Optimal Retirement Policies with Private Annuities*

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[PRELIMINARY AND INCOMPLETE]

Abstract

This paper examines the optimal design of retirement policies that rely on private insurance markets while accounting for financial frictions within these markets. We analyze the impact of uninsurable aggregate interest rate shocks in a life-cycle model where the government screens productivity, longevity, and altruism. We show that observing retirees' annuitization choices helps separate individual types. However, private annuity markets face supply-side inefficiencies due to interest rate risk. To improve market efficiency, the government can issue long-term bonds, supported by interest rate-contingent taxes. Therefore, full privatization can only be optimal without interest rate shocks; otherwise, Social Security remains necessary.

JEL Codes: E21; E62; G32; H21; H55

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> "*I believe that the ongoing discussion about privatizing Social Security would benefit from paying more attention to fundamentals, rather than dwelling simply on nuts and bolts of privatization.*"
>
> — Milton Friedman, *Social Security Chimeras*, New York Times, January 11, 1999

1 Introduction

Over the past forty years, demographic shifts, medical advances, and declining long-term interest rates have placed significant strain on tax-funded social insurance programs, such as Social Security in the United States. This strain threatens both the sustainability of the system and the welfare of low-income individuals who rely on public pensions and Social Security income during retirement. In response, many policy proposals seek to increase retirees' reliance on their own savings and private-market solutions—essentially reducing public social insurance while encouraging self-insurance.

Proposals to cutback or eliminate Social Security in the U.S., such as President George W. Bush's 2005 plan to privatize the system with individual investment accounts or Paul Ryan's Roadmap for America's Future, which phases out benefits by raising the retirement age and means-testing, illustrate this shift towards increased self-insurance. However, these proposals, like much of the existing research, assume financial markets can efficiently supply the necessary private insurance, such as annuities or other forms of longevity coverage. In practice, private insurance markets face frictions—e.g., adverse selection and interest rate risk—that may limit the effectiveness of policies that rely heavily on self-insurance. Understanding the welfare implications of reducing public annuities, therefore, requires modeling these market imperfections explicitly.

Rather than focusing on the transition from pay-as-you-go systems to fully-funded accounts, as much of the Social Security reforms literature does, this paper examines an environment that is already fully funded. In doing so, we isolate the role of financial frictions—specifically, the challenges private insurers face when offering long-term annuities. We show that these frictions, which have been largely overlooked in the literature, are critical for evaluating the welfare implications of market-based retirement systems. When these fric-

tions are present, market-based solutions like private annuities require careful government intervention. In this context, public annuities such as Social Security continue to play a significant role in the optimal retirement system.

To formalize our analysis, we examine a life-cycle model with sequential, multidimensional private information and aggregate risk. In our model, individuals privately learn their productivity during their working years, followed by their longevity and level of altruism upon retiring. The economy faces an uninsurable aggregate interest rate shock after agents retire, creating incomplete markets. This shock is the source of our financial friction and demonstrates how interest rate risk increases the inefficiency of private annuities.

To derive the optimal private-annuity-based retirement policy, we develop a novel adaptation of the Mirrlees framework. In standard Mirrlees analysis, one first finds an abstract optimum and then determines an implementation. By contrast, we look for the optimum under a particular implementation. Specifically, we consider a retirement system that features three sources of income: private savings, Social Security, and private annuities, which we call the three-legged stool.[1] We show that this specific decentralization fails to implement the constrained-efficient optimum when life insurers face costly interest rate risk management. Thus, our three-legged-stool implementation can modify the classic Mirrlees planning problem by introducing new constraints or altering existing ones. Our approach resembles the literature on Mirrlees settings with exogenous policy restrictions, such as Heathcote and Tsujiyama (2021), who impose different parametric income tax functions. This paper, however, focuses on a specific implementation strategy rather than exogenous tax policies. In other words, although we restrict attention to a particular set of retirement income sources, we impose no restrictions on tax policy (i.e., income, savings, or bequest taxes). This flexibility allows us to formulate the optimal taxes and compute the maximal welfare achievable under the three-legged stool framework.

In the first part of our analysis, we abstract from interest rate risk to highlight the role of private annuities in screening for longevity risk and altruism types. In contrast to the existing literature, income-contingent Social Security and private savings alone are insufficient

[1]This implementation reflects the long-standing principle of balancing these three income sources, often referred to as the three-legged stool of retirement income (DeWitt, 1996).

to separate longevity and altruism types in our environment with multidimensional private information.[2] Intuitively, annuitization demand rises with longevity risk and falls with altruism levels. Consequently, policies that are contingent on both income and annuitization can achieve the optimum in this setting. Thus, our framework naturally supports reliance on private markets and does not disadvantage privatization in any way. Indeed, without the interest rate shock, only private savings and private annuities are required to implement the optimum; Social Security is not necessary, providing a theoretical justification for full privatization. Intuitively, despite the presence of adverse selection in the private annuity market, the government does not possess more information on longevity risk than private insurers, so public annuities such as Social Security offer no efficiency benefits. This result does not hold when private annuity insurers face interest rate risk.

Before formulating the optimal policy with interest rate risk, the second part of our analysis builds a realistic model of annuity insurers that encompasses adverse selecton and the cost of managing interest rate risk. We adopt the framework of Verani and Yu (2024), where competitive life insurers manage both interest rate risk and adverse selection. Life insurers manage interest rate risk by investing the proceeds from their annuity sales in an optimal portfolio of bonds with different maturities and choosing an optimal annuity price and capital structure—i.e., a level of net worth or equity. If the returns on long-term bonds are low due to frictions in the bond market, the duration of life insurers' assets is lower than the duration of their insurance liabilities. Consequently, a low realization of the interest rate shock during retirement increases the present value of an insurer's annuity liabilities more than the present value of its bond holdings, risking insolvency. To mitigate this, insurers optimally balance bond portfolios and capital structures to manage risk, holding precautionary net worth buffers and setting annuity prices above the actuarially fair value. This dynamic leads to an adverse feedback loop, where higher prices attract annuitants with longer life expectancies that increase the annuity interest rate risk.

We then explore how a government can mitigate financial frictions in the private annuity market with its debt maturity management. Intuitively, higher long-term government bond

[2] Lockwood (2018) showed that large observed savings can reflect either high longevity risk or a strong desire to leave a bequest.

issuance reduces interest rate risk for insurers, enabling them to offer annuities at lower prices. This is because long-term government bonds with high yields help life insurers hedge the interest rate risk associated with annuities. Lower annuity prices, in turn, decrease the cost of using private annuity demand to screen longevity and altruism types. However, long-term bond issuance exposes the government to interest rate risk. We show that, to absorb the interest rate risk in a fully-funded system, the government can back its long-term debt with interest rate-dependent lump-sum taxes on the retirees—i.e., agents are exposed to volatile post-retirement taxes.

In the third and main part of our analysis, we examine the optimal retirement policy that is decentralized through a private annuity market facing costly interest rate risk management. We start by formulating the social cost of private annuities, which encompasses both the insurers' risk management strategies and the government's debt maturity management. Then, taking inspiration from the Ramsey taxation approach, we adopt a novel approach to derive optimal policies that are decentralized through a private annuity market. Due to the financial frictions inherent in the private market, we find that the optimal approach is for the government to rely entirely on public annuities, redesigning them to be contingent on longevity risk and altruism types. However, given the challenges of reforming public insurance, our focus is on a setting where private annuity demand still plays a significant role in screening longevity and altruism types. In this third-best environment, the optimal retirement policy depends on private annuity demand. At the optimum, to minimize the social cost of managing interest rate risk, the annuity market is kept small, with policies incentivizing lower longevity types to purchase more annuities than higher longevity types—opposite to what would occur in a laissez-faire market. Additionally, standard results, such as the inverse Euler equation, no longer apply.

[Insert results from quantitative analysis here.]

1.1 Relation to the literature

The largely unexplored question we tackle in this paper concerns optimal retirement policies seeking to promote self-insurance over social insurance when financial market are not frictionless. In particular, we focus on the impact of financial frictions on the supply of

private annuities when designing optimal retirement policies—a gap in the existing literature on optimal retirement policies. Previous studies that demonstrated the welfare-enhancing role of Social Security, such as Hong and Ríos-Rull (2007) and Hosseini (2015), have generally ignored supply-side frictions, assuming frictionless financial markets where life insurers hedge interest rate risk at no cost and capital structure is irrelevant. Consequently, government debt management does not affect the supply of private annuities. Other related works, such as Hubbard and Judd (1987) and İmrohoroglu et al. (1995), have focused on the insurance properties of Social Security when private annuity markets are absent for exogenous reasons.

In contrast, our analysis emphasizes how financial frictions influence retirement policy reforms through their effect on the supply of private annuities. Recent studies have identified inefficiencies in the supply of private annuities (Koijen and Yogo, 2015; Verani and Yu, 2024). In particular, Verani and Yu (2024) show that supply-side inefficiencies exacerbate well-known market distortions from adverse selection (Friedman and Warshawsky, 1990; Finkelstein and Poterba, 2004). Our key contribution in this paper is the integration of a realistic model of life annuity insurers into an optimal policy framework. These insurers confront adverse selection and internalize the costs of managing interest rate risk associated with annuities. Given that long-term fixed coupon bonds beyond a two-year maturity are largely non-existent (Gajek and Ostaszewski, 2004) and private annuity markets are small (Mitchell et al., 1999), we show that government involvement in supplying debt facilitates idiosyncratic and aggregate risk sharing by aiding insurers in hedging interest rate risk.

This paper is also related to the extensive literature on Social Security reforms. A significant body of work has examined the transition from the current pay-as-you-go system to investment-based retirement accounts.[3] A key concern in this literature is the distribution of welfare gains across generations: Future generations typically benefit from the more efficient fully-funded system, often at the expense of current generations (Auerbach and Kotlikoff, 1987; Conesa and Krueger, 1999; Nishiyama and Smetters, 2007; McGrattan and Prescott, 2017). Our paper departs from this literature in two important ways. First, rather than focusing on transition dynamics, we analyze an environment that is already fully funded.

[3] Feldstein (2005a) and Feldstein (2005b) provide an overview of the arguments and concerns surrounding such transitions.

Thus, our work is not concerned with reforming the current system but instead examines the financial frictions—largely overlooked in the reforms literature—that could reduce the potential welfare gains of future policy changes. Second, our paper formulates the optimal policy, while much of the reforms literature—with the notable exception of Conesa and Garriga (2008) and Hosseini and Shourideh (2019)—has centered on the welfare implications of specific policy changes.

Among the optimal retirement policy design literature, the papers closest to ours are Hosseini and Shourideh (2019), Park (2021), and Bellofatto (2023). These papers, instead of conducting quantitative policy experiments on different proposals (like the aforementioned papers on Social Security reforms), adopt the Mirrleesian approach and focus on formulating the optimal policy without any constraints on the instruments. Hosseini and Shourideh (2019) explores Pareto improving retirement reforms that maintain at least status quo welfare for each agent, finding that savings taxes enhance production efficiency when productivity correlates with longevity risk, while a savings subsidy mitigates the absence of annuity markets. Park (2021) and Bellofatto (2023) also characterize optimal Mirrlees policies in the presence of longevity risk and bequest motives. However, in all three papers, agents' productivity is perfectly correlated with their longevity risk, rendering private annuities non-essential. In contrast, our sequential-multidimensional screening environment highlights the unique role of private annuities in the decentralization process, emphasizing supply-side frictions in annuity markets and its connection to the government's debt maturity structure.

Our paper is also related to the literature on optimal public debt maturity, including Angeletos (2002) and Buera and Nicolini (2004), which utilizes the Ramsey approach to characterize optimal debt maturity structures in incomplete markets with government expenditure shocks. Aparisi de Lannoy et al. (2022) emphasizes interest rate risk as a determinant of optimal government debt maturity structure, suggesting that the U.S. government should increase the duration of its debt to minimize rollover risk. In contrast, we focus on the issuance of long-term government bonds to mitigate inefficiencies in the private annuity market stemming from costly interest rate risk management, rather than raising government revenue. Additionally, while previous literature has concentrated on infinite horizon models, we restrict our attention to a single generation of finitely lived agents to formulate an opti-

mal fully-funded retirement system. Consequently, the government's balance sheet becomes crucial: the government must invest the proceeds from long-term bond sales in short-term assets to avoid intergenerational transfers.

Lastly, our paper relates to the literature studying the low annuitization rate in the U.S. This literature attributes low annuitization rate to bequest motives (Lockwood, 2018), behavioral biases (Brown et al., 2017; Gottlieb, 2018), and high annuity prices due to adverse selection (Eichenbaum and Peled, 1987; Friedman and Warshawsky, 1990; Finkelstein and Poterba, 2004) and other costs (Mitchell et al., 1999). Recent findings by Illanes and Padi (2021) indicate that high annuitization rates in Chile are a consequence of the absence of social insurance, such as Social Security. In contrast, our normative approach emphasizes the crucial role of both private and public annuities in enabling the government to better screen individuals and improve the efficiency-equity trade-off of a retirement system. In our policy framework, private savings and annuities are not meant to replace Social Security but to work alongside it.

2 Economic environment

We formulate a fully-funded retirement system centered around a single generation of agents. We consider a life-cycle model where agents are alive for at most four periods: $t = 0, 1, 2, 3$, and die with certainty at the end of the fourth period. Agents work in $t = 0$ and retire at $t = 1$. It is important to note that the final three periods—each period after retirement—is of the same length, but $t = 0$—the employment phase—is considerably longer.

At $t = 0$, agents privately learn their productivity $\theta \in \left[\underline{\theta}, \overline{\theta}\right] \subset \mathbb{R}_+$ and proceed to enter the labor force. Productivity θ is drawn from a differentiable distribution $F(\theta)$ with density $f(\theta)$. An agent with productivity θ who provides work effort n produces output $y = \theta n$ at $t = 0$. The government observes output y, but not productivity θ nor labor supply n. We refer to θ as the *ex-ante* private information.

Upon retirement at $t = 1$, agents privately learn their longevity risk $\mu \in \left[\underline{\mu}, \overline{\mu}\right] \subset \mathbb{R}$. $S_t(\mu)$ denotes the probability that agents survive to $t \geq 1$ at $t = 0$. We assume that $1 = S_1(\mu) > S_2(\mu) > S_3(\mu) > S_4(\mu) = 0$ for all μ. In other words, all agents survive till

the end of $t = 1$ and all of them die at the end of $t = 3$ with mortality risk in between. Furthermore, agents with higher longevity μ expect to live longer: $S_t(\mu)$ is strictly increasing in μ for $t = 2, 3$. Since productivity and longevity are correlated, we follow the sequential screening setup of Courty and Li (2000): μ is drawn from a distribution conditional on productivity $G(\mu|\theta)$ with density $g(\mu|\theta)$. Agents who are more productive are also likely to live longer: $\frac{\partial G(\mu|\theta)}{\partial \theta} < 0$ (Kitagawa and Hauser, 1973; Cristia, 2009).

At $t = 1$, agents also privately learn their altruism type $\alpha \in \{0, 1\}$ with probability $p \in (0, 1)$ that $\alpha = 1$. Therefore, agents possess multidimensional private information at $t = 1$. Notice that altruism α is independent of productivity θ and longevity risk μ. This implies that longevity risk μ is more informative of productivity θ than altruism type, raising the importance of screening the two different motivations for post-retirement savings. Upon death at the end of t, agents receive a warm glow of $\alpha v(x_t)$ from leaving a bequest of $x_t \geq 0$. The function v satisfies: $v', -v'' > 0$, with $\lim_{x\to 0} v(x) = -\infty$ and $\lim_{x\to 0} v'(x) = \infty$. We refer to agents with $\alpha = 1$ as *altruistic*, and those with $\alpha = 0$ as *non-altruistic*. Since the government does not observe longevity risk μ and altruism type α, we will refer to them as the *ex-post* private information.

Utility is additively separable across time and between consumption and leisure. Specifically, θ-agents at $t = 0$ have a flow utility of $u(c_0) - h\left(\frac{y}{\theta}\right)$, and agents continue to enjoy utility $u(c_t)$ from consumption as long as they live. Utility from consumption u is twice differentiable, strictly increasing and strictly concave $(u', -u'' > 0)$, and $u(0) = 0$. Disutility from working h is twice differentiable, strictly increasing and strictly convex $(h', h'' > 0)$, and $h(0) = 0$.

The length of period $t = 0$ is longer than the other periods, so the discount factor at $t = 0$ for future periods would also be different. Let $\beta_1 \in (0, 1)$ denote the discount factor at $t = 0$ and $\beta \in (0, 1)$ denotes the discount factor in other periods, while β_0 normalizes the utility in $t = 0$. All discount factors are determined by the annual discount factor and the number of years in each period.

We focus on an economy with exogenous (one-period) interest rates.[4] The return on

[4]The exogenous interest rates can be micro-founded by the existence of a one-period storage technology whose supply is perfectly elastic.

savings made after retirement at t is denoted as R_{t+1}. The return $R_3 \in \mathcal{R} = \left[1, \bar{R}\right]$ realized in $t = 2$ is a one-time aggregate shock drawn from distribution $\Pi(R_3)$ with density $\pi(R_3)$ and has an expected return of $\mathbb{E}(R_3)$. State-contingent debt is unavailable, so the economy is unable to insure against the interest rate shock. For simplicity, the other returns are known with $R_4 = (1-\omega)\mathbb{E}(R_3) + \omega R_3$ where $\omega \in [0,1]$ is the persistence of the interest rate shock. In the next section, we examine a model without the aggregate interest rate shock in $t = 2$, so $\Pi(R_3)$ is degenerate.

The discounted lifetime utility of θ-agents at $t = 0$ is

$$U_0(c_0, y; \theta) = \beta_0 \left[u(c_0) - h\left(\frac{y}{\theta}\right)\right] + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \begin{bmatrix} pU_1\left(\{c_t, x_t\}_{t\geq 1}; \mu, \alpha = 1\right) \\ +(1-p)U_1\left(\{c_t, x_t\}_{t\geq 1}; \mu, \alpha = 0\right) \end{bmatrix} dG(\mu|\theta),$$

where

$$U_1\left(\{c_t, x_t\}_{t\geq 1}; \mu, \alpha\right) = \int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu)\, w(c_t, x_t; \mu, \alpha)\right] d\Pi(R_3)$$

and

$$w(c_t, x_t; \mu, \alpha) = u(c_t) + \beta\left[1 - \frac{S_{t+1}(\mu)}{S_t(\mu)}\right] \alpha v(x_t).$$

Note that $1 - \frac{S_{t+1}(\mu)}{S_t(\mu)}$ is the probability of dying at the beginning of $t+1$ conditional on surviving till t for agents with longevity type μ.

The remainder of this section presents the incentive compatibility constraints and the planning problem. While we largely follow the Mirrlees framework, our focus on an implementation that relies on an inefficient annuity market introduces modifications to both the direct mechanism and the planning problem when insurers face costly interest rate risk management. Although we do not present these modifications in full detail here, we will indicate the components that are affected. A complete discussion of how costly interest rate risk management alters the government's problem is deferred to Section 6.

2.1 Incentive compatible direct mechanism

To characterize the constrained efficient allocation, we first formulate a direct truth-telling mechanism, which we use as a blueprint to decentralize the optimum in an economy with a

competitive private annuity market. The government has full commitment and designs the direct mechanism P defined as

$$P = \left\{ \begin{array}{c} [c_0(\theta), y(\theta)], \\ [c_1(\theta,\mu,\alpha), c_2(\theta,\mu,\alpha,R_3), c_3(\theta,\mu,\alpha,R_3)], \\ [x_1(\theta,\mu,\alpha,R_3), x_2(\theta,\mu,\alpha,R_3), x_3(\theta,\mu,\alpha,R_3)] \end{array} \right\}_{\theta\in[\underline{\theta},\overline{\theta}];\mu\in[\underline{\mu},\overline{\mu}];\alpha\in\{0,1\};R_3\in\mathcal{R}}.$$

In Section 6, we show that when life insurers face costly interest rate risk management, there exists an optimal balance between public and private annuities. Consequently, in a private-annuity-based implementation, the government also assigns both Social Security benefits and life annuities to agents in a direct mechanism. However, this is not relevant for the purposes of introducing the incentive compatibility constraints.

To simplify notation, we will express the utilities U_0 and U_1 as functions of an agent's reports and type. Let the utility of a (θ,μ,α)-agent who reports $\mu' \in [\underline{\mu},\overline{\mu}]$ and $\alpha' \in \{0,1\}$ in $t=1$ be denoted as

$$U_1(\mu',\alpha';\theta,\mu,\alpha) = \int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu)\, w_t(\mu',\alpha';\theta,\mu,\alpha,R_3) \right] d\Pi(R_3),$$

where $w_t(\mu',\alpha';\theta,\mu,\alpha,R_3) = w(c_t(\theta,\mu',\alpha',R_3), x_t(\theta,\mu',\alpha',R_3);\mu,\alpha)$. Furthermore, denote the utility from truth-telling at $t=1$ as $U_1(\theta',\mu,\alpha) = U_1(\mu,\alpha;\theta',\mu,\alpha)$. Note that at $t=1$, consumption c_1 is not a function of R_3, but bequest x_1 may depend on R_3. Denote the utility in $t=0$ of θ-agents who reported productivity θ' as

$$U_0(\theta';\theta) = \beta_0 \left[u(c_0(\theta')) - h\left(\frac{y(\theta')}{\theta}\right) \right] + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \left[pU_1(\theta',\mu,1) + (1-p)U_1(\theta',\mu,0) \right] dG(\mu|\theta).$$

Since agents learn their private information sequentially, it is without loss in requiring P to be incentive compatible for each period. There are two sets of incentive compatibility constraints. The first set of incentive constraints are the *ex-post incentive compatibility constraints*, which ensure the agents report their longevity risk μ and altruism α truthfully:

for any $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$, $\mu, \mu' \in \left[\underline{\mu}, \overline{\mu}\right]$, $\alpha, \alpha' \in \{0, 1\}$,

$$U_1(\theta, \mu, \alpha) \geq U_1(\mu', \alpha'; \theta, \mu, \alpha). \tag{1}$$

The ex-post incentive compatibility constraints (1) characterize the multidimensional screening problem in $t = 1$. By the dynamic revelation principle, the ex-post incentive constraints (1) only require truth-telling in $t = 1$ after truth-telling in $t = 0$ (Myerson, 1986).

At the optimum, the bequest for non-altruistic agents ($\alpha = 0$) is zero, i.e., $x_t(\theta, \mu, 0, R_3) = 0$ for all t. Since non-altruistic agents do not receive a warm glow from bequests, there is no gain in having them leave a strictly positive bequest. Also, by specifying the lowest possible bequest for non-altruistic agents, the ex-post incentive constraints (1) are relaxed.

The second set of incentive constraints are the *ex-ante incentive compatibility constraints*. They ensure that the agents report productivity θ truthfully at $t = 0$: for any $\theta, \theta' \in \left[\underline{\theta}, \overline{\theta}\right]$,

$$U_0(\theta) = U_0(\theta; \theta) \geq U_0(\theta'; \theta). \tag{2}$$

We say that the set of allocations P is incentive compatible if it satisfies both ex-ante and ex-post incentive compatibility constraints.

Appendix A shows how standard mechanism design tools can simplify the set of complicated ex-ante and ex-post incentive constraints by reducing them to their necessary conditions. It also shows how to simplify the post-retirement multidimensional screening problem.

2.2 Planning problem

Given our focus on implementations involving private annuity markets, the planning problem in our setting must account for the government's policy toward these markets. In contrast to most planning problems in the literature, ours maximizes welfare subject to the usual constraints while also choosing the policy towards the private annuity market.

The utilitarian government chooses the set of allocations P to maximize $\int_{\underline{\theta}}^{\overline{\theta}} U_0(\theta)\, dF(\theta)$ subject to the ex-post incentive constraints (1), the ex-ante incentive constraints (2), and

the resource constraints: for any $R_3 \in \mathcal{R}$,

$$\int_{\underline{\theta}}^{\overline{\theta}} \left\{ \frac{y(\theta) - c_0(\theta)}{R_0} - \int_{\underline{\mu}}^{\overline{\mu}} \begin{bmatrix} pK(\theta, \mu, 1, R_3) \\ +(1-p) K(\theta, \mu, 0, R_3) \end{bmatrix} dG(\mu|\theta) \right\} dF(\theta) \geq \frac{\Psi(R_3)}{\prod_{s=1}^{2} R_s},$$

where R_0 normalizes the allocations in $t = 0$ and R_{t+1} denotes the gross rate of return on savings made in t. Also, $K(\theta, \mu, \alpha, R_3)$ denotes the total discounted consumption for (θ, μ, α)-type agents after retirement and interest rate shock R_3 :

$$K(\theta, \mu, \alpha, R_3) = \sum_{t=1}^{3} \frac{S_t(\mu)}{\prod_{s=1}^{t} R_s} \left[c_t(\theta, \mu, \alpha, R_3) + \frac{1}{R_{t+1}} \left(1 - \frac{S_{t+1}(\mu)}{S_t(\mu)} \right) x_t(\theta, \mu, \alpha, R_3) \right].$$

Recall that R_3 is an aggregate uninsurable shock, so we require the resource constraint to hold for each realization of the interest rate shock R_3.

As part of the government's policy towards annuity markets, $\Psi(R_3)$ contains the lump-sum tax the government uses to finance the issuance of long-term debt, thereby lowering the cost of interest rate risk management for insurers, and the positive net worth the insurers use to manage its interest rate risk. In Section 6, we show that $\Psi(R_3)$ can be interpreted as a measure of welfare deviation from the constrained efficient optimum. In essence, it reflects the welfare cost associated with enhancing the efficiency of private annuity markets caused by financial frictions. Crucially, $\Psi(R_3)$ depends on the size of the annuity market, which is endogenously determined by our policy decentralization. Our implementation with private annuities is presented in Section 4 in a setting without interest rate shock R_3, serving as a benchmark since government debt policies cannot improve annuity market efficiency in the absence of aggregate shocks. Section 6 examines the setting with the shock R_3.

Furthermore, since we focus on an implementation that relies on an inefficient annuity market, the planning problem can include additional constraints that reflects the market frictions. Specifically, since $\Psi(R_3)$ is a function of annuity demand when insurers face costly interest rate risk management, the set of annuities that the planner can assign to agents may be restricted. By contrast, if the optimum is implemented through policies that do not rely on private annuities, then these additional constraints could potentially be avoided. This situation closely parallels the appearance of implementability constraints in classic Ramsey

taxation, which likewise arise when policy instruments are restricted to linear taxes; if tax instruments were unrestricted, those constraints would cease to bind. Section 6 derives the additional constraints that augment the planning problem.

To simplify the problem, we follow the literature and examine the relaxed planning problem by replacing the incentive constraints (1) and (2) with (16), (18), and(19), which are detailed in Appendix A. Given that the envelope conditions (18) and (19) are only necessary, we numerically verify the incentive compatibility of the optimum under the relaxed problem in Section 7.

2.3 Wedges

The optimum is characterized by the intertemporal and labor wedges, which provide the foundations for our optimal taxes. For the intertemporal wedges, we can define the *savings wedges* τ_t^s and *bequest wedges* τ_t^d :

$$\tau_t^s = 1 - \frac{u'(c_t)}{\beta_{t+1} R_{t+1} \mathbb{E}_t \left[u'(c_{t+1})\right]} \text{ and } \tau_t^d = 1 - \frac{u'(c_t)}{\alpha \beta_{t+1} R_{t+1} \mathbb{E}_t \left[v'(x_t)\right]}.$$

If $\tau_t^s > 0$ ($\tau_t^s < 0$), then savings in t is inefficiently low (high). Similarly, if $\tau_t^d > 0$ ($\tau_t^d < 0$), then bequests in t is inefficiently low (high). The savings and bequest wedges provide us with the implicit taxes on savings and bequests *conditional* on surviving and dying respectively.[5] We refer to the savings wedges τ_t^s after $t = 1$ as the *retirement savings wedges*.

The labor wedge in $t = 0$ characterizes the optimal distortions in the marginal rate of substitution between consumption and leisure, and it is defined as

$$\tau^w(\theta) = 1 - \frac{h'\left(\frac{y(\theta)}{\theta}\right)}{\theta u'\left(c_0(\theta)\right)}.$$

Since the equilibrium wage is equal to productivity θ in a competitive labor market, if $\tau^w > 0$ ($\tau^w < 0$), then there is an under-supply (over-supply) of labor given the market wage. The labor wedge also provides us with the implicit tax on income.

[5] Our definitions of the savings and bequest wedges follow from the ex-post intertemporal wedges of Park (2021) and Bellofatto (2023). Note that our definition is different from the savings wedge in Hosseini and Shourideh (2019), where it is not conditional on survival outcomes.

3 The three-legged stool retirement system

This section presents an implementation of a version of the three-legged stool: private annuities, private savings, and Social Security. The rest of the paper focuses on an optimum that can be decentralized using these three sources of retirement income. Section 4 characterizes the optimum in a setting without the interest rate shock R_3, while Section 6 examines the environment with the interest rate shock R_3. Our model of annuity insurers depends on whether the interest rate shock is present. We therefore defer its presentation until later, introducing one model for the case without the shock and another model, with explicit interest rate risk management, for the case with the shock in Section 5.

The fully-funded Social Security benefits only depend on lifetime income y and is primarily used as a post-retirement redistribution tool. Meanwhile, private annuities combined with private savings in the form of bonds help the government screen longevity risk μ and altruism type α. For our implementation, altruistic agents ($\alpha = 1$) only save through bonds while non-altruistic agents ($\alpha = 0$) purchase annuities. Our implementation highlights the unique role of each leg in the three-legged stool.

Upon retirement at $t = 1$, agents start receiving tax-funded income-contingent Social Security benefits $z(y) \geq 0$ in every period as long as they are alive. For simplicity, we focus on an implementation where agents only purchase one-period government bonds. This is analogous to an implementation with a sufficiently large tax on the returns to multi-period bonds held by agents.[6] In each period, agents can purchase one-period bonds denoted as b. One unit of the one-period bond returns $R_t \geq 1$ in t. Another financial asset available to the agents is annuities. Agents can also purchase private annuities at $t = 1$.[7] An annuity contract pays one unit of pre-tax consumption in each subsequent period the contract holder is alive in exchange for a lump-sum payment of q in $t = 1$.

Let b_{t+1} denote an agent's bond purchases or private savings in t. Social Security benefits, returns from private savings, and annuity payments at $t \geq 2$ are subject to a non-linear post-retirement income tax rate of $\tau_t^K\left(y, a, \{b_s,\}_{s=2}^t, R_3\right)$, which depends on pre-retirement

[6] In Section 5, we introduce two-period bonds and demonstrate their crucial role in helping life insurers manage interest rate risk. We also show that this allows the government to reduce annuity market inefficiencies through strategic debt management.

[7] We assume agents are restricted to purchasing annuities at $t = 1$.

income y, annuity holding a, the history of post-retirement private savings $\{b_s\}_{s=2}^{t}$, and the interest rate shock R_3 if $t \geq 2$. If agents are dead at t, their private savings are treated as bequests and are subject to a non-linear bequest tax rate of $\tau_t^B\left(y, a, \{b_s\}_{s=2}^{t}, R_3\right)$. To summarize, the retirement policy at $t = 1$ is

$$\hat{P}_1 = \left\{ z(y), \left[\tau_t^K\left(y, a, \{b_s\}_{s=2}^{t}, R_3\right), \tau_t^B\left(y, a, \{b_s\}_{s=2}^{t}, R_3\right)\right]_{t\geq 2, R_3 \in \mathcal{R}} \right\}.$$

Let $\tau_1^K(y)$ denote the tax rate on the returns to pre-retirement savings. Then, given income y and $\hat{P}_1$, an agent with ex-post type (μ, α) at $t = 1$ and $\left[1 - \tau_1^K(y)\right] R_1 b_1$ in the retirement savings account takes interest rates and annuity price q as given and solves

$$\max_{\{c_t, x_t\}_{t=1}^{3}, \{b_t\}_{t=2}^{4}, a} \int_{\mathcal{R}} \sum_{t=1}^{3} \beta^{t-1} S_t(\mu) \left[u(c_t) + \beta \left(1 - \frac{S_{t+1}(\mu)}{S_t(\mu)}\right) \alpha v(x_t) \right] d\Pi(R_3), \qquad \text{(P1)}$$

subject to

$$c_1 + b_2 + qa \leq \left[1 - \tau_1^K(y)\right] \left[R_1 b_1 + z(y)\right],$$

$$c_t + b_{t+1} \leq \left[1 - \tau_t^K\left(y, a, \{b_s\}_{s=2}^{t}, R_3\right)\right] \left[R_t b_t + a + z(y)\right], \text{ for } t \geq 2 \text{ and any } R_3 \in \mathcal{R},$$

$$x_t = \left[1 - \tau_{t+1}^B\left(y, a, \{b_s\}_{s=2}^{t+1}, R_3\right)\right] R_{t+1} b_{t+1}, \text{ for } t \geq 1 \text{ and any } R_3 \in \mathcal{R},$$

$$a \geq 0.$$

Agents are not allowed to sell annuities, i.e., annuity demand is positive. The value function of the decentralized problem at $t = 1$ is denoted as $V_1\left(y, b_1, \hat{P}_1; \mu, \alpha\right)$, which also depends on prices (R_2, q) but is omitted to streamline notation.

Savings b_1 at $t = 0$ are placed in a retirement savings account. Retirement account contributions come from pretax income (similar to a traditional 401(k)) and are only taxed upon withdrawal at a rate of τ_1^K in $t = 1$. Prior to retirement, agents face a non-linear labor income tax of $T^L(y - b_1)$. Let $\hat{P}_0$ denote the set of policies agents face while working in $t = 0 : \hat{P}_0 = \left[T^L(y - b_1), \tau_1^K(y, a, b_1)\right]$. Therefore, at $t = 0$, agents with productivity θ solve

$$\max_{c_0, y, b_1} \beta_0 \left[u(c_0) - h\left(\frac{y}{\theta}\right)\right] + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \begin{bmatrix} pV_1\left(y, b_1, \hat{P}_1; \mu, 1\right) \\ + (1-p) V_1\left(y, b_1, \hat{P}_1; \mu, 0\right) \end{bmatrix} dG(\mu|\theta), \qquad \text{(P0)}$$

subject to

$$c_0 + b_1 \leq y - T^L(y - b_1).$$

Let $\hat{P} = \left(\hat{P}_0, \hat{P}_1\right)$ denote the set of tax policies on agents.

An important observation is that although we focus on the three-legged stool implementation, our tax policies remain very general. In essence, for any optimal allocation of consumption, output, and bequests, we can construct a corresponding tax system that implements it. This generality allows us to isolate the effects of the interest rate shock on the optimum and understand how frictions in the annuity market—specifically, insurers' costly interest rate risk management—shape the equity-efficiency tradeoff and influence the structure of optimal taxes. Crucially, it also provides an upper bound on the welfare attainable under our three-legged stool retirement system.

Finally, as noted earlier, the supply of private annuities depends on the presence of the interest rate shock. Therefore, we introduce a separate model of annuity insurers for each setting: one with the interest rate shock and one without. The next section presents the optimum in an environment without the interest rate shock.

4 Optimal policy without interest rate shocks

To focus on how the sequential and multidimensional private information affects the optimum, this section examines a setting without aggregate shocks, where $\Pi(R_3)$ is degenerate. We focus on the implementation with private annuities introduced in Section 3. By abstracting from the interest rate shock, we can cleanly demonstrate how policies that depend on annuity demand help with the multidimensional screening problem. We can also demonstrate that a fully-funded retirement system relying exclusively on the private market can achieve the optimum in this specialized environment. For notational simplicity, we drop the dependence on R_3 for allocations in $t \geq 2$. Furthermore, in this section, $\Psi(R_3) = 0$ for all $R_3 \in \mathcal{R}$.

4.1 Theoretical characterization

This subsection outlines the properties of the optimum in an environment without aggregate shocks. The results here serve as a benchmark for the main results in Section 6, where we examine the setting with interest rate shocks. While we do not provide a full characterization of the wedges here, we relate the optimum to established results in the literature while highlighting our unique features. See Appendix C for a complete characterization and detailed analysis of the wedges.

4.1.1 Pre-retirement optimum

In our model, agents work for a single period, and their productivity is not dependent on any other private information. Consequently, the qualitative properties of the labor wedge mirror those in static Mirrlees models, where productivity is the only private information held by agents (Mirrlees, 1971; Diamond, 1998; Saez, 2001).

The optimal intertemporal distortions at $t = 0$ are governed by the inverse Euler equation (Golosov et al., 2003, 2006), which is a standard feature in models with dynamic private information. The inverse Euler equation implies that all agents' savings in $t = 0$ are taxed to reduce the cost of screening future private information, i.e., $\tau_0^s > 0$.

In Section 6, we examine the optimum when the government decentralizes through private annuity insurers with costly interest rate risk management. We argue that while the theoretical characterization of the labor wedges remains unchanged, the inverse Euler equation no longer holds in this environment.

4.1.2 Post-retirement optimum

Before discussing the bequest and retirement savings wedges, first consider two benchmarks: when only ex-post private information (μ, α) is observable, and when only ex-ante private information θ is observable.

If ex-post private information (μ, α) is observable, the uniform-taxation result of Atkinson and Stiglitz (1976) holds. The government does not need to introduce any distortions in the

intertemporal wedges after retirement, i.e., $\tau_t^s = \tau_t^d = 0$ for all agents at any $t \geq 1$.[8]

If ex-ante private information θ is observable, the full-information efficient optimum is implementable despite the fact that μ and α are unobservable. The full-information efficient optimal allocation fully insures all agents against differences in longevity and altruism: All agents share the same consumption and bequest in any t. Similar to Harris and Raviv (1978), retirement savings and bequests are undistorted when only productivity is publicly known. These two benchmarks imply that post-retirement intertemporal distortions are caused by the presence of dynamic private information.

Part of the distortions to retirement savings at the optimum are driven by ex-ante incentive constraints (2). Recent studies have demonstrated how distortions in retirement savings can reduce the cost of screening productivity θ (Hosseini and Shourideh, 2019; Park, 2021; Bellofatto, 2023).[9] Specifically, since more productive agents are expected to live longer, they prefer a more back-loaded consumption stream. As a result, it is optimal to front-load the retirement consumption of less-productive agents in order to discourage productive types from under-reporting their productivity in $t = 0$.

While recent studies have examined similar environments, the literature generally assumes a simpler framework where productivity θ, longevity μ, and altruism α are perfectly correlated. By contrast, our model diverges from this assumption by exploring a sequential screening setting in which longevity is positively, but not perfectly, correlated with productivity. As a result, retirement consumption varies due to differences in ex-post private information (μ, α) among agents with the same productivity θ. Since agents are risk averse, these differences tighten the ex-ante incentive constraints (2), which introduces an additional economic force that works to reduce consumption differences after retirement.

Another source of distortion in retirement savings arises from the post-retirement multidimensional screening—a distinct feature of our model. In other words, the retirement savings wedges are also shaped by ex-post incentive constraints on altruism (16), which are meant to

[8]The absence of intertemporal distortions after retirement does not imply insurance against longevity differences. Consumption and bequest differ across longevity types, since longevity is correlated with productivity. However, consumption is the same across altruism types for fixed productivity and longevity, because altruism is independent of productivity.

[9]In a related paternalistic setting, Paluszynski and Yu (2023) showed that distortions in savings during the employment phase can incentivize present-biased agents to invest in education before entering the workforce.

prevent non-altruistic agents from misreporting as altruistic. Consequently, for a fixed pair of productivity and longevity types, the retirement savings wedges for non-altruistic agents are smaller than the ones for altruistic agents.

The same logic that applies to retirement savings wedges extends to the bequest wedges of altruistic agents. As Park (2021) and Bellofatto (2023) show, bequests are also distorted to lower the cost of incentivizing pre-retirement labor supply. Specifically, more productive agents, who tend to live longer, assign less value to bequests. Therefore, optimal bequest wedges complement the retirement savings wedges: When retirement savings are taxed ($\tau_t^s > 0$), bequests are subsidized ($\tau_t^d < 0$). Furthermore, due to multidimensional screening, bequests may be further subsidized to discourage non-altruistic agents—who place no value on bequests—from pretending to be altruistic.

In our model, the planner assigns zero weight to future generations of agents, so the distortions in bequests are not driven by the planner's welfare criterion. This contrasts with other works that abstract from longevity risk and attribute bequest taxes or subsidies to variations in generational-Pareto weights or differing welfare criteria (Farhi and Werning, 2010, 2013; Piketty and Saez, 2013).[10]

4.2 Decentralization

This subsection continues the three-legged-stool implementation presented in Section 3. First, we present a model of the annuity insurers in a setting without interest rate risk. Then, we define the tax-distorted competitive equilibrium for our implementation and show that it can decentralize the optimum. Finally, we explain why public annuities or Social Security can be eliminated without any welfare loss in the setting without interest rate shock.

4.2.1 Supply of life annuities

Consider a model with multiple identical life insurers supplying annuities. The annuity market is competitive and life insurers compete over annuity prices, i.e., Bertrand competition. Let $a(\theta, \mu, \alpha, q)$ denote the endogenously determined demand for annuities of

[10] Pavoni and Yazici (2017) is a notable exception. They showed that bequests are taxed in cases when parents are more patient than their offspring.

type (θ, μ, α) agents at $t = 1$ when annuity price is q, and $\hat{a}(\theta, \mu, q) = pa(\theta, \mu, 1, q) + (1-p)\, a(\theta, \mu, 0, q)$. Also, let $L_t(q) = \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\mu}}^{\overline{\mu}} S_t(\mu)\, \hat{a}(\theta, \mu, q)\, dG(\mu|\theta)\, dF(\theta)$ denote the annuity liability in t for a given annuity price of q.

Without the interest rate shock R_3, life insurers do not face any uncertainty. Therefore, similar to Einav and Finkelstein (2011) and Hosseini (2015), the competitive annuity price $\tilde{q}$ is characterized by

$$\tilde{q} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\mu}}^{\overline{\mu}} \hat{a}(\theta, \mu, \tilde{q})\, dG(\mu|\theta)\, dF(\theta) - \frac{1}{R_2}\left[L_2(\tilde{q}) + \frac{L_3(\tilde{q})}{R_3}\right] = 0. \tag{3}$$

From (3), the annuity price $\tilde{q}$ is equal to the average cost of issuing an annuity contract. Importantly, note that the price $\tilde{q}$ includes the cost of adverse selection.

In Section 5, we introduce the interest rate shock R_3 and present a model of life insurers managing the aggregate risk in an incomplete market by choosing the appropriate capital structure and portfolio of bonds.

4.2.2 Competitive equilibrium

The following definition of the competitive equilibrium for our environment is standard.

Definition 1 *The annuity market competitive equilibrium without interest rate risk is a set of allocations* $\left\{y, [c_t]_{t\geq 0}, [x_t]_{t\geq 1}, a, [b_t]_{t\geq 1}\right\}$, *a set of taxes* $\hat{P}$, *and a set of prices* $\left\{[R_t]_{t\geq 1}, q\right\}$ *such that: (i) given policies and prices, agents choose allocations to solve (P0) and (P1), (ii) given policies and prices, life insurers choose a competitive annuity price* q *according to (3), and (iii) the goods, labor, and annuity markets clear.*

By Walras' law, the government budget constraint holds when all markets clear, so it is omitted from Definition 1. The next proposition shows that the constrained efficient optimum without interest rate shock can be decentralized in a competitive equilibrium with retirement policy $\hat{P}$.

Proposition 1 *Tax policy* $\hat{P}$ *implements the constrained efficient optimum without interest rate shock in a competitive equilibrium.*

4.2.3 Optimality of full privatization

A closer inspection of (P1) reveals that adjustments to the post-retirement savings taxes $\tau_t^K\left(y, a, \{b_s\}_{s=2}^t\right)$ for $t \geq 2$ can ensure that the tax policy $\hat{P}$ continues to implement the constrained efficient optimum in a competitive equilibrium under a fully privatized retirement system. Intuitively, Social Security benefits $z(y)$ crowd out private savings b_t for altruistic agents and reduce the purchase of private annuities a by non-altruistic agents. Consequently, when benefits $z(y)$ are removed, agents reoptimize their choices for private savings and private annuities. Post-retirement savings taxes can be adjusted simultaneously to account for these changes, ensuring that the optimum is still implemented.

Corollary 1 *Tax policy $\hat{P}$ with $z(y) = 0$ for all income y implements the constrained efficient optimum without interest rate shock in a competitive equilibrium.*

Corollary 1 demonstrates that in an environment free from interest rate risk or financial frictions, reforms that eliminate Social Security can implement the constrained efficient optimum in the long run. In essence, a carefully designed set of retirement policies can achieve the optimum within a fully-funded system that relies entirely on private markets. Importantly, it shows that the presence of adverse selection in the annuity market does not prevent full privatization. The reason is that the government does not have better information on the agents' longevity risk than the insurers. Hence, as long as the annuity markup is zero, the optimum can be decentralized without relying on public annuities. However, this result breaks down when we introduce an incomplete market environment with interest rate shocks.

4.3 Importance of observing annuity demand

This subsection illustrates how post-retirement multidimensional private information naturally justifies focusing on implementations that use private annuities as a source of retirement income. This implies that our model does not disadvantage policy reforms aimed at increasing self-reliance in any way. In fact, it identifies an economic role for private annuities, rather than relying on ideological reasons for favoring market-based implementations.

To illustrate the importance of annuity demand in the design of optimal retirement policies, consider an implementation with only private savings. In this scenario, agents

differing in longevity and altruism types may choose to save the same amount. Therefore, the government is unable to screen longevity types by only observing retirement savings.

Consider agents who earned the same income, so they have the same amount of after-tax wealth (including Social Security) in $t = 1$, which is denoted by Y. The only savings instrument available is the one-period bond b_2. Let the tax rates on savings be denoted as $\tau_t^K\left(\{b_s\}_{s=2}^t\right)$ and bequests as $\tau_t^B\left(\{b_s\}_{s=2}^t\right)$. The taxes are general and are only required to be differentiable.

Let (c_1^*, b_2^*) denote the optimal consumption-savings decision for non-altruistic agents with longevity risk μ in $t = 1$, which satisfies the Euler equation

$$u'(c_1^*) = \beta S_2(\mu) W'_\mu(b_2^*),$$

where $W_\mu(b_2^*)$ is the value function of non-altruistic agents with longevity risk μ in $t = 2$.

Now, we will show that (c_1^*, b_2^*) also solves the utility maximization problem of altruistic agents with longevity risk $\tilde{\mu} \neq \mu$. If (c_1^*, b_2^*) is the optimal consumption-savings decision, then it must satisfy the altruistic agents' Euler equation

$$u'(c_1^*) = \beta S_2(\tilde{\mu}) \hat{W}'_{\tilde{\mu}}(b_2^*) + \beta\left[1 - S_2(\tilde{\mu})\right] R_2 \left\{ b_2^* \frac{\partial \tau_2^B(b_2^*)}{\partial b_2} - \left[1 - \tau_2^B(b_2^*)\right] \right\} v'(x_1^*),$$

where $x_1^* = \left[1 - \tau_2^B(b_2^*)\right] R_2 b_2^*$ and $\hat{W}_{\tilde{\mu}}(b_2^*)$ is the value function of altruistic agents with longevity risk $\tilde{\mu}$ in $t = 2$. As a result, from the non-altruistic and altruistic agents' Euler equations, there may exist altruistic agents (if $\tilde{\mu} \in \left[\underline{\mu}, \overline{\mu}\right]$) who share the same amount of retirement savings as non-altruistic agents with a different longevity risk.

As shown by Dynan et al. (2002, 2004) and Lockwood (2018), savings through bonds or other assets that do not depend on survival outcomes can serve both as a bequest and as precautionary savings for longevity. Consequently, agents with low longevity risk but a high preference for altruism may have the same savings as agents with high longevity risk and a low preference for altruism. Since productivity is correlated with longevity, the government can enhance production efficiencies by screening agents' longevity types through properly designed longevity-dependent retirement policies. This follows from the well-known principle

that, to incentivize highly productive agents to work, a distortion should be introduced on goods preferred by them but consumed by less productive agents (Golosov et al., 2013).[11]

On the other hand, if a private annuity market was present, then agents with high longevity risk and a low preference for altruism would prefer to buy more annuities than bonds. Meanwhile, agents with low longevity risk and a high preference for altruism would prefer bonds over annuities. Therefore, by observing the demand for private annuities and bonds, the government can identify the motives for saving and screen the longevity and altruism types to provide better work incentives.

More generally, the government needs to observe the demand of at least two distinct savings instruments to implement the constrained-efficient optimum. Even if none of the savings instruments are survival contingent, following Kopczuk (2003), the government can design the estate taxes in such a way that one of the assets effectively becomes survival-contingent. However, the choice of which asset to make survival-contingent would be arbitrary, especially since a market for life annuities already exists. Alternatively, a reform to the public pension system to allow for a choice in the annuitization amount could also achieve the optimum, an option we explore in Section F.1.

A potential concern with a decentralization that relies on private annuities is the inefficiency of the annuity market, which likely contributes to its small size. The next section examines a life annuity model with interest rate risk, where the annuity markup includes the costs of adverse selection and insurers' interest rate risk management. We will show how government debt policies can mitigate the costs associated with managing interest rate risk.

5 Private life annuity market with interest rate risk

In this section, we turn to the environment with the interest rate shock R_3 and examine how it affects the annuity market. We first present a model of private annuities which features life insurers using non-contingent bonds of different maturities to manage interest rate risk.

[11] Hosseini and Shourideh (2019), Park (2021), and Bellofatto (2023) have used this principle to characterize the optimal retirement policies. In these models, the longevity risk and preference for bequests are perfectly correlated with productivity. As a result, by solely observing the bond investments, the government can clearly distinguish the agents' motives for saving.

Next, we show how the issuance of long-term government bonds determines the interest rate risk sharing scheme between the government and private annuity insurers, providing a novel theory of government debt maturity management.

While the results of the previous section highlighted the importance of private annuity markets, this section will demonstrate that decentralizing through these markets is costly due to the burden of managing interest rate risk. Our novel insights on the private annuity market form the foundation for our main analysis in Section 6.

5.1 Supply of life annuities and insurer's balance sheet

Following Verani and Yu (2024), we extend the setup of 4.2.1 and present a portable annuity model with interest rate risk management.

Life insurers use bonds to transfer wealth across periods. The bonds available are one-period bonds b_t, and zero-coupon two-period bonds l_3. For simplicity, we abstract from corporate bonds and focus on an economy where the government is the sole supplier of bonds. Life insurers do not short-sell or issue bonds, so $b_t \geq 0$ and $l_3 \geq 0$. Recall that the short rates R_t are exogenously determined while the long rate R_l is endogenous, but we restrict attention to the case where $R_l \geq R_2$.[12] The bond market is considered *constrained* when $\frac{1}{R_l} > \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$, i.e., the return from the two-period bond is low, and *unconstrained* when $\frac{1}{R_l} \leq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$.

Life insurers are protected by limited liability, which means that the owners of a life insurer are not liable for corporate losses in excess of the value of the insurer's assets. This friction implies that insurers may fail to honor their annuity payments if they become insolvent. Specifically, if the two-period bond market is constrained, then the duration of annuity liabilities may exceed the duration of insurers' assets. This duration mismatch can lead to insolvency when interest rate shock R_3 is sufficiently low. Verani and Yu (2024) showed how life insurers manage this interest rate risk by financing a positive net worth, which acts as a buffer, with the annuity markup.

For simplicity, our analysis will focus on a representative life insurer. At $t = 1$, the

[12] Note that if the long-term bond returns are too low, $R_l < R_2$, then the insurers would never purchase long-term bonds to manage interest rate risk.

representative insurer invests its annuity considerations in a portfolio of bonds:

$$b_2 + l_3 = q \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\mu}}^{\overline{\mu}} \hat{a}(\theta, \mu, q)\, dG(\mu|\theta)\, dF(\theta). \tag{4}$$

The insurer's balance sheet at $t = 1$ is given by

$$b_2 + l_3 = \frac{1}{R_2}\left[L_2(q) + \mathbb{E}\left(\frac{1}{R_3}\right) L_3(q)\right] + NW_1, \tag{5}$$

where the first two terms on the right-hand side of (5) are the present value of expected annuity liabilities and NW_t denotes the net worth at t. At $t = 2$, the aggregate shock R_3 is realized, and the insurer's balance sheet becomes:

$$b_3(R_3) = \frac{L_3(q)}{R_3} + NW_2(R_3), \tag{6}$$

where

$$b_3(R_3) = R_2 b_2 + \frac{R_l l_3}{R_3} - L_2(q). \tag{7}$$

Note that the insurer's demand for one-period bonds and its net worth in $t = 2$ depend on the realization of the interest rate R_3. Equation (7) shows that the insurer finances the purchase of one-period bonds $b_3(R_3)$ using the proceeds from the returns of the initial bond holdings net of the annuity payments made to the surviving individuals in $t = 2$.

Note how (6) and (7) show that the insurer risks becoming insolvent in $t = 2$ if the present value of its liabilities exceeds the present value of its assets for certain realizations of R_3. We assume that an insurance regulator or a credit rating agency requires the insurer to remain solvent for all possible future interest rates. Therefore, the representative limited-liability life insurer has an incentive to manage interest rate risk by choosing an asset portfolio, an annuity price, and a capital structure, such that $NW_2(R_3) \geq 0$ for all R_3.

5.1.1 Interest rate risk management

This section shows how the representative life insurer constructs asset portfolios and capital structure to manage interest rate risk. The following definition defines the insurer's

optimal interest rate risk management strategy in a competitive annuity market.

Definition 2 *A competitive interest rate risk management strategy provides a set of asset allocation* $\{b_2, l_3, b_3(R_3)\}$ *and capital structure* $\{NW_1, NW_2(R_3)\}$ *for all* R_3 *such that (i) insurers remain solvent:* $NW_t \geq 0$*, and (ii)* NW_1 *is minimized.*

By Definition 2, an interest rate risk management strategy specifies the asset allocation and capital structure. The first part of Definition 2 ensures that the interest rate risk management strategy keeps the insurer solvent. Alternatively, $NW_t \geq 0$ can be interpreted as a limited liability constraint. The second part of Definition 2 ensures that the insurer remains competitive in a Bertrand environment. From (4) and (5), the insurer can lower the annuity price when it decreases NW_1. The following proposition characterizes the unique competitive interest rate risk management strategy.

Proposition 2 *For a given annuity price* q*, the unique competitive interest rate risk management strategy satisfies:*

i. Asset portfolio:

$$l_3 = \frac{L_3(q)}{R_l},\ b_2 = \frac{L_2(q)}{R_2} \text{ and } b_3(R_3) = \frac{L_3(q)}{R_3}.$$

ii. Capital structure:

$$NW_1 = \left[\frac{1}{R_l} - \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3(q) \text{ and } NW_2(R_3) = 0.$$

In particular, $NW_t = 0$ *for all* t *when the bond market is unconstrained and* $NW_1 > 0$ *otherwise.*

Proposition 2 provides us with two important lessons on interest rate risk management. The first lesson of Proposition 2 is that the insurer prefers to use more net worth to manage interest rate risk when the long-term bond yield R_l is low. In essence, the insurer requires higher net worth in a more constrained bond market. The second lesson is that, when the bond market is constrained, the insurer maintains a strictly positive level of net worth to decrease the duration of its total liabilities (because net worth has a lower duration than the

insurance liabilities) and eliminate its duration gap. Furthermore, Proposition 2 shows that the competitive interest rate risk management strategy is unique.

5.1.2 Annuity pricing

Life insurers engage in Bertrand competition. Therefore, by (4) and (5) and Proposition 2, the competitive equilibrium annuity price with interest rate risk management q^* is characterized by

$$q^* = \frac{\frac{L_2(q^*)}{R_2} + \frac{L_3(q^*)}{R_l}}{\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\mu}}^{\overline{\mu}} \hat{a}\left(\theta, \mu, q^*\right) dG\left(\mu|\theta\right) dF\left(\theta\right)}. \tag{8}$$

The annuity price q^* differs from the price $\tilde{q}$ in a standard adverse selection model without financial frictions, as shown in (3). However, when the bond market is unconstrained, the optimal net worth is zero. This implies that the cost of interest rate risk management is also zero, making the annuity price q^* equivalent to that in a standard adverse selection model without financial frictions. In essence, similar to Einav and Finkelstein (2011) and Hosseini (2015), the annuity price $q^* = \tilde{q}$: The competitive equilibrium annuity price equals the average cost of issuing an annuity contract in an unconstrained bond market.

In contrast, in a constrained bond market, the annuity price q^* differs from $\tilde{q}$ because insurers must cover the average cost in the form of expected annuity liability per policyholder plus the strictly positive average net worth, which is absent in traditional annuity pricing models. In the next section, we will show how the government can issue more long-term debt to lower the cost of interest rate risk management for insurers.

5.2 Government debt maturity management

This section presents a novel theory for government debt maturity management based on the interaction between government debt and frictional markets. Specifically, we show how the government can alleviate the private annuity insurers' interest rate risk through the supply of long-term bonds. However, the management of interest rate risk is costly for the government as well. This implies that there exists an efficient interest rate risk sharing scheme between the government and the insurers, which is summarized by the long-term

bond yield. We will theoretically analyze the impact of the government's debt maturity management in the next section, and its effects on welfare in the quantitative section.

We adopt the term structure model of Vayanos and Vila (2021). Recall that the short rates (one-period bond returns R_t) are exogenous, and the long rate (two-period bond return R_l) is endogenously determined. For simplicity, we assume that there are only two types of long-term bond investors: preferred-habitat investors and foreign arbitrageurs.[13] Preferred-habitat investors have preferences for specific maturities. The annuity life insurers are the only preferred-habitat investors in our economy. Their total two-period bond demand, given by Proposition 2, is $\frac{1}{R_l} L_3(q^*)$.

A set of risk-averse foreign arbitrageurs with mass Ω reside outside of the economy, so they are not considered in the welfare calculations. The foreign arbitrageurs invest in both long- and short-term bonds. We assume that, given the long and short rates, the arbitrageurs have long-term bond demand $H(R_l)$, which is differentiable and strictly increasing: $H'(R_l) > 0$. Arbitrageurs can buy or short sell long-term bonds, i.e., H can be positive or negative. Specifically, if we assume that R_l is higher (lower) than the expected return of rolling over short-term bonds, then arbitrageurs purchase (short) long-term bonds: $H(R_l) > 0$ when $R_l > R_2\mathbb{E}(R_3)$ and $H(R_l) < 0$ when $R_l < R_2\mathbb{E}(R_3)$.

Note that the government does not face a limited liability constraint, so it has an advantage over the representative life insurer in managing interest rate risk. Instead, we assume the government can immunize itself against interest rate risk through an interest-rate dependent lump-sum tax $T(R_3)$ at $t = 2$.[14] Specifically, immediately after the interest rate shock is realized, the government levies a tax $T(R_3)$ on agents and invests the proceeds to pay off the long-term debt in $t = 3$. Therefore, the government's bond revenue constraint at $t = 3$ satisfies the following: for all $R_3 \in \mathcal{R}$,

$$R_3 T(R_3) + \Lambda(R_2 R_3 - R_l) \geq 0, \tag{9}$$

where Λ is the supply of long-term government bond. Note that if $\Lambda > 0$, the government

[13] As in Section 3, we restrict attention to an implementation where retired agents only purchase annuities or short-term bonds.

[14] Although the government can fulfill its debt obligation by selling more long-term bonds, we restrict attention to taxes since our implementation focuses on a fully-funded retirement system.

issues long-term bonds, while the government purchases long-term bonds when $\Lambda < 0$.

Finally, in equilibrium, the long-term bond return R_l is determined by a standard bond market clearing condition at $t = 1$:

$$\Omega H\left(R_l\right) + \frac{L_3\left(q^*\right)}{R_l} = \Lambda, \tag{10}$$

where the equilibrium annuity price q^* is defined by (8).

As a demonstration, Appendix D follows Greenwood et al. (2010) and Vayanos and Vila (2021) to derive the equilibrium long-term bond return R_l when the supply Λ is exogenous and arbitrageurs have mean-variance preferences. Also, Appendix E shows how the government's debt policy can help reduce the insurers' interest rate risk management cost in a simplified setting without considering the agents' welfare. The next section explores how the government's debt maturity structure and its welfare optimization problem interact.

6 Retirement policy with interest rate shock

This section revisits the planning problem with interest rate risk. We aim to characterize the additional welfare loss that arises from the costly interest rate risk management by private life insurers.

6.1 Social cost of insurers' interest rate risk management

This section presents the deadweight loss associated with decentralizing through private annuity markets, i.e., the social cost of managing interest rate risk. From Section 5, the deadweight loss from managing interest rate risk in an incomplete market setting consists of the life insurers' net worth NW_1, and the lump-sum taxes $T\left(R_3\right)$ raised by the government in $t = 2$ to finance long-term bonds. Using notation from Section 2, this is expressed as $\Psi\left(R_3\right) = R_2 NW_1 + T\left(R_3\right)$.

The optimal net worth is given by Proposition 2. Meanwhile, by (9) and the long-term bond market clearing condition (10), the lump-sum tax can be rewritten as $T\left(R_3\right) = \left(\frac{R_l}{R_3} - R_2\right)\left[\Omega H\left(R_l\right) + \frac{L_3\left(q^*\right)}{R_l}\right]$, where the equilibrium annuity price q^* is given by (8). Note

that since only non-altruistic agents buy annuities in our implementation, the annuity liability in $t=3$ is $L_3(q^*) = (1-p)\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\mu}}^{\overline{\mu}} S_3(\mu)\, a(\theta,\mu,0,q^*)\, dG(\mu|\theta)\, dF(\theta)$. To understand the tax $T(R_3)$, recall that the total demand for long-term government bonds comprises the demand from arbitrageurs $\Omega H(R_l)$ and life insurers $\frac{L_3(q^*)}{R_l}$, while $\frac{R_l}{R_3} - R_2$ measures the relative attractiveness of long-term bonds compared to short-term bonds given R_3.

As a result, the social cost of insurers' interest rate risk management is

$$\Psi(R_3) = \begin{cases} \left(\frac{R_l}{R_3} - R_2\right)\Omega H(R_l) + \left[\frac{1}{R_3} - \mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3(q^*) & \frac{R_2}{R_l} \geq \mathbb{E}\left(\frac{1}{R_3}\right) \\ \left(\frac{R_l}{R_3} - R_2\right)\Omega H(R_l) + \left(\frac{1}{R_3} - \frac{R_2}{R_l}\right) L_3(\tilde{q}) & \frac{R_2}{R_l} < \mathbb{E}\left(\frac{1}{R_3}\right) \end{cases}. \tag{11}$$

Theoretically characterizing the optimum is challenging since the social cost $\Psi(R_3)$ is non-differentiable in long-term bond return R_l. The next lemma simplifies $\Psi(R_3)$ by focusing on the case when the population of arbitrageurs Ω is sufficiently small.

Lemma 1 *If Ω is sufficiently small, then at the optimum: (i) $\frac{R_2}{R_l} \geq \mathbb{E}\left(\frac{1}{R_3}\right)$, and (ii) $\Psi(R_3) = \left(\frac{R_l}{R_3} - R_2\right)\Omega H(R_l) + \left[\frac{1}{R_3} - \mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3(q^*)$.*

To understand Lemma 2, note that increasing R_l beyond $R_2/\mathbb{E}\left(\frac{1}{R_3}\right)$ does not reduce the financial distortions caused by private annuity insurers. This is because, by Proposition 2, the equilibrium annuity price q^* remains at the actuarially fair level $\tilde{q}$, and annuity liability $L_3(q^*)$ stays unchanged as R_l increases. Instead, such an increase primarily raises the lump-sum taxes needed to cover the larger long-term debt. While higher R_l could provide some benefits through arbitrageurs, these gains are outweighed by the costs imposed by insurers when the population Ω of arbitrageurs is small.

6.2 Competitive equilibrium and implementability

Since the social cost of interest rate risk depends on the size of the annuity market, the optimum is sensitive to the government's annuity policy. To illustrate the impact of costly interest rate risk management on the optimum, we choose to characterize the optimum under an extension of the decentralization presented in Section 3. In particular, it is important to recall that only non-altruistic agents purchase private annuities in our decentralization.

As in our previous theoretical analysis, this section employs a direct truth-telling mechanism to characterize the optimum. The social cost $\Psi(R_3)$ creates a natural trade-off between public and private annuities within our three-legged stool implementation. Thus, we need to augment the direct mechanism P to include private annuity demand of non-altruistic agents $a(\theta,\mu)$ and the income-contingent Social Security benefit $z(\theta)$.[15] In other words, the costly interest rate risk management faced by insurers imposes constraints on the amount of private annuities and Social Security benefits the government can assign to each type of agent.[16]

Specifically, the choice of $a(\theta,\mu)$ is constrained by the competitive market. Thus, before examining the direct mechanism, we present the following definition of the competitive equilibrium based on the implementation in Section 3, which incorporates the interest rate risk management strategy chosen by competitive life insurers from Section 5.

Definition 3 *The annuity market competitive equilibrium with interest rate risk is a set of allocations* $\left\{y,[c_t]_{t\geq 0},[x_t]_{t\geq 1},a,[b_t]_{t\geq 1}\right\}$, *the insurers' interest rate risk management strategy* $\left\{b_2,l_3,NW_1,[b_3(R_3),NW_2(R_3)]_{R_3\in\mathcal{R}}\right\}$, *a set of tax and long-term government bond policies* $\left\{\hat{P},\Lambda,[T(R_3)]_{R_3\in\mathcal{R}}\right\}$, *and a set of prices* $\left\{[R_t]_{t\geq 1},R_l,q\right\}$ *such that: (i) given policies and prices, agents choose allocations to maximize their utility in* $t=0$ *and* $t=1$, *(ii) given policies and prices, life insurers choose a competitive interest rate risk management strategy and set price* q *according to (8), and (iii) the goods, labor, annuity, and bond markets clear.*

From Definition 3, in an annuity market competitive equilibrium, the annuity demand of non-altruistic agents satisfy the following constraints: for all $\theta\in\left[\underline{\theta},\overline{\theta}\right]$ and $\mu\in\left[\underline{\mu},\overline{\mu}\right]$,

$$qu'\left(c_1(\theta,\mu,0)\right)\left[a(\theta,\mu)+z(\theta)\right] = \beta\int_1^{\overline{R}}\sum_{t=2}^{3}\beta^{t-2}S_t(\mu)\,u'\left(c_t(\theta,\mu,0,R_3)\right)c_t(\theta,\mu,0,R_3)\,d\Pi(R_3). \quad (12)$$

Simple manipulations of the first-order conditions from the non-altruistic agents' utility

[15] We drop the dependence of annuity demand on price q in the direct mechanism.

[16] Unlike Golosov and Tsyvinski (2007), the government observes annuity purchases. Consequently, in our direct mechanism, the government can determine the amount annuitized through the private market based on the reported types. Indeed, recall that the rationale for adopting an implementation that incorporates private annuities is precisely because, by observing private annuity demand, the government can effectively screen for longevity and altruism types.

maximization problem, combined with the sequential budget constraints, yield (12), where annuity price q satisfies (8).

We refer to (12) as the *implementability constraint*, since it is similar in spirit to the well-known constraint in Ramsey taxation.[17] The implementability constraint in Ramsey taxation places a restriction on the set of allocations that can be implemented in a competitive equilibrium taxes are linear. Here, the implementability constraints limit the set of allocations that can be decentralized through the private annuity market, not only due to the social cost $\Psi(R_3)$ from private annuities but also, as we will see, because Social Security benefits are independent of longevity risk μ. Also, since private and public annuities are both subject to the same taxes, the implementability constraint contains the income- or productivity-dependent Social Security benefit $z(\theta)$.

A crucial aspect of the implementation with private annuities, which is not captured by the implementability constraints (12), is that the non-linear retirement savings taxes $\left\{\tau_t^K\right\}_{t\geq 1}$ depend on annuity demand. Annuity demand reflects the underlying longevity type μ. Therefore, in addition to the implementability constraints, the optimum must also satisfy a set of *separation constraints*: for all $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and $\mu, \mu' \in \left[\underline{\mu}, \overline{\mu}\right]$,

$$a(\theta, \mu) \neq a(\theta, \mu'). \tag{13}$$

These separation constraints (13) require non-altruistic agents with the same productivity but different longevity types to have distinct demands for private annuities. However, the constraints are weak in the sense that they do not restrict the annuity demand $a(\theta, \mu)$ to be strictly increasing or decreasing with respect to longevity risk μ. The next lemma shows that the solution to the planning problem with the new constraints can be decentralized in an annuity market competitive equilibrium with interest rate risk.

Lemma 2 *Given allocations that satisfy the incentive compatibility constraints, (1) and (2), the implementability constraints (12), the separation constraints (13), and the resource constraints, we can construct policies and prices, which together with the given allocations,*

[17] In contrast to the standard implementability constraint, where the taxes and prices in the consumer budget constraint are substituted out by using the marginal rate of substitutions, (12) uses the sequential budget constraint to substitute out the taxes in the first-order conditions.

constitute an annuity market competitive equilibrium with interest rate risk.

Lemma 2 follows directly from the taxation principle. It demonstrates that the optimal solution under our three-legged stool retirement system can be obtained by analyzing the original planning problem with an augmented direct mechanism that incorporates private annuities for non-altruistic agents $a(\theta, \mu)$ and income-contingent Social Security benefits $z(\theta)$. Our new planning problem also includes the implementability constraints (12), the separation constraints (13), and the new resource constraints. The lemma further confirms that reliance on the private annuity market introduces additional distortions stemming from the new constraints and the social cost $\Psi(R_3)$, which reflects the insurers' costly interest rate risk management.

The separation constraints are challenging to handle analytically, so we will proceed without explicitly incorporating them in our theoretical analysis. However, in our quantitative exercise, we will verify that these constraints are automatically satisfied.

In Appendix F, we consider two alternative implementations that avoid the cost of interest rate risk management by private annuity insurers. In the first, the interest rate risk management cost is avoided by relying entirely on a reformed public annuity that depends on both income and longevity risk. In the second, insurers transfer the interest rate risk to consumers by offering variable life benefits, i.e., annuity benefits that adjust with the interest rate shock. Appendix F shows that both implementations lead to non-binding implementability constraints and zero social cost ($\Psi(R_3) = 0$), thereby achieving higher welfare than our three-legged stool retirement system. However, both implementations have drawbacks, which we discuss in the appendix.

6.3 Theoretical characterization

This section characterizes the optimal demand for private annuities, the long-term government bond yield, and the optimal savings wedges. Let the shadow price on the resource constraint be an interest rate-dependent multiplier $\phi(R_3)\pi(R_3)$, reflecting market incompleteness, with $\Phi = \mathbb{E}[\phi(R_3)]$.

6.3.1 Optimal private annuity demand

When decentralizing through private annuities, there is an optimal degree of private annuitization. To understand how the optimal annuity demand is determined, it is useful to define the following objects:

$$\hat{q}(\mu) = \frac{S_2(\mu)}{R_2} + \frac{S_3(\mu)}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right) \text{ and } \Delta(R_3) = \frac{\frac{1}{R_3} - \mathbb{E}\left(\frac{1}{R_3}\right)}{\mathbb{E}\left(\frac{1}{R_3}\right)}.$$

Note that $\hat{q}(\mu)$ is the efficient annuity price, i.e, the price that competitive life insurers charge when longevity type μ is observable and when the bond market is unconstrained. Finally, $\Delta(R_3)$ is the percentage change in the interest rate from its mean.

Theorem 1 *For any productivity θ and longevity risk μ, if $a(\theta,\mu) > 0$, the optimal demand for private annuities of non-altruistic agents satisfies*

$$q\beta_1 R_1 u'(c_1(\theta,\mu,0))\psi(\theta,\mu) + \eta(1-p)[\hat{q}(\mu) - q] = \frac{(1-p)S_3(\mu)}{R_l}\mathbb{E}\left[\frac{\phi(R_3)}{\Phi}\Delta(R_3)\right], \quad (14)$$

where q satisfies (8),

$$\eta = \frac{\beta_1 R_1 \mathbb{E}[\psi(\theta',\mu')u'(c_1(\theta',\mu',0))a(\theta',\mu')]}{(1-p)\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\mu}}^{\overline{\mu}} a(\theta',\mu')dG(\mu'|\theta')dF(\theta')} > 0,$$

and $\psi(\theta,\mu)$ satisfies $\mathbb{E}[u'(c_1(\theta,\mu',0))\psi(\theta,\mu')|\theta] = 0$.

Theorem 1 shows that the optimal annuity demand for non-altruistic (θ,μ) agents is shaped by two key factors: the marginal effect of private annuities on the social cost $\Psi(R_3)$ and the binding implementability constraints. Both factors are driven by the interest rate shock and the absence of longevity-dependent benefits z.[18]

To see how social cost $\Psi(R_3)$ affects annuity demand, by Lemma 1, the right-hand side of (14) represents the rise in expected social cost from a marginal increase in the annuity demand of non-altruistic (θ,μ) types. Since the aggregate interest rate shock R_3 is uninsurable, the

[18]From Theorem 1, in the absence of an interest rate shock ($\Delta(R_3) = 0$), the implementability constraint becomes non-binding ($\psi(\theta,\mu) = 0$). Consequently, Theorem 1 reduces to the results presented in Section 4.

marginal social cost is weighted by the interest rate-dependent shadow price of the resource constraint $\phi(R_3)$. Crucially, since a fall in R_3 raises both social cost $\Psi(R_3)$ and the present value of consumption, the resource constraint is tighter for low interest rate realizations: $\phi(R_3)$ is decreasing in R_3. This implies that $\mathbb{E}\left[\frac{\phi(R_3)}{\Phi}\Delta(R_3)\right]$ is strictly positive. As a result, an increase in annuity demand from any type of agents would raise the expected social cost of interest rate risk management. However, note that the marginal social cost is lower for low-longevity types, so the next corollary shows that the optimal annuity demand is not monotonically increasing in longevity risk μ.

Corollary 2 *For some productivity θ and longevity types $\mu > \mu'$, $a(\mu, \theta) < a(\mu', \theta)$.*

In standard models of private annuity insurance with adverse selection, agents with higher longevity risk (higher μ) purchase more annuities than those with lower longevity risk (Eichenbaum and Peled, 1987; Finkelstein and Poterba, 2004; Hosseini, 2015). Verani and Yu (2024) demonstrated that adverse selection increases insurers' interest rate risk management costs. Specifically, as adverse selection raises the expected survival probability of the annuitant pool, the duration of annuity liabilities lengthens, leading to higher costs in managing interest rate risk. Since the government is absorbing some of the risk management cost on behalf of the insurers through issuing long-term government bonds, Corollary 2 indicates that the government reduces the social cost of interest rate risk management by incentivizing some low-longevity types to buy more annuities than the high-longevity types. Despite the surprising result in Corollary 2, it is important to recall that as long as annuity demands differ across longevity types, the separation constraints (13) are satisfied.

The left-hand side of (14) represents the effect of the implementability constraint (12) on annuity demand. The Lagrange multiplier on (12) is $\psi(\theta, \mu)$, which captures the utility cost for non-altruistic (θ, μ) types when the government decentralizes through the private annuity market without longevity-dependent benefits z. By Theorem 1, $\psi(\theta, \mu)$ can be positive or negative. A positive multiplier $\psi(\theta, \mu) > 0$ indicates that, if the implementability constraint (12) can be relaxed, the government would increase the annuity demand of non-altruistic (θ, μ) agents, which presumably lowers the annuity demand of other non-altruistic agents with the same productivity θ but different longevity μ. Conversely, if $\psi(\theta, \mu) < 0$, the

government would reduce the annuity demand of non-altruistic (θ, μ) agents. In summary, the non-zero multiplier $\psi(\theta, \mu)$ arises because benefits $z(\theta)$ are the same for agents with the same productivity, regardless of their ex-post types.[19]

Specifically, the left-hand side of (14) shows that the implementability constraint (12) affects annuity demand through two channels. The first term on the left-hand side represents the direct effect, and the second term captures the indirect effect through equilibrium annuity price q. Following the logic outlined above, the direct effect recommends a decrease in the annuity demand of non-altruistic (θ, μ) agents when $\psi(\theta, \mu)$ is negative and an increase when $\psi(\theta, \mu)$ is positive. The indirect effect represents the marginal change in the equilibrium annuity price when the annuity demand of non-altruistic (θ, μ) agents change. The magnitude of the indirect effect is governed by $\eta > 0$, which represents the average total utility cost of decentralizing through the private annuity market without longevity-contingent public benefits. The following corollary provides more structure to the relationship between annuity demand and $\psi(\theta, \mu)$.

Corollary 3 *For any productivity θ, there exists $\mu^h, \mu^l \in \left(\underline{\mu}, \overline{\mu}\right]$ such that $\mu^h > \mu^l$ with (i) $\psi(\theta, \mu) > 0$ for all $\mu \leq \mu^l$, and (ii) if μ is such that $\psi(\theta, \mu) \leq 0$, then $\mu > \mu^h$.*

Corollary 3 follows directly from (14) and the fact that the efficient annuity price $\hat{q}(\mu)$ is strictly increasing while $\mathbb{E}\left[\frac{\phi(R_3)}{\Phi}\Delta(R_3)\right]$ is strictly positive. Corollary 3 illustrates how the relationship between the efficient annuity price $\hat{q}(\mu)$ and equilibrium price q determines the utility cost of implementation $\psi(\theta, \mu)$. Specifically, for low-longevity types $(\hat{q}(\mu) < q)$, the utility cost of implementation is strictly positive $(\psi(\theta, \mu) > 0)$. As mentioned before, this implies that the government wishes to increase the annuity demand of the low-longevity types for reasons related to the result in Corollary 2. Conversely, the utility cost of implementation can be strictly negative $(\psi(\theta, \mu) < 0)$ only for sufficiently high longevity types, indicating the government aims to decrease their demand.

[19] The logic here is similar to the dynamic implementability constraints in Aiyagari et al. (2002), where they derive the optimal Ramsey taxes in an incomplete market setting.

6.3.2 Optimal long-term government bond yield

Underlying the optimal annuity demand is the delicate balance between public and private annuities for non-altruistic agents, i.e., $a(\theta,\mu)$ and $z(\theta)$. On the one hand, private annuities are useful for screening the longevity risk of non-altruistic agents. On the other hand, an increase in annuity demand increases interest rate risk management cost which in turn raises the social cost. Therefore, private annuities are essential in our decentralization, but they cannot fully substitute for public longevity insurance, such as Social Security.

Theorem 2 shows how the government can strike the optimal balance between private and public longevity insurance through the long-term bond return R_l.

Theorem 2 *The optimal long-term bond return R_l satisfies*

$$\left\{\frac{1}{R_l}-\frac{1}{R_2}\mathbb{E}\left[\frac{\phi(R_3)}{\Phi R_3}\right]\right\}\epsilon^H=\frac{1}{R_2}\mathbb{E}\left[\frac{\phi(R_3)}{\Phi R_3}\right]+\frac{q\beta_1 R_1\mathbb{E}\left[\psi(\theta',\mu')\,u'(c_1(\theta',\mu',0))\,a(\theta',\mu')\right]}{\Omega R_l H(R_l)}\epsilon^q,$$

where q satisfies (8) and the elasticities are defined as $\epsilon^H=\frac{R_l H'(R_l)}{H(R_l)}$ and $\epsilon^q=-\frac{\partial q}{\partial R_l}\frac{R_l}{q}$.

To understand Theorem 2, it is crucial to first note that Lemma 1 establishes that, in equilibrium, the return on long-term bonds satisfies $R_l < R_2\mathbb{E}(R_3)$, which implies arbitrageurs are shorting the long-term bond: $H(R_l) < 0$.[20] The underlying intuition is as follows: By lowering R_l, the government not only reduces its debt obligations to arbitrageurs but also induces arbitrageurs to short-sell long-term bonds when $R_l < R_2\mathbb{E}(R_3)$. This short-selling behavior reduces the government's pressure to supply the long-term assets used by life insurers to hedge their interest rate risk. As a result, this economic force decreases the long-term bond yield R_l. However, there is a trade-off: A lower R_l also reduces the aggregate supply of long-term bonds Λ available in the market, thereby limiting the government's borrowing capacity. Theorem 2 formalizes this trade-off by showing that even in the absence of life insurers, there exists an optimal long-term bond yield R_l, determined by the arbitrageurs' elasticity of bond demand ϵ^H with respect to R_l. Intuitively, a higher elasticity ϵ^H reflects more responsive arbitrageurs, which reduces the downward pressure on R_l, leading to a higher equilibrium bond yield.

[20] Since Lemma 1 implies that $\frac{1}{R_l}\geq\frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$, we have $R_l < R_2\mathbb{E}(R_3)$ by Jensen's inequality.

Theorem 2 also demonstrates how decentralizing through the private annuity market further reduces the long-term bond yield R_l. In our framework, the primary role of the private annuity market is to identify the longevity types of non-altruistic agents. Otherwise, the substantial costs associated with managing interest rate risk make decentralizing through the private annuity market undesirable. Theorem 2 shows that the optimal R_l is pushed further downwards to increase the equilibrium annuity price q. A higher q reduces the demand for private annuities, allowing the government to partially mitigate the welfare loss caused by financial frictions. Intuitively, the effect is more pronounced when the elasticity of the annuity price with respect to the long-term interest rate ϵ^q is high, as it leads to a greater reduction in R_l.

6.3.3 Optimal wedges with private annuities

The wedges in our environment are different from those in Section 4.1—the setup without aggregate shocks. Although the labor wedge and the post-retirement wedges for altruistic agents remain mostly unchanged, it is important to remember that—for our setup with interest rate risk—the allocations after $t = 1$ depend on the interest rate R_3.

Due to the financial frictions in the private annuity market, the inverse Euler equation no longer holds and the retirement savings wedges for non-altruistic agents differ from the ones in Section 4.1. The deviation from the inverse Euler equation depends on the the type-dependent consumption-based pricing of the annuity for non-altruistic agents, which is defined as

$$P(\theta, \mu, 0) = \int_1^{\overline{R}} \sum_{t=1}^{2} m_t(\theta, \mu, 0, R_3)\, c_{t+1}(\theta, \mu, 0, R_3)\, d\Pi(R_3),$$

where

$$m_t(\theta, \mu, 0, R_3) = \beta^t S_{t+1}(\mu) \frac{u'(c_{t+1}(\theta, \mu, 0, R_3))}{u'(c_1(\theta, \mu, 0))}$$

denotes the t-period ahead stochastic discount factor at $t = 1$ for a non-altruistic agent with productivity θ and longevity risk μ under interest rate R_3. The price $P(\theta, \mu, 0)$ provides the value of uncertain retirement consumption flow $\{c_{t+1}\}_{t\geq 1}$ to non-altruistic agents of type (θ, μ) at $t = 1$. The following theorem presents the annuity-distorted inverse Euler equation.

Theorem 3 *At $t = 0$, for any productivity θ, the optimum satisfies the annuity-distorted inverse Euler equation:*

$$\frac{1}{\beta_0 R_0 u'\left(c_0\left(\theta\right)\right)} = \frac{1}{\beta_1 R_1}\mathbb{E}\left[\frac{1}{u'\left(c_1\left(\theta,\mu,\alpha\right)\right)}\middle|\theta\right] - \mathbb{E}\left[\psi\left(\theta,\mu\right)P\left(\theta,\mu,0\right)\middle|\theta\right]. \tag{15}$$

By Theorem 1, if private annuity demand is strictly positive $a\left(\theta,\mu\right) > 0$, then the utility cost of implementation is non-zero $\psi\left(\theta,\mu\right) \neq 0$. Consequently, Theorem 3 implies that the inverse Euler equation does not hold in our environment. The inverse marginal utility of consumption for θ-agents is either a strict *supermartingale*:

$$\frac{1}{\beta_0 R_0 u'\left(c_0\left(\theta\right)\right)} > \frac{1}{\beta_1 R_1}\mathbb{E}\left[\frac{1}{u'\left(c_1\left(\theta,\mu,\alpha\right)\right)}\middle|\theta\right],$$

or a strict *submartingale*:

$$\frac{1}{\beta_0 R_0 u'\left(c_0\left(\theta\right)\right)} < \frac{1}{\beta_1 R_1}\mathbb{E}\left[\frac{1}{u'\left(c_1\left(\theta,\mu,\alpha\right)\right)}\middle|\theta\right].$$

In the case of a supermartingale, consumption for θ-agents is more front-loaded than what the standard inverse Euler equation suggests. Conversely, with a submartingale, consumption for θ-agents is relatively back-loaded compared to the standard inverse Euler equation.[21]

Theorem 3 illustrates that there are two factors that determine the degree of deviation from the inverse Euler equation. The first is the consumption-based price $P\left(\theta,\mu,0\right)$, and the second is the utility cost $\psi\left(\theta,\mu\right)$ of implementing through the private annuity market.

Recall that Corollary 3 established the existence of a threshold longevity type μ^h such that all negative utility costs $\psi\left(\theta,\mu\right)$ originate from agents facing greater longevity risk. Since more productive agents have a higher probability of drawing a high longevity type, their utility cost $\psi\left(\theta,\mu\right)$ is more likely to be negative. Consequently, agents who are sufficiently productive have their inverse marginal utilities behave as a supermartingale. Conversely, by Corollary 3, there exists a threshold longevity type μ^l such that $\psi\left(\theta,\mu\right)$ is positive for all

[21] It is well-known that the inverse Euler equation usually does not hold when consumption and leisure are non-separable. See Hellwig (2021) for a characterization of the optimum with non-separable preferences. In setups with separable preferences, Grochulski and Piskorski (2010) found that the inverse marginal utility of consumption is a strict supermartingale when human capital investments are unobservable, and Paluszynski and Yu (2023) showed that the inverse Euler equation does not hold when agents are time inconsistent.

agents facing less longevity risk. Therefore, less productive agents are more likely to have a positive utility cost $\psi(\theta, \mu)$, making the inverse marginal utilities of sufficiently unproductive agents behave as a submartingale. As a result, Theorem 3 illustrates that more productive agents face higher savings taxes before retirement.

In a setup with dynamic private information, the intertemporal wedge is typically characterized by the inverse Euler equation, which implies a front-loaded consumption to reduce the cost of eliciting future private information. However, when decentralization occurs through a private annuity market with frictions, Theorem 3 reveals an additional factor that further front-loads the consumption of highly productive agents, while slightly back-loading the consumption of less productive agents. The underlying reason is that in an unregulated annuity market, agents with higher longevity types would naturally purchase more annuities. However, as Corollary 2 demonstrates, this is suboptimal due to the fact that the marginal social cost is increasing with longevity types. To mitigate this, higher pre-retirement savings taxes are imposed on more productive agents to lower the annuity demand of those more likely to face greater longevity risk. Finally, the savings tax is positively related to the type-dependent consumption-based annuity price $P(\theta, \mu, 0)$. Since this price reflects the attractiveness of purchasing an annuity for any given type, the pre-retirement savings tax increases with it to effectively lower the annuity demand of more productive agents.

The retirement savings wedges for non-altruistic agents in $t = 1$ and $t = 2$ are complex. Given this complexity, we will explore these wedges quantitatively. Nevertheless, it is important to note that the innovative economic forces operating through the stochastic discount factor of private annuity returns, as captured by the new inverse Euler equation, continue to influence the retirement savings wedges in both $t = 1$ and $t = 2$.

7 Quantitative analysis [in progress]

7.1 Benchmark calibration

The model $t = 0$ working period is 40 years and the three retirement period $t \in \{1, 2, 3\}$ are each 10 years. Individuals enter the model at age 25, and no one survives past 95.

Individuals draw their productivity type θ when they are 25 years old and their longevity μ and altruism α type upon retiring at 65 years old. The utility function is constant relative risk aversion with coefficient of risk aversion γ over consumption and bequests and ρ over labor input

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \text{ and } v(x) = \chi \frac{x^{1-\gamma}}{1-\gamma} \text{ and } h\left(\frac{y}{\theta}\right) = \frac{1}{\rho}\left(\frac{y}{\theta}\right)^{\rho}.$$

The parameter $\chi >= 0$ is the weight on the bequest in the utility function and is identical for every individual. Individuals draw their idiosyncratic productivity θ and longevity μ parameter from a bivariate log-normal distribution and draw their altruism type $\alpha \in \{0, 1\}$ from an independent distribution with $P(\alpha = 1) = p$. The bequest weight parameter is $\chi = 1$ with probability p and $\chi = 0$ with probability $1-p$. The interest rate shock is defined as $R_3(r3) = 1 + r_3$, where r_3 is a random variable, and $R_4(R3) = (1-\psi)\int R_3(r_3)dr_3 + \psi R3$ where ψ is the persistence of the random interest rate.

Table 1: **Parameter values**

Parameter	**Symbol**	**Value**	**Explanations**
Consumption and bequest utility	γ	2	
Labor disutility	ρ	3	Chetty (2012)
Survival probabilities	S_1	1	
	$S_2(\mu)$	$0.9\frac{\mu}{\max(\mu)}$	
	$S_3(\mu)$	$0.4\frac{\mu}{\max(\mu)}$	
	S_4	0	
Productivity and longevity	θ and μ	Bivariate log-normal	$corr(\theta, \mu) = 0.8$
Altruism probability	p	0.5	
Bequest weight	$\chi \in \{0, 1\}$	$\chi = 1$ if $\alpha = 1$	
Discount rates	β_0	1	
	β_1	0.95	
	β	0.97	
Interest rates	R_0	1	
	R_1	β_1^{-1}	
	R_2	β^{-1}	
	$r3 \in \{0, 0.025, 0.05\}$	$p(r_3) = \frac{1}{3} \forall r_3$	
	ψ	0.2	
Arbitragers' risk aversion	γ_a	4	
Mass of arbitragers	Ω	0.1	

The disutility of labor parameter is set to $\rho = 3$ and the consumption and bequest utility

parameter is set to $\gamma = 2$. [[Average productivity is between 0.796 and 4.604.]] Individual draw their altruism type $\alpha \in \{0, 1\}$ from an independent distribution with $P(\alpha = 1) = 0.5$. The bequest weight parameter is $\chi = 1$ with probability $P(\alpha = 1)$ and $\chi = 0$ otherwise. The arbitrager risk aversion parameter is set to $\gamma_a = 4$ and the mass of arbitrager is $\Omega = 1e - 1$. The survival probabilities are set as $S_1 = 1$, $S_2(\mu) = 0.9\frac{\mu}{\max(\mu)}$, $S_3(\mu) = 0.4\frac{\mu}{\max(\mu)}$, and $S_4 = 0$.

7.2 Welfare calculations and optimal decision rules

We analyze five different economies each with incomplete markets due to interest rate risk. The first best ignore all incentive compatibility constraints and maximizes total utility subject to the resource constraints. In the second-best, all incentive compatibility constraints are included in the welfare maximization problem. In the third best, equation (16) is added as constraint and q and R_l are endogenous prices supporting the decentralization. The optimal decision rules for these eocnomies are reported in Appendix F, G, and H. The fourth best economy with an exogenously low R_l and the fifth best in which social security benefits are canceled is work in progress.

Table 2: **Welfare loss from private information and incomplete financial market frictions**

Economy	**Welfare normalized by first best level**
1. First-best with uninsurable interest rate risk	100
2. Second-best with fully revealed private information	97.093
3. Third-best: Endogenous private insurance and bond supply Endogenous long-term bond yield: $R_l - R_2E(R_3) = -2.2976e^{-5}$ Endogenous annuity price: $q = 0.51326$	97.081
4. Fourth-best: Exogenous bond convenience yield Exogenous long-term bond yield: $R_l - R_2E(R_3) = -100bps$ Endogenous annuity price: $q = TBD$	TBD
5. Fifth-best: Exogenous bond convenience yield and no Social Security Exogenous benefit: $z(\theta) = 0$ Exogenous long-term bond yield: $R_l - R_2E(R_3) = -100bps$ Endogenous annuity price: $q = TBD$	TBD

8 Concluding remarks

We derived the optimal fully-funded retirement system in a life-cycle model featuring sequential and multidimensional screening. Our analysis focuses on policy decentralization that prioritizes private annuities and private savings. The simultaneous screening of longevity risk and altruism types necessitates both intertemporal instruments to separate post-retirement types effectively. While our model highlights the importance of private markets and provides rationale for a fully privatized retirement system, aggregate uninsurable interest rate shocks render full privatization suboptimal. In an incomplete market with inefficient long-term bond supply, private annuity insurers must hedge interest rate risk through annuity markups financed by positive net worth. Therefore, the optimal decentralization includes government-issued long-term bonds supported by taxes contingent on interest rates. Consequently, full privatization becomes costly in the presence of aggregate shocks, and Social Security remains necessary.

Our work contributes to the ongoing debate about Social Security reform. In his 1999 op-ed "Social Security Chimeras," Milton Friedman critiques the fundamental flaws of the U.S. Social Security system, particularly its pay-as-you-go structure, which he argues imposes an unsustainable burden on future generations (Friedman, 1999). He advocates for a paradigm shift toward a fully funded system, empowering individuals to manage their retirement savings entirely through private accounts. Similarly, in his American Economic Association presidential address, "Rethinking Social Insurance," Martin Feldstein critiques the inefficiency and inequity of current social insurance programs, attributing these flaws to their reliance on the outdated pay-as-you-go model (Feldstein, 2005a). Unlike Friedman, Feldstein proposes a mixed system that incorporates personal retirement accounts, allowing individuals to invest a portion of their contributions in private markets while maintaining a smaller Social Security system. Our paper demonstrates that Friedman's proposal is unlikely to be optimal, while providing additional rationale for Feldstein's balanced approach.

In sum, any retirement policy proposal that seeks to enhance self-insurance while overlooking the financial frictions affecting the supply of private longevity insurance is unlikely to improve welfare. Consider the Chilean experience as an example. It reveals the pitfalls of

full privatization, where shifting to individual retirement accounts exposed the limitations of relying solely on private markets for retirement income.[22] Private insurers in Chile faced significant interest rate risks when investing annuity premiums, consistent with the frictions highlighted by our model. To mitigate these risks, the Chilean government substantially increased long-term bond issuance post-reform. However, our analysis suggests that such measures remain insufficient without further government support—specifically through the issuance of long-term bonds financed by interest rate-contingent taxes. When aggregate interest rate risk cannot be costlessly hedged by private insurers due to distortions in the bond market, increasing reliance on private annuities necessitates government intervention.

References

Aiyagari, S. Rao, Albert Marcet, Thomas J. Sargent, and Juha Seppälä, "Optimal Taxation without State-Contingent Debt," *Journal of Political Economy*, 2002, *110* (6), 1220–1254.

Angeletos, George-Marios, "Fiscal Policy with Noncontingent Debt and the Optimal Maturity Structure," *The Quarterly Journal of Economics*, 2002, *117* (3), 1105–1131.

Armstrong, Mark, "Multiproduct Nonlinear Pricing," *Econometrica*, 1996, *64*, 51–75.

_ **and Jean-Charles Rochet**, "Multi-dimensional screening: A user's guide," *European Economic Review*, 1999, *43*, 959–979.

Aryal, Gaurab, Eduardo Fajnzylber, M Florencia Gabrielli, and Manuel Willington, "Auctioning Annuities," *Available at SSRN 3757206*, 2021.

Atkinson, Anthony and Joseph Stiglitz, "The Design of Tax Structure: Direct versus Indirect Taxation," *Journal of Public Economics*, 1976, *6*, 55–75.

Auerbach, Alan and Lawrence Kotlikoff, *Dynamic Fiscal Policy*, Cambridge University Press, 1987.

Baron, David and David Besanko, "Regulation and Information in a Continuing Relationship," *Information Economics and Policy*, 1984, *1*, 267–302.

[22] See Aryal et al. (2021) for more information about the Chilean pension reform.

Bellofatto, Antonio Andres, "Wealth Taxation and Life Expectancy," *Working Paper*, 2023.

Boerma, Job, Aleh Tsyvinski, and Alexander Zimin, "Bunching and Taxing Multidimensional Skills," *Working Paper*, 2022.

Brown, Jeffrey, Arie Kapteyn, Erzo F.P. Luttmer, and Olivia Mitchell, "Cognitive Constraints on Valuing Annuities," *Journal of European Economic Association*, 2017, *15* (2), 429–462.

Buera, Francisco and Juan Pablo Nicolini, "Optimal Maturity of Government Debt Without State Contingent Bonds," *Journal of Monetary Economics*, 2004, pp. 531–554.

Conesa, Juan Carlos and Carlos Garriga, "Optimal Fiscal Policy in the Design of Social Security Reforms," *International Economic Review*, 2008, *49* (1), 291–318.

_ **and Dirk Krueger**, "Social Security Reform with Heterogeneous Agents," *Review of Economic Dynamics*, 1999, *2*, 757–795.

Courty, Pascal and Hao Li, "Sequential Screening," *Review of Economic Studies*, 2000, *67*.

Cristia, Julian, "Rising Mortality and Life Expectancy Differentials by Lifetime Earnings in the United States," *Journal of Health Economics*, 2009, *28* (5), 984–995.

de Lannoy, Leo Aparisi, Anmol Bhandari, Davind Evans, Mikhail Golosov, and Thomas Sargent, "Managing Public Portfolios," 2022.

DeWitt, Larry, "Origins of the Three-Legged Stool Metaphor for Social Security," *Research Notes & Special Studies by the Historian's Office*, 1996, (Research Note No. 1).

Diamond, Peter, "Optimal Income Taxation: An Example with a U-Shaped Pattern of Optimal Marginal Tax Rates," *American Economic Review*, 1998, *96* (3), 694–719.

Drezner, Zvi and Dawit Zerom, "A Simple and Effective Discretization of a Continuous Random Variable," *Communications in Statistics-Simulation and Computation*, 2016, *45* (10), 3798–3810.

Dynan, Karen, Jonathan Skinner, and Stephen Zeldes, "The Importance of Bequests and Life-Cycle Saving in Capital Accumulation: A New Answer," *American Economic Review, Papers and Proceedings*, 2002, *92*, 274–278.

_ , _ , **and** _ , "Do the Rich Save More?," *Journal of Political Economy*, 2004, *112*, 397–444.

Eichenbaum, Martin and Dan Peled, "Capital Accumulation and Annuities in an Adverse Selection Economy," *Journal of Political Economy*, 1987, *95* (2), 334–354.

Einav, Liran and Amy Finkelstein, "Selection in Insurance Markets: Theory and Empirics in Pictures," *Journal of Economic Perspectives*, 2011, *25* (1), 115–38.

Farhi, Emmanuel and Ivan Werning, "Progressive Estate Taxation," *Quarterly Journal of Economics*, 2010, *125* (2).

_ **and** _ , "Estate Taxation with Altruism Heterogeneity," *American Economic Review: Papers and Proceedings*, 2013, *103*, 489–495.

Feldstein, Martin, "Rethinking Social Insurance," *American Economic Review*, 2005, *95* (1), 1–24.

_ , "Structural Reform of Social Security," *Journal of Economic Perspectives*, 2005, *19* (2), 33–55.

Finkelstein, Amy and James Poterba, "Adverse Selection in Insurance Markets: Policyholder Evidence from the U.K. Annuity Market," *Journal of Political Economy*, 2004, *112* (1).

Friedman, Benjamin and Mark Warshawsky, "The Cost of Annuities: Implications for Saving Behavior and Bequests," *The Quarterly Journal of Economics*, 1990, *105* (1), 135–154.

Friedman, Milton, "Social Security Chimeras," *The New York Times*, January 1999.

Gajek, Leslaw and Krzysztof M Ostaszewski, *Financial Risk Management for Pension Plans*, Elsevier, 2004.

Golosov, Mikhail, Aleh Tsyvinski, and Ivan Werning, "New Dynamic Public Finance: A User's Guide," *NBER Macroeconomics Annual 2006*, 2006.

_ **and** _ , "Optimal Taxation with Endogenous Insurance Markets," *The Quarterly Journal of Economics*, 2007, *122* (2), 487–534.

_ **, Maxim Troshkin, Aleh Tsyvinski, and Matthew Weinzierl**, "Preference Heterogeneity and Optimal Capital Income Taxation," *Journal of Public Economics*, 2013, *97*, 160–175.

_ , **Narayana Kocherlakota, and Aleh Tsyvinski**, "Optimal Indirect and Capital Taxation," *Review of Economic Studies*, 2003, *70* (3), 569–587.

Gottlieb, Daniel, "Prospect Theory, Life Insurance, and Annuities," *Working Paper*, 2018.

Greenwood, Robin, Samuel Hanson, and Jeremy Stein, "A Gap-Filling Theory of Corporate Debt Maturity Choice," *Journal of Finance*, 2010, *65*, 993–1028.

Grochulski, Borys and Tomasz Piskorski, "Risky Human Capital and Deferred Capital Income Taxation," *Journal of Economic Theory*, 2010, *145*, 908–943.

Harris, Milton and Artur Raviv, "Some Results on Incentive Contracts with Applications to Education and Employment, Health Insurance, and Law Enforcement," *American Economic Review*, 1978, *68*, 20–30.

Heathcote, Jonathan and Hitoshi Tsujiyama, "Optimal Income Taxation: Mirrlees Meets Ramsey," *Journal of Political Economy*, 2021, *129* (11), 3141–3184.

Hellwig, Christian, "Static and Dynamic Mirrleesian Taxation with Non-separable Preferences: A Unified Approach," *Working Paper*, 2021.

Hong, Jay H and José-Víctor Ríos-Rull, "Social Security, Life Insurance and Annuities for Families," *Journal of Monetary Economics*, 2007, *54* (1), 118–140.

Hosseini, Roozbeh, "Adverse Selection in the Annuity Market and the Role for Social Security," *Journal of Political Economy*, 2015, *123* (4), 941–984.

_ **and Ali Shourideh**, "Retirement Financing: An Optimal Reform Approach," *Econometrica*, 2019, *87* (4), 1205–1265.

Hubbard, R Glenn and Kenneth L Judd, "Social Security and Individual Welfare: Precautionary Saving, Borrowing Constraints, and the Payroll Tax," *The American Economic Review*, 1987, pp. 630–646.

Illanes, Gaston and Manisha Padi, "Retirement Policy and Annuity Market Equilibria: Evidence from Chile," *Working Paper*, 2021.

İmrohoroglu, Ayşe, Selahattin Imrohoroglu, and Douglas H Joines, "A Life Cycle Analysis of Social Security," *Economic theory*, 1995, *6* (1), 83–114.

Judd, Kenneth, Ding Ma, Michael Saunders, and Che-Lin Su, "Optimal Income Taxation with Multidimensional Taxpayer Types," *Working Paper*, 2017.

Kitagawa, Evelyn and Philip Hauser, *Differential Mortality in the United States: A Study in Socioeconomic Epidemiology*, Harvard University Press, Cambridge, MA, 1973.

Koijen, Ralph SJ and Motohiro Yogo, "The Cost of Financial Frictions for Life Insurers," *American Economic Review*, 2015, *105* (1), 445–75.

Kopczuk, Wojciech, "The Trick Is to Live: Is the Estate Tax Social Security for the Rich?," *Journal of Political Economy*, 2003, *111*, 1318–1341.

Lockwood, Lee M, "Incidental Bequests and the Choice to Self-Insure Late-Life Risks," *American Economic Review*, 2018, *108* (9), 2513–50.

McGrattan, Ellen and Edward Prescott, "On Financing Retirement with an Aging Population," *Quantitative Economics*, 2017, pp. 75–115.

Milgrom, Paul and Ilya Segal, "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 2002, *70* (2).

Mirrlees, James, "An Exploration in the Theory of Optimal Income Taxation," *Review of Economic Studies*, 1971, *38*.

Mitchell, Olivia S, James M Poterba, Mark J Warshawsky, and Jeffrey R Brown, "New Evidence on the Money's Worth of Individual Annuities," *American economic review*, 1999, *89* (5), 1299–1318.

Moser, Christian and Pedro Olea de Souza e Silva, "Paternalism vs Redistribution: Designing Retirement Savings Policies with Behavioral Agents," *Working Paper*, 2019.

Myerson, Roger, "Multistage Games with Communication," *Econometrica*, 1986, *54* (2).

Nishiyama, Shinichi and Kent Smetters, "Does social security privatization produce efficiency gains?," *Quarterly Journal of Economics*, 2007, pp. 1677–1719.

Paluszynski, Radoslaw and Pei Cheng Yu, "Efficient Consolidation of Incentives for Education and Retirement Savings," *American Economic Journal: Macroeconomics*, 2023, *15*, 153–190.

Park, Yena, "Optimal Taxation of Inheritance and Retirement Savings," *Working Paper*, 2021.

Pavoni, Nicola and Hakki Yazici, "Intergenerational Disagreement and Optimal Taxation of Parental Transfers," *Review of Economic Studies*, 2017, *84* (3).

Piketty, Thomas and Emmanuel Saez, "A Theory of Optimal Inheritance Taxation," *Econometrica*, 2013, *81*, 1851–1886.

Rochet, Jean-Charles and Philippe Chone, "Ironing, Sweeping, and Multidimensional Screening," *Econometrica*, 1998, *66*, 783–826.

Saez, Emmanuel, "Using Elasticities to Derive Optimal Income Tax Rates," *Review of Economic Studies*, 2001, *68*, 205–229.

Spiritus, Kevin, Etienne Lehmann, Sander Renes, and Floris Zoutman, "Optimal Taxation with Multiple Incomes and Types," *Working Paper*, 2022.

Vayanos, Dimitri and Jean-Luc Vila, "A Preferred-Habitat Model of the Term Structure of Interest Rates," *Econometrica*, 2021, *89* (1), 77–112.

Verani, Stephane and Pei Cheng Yu, "What's Wrong with Annuity Markets?," *Journal of the European Economic Association*, 2024, *22* (4), 1981–2024.

Yu, Pei Cheng, "Optimal Retirement Policies with Present-Biased Agents," *Journal of European Economic Association*, 2021, *19* (4), 2085–2130.

Optimal Retirement Policies with Private Annuities

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APPENDIX FOR ONLINE PUBLICATION ONLY

A Simplifying the incentive compatibility constraints

The large number of ex-ante and ex-post incentive constraints complicates our planning problem. This section uses standard tools from mechanism design to simplify it.

It is important to note that multidimensional screening problems are notoriously difficult to characterize (Armstrong, 1996; Rochet and Chone, 1998).[23] The standard first-order approach is often applied to unidimensional screening problems, which we do for the ex-ante incentive constraints, because local incentive compatibility typically implies global incentive compatibility via the single-crossing condition. However, this may not be true with multidimensional screening. Another difficulty comes from the fact that, even with discrete types in each dimension, the complexity of the problem makes it hard to determine the relevant binding incentive constraints (Armstrong and Rochet, 1999). The setup of our model tries to minimize these difficulties.

First, since the optimal bequest for non-altruistic agents is zero. By focusing on binary altruism types and assuming that $\lim_{x\to 0} v(0) = -\infty$, altruistic agents will never misreport as non-altruistic. Therefore, we only need to concentrate on deviations from non-altruistic to altruistic agents:

$$U_1(\theta, \mu, 0) \geq \max_{\hat{\mu}} U_1(\hat{\mu}, 1; \theta, \mu, 0). \tag{16}$$

Note that (16) accounts for double deviations, i.e., a non-altruistic agent who mimics an altruistic agent may also misreport the longevity risk μ. Therefore, the ex-post incentive constraints (1) can be replaced with (16) and incentive constraints that only deter agents

[23] Several recent papers have attempted to characterize the optimal policy in a setting with multidimensional heterogeneity (Judd et al., 2017; Boerma et al., 2022; Spiritus et al., 2022). Notably, Moser and de Souza e Silva (2019) and Yu (2021) focused on the design of retirement savings policies where agents differ in skills and time preferences. However, they focused on a paternalistic setting with present-biased agents, so only the heterogeneity in productivity entered the welfare criteria but not the agents' bias, which simplifies the problem.

from misreporting their longevity risk μ, i.e., for any θ, $\mu, \mu' \in \left[\underline{\mu}, \overline{\mu}\right]$, and α,

$$U_1(\theta, \mu, \alpha) \geq U_1(\mu', \alpha; \theta, \mu, \alpha). \tag{17}$$

Next, notice that for a fixed productivity θ, both altruistic and non-altruistic agents with higher longevity types prefer more back-loaded consumption paths at $t = 1$. On the other hand, at $t = 1$, altruistic agents with higher longevity types prefer consumption over bequests. In essence, our setup satisfies the single-crossing property.[24] Assuming (16) holds, this allows us to focus on local incentive compatibility for longevity risk μ which greatly simplifies the ex-post incentive constraints. The following lemma provides simple necessary conditions for the set of allocations to be incentive compatible.

Lemma 3 *For any θ and α, if (16) is satisfied, then $U_1(\theta, \mu, \alpha)$ is absolutely continuous in μ so it is differentiable almost everywhere in μ with*

$$\frac{\partial U_1(\theta, \mu, \alpha)}{\partial \mu} = \int_{\mathcal{R}} \sum_{t=1}^{3} \beta^{t-1} S_t'(\mu) \Big\{ u(c_t(\theta, \mu, \alpha, R_3)) + \beta \left[1 - \frac{S_{t+1}'(\mu)}{S_t'(\mu)}\right] \alpha v(x_t(\theta, \mu, \alpha, R_3)) \Big\} d\Pi(R_3). \tag{18}$$

If P is ex-ante incentive compatible, then $U_0(\theta)$ is absolutely continuous, so it is differentiable almost everywhere with

$$U_0'(\theta) = \beta_0 \frac{y(\theta)}{\theta^2} h'\left(\frac{y(\theta)}{\theta}\right) + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \begin{bmatrix} pU_1(\theta, \mu, 1) \\ +(1-p)\beta_1 U_1(\theta, \mu, 0) \end{bmatrix} \frac{\partial g(\mu|\theta)}{\partial \theta} d\mu. \tag{19}$$

Proof First, for a fixed (θ, μ, α), (17) can be rewritten as

$$U_1(\theta, \mu, \alpha) = \max_{\mu' \in \left[\underline{\mu}, \overline{\mu}\right]} \int_{\mathcal{R}} \sum_{t=1}^{3} \beta^{t-1} S_t(\mu) \Big\{ u(c_t(\theta, \mu', \alpha, R_3)) + \beta \left[1 - \frac{S_{t+1}(\mu)}{S_t(\mu)}\right] \alpha v(x_t(\theta, \mu', \alpha, R_3)) \Big\} d\Pi(R_3).$$

[24] Note that $\frac{\partial^2 U_1}{\partial c_1 \partial \mu} = 0$, $\mathbb{E}\left[\frac{\partial^2 U_1}{\partial c_t \partial \mu}\right] > 0$ for $t \geq 2$, and $\mathbb{E}\left[\frac{\partial^2 U_1}{\partial x_1 \partial \mu}\right] < 0$, which implies the single-crossing condition.

Theorem 2 of Milgrom and Segal (2002) applies, so $U_1(\theta, \mu, \alpha)$ is absolutely continuous in μ. Finally, since $U_1(\theta, \mu, \alpha)$ is absolutely continuous in μ it is differentiable in μ almost everywhere, which gives us the envelope conditions in $t = 1$. Finally, the necessary condition for ex-ante incentive compatibility follows from the same argument. ■

The standard approach typically solves a relaxed planning problem by replacing the incentive constraints (2) and (17) with the envelope conditions in Lemma 3, and then check whether the ex-ante and ex-post incentive constraints are satisfied numerically. Our paper follows this standard approach.

B Proofs

Before proving Proposition 1, the next lemma shows how income-contingent policies are inherently dependent on productivity.

Lemma 4 *The optimal allocations for any reported productivity θ are also functions of income $y(\theta)$.*

Proof Suppose there exists $\tilde{\theta}$ and $\hat{\theta}$ such that $y\left(\tilde{\theta}\right) = y\left(\hat{\theta}\right)$. Let $\Gamma(\theta) = \beta_0 u(c_0(\theta)) + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \left[pU_1(\theta, \mu, 1) + (1-p) U_1(\theta, \mu, 0)\right] g(\mu|\theta)\, d\mu$. There are two cases to consider. First, suppose $\Gamma\left(\tilde{\theta}\right) \neq \Gamma\left(\hat{\theta}\right)$, then clearly the allocations are not incentive compatible. Next, suppose $\Gamma\left(\tilde{\theta}\right) = \Gamma\left(\hat{\theta}\right)$, and without loss of generality $c_0\left(\tilde{\theta}\right) > c_0\left(\hat{\theta}\right)$ and

$$\int_{\underline{\mu}}^{\overline{\mu}} \left[pU_1\left(\tilde{\theta}, \mu, 1\right) + (1-p) U_1\left(\tilde{\theta}, \mu, 0\right)\right] g(\mu|\theta)\, d\mu$$
$$< \int_{\underline{\mu}}^{\overline{\mu}} \left[pU_1\left(\hat{\theta}, \mu, 1\right) + (1-p) U_1\left(\hat{\theta}, \mu, 0\right)\right] g(\mu|\theta)\, d\mu.$$

Let $\tilde{\omega}$ and $\hat{\omega}$ denote the measure of agents with productivity $\tilde{\theta}$ and $\hat{\theta}$ respectively and assign these agents the average utility of the two types of agents in every period such that, for example, $\bar{u}_0 = \frac{1}{\tilde{\omega}+\hat{\omega}} \left[\tilde{\omega} u\left(c_0\left(\tilde{\theta}\right)\right) + \hat{\omega} u\left(c_0\left(\hat{\theta}\right)\right)\right]$. By assigning these agents the average utility, the total welfare is unchanged and incentive compatibility is preserved. However,

since u is strictly concave, the consumption level that gives $\bar{u}_t$ for every t relaxes the resource constraint. This means that it is optimal for consumption at $t = 0$ to be equivalent for agents of the same level of income. A similar argument can be applied to consumption in other periods. ■

Proof of Proposition 1:

Let the constrained efficient optimum be denoted as

$$P^* = \left\{ \begin{array}{c} [c_0^*(\theta), y^*(\theta)], \\ [c_1^*(\theta,\mu,\alpha), c_2^*(\theta,\mu,\alpha), c_3^*(\theta,\mu,\alpha)], \\ [x_1^*(\theta,\mu,\alpha), x_2^*(\theta,\mu,\alpha), x_3^*(\theta,\mu,\alpha)] \end{array} \right\}_{\theta\in[\underline{\theta},\overline{\theta}];\mu\in[\underline{\mu},\overline{\mu}];\alpha\in\{0,1\}}.$$

By Lemma 4, we can replace the dependence on θ with income $y^*(\theta)$. Define

$$\mathbb{Y} = \left\{ y | y = y^*(\theta) \text{ and } \theta \in [\underline{\theta}, \overline{\theta}] \right\}.$$

We consider an implementation where agents with $\alpha = 0$ only buy annuities, and agents with $\alpha = 1$ only save through one-period bonds. Thus, each unique income-longevity-altruism type (y, μ, α) maps into a different retirement asset demand profile $\left(a, \{b_t\}_{t\geq 2}\right)$ for a given set of prices. We verify after presenting the implementation that it is indeed the case that we can replace (μ, α) with $\left(a, \{b_t\}_{t\geq 2}\right)$.

We focus on an implementation with strictly positive savings, $b_t > 0$ for all t, for agents with $\alpha = 1$ and strictly positive demand for annuities $a > 0$ for agents with $\alpha = 0$. To do this, we set the income-contingent social security benefits $z(y)$ to be the lowest consumption level for any agent in any retirement period with income y. In essence,

$$z(y) = \min_{t\geq 1} \left\{ \min_{a,b_2} c_t^*(y, a, b_2) \right\} - \epsilon,$$

with sufficiently small $\epsilon > 0$ such that $z(y) > 0$ for all y. This setup also ensures that social security benefits are only a function of income y and are constant across time, which is consistent with current policies.

Next, we construct the post-retirement taxes. Set the post-retirement savings income tax τ_t^K for $t = 2$ as

$$1 - \tau_2^K(y, a, b_2) = \begin{cases} \frac{c_2^*(y,a,0)}{a+z(y)} & a > 0 \text{ and } b_2 = 0 \\ \frac{u'\left(c_1^*(y,0,b_2)\right)}{\beta R_2 u'\left(c_2^*(y,0,b_2,b_3)\right)} & a = 0 \text{ and } b_2 > 0 \\ 0 & a > 0 \text{ and } b_2 > 0 \end{cases},$$

and for $t = 3$ as

$$1 - \tau_3^K(y, a, b_2, b_3) = \begin{cases} \frac{c_3^*(y,a,0,0)}{a+z(y)} & a > 0 \text{ and } b_2 = b_3 = 0 \\ \frac{u'\left(c_2^*(y,0,b_2,b_3)\right)}{\beta R_3 u'\left(c_3^*(y,0,b_2,b_3)\right)} & a = 0 \text{ and } b_3 > 0 \\ 0 & a > 0 \text{ and } b_3 > 0 \end{cases}.$$

The bequest taxes are

$$1 - \tau_{t+1}^B\left(y, a, \{b_s\}_{s=2}^{t+1}\right) = \begin{cases} \frac{u'\left(c_t^*\left(y,0,\{b_s\}_{s=2}^{t+1}\right)\right)}{\alpha\beta R_{t+1} v'\left(x_t^*\left(y,0,\{b_s\}_{s=2}^{t+1}\right)\right)} & a = 0 \\ 0 & a > 0 \end{cases} \text{ for } t \geq 1.$$

Hence, during retirement, agents with $\alpha = 0$ only purchase annuities while agents with $\alpha = 1$ only purchase bonds. Note that for any $y \in \mathbb{Y}$, agents with $\alpha = 0$ choose a to maximize their utility in $t = 1$ given their longevity type μ. Similarly, for any $y \in \mathbb{Y}$, agents with $\alpha = 1$ choose b_{t+1} to maximize their utility in t given their longevity type μ.

For instruments $\hat{P}_0$ in $t = 0$ to implement P^*, τ_1^K satisfies $\tau_1^K(y) = 1 - \frac{\beta_0 u'\left(c_0^*(y)\right)}{\beta_1 R_1 \mathbb{E}\left[u'\left(c_1^*(y,a,b_2)\right)\right]}$ from the first-order conditions of the decentralized optimization problem, where the expectation is taken over μ and α since annuity demand a and private savings b_2 are functions of μ and α. We set the income tax to be $T(y - b_1) = y - b_1 - c_0^*(y)$ for any $y \in \mathbb{Y}$ and $T(y - b_1) = y - b_1$ otherwise.

Then, we check that the policy instruments implement the optimum. First, notice that all agents would choose $y \in \mathbb{Y}$, otherwise $c_0 = 0$. By the taxation principle, agents with productivity θ choose $y = y(\theta)$. Finally, to see that the demand for annuities is unique for

agents with $\alpha = 0$ and each income-longevity pair (y, μ), suppose that there exists agents with the same income y and different longevity but with the same annuity demand. Then, given the decentralized optimization problem, they would share the same consumption path, which is not optimal by the taxation principle. Hence, annuity demand uniquely identifies agents' longevity μ when $\alpha = 0$. A similar argument holds for agents with $\alpha = 1$. ■

Proof of Corollary 1: The result follows by following the same proof for Proposition 1 but with $z(y) = 0$ for all $y \in \mathbb{Y}$. ■

Proof of Proposition 2: Competitive limited-liability insurers finance just enough net worth so that $NW_2(R_3) = 0$ when $R_3 = 1$ and $NW_2(R_3) \geq 0$ when $R_3 > 1$. From (6), this implies that $b_3(1) = L_3(q)$, which is the minimal amount of assets the insurers must purchase in $t = 2$ to remain solvent when the worst interest rate shock occurs. By backward induction, this implies that the insurers must purchase sufficient assets (b_2, l_3) in $t = 1$. In essence, by (7), insurers must purchase one-period bonds $b_2(l_3)$ given the amount of two-period bonds in the insurer's portfolio:

$$b_2(l_3) = \frac{1}{R_2}\left[L_2(q) + L_3(q) - R_l l_3\right]. \tag{20}$$

Substituting the demand $b_2(l_3)$ into (5) yields

$$NW_1 + \left(\frac{R_l}{R_2} - 1\right) l_3 = \frac{1}{R_2}\left[1 - \mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3(q). \tag{21}$$

Next, we consider the following three cases: (i) $NW_2(R_3) = 0$ for all R_3, (ii) $NW_1 = 0$, and (iii) $NW_1 > 0$ with $NW_2(R_3) > 0$ for some R_3.

For case (i), we have $b_3(R_3) = \frac{L_3(q)}{R_3}$ when $NW_2(R_3) = 0$ for all R_3 from (6). By (7) and (20), we get $l_3 = \frac{L_3(q)}{R_l}$ and $b_2 = \frac{L_2(q)}{R_2}$. Therefore, when the bond market is constrained, equation (21) yields $NW_1 = \left[\frac{1}{R_l} - \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3(q) > 0$, and $NW_1 = 0$ when the bond market is unconstrained due to limited liability.

For case (ii), when $NW_1 = 0$, equation (21) gives us $l_3 = \left[\frac{1-\mathbb{E}\left(\frac{1}{R_3}\right)}{R_l - R_2}\right] L_3(q)$. Therefore,

by (6), (7), and (20), $NW_2(R_3) = \left(1 - \frac{1}{R_3}\right)\left[1 - \frac{R_l\left(1-\mathbb{E}\left(\frac{1}{R_3}\right)\right)}{R_l - R_2}\right] L_3(q)$, which is strictly negative when $\frac{1}{R_l} > \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$. Therefore, $NW_1 = 0$ is not optimal when the bond market is constrained.

For case (iii), if $NW_2\left(\tilde{R}_3\right) > 0$ for some $\tilde{R}_3 > 1$, then $R_2 b_2 + \frac{R_l l_3}{\tilde{R}_3} > L_2(q) + \frac{L_3(q)}{\tilde{R}_3}$ by (6) and (7). Consider a new asset allocation $\left(\tilde{b}_2, \tilde{l}_3\right) = \left(b_2 - \epsilon, l_3 + \frac{R_2}{R_l}\epsilon\right)$, so that $NW_2(1) = 0$ under the new allocation and $R_2\tilde{b}_2 + \frac{R_l\tilde{l}_3}{\tilde{R}_3} = R_2 b_2 + \frac{R_l l_3}{\tilde{R}_3} - R_2\left(1 - \frac{1}{\tilde{R}_3}\right)\epsilon$. Hence, for sufficiently small ϵ, $NW_1\left(\tilde{R}_3\right) > 0$. However, $\tilde{b}_2 + \tilde{l}_3 = b_2 + l_3 - \left(1 - \frac{R_2}{R_l}\right)\epsilon$, so by (4), the insurer can lower the annuity price under the new asset portfolio since $R_l \geq R_2$. Therefore, case (iii) is inconsistent with a competitive annuity market.

Finally, we show that the optimal IRM is unique in a competitive equilibrium. Let $\{b_2, l_3, b_3(R_3), NW_1, NW_2(R_3)\}$ denote the optimal asset portfolio and capital structure for a given annuity price q. Notice that b_2 is uniquely pinned down by (4), NW_1 is uniquely pinned down by (5), $NW_2(R_3)$ is uniquely pinned down by (6), and $b_3(R_3)$ is uniquely pinned down by (7). Therefore, to show uniqueness, it is sufficient to show that at the optimum, $l_3 = \frac{L_3(q)}{R_l}$. Taking the original annuity price q as given, first suppose an insurer deviates and chooses $\hat{l}_3 = l_3 - \epsilon$, where $\epsilon \in (0, l_3]$. Then, by (4), the new short-term bond demand at $t = 1$ is $\hat{b}_2 = b_2 + \epsilon$. This implies that the new short-term bond demand at $t = 2$ is $\hat{b}_3(R_3) = b_3(R_3) + \frac{R_2}{R_3}\left(R_3 - \frac{R_l}{R_2}\right)\epsilon$ by (7). Hence, by (6), the new net worth at $t = 2$ is $\hat{NW}_2(R_3) = \frac{R_2}{R_3}\left(R_3 - \frac{R_l}{R_2}\right)\epsilon$. If returns are such that $\hat{NW}_2(1) < 0$, then the new allocation is not optimal since insurers can become insolvent. If returns are such that $\hat{NW}_2(R_3) \geq 0$ for all R_3, then NW_2 is strictly positive for large R_3. This allocation is also not optimal since the insurer can perform IRM with less net worth and charge a lower price. Also, by the same argument as before, the insurer can become insolvent if it deviates and chooses $\hat{l}_3 = l_3 + \epsilon$, where $\epsilon > 0$. Hence, it is optimal for $l_3 = \frac{L_3(q)}{R_l}$. This outcome proves uniqueness. ■

Proof of Lemma 1: Note that R_l only affects the planning problem in Section 2.2 through $\Psi(R_3)$. Suppose $\frac{R_2}{R_l} < \mathbb{E}\left(\frac{1}{R_3}\right)$, then $q^* = \tilde{q}$ with $NW_1 = 0$ and $L_3(q^*)$ is constant with respect to changes in R_l. This implies that $\left(\frac{1}{R_3} - \frac{R_2}{R_l}\right) L_3(q^*)$ is strictly increasing in R_l for any R_3. Hence, by (11), $\Psi(R_3)$ is increasing in R_l when Ω is suffi-

ciently small. As a result, for sufficiently small Ω, $\frac{R_2}{R_l} \geq \mathbb{E}\left(\frac{1}{R_3}\right)$ and it follows that $\Psi(R_3) = \left(\frac{R_l}{R_3} - R_2\right)\Omega H(R_l) + \left[\frac{1}{R_3} - \mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3(q^*)$. ■

Proof of Lemma 2: By (13), annuity demand is unique for each μ for any given θ and $\alpha = 0$. The retirement policies can be constructed following the proof for Proposition 1. Given $a(\theta, \mu)$, q is set according to (8) and the insurers' interest rate risk management strategy satisfies Proposition 2. By Lemma 1, the bond policies satisfy $\frac{R_2}{R_l} \geq \mathbb{E}\left(\frac{1}{R_3}\right)$, and given annuity demand $a(\theta, \mu)$, $[T(R_3), \Lambda]$ satisfy (9) and (10). ■

To prove the theorems in Section 6.2, we first present the relaxed planning problem:

$$\max_P \int_{\underline{\theta}}^{\overline{\theta}} U_0(\theta) f(\theta) d\theta.$$

subject to

$$U_0(\theta) = \beta_0 \left[u(c_0(\theta)) - h\left(\frac{y(\theta)}{\theta}\right)\right] + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \left[pU_1(\theta, \mu, 1) + (1-p) U_1(\theta, \mu, 0)\right] dG(\mu|\theta), \quad (22)$$

$$U_1(\theta, \mu, \alpha) = \int_1^{\overline{R}} \sum_{t=1}^{3} \beta^{t-1} S_t(\mu) w(c_t(\theta, \mu, \alpha, R_3), x_t(\theta, \mu, \alpha, R_3); \mu, \alpha) d\Pi(R_3), \quad (23)$$

$$U_1(\theta, \mu, 0) \geq u(c_1(\theta, \mu', 1)) + \int_1^{\overline{R}} \sum_{t=2}^{3} \beta^{t-1} S_t(\mu) u(c_t(\theta, \mu', 1, R_3)) d\Pi(R_3), \quad (24)$$

(18), (19), (8), (12), and the resource constraints. In addition to the multipliers introduced in the proof of Proposition 3 in Appendix C, let $\left[\beta_1 \psi(\theta, \mu) \Phi g(\mu|\theta) f(\theta), \frac{\Phi\eta}{R_1}, \phi(R_3) \pi(R_3)\right]$ be the multipliers on (12), (8), and the resource constraints respectively, where $\Phi = \mathbb{E}[\phi(R_3)]$.

Proof of Theorem 1: The first-order condition on $a(\theta, \mu)$ is the same as (51). When $a(\theta, \mu) > 0$, $\frac{\partial \Psi(R_3)}{\partial a(\theta, \mu)}$ is given by (52), so (14) follows immediately. Next, from the first-order condition on $z(\theta)$, we have

$$\mathbb{E}[\psi(\theta, \mu) u'(c_1(\theta, \mu, 0)) | \theta] = 0. \quad (25)$$

Next, from the first-order condition on q, we have

$$\eta(1-p)\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\mu}}^{\overline{\mu}} a(\theta,\mu)\,dG(\mu|\theta)\,dF(\theta) = \beta_1 R_1 \mathbb{E}\left[\psi(\theta,\mu)\,u'(c_1(\theta,\mu,0))\left[a(\theta,\mu)+z(\theta)\right]\right],$$

where the expectation is over θ and μ. By (25), the condition above can be rewritten as

$$\eta(1-p)\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\mu}}^{\overline{\mu}} a(\theta,\mu)\,dG(\mu|\theta)\,dF(\theta) = \beta_1 R_1 \mathbb{E}\left[\psi(\theta,\mu)\,u'(c_1(\theta,\mu,0))\,a(\theta,\mu)\right],$$

which gives us the expression for η.

To see why $\eta > 0$, multiply both sides of (14) by $a(\theta,\mu)\,g(\mu|\theta)\,f(\theta)$ and integrate over μ and θ. By (8), we have

$$q\beta_1 R_1 \mathbb{E}\left[\psi(\theta,\mu)\,u'(c_1(\theta,\mu,0))\,a(\theta,\mu)\right] = \frac{L_3(q)}{R_l}\mathbb{E}\left[\frac{\phi(R_3)}{\Phi}\Delta(R_3)\right].$$

As shown in the proof for Theorem 7, the right-hand side of the above equation is strictly positive, so $\mathbb{E}\left[\psi(\theta,\mu)\,u'(c_1(\theta,\mu,0))\,a(\theta,\mu)\right] > 0$, which implies $\eta > 0$. ■

Proof of Corollary 2: By Theorem 1, multiplying (14) by $g(\mu|\theta)\,f(\theta)$ and integrating over (θ,μ) yields

$$\eta(1-p)\left\{\mathbb{E}\left[\hat{q}(\mu)\right]-q\right\} = \frac{(1-p)\,\mathbb{E}\left[S_3(\mu)\right]}{R_l}\mathbb{E}\left[\frac{\phi(R_3)}{\Phi}\Delta(R_3)\right],$$

which implies $\mathbb{E}\left[\hat{q}(\mu)\right] > q$. Note that (8) can be rewritten as

$$\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\mu}}^{\overline{\mu}} \frac{\hat{q}(\mu)}{q}\frac{a(\theta,\mu)}{\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\mu}}^{\overline{\mu}} a(\theta',\mu')\,dG(\mu'|\theta')\,dF(\theta')}dG(\mu|\theta)\,dF(\theta) = 1.$$

Since $\hat{q}(\mu)$ is strictly increasing in μ and $\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\mu}}^{\overline{\mu}}\frac{\hat{q}(\mu)}{q}dG(\mu|\theta)\,dF(\theta) > 1$, to satisfy the equation above, $a(\theta,\mu)$ cannot be weakly increasing in μ for all θ. ■

Proof of Corollary 3: First, note that (3) can be expressed as

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\mu}}^{\overline{\mu}} \left[\tilde{q} - \hat{q}(\mu)\right] a(\theta, \mu)\, dG(\mu|\theta)\, dF(\theta) = 0.$$

Since $\hat{q}(\mu)$ is strictly increasing in μ, there exists $\mu^l \in \left(\underline{\mu}, \overline{\mu}\right]$ such that $\tilde{q} \geq \hat{q}(\mu)$ for all $\mu \leq \mu^l$. Since the equilibrium annuity price q is weakly greater than $\tilde{q}$, it follows immediately that $\hat{q}(\mu) \leq q$ for all $\mu \leq \mu^l$. Since the right-hand side of (14) is strictly positive, we have $\psi(\theta, \mu) > 0$ for all $\mu \leq \mu^l$.

Finally, following a similar reasoning, if $\psi(\theta, \mu) \leq 0$, then it must be the case that $\hat{q}(\mu) > q$. This can only occur if μ is sufficiently large, i.e., $\mu > \mu^h$ for some $\mu^h \in \left(\underline{\mu}, \overline{\mu}\right]$. ■

Proof of Theorem 2: By Lemma 1, $\Psi(R_3) = \left(\frac{R_l}{R_3} - R_2\right) \Omega H(R_l) + \left[\frac{1}{R_3} - \mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3(q^*)$ with $\frac{R_2}{R_l} \geq \mathbb{E}\left(\frac{1}{R_3}\right)$. The first-order condition on R_l is

$$\Omega \left\{ H'(R_l) - \frac{1}{R_2}\left[H(R_l) + R_l H'(R_l)\right] \mathbb{E}\left[\frac{\phi(R_3)}{\Phi R_3}\right] \right\} = \frac{\eta(1-p)}{R_l^2} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\mu}}^{\overline{\mu}} S_3(\mu)\, a(\theta, \mu)\, dG(\mu|\theta)\, dF(\theta).$$

Substituting the expression for η in Theorem 1 and by the definition of ϵ^H and ϵ^q, yields the result. ■

Proof of Theorem 3: The following proof uses results from Appendix C. The first-order condition on $c_1(\theta, \mu, 1)$ is characterized by (34), while the first-order condition on $c_1(\theta, \mu, 0)$ yields

$$\lambda_1(\theta, \mu, 0) = \frac{(1-p)\,\Phi g(\mu|\theta) f(\theta)}{R_1 u'(c_1(\theta, \mu, 0))} - q\beta_1 \psi(\theta, \mu)\, \Phi\left[a(\theta, \mu) + z(\theta)\right] g(\mu|\theta) f(\theta).$$

Substitute the first-order conditions on c_1 into (32) and (33), which remain the same in our

new planning problem, and sum them together, then integrate over the range of μ to yield

$$\frac{1}{\beta_0 R_0 u'\left(c_0\left(\theta\right)\right)}=\frac{1}{\beta_1 R_1}\mathbb{E}\left[\frac{1}{u'\left(c_1\left(\theta,\mu,\alpha\right)\right)}\middle|\theta\right]-q\mathbb{E}\left[\psi\left(\theta,\mu\right)\left[a\left(\theta,\mu\right)+z\left(\theta\right)\right]|\theta\right],$$

where the first expectation is taken over μ and α, and the second expectation is taken over μ. Finally, substituting out $q\left[a\left(\theta,\mu\right)+z\left(\theta\right)\right]$ by (12) yields the result. ■

C Optimal wedges without interest rate shock

C.1 Pre-retirement wedges

In this section, we characterize the constrained efficient optimum before retirement. To separate the economic forces that determine the labor wedge, we define

$$A(\theta)=\frac{1-F\left(\theta\right)}{\theta f\left(\theta\right)},$$

$$B(\theta)=1+\frac{\frac{y(\theta)}{\theta}h''\left(\frac{y(\theta)}{\theta}\right)}{h'\left(\frac{y(\theta)}{\theta}\right)},$$

$$C(\theta)=\int_{\theta}^{\bar{\theta}}\frac{u'\left(c_0\left(\theta\right)\right)}{u'\left(c_0\left(\theta'\right)\right)}\left[1-\frac{\beta_0 R_0 u'\left(c_0\left(\theta'\right)\right)}{\Phi}\right]\frac{f\left(\theta'\right)}{1-F\left(\theta\right)}d\theta',$$

where $\Phi>0$ is the shadow price on the resource constraint.[25]

Proposition 3 *At $t=0$, for any productivity θ, the constrained efficient allocation satisfies: (i) the inverse Euler equation: $\frac{1}{\beta_0 R_0 u'(c_0(\theta))}=\mathbb{E}\left[\frac{1}{\beta_1 R_1 u'(c_1(\theta,\mu,\alpha))}|\theta\right]$, and (ii) the Mirrlees-Diamond-Saez formula: $\frac{\tau^w(\theta)}{1-\tau^w(\theta)}=A\left(\theta\right)B\left(\theta\right)C\left(\theta\right)$, where $\frac{1}{\Phi}=\int_{\underline{\theta}}^{\bar{\theta}}\frac{f(\theta)}{\beta_0 R_0 u'(c_0(\theta))}d\theta$.*

Proof Given Lemma 3, the relaxed optimal tax problem is

$$\max_{P}\int_{\underline{\theta}}^{\bar{\theta}}U_0\left(\theta\right)f\left(\theta\right)d\theta.$$

[25] The government faces a single resource constraint in a setting without interest rate shock R_3.

subject to

$$U_0(\theta) = \beta_0 \left[u(c_0(\theta)) - h\left(\frac{y(\theta)}{\theta}\right)\right] + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \left[pU_1(\theta,\mu,1) + (1-p)U_1(\theta,\mu,0)\right] dG(\mu|\theta), \quad (26)$$

$$U_1(\theta,\mu,\alpha) = \sum_{t=1}^{3} \beta^{t-1} S_t(\mu)\, w(c_t(\theta,\mu,\alpha), x_t(\theta,\mu,\alpha); \mu, \alpha), \quad (27)$$

(16), (18), (19), and the resource constraint. We rewrite (16) as, for any $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ and for any $\mu, \mu' \in \left[\underline{\mu}, \overline{\mu}\right]$,

$$U_1(\theta,\mu,0) \geq u(c_1(\theta,\mu',1)) + \beta\left[S_2(\mu)\, u(c_2(\theta,\mu',1)) + \beta S_3(\mu)\, u(c_3(\theta,\mu',1))\right]. \quad (28)$$

Let

$$\left[\lambda_0(\theta), \xi(\theta), \lambda_1(\theta,\mu,\alpha), \xi^{\mu}(\theta,\mu,\alpha), \xi^{\alpha}(\mu';\theta,\mu), \Phi\right]$$

be the multipliers on (26), (19), (27), (18), (28), and resource constraint respectively. Using standard Hamiltonian techniques, we derive the following necessary conditions for optimality: for any $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$,

$$\lambda_0(\theta) = f(\theta) - \xi'(\theta), \quad (29)$$

$$\lambda_0(\theta)\, \beta_0 u'(c_0(\theta)) = \frac{\Phi}{R_0} f(\theta), \quad (30)$$

$$\lambda_0(\theta)\, \beta_0 \frac{1}{\theta} h'\left(\frac{y(\theta)}{\theta}\right) + \xi(\theta)\, \beta_0 \left[\frac{1}{\theta^2} h'\left(\frac{y(\theta)}{\theta}\right) + \frac{y(\theta)}{\theta^3} h''\left(\frac{y(\theta)}{\theta}\right)\right] = \frac{\Phi}{R_0} f(\theta), \quad (31)$$

for any type $(\theta,\mu,\alpha) \in \left[\underline{\theta}, \overline{\theta}\right] \times \left[\underline{\mu}, \overline{\mu}\right] \times \{0,1\}$,

$$\lambda_1(\theta,\mu,1) = p\beta_1 \left[\lambda_0(\theta)\, g(\mu|\theta) - \xi(\theta) \frac{\partial g(\mu|\theta)}{\partial \theta}\right] - \frac{\partial \xi^{\mu}(\theta,\mu,1)}{\partial \mu}, \quad (32)$$

$$\lambda_1(\theta,\mu,0) = (1-p)\,\beta_1 \left[\lambda_0(\theta)\, g(\mu|\theta) - \xi(\theta) \frac{\partial g(\mu|\theta)}{\partial \theta}\right] - \frac{\partial \xi^{\mu}(\theta,\mu,0)}{\partial \mu} + \int_{\underline{\mu}}^{\overline{\mu}} \xi^{\alpha}(\mu';\theta,\mu)\, d\mu', \quad (33)$$

$$\left[\lambda_1(\theta,\mu,1) - \int_{\underline{\mu}}^{\overline{\mu}} \xi^{\alpha}(\mu;\theta,\tilde{\mu})\, d\tilde{\mu}\right] u'(c_1(\theta,\mu,1)) = \frac{p\Phi}{R_1} g(\mu|\theta)\, f(\theta), \quad (34)$$

$$\lambda_1(\theta,\mu,0)\,u'(c_1(\theta,\mu,0)) = \frac{(1-p)\,\Phi}{R_1} g(\mu|\theta)\,f(\theta), \tag{35}$$

and the boundary conditions are satisfied: $\xi(\underline{\theta}) = \xi(\overline{\theta}) = 0$ and $\xi^{\mu}(\theta,\underline{\mu},\alpha) = \xi^{\mu}(\theta,\overline{\mu},\alpha) = 0$ for any θ and any α.

First, we derive the inverse Euler equation. From (30), (34), and (35), we have

$$\begin{aligned}
\lambda_0(\theta) &= \frac{\Phi f(\theta)}{\beta_0 R_0 u'(c_0(\theta))},\\
\lambda_1(\theta,\mu,1) &= \frac{p\Phi g(\mu|\theta)\,f(\theta)}{R_1 u'(c_1(\theta,\mu,1))} + \int_{\underline{\mu}}^{\overline{\mu}} \xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu},\\
\lambda_1(\theta,\mu,0) &= \frac{(1-p)\,\Phi g(\mu|\theta)\,f(\theta)}{R_1 u'(c_1(\theta,\mu,0))}.
\end{aligned}$$

Substitute these expressions into (32) and (33) and sum them together, then integrate over the whole range of μ to yield

$$\frac{1}{\beta_0 R_0 u'(c_0(\theta))} - \frac{\xi(\theta)}{\Phi f(\theta)} \int_{\underline{\mu}}^{\overline{\mu}} \frac{\partial g(\mu|\theta)}{\partial\theta} d\mu = \frac{1}{\beta_1 R_1} \mathbb{E}\left[\frac{1}{u'(c_1(\theta,\mu,\alpha))}\middle|\theta\right],$$

where the expectation is taken over μ and α. Since $\int_{\underline{\mu}}^{\overline{\mu}} \frac{\partial g(\mu|\theta)}{\partial\theta} d\mu = \frac{\partial G(\mu|\theta)}{\partial\theta}\Big|_{\underline{\mu}}^{\overline{\mu}} = 0$, the inverse Euler equation characterizes the savings distortion in $t=0$. Alternatively, we can also derive the inverse Euler equation by using the perturbation method (Golosov et al., 2003).

Next, we derive the labor wedge. Since $\lambda_0(\theta) = \frac{\Phi f(\theta)}{\beta_0 R_0 u'(c_0(\theta))}$, from (29) and (30), we have

$$\xi(\theta) = \int_{\theta}^{\overline{\theta}} \left[\frac{\Phi}{\beta_0 R_0 u'(c_0(\theta'))} - 1\right] f(\theta')\,d\theta'.$$

Therefore, we can rewrite (31) as

$$\begin{aligned}
&\frac{\Phi f(\theta)}{\beta_0 R_0}\left[1 - \frac{\frac{1}{\theta}h'\left(\frac{y(\theta)}{\theta}\right)}{u'(c_0(\theta))}\right]\\
&\qquad = \left[\frac{1}{\theta^2}h'\left(\frac{y(\theta)}{\theta}\right) + \frac{y(\theta)}{\theta^3}h''\left(\frac{y(\theta)}{\theta}\right)\right]\int_{\theta}^{\overline{\theta}}\left[\frac{\Phi}{\beta_0 R_0 u'(c_0(\theta'))} - 1\right] f(\theta')\,d\theta'.
\end{aligned}$$

Let $A(\theta)=\frac{1-F(\theta)}{\theta f(\theta)}$ and $B(\theta)=1+\frac{\frac{y(\theta)}{\theta}h''\left(\frac{y(\theta)}{\theta}\right)}{h'\left(\frac{y(\theta)}{\theta}\right)}$. Dividing both sides by $\frac{1}{\theta}h'\left(\frac{y(\theta)}{\theta}\right)\frac{\Phi f(\theta)}{\beta_0 R_0}$ yields

$$\frac{1}{\frac{1}{\theta}h'\left(\frac{y(\theta)}{\theta}\right)}-\frac{1}{u'\left(c_0\left(\theta\right)\right)}=A\left(\theta\right)B\left(\theta\right)\int_{\theta}^{\overline{\theta}}\left[\frac{1}{u'\left(c_0\left(\theta'\right)\right)}-\frac{\beta_0 R_0}{\Phi}\right]\frac{f\left(\theta'\right)}{1-F\left(\theta\right)}d\theta'.$$

By definition $\frac{1}{\theta}h'\left(\frac{y(\theta)}{\theta}\right)=\left(1-\tau^w(\theta)\right)u'\left(c_0\left(\theta\right)\right)$, so we have

$$\frac{1}{u'\left(c_0\left(\theta\right)\right)}\left(\frac{\tau^w\left(\theta\right)}{1-\tau^w\left(\theta\right)}\right)=A\left(\theta\right)B\left(\theta\right)\int_{\theta}^{\overline{\theta}}\left[\frac{1}{u'\left(c_0\left(\theta'\right)\right)}-\frac{\beta_0 R_0}{\Phi}\right]\frac{f\left(\theta'\right)}{1-F\left(\theta\right)}d\theta'.$$

Define $C(\theta)=\int_{\theta}^{\overline{\theta}}\frac{u'(c_0(\theta))}{u'(c_0(\theta'))}\left[1-\frac{\beta_0 R_0 u'(c_0(\theta'))}{\Phi}\right]\frac{f(\theta')}{1-F(\theta)}d\theta'$, then multiply both sides by $u'\left(c_0\left(\theta\right)\right)$ to get the Mirrlees-Diamond-Saez formula.

Finally, by (29) and the boundary conditions, we have $\int_{\underline{\theta}}^{\overline{\theta}}\lambda_0\left(\theta\right)d\theta=1$. Then, by (30), integrating the whole range of θ yields $\frac{1}{\Phi}=\int_{\underline{\theta}}^{\overline{\theta}}\frac{f(\theta)}{\beta_0 R_0 u'(c_0(\theta))}d\theta$. ■

From Proposition 3, the optimal intertemporal distortion at $t=0$ satisfies the standard inverse Euler equation (Golosov et al., 2003, 2006). By Jensen's inequality, the inverse Euler equation implies $\beta_0 R_0 u'\left(c_0\left(\theta\right)\right)<\beta_1 R_1 \mathbb{E}\left[u'\left(c_1\left(\theta,\mu,\alpha\right)\right)|\theta\right]$ for any productivity θ. Due to informational constraints, the transfer of consumption from $t=0$ to $t=1$ is restricted regardless of the agents' productivity. By restricting savings, the government can induce truth-telling of ex-post private information (μ,α) at a lower cost, which relaxes the ex-post incentive constraints (1). Thus, the savings wedge in $t=0$ is strictly positive for any θ.

Proposition 3 also shows that the labor wedge is characterized by the economic forces found in static Mirrlees taxation models, such as Mirrlees (1971), Diamond (1998), and Saez (2001). First note that, when distorted, the labor supply of θ-agents changes according to their Frisch elasticity, which is $B\left(\theta\right)$. Furthermore, $A\left(\theta\right)$ captures the trade-off between the loss of total output in proportion of $\theta f\left(\theta\right)$ from an increase in labor distortions for θ-agents and the relaxation of incentive constraints for higher productivity agents of mass $1-F\left(\theta\right)$. The labor wedge also consists of $C\left(\theta\right)$, which captures the strength of the government's insurance motive against productivity differences. Since the inverse marginal utility is the cost of a marginal increase in utility in consumption terms, the cost of a marginal increase in

average utility in $t = 1$ is $\frac{1}{\Phi}$. Note that if the cost of increasing average utility is small relative to the cost of increasing the utility of θ'-agents, i.e., $\frac{1}{\Phi} < \frac{1}{\beta_0 R_0 u'(c_0(\theta'))}$, then the benefits of increasing the labor wedge of θ-agents to relax the ex-post incentive constraints of higher productivity agents ($\theta' \geq \theta$) outweighs the cost. Intuitively, the degree of labor distortion also increases with consumption inequality, which is represented by $\frac{u'(c_0(\theta))}{u'(c_0(\theta'))}$.

C.2 Post-retirement wedges

This section presents the optimal retirement savings wedges and the bequest wedges in a setting without aggregate shocks. In contrast to the pre-retirement wedges, the post-retirement distortions are shaped by both sequential and multidimensional screening.

C.2.1 Retirement savings

This section presents the retirement savings wedges. The ex-ante incentive constraints affect retirement savings through the *intertemporal component* $D(\theta, \mu)$ and the *intratemporal component* $E(\theta, \mu, \alpha)$, defined as

$$D(\theta, \mu) = \frac{-\frac{\partial G(\mu|\theta)}{\partial \theta}}{g(\mu|\theta)} \frac{1 - F(\theta)}{f(\theta)} C(\theta),$$

$$E(\theta, \mu, \alpha) = \frac{1 - G(\mu|\theta)}{g(\mu|\theta)} \left\{ \frac{\int_{\mu}^{\overline{\mu}} \left[\frac{1}{u'(c_1(\theta, \mu', \alpha))} \right] \frac{g(\mu'|\theta)}{1 - G(\mu|\theta)} d\mu'}{\mathbb{E}\left[\frac{1}{u'(c_1(\theta, \tilde{\mu}, \tilde{\alpha}))} | \theta \right]} - 1 \right\}.$$

Unique to our multidimensional setting, distortions in retirement savings have an additional role in relaxing the incentive constraints on altruism (16). This novel economic force is

represented by the following terms

$$I_1(\theta,\mu,1)=\frac{R_1u'(c_1(\theta,\mu,1))}{\Phi pg(\mu|\theta)f(\theta)}\left[\frac{S_2'(\mu)}{S_2(\mu)}\int_{\mu}^{\overline{\mu}}\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\mu';\theta,\tilde{\mu})\,d\tilde{\mu}d\mu'-\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_2(\tilde{\mu})}{S_2(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}\right],$$

$$I_1(\theta,\mu,0)=-\frac{S_2'(\mu)}{S_2(\mu)}\frac{R_1u'(c_1(\theta,\mu,0))}{\Phi(1-p)g(\mu|\theta)f(\theta)}\int_{\mu}^{\overline{\mu}}\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\tilde{\mu};\theta,\mu')\,d\tilde{\mu}d\mu',$$

$$I_2(\theta,\mu,1)=\frac{R_1u'(c_1(\theta,\mu,1))}{\Phi pg(\mu|\theta)f(\theta)}\int_{\underline{\mu}}^{\overline{\mu}}\frac{S_3'(\mu)}{S_3(\mu)}\left[\frac{S_3(\mu)-S_3(\tilde{\mu})}{S_3'(\mu)}-\frac{S_2(\mu)-S_2(\tilde{\mu})}{S_2'(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu},$$

where $\xi^{\alpha}\geq 0$ is the shadow price on (16). Precisely, $\xi^{\alpha}(\mu';\theta,\mu)$ represents the shadow price of deterring a non-altruistic agent with productivity θ and longevity type μ from misreporting as an altruistic agent with longevity risk μ'. The next proposition shows how these economic forces jointly determine the distortions to retirement savings.

Proposition 4 *For any type (θ,μ,α), the retirement savings wedge at $t=1$ satisfies*

$$\tau_1^s(\theta,\mu,\alpha)=\frac{S_2'(\mu)}{S_2(\mu)}\mathbb{E}\left[\frac{u'(c_1(\theta,\mu,\alpha))}{u'(c_1(\theta,\tilde{\mu},\tilde{\alpha}))}\middle|\theta\right][D(\theta,\mu)+E(\theta,\mu,\alpha)]+I_1(\theta,\mu,\alpha),\tag{36}$$

where the expectation is over $(\tilde{u},\tilde{\alpha})$, and the retirement savings wedge at $t=2$ satisfies

$$\tau_2^s(\theta,\mu,\alpha)=\left[\frac{S_3'(\mu)/S_3(\mu)}{S_2'(\mu)/S_2(\mu)}-1\right]\frac{\tau_1^s(\theta,\mu,\alpha)}{1-\tau_1^s(\theta,\mu,\alpha)}-\mathbf{1}_{\alpha=1}\frac{I_2(\theta,\mu,1)}{1-\tau_1^s(\theta,\mu,\alpha)},\tag{37}$$

where $\mathbf{1}_{\alpha=1}=1$ when $\alpha=1$ and zero otherwise.

Proof Let $\Pr(\alpha)$ denote the probability that altruism type is α, so $\Pr(1)=p$. To simplify notation, let $\Lambda_t(\theta,\mu,\alpha)=\lambda_1(\theta,\mu,\alpha)-\mathbf{1}_{\alpha=1}\int_{\underline{\mu}}^{\overline{\mu}}\frac{S_t(\tilde{\mu})}{S_t(\mu)}\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}$, where $\mathbf{1}_{\alpha=1}=1$ when $\alpha=1$ and zero for all other values of α.

First, to characterize the retirement savings wedges at $t=1$, note that the first-order necessary condition for $c_2(\theta,\mu,\alpha)$ is

$$\Lambda_2(\theta,\mu,\alpha)-\frac{S_2'(\mu)}{S_2(\mu)}\xi^{\mu}(\theta,\mu,\alpha)=\frac{\Pr(\alpha)\Phi g(\mu|\theta)f(\theta)}{\beta R_1R_2u'(c_2(\theta,\mu,\alpha))}.\tag{38}$$

We will first derive (36). By (34) and (35),

$$\Lambda_t(\theta,\mu,\alpha)=\frac{\Pr(\alpha)\Phi g(\mu|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu,\alpha))}+\mathbf{1}_{\alpha=1}\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_t(\tilde{\mu})}{S_t(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu},$$

which we substitute into (38) to get

$$\frac{R_1\left\{\mathbf{1}_{\alpha=1}\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_2(\tilde{\mu})}{S_2(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}-\frac{S_2'(\mu)}{S_2(\mu)}\xi^{\mu}(\theta,\mu,\alpha)\right\}}{\Pr(\alpha)\Phi g(\mu|\theta)f(\theta)}$$
$$+\frac{1}{u'(c_1(\theta,\mu,\alpha))}=\frac{1}{\beta R_2u'(c_2(\theta,\mu,\alpha))}.$$

Since $\tau_1^s(\theta,\mu,\alpha)=1-\frac{u'(c_1(\theta,\mu,\alpha))}{\beta R_2u'(c_2(\theta,\mu,\alpha))}$ by definition, we have

$$\tau_1^s(\theta,\mu,\alpha)=\frac{R_1u'(c_1(\theta,\mu,\alpha))}{\Pr(\alpha)\Phi g(\mu|\theta)f(\theta)}\left\{\frac{S_2'(\mu)}{S_2(\mu)}\xi^{\mu}(\theta,\mu,\alpha)\right.$$
$$\left.-\mathbf{1}_{\alpha=1}\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_2(\tilde{\mu})}{S_2(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}\right\}. \quad (39)$$

From (29) and (30), we have $\lambda_0(\theta)=\frac{\Phi f(\theta)}{\beta_0R_0u'(c_0(\theta))}$, $\xi(\theta)=\int_{\theta}^{\overline{\theta}}\left[\frac{\Phi}{\beta_0R_0u'(c_0(\theta'))}-1\right]f(\theta')\,d\theta'$, and by (34) and (35), $\lambda_1(\theta,\mu',\alpha)=\frac{\Pr(\alpha)\Phi g(\mu'|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu',\alpha))}+\mathbf{1}_{\alpha=1}\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\mu';\theta,\tilde{\mu})\,d\tilde{\mu}$. Thus, by (32) and (33), we have

$$\xi^{\mu}(\theta,\mu,\alpha)=\frac{\Pr(\alpha)\Phi f(\theta)}{R_1}\left\{\int_{\mu}^{\overline{\mu}}\frac{g(\mu'|\theta)}{u'(c_1(\theta,\mu',\alpha))}d\mu'-\frac{\beta_1R_1}{\beta_0R_0u'(c_0(\theta))}[1-G(\mu|\theta)]\right.$$
$$\left.-\beta_1R_1\frac{\partial G(\mu|\theta)}{\partial\theta}\int_{\theta}^{\overline{\theta}}\left[\frac{1}{\beta_0R_0u'(c_0(\theta'))}-\frac{1}{\Phi}\right]\frac{f(\theta')}{f(\theta)}d\theta'\right\}$$
$$+\mathbf{1}_{\alpha=1}\int_{\mu}^{\overline{\mu}}\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\mu';\theta,\tilde{\mu})\,d\tilde{\mu}d\mu'-\mathbf{1}_{\alpha=0}\int_{\mu}^{\overline{\mu}}\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\tilde{\mu};\theta,\mu')\,d\tilde{\mu}d\mu',$$

where $\mathbf{1}_{\alpha=0}=1$ when $\alpha=0$ and zero for all other values of α. Since by definition, $C(\theta)=$

$\int_{\theta}^{\overline{\theta}} \frac{u'(c_0(\theta))}{u'(c_0(\theta'))}\left[1-\frac{\beta_0 R_0 u'(c_0(\theta'))}{\Phi}\right]\frac{f(\theta')}{1-F(\theta)}d\theta'$, we have

$$\xi^{\mu}(\theta,\mu,\alpha)=\frac{\beta_1 \Pr(\alpha)\Phi f(\theta)}{\beta_0 R_0 u'(c_0(\theta))}\left\{\int_{\mu}^{\overline{\mu}}\frac{\beta_0 R_0 u'(c_0(\theta))\, g(\mu'|\theta)}{\beta_1 R_1 u'(c_1(\theta,\mu',\alpha))}d\mu'-[1-G(\mu|\theta)]\right.$$
$$\left.-\frac{\partial G(\mu|\theta)}{\partial\theta}\frac{1-F(\theta)}{f(\theta)}C(\theta)\right\}+\Xi^{\alpha}(\theta,\mu,\alpha),$$

where $\Xi^{\alpha}(\theta,\mu,\alpha) = \mathbf{1}_{\alpha=1}\int_{\mu}^{\overline{\mu}}\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\mu';\theta,\tilde{\mu})\,d\tilde{\mu}d\mu' - \mathbf{1}_{\alpha=0}\int_{\mu}^{\overline{\mu}}\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\tilde{\mu};\theta,\mu')\,d\tilde{\mu}d\mu'$. Finally, substituting the above expression for $\xi^{\mu}(\theta,\mu,\alpha)$ into (39), and using the inverse Euler equation, yields (36).

Finally, to derive the retirement savings wedge τ_2^s, the first-order necessary condition for $c_3(\theta,\mu,\alpha)$ is

$$\Lambda_3(\theta,\mu,\alpha)-\frac{S_3'(\mu)}{S_3(\mu)}\xi^{\mu}(\theta,\mu,\alpha)=\frac{\Pr(\alpha)\Phi g(\mu|\theta)f(\theta)}{\beta^2 R_1 R_2 R_3 u'(c_3(\theta,\mu,\alpha))}. \tag{40}$$

To derive (37), note that (38) and (40) imply

$$\frac{u'(c_2(\theta,\mu,\alpha))}{\beta R_3 u'(c_3(\theta,\mu,\alpha))}=\frac{\Lambda_3(\theta,\mu,\alpha)-\xi^{\mu}(\theta,\mu,\alpha)\frac{S_3'(\mu)}{S_3(\mu)}}{\Lambda_2(\theta,\mu,\alpha)-\xi^{\mu}(\theta,\mu,\alpha)\frac{S_2'(\mu)}{S_2(\mu)}}.$$

Since $\tau_2^s(\theta,\mu,\alpha)=1-\frac{u'(c_2(\theta,\mu,\alpha))}{\beta R_3 u'(c_3(\theta,\mu,\alpha))}$, we have

$$1-\tau_2^s(\theta,\mu,\alpha)=\frac{\Lambda_3(\theta,\mu,\alpha)-\xi^{\mu}(\theta,\mu,\alpha)\frac{S_3'(\mu)}{S_3(\mu)}}{\Lambda_2(\theta,\mu,\alpha)-\xi^{\mu}(\theta,\mu,\alpha)\frac{S_2'(\mu)}{S_2(\mu)}}. \tag{41}$$

Note that $\Lambda_t(\theta,\mu,\alpha)=\frac{\Pr(\alpha)\Phi g(\mu|\theta)f(\theta)}{R_1 u'(c_1(\theta,\mu,\alpha))}+\mathbf{1}_{\alpha=1}\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_t(\tilde{\mu})}{S_t(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}$ by (34) and (35), and by (39),

$$\xi^{\mu}(\theta,\mu,\alpha)=\frac{1}{\frac{S_2'(\mu)}{S_2(\mu)}}\left[\frac{\Phi\Pr(\alpha)\,g(\mu|\theta)\,f(\theta)}{R_1 u'(c_1(\theta,\mu,\alpha))}\tau_1^s(\theta,\mu,\alpha)+\mathbf{1}_{\alpha=1}\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_2(\tilde{\mu})}{S_2(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}\right].$$

Substituting the above expressions for $\Lambda_t(\theta,\mu,\alpha)$ and $\xi^{\mu}(\theta,\mu,\alpha)$ into (41) yields (37). ■

Proposition 4 presents the retirement savings wedges and characterizes the economic forces that relax the ex-ante incentive constraints and the ex-post incentive constraints. We first examine the retirement savings wedge τ_1^s in (36).

The intertemporal component $D(\theta, \mu)$ introduces savings distortions after retirement to relax the ex-ante incentive constraints (2). To see this, notice that $D(\theta, \mu)$ is increasing in the insurance motive $C(\theta)$ from $t = 0$. Recall that when $C(\theta)$ is large, then the government has an incentive to increase the labor wedge of θ-agents to raise redistribution. From (36), the consumption path for θ-agents is more front-loaded when $C(\theta)$ is large. To understand this result, note that agents with productivity higher than θ are likely to live longer (higher μ) by assumption, so they prefer a consumption path that is relatively more back-loaded. Therefore, the front-loading of consumption for θ-agents helps relax the ex-ante incentive constraints (2) of more productive agents. Intuitively, the intertemporal component $D(\theta, \mu)$ shows how the benefits from this distortion increases with the mass of agents whose ex-ante incentive constraints are relaxed, $1 - F(\theta)$, and decreases with the population of agents who are made worse off, $f(\theta)$. By Baron and Besanko (1984), $-\frac{\partial G(\mu|\theta)}{\partial \theta}/g(\mu|\theta)$ is referred to as the *informativeness measure*. A higher informativeness measure implies a stronger link between productivity θ and longevity risk μ, which in turn means that distortions in retirement consumption paths would be more effective at incentivizing work in $t = 0$. Thus, the informativeness measure increases the intertemporal component $D(\theta, \mu)$.

Next, $I_1(\theta, \mu, \alpha)$ captures the economic force that deters non-altruistic agents from mimicking altruistic agents. The second component of $I_1(\theta, \mu, 1)$, $\int_{\underline{\mu}}^{\overline{\mu}} \left[1 - \frac{S_2(\tilde{\mu})}{S_2(\mu)}\right] \xi^{\alpha}(\mu; \theta, \tilde{\mu})\, d\tilde{\mu}$, affects $I_1(\theta, \mu, 1)$ based on its sign. This component is negative when the tendency for non-altruistic agents who live longer to pretend to be $(\mu, 1)$ is stronger, and positive when non-altruistic agents who live shorter pretend to be $(\mu, 1)$. Intuitively, a negative component increases $I_1(\theta, \mu, 1)$ thereby front-loading the consumption path to deter such deviations, while a positive component decreases $I_1(\theta, \mu, 1)$, leading to less front-loaded consumption path. Furthermore, the ex-post incentive constraints on altruism (16) impact the ex-post incentive constraints on longevity (17). Specifically, the additional distortion to deter misreports on altruism may incentivize misreports on longevity within a given productivity and altruism type. As a result, the first component of $I_1(\theta, \mu, 1)$ shows that if the multiplier

$\xi^{\alpha}(\mu';\theta,\tilde{\mu})$ for $\mu' > \mu$ is strictly positive, then the retirement savings wedges for agents with lower longevity types must also be distorted upwards. These additional distortions to retirement savings are necessary to maintain incentive compatibility on longevity for a fixed productivity and altruism type. The same is true for non-altruistic agents.

Since both $D(\theta,\mu)$ and $I_1(\theta,\mu,\alpha)$ introduce inequality in consumption among agents of the same productivity, the intratemporal component $E(\theta,\mu,\alpha)$ offsets some of this inequality. Since agents are risk averse, the ex-ante incentive constraint for θ-agents tightens when their consumption inequality during retirement is increased relative to other productivity types. Notice the negative (positive) intratemporal component for (θ,μ,α)-type agents, wherein agents with identical productivity θ and altruism α but facing higher longevity risk $\mu' > \mu$ consume less (more) in period $t = 1$ compared to the average θ-agent. To understand its effects, note that for a fixed type (θ,μ,α), their consumption paths are more front-loaded relative to the agents who share the same productivity and altruism type but expect to live longer. In this case, from (36), a negative intratemporal component $E(\theta,\mu,\alpha)$ offsets the front-loaded consumption, which improves post-retirement equality. At the extreme, $E(\theta,\mu,\alpha)$ is zero when all θ-agents consume the same after retirement, i.e., retirement consumption is insured against differences in (μ,α).

From (36), the retirement savings wedge τ_1^s also increases with the survival probability's sensitivity to longevity types: $\frac{S_2'(\mu)}{S_2(\mu)}$. Intuitively, post-retirement savings distortions are more useful in screening both productivity in $t = 0$ and altruism in $t = 1$ when the survival probabilities are sensitive to μ. Also, by (36), the degree of savings distortion in τ_1^s for (θ,μ,α)-type agents is magnified when they consume less than the average population in $t = 1$.

Finally, by (37), τ_2^s is increasing with the previous period's savings distortion τ_1^s if the one-period survival probability at $t = 2$ (probability of surviving to $t = 3$ conditional on surviving to $t = 2$) is strictly increasing in μ, i.e., $\frac{S_3(\mu)}{S_2(\mu)}$ is strictly increasing in μ.[26] Note that if $\frac{S_3(\mu)}{S_2(\mu)}$ is strictly increasing in μ, then the single-crossing property also holds in $t = 2$. In essence, agents with higher longevity types prefer more back-loaded consumption paths in $t = 2$ as well, and they also prefer consumption over bequests. Therefore, the logic for the distortions

[26] If $\frac{S_3(\mu)}{S_2(\mu)}$ is strictly increasing, then $S_3'(\mu)S_2(\mu) > S_2'(\mu)S_3(\mu)$, which implies $\frac{S_3'(\mu)/S_3(\mu)}{S_2'(\mu)/S_2(\mu)} > 1$.

in τ_1^s extends to the retirement savings wedge τ_2^s. It is worth noting that if $\frac{S_3(\mu)}{S_2(\mu)}$ is decreasing in μ, then the sign of τ_2^s is the opposite of τ_1^s so the post-retirement consumption path is either hump-shaped or U-shaped. Meanwhile, $I_2(\theta,\mu,1)$ makes additional adjustments to τ_2^s to deter non-altruistic agents from misreporting as altruistic agents.

C.2.2 Bequests

Since it is optimal for non-altruistic agents ($\alpha=0$) to leave no bequests, this section characterizes the bequest wedges τ_t^d for altruistic agents ($\alpha=1$). Denote the mass of agents dying in t as $M_t(\mu)$, i.e., $M_2(\mu)=1-S_2(\mu)$, $M_3(\mu)=S_2(\mu)-S_3(\mu)$, and $M_4(\mu)=S_3(\mu)$. Let

$$J_t(\theta,\mu,1)=\frac{R_1u'(c_1(\theta,\mu,1))}{p\Phi g(\mu|\theta)f(\theta)}\int_{\underline{\mu}}^{\overline{\mu}}\frac{M_t(\tilde{\mu})}{M_t(\mu)}\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}$$

denote a novel component in the bequest wedges that deter non-altruistic agents from misreporting as altruistic agents. The following proposition presents the optimal bequest wedges.

Proposition 5 *For any type $(\theta,\mu,1)$, the bequest wedges satisfy*

$$\tau_1^d(\theta,\mu,1)=-\frac{S_2(\mu)}{M_2(\mu)}\tau_1^s(\theta,\mu,1)-J_2(\theta,\mu,1), \tag{42}$$

$$\tau_2^d(\theta,\mu,1)=-\frac{S_3(\mu)}{M_3(\mu)}\tau_2^s(\theta,\mu,1)-\frac{J_3(\theta,\mu,1)}{1-\tau_1^s(\theta,\mu,1)}, \tag{43}$$

$$\tau_3^d(\theta,\mu,1)=-\frac{J_4(\theta,\mu,1)}{[1-\tau_1^s(\theta,\mu,1)][1-\tau_2^s(\theta,\mu,1)]}. \tag{44}$$

Proof Recall that $x_t(\theta,\mu,0)=0$, so we focus on the bequest wedge for $\alpha=1$. First, we derive the bequest wedge at $t=1$. The first-order necessary condition for $x_1(\theta,\mu,1)$ is

$$\lambda_1(\theta,\mu,1)+\frac{S_2'(\mu)}{1-S_2(\mu)}\xi^{\mu}(\theta,\mu,1)=\frac{p\Phi g(\mu|\theta)f(\theta)}{\beta R_1R_2v'(x_1(\theta,\mu,1))}. \tag{45}$$

By (34) and (35), $\lambda_1(\theta,\mu,1)=\frac{p\Phi g(\mu|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu,1))}+\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}$, which we substitute into (45)

to get

$$\frac{1}{u'\left(c_1\left(\theta,\mu,1\right)\right)}+\frac{R_1}{p\Phi g\left(\mu|\theta\right)f\left(\theta\right)}\left[\frac{S_2'\left(\mu\right)}{1-S_2\left(\mu\right)}\xi^{\mu}\left(\theta,\mu,1\right)+\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}\left(\mu;\theta,\tilde{\mu}\right)d\tilde{\mu}\right]$$
$$=\frac{1}{\beta R_2 v'\left(x_1\left(\theta,\mu,1\right)\right)}.$$

Since $\tau_1^d\left(\theta,\mu,1\right)=1-\frac{u'(c_1(\theta,\mu,1))}{\beta R_2 v'(x_1(\theta,\mu,1))}$ by definition, we have

$$\tau_1^d\left(\theta,\mu,1\right)=-\frac{R_1u'\left(c_1\left(\theta,\mu,1\right)\right)}{p\Phi g\left(\mu|\theta\right)f\left(\theta\right)}\left[\frac{S_2'\left(\mu\right)}{1-S_2\left(\mu\right)}\xi^{\mu}\left(\theta,\mu,1\right)+\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}\left(\mu;\theta,\tilde{\mu}\right)d\tilde{\mu}\right]. \tag{46}$$

By (39), (46) can be rewritten as

$$\tau_1^d\left(\theta,\mu,1\right)=-\frac{S_2\left(\mu\right)}{1-S_2\left(\mu\right)}\tau_1^s\left(\theta,\mu,1\right)-\frac{R_1u'\left(c_1\left(\theta,\mu,1\right)\right)}{p\Phi g\left(\mu|\theta\right)f\left(\theta\right)}\int_{\underline{\mu}}^{\overline{\mu}}\left[\frac{1-S_2\left(\tilde{\mu}\right)}{1-S_2\left(\mu\right)}\right]\xi^{\alpha}\left(\mu;\theta,\tilde{\mu}\right)d\tilde{\mu}.$$

Finally, we derive the bequest wedge $\tau_2^d\left(\theta,\mu,1\right)$. The first-order necessary conditions for $x_2\left(\theta,\mu,1\right)$ and $x_3\left(\theta,\mu,1\right)$ are

$$\lambda_1\left(\theta,\mu,1\right)-\left[\frac{S_2'\left(\mu\right)-S_3'\left(\mu\right)}{S_2\left(\mu\right)-S_3\left(\mu\right)}\right]\xi^{\mu}\left(\theta,\mu,1\right)=\frac{p\Phi g\left(\mu|\theta\right)f\left(\theta\right)}{\beta^2R_1R_2R_3v'\left(x_2\left(\theta,\mu,1\right)\right)}, \tag{47}$$

$$\lambda_1\left(\theta,\mu,1\right)-\frac{S_3'\left(\mu\right)}{S_3\left(\mu\right)}\xi^{\mu}\left(\theta,\mu,1\right)=\frac{p\Phi g\left(\mu|\theta\right)f\left(\theta\right)}{\beta^3R_1R_2R_3R_4v'\left(x_3\left(\theta,\mu,1\right)\right)}. \tag{48}$$

To derive (44), note that since $\lambda_1\left(\theta,\mu,1\right)=\frac{p\Phi g(\mu|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu,1))}+\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}\left(\mu;\theta,\tilde{\mu}\right)d\tilde{\mu}$, by (38), (47) and the definition of τ_t^d, we have

$$1-\tau_2^d\left(\theta,\mu,1\right)=\frac{\frac{p\Phi g(\mu|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu,1))}+\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}\left(\mu;\theta,\tilde{\mu}\right)d\tilde{\mu}-\xi^{\mu}\left(\theta,\mu,1\right)\left[\frac{S_2'(\mu)-S_3'(\mu)}{S_2(\mu)-S_3(\mu)}\right]}{\frac{p\Phi g(\mu|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu,1))}+\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_2(\tilde{\mu})}{S_2(\mu)}\right]\xi^{\alpha}\left(\mu;\theta,\tilde{\mu}\right)d\tilde{\mu}-\xi^{\mu}\left(\theta,\mu,1\right)\frac{S_2'(\mu)}{S_2(\mu)}}. \tag{49}$$

By (39), (49) can be rewritten as

$$\tau_2^d(\theta,\mu,1)=\left\{\frac{S_2(\mu)}{S_2'(\mu)}\left[\frac{S_2'(\mu)-S_3'(\mu)}{S_2(\mu)-S_3(\mu)}\right]-1\right\}\frac{\tau_1^s(\theta,\mu,1)}{1-\tau_1^s(\theta,\mu,1)} - \frac{\frac{R_1u'(c_1(\theta,\mu,1))}{p\Phi g(\mu|\theta)f(\theta)}\int_{\underline{\mu}}^{\overline{\mu}}\left\{1-\frac{[S_2(\mu)-S_2(\tilde{\mu})]\left[S_2'(\mu)-S_3'(\mu)\right]}{S_2'(\mu)[S_2(\mu)-S_3(\mu)]}\right\}\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}}{1-\tau_1^s(\theta,\mu,1)}.$$

By Proposition 4, $\frac{\tau_1^s(\theta,\mu,1)}{1-\tau_1^s(\theta,\mu,1)}=\frac{\tau_2^s(\theta,\mu,1)+\frac{\hat{I}(\theta,\mu,1)}{1-\tau_1^s(\theta,\mu,1)}}{\frac{S_3'(\mu)/S_3(\mu)}{S_2'(\mu)/S_2(\mu)}-1}$, and substituting this into the expression above yields

$$\tau_2^d(\theta,\mu,1)=-\frac{S_3(\mu)}{S_2(\mu)-S_3(\mu)}\tau_2^s(\theta,\mu,1)-\frac{\frac{R_1u'(c_1(\theta,\mu,1))}{p\Phi g(\mu|\theta)f(\theta)}\int_{\underline{\mu}}^{\overline{\mu}}\left[\frac{S_2(\tilde{\mu})-S_3(\tilde{\mu})}{S_2(\mu)-S_3(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}}{1-\tau_1^s(\theta,\mu,1)}.$$

We follow a similar procedure to derive τ_3^d. From (40) and (48), the definition of τ_t^d yields

$$1-\tau_3^d(\theta,\mu,1)=\frac{\frac{p\Phi g(\mu|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu,1))}+\int_{\underline{\mu}}^{\overline{\mu}}\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}-\xi^{\mu}(\theta,\mu,1)\frac{S_3'(\mu)}{S_3(\mu)}}{\frac{p\Phi g(\mu|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu,1))}+\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_3(\tilde{\mu})}{S_3(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}-\xi^{\mu}(\theta,\mu,1)\frac{S_3'(\mu)}{S_3(\mu)}}. \tag{50}$$

By (39),

$$\xi^{\mu}(\theta,\mu,1)=\frac{1}{\frac{S_2'(\mu)}{S_2(\mu)}}\left\{\frac{\Phi pg(\mu|\theta)f(\theta)}{R_1u'(c_1(\theta,\mu,1))}\tau_1^s(\theta,\mu,1)+\int_{\underline{\mu}}^{\overline{\mu}}\left[1-\frac{S_2(\tilde{\mu})}{S_2(\mu)}\right]\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}\right\},$$

and by (41),

$$[1-\tau_2^s(\theta,\mu,1)]\left[\Lambda_2(\theta,\mu,1)-\xi^{\mu}(\theta,\mu,1)\frac{S_2'(\mu)}{S_2(\mu)}\right]=\Lambda_3(\theta,\mu,1)-\xi^{\mu}(\theta,\mu,1)\frac{S_3'(\mu)}{S_3(\mu)},$$

substituting into (50) yields

$$\tau_3^d(\theta,\mu,1)=-\frac{\frac{R_1u'(c_1(\theta,\mu,1))}{p\Phi g(\mu|\theta)f(\theta)}\int_{\underline{\mu}}^{\overline{\mu}}\frac{S_3(\tilde{\mu})}{S_3(\mu)}\xi^{\alpha}(\mu;\theta,\tilde{\mu})\,d\tilde{\mu}}{[1-\tau_1^s(\theta,\mu,1)][1-\tau_2^s(\theta,\mu,1)]}.$$

This completes the proof. ■

The relevant incentive constraint on altruism is to deter non-altruistic agents from pretending to be altruistic. Proposition 5 shows how it is optimal to subsidize bequests to deter such deviations, since non-altruistic agents do not value bequests. This is captured by the component $J_t(\theta, \mu, 1) \geq 0$.

Proposition 5 also shows how the optimal bequest wedges complement the retirement savings wedges. First, for any type $(\theta, \mu, 1)$, the bequest wedges have the opposite sign of the retirement savings wedges. Second, bequest wedges are increasing in the survivor-to-casualty ratio: $\frac{S_2(\mu)}{M_2(\mu)}$ for $t = 2$ and $\frac{S_3(\mu)}{M_3(\mu)}$ for $t = 3$. These features, together with the retirement savings wedges in Proposition 4, help relax the ex-ante incentive constraints on productivity (2). More productive agents likely live longer, so they place less weight on the warm glow from leaving bequests. Therefore, if the retirement savings wedges are positive, then the bequest wedges are negative, i.e., bequests are subsidized. In other words, since more productive agents are less likely to value bequests, a bequest subsidy discourages downward misreports in productivity.

D Characterization with mean-variance arbitrageurs

This appendix follows Greenwood et al. (2010) and presents a characterization of the arbitrageurs' long-term bond demand with mean-variance preferences, and the equilibrium long-term bond return R_l when supply Λ is exogenous.

First, we derive the arbitrageurs' long-term bond demand. At $t = 1$, arbitrageurs maximize wealth by buying and selling one- and two-period bonds. Specifically, arbitrageurs have an initial wealth of zero at $t = 1$, and they maximize a mean-variance objective over their terminal wealth at $t = 3$. Following Vayanos and Vila (2021), arbitrageurs choose their two-period bond demand H to maximize $\mathbb{E}(w) - \frac{\gamma}{2}\mathbb{V}(w)$, subject to $w = H(R_l - R_2 R_3)$, where γ is the arbitrageur's risk tolerance and $\mathbb{V}(w)$ is the variance of terminal wealth. Thus, the arbitrageurs' two period bond demand is

$$H(R_l) = \frac{R_l - R_2 \mathbb{E}(R_3)}{\gamma R_2^2 \mathbb{V}(R_3)}.$$

Intuitively, arbitrageurs borrow short and invest long when $R_l > R_2\mathbb{E}(R_3)$, and they borrow long and invest at the short rate when $R_l < R_2\mathbb{E}(R_3)$.

Finally, we determine the equilibrium long-term bond return. Let Λ denote the exogenous supply of two-period government bonds. Then, given $H(R_l)$, the market clearing condition is $\Lambda = \Omega H(R_l) + \frac{1}{R_l} L_3(q^*)$. Notice immediately that the equilibrium return R_l is a function of the supply Λ and the equilibrium annuity price q^*. Intuitively, R_l should increase with supply Λ and equilibrium annuity price q^* (a low annuity price implies a smaller demand for long-term bonds by the insurers). Therefore, the solution is

$$R_l(\Lambda, q^*) = \frac{\left[\Lambda + \frac{\Omega\mathbb{E}(R_3)}{\gamma R_2 \mathbb{V}(R_3)}\right] + \sqrt{\left[\Lambda + \frac{\Omega\mathbb{E}(R_3)}{\gamma R_2 \mathbb{V}(R_3)}\right]^2 - \frac{4\Omega L_3(q^*)}{\gamma R_2^2 \mathbb{V}(R_3)}}}{2\Omega\left[\gamma R_2^2 \mathbb{V}(R_3)\right]^{-1}}.$$

In contrast to this appendix, our paper considers a government that enodgenizes the supply of long-term government bonds Λ in a welfare-maximizing manner, as well as a more general demand function H for arbitrageurs.

E Interest rate risk sharing without welfare concerns

In this appendix, we determine the efficient interest rate risk sharing scheme between the government and private annuity insurers, independent of the welfare of the retirees. The purpose is to demonstrate how the government's debt maturity structure can influence the insurers' interest rate risk management cost.

By Proposition 2, the total cost of interest rate risk management in the economy at $t = 0$ is $\frac{NW_1}{R_1} + \mathbb{E}\left[\frac{T(R_3)}{R_1 R_2}\right]$. Therefore, to implement the risk sharing scheme that minimizes the cost of interest rate risk, the government chooses long-term bond supply Λ, return R_l, lump-sum tax $T(R_3)$, and equilibrium annuity price q^* to solve the following problem:

$$\min_{T,\Lambda,R_l,q^*} \mathbf{1}_{\frac{1}{R_l} \geq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)} \left[\frac{1}{R_l} - \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3(q^*) + \mathbb{E}\left[\frac{T(R_3)}{R_2}\right],$$

subject to (8), the long-term bond revenue constraint (9), the market clearing condition (10), and $R_l \geq R_2$. The indicator function is such that $\mathbf{1}_{\frac{1}{R_l} \geq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)} = 1$ when $\frac{1}{R_l} \geq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$

and zero otherwise.

Proposition 6 *The optimal interest rate risk sharing scheme satisfies:*

i. $NW_1 = 0$ *and* $q^* = \tilde{q}$,

ii. There exists $\bar{\Omega}$ *such that the optimal long-term bond return* R_l *satisfies* $\frac{1}{R_l} = \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$ *when* $\Omega \leq \bar{\Omega}$ *and* $R_l \in \left(\frac{R_2}{\mathbb{E}\left(\frac{1}{R_3}\right)}, R_2\mathbb{E}\left(R_3\right)\right)$ *when* $\Omega > \bar{\Omega}$, *and*

iii. $\mathbb{E}\left[T\left(R_3\right)\right] = 0$ *when* $\Omega \leq \bar{\Omega}$ *and* $\mathbb{E}\left[T\left(R_3\right)\right] < 0$ *when* $\Omega > \bar{\Omega}$.

iv. $\Lambda \geq 0$ *when* $\Omega \leq \bar{\Omega}$ *and* $\Lambda < 0$ *when* $\Omega > \bar{\Omega}$.

Proof At the optimum, (9) binds, so the optimization problem can be rewritten as

$$\min_{R_l, q^*} \mathbf{1}_{\frac{1}{R_l} \geq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)} \left[\frac{1}{R_l} - \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)\right] L_3\left(q^*\right) + \left[\frac{R_l}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right) - 1\right]\left[\Omega H\left(R_l\right) + \frac{L_3\left(q^*\right)}{R_l}\right],$$

subject to (8) and $R_l \geq R_2$. Consider two cases: $\frac{1}{R_l} \geq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$ and $\frac{1}{R_l} < \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$.

First, if $\frac{1}{R_l} \geq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$, then $R_l < R_2\mathbb{E}\left(R_3\right)$ by Jensen's inequality. Hence, $H\left(R_l\right) < 0$, i.e., arbitrageurs short the long-term bond. The optimization problem can be rewritten as

$$\min_{R_l, q^*} \Omega\left[\frac{R_l}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right) - 1\right] H\left(R_l\right),$$

subject to (8) and $R_l \geq R_2$. Since $H\left(R_l\right) < 0$ and $\frac{1}{R_l} \geq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$, the cost of interest rate risk management is minimized when $\frac{1}{R_l} = \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$.

Next, we consider $\frac{1}{R_l} \leq \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$, so the optimization problem can be rewritten as

$$\min_{R_l, q^*} \left[\frac{R_l}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right) - 1\right]\left[\Omega H\left(R_l\right) + \frac{L_3\left(q^*\right)}{R_l}\right],$$

subject to (8). Let $\hat{R}_l$ denote the long-term interest rate such that $\frac{1}{\hat{R}_l} = \frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)$. Since we are considering the case with $R_l \geq \hat{R}_l$, then $\frac{R_l}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right) - 1 \geq 0$. Hence, if the objective function is increasing (decreasing) at $\hat{R}_l$, then the minimum occurs at $R_l = \hat{R}_l$ ($R_l > \hat{R}_l$).

By (8), q^* is defined as a function of R_l. Differentiating the objective at $R_l = \hat{R}_l$ yields

$$\frac{1}{R_2}\mathbb{E}\left(\frac{1}{R_3}\right)\left[\Omega H\left(\hat{R}_l\right) + \frac{L_3\left(q^*\left(\hat{R}_l\right)\right)}{\hat{R}_l}\right].$$

Note that $H(R_l) < 0$ at the optimum. Suppose $H(R_l) \geq 0$ instead, then the minimum is achieved by setting $R_l = \hat{R}_l$, which implies that $R_l < R_2\mathbb{E}(R_3)$ so $H(R_l) < 0$, which is a contradiction. Set

$$\bar{\Omega} = -\frac{\frac{L_3\left(q^*\left(\hat{R}_l\right)\right)}{\hat{R}_l}}{H\left(\hat{R}_l\right)} > 0,$$

so that the objective function is increasing when $\Omega \leq \bar{\Omega}$ and strictly decreasing when $\Omega > \bar{\Omega}$. Furthermore, when $\Omega > \bar{\Omega}$, then there exists $\tilde{R}_l > \hat{R}_l$ such that the objective function is zero. As a result, by Rolle's Theorem, part (ii) of the proposition follows when $\Omega > \bar{\Omega}$. Also, part (iv) of the proposition follows from the definition of $\bar{\Omega}$ and the fact that the argument above implies that $\Omega H\left(\hat{R}_l\right) + \frac{L_3\left(q^*\left(\hat{R}_l\right)\right)}{\hat{R}_l}$ is negative at the optimum when $\Omega > \bar{\Omega}$.

Finally, it follows immediately that $NW_1 = 0$ at the optimum, which implies that $q^* = \tilde{q}$ by (8). Also, since the expected lump-sum tax is $\mathbb{E}(T(R_3)) = \Lambda\left[R_l\mathbb{E}\left(\frac{1}{R_3}\right) - R_2\right]$, it follows from part (ii) of the proposition that $\mathbb{E}(T(R_3)) = 0$ when $\Omega \leq \bar{\Omega}$. While if $\Omega > \bar{\Omega}$, then the market clearing condition (10) and part (ii) of the proposition imply that $\Lambda = \Omega H(R_l) + \frac{L_3(q^*)}{R_l} < 0$ so $\mathbb{E}(T(R_3)) < 0$. This completes the proof. ■

Proposition 6 shows how the government fully absorbs the interest rate risk of the insurers, so that the insurers' capital structure is $NW_1 = 0$. This implies that the bond market is unconstrained and the equilibrium annuity price is at the actuarially fair level: $q^* = \tilde{q}$. This result follows from the fact that the government can lower the interest rate risk management cost of the private annuity insurers through the issuance of long-term bonds. Though the issuance of long-term bonds also exposes the government to interest rate risk, the government has the advantage of backing its long-term debt with an inelastic tax base and levy lump-sum taxes that depend on the interest rate realization.

Proposition 6 also presents a novel theory of the optimal term structure of interest rates.

In particular, the optimal term structure is influenced by the relative size of the arbitrageurs. Note that Proposition 6 shows arbitrageurs short selling the long-term government bond. This is due to the fact that, to mitigate the cost of interest rate risk management, the government limits the supply of long-term bonds Λ. This not only flattens the yield curve but also incentivizes the arbitrageurs to buy short-term bonds financed with long-term borrowing from the insurers. In essence, the insurers' need for long-term assets is not solely dependent on the government's long-term bond supply. Instead, it is also catered to by the arbitrageurs, who essentially act as a parallel supply channel for these assets. In fact, when the population of arbitrageurs Ω is greater than $\bar{\Omega}$, then the government limits the supply of long-term bonds to the point that net supply is negative: $\Lambda < 0$.

Note that other factors can also influence the total long-term bond demand of arbitrageurs. In Appendix D, we show that when arbitrageurs have mean-variance preferences, then the demand for long-term bonds is positively related to the arbitrageurs' risk tolerance and inversely related to the variance of the interest rate shock. Instead of a cap on the population size of the arbitrageurs $\bar{\Omega}$, Proposition 6 can be rewritten with limits on the variance of the interest rate shock $\mathbb{V}(R_3)$ and the arbitrageurs' risk tolerance.

F Alternative implementations

This appendix presents two alternative implementations that fully relax the implementability constraints, effectively avoiding the cost of hedging interest rate risk. First, we consider an implementation with a reformed Social Security benefit that depends on both income and longevity risk. Second, we examine an implementation that relies on private annuities with variable life benefits, where annuity payouts depend on the interest rate shock.

F.1 Public versus private annuities

This subsection shows that public annuities can be more efficient than private annuities in the presence of interest rate risk. This stands in stark contrast to our findings without the interest rate shock, where we demonstrated that public and private annuities can achieve the same welfare outcomes. That said, the higher efficiency of public annuities with interest

rate risk—the Social Security benefit in our implementation—crucially depends on a government's ability to make them contingent on productivity θ and longevity risk μ. Obviously, this may not be feasible in practice. And even if a government could design optimal productivity and longevity risk contingent benefits, the political majority may prefer a smaller government and, therefore, favor an implementation with private annuities. With that said, formulating optimal public annuity policies without the private annuity market highlights the additional costs of implementing third-best allocations, which rely on private annuities to screen longevity risk.

Proposition 7 *If $z(\theta,\mu)$ is a function of productivity θ and longevity risk μ, then $a(\theta,\mu)=0$ for all $\theta\in\left[\underline{\theta},\overline{\theta}\right]$ and $\mu\in\left[\underline{\mu},\overline{\mu}\right]$ and $\Psi(R_3)=0$ for all R_3 at the optimum.*

Proof First, we replace the social security benefit $z(\theta)$ with $z(\theta,\mu)$. Next, the first-order condition on $z(\theta,\mu)$ yields $\psi(\theta,\mu)=0$ for all θ and μ. The first-order conditions on q and $a(\theta,\mu)$ are

$$\eta(1-p)\int_{\underline{\theta}}^{\overline{\theta}}\int_{\underline{\mu}}^{\overline{\mu}}a(\theta,\mu)\,dG(\mu|\theta)\,dF(\theta)=\beta_1R_1\mathbb{E}\left[\psi(\theta,\mu)\,u'\left(c_1(\theta,\mu,0)\right)\left[a(\theta,\mu)+z(\theta,\mu)\right]\right],$$

where the expectation is taken over μ and θ, and

$$\psi(\theta,\mu)\,q\beta_1R_1u'\left(c_1(\theta,\mu,0)\right)+\eta(1-p)\left[\frac{S_2(\mu)}{R_2}+\frac{S_3(\mu)}{R_l}-q\right]=\frac{\mathbb{E}\left[\phi(R_3)\frac{\partial\Psi(R_3)}{\partial a(\theta,\mu)}\right]}{R_2\Phi g(\mu|\theta)\,f(\theta)}. \quad (51)$$

Immediately, we have $\eta=0$ and $\mathbb{E}\left[\phi(R_3)\frac{\partial\Psi(R_3)}{\partial a(\theta,\mu)}\right]=0$. Since the resource constraint is binding and more relaxed when R_3 increases, $\phi(R_3)>0$ and strictly decreasing in R_3. Furthermore, if $a(\theta,\mu)>0$, Lemma 1 implies

$$\frac{\partial\Psi(R_3)}{\partial a(\theta,\mu)}=\left[\frac{1}{R_3}-\mathbb{E}\left(\frac{1}{R_3}\right)\right](1-p)\,S_3(\mu)\,g(\mu|\theta)\,f(\theta), \quad (52)$$

which is strictly decreasing in R_3 with $\mathbb{E}\left[\frac{\partial\Psi(R_3)}{\partial a(\theta,\mu)}\right]=0$. This implies that $\mathbb{E}\left[\phi(R_3)\frac{\partial\Psi(R_3)}{\partial a(\theta,\mu)}\right]>0$ when $a(\theta,\mu)>0$. Therefore, it must be the case that $\Psi(R_3)=0$ for all R_3 and $a(\theta,\mu)=0$ for all θ and μ. ■

Proposition 7 is a direct consequence of the fact that, for any retirement consumption for non-altruistic agents, the government is able to design benefits $z(\theta, \mu)$ such that the implementability constraints (12) hold with zero demand for private annuities. Hence, the implementability constraints also reflect the fact that traditional Social Security benefits are not contingent on longevity risk.

Proposition 7 highlights a key drawback of private annuities in an incomplete market setting. While private annuities provide insurance against longevity risk, they also cover aggregate risks, such as interest rate fluctuations. In an incomplete market, this dual coverage by private annuities becomes suboptimal. When private insurers offer annuities that hedge against both longevity and aggregate risks, inefficiencies arise because these insurers cannot perfectly hedge against aggregate shocks if the supply of long-term bonds is limited. As a result, private insurers accumulate net worth as a buffer against adverse interest rate movements, which is inefficient because this net worth is financed by positive annuity markups. Thus, it is optimal for the government to issue more long-term bonds to increase the efficiency of the private annuity market. However, this in turn exposes the government to interest rate risk, which it can manage by imposing additional taxes that depend on the interest rate.

In contrast, public annuities eliminate the intermediary step where private insurers transfer interest rate risk to the government through long-term bonds. Specifically, the government can avoid issuing long-term bonds and exposing itself to interest rate risk, thereby avoiding the need for lump-sum taxes $T(R_3)$. Furthermore, if the government can offer public annuities with payments contingent on the reported longevity type μ, then public annuities dominate private annuities. As an example, the government can offer income- and longevity-dependent public annuities to non-altruistic agents, while altruistic agents save through bonds. This allows the government to screen longevity and altruism without relying on private annuities. As a result, unlike the implementation with private annuities, $\Psi(R_3) = 0$ for all R_3 under public annuities.

Despite the theoretical advantages of public annuities, reforming the current Social Security system is notoriously difficult, let alone redesigning it to be contingent on both income

and longevity. Real-world examples of reasonable but ultimately failed reforms to public annuity systems highlight this difficulty. In the United Kingdom, the *Pension Commission* in the early 2000s sought to expand access to private retirement savings through stakeholder pensions, aiming to reduce reliance on state pensions. However, low uptake—due to insufficient incentives and poor public understanding—diminished the impact of the reform. Similarly, Italy's 1990s pension reforms aimed to enhance sustainability by linking benefits to contributions, but failed to account for the country's high informal labor market and political resistance to reduced benefits, resulting in partial rollback. These examples demonstrate how well-intentioned, technically sound reforms can falter when structural economic issues, political opposition, or public engagement are neglected. These challenges underscore the importance of our focus on optimal retirement policies involving private annuity markets, as they offer a viable avenue for reform in a politically and fiscally constrained environment.

F.2 Variable Life Benefits

In this subsection, we consider life annuities with benefits that vary with interest rate shock R_3. Typically, life annuity contracts not only insure against longevity risk, but they also insure against aggregate shocks, such as changes in interest rate. In essence, annuity consumers who are alive receive a fixed payment independent of the economic environment as long as the insurer is solvent. In contrast, this section considers annuity contracts with variable life benefits, so they only insure against longevity risk.

The private annuity market is the same as in the previous section. An annuity contract pays 1 unit of pre-tax consumption in $t = 2$, but it pays $\iota(R_3)$ units in $t = 3$. In essence, the benefit in $t = 3$ is a function of the realized interest rate shock. The following proposition is analogous to Proposition 2 but for variable life benefits.

Proposition 8 *For a given annuity price q, the following competitive interest rate risk management strategy achieves zero annuity markup when $\iota(R_3) = R_3$:*

i. Asset portfolio:

$$l_3 = 0,\ b_2 = \frac{L_2(q) + L_3(q)}{R_2},\ \textit{and}\ b_3(R_3) = L_3(q).$$

ii. Capital structure:

$$NW_1 = NW_2(R_3) = 0.$$

In particular, $\Psi(R_3) = 0$ *for all* R_3.

Proof With variable life benefits $\iota(R_3)$, the balance sheet equations are

$$b_2 + l_3 = \frac{1}{R_2}\left[L_2(q) + \mathbb{E}\left(\frac{\iota(R_3)}{R_3}\right) L_3(q)\right] + NW_1, \tag{53}$$

and

$$b_3(R_3) = \frac{\iota(R_3) L_3(q)}{R_3} + NW_2(R_3). \tag{54}$$

Next, when $\iota(R_3) = R_3$, (54) yields $b_3(R_3) = L_3(q)$ if $NW_2(R_3) = 0$ for all R_3. Also, we can rewrite (53) as $b_2 + l_3 = \frac{1}{R_2}[L_2(q) + L_3(q)]$ if $NW_1 = 0$. With (7), this implies that $l_3 = 0$ and $b_2 = \frac{1}{R_2}[L_2(q) + L_3(q)]$. Thus, by Definition 2, the asset portfolio and capital structure constitute a competitive interest rate risk management strategy. Furthermore, with $NW_1 = 0$, the annuity markup is zero.

Finally, since $NW_1 = l_3 = 0$, the optimal long-term government bond supply $\Lambda = 0$, so $\Psi(R_3) = 0$. ■

Proposition 8 shows how insurers can transfer their interest rate risk management cost to agents by introducing variable life benefits. With $\iota(R_3) = R_3$, annuity payments in $t = 3$ increase with the realized interest rate, so the agents fully absorb the interest rate shock. Proposition 8 also shows that insurers no longer need to purchase long-term bonds, so the condition of the bond market is irrelevant. As a result, insurers do not need to manage interest rate risk with variable life benefits, and the annuity markup is zero.[27] This provides an alternative interpretation of the annuity markup in Section 5.1: Agents pay for insurance against aggregate risk through higher annuity markups.

Importantly, for our policy exercise, Proposition 8 implies that the government can maintain the efficiency of private annuity markets in the presence of interest rate shocks by imple-

[27] Since variable life benefits are a different product from the annuity contracts considered in Section 5.1, the actuarially fair price of variable life benefits not the same as (3).

menting private annuity contracts with payments that depend on interest rate realizations. Therefore, the government no longer needs to issue long-term bonds, and the social cost of interest rate risk is zero ($\Psi(R_3) = 0$). Variable life benefits provide an alternative implementation to the one with public annuities that depend on longevity risk μ and altruism type α discussed in the last subsection.

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G Solving the model numerically

G.1 Discretization of the bivariate log-normal distribution

To calculate the discretized conditional distribution of μ given θ, we exploit the properties of the bivariate log-normal distribution. Let X and Y be two log-normal random variables with means μ_X, μ_Y and standard deviations σ_X, σ_Y, respectively. The correlation between $\log(X)$ and $\log(Y)$ is denoted as ρ.

We first generate samples from the bivariate normal distribution for $\log(X)$ and $\log(Y)$ with the covariance matrix:

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}$$

The generated samples are then exponentiated to obtain samples from the bivariate log-normal distribution. The conditional distribution of Y given $X = x$ is also log-normal with:

$$\mu_{Y|X=x} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(\log(x) - \mu_X)$$

$$\sigma_{Y|X} = \sigma_Y\sqrt{1-\rho^2}$$

To discretize θ, we first optimally discretize the marginal distribution of θ into `disc_step` intervals using the algorithm of Drezner and Zerom (2016), which efficiently discretizes continuous random variables. The aim is to capture key distributional properties—such as skewness and heavy tails—with a finite number of points.

For each value of θ, we compute the conditional distribution of μ given θ by solving for the cumulative distribution function (CDF) at each discretized point of μ, which are stored in the vector $\mu_{\text{val_disc}}$. The CDF of μ conditional on $\theta = \theta_i$ is computed as:

$$\text{CDF}_{\mu|\theta_i}(u) = \Phi\left(\frac{\log(\mu_{\text{val_disc}[j]}) - \mu_{Y|\theta_i}}{\sigma_{Y|\theta_i}}\right)$$

where Φ is the cumulative distribution function of the standard normal distribution.

Finally, the discretized probability mass function (PMF) is obtained by differencing the CDF values:

$$\text{PMF}_{\mu|\theta_i}(u) = \text{CDF}_{\mu|\theta_i}(u) - \text{CDF}_{\mu|\theta_i}(u-1)$$

The resulting discretized conditional distribution is then normalized such that the sum of probabilities equals 1.

G.2 Discretization of the incentive compatibility constraints

We discretize the incentive compatibility constraints as follows. Let there be N productivity types: $\theta \in \{\theta_1, \theta_2, \ldots, \theta_N\}$ and M longevity types: $\mu \in \{\mu_1, \mu_2, \ldots, \mu_M\}$. The ex-post incentive compatibility constraints on longevity are, for any (θ, α) and μ_j and μ_k :

$$\begin{aligned}
&\int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu_j)\, u\left(c_t(\theta, \mu_j, \alpha, R_3)\right) + \beta\left[1 - \frac{S_{t+1}(\mu_j)}{S_t(\mu_j)}\right] \alpha v\left(x_t(\theta, \mu_j, \alpha, R_3)\right)\right] d\Pi(R_3) \\
&\geq \int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu_j)\, u\left(c_t(\theta, \mu_k, \alpha, R_3)\right) + \beta\left[1 - \frac{S_{t+1}(\mu_j)}{S_t(\mu_j)}\right] \alpha v\left(x_t(\theta, \mu_k, \alpha, R_3)\right)\right] d\Pi(R_3).
\end{aligned}$$

Similar to Moser and de Souza e Silva (2019), we can simplify this by focusing on neighboring deviations. Therefore, for a fixed type (θ, μ_j, α), we need to deter adjacent upward deviation:

$$\begin{aligned}
&\int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu_j)\, u\left(c_t(\theta, \mu_j, \alpha, R_3)\right) + \beta\left[1 - \frac{S_{t+1}(\mu_j)}{S_t(\mu_j)}\right] \alpha v\left(x_t(\theta, \mu_j, \alpha, R_3)\right)\right] d\Pi(R_3) \\
&\geq \int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu_j)\, u\left(c_t(\theta, \mu_{j+1}, \alpha, R_3)\right) + \beta\left[1 - \frac{S_{t+1}(\mu_j)}{S_t(\mu_j)}\right] \alpha v\left(x_t(\theta, \mu_{j+1}, \alpha, R_3)\right)\right] d\Pi(R_3)
\end{aligned}$$

and deter adjacent downward deviation:

$$\int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu_j)\, u\left(c_t(\theta, \mu_j, \alpha, R_3)\right) + \beta \left[1 - \frac{S_{t+1}(\mu_j)}{S_t(\mu_j)}\right] \alpha v\left(x_t(\theta, \mu_j, \alpha, R_3)\right) \right] d\Pi(R_3)$$
$$\geq \int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu_j)\, u\left(c_t(\theta, \mu_{j-1}, \alpha, R_3)\right) + \beta \left[1 - \frac{S_{t+1}(\mu_j)}{S_t(\mu_j)}\right] \alpha v\left(x_t(\theta, \mu_{j-1}, \alpha, R_3)\right) \right] d\Pi(R_3).$$

For each (θ, α) we have $2 \times (M-1) + 2$ of these ICs. Note that there are $M-1$ number of longevity types that have two adjacent ICs, plus two types (the highest and lowest types) that each have only one adjacent IC. There are N productivity types and 2 altruism types. Therefore, there is a total of $4 \times N \times M$ number of ex-post ICs on longevity.

Lemma 1 shows that we can rewrite the ex-post IC on altruism as, for any (θ, μ),

$$\int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu)\, u\left(c_t(\theta, \mu, 1, R_3)\right) + \beta \left[1 - \frac{S_{t+1}(\mu)}{S_t(\mu)}\right] v\left(x_t(\theta, \mu, 1, R_3)\right) \right] d\Pi(R_3)$$
$$\geq \int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu)\, u\left(c_t(\theta, \mu, 0, R_3)\right) \right] d\Pi(R_3).$$

We will first ignore the other IC, where the non-altruistic agent pretends to be altruistic. Then, we will check whether it holds given the solutions. There are $N \times M$ of them!

The ex-ante IC on productivity is, for any θ_i and θ_j,

$$\beta_0 \left[u\left(c_0(\theta_i)\right) - h\left(\frac{y(\theta_i)}{\theta_i}\right) \right] + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \begin{bmatrix} pU_1(\theta_i, \mu, 1) \\ +(1-p)\, U_1(\theta_i, \mu, 0) \end{bmatrix} dG(\mu|\theta)$$
$$\geq \beta_0 \left[u\left(c_0(\theta_k)\right) - h\left(\frac{y(\theta_k)}{\theta_i}\right) \right] + \beta_1 \int_{\underline{\mu}}^{\overline{\mu}} \begin{bmatrix} pU_1(\theta_k, \mu, 1) \\ +(1-p)\, U_1(\theta_k, \mu, 0) \end{bmatrix} dG(\mu|\theta),$$

where

$$U_1(\theta, \mu, \alpha) = \int_{\mathcal{R}} \left[\sum_{t=1}^{3} \beta^{t-1} S_t(\mu)\, u\left(c_t(\theta, \mu, \alpha, R_3)\right) + \beta \left[1 - \frac{S_{t+1}(\mu)}{S_t(\mu)}\right] \alpha v\left(x_t(\theta, \mu, \alpha, R_3)\right) \right] d\Pi(R_3).$$

In essence, the ex-ante IC on productivity assumes that the agent will tell the truth in the future, i.e., the ex-post ICs work. It is a classic result that the ex-ante IC can be simplified

by only focusing on adjacent downward deviations. Therefore, we can rewrite the ex-ante IC on productivity as, for any θ_i,

$$\beta_0\left[u\left(c_0\left(\theta_i\right)\right)-h\left(\frac{y\left(\theta_i\right)}{\theta_i}\right)\right]+\beta_1\int_{\underline{\mu}}^{\overline{\mu}}\left[\begin{array}{c}pU_1\left(\theta_i,\mu,1\right)\\+\left(1-p\right)U_1\left(\theta_i,\mu,0\right)\end{array}\right]dG\left(\mu|\theta\right)$$
$$\geq\beta_0\left[u\left(c_0\left(\theta_{i-1}\right)\right)-h\left(\frac{y\left(\theta_{i-1}\right)}{\theta_i}\right)\right]+\beta_1\int_{\underline{\mu}}^{\overline{\mu}}\left[\begin{array}{c}pU_1\left(\theta_{i-1},\mu,1\right)\\+\left(1-p\right)U_1\left(\theta_{i-1},\mu,0\right)\end{array}\right]dG\left(\mu|\theta\right).$$

There are N productivity types, and we only have $N-1$ adjacent downward ICs. So we only need to look at $N-1$ ex-ante IC constraints.

G.2.1 Decision variable upper bounds

We find the upper bound on all decision variables so that we can bound the solution. Consider the optimality conditions for the first-best case. From the first-order conditions, we get

$$y_0\left(\theta\right)=\theta^{\frac{\rho}{\rho-1}}c_0^{-\frac{\gamma}{\rho-1}}$$
$$c_1=\left(\frac{\beta_1R_1}{\beta_0R_0}\right)^{\frac{1}{\gamma}}c_0=\left(\frac{\theta\beta_1R_1}{\beta_0R_0}\right)^{\frac{1}{\gamma}}\left(\frac{y_0\left(\theta\right)}{\theta}\right)^{-\frac{\rho-1}{\gamma}}$$

Note that c_0 and c_1 are constant across types $\left(\theta,\mu,\alpha\right)$, but y changes with respect to θ. The rest of the allocations depend on R_3

$$\mathbb{E}\left(c_2\left(R_3\right)^{-\gamma}\right)=\sum_{R_3}\pi\left(R_3\right)c_2\left(R_3\right)^{-\gamma}=\frac{\beta_0R_0}{\beta_1R_1\beta R_2}c_0^{-\gamma}$$

$$c_3\left(R_3\right)=\left(\beta R_3\right)^{\frac{1}{\gamma}}c_2\left(R_3\right)$$

$$x_1\left(R_3\right)=\chi^{\frac{1}{\gamma}}c_2\left(R_3\right)$$

$$x_2\left(R_3\right)=\left(\frac{\chi}{\beta R_3}\right)^{\frac{1}{\gamma}}c_2\left(R_3\right)$$

$$x_3\left(R_3\right)=\left(\frac{\chi}{\beta^2R_3R_4}\right)^{\frac{1}{\gamma}}c_2\left(R_3\right)$$

Again, note that though the allocations depend on R_3, they are constant across types (θ, μ, α).

Take the allocations above and plug it in the resource constraint. Solve for c_0.

Given c_0, we have $y_0(\theta)$ for all θ. The maximum consumption and bequest in any period cannot be larger than the total first-best output $\int y_0(\theta)\, d\theta$.

G.3 Solving the Model

Solving our public policy model requires solving a large-scale nonlinear programming (NLP) problem due to the high dimensionality of the decision vector, the presence of thousands of equality and inequality constraints, and the nonlinear structure of both the objective function and the constraints, which arise from the dynamic, multi-period nature of the model with multiple state variables and heterogeneity across agents. In the third-best version of the model with 30 μ and 30 θ discrete points is characterized by the following statistics:

- **Number of Variables:** 45,903
- **Number of Equality Constraints:** 27,723
- **Number of Inequality Constraints:** 21,351
- **Number of Nonzeros in the Equality Constraint Jacobian:** 68,039
- **Number of Nonzeros in the Inequality Constraint Jacobian:** 324,561
- **Number of Nonzeros in the Lagrangian Hessian:** 5,205,060

We solve the optimization problem using **Julia JuMP** with the **Ipopt 3.14.17** optimizer running with linear solver **MUMPS 5.7.3**. The approach assigns a decision variable for each state of the model and applies equality constraints across states where variables must be identical. This method streamlines the coding of the objective function and the discretized incentive compatibility constraints. It also leverages **JuMP's** constraint macros, automatic differentiation, function tracing, and sparse matrix handling. Essentially, **JuMP** creates a symbolic representation of the model, including derivatives, before passing it to **Ipopt** for solving and the linear algebra is handled in parallel by **MUMPS** at each iteration.

The methodology involves constructing the optimization problem on a state grid, specifying constraints and objectives, and employing a sequential strategy to refine the solution.

State Representation and Grid Setup

The model considers multiple types of heterogeneity across agents and states:

- θ: Productivity type, drawn from the bivariate distribution in period $t = 0$.
- μ: Longevity type, drawn from the conditional distribution in period $t = 1$ alongside altruism type α.
- r_3: Interest rate shock, realized in period $t = 2$.

The state space is discretized into a structured grid using the Cartesian product of the sets of values for θ, μ, and r_3. Each grid point corresponds to a unique combination of state variables, which ensures a structured representation of the problem suitable for vectorized operations. The grid is generated using the `IterTools.product` package in Julia and stored in a `DataFrame named grid`. Each variable in the decision vector corresponds to a control variable for a specific grid state.

Objective Function and Utility Specification

The objective function maximizes lifetime utility across all state realizations of the form:

$$U = \sum_{\theta,\mu,r_3} p(\theta, \mu, r_3) \times \text{utility}(c, y, x)$$

The utility functions are defined as:

- **Consumption utility:** $u(c) = \frac{c^{1-\gamma_u}}{1-\gamma_u}$,
- **Labor disutility:** $h(y, \theta) = \frac{1}{\gamma_h} \left(\frac{y}{\theta}\right)^{\gamma_h}$,
- **Bequest utility:** $v(b) = \xi \frac{b^{1-\gamma_u}}{1-\gamma_u}$.

The objective function aggregates these utilities across states and control variables, weighted by the appropriate probabilities for the different state variables. It is implemented as a summation loop over the grid.

Constraints

The optimization problem includes several types of constraints, all defined as nonlinear expressions:

- **Resource constraints:** Ensuring total consumption and savings across periods respect the endowment and interest rate conditions.
- **Incentive constraints:** Representing restrictions based on productivity and longevity types to ensure truthful reporting.
- **Feasibility constraints:** Ensuring non-negativity and upper bounds on control variables.

Each constraint is defined as a nonlinear expression using functions such as `rec_cons`, `prod_ic`, `alt_ic`, and `long_ic`. These constraints are enforced by adding them to the JuMP model using the `@constraint` macro.

Numerical Strategy: Sequential Optimization and Initialization

The model is solved using a stepwise optimization strategy to avoid numerical instabilities in this large-scale nonlinear program:

1. **Initialization:** The model is first solved with relaxed constraints to find a feasible point.
2. **Progressive Refinement:** The solution from each stage is used as a warm start for the next stage, with tighter constraints and a lower tolerance.
3. **Warm Start:** Each iteration reuses the previous solution as the starting point with the command

`set_start_value` and `set_optimizer_attribute(model, "warm_start_init_point", "yes")`.

4. **Regularization:** A regularization term $\lambda \sum x_i^2$ is added to the objective function to help stabilize convergence.

Hessian Approximation and Memory Management

To handle the large number of variables and constraints:

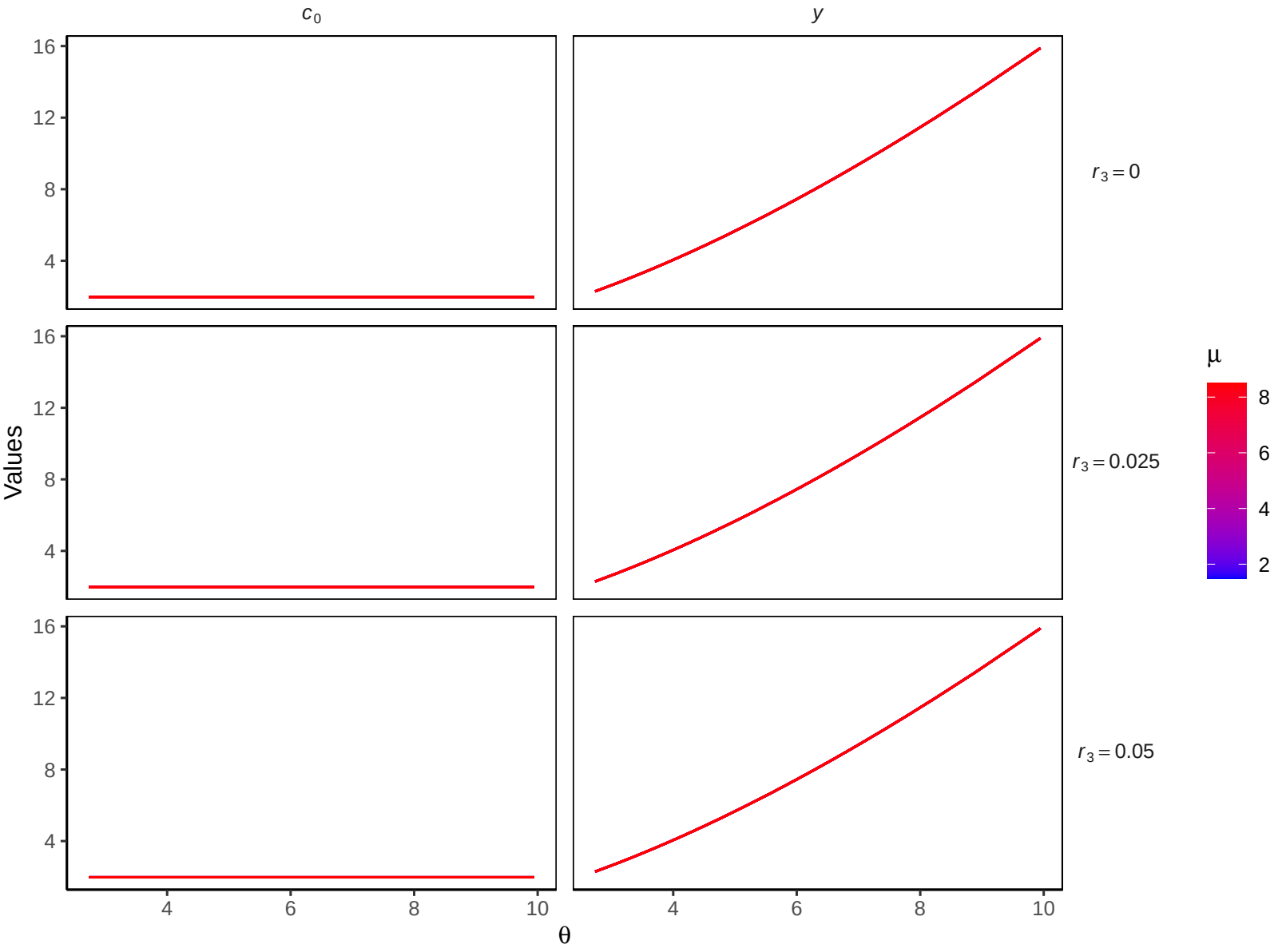
- **Limited-memory Hessian:** The option `set_optimizer_attribute(model, "hessian_approximat` `"limited-memory")` reduces memory usage by approximating the Hessian matrix with a quasi-Newton approach.
- **Sparse Matrices:** The constraint matrices are converted into sparse format using the `SparseArrays` library.
- **Linear Solver:** The default MUMPS solver is used for numerical stability with `set_optimizer_attribute(model, "linear_solver", "mumps")`.

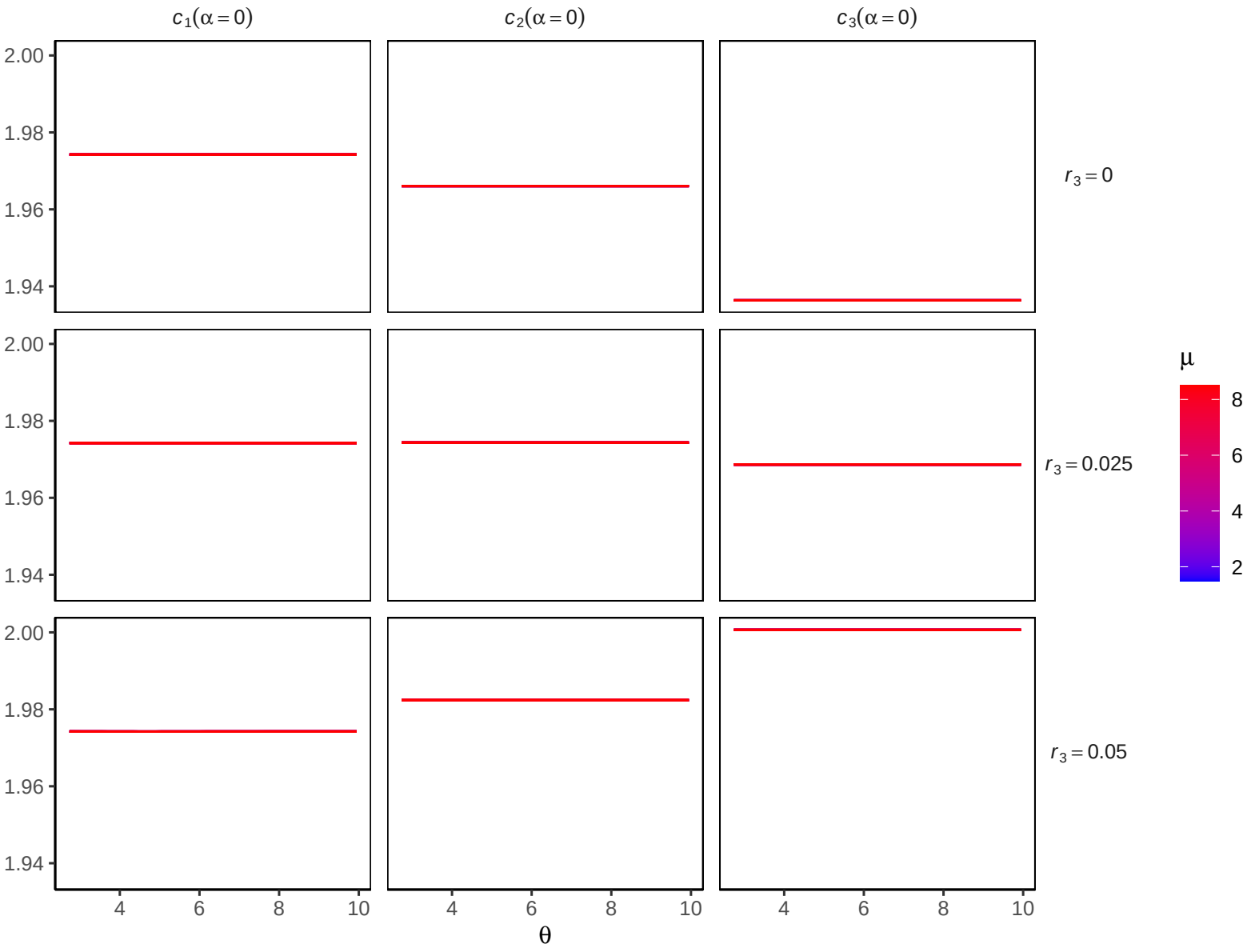
Computational Considerations

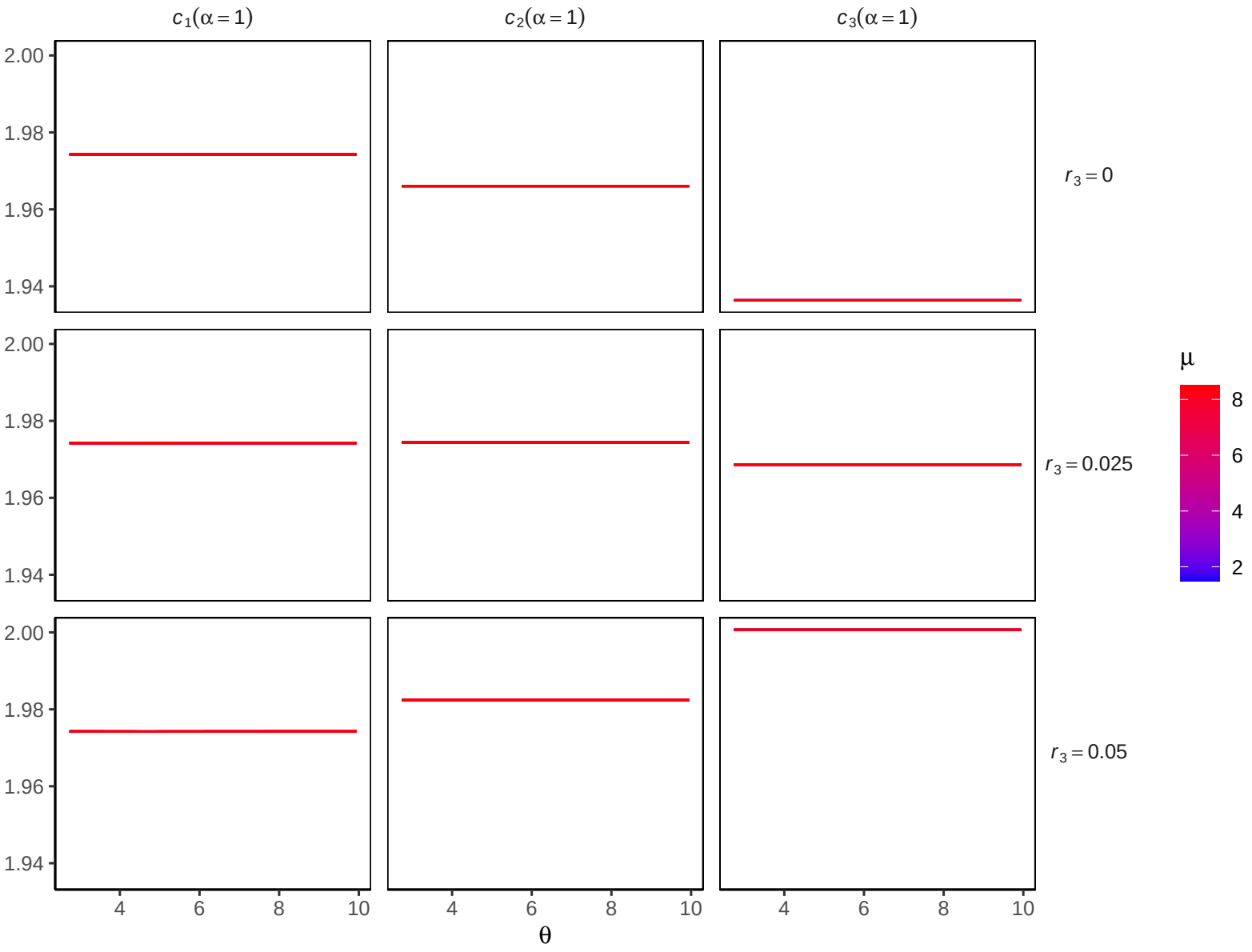
Given the problem size (over 45,900 variables and millions of nonzeros in the Jacobian and Hessian), the following optimizations were implemented:

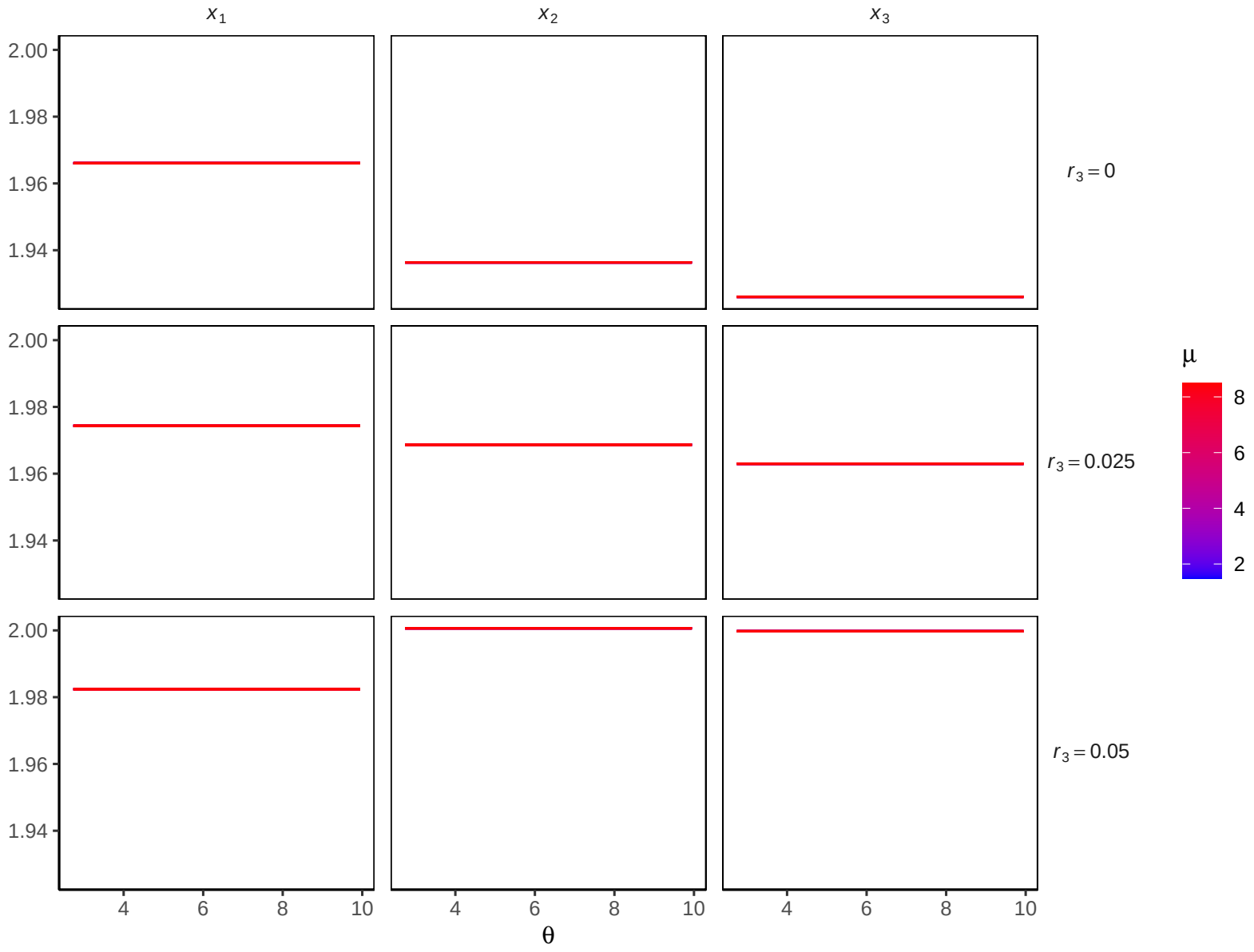
- **Multithreading:** BLAS parallelization is set using `BLAS.set_num_threads(10)` to utilize less than the number of available CPU cores on a standard MacBook Pro M2 Pro.
- **Parallel Linear Solver Scaling:** The option `set_optimizer_attribute(model, "mumps_scaling", 77)` activates parallel scaling for the MUMPS linear solver, balancing the workload across multiple threads to improve performance on large-scale nonlinear optimization problems.
- **Regularization:** A small regularization parameter $\lambda = 10^{-9}$ helps with numerical stability while preserving solution accuracy.

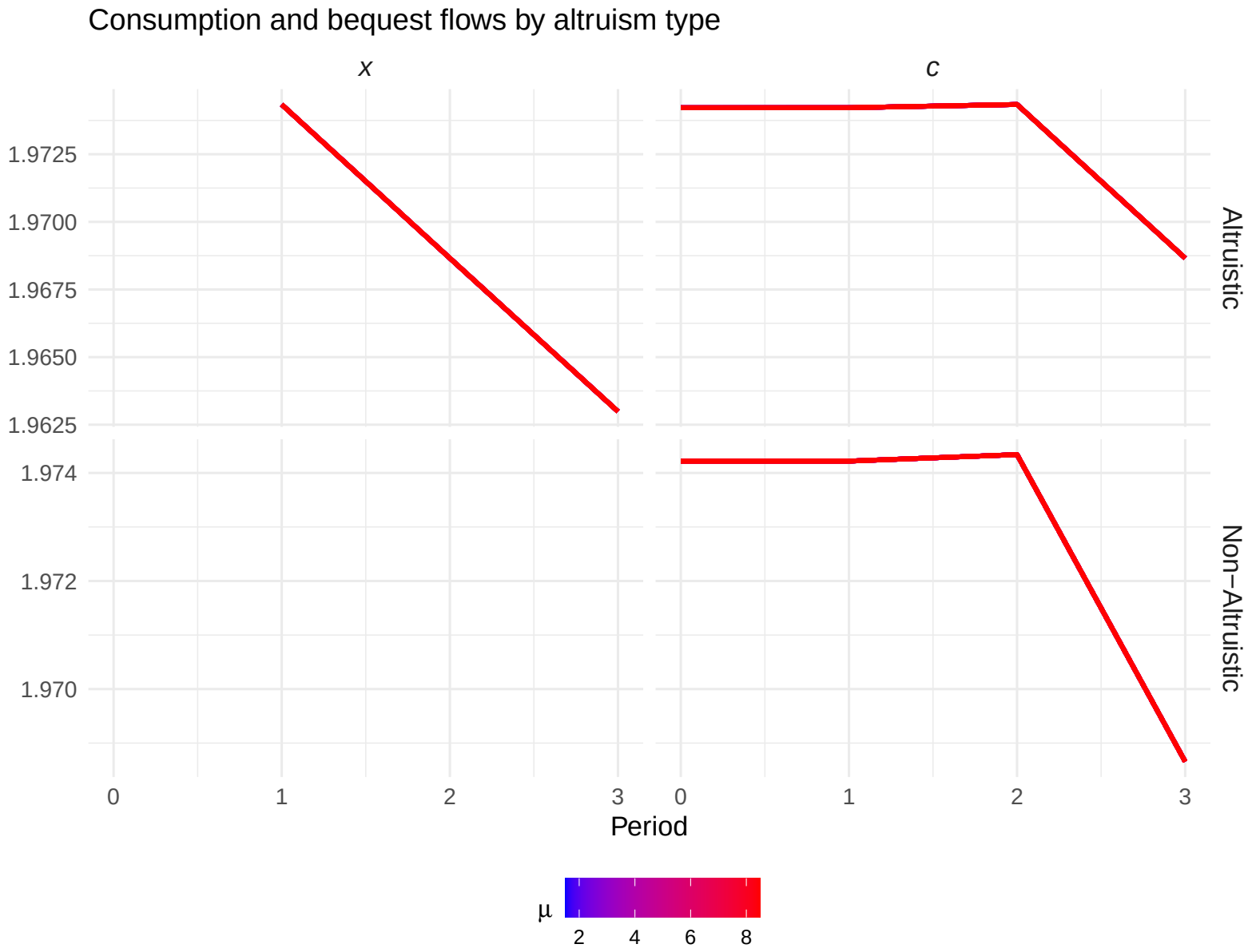
H First best optimal decision rules

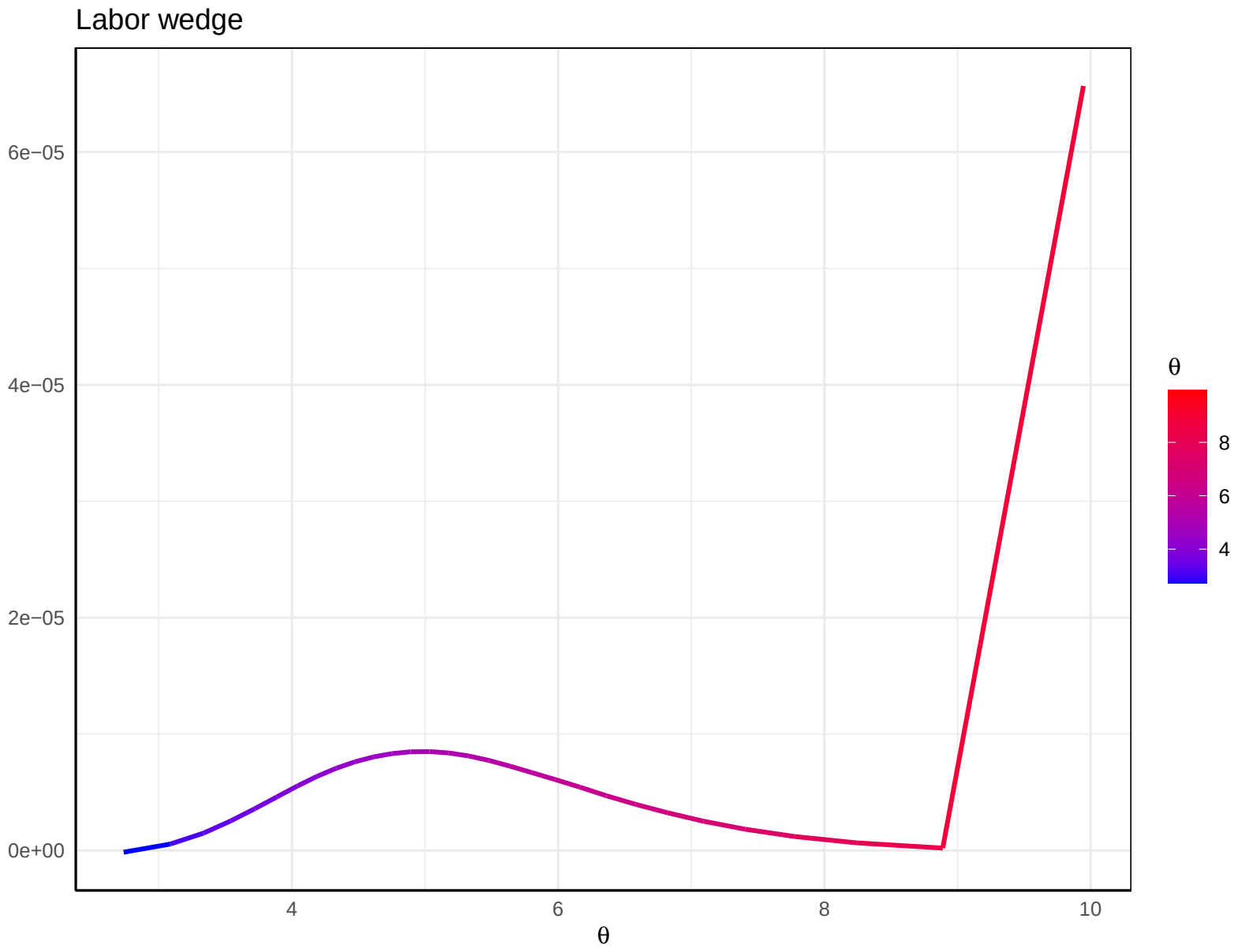

c_0
y
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
Values
θ
μ
16
12
8
4
2
6
10

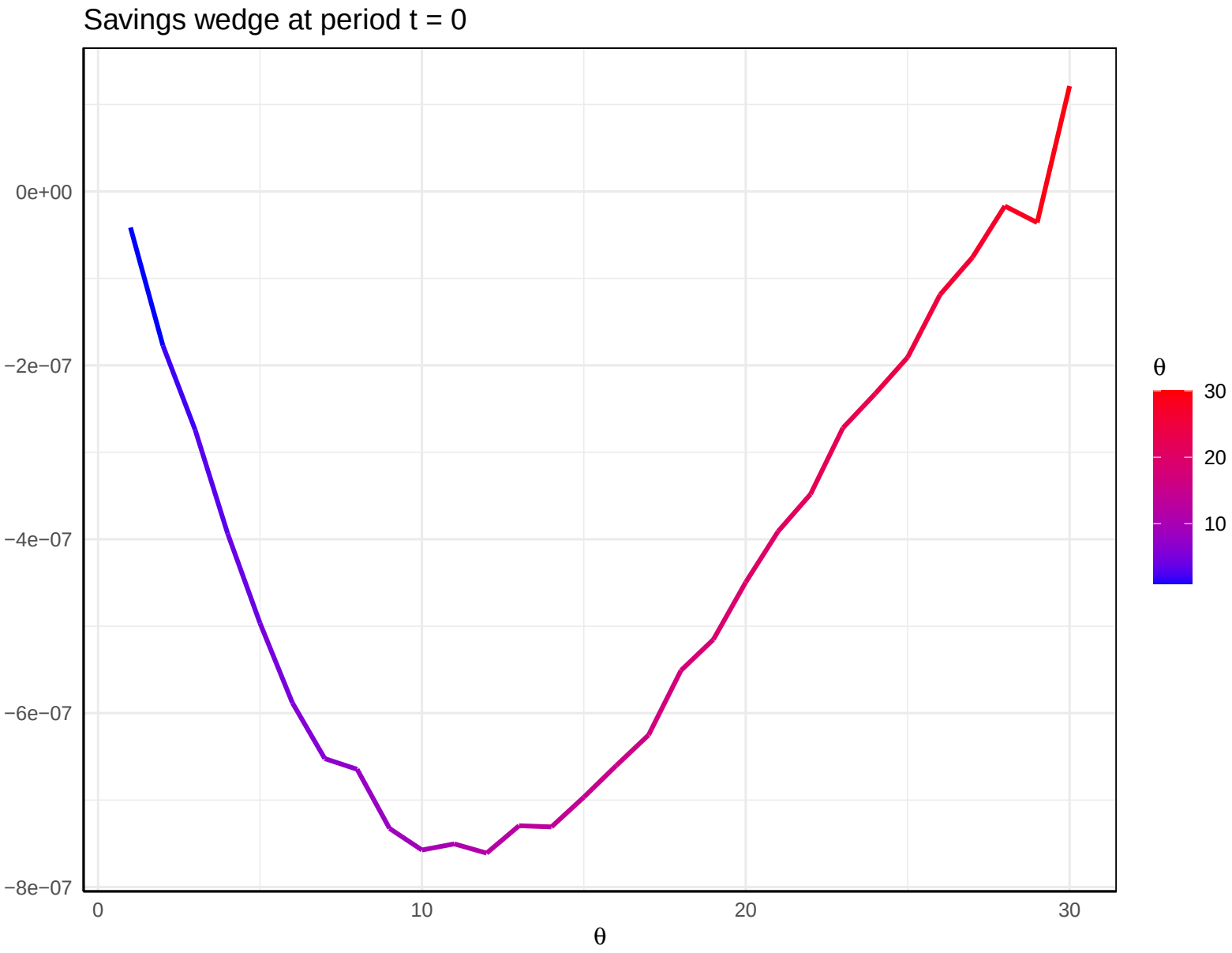

$c_1(\alpha = 0)$
$c_2(\alpha = 0)$
$c_3(\alpha = 0)$
2.00
1.98
1.96
1.94
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
4
6
8
10
θ
μ
8
6
4
2

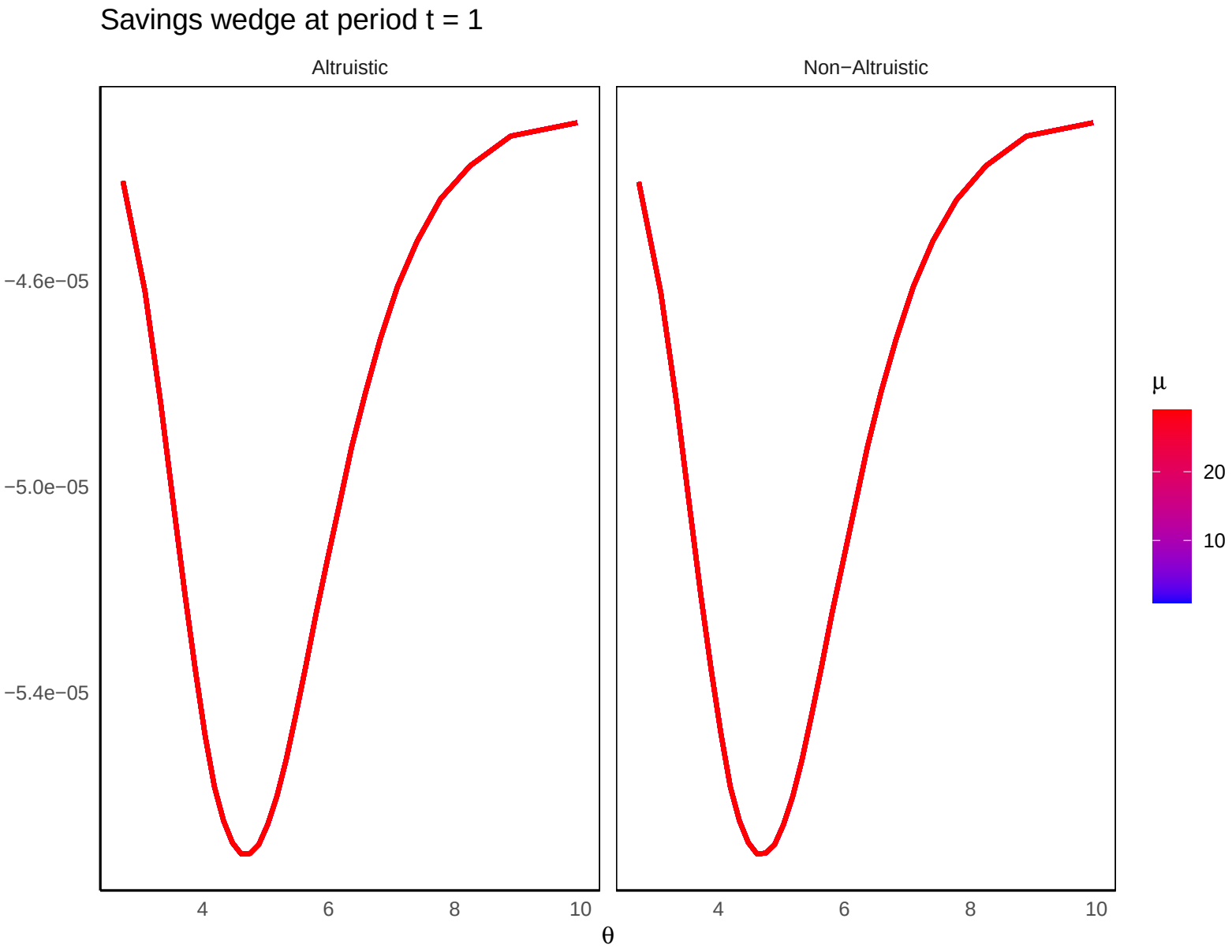

$c_1(\alpha = 1)$
$c_2(\alpha = 1)$
$c_3(\alpha = 1)$
2.00
1.98
1.96
1.94
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
4
6
8
10
θ
μ
8
6
4
2

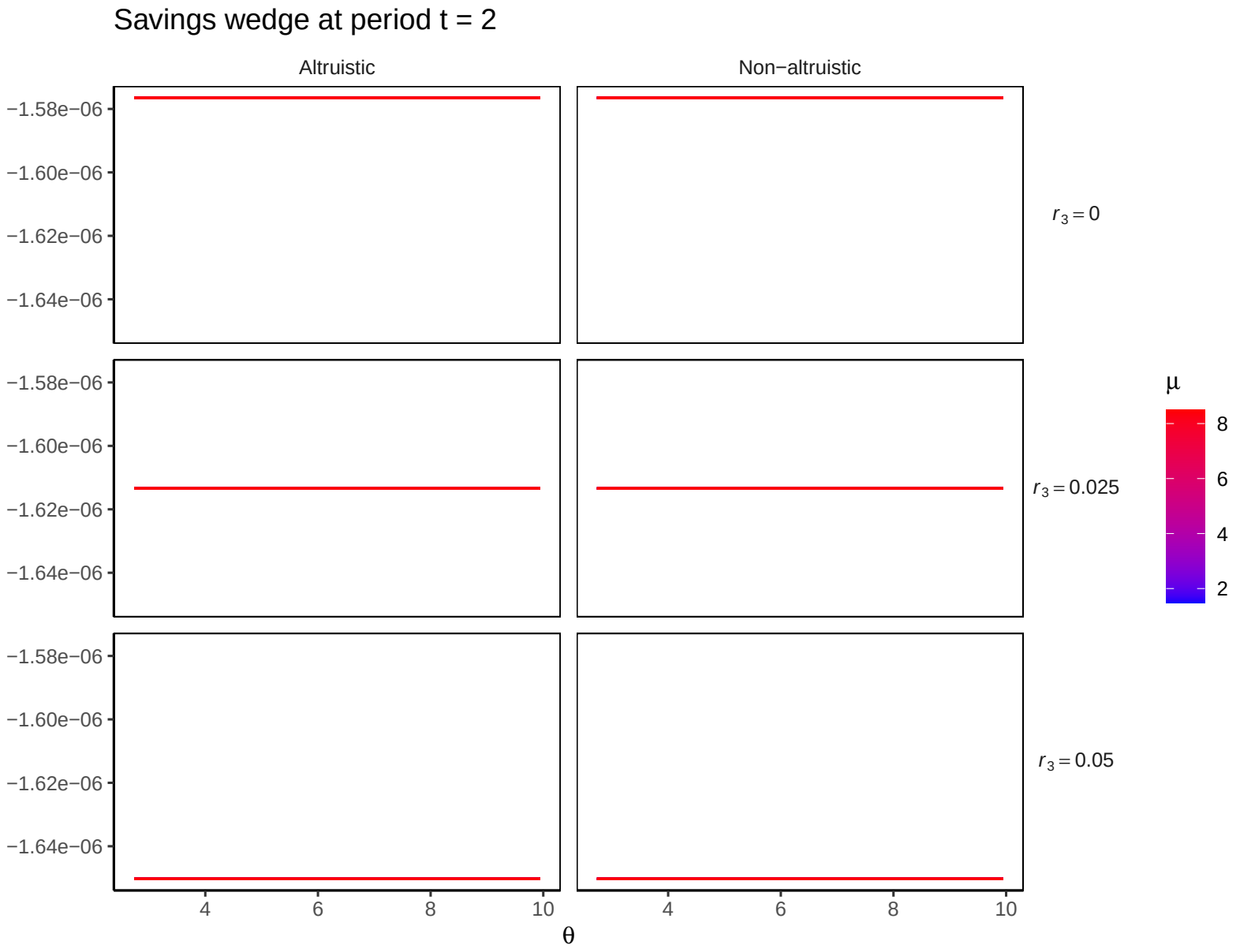

x_1
x_2
x_3
2.00
1.98
1.96
1.94
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
4
6
8
10
θ
μ
8
6
4
2

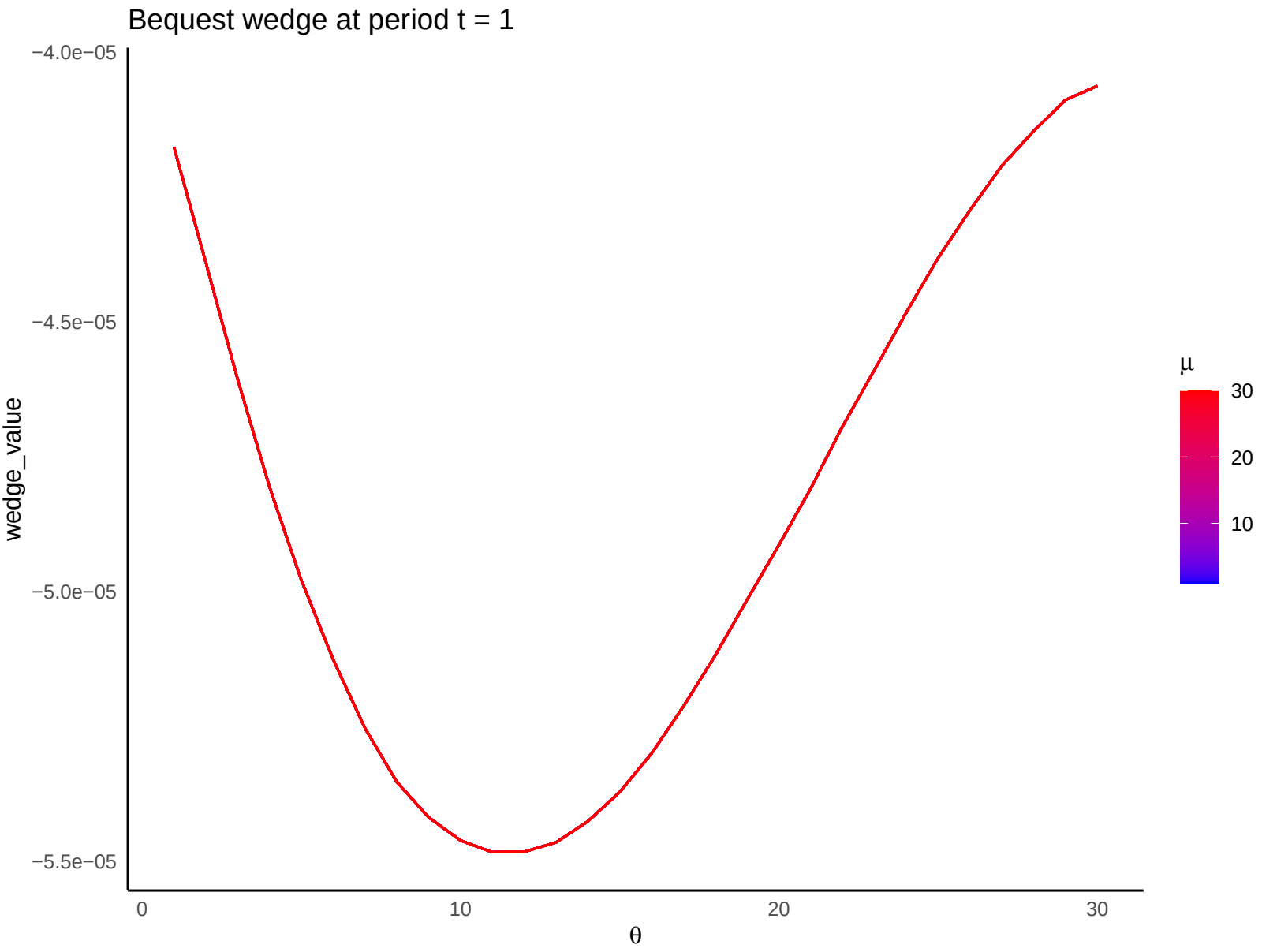

Consumption and bequest flows by altruism type
x
c
1.9725
1.9700
1.9675
1.9650
1.9625
1.974
1.972
1.970
Altruistic
Non−Altruistic
0
1
2
3
Period
μ
2
4
6
8

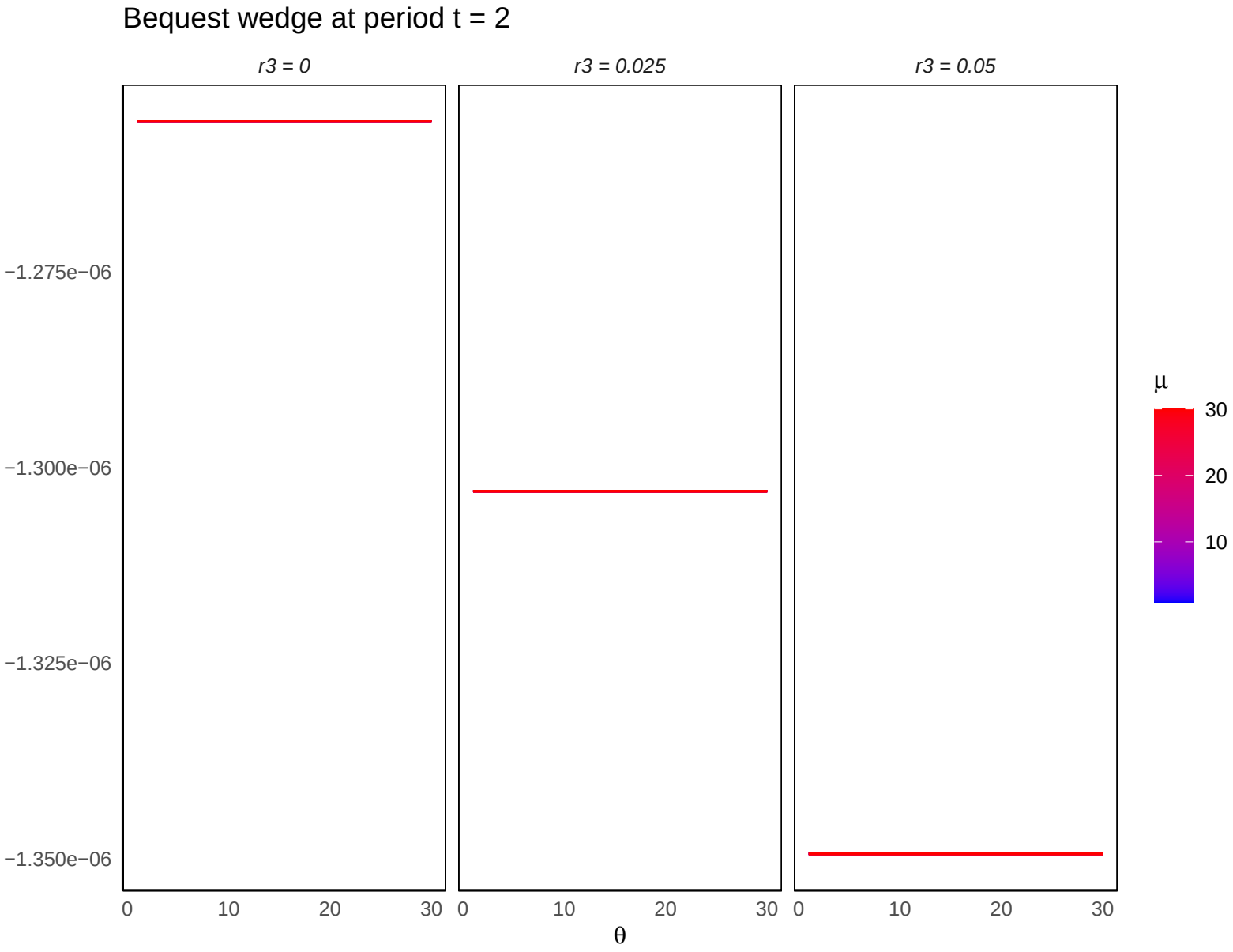

Labor wedge
6e−05
4e−05
2e−05
0e+00
4
6
8
10
θ
θ
8
6
4

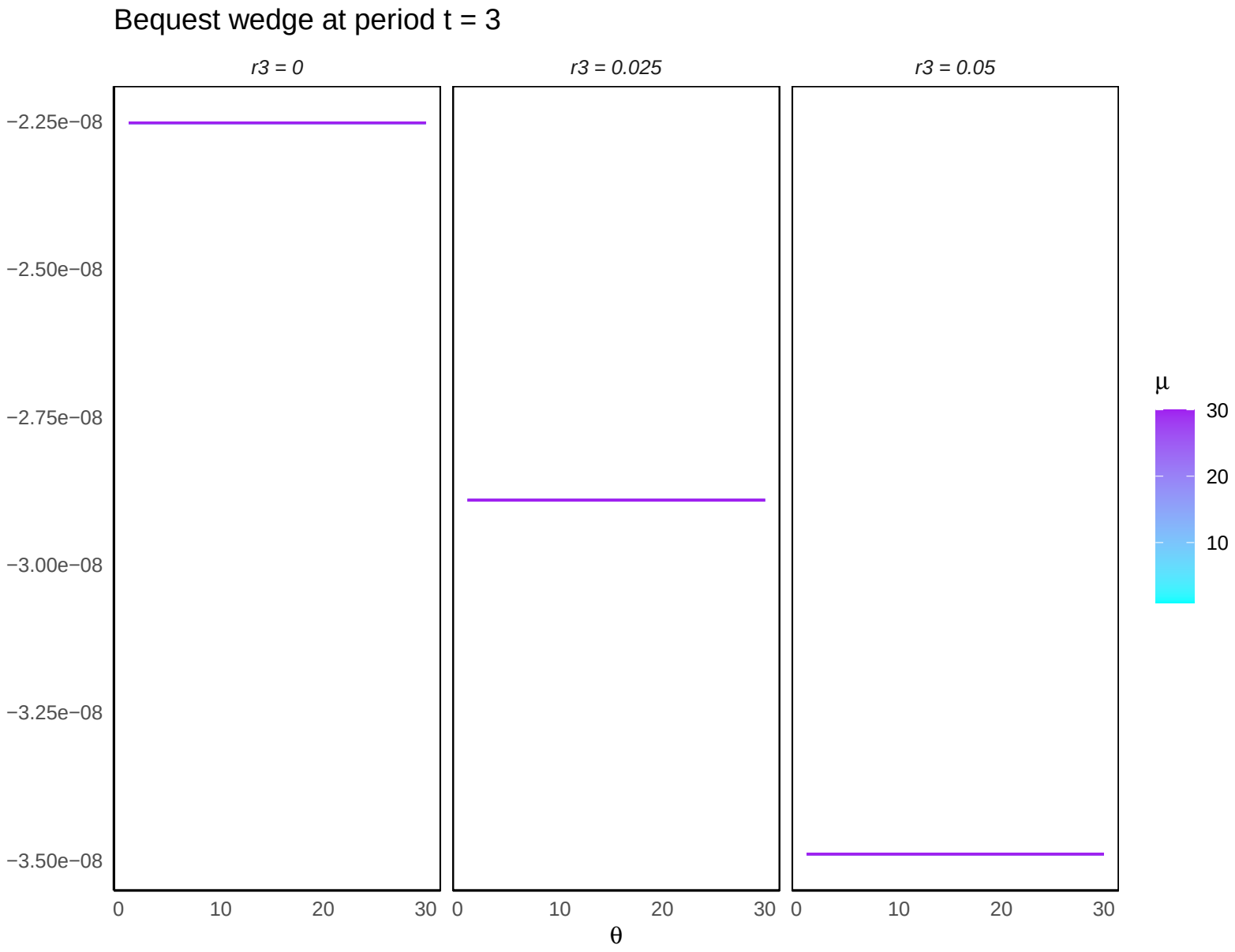

Savings wedge at period t = 0
0e+00
−2e−07
−4e−07
−6e−07
−8e−07
0
10
20
30
θ
θ
30
20
10


Savings wedge at period t = 1
Altruistic
Non−Altruistic
−4.6e−05
−5.0e−05
−5.4e−05
4
6
8
10
θ
μ
20
10

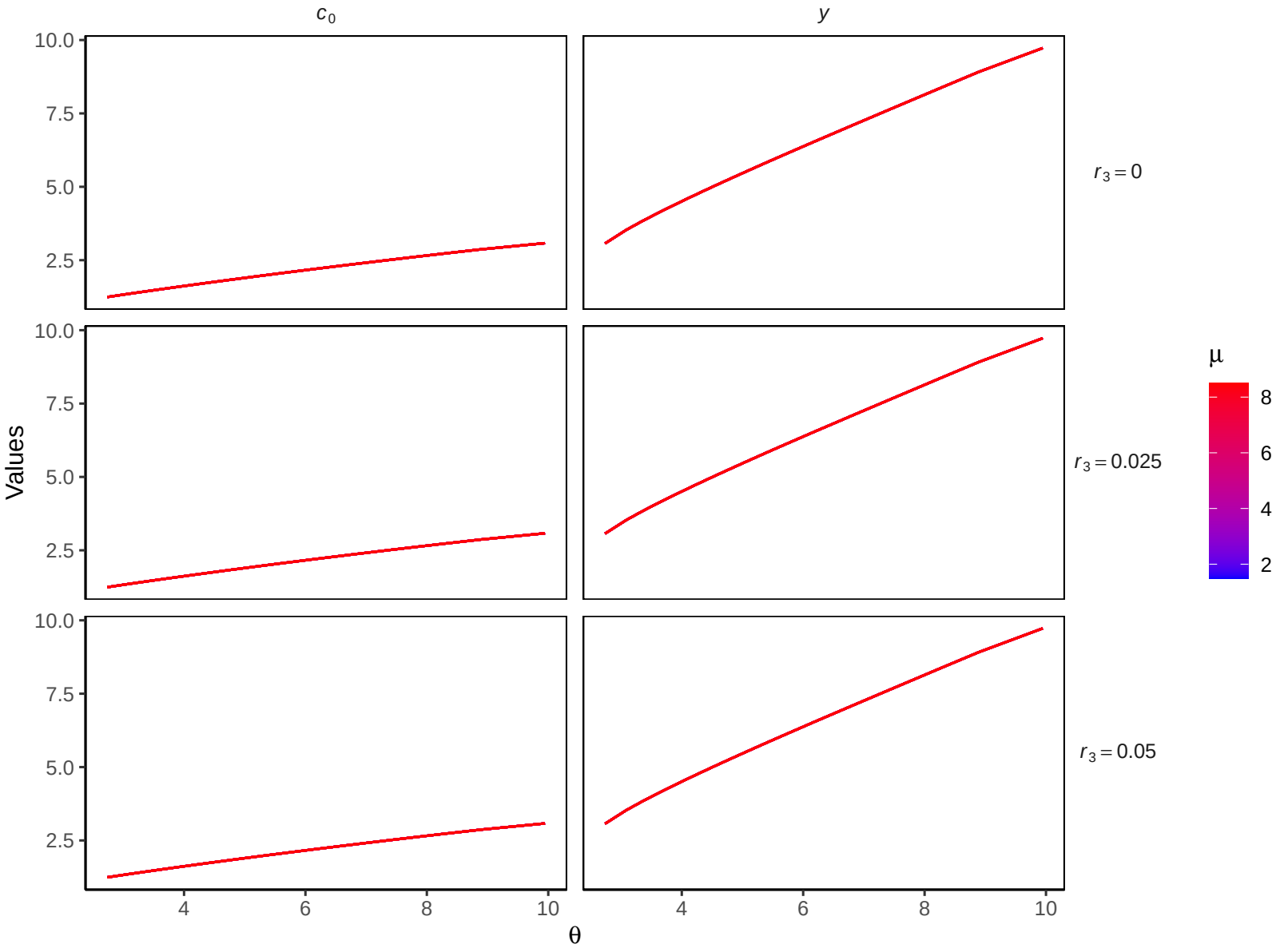

Savings wedge at period t = 2
Altruistic
Non−altruistic
−1.58e−06
−1.60e−06
−1.62e−06
−1.64e−06
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
4
6
8
10
θ
μ
8
6
4
2

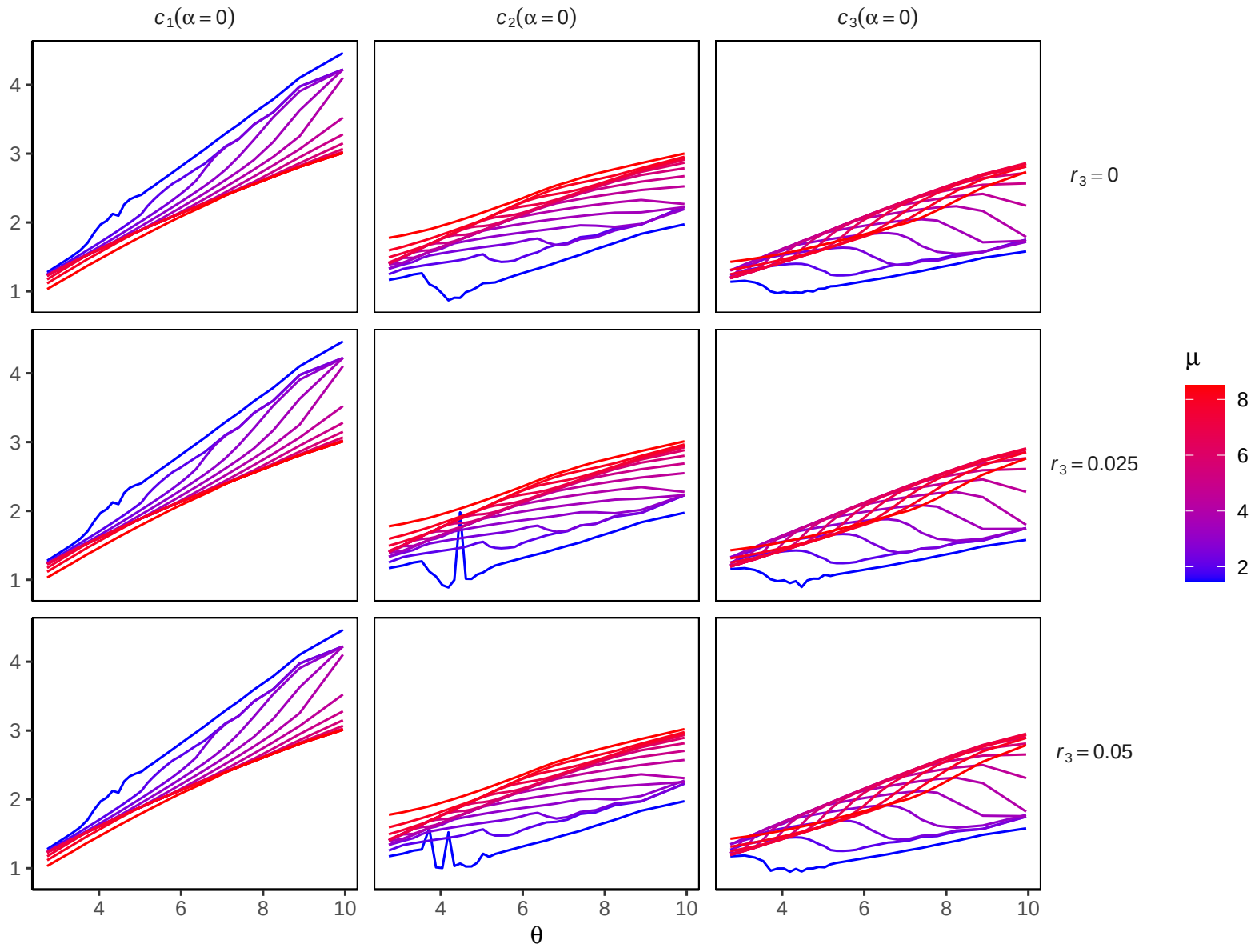

Bequest wedge at period t = 1
−4.0e−05
−4.5e−05
−5.0e−05
−5.5e−05
wedge_value
0
10
20
30
θ
μ
30
20
10

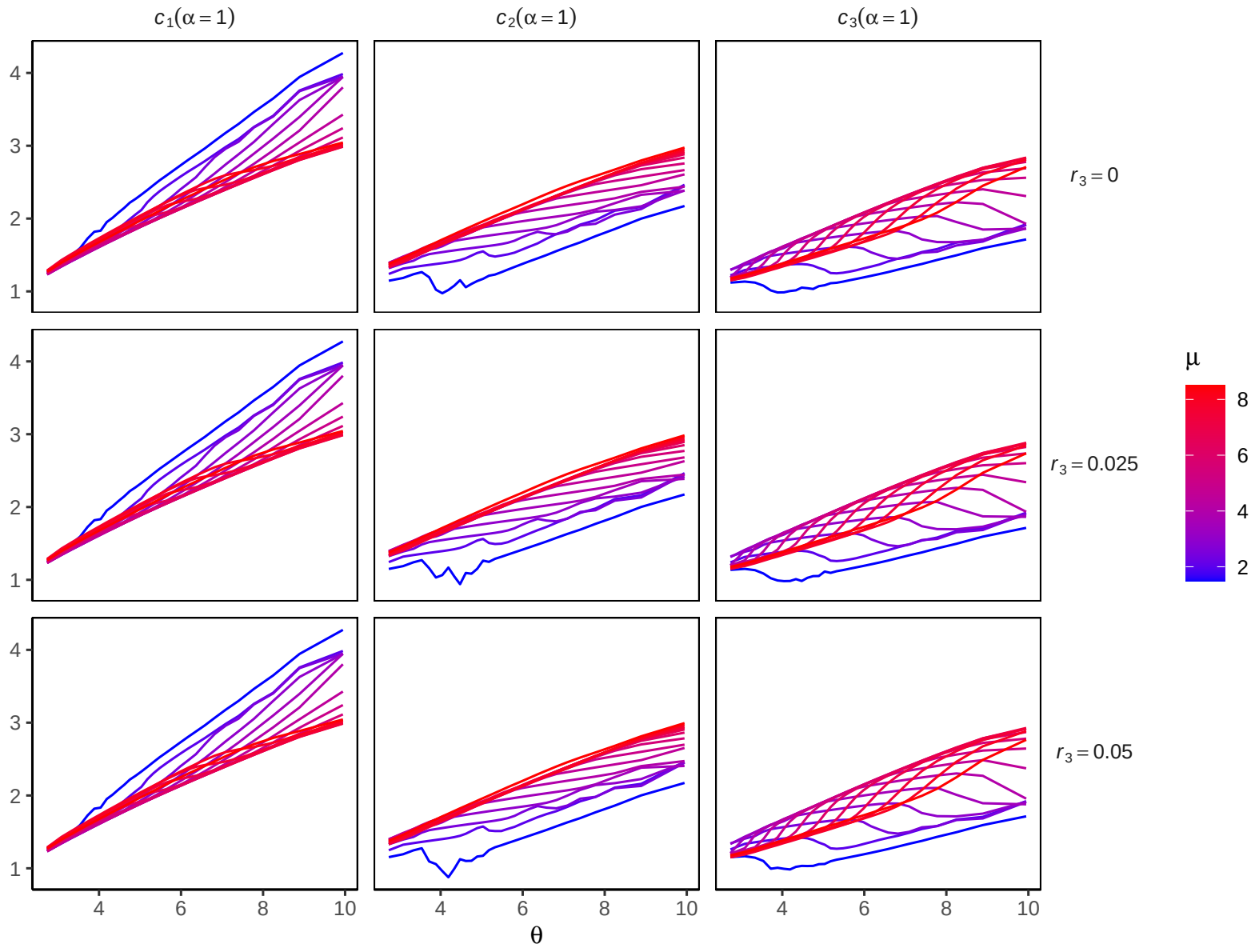

Bequest wedge at period t = 2
r3 = 0
r3 = 0.025
r3 = 0.05
−1.275e−06
−1.300e−06
−1.325e−06
−1.350e−06
0
10
20
30
θ
μ
30
20
10

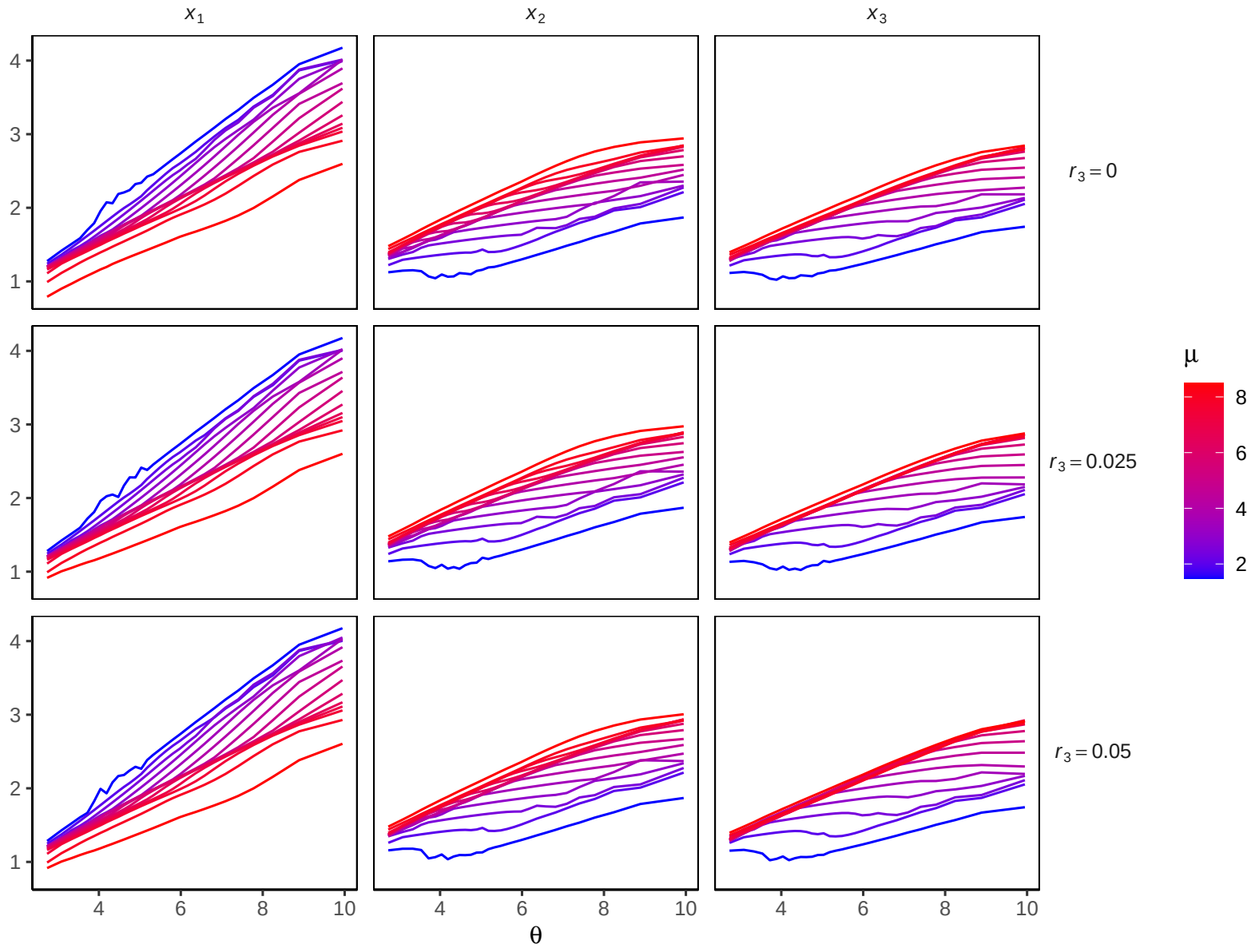

Bequest wedge at period t = 3
r3 = 0
r3 = 0.025
r3 = 0.05
−2.25e−08
−2.50e−08
−2.75e−08
−3.00e−08
−3.25e−08
−3.50e−08
0
10
20
30
θ
μ
30
20
10

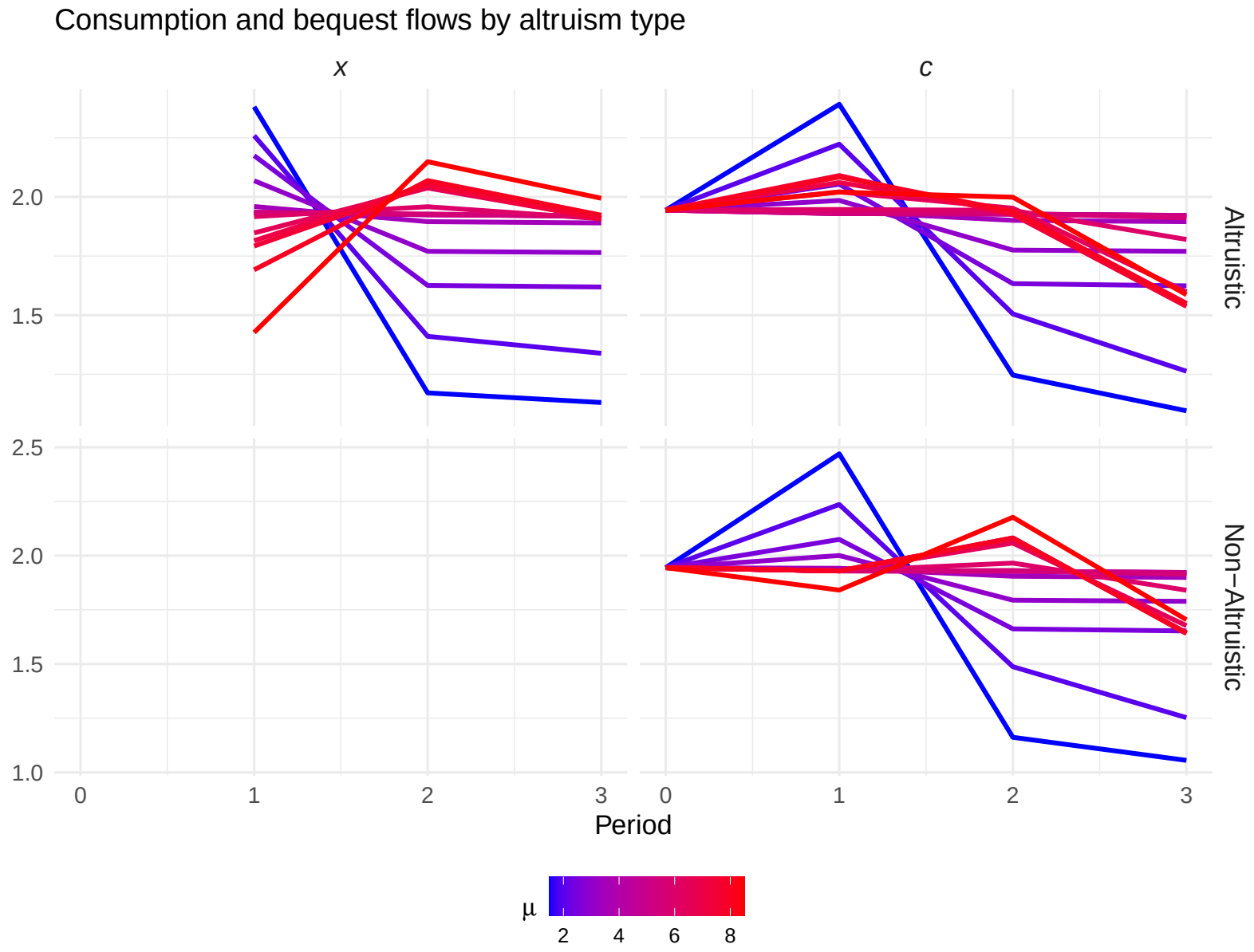
I Second best optimal decision rules

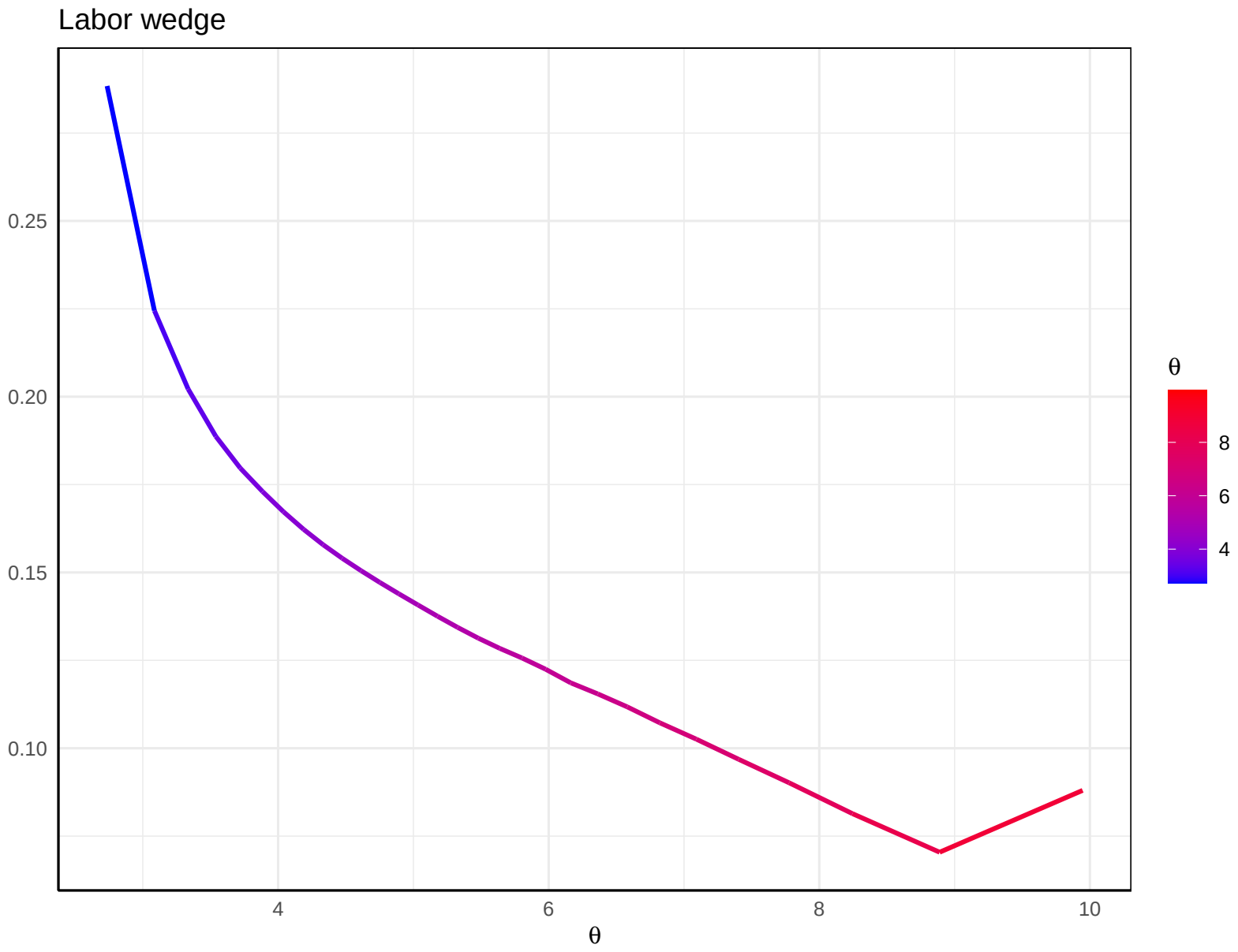

c_0
y
10.0
7.5
5.0
2.5
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
Values
4
6
8
10
θ
μ
8
6
4
2

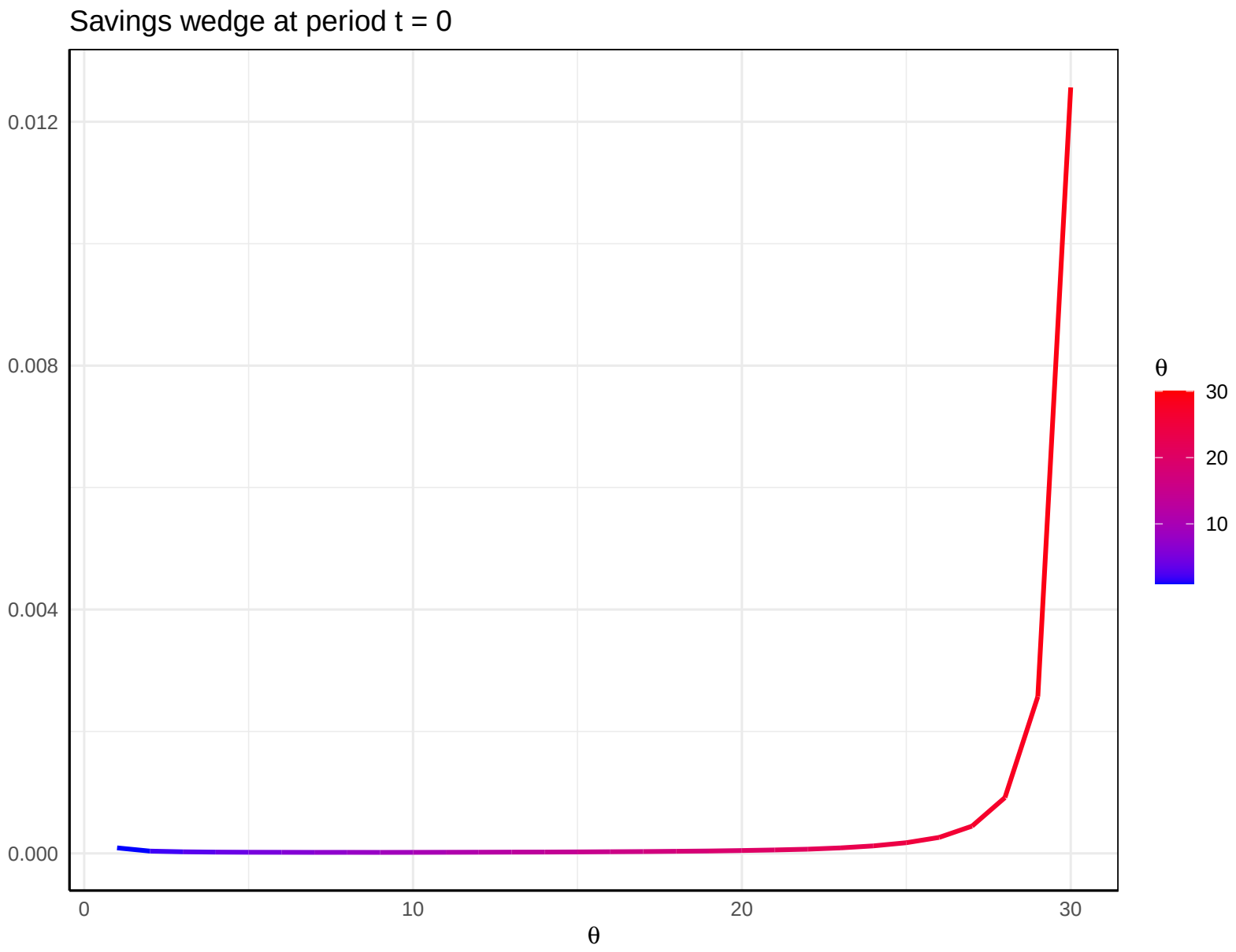

$c_1(\alpha = 0)$
$c_2(\alpha = 0)$
$c_3(\alpha = 0)$
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
μ
8
6
4
2
θ
1
2
3
4
4
6
8
10

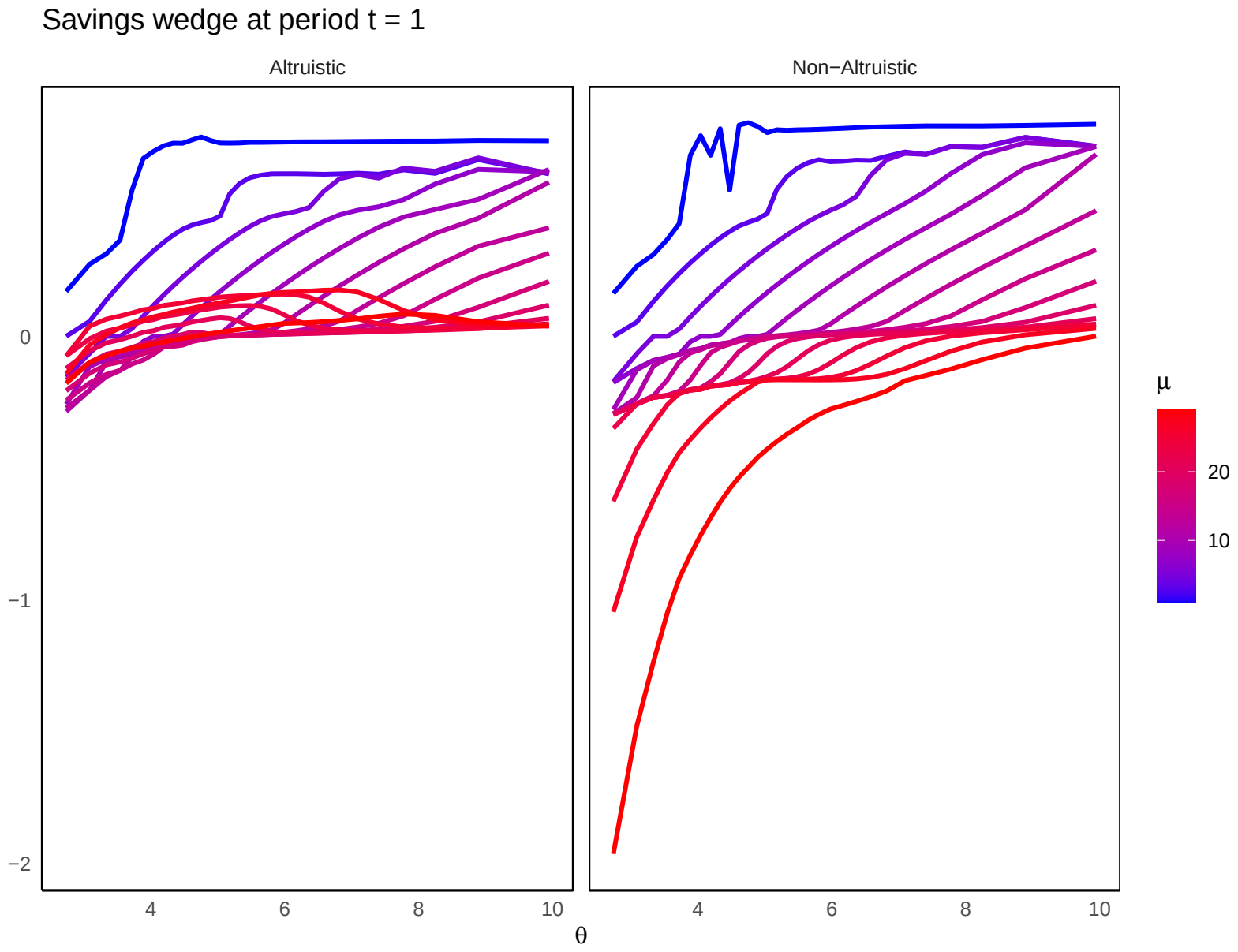

$c_1(\alpha = 1)$
$c_2(\alpha = 1)$
$c_3(\alpha = 1)$
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
μ
8
6
4
2
θ
1
2
3
4
6
8
10

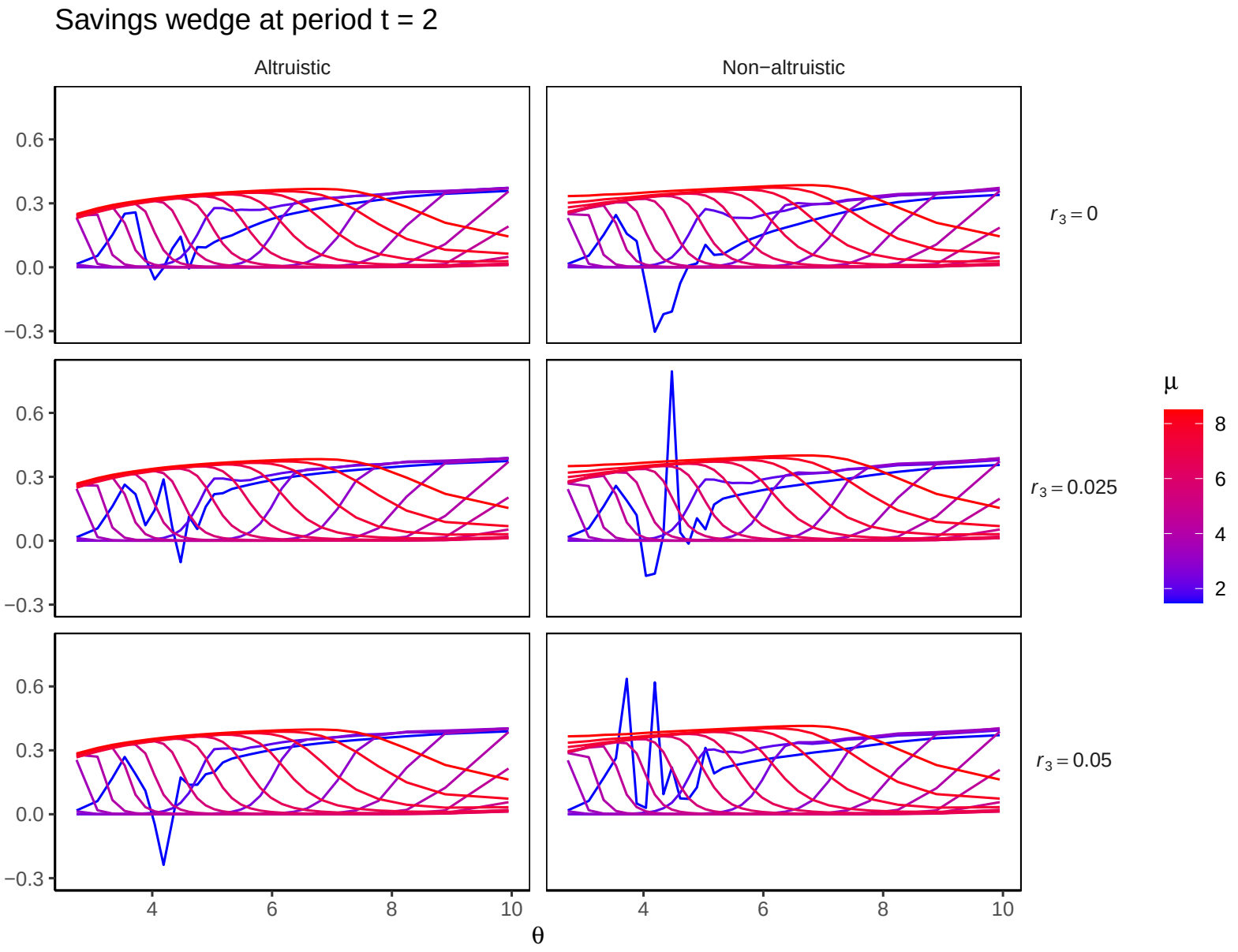

x_1
x_2
x_3
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
μ
8
6
4
2
1
2
3
4
4
6
8
10
θ

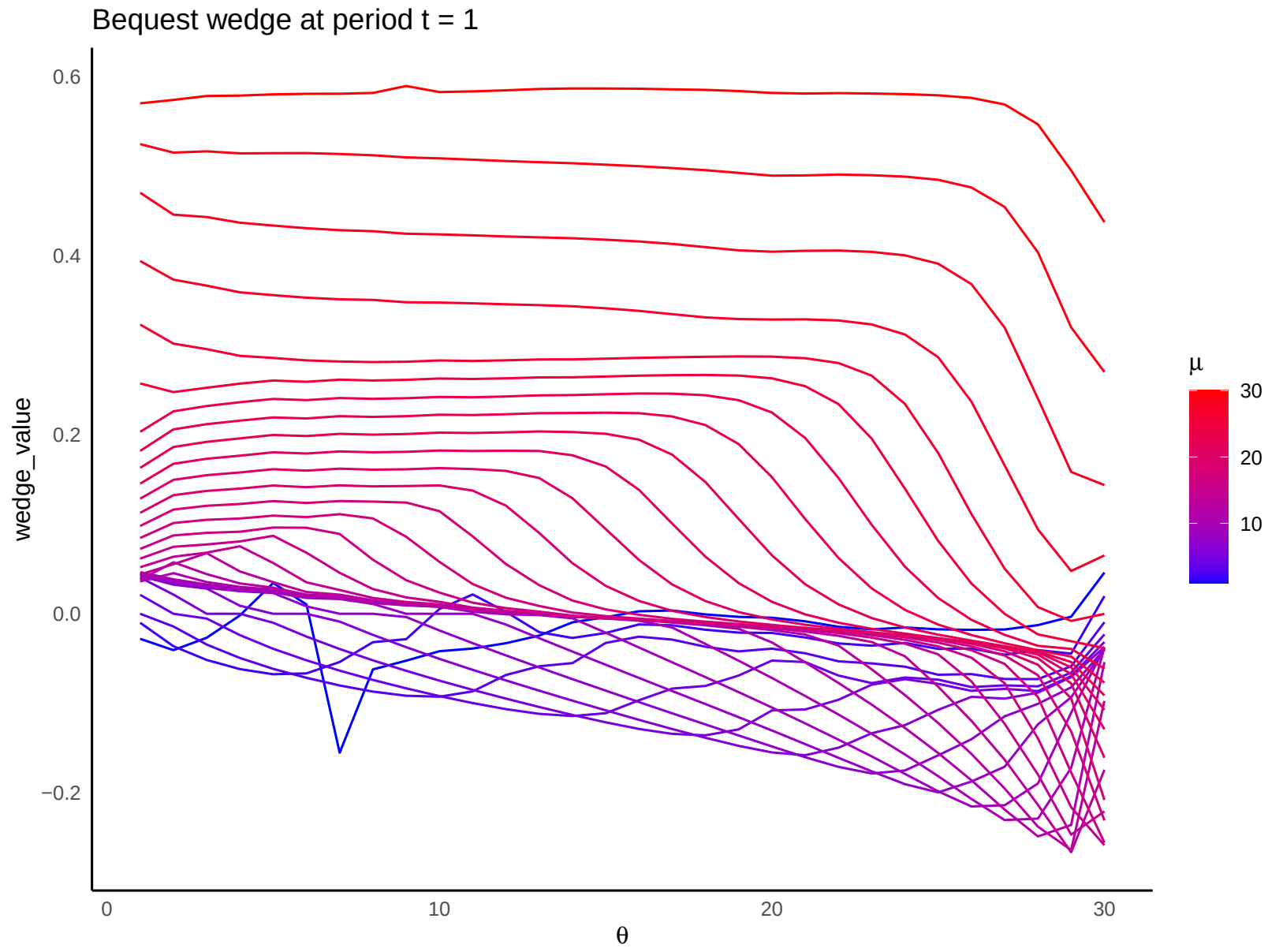

Consumption and bequest flows by altruism type
x
c
Altruistic
Non−Altruistic
2.0
1.5
2.5
2.0
1.5
1.0
0
1
2
3
0
1
2
3
Period
μ
2
4
6
8

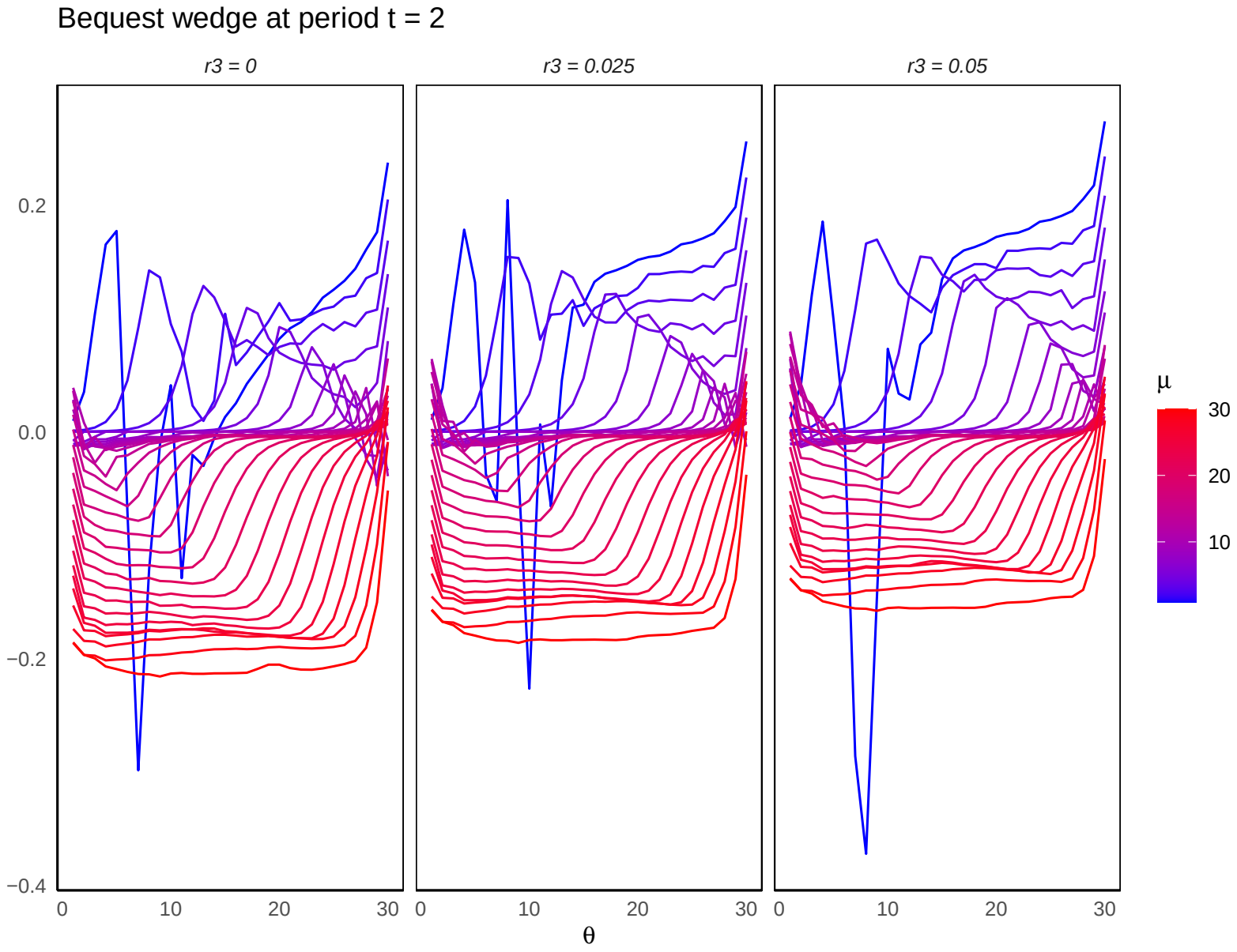

Labor wedge
0.25
0.20
0.15
0.10
4
6
8
10
θ
θ
8
6
4

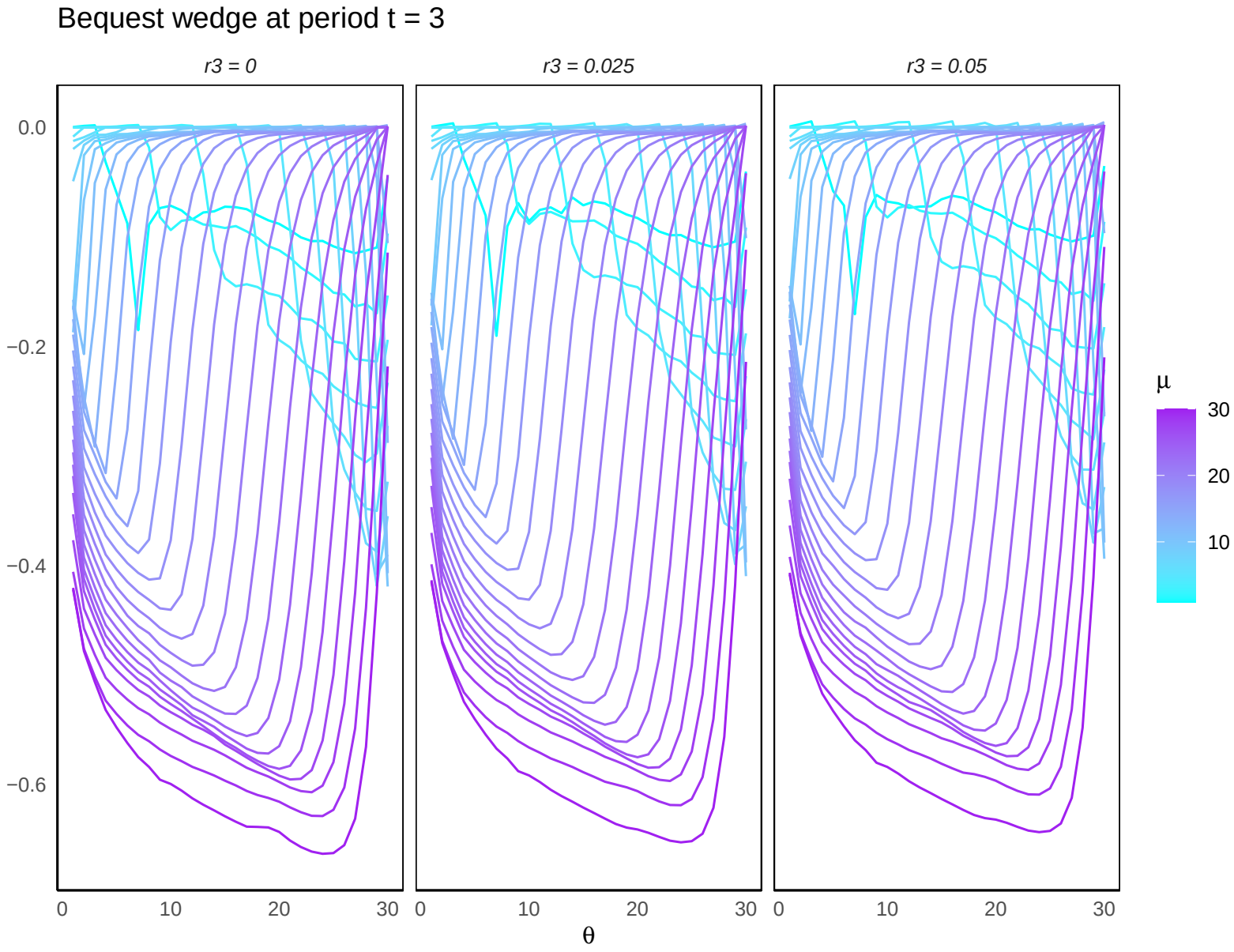

Savings wedge at period t = 0
0.012
0.008
0.004
0.000
0
10
20
30
θ
θ
30
20
10


Savings wedge at period t = 1
Altruistic
Non−Altruistic
0
−1
−2
4
6
8
10
θ
μ
20
10

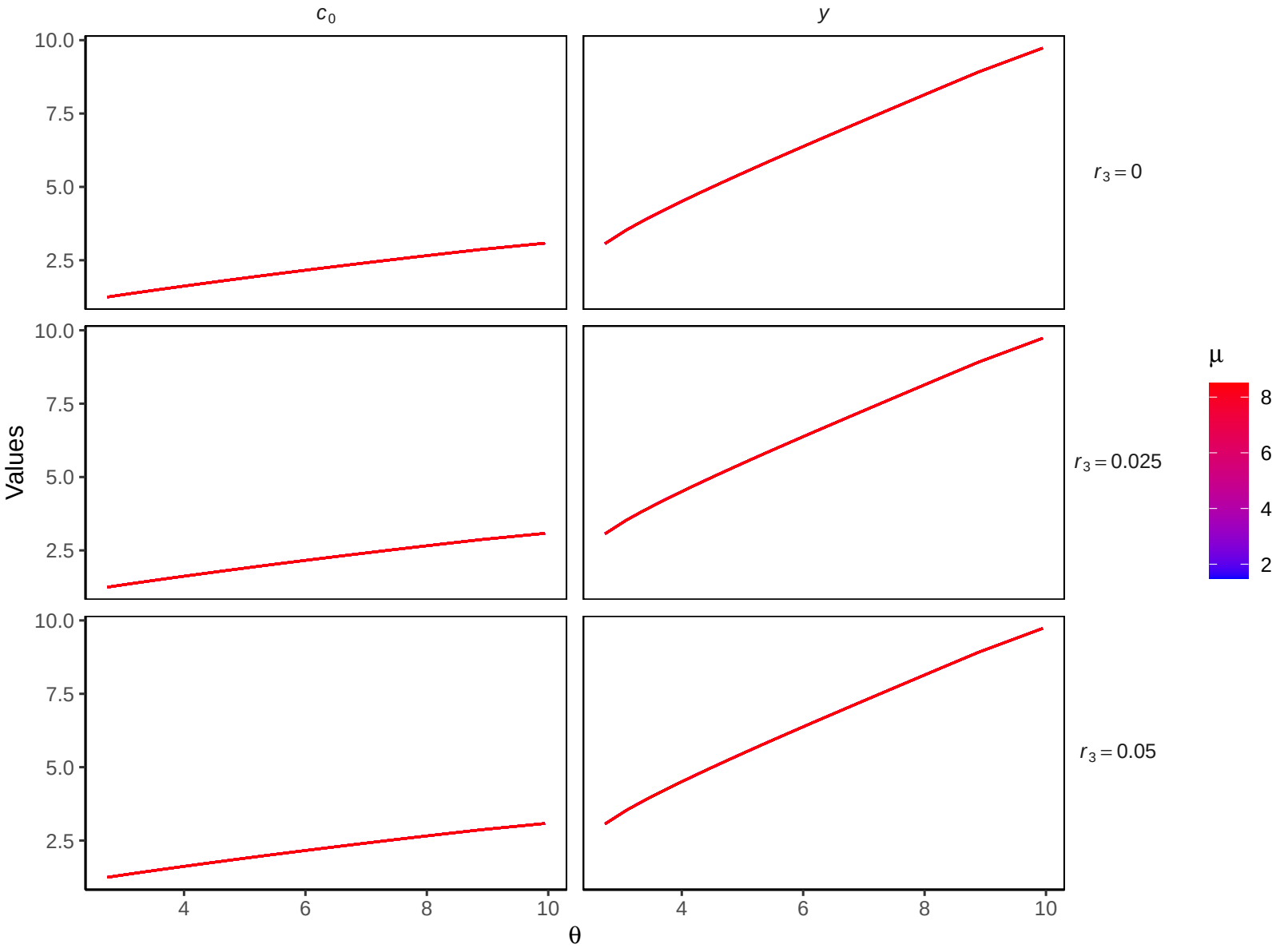

Savings wedge at period t = 2
Altruistic
Non−altruistic
0.6
0.3
0.0
−0.3
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
4
6
8
10
θ
μ
8
6
4
2

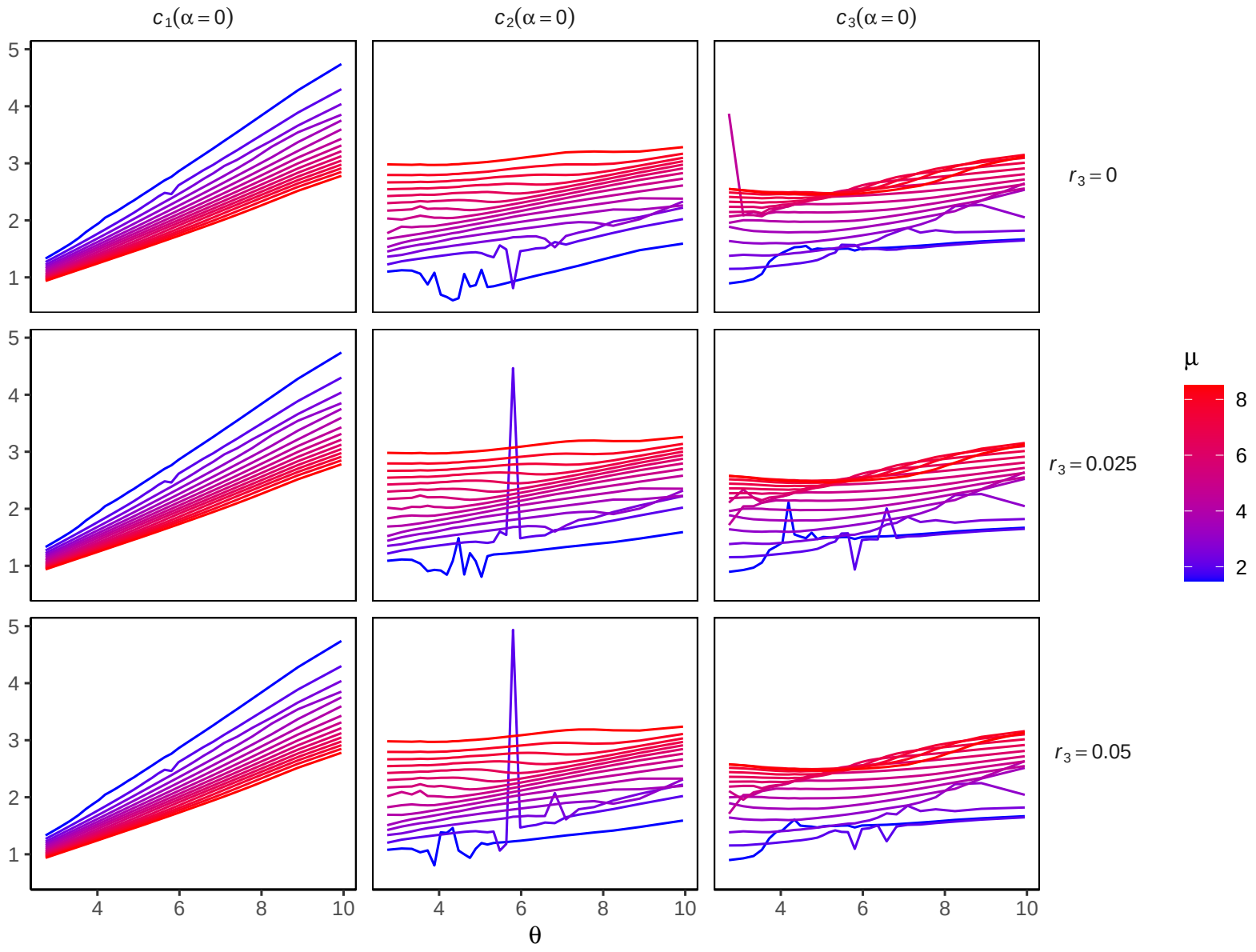

Bequest wedge at period t = 1
wedge_value
0.6
0.4
0.2
0.0
−0.2
0
10
20
30
θ
μ
30
20
10

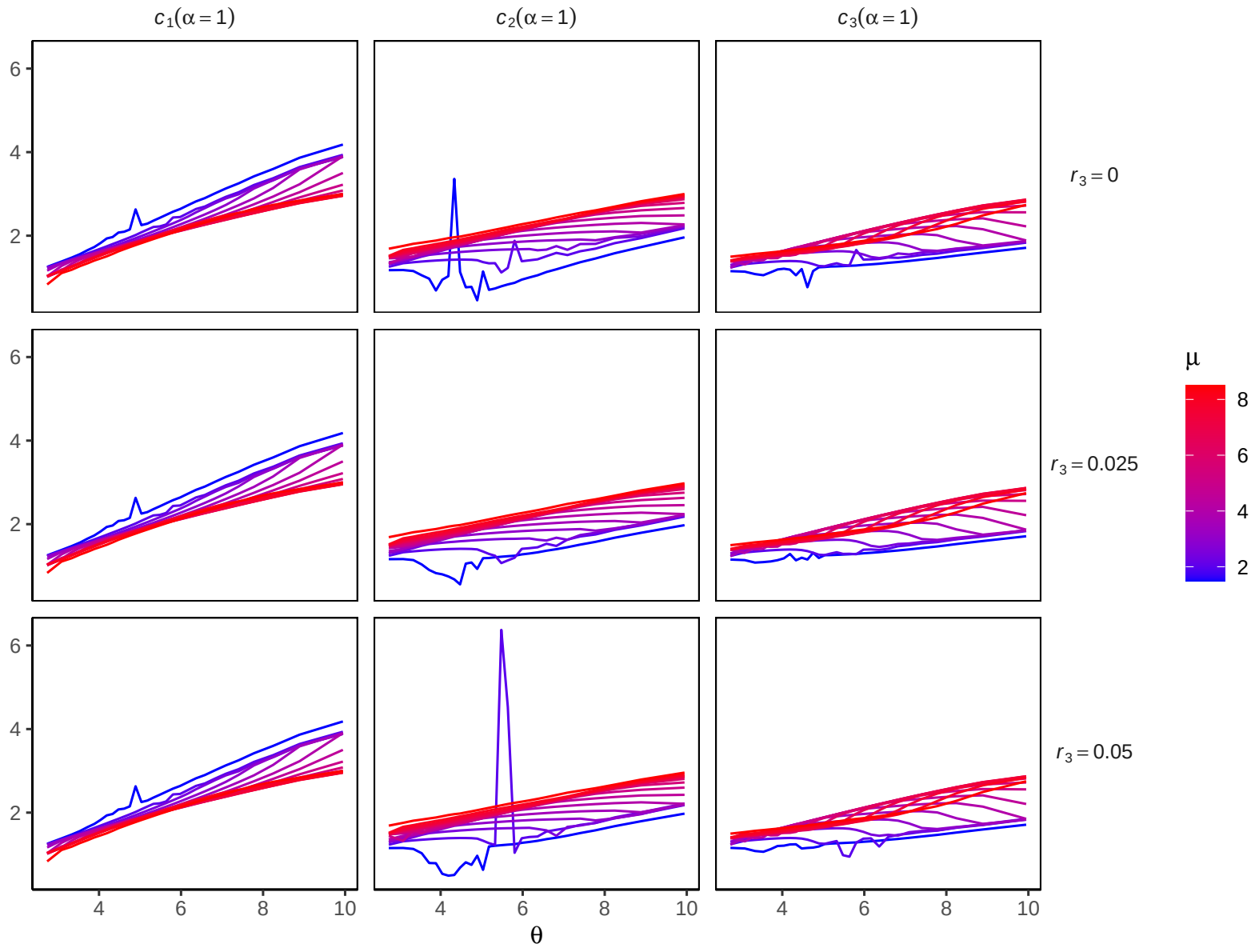

Bequest wedge at period t = 2
r3 = 0
r3 = 0.025
r3 = 0.05
0.2
0.0
−0.2
−0.4
0
10
20
30
θ
μ
30
20
10

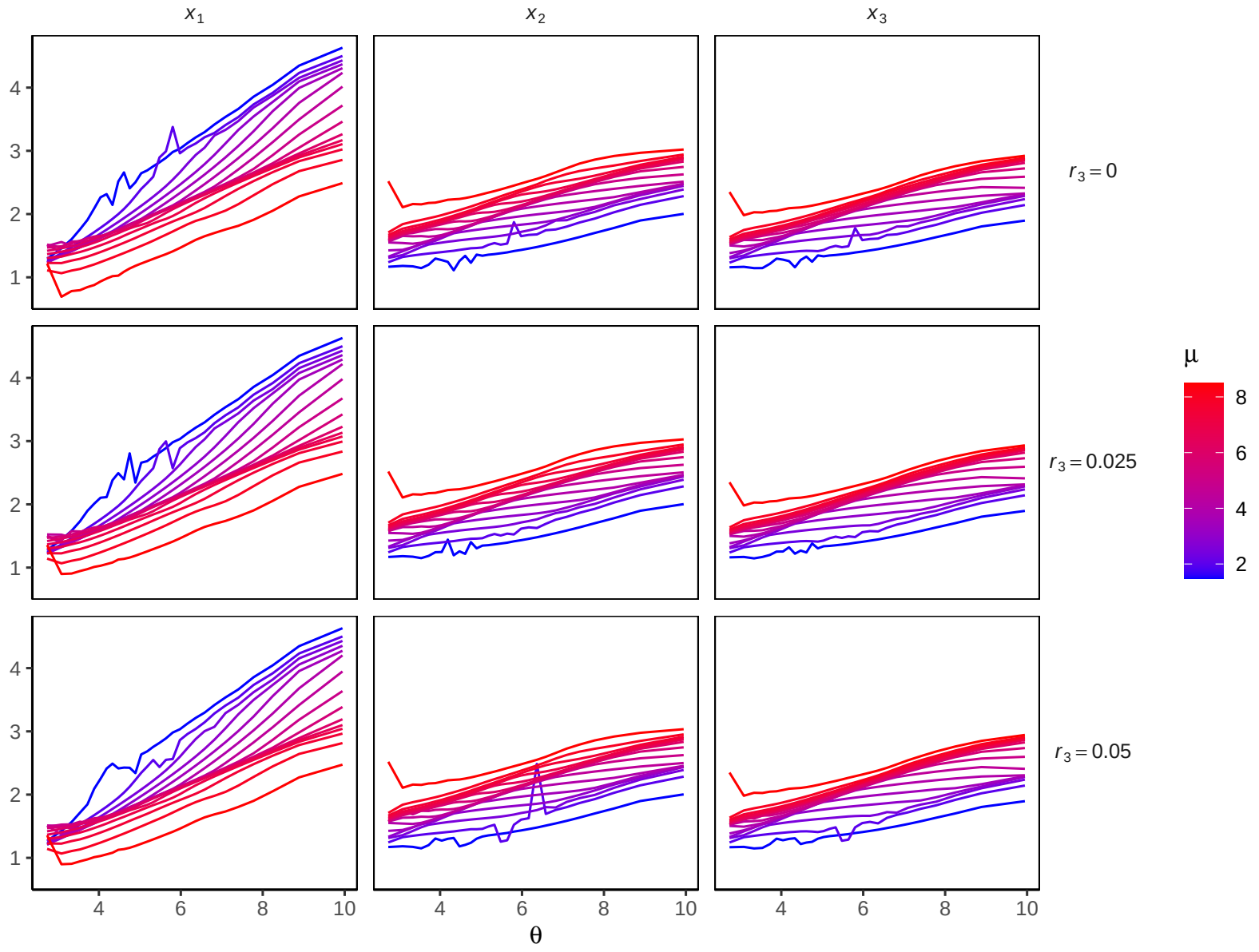

Bequest wedge at period t = 3
r3 = 0
r3 = 0.025
r3 = 0.05
0.0
−0.2
−0.4
−0.6
0
10
20
30
θ
μ
30
20
10

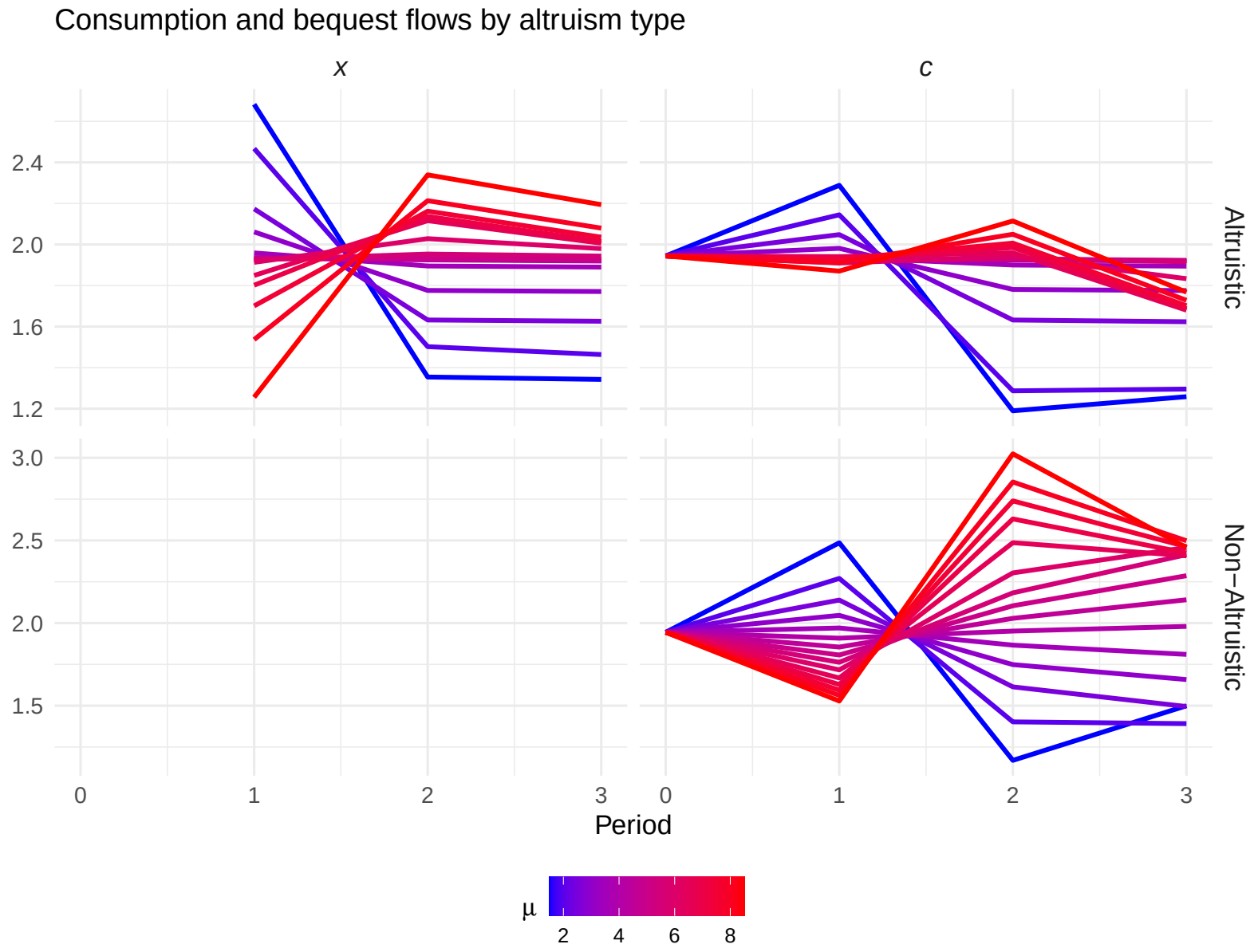
J Third best optimal decision rules

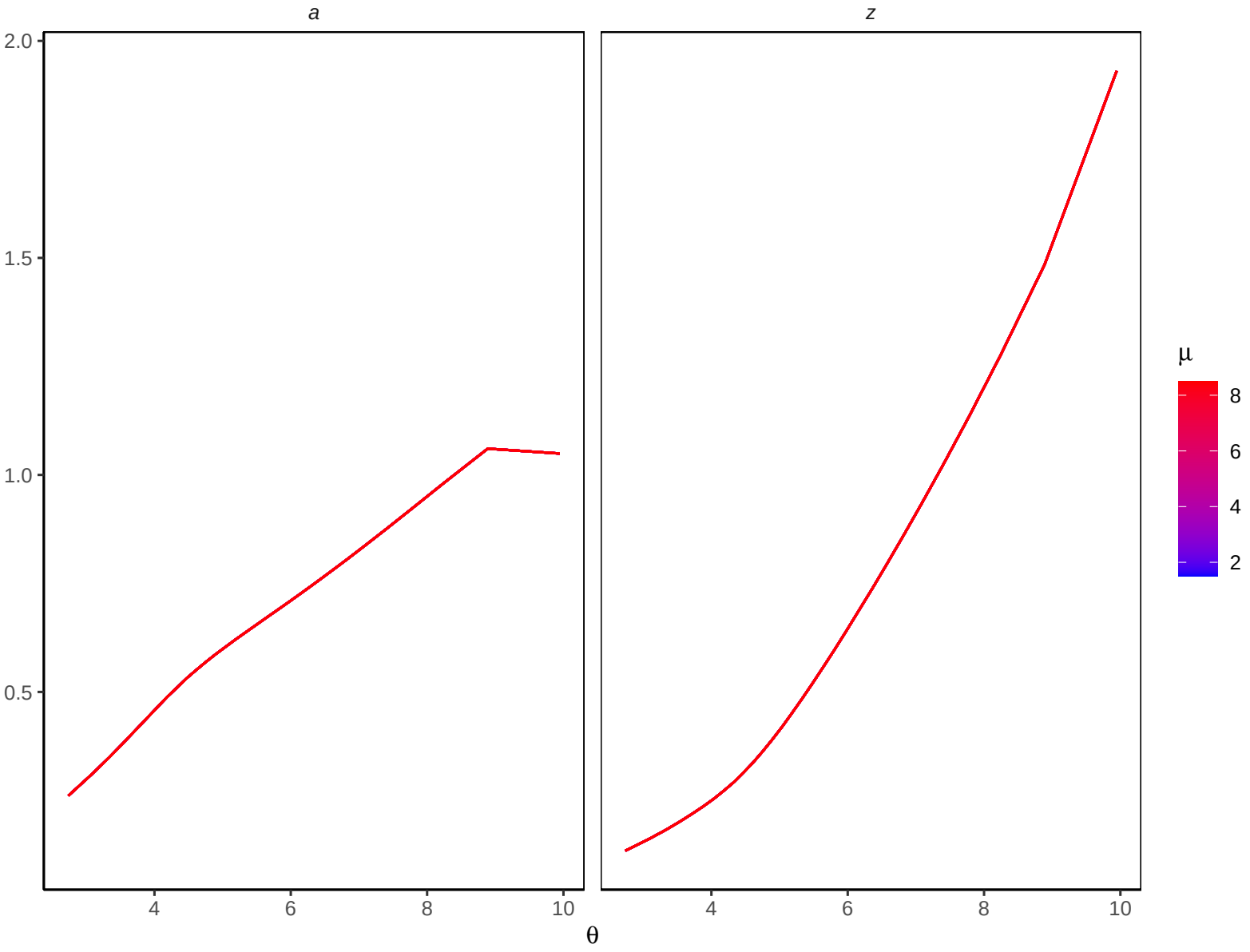

c_0
y
10.0
7.5
5.0
2.5
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
Values
4
6
8
10
θ
μ
8
6
4
2

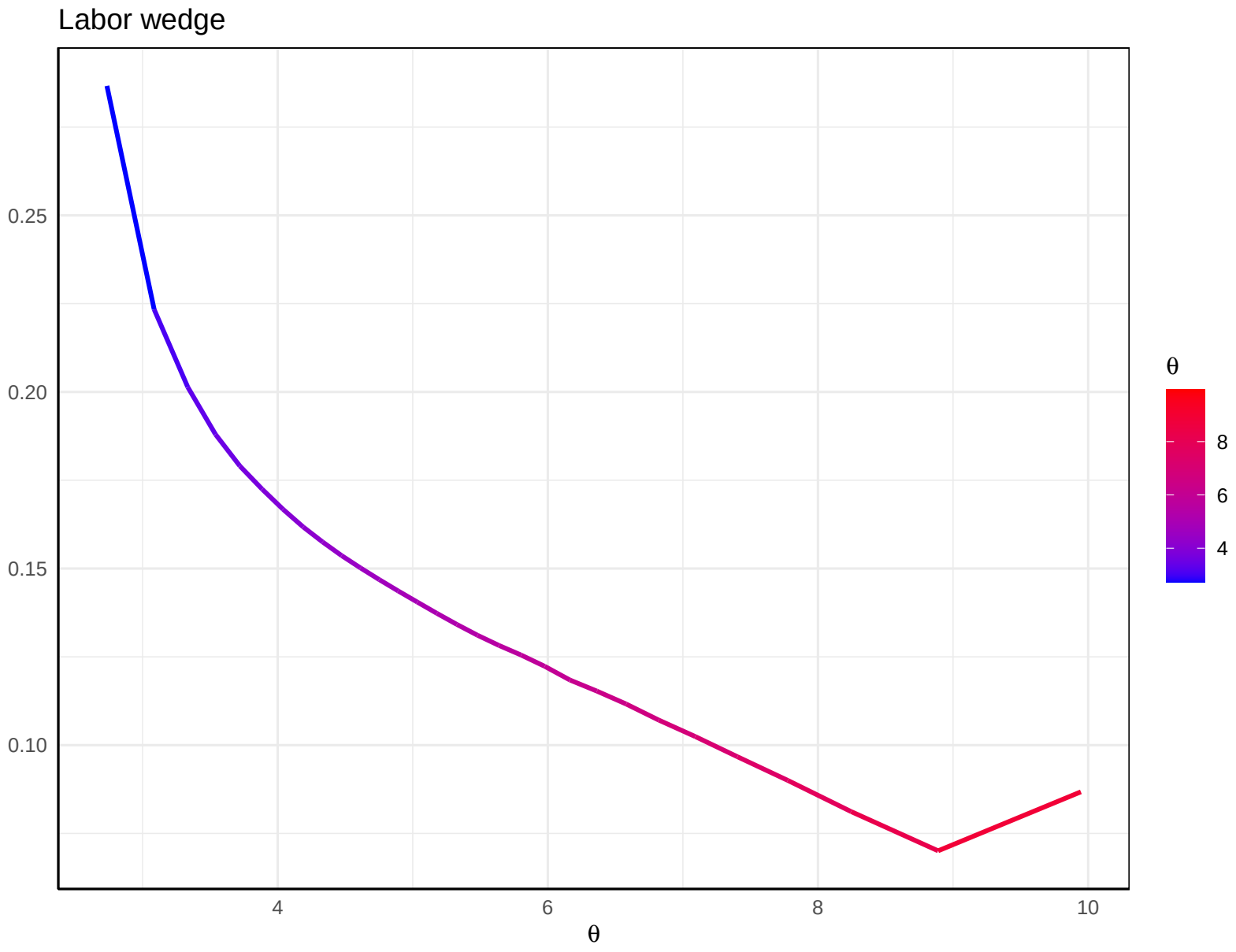

$c_1(\alpha = 0)$
$c_2(\alpha = 0)$
$c_3(\alpha = 0)$
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
μ
8
6
4
2
θ
1
2
3
4
5
4
6
8
10

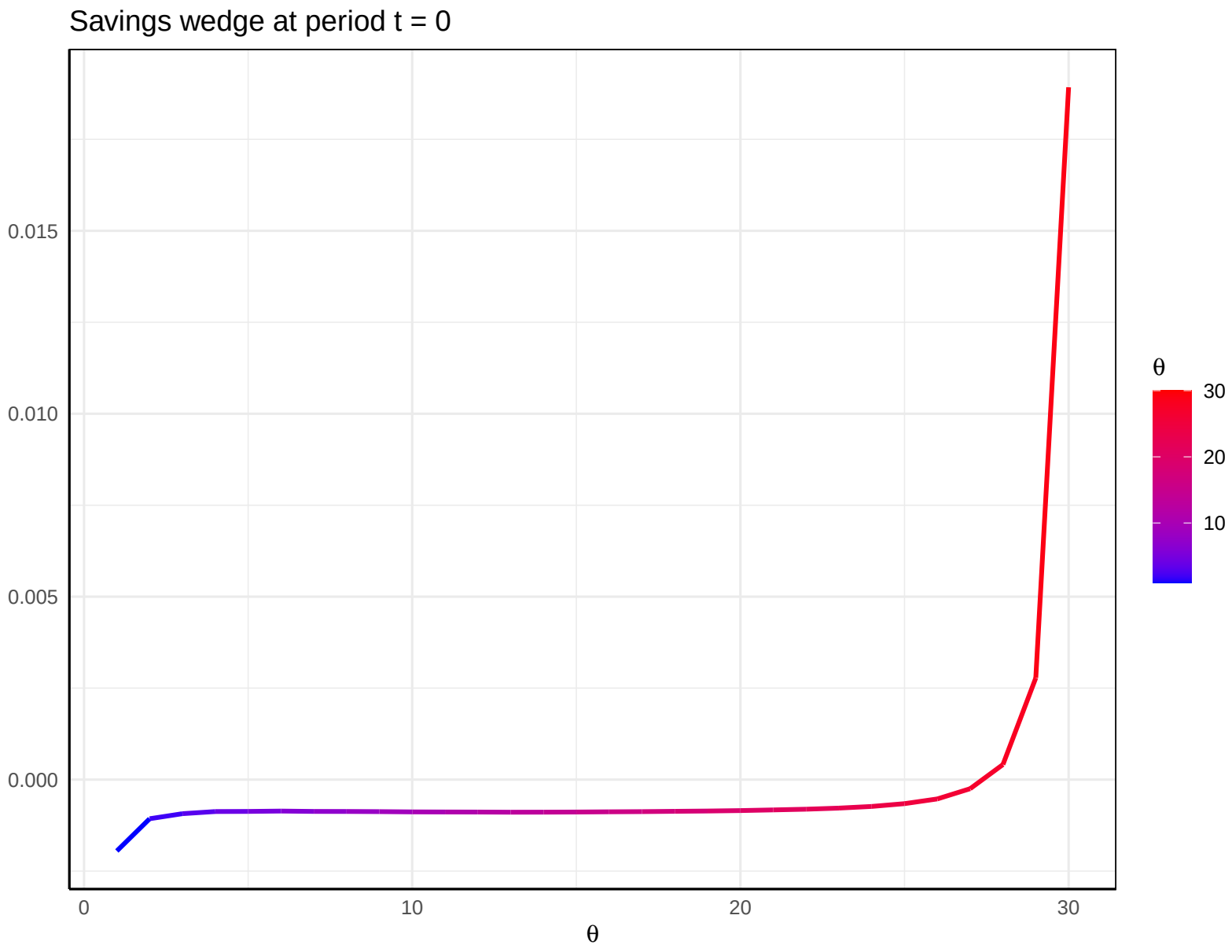

$c_1(\alpha = 1)$
$c_2(\alpha = 1)$
$c_3(\alpha = 1)$
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
μ
8
6
4
2
θ
4
6
8
10

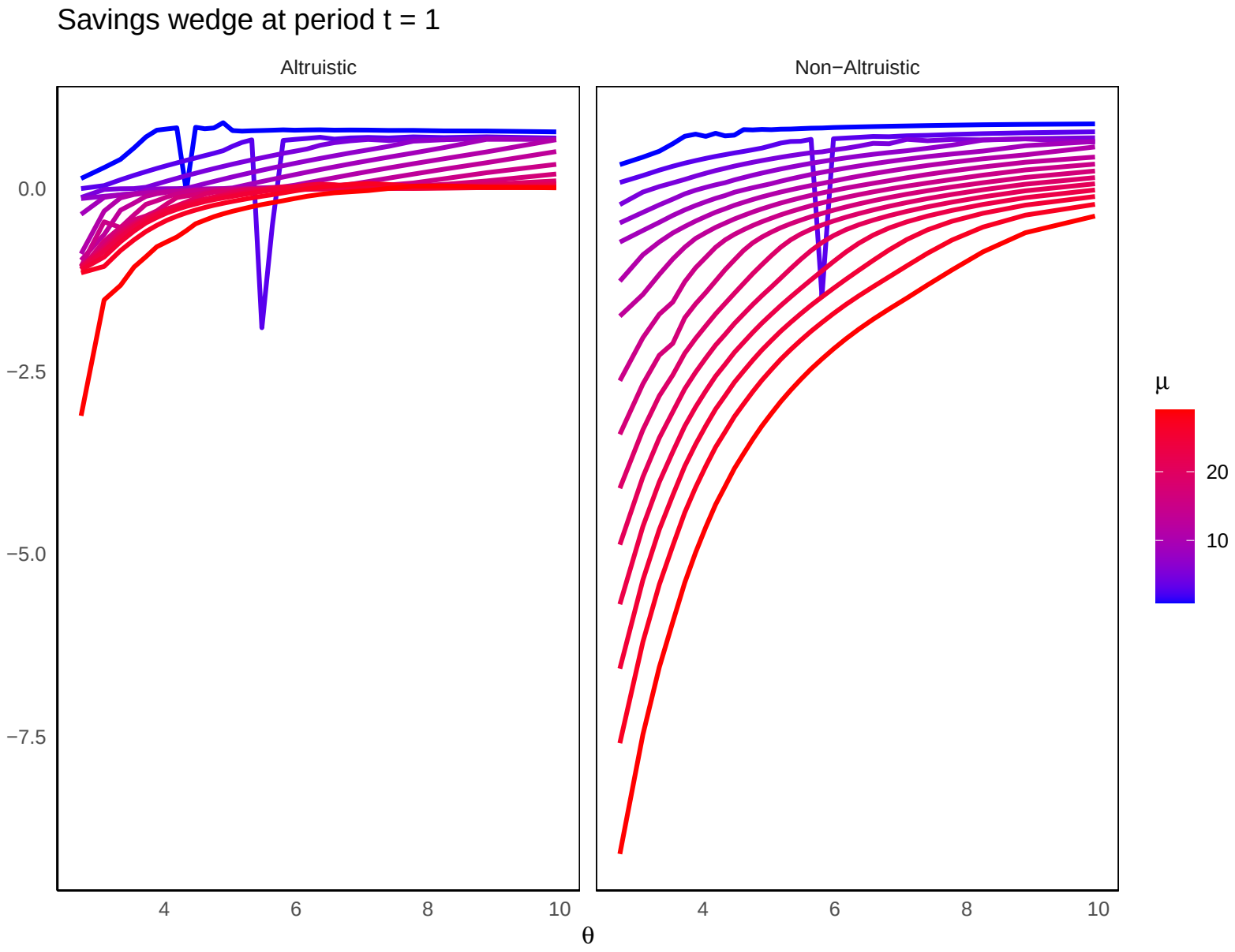

x_1
x_2
x_3
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
μ
8
6
4
2
1
2
3
4
4
6
8
10
θ

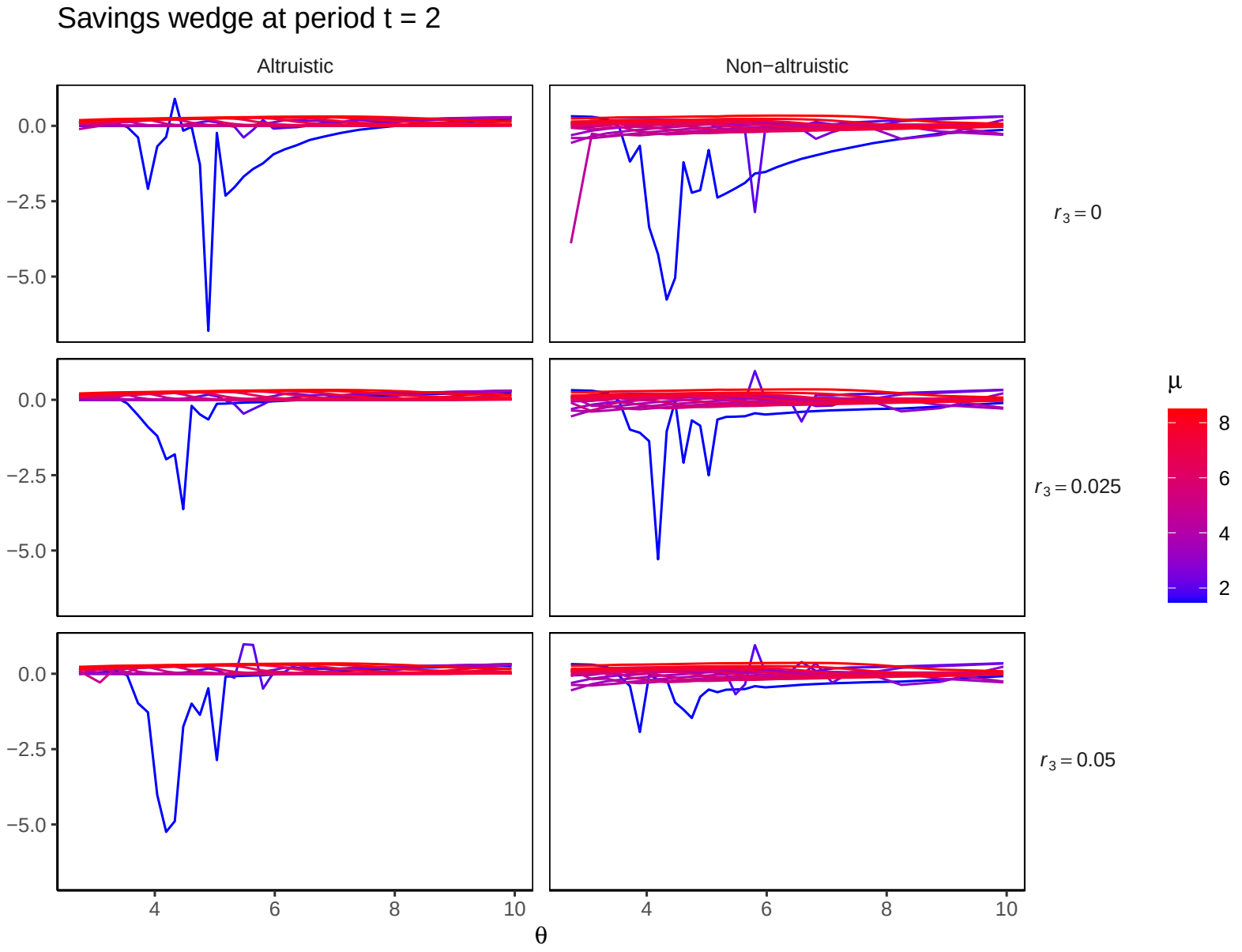

Consumption and bequest flows by altruism type
x
c
Altruistic
Non−Altruistic
2.4
2.0
1.6
1.2
3.0
2.5
2.0
1.5
0
1
2
3
Period
μ
2
4
6
8

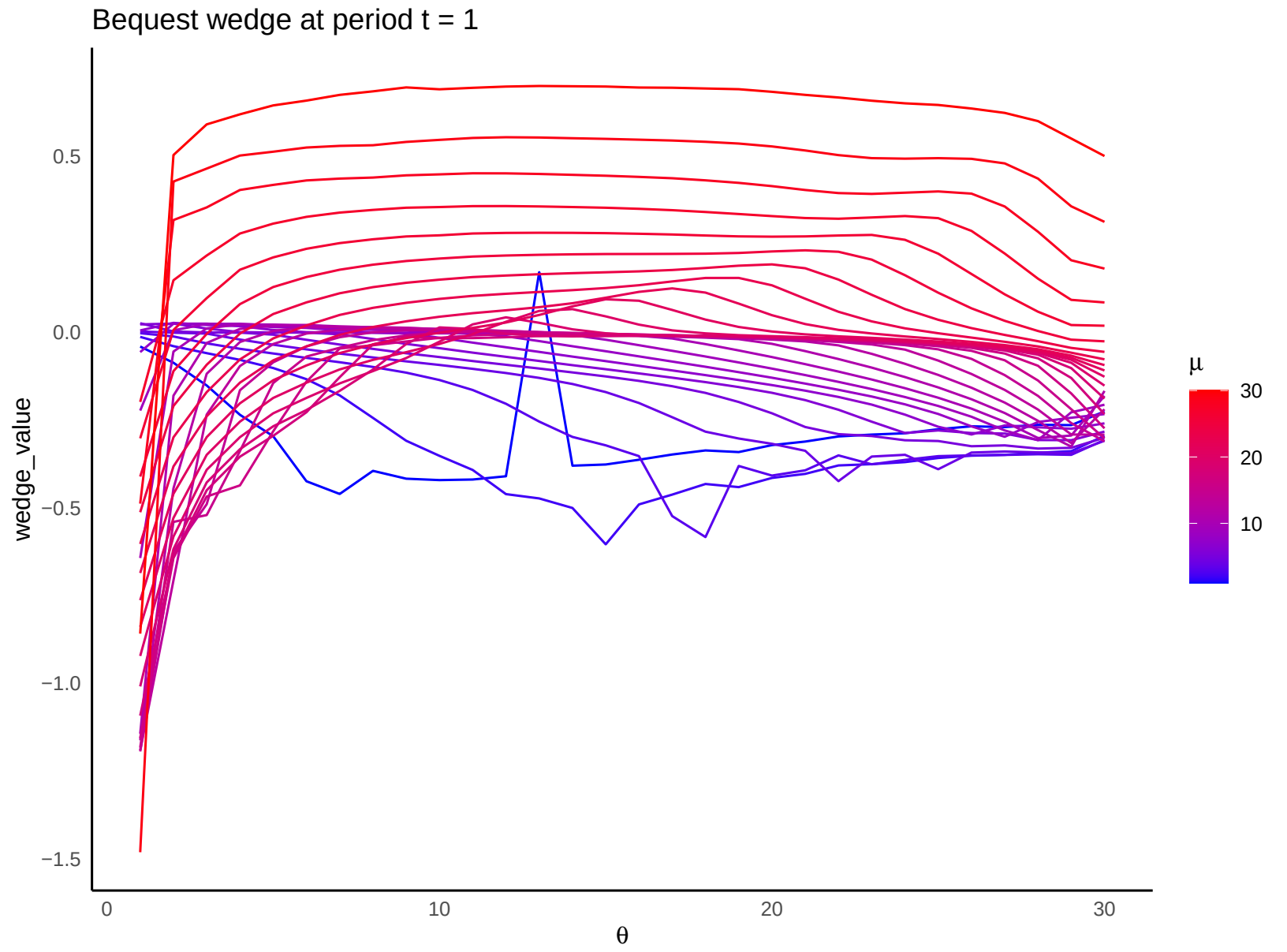

a
z
2.0
1.5
1.0
0.5
4
6
8
10
θ
μ
8
6
4
2

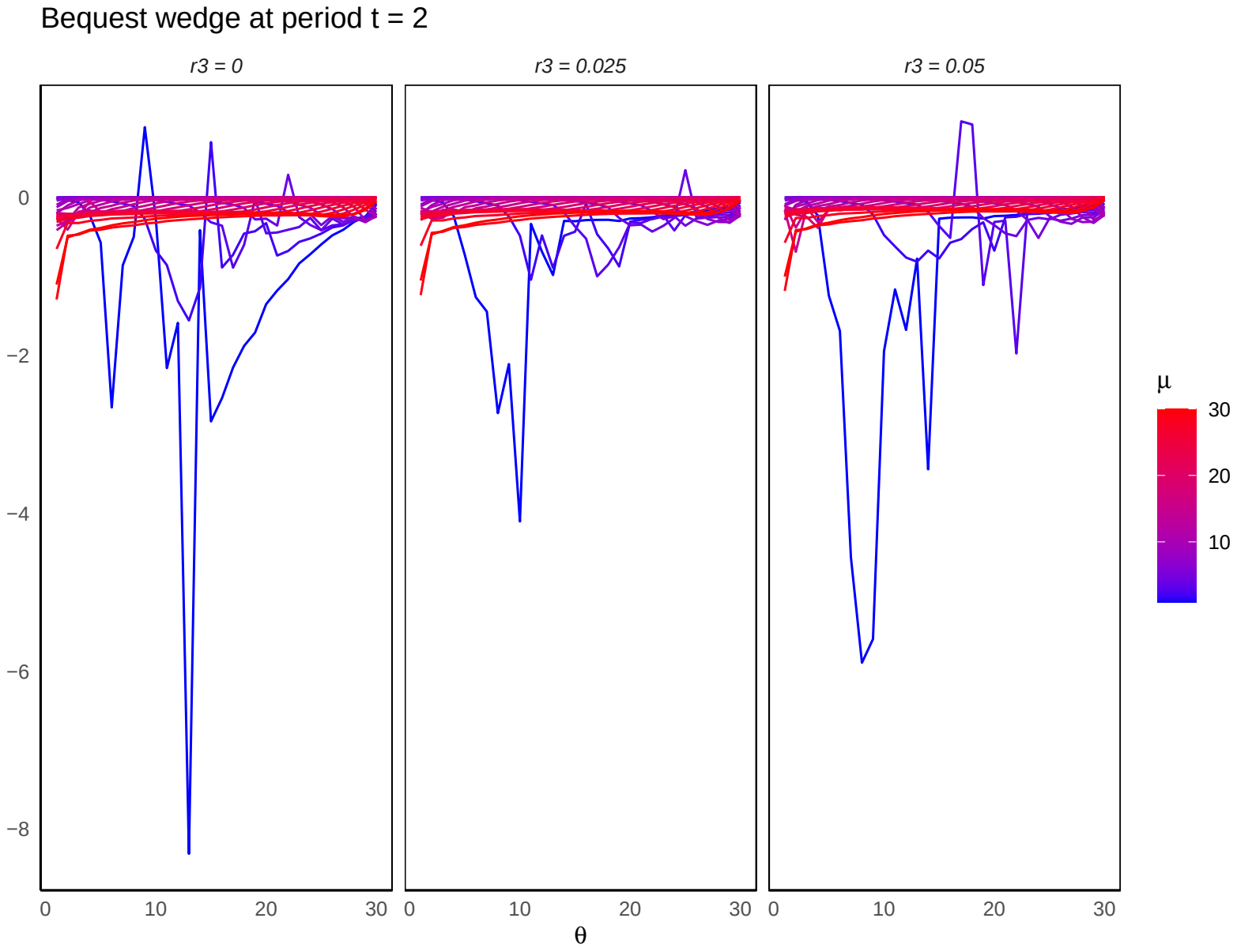

Labor wedge
0.25
0.20
0.15
0.10
4
6
8
10
θ
θ
8
6
4

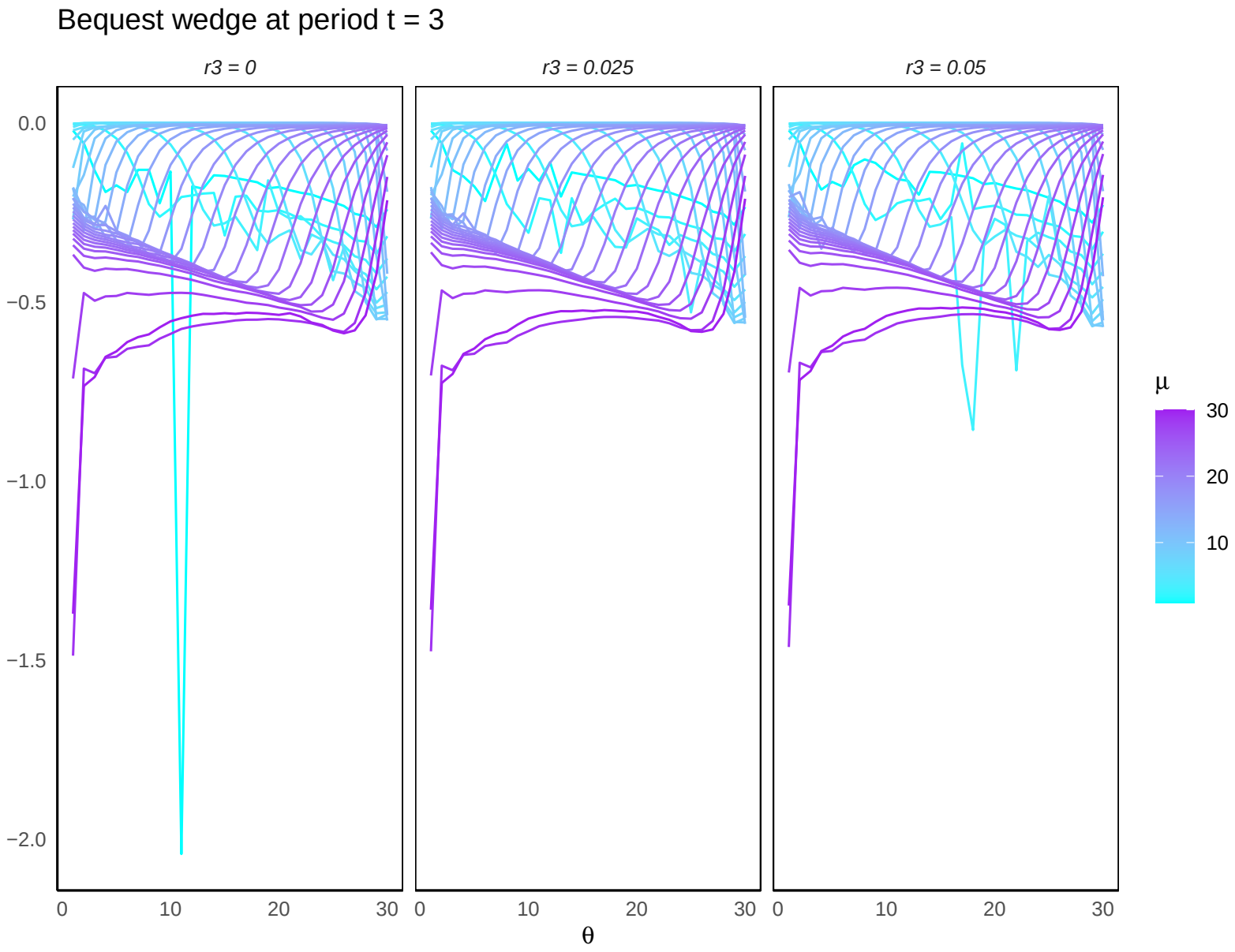

Savings wedge at period t = 0
0.015
0.010
0.005
0.000
0
10
20
30
θ
θ
30
20
10


Savings wedge at period t = 1
Altruistic
Non−Altruistic
0.0
−2.5
−5.0
−7.5
4
6
8
10
θ
μ
20
10


Savings wedge at period t = 2
Altruistic
Non−altruistic
0.0
−2.5
−5.0
$r_3 = 0$
$r_3 = 0.025$
$r_3 = 0.05$
4
6
8
10
θ
μ
8
6
4
2


Bequest wedge at period t = 1
wedge_value
0.5
0.0
−0.5
−1.0
−1.5
0
10
20
30
θ
μ
30
20
10


Bequest wedge at period t = 2
r3 = 0
r3 = 0.025
r3 = 0.05
0
−2
−4
−6
−8
0
10
20
30
θ
μ
30
20
10


Bequest wedge at period t = 3
r3 = 0
r3 = 0.025
r3 = 0.05
0.0
−0.5
−1.0
−1.5
−2.0
0
10
20
30
θ
μ
30
20
10