their grouping cohesive.
Consumer & Industrial	Consumer Discretionary, Industrials	Consumer Products, Food, Retail, Aerospace, Defense, Industrial Machinery	Both sectors produce tangible goods, whether they're non-essential consumer items or goods used in construction and manufacturing. Their tangible output and close ties to consumer behavior and industrial activities underpin this grouping.
Diversified	Diversified	Multiple industries span this category.	As a standalone category, "Diversified" likely includes entities that span multiple industries, making it difficult to classify them under a specific sector.

expressed a pessimistic sentiment. In contrast, the Finance and Real Estate industry, despite its limited count of only 15 funds, had 40% of its funds characterized as optimistic. This is the only industry where optimistic funds outnumber both neutral and pessimistic funds. For other

industries, the sentiment analysis shows that investors are primarily neutral, with a slight leaning towards pessimism.

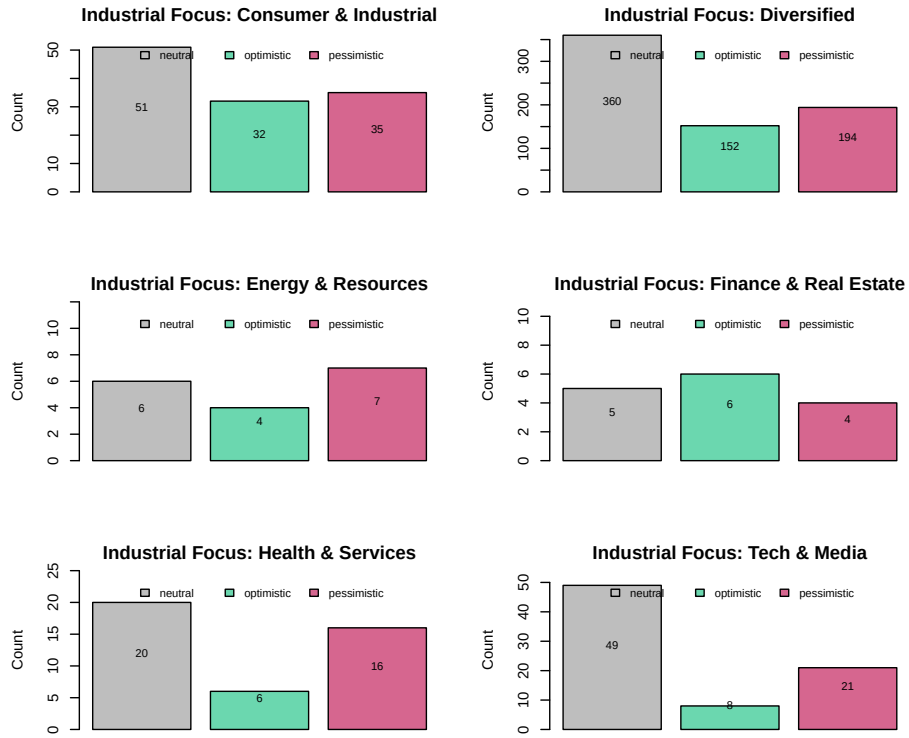


Figure D.17: Sentiment Distribution by Industrial Focus. The figure plots sentiment distributions (Neutral, Optimistic, and Pessimistic) across various enlarged industrial categories.

Further investigation into the Finance and Real Estate industry required an exploration of the temporal dynamics of pessimism-optimism patterns. These patterns are detailed in Figure D.18. The trends are quite telling. All years before the GFC were characterized by the inception of optimistic and neutral funds. However, 2008 saw a stark shift, and in the aftermath of the financial crisis, investors grew markedly pessimistic about the financial industry — a reaction that aligns with general sentiment. Notably, no funds initiated in the Finance and Real Estate industry after 2008 were identified as optimistic.

In conclusion, estimating investors' subjective beliefs on a fund-by-fund basis provides a fresh perspective distinct from the primary analysis. This approach permits a more detailed temporal exploration, as well as nuanced geographical and industrial analyses of the beliefs. These insights afford a deeper comprehension of the PE belief formation process and its implications for PE funds' cash flows. However, this novel subjective belief estimator may introduce a greater degree of noisiness compared to the estimator employed in the primary analysis. Nonetheless, this chapter

presents robust evidence that, at the aggregate level, the implications of beliefs do not critically diverge. Therefore, cross-sectional results can be seen as informative to a certain degree.

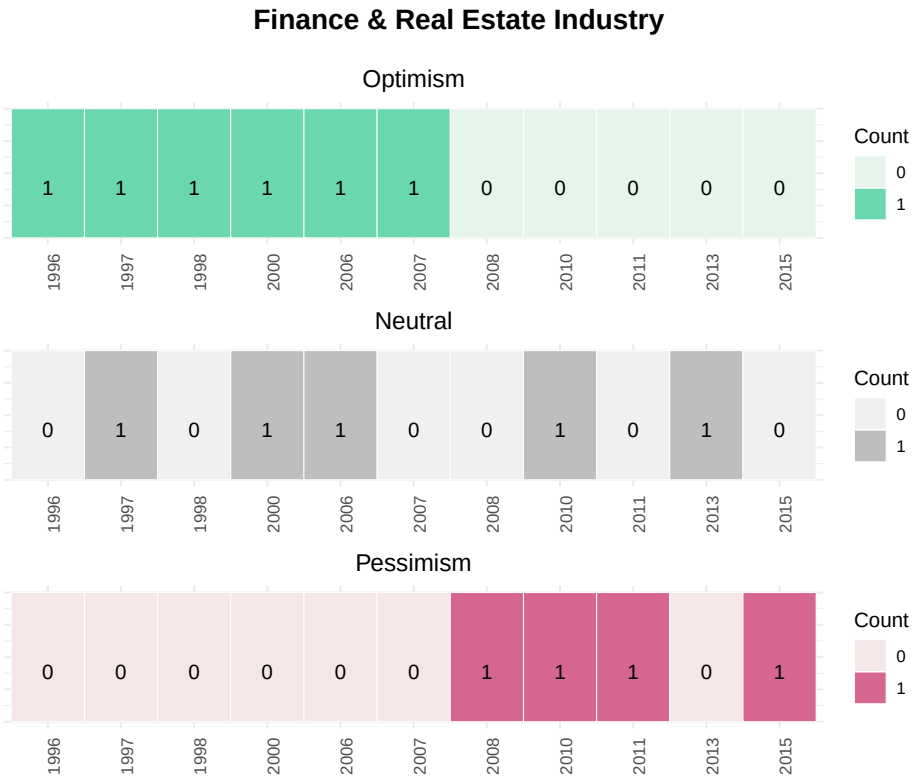


Figure D.18: Sentiment Distribution for 'Finance & Real Estate' Industry Across Vintage Years. The figure plots the sentiment distributions (Optimism, Neutral, and Pessimism) within the 'Finance & Real Estate' industry on a yearly basis. The distinct tiles for each year represent the count of funds, color-coded by their respective sentiment.