Multi-pollutant Cap-and-Trade Systems with Heterogeneous Firms: An Optimal Transport Approach[1]

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Abstract

The simultaneous exposure to multiple air pollutants has become a pressing concern for policymakers seeking integrated market-based instruments to address this challenge effectively. In this paper, we develop and analyze a multi-pollutant cap-and-trade system encompassing a diverse array of polluting firms and an independent pollution control authority responsible for issuing multi-pollutant permits. Each multi-pollutant permit encapsulates distinct pollutant rights, reflecting a spectrum of types and quantities of pollutants bundled together. We assume that the control authority agency possesses knowledge of the aggregate heterogeneity distribution of polluting firms, which accommodates situations where it may lack granular information on individual firm pollution profiles (private data). The crux of the challenge resides in efficiently allocating the diverse array of multi-pollutant permits across the heterogeneous pool of polluting firms. To address this, we leverage optimal mass transportation techniques to establish the presence of an optimal allocation strategy for multi-pollutant emissions permits among heterogeneous polluting firms. This allocation process is found to be decentralized, with equilibrium prices of multi-pollutant permits enabling us to derive a geometric characterization of the firms involved based on their self-selection of permits. The intricate interplay between the distribution of polluting firms, the spectrum of multi-pollutant permits, and the firms' heterogeneity in payoff reveals strategic choices in multi-pollutant permit selection and the emergence of market segmentation within the cap-and-trade framework. We also address the question of whether separated markets for individual pollutants can replicate the outcomes of a multi-pollutant cap-and-trade market. For illustration purposes, numerical simulations employing computational geometry and optimal transport techniques elucidate the determination of permit prices and the optimal demand partition within a multi-pollutant cap-and-trade market.

Keywords: Multi-pollutant Permits, Cap-and-Trade Markets, Environmental Policy, Heterogeneous Firms, Matching, Optimal Mass Transport.
JEL Classification: Q30, G12, Q4, C52

1 Introduction

Addressing simultaneous exposure to multiple pollutants is one of the most pressing environmental challenges of our time. Pollutants in the air do not exist in isolation—they mix, interact, and amplify their harmful effects on human health and ecosystems. Traditional regulatory and market-based approaches have largely treated pollutants separately, despite growing evidence highlighting the need for strategies that account for their combined impacts (Council, 2004; Dominici et al., 2010; Hidy and Pennell, 2010; Cohan et al., 2007; Sun et al., 2014; Masiol et al., 2014; Godzinski and Castillo, 2021). Among market-based solutions, cap-and-trade systems provide a promising tool by creating incentives for firms to reduce emissions efficiently.[1] However, existing frameworks typically regulate single pollutants in isolation, overlooking the complex interdependencies among them. For instance, multiple pollutants, including carbon monoxide (CO), particulate matter (PM), nitrogen oxides (NO_x), sulfur dioxide (SO_2), and ozone (O_3), originate from overlapping sources and collectively contribute to environmental degradation and adverse health outcomes. These pollutants often co-occur with greenhouse gases such as carbon dioxide (CO_2), methane (CH_4), and nitrous oxide (N_2O), and their combined effects on health and the environment are often greater than the sum of their individual impacts (Billionnet et al., 2012; Winquist et al., 2014).[2] Recognizing these challenges, the National Research Council (NRC) in 2004 called for a regulatory framework that addresses multiple pollutants simultaneously (Council, 2004). In response, the Environmental Protection Agency (EPA) advanced multi-pollutant air quality management in 2010 and, more recently, released an *Interim Framework for Advancing Consideration of Cumulative Impacts* (U.S. Environmental Protection Agency, 2024), emphasizing the need for holistic approaches.

While greenhouse gas mitigation policies can generate co-benefits in air pollutant reduction Burtraw et al. (2003), there is a growing need to design market-based systems that directly account for joint exposure to multiple pollutants. By simultaneously placing a cap on all pollutants, such systems can maximize public health benefits and address the complex interactions between different emissions. This approach is particularly important given the interconnected nature of pollutants, which often have cumulative and compounding effects on human health and the environment. The availability of comprehensive emissions datasets underscores the importance of addressing multiple pollutants simultaneously. Programs such as the EPA Clean Air Markets Division (CAMD), the National Emissions Inventory (NEI), the EPA Greenhouse Gas Reporting Program (GHGRP), and the Toxics Release Inventory (TRI) provide detailed insights into the sources and impacts of emissions. These datasets reveal how pollutants from different sources interact, highlighting the need for integrated strategies that consider their combined effects rather than treating them in isolation. This global recognition of the interconnected nature of air pollution is further

[1] John Dales pioneered the concept of marketable permits for pollution regulation (Dales, 1974). Since then, cap-and-trade systems have gained recognition as promising market-based tools for addressing environmental challenges, as highlighted by Stavins (2001), Newell et al. (2014), and Fischer and Jacobsen (2021).

[2] For clarity, even if two greenhouse gases are carbon equivalent in terms of their contribution to climate change, they may differ significantly in terms of their individual health impacts or joint effects, as their toxicity, atmospheric interactions, and exposure pathways can vary widely.

reflected in the development of air quality indices. For example, the U.S. Air Quality Index (AQI), Canada's Air Quality Health Index (AQHI), China's Ambient Air Quality Standards (CAAQS), and India's air quality index (IQair) all rely on comprehensive datasets to monitor and regulate multiple pollutants simultaneously.

Mounting evidence confirms that simultaneous exposure to multiple air pollutants poses significantly greater risks than exposure to single pollutants (Godzinski and Castillo, 2021; Greenbaum and Shaikh, 2010; Dominici et al., 2010; Wu et al., 2023; Billionnet et al., 2012; Zhu et al., 2022; Yu et al., 2022; Hu et al., 2015; Stafoggia et al., 2017; Leirião et al., 2023; Huang et al., 2023; Kortenkamp and Faust, 2018). This underscores the limitations of existing cap-and-trade programs, which largely operate in isolation and fail to account for cross-pollutant interactions and their compounded impacts on air quality, public health, and the environment. Some models attempt to connect separate markets for distinct pollutants through predefined permit trading rules (Montero, 2001; Ambec and Coria, 2018). However, such approaches often lead to inefficiencies, as they ignore the combined effects of pollutants on human well-being.[3]

To address these limitations, this study develops a novel framework that integrates economic and mathematical modeling to improve the design of cap-and-trade systems. Our approach combines optimal transport theory —a mathematical tool for efficiently transforming probability distributions — with environmental economics, which focuses on market-based solutions to environmental challenges. Optimal transport theory, with roots in Gaspard Monge's 18th-century work and later advancements by Nobel laureate Leonid Kantorovich, has broad applications in data science, economics, and machine learning (Chiappori et al., 2010; Ekeland, 2010; Monge, 1781; Galichon, 2021b; Kantorovich, 1942; Figalli et al., 2011; Galichon, 2016, 2021a). In the context of cap-and-trade systems, it addresses matching problems and multidimensional complexities in regulating multiple pollutants. By integrating this theory into policy design, we enable an efficient allocation of multi-pollutant permits that accounts for the distribution of emission profiles across firms. This approach can be viewed as grandfathering in the permit design, ensuring that the diverse characteristics of polluting firms are reflected in the portfolio of permits.

Our framework supports the design of market-based mechanisms that account for the multidimensional heterogeneity of environmental impacts. Using optimal transport theory, we frame the cap-and-trade problem for multiple pollutants as a matching problem within a multidimensional pollution space that captures both firm characteristics and the structure of multi-pollutant permits. This approach allows a regulatory agency to allocate permits effectively without needing detailed firm-level pollution data, relying instead on the statistical distribution of firm types. By modeling permits as multidimensional vectors that encapsulate multiple pollutants, the system remains flexible and adaptable to real-world emissions complexities. The existence and stability of

[3]For example, Godzinski and Castillo (2021) finds that increased concentrations of ozone and sulfur dioxide correlate with higher rates of respiratory-related emergency admissions, particularly among children and the elderly. This underscores the need to recognize pollutant interdependencies, as overlooking these relationships can lead to misleading conclusions about public health risks. The study by Fann et al. (2011) further demonstrates that adopting a multi-pollutant emission reduction approach can yield twice the total monetized benefits of traditional pollutant-by-pollutant regulation, even at a modestly higher cost.

a pricing system for multi-pollutant permits provide a strong theoretical foundation for effective cap-and-trade markets.

Our work contributes to the growing literature that connects optimal transport approaches with hedonic models in multidimensional settings (Rosen, 1974; Epple, 1987; Ekeland et al., 2004; McFadden, 1974; Rochet and Choné, 1998), while uniquely addressing issues related to multi-pollutant cap-and-trade mechanisms. We address five central questions: (1) Can a pricing system match heterogeneous firms with multi-pollutant permits? (2) What is the socially optimal allocation of permits? (3) Does the market mechanism facilitate this allocation? (4) How stable is the pricing system under changes in pollutant weighting, composition, or firm distribution? (5) Can this framework inform numerical simulations for multi-pollutant markets?

The subsequent sections of this paper are organized as follows: In Section 2, we introduce the multi-pollutant cap-and-trade model. Section 3 explores the competitive equilibrium in a multi-pollutant cap-and-trade market, delving into the properties of the demand for multi-pollutant permits. Moving on to Section 4, we present the socially efficient allocation of multi-pollutant permits through the optimal transport approach and its dual formulation, establishing the connection between the socially optimal allocation of permits and the existence of a system of shadow prices. In Section 5, we show that the socially optimal allocation of permits to firms can be decentralized by a system of market prices. In Section 6, we examine the stability of multi-pollutant cap-and-trade markets by assessing the consequences of specific policy changes on the mix of multi-pollutants. This section analyzes the intricate relationship between pricing mechanisms and the efficient matching of polluting firms to multi-pollutant permits, considering varying market conditions and firm characteristics. In Section 7, numerical simulations of prices and optimal demand partition in the multi-pollutant cap-and-trade problem are performed for illustration, employing stochastic gradient descent (SGD) methods, which are presented in Appendix D. In Section 8, we show that there is no strict equivalence between primary markets for multi-pollutant permits and secondary markets for individual pollutant rights. Finally, in Section 9, we offer concluding remarks, and some of the results are provided in the Appendix.

2 Multi-pollutant cap and trade matching model

2.1 Overview

This paper introduces a new framework for managing pollution through a multi-pollutant cap-and-trade system. The goal is to reduce overall pollution by efficiently matching a diverse set of firms—each with unique pollution profiles—to a portfolio of multi-pollutant permits. These permits, issued by a government agency (such as the Environmental Protection Agency (EPA) in the United States or Environment and Climate Change Canada), vary in terms of the types and amounts of pollutants they allow firms to emit. The government agency does not need detailed information about each firm's pollution characteristics; instead, it relies on the overall distribution of firm types. Each firm, in turn, selects permits that best align with its needs while contributing to a lower-pollution economy. This framework addresses key questions: How can multi-pollutant permits be

matched to firms in a way that maximizes societal benefits? Can a system of prices ensure that the available permits are allocated efficiently? How stable are these prices under changing regulatory conditions? Finally, can separate markets for individual pollutants achieve the same outcomes as a unified multi-pollutant market?

2.2 Theoretical model

2.2.1 Heterogeneous distribution of polluting firms

In this multi-pollutant cap and trade market, we have an atomless continuum of polluting firms, each characterized by a continuous distribution of types denoted as $d\mu(x) = f(x)dx$, where $x \in \Omega \subset \mathbb{R}^d$ represents a bounded domain in a multidimensional space. The continuous distribution of firm types captures the heterogeneity among the polluting firms, with each firm's characteristics being represented by its characteristic vector x within the domain Ω. In this setting, the pollution space is described by the continuous variation of firm characteristics, and each firm is uniquely identified by its specific vector x in Ω. This continuous representation allows for a comprehensive understanding of the diverse characteristics and behaviors of the polluting firms within the market, adding richness and complexity to the multi-pollutant space.

2.2.2 Portfolio of Multi-pollutant Permits Issued by the Regulator

The government agency issues a portfolio of multi-pollutant permits, denoted as

$$\{(y_1, S_1), \ldots, (y_M, S_M)\}, \tag{1}$$

where each permit y_j is a multidimensional vector representing the rights to emit specific quantities of N pollutants, and S_j represents the proportion of permit y_j in the portfolio. Each permit y_j is defined as:

$$y_j = \begin{bmatrix} y_{j1} \\ y_{j2} \\ \vdots \\ y_{jN} \end{bmatrix} \in \mathbb{R}^N_+, \quad j = 1, \ldots, M, \tag{2}$$

where $y_{jk} \geq 0$ represents the quantity of pollutant k (e.g., CO_2, SO_2, NO_2) allowed under permit y_j. Without loss of generality, we can assume that $k = 1, 2, ..., N$ are associated with pollutants CO_2, SO_2, and NO_2 based on the permit's dimensionality set up by the regulator. In this multi-pollutant cap-and-trade framework, firms aim to maximize their value by selecting from the portfolio $\{(y_1, S_1), \ldots, (y_M, S_M)\}$, which consists of permits with varying attribute levels for each pollutant type. The set of M multi-pollutant permits can be seen as heterogeneous commodities, each defined as a distinct commodity differentiated based on various right attributes, such as the

type of pollutants and their concentration levels.[4]

The proportions S_j assigned to each permit y_j ensure that the portfolio is complete:

$$\sum_{j=1}^{M} S_j = 1. \tag{3}$$

This proportional allocation ensures that the portfolio $\{(y_1, S_1), \ldots, (y_M, S_M)\}$ represents a comprehensive set of permits, covering the full spectrum of pollutant types and their respective quantities. The portfolio of multi-pollutant permits, denoted by ν, is represented as a weighted sum of Dirac masses:

$$\nu = \sum_{j=1}^{M} S_j \delta_{y_j}, \tag{4}$$

where δ_{y_j} represents the Dirac distribution centered at permit y_j, and S_j is its share in the portfolio. This formulation ensures the portfolio captures the full range of permit types and their relative proportions, providing a flexible and adaptable framework for managing emissions.[5]

To enforce environmental sustainability, the regulator sets caps Cap_k on the total emissions of each pollutant k. The set of caps $(Cap_k)_{k=1,\ldots,N}$ can be determined through a concerted agreement in a multidisciplinary committee of experts, including health experts, toxicologists, economists, epidemiologists, mathematicians, and others.[6]

These caps constrain the portfolio $\{(y_1, S_1), \ldots, (y_M, S_M)\}$ such that:

$$\sum_{j=1}^{M} S_j y_j = S_1 y_1 + S_2 y_2 + \cdots + S_M y_M \leq \begin{bmatrix} Cap_1 \\ \vdots \\ Cap_k \\ \vdots \\ Cap_N \end{bmatrix}. \tag{5}$$

This ensures the total emissions across all permits do not exceed the prescribed limits for each pollutant, aligning with the overall environmental and policy objectives set by the regulator. By

[4] In addressing the multidimensional heterogeneity among polluting firms, the regulator takes into account a range of permits. This collection of permits can be viewed as an example of a commodity exhibiting multidimensional heterogeneity. For an in-depth exploration of the concept of commodity heterogeneity in the context of analyzing environmental demand concerns, refer to (Phaneuf, 2013).

[5] The permit portfolio designed as a discrete distribution simplifies administrative tasks like tracking and enforcement by categorizing permits into a finite number of types, each with a fixed supply. It also offers flexibility, allowing regulators to adapt the portfolio to changing environmental conditions or policy goals by adding new permit categories. While the portfolio is exogenously issued by the government, its design combines bundling—integrating multiple pollution rights into a single permit type—and discrimination—tailoring permits to firms based on their pollution characteristics.

[6] From a pollution science perspective, existing techniques in decomposing multi-pollutant mixtures into their individual components-such as mass spectrometry, gas spectrometry, and remote sensing-can be leveraged to determine pollutant-specific caps. These methods provide scientific grounding for setting emission caps that account for their individual contributions to environmental degradation and public health risks.

managing the composition of the permit portfolio and complying with the caps, the government agency can effectively control and regulate pollution levels in the market, promoting sustainable practices and responsible environmental management.[7]

Furthermore, the portfolio-based approach enables the regulator to address the multidimensional heterogeneity among polluting firms. By offering a diverse range of permits with varying pollutant attributes, the regulator can better match the needs of firms with different emission profiles. This not only enhances the efficiency of the cap-and-trade system but also promotes equity by ensuring that firms with varying levels of pollution can participate in the market on fair terms. In the spot market for multi-pollutant permits, a large number of firms and the regulatory agency trade fixed quantities of contracts (multi-pollutant permits), and the value of these multi-pollutant permits to firms depends on their quantifiable characteristics. The characteristics of multi-pollutant permits are known to the firms at the time of the transaction, which ultimately influences the equilibrium market price. This unique aspect of the multi-pollutant cap-and-trade market allows for a hedonic analysis, considering the different attributes of permits and their impact on market outcomes.

2.2.3 Value of Matching Heterogeneous Polluting Firms with Multi-Pollutant Permits

The gross green payoff for a firm of type x using a multi-pollutant emission permit y_j is represented by the multivariable concave function $V(x, y_j)$. This function captures the overall gains a firm achieves by transitioning to the new emission level y_j. It also ensures that the global cap, which allows moving toward a lower aggregate level of all pollutants, is attained. Importantly, the value function $V(x, y_j)$ can account for pollution mixture effects, which arise when the interaction of multiple pollutants leads to nonlinear environmental or economic impacts.

One example of function function $V(x, y_j)$ can be expressed as a multidimensional matching index between firms and multipollutant permmits as follows:

$$V(x, y_j) = \sum_{k=1}^{N} \sum_{l=1}^{N} A_{kl} x_k y_{jl}, \tag{6}$$

where: x_k represents the characteristics of the polluting firm, y_{jl} represents the characteristics of the emission permit, and $A = (A_{kl})_{k,l=1,\ldots,N}$ is a matrix that captures interactions between these characteristicss. When A is not diagonal, the off-diagonal terms A_{kl} (for $k \neq l$) capture the cross-effects of different pollutants. These terms allow the value function to reflect the synergistic or antagonistic effects of pollution mixtures. For instance: If two pollutants k and l interact in a way that amplifies environmental damage, the corresponding A_{kl} term would be positive. If the interaction mitigates damage, A_{kl} could be negative. This flexibility makes the value function a

[7] Representing caps and permits as vectors aligns with the "bundle of rights" concept Ostrom (2011), where ownership consists of distinct rights. Similarly, achieving cleaner air, water, and soil quality can be viewed as defining acceptable limits for multiple pollutants through a combination of rights. This approach addresses the complexity of managing multiple pollutants holistically, akin to multi-quota fisheries or multi-band spectrum allocation, where multidimensional aspects are integrated to achieve system-wide goals.

flexible tool for modeling complex interactions.[8] In the particular case where A is an identity matrix (i.e., diagonal entries are 1 and off-diagonal entries are 0), the value function reduces to:

$$V(x, y_j) = \sum_{k=1}^{N} x_k y_{jk}. \tag{7}$$

This represents a case where the payoff is simply the sum of the products of corresponding firm and permit characteristics. More generally, the value function ((6)) can be simplified to the form ((7)) by summarizing the matching information using weighted combinations of the firm and permit characteristics. This simplification is particularly useful for identifying the most important factors driving the interaction between firms and permits, as well as measuring their relative significance.[9]

The matching function $V(x, y_j)$ is assumed to satisfy the Spence-Mirrlees condition, also known as the twist condition:[10]

$$\frac{\partial V(x, y_j)}{\partial x} \neq \frac{\partial V(x, y_k)}{\partial x}, \quad \text{for all } x \text{ and } y_j \neq y_k. \tag{8}$$

In simpler terms, the twist condition means that the additional benefit a firm receives from a small increase in its characteristics x depends on the specific multi-pollutant permit y_j it is matched with. This ensures that each firm can uniquely rank the M multi-pollutant permits based on how well they align with the firm's specific characteristics. For the case where the value function is given by (6) the twist condition is satisfied if the matrix $A = (A_{kl})$ the rows of A corresponding to different permits y_j are linearly independent.[11] The twist condition plays a crucial role in a multi-pollutant matching model[12] Furthermore, the twist condition, extending the single-crossing property to a multi-pollutant cap-and-trade system, generalizes the notion of assortative matching to a multi-dimensional setting, following the arguments of Chiappori, McCann, and Nesheim (2010). In

[8] Another way to specify the value function is:

$$V(x, y_j) = R\left(\sum_{k=1}^{M} A_k x_k y_{jk}\right),$$

where R is an increasing and concave function, and A_k (for $k = 1, \ldots, M$) are parameters that weight the contributions of different characteristics. This formulation allows for nonlinear aggregation of the firm and permit characteristics, reflecting realistic environmental and economic trade-offs. For further discussion on convex functions in this context, see Moulin and Sprumont (2007).

[9] For example, by breaking down the interaction matrix A using mathematical techniques like Singular Value Decomposition (SVD), the value function can be expressed as a weighted combination of firm and permit characteristics, where each combination represents a distinct aspect of their interaction. This approach helps identify the most important factors driving the matching process and simplifies the analysis by focusing on the key dimensions.

[10] In one dimension, the single crossing property ensures that two functions intersect at most once, creating a clear and unambiguous preference order. In a multidimensional setting, the twist condition extends this idea by requiring that the functions "twist" or change in a specific way as variables vary. This ensures stability and well-defined economic properties in complex problems, such as the multi-pollutant cap and trade framework.

[11] In other words, this requires that the rank of A equals the number of permits M. This guarantees that firms can uniquely rank permits based on their preferences.

[12] Empirical matching literature, such as studies by Galichon and Salanie (2012), Dupuy and Galichon (2014), and Lindenlaub (2015), has employed similar conditions.

summary, the twist condition allows us to establish a robust and interpretable framework for understanding how polluting firms are matched with specific multi-pollutant permits based on their distinct characteristics.

3 Competitive Equilibrium and Market Segmentation in a Multi-pollutant Cap-and-Trade Market

3.1 Defining the Multi-pollutant Equilibrium Market

A multi-pollutant competitive equilibrium is characterized by a price system $p = (p_1, \ldots, p_M)$ for a set of multi-pollutant permits $Y = \{y_1, \ldots, y_M\}$ and a partition $\Omega = \bigcup_{j=1}^{M} D_j(Y, p)$ that satisfies the following conditions:

- (i) **Firm Optimization**: Given the price system $p = (p_1, \ldots, p_M)$, each firm x chooses the multi-pollutant permit that maximizes its value:

$$\pi(x, p) = \max_{j=1,\ldots,M} \left[V(x, y_j) - p_j \right]. \tag{9}$$

  Here, $V(x, y_j)$ represents the value that firm x derives from permit y_j, and p_j is the price of that permit.

- (ii) **Market Clearing**: Given the price system $p = (p_1, \ldots, p_M)$ and the masses of supplied permits $(S_1, \ldots, S_M)$, the market-clearing conditions are satisfied for all permits. Specifically, the sorting of firms is such that the measure of firms choosing permit y_j equals the mass of permits supplied:

$$\mu(D_j(Y, p)) = S_j, \quad j = 1, \ldots, M. \tag{10}$$

In the multi-pollutant cap-and-trade market, firms purchase permits that are effectively bundles of pollution rights, each carrying its own market price. This study elucidates how the competitive equilibrium allows firms to make rational decisions based on a cost-benefit analysis. Firms assess their specific pollution characteristics and the associated payoffs, selecting permits that optimally align with their unique needs and the prevailing market prices. This process leads to market segmentation, where firms sort themselves into distinct groups based on their pollution profiles and permit preferences.

3.1.1 Decentralized Equilibrium in a Multi-pollutant Cap-and-Trade Market

Let p_j denote the price (in monetary terms) of the multi-pollutant emission permit $y_j = (y_{j1}, y_{j2}, \ldots, y_{jN}) \in \mathbb{R}^N$. Given a system of prices $p = (p_1, \ldots, p_M)$, a polluting firm of type $x \in \Omega \subset \mathbb{R}^N$ seeks to maximize its value by selecting a permit y_j that solves the following optimization problem:

$$\pi(x, p) = \max_{j=1,\ldots,M} \left\{ V(x, y_j) - p_j \right\}. \tag{11}$$

Here, $V(x, y_j)$ represents the value that a firm of type x derives from using permit y_j, and p_j is the price of that permit. Firms aim to balance the value they gain from a permit against its cost, choosing the permit y_j that maximizes the net benefit $V(x, y_j) - p_j$.

This decision-making process leads to market segmentation, where firms sort themselves based on their pollution characteristics x and their preferred permits. Geometrically, the aggregate behavior of firms can be viewed as a market tessellation, with each region representing the set of firms that prefer a specific permit y_j over others. Firms group into smaller, more homogeneous segments based on similar (though not identical) pollution characteristics as they select permits. As a result, the market organizes itself into distinct groups, each aligned with the permit that best matches their pollution profile and economic incentives.

Changing the price p_j of a permit y_j influences the demand for that permit and, consequently, its market share $\mu(D_j(Y, p))$. Although the relationship between prices and market share is complex, it is well-behaved. It can be shown that there exists a unique set of prices p that aligns the market demand $\mu(D_j(Y, p))$ for each permit with the government-regulated supply ν_j. This equilibrium ensures that the market demand for emission permits matches the supply issued by the government. At this equilibrium, the assignment of permits to firms, defined by the market segmentation resulting from the prices p, corresponds to an *optimal transport map*. This map allocates permits to firms in a way that balances supply and demand.

Proposition 3.1 ***Deriving Permit Demand from a Submodular Welfare Function***

For a given system of permit prices $p = (p_1, \ldots, p_M)$, let $D_j(Y, p)$ denote the set of firms that choose y_j as the payoff-maximizing multi-pollutant permit:

$$D_j(Y, p) = \left\{ x \in \Omega \;\middle|\; V(x, y_j) - p_j = \max_{k=1,\ldots,M} \{V(x, y_k) - p_k\} \right\}.$$

The demand for permit j, measured by $\mu(D_j(Y, p))$, can be expressed as the negative partial derivative of a submodular welfare function $W(p)$ with respect to price p_j:

$$\underbrace{-\frac{\partial W(p)}{\partial p_j}}_{\text{Marginal change in welfare with respect to price } p_j} = \underbrace{\int_{D_j(Y,p)} f(x)\, dx}_{\text{Mass of firms choosing permit } y_j} = \mu(D_j(Y, p)), \tag{12}$$

where $W(p)$ is the aggregate welfare of firms, defined as:

$$W(p) = \int_\Omega \pi(x, p) f(x)\, dx = \int_\Omega \left[\max_{j=1,\ldots,M} \left\{ V(x, y_j) - p_j \right\} \right] f(x)\, dx. \tag{13}$$

In addition, the second-order partial derivatives exhibit the following signs:

$$\begin{cases} \dfrac{\partial^2 W}{\partial p_i \partial p_j}(p) \leq 0 \quad for \quad i \neq j, \\ \dfrac{\partial^2 W}{\partial p_i^2}(p) \geq 0 \quad for \quad i = j. \end{cases} \tag{14}$$

Proof. See Appendix A and B.

The relationship (12) establishes that the demand for permits is directly tied to changes in the aggregate welfare of firms, modulated by the price system p. The submodular nature of $W(p)$, as shown in (14), indicates diminishing marginal returns, which influence the aggregate demand for permits. This effect varies across firms: some exhibit high elasticity in their demand responses to price changes, while others are less sensitive. These differences introduce strategic dynamics into the market. An equilibrium is achieved when no firm can improve its payoff by unilaterally changing its permit choice, reflecting a balance of strategic responses across the market.

How does the demand for permits itself change with variation of permits prices in the multi-pollutant emission cap and trade market ? The following proposition give us the answer to that question.

Proposition 3.2 ***Segmentation and Elasticities in Multi-pollutant Cap-and-Trade Systems***

- ***Law of Demand in Market Segmentation:*** *A multi-pollutant cap-and-trade system generates a market segmentation where the law of demand holds for each permit type. Specifically, for permit type j, there exists a negative correlation between the size of the market segment and the permit price, as represented by:*

$$\frac{\partial \mu(D_j(Y,p))}{\partial p_j} < 0, \quad j = 1, \ldots, M. \tag{15}$$

- ***Cross-Price Effects on Permit Demand:*** *The effect of a marginal price adjustment in permit i on its demand set can be determined by aggregating the effects of marginal price changes in all other permits $k \neq i$:*

$$\underbrace{\frac{\partial \mu(D_i(Y,p))}{\partial p_i}}_{\text{Internal effect on demand for permit } i} = \underbrace{-\sum_{\substack{k=1 \\ k \neq i}}^{M} \frac{\partial \mu(D_i(Y,p))}{\partial p_k}}_{\text{External effect on demand for permit } i}. \tag{16}$$

In simpler terms, the number of firms switching to permit y_i due to a price increase in other permits y_k ($k \neq i$) equals the number of firms switching away from permit y_i due to a marginal increase in its own price p_i.

- ***Price Elasticity of Demand:*** *The price elasticity of demand for permit i can be expressed as a weighted sum of cross-price elasticities:*

$$\varepsilon_{ii} = -\sum_{\substack{k=1 \\ k\neq i}}^{M} \varepsilon_{ik} \frac{p_i}{p_k}, \tag{17}$$

where $\varepsilon_{ik} = \frac{\partial \mu(D_i(Y,p))}{\partial p_k} \frac{p_i}{\mu(D_k(Y,p))}$.

- ***Substitution Between Permits:*** *An increase in the price of a specific permit type leads to an increase in the demand for alternative permits within the multi-pollutant cap-and-trade system. This relationship is expressed as:*

$$\frac{\partial \mu(D_j(Y,p))}{\partial p_i} > 0, \quad i \neq j. \tag{18}$$

- ***Mutual Substitution Effect:*** *The mutual substitution effect between permits is captured by the symmetry in cross-price derivatives:*

$$\frac{\partial \mu(D_j(Y,p))}{\partial p_i} = \frac{\partial \mu(D_i(Y,p))}{\partial p_j}, \quad i \neq j. \tag{19}$$

This equation highlights the interconnectedness of permit prices and their effects on demand. A change in the price of one permit influences the demand for other permits, reflecting the dynamic interactions within a multi-pollutant cap-and-trade market.

Proof. See Appendix B

We can understand this as follows: in the multi-pollutant emission cap and trade market where several different pollutants are regulated simultaneously, the different types of emission permits are substitute goods for achieving a given regulated multi-pollutant cap (target) while maximizing each firm objective. The fact that one multi-pollutant emission permit is substitutable for another has immediate economic consequences: insofar as one multi-pollutant emission permit can be substituted for another, the demands for all the multi-pollutant emission permits will be interrelated by the fact that value-maximizing firms can trade off one multi-pollutant emission permit for the other if it becomes advantageous to do so. Equation (67) expresses the law of demand, which suggests that the higher the price ψ_i of permit y_i the lower the quantity demanded of the multi-pollutant emission permit y_i. Holding all else fixed, equilibrium prices in the multi-pollutant framework will depend on the own-price elasticities of residual demand. Equilibrium further depends on how a change in the price of one permit affects the quantities sold of others, i.e., on cross-price demand elasticities.

Proposition 3.3 ***Market Clearing in Multi-pollutant Cap-and-Trade Systems***

The multi-pollutant cap-and-trade market clears by matching the heterogeneous distribution of polluting firms, represented by the density $d\mu = f(x)dx$ *over the space* $\Omega \subset \mathbb{R}^d$*, with the portfolio*

of multi-pollutant emission permits, represented by the weighted Dirac distribution:

$$\nu = \sum_{j=1}^{M} S_j \delta_{y_j},$$

where $\sum_{j=1}^{M} S_j y_j \leq Cap$. *Here:*

- S_j *represents the share of permit* y_j *in the portfolio,*
- $Cap = (Cap_1, \ldots, Cap_N)$ *is the vector of pollutant caps, and*
- δ_{y_j} *denotes the Dirac distribution centered at permit* y_j.

The system of permit prices $p = (p_1, \ldots, p_M)$ *must satisfy the following equilibrium conditions, driven by the submodular function* $W(p_1, \ldots, p_M)$*:*

$$\begin{cases} \dfrac{\partial W(p)}{\partial p_1} + S_1 = 0, \\ \dfrac{\partial W(p)}{\partial p_2} + S_2 = 0, \\ \qquad \vdots \\ \dfrac{\partial W(p)}{\partial p_M} + S_M = 0. \end{cases} \tag{20}$$

These conditions characterize the prices that align the market demand for permits with the government-regulated supply. However, the existence of such a price system is not yet proven at this stage.

Proof. A competitive market equilibrium consists of a set of prices $(p_j)_{j=1,\ldots,M}$ and a set of demands $(D_j(Y,p))_{j=1,\ldots,M}$ such that:

- Each firm maximizes its net value, and
- Markets clear, implying:

$$\mu(D_j(Y,p)) = S_j, \quad j = 1, \ldots, M. \tag{21}$$

From the firm's maximization problem, as shown in (12), we have:

$$\mu(D_j(Y,p)) = -\frac{\partial W(p)}{\partial p_j}. \tag{22}$$

Combining (21) and (22), the market equilibrium requires that prices and quantities of the

proportion distribution of multi-pollutant permits simultaneously satisfy:

$$\begin{cases} \dfrac{\partial W(p)}{\partial p_1} + S_1 = 0, \\ \dfrac{\partial W(p)}{\partial p_2} + S_2 = 0, \\ \qquad\qquad \vdots \\ \dfrac{\partial W(p)}{\partial p_M} + S_M = 0. \end{cases} \tag{23}$$

Proposition 3.4 ***Existence of Equilibrium Prices in a Multi-pollutant Cap-and-Trade Market*** *There exists a system of permit prices $p = (p_1, \ldots, p_M)$ in the multi-pollutant cap-and-trade market that clears the market by matching polluting firms with multi-pollutant permits. These equilibrium prices satisfy the following system of equations:*

$$\begin{cases} \dfrac{\partial W(p)}{\partial p_1} + S_1 = 0, \\ \dfrac{\partial W(p)}{\partial p_2} + S_2 = 0, \\ \qquad\qquad \vdots \\ \dfrac{\partial W(p)}{\partial p_M} + S_M = 0, \end{cases}$$

where:

- *$W(p)$ is the welfare function, representing the aggregate payoff of firms in the market, and*
- *$S = (S_1, \ldots, S_M)$ is the vector of mass supply for multi-pollutant permits, representing the total share of each permit type issued by the regulator.*

The existence of such a price system ensures that the demand for permits aligns with the supply, achieving market equilibrium.

Proof. Let $\eta = (\eta_1, \ldots, \eta_M) \in \mathbb{R}^M$ be a multidimensional vector, and define the vector field

$$G(\eta) = \left(-\frac{\partial W(\eta)}{\partial \eta_1}, \ldots, -\frac{\partial W(\eta)}{\partial \eta_M} \right).$$

Let $S = (S_1, \ldots, S_M)$ represent the vector of mass supply for multi-pollutant permits. The vector field $G(\eta) - S$ represents the net demand for permits at a given price vector η, where $-\frac{\partial W(\eta)}{\partial \eta_j}$ captures the marginal benefit of permit type j to firms, and S_j is the fixed supply of permits issued by the regulator.

Consider the line integral:

$$E(p) = \int_0^p \sum_{j=1}^{M} \left(G_j(\eta) - S_j\right) d\eta_j, \tag{24}$$

where $p = (p_1, \ldots, p_M)$ is a vector of prices. This integral measures the cumulative net demand across all permit types as prices vary from 0 to p.

The expression (24) represents the line integral of the continuous vector field $G(\eta) - S$ along a smooth path $\gamma = (\gamma_1, \ldots, \gamma_M) : [a,b] \to \mathbb{R}^M$ in a multidimensional space, with $\gamma(a) = 0$ and $\gamma(b) = p$. Equivalently, (24) can be expressed as:

$$E(p) = \int_a^b \sum_{j=1}^{M} \left(G_j(\gamma(h)) - S_j\right) \gamma_j'(h)\, dh. \tag{25}$$

Using the symmetry condition:

$$\frac{\partial \left[G_j(\eta) - S_j\right]}{\partial \eta_i} = \frac{\partial \left[G_i(\eta) - S_i\right]}{\partial \eta_j}, \tag{26}$$

we observe that the 1-form $\eta \mapsto \sum_{j=1}^{M} \left(G_j(\eta) - S_j\right) d\eta_j$ is closed. By the Poincaré lemma, this implies the existence of a well-defined function $E(p)$ such that:

$$E(p) = \int_0^p \sum_{j=1}^{M} \left(G_j(\eta) - S_j\right) d\eta_j. \tag{27}$$

The partial derivatives of $E(p)$ are given by:

$$\frac{\partial E(p)}{\partial p_j} = G_j(p) - S_j. \tag{28}$$

The Hessian matrix of $E(p)$, which captures the second-order partial derivatives of $E(p)$, is given by:

$$H(p) = \begin{bmatrix} \frac{\partial^2 E}{\partial p_1^2} & \frac{\partial^2 E}{\partial p_1 \partial p_2} & \cdots & \frac{\partial^2 E}{\partial p_1 \partial p_M} \\ \frac{\partial^2 E}{\partial p_2 \partial p_1} & \frac{\partial^2 E}{\partial p_2^2} & \cdots & \frac{\partial^2 E}{\partial p_2 \partial p_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 E}{\partial p_M \partial p_1} & \frac{\partial^2 E}{\partial p_M \partial p_2} & \cdots & \frac{\partial^2 E}{\partial p_M^2} \end{bmatrix}.$$

The second partial derivatives are:

$$\frac{\partial^2 E(p)}{\partial p_i \partial p_j} = \frac{\partial \left[G_j(p) - S_j\right]}{\partial p_i} = \frac{\partial \left[G_i(p) - S_i\right]}{\partial p_j} = -\frac{\partial^2 W(p)}{\partial p_i \partial p_j} \tag{29}$$

which implies, using (14), that

$$\begin{cases} \dfrac{\partial^2 E}{\partial p_i \partial p_j}(p) \geq 0 \quad \text{for} \quad i \neq j, \\ \dfrac{\partial^2 E}{\partial p_i^2}(p) \leq 0 \quad \text{for} \quad i = j, \end{cases} \tag{30}$$

indicating that the Hessian matrix $H(p)$ is negative semi-definite. This property reflects the concavity of $E(p)$.

Thus, there exists a system of prices $p = (p_1, \ldots, p_M)$ that maximizes $E(p)$, and this system of prices satisfies the first-order condition:

$$\frac{\partial E(p)}{\partial p_j} = G_j(p) - S_j = 0 \quad \text{for} \quad j = 1, \ldots, M. \tag{31}$$

Since $G_j(p) = -\frac{\partial W(p)}{\partial p_j}$ for $j = 1, \ldots, M$, it follows that the equilibrium prices p satisfy:

$$\begin{cases} \dfrac{\partial W(p)}{\partial p_1} + S_1 = 0, \\ \dfrac{\partial W(p)}{\partial p_2} + S_2 = 0, \\ \qquad \vdots \\ \dfrac{\partial W(p)}{\partial p_M} + S_M = 0. \end{cases} \tag{32}$$

There is a strategic element in how the multi-pollutant cap-and-trade market equilibrium works. The system of price equations (3.4) reveals an implicit Nash equilibrium within the topology of demands for multi-pollutant permits $(D_1, \ldots, D_M)$. This equilibrium arises when firms make value-maximizing permit choices, and, given the choices of other firms, no firm has an incentive to switch to a different multi-pollutant permit. With a fixed supply of permits for each type, the permit selection behavior determines three key outcomes: the equilibrium prices $(p_1, \ldots, p_M)$, the demand partition sets $\{D_1(Y, p), \ldots, D_M(Y, p)\}$, and the market demand proportions $\mu(D_1(Y, p)), \ldots, \mu(D_M(Y, p))$. The market equilibrium is also shaped by the statistical distribution $d\mu(x) = f(x)dx$ of firms' heterogeneity in the multidimensional pollution space, which captures the diversity of firms' pollution profiles and their preferences for different permit types. This framework highlights how firms' decisions and the distribution of their characteristics shape the equilibrium outcomes in multi-pollutant cap-and-trade markets.

Remark 1 *The system of pricing equations* (3.4) *can be expressed succinctly as* $G(p) = S$*, where* $G(p) == \left(-\frac{\partial W(p)}{\partial p_1}, \ldots, -\frac{\partial W(p)}{\partial p_M}\right)$ *represents the demand for multi-pollutant permits as a function of prices, and S denotes the supply of permits issued by the regulator. This reformulation simplifies the equilibrium sorting problem in the multi-pollutant market into a compact set of nonlinear*

equations. The behavior of the equilibrium price depends on the smoothness and predictable behavior of the demand mapping $G(p)$, which is influenced by the characteristics of the matching function $V(x,y)$ and the regularity of polluting firms' heterogeneity. In practical terms, this means that the stability and efficiency of the market rely on how well permit design aligns with firms' needs and how smoothly the market adjusts to changes in supply and demand. These insights will be explored further in the following sections.

Remark 2 *The concept of line integrals in multidimensional spaces has been used to develop the proof of Proposition 3.4. While line integrals are uncommon in economics, there are notable instances where this mathematical tool has proven invaluable. For example, it was first introduced by Hotelling (1938) in the context of welfare analysis for railways and utilities, and later extended by Carey (1977), Samuelson (1950), and Silberberg (1972) to address a wide range of economic problems. In this work, the application of line integrals to establish the existence of permit pricing systems in multi-pollutant cap-and-trade markets provides a striking example of how abstract mathematical tools can address economic challenges. In the proof above, the vector field $G(\eta)-S$, representing the net demand for permits, and its integral $E(p)$, which quantifies the cumulative imbalance between demand and supply, offer an economic interpretation of the market-clearing process. By maximizing $E(p)$, we identify the price vector p where net demand is zero for all permit types, ensuring that the market clears and resources are allocated efficiently. The use of line integrals in this context demonstrates a deep connection between abstract mathematical tools and practical policy design, highlighting their potential to address complex economic problems.*

4 Socially Efficient Allocations of Permits in the Multi-pollutant Cap and Trade System

4.1 The multi-pollutant optimal transport problem

In the multi-pollutant cap and trade system, our aim is to determine the most socially efficient allocation of permits. To achieve this, we need to identify a distribution of permits among polluting firms that maximizes overall social welfare. This distribution should consider the unique emission characteristics of each firm and the availability of different permits.

We start by formulating a problem called the multi-pollutant optimal transport problem, which focuses on maximizing social welfare by efficiently matching firms' emission characteristics with the corresponding emission permits. The problem involves finding an optimal transport map that moves each firm's emission characteristics to the appropriate emission permit, considering factors like demand and pollutant quantities.

$$\underbrace{\Phi(\mu,\nu,V)}_{\text{Social Value of Policy}} = \begin{cases} \sup\limits_{T} & \int_\Omega V(x,T(x))d\mu(x) \quad \text{Social Welfare Maximization} \\ \text{s.t.} & d\mu(x) = f(x)dx \quad \textit{Distribution of Firms' Heterogeneity} \\ & \nu(y) = \sum_{i=1}^{N} S_j \delta_{y=y_j} \quad \textit{Distribution of multi-pollutant permits} \\ & T\#\mu = \nu. \quad \textit{Policy matching firms' distribution and permits.} \end{cases} \tag{33}$$

The social welfare function, denoted by $V(x,T(x))$, represents the value derived from matching firms' emissions with the corresponding permits. In the abovementioned multi-pollutant optimal transport problem, the objective is to find an optimal transport map, denoted by T, that moves each point x in the emission characteristics space of firms to the emission permit point y_j associated with the demand set D_j. More precisely, the social planner aims to maximize the total value obtained from this mapping, which is captured by the social welfare function $V(x,T(x))$. To ensure a realistic allocation of permits, the problem introduces two essential constraints. Firstly, the distribution of firms' heterogeneity, represented by $\mu(x)$, must be taken into account in the mapping process. Secondly, the cap distribution of permits, denoted by $\nu(y)$, is a weighted sum of Dirac masses, reflecting the fixed quantity of permits available for each emission type. The optimal transport map T must adhere to these constraints while maximizing the overall social welfare. The multi-pollutant optimal transport considers the unique characteristics of each company's emissions and the availability of different permits. By considering the distribution of firms' heterogeneity and the cap distribution of permits, the problem provides a comprehensive approach to address the challenges of multi-pollutant cap-and-trade systems, enabling better policy design and environmental management.

The multi-pollutant optimal transport problem (33) can also be interpreted as an optimal partitioning of polluting firms. This partitioning aims to allocate multi-pollutant permits to the firms in a way that optimally aligns with their respective pollution profiles, as elaborated below.[13]

$$\underbrace{\Phi(\mu,\nu,V)}_{\text{Social Value of Policy}} = \begin{cases} \sup\limits_{\{D_j\}_{j=1}^M} & \sum_{j=1}^M \int_{D_j} V(x,y_j) f(x)dx, \quad \text{Social Welfare Maximization} \\ \text{s.t.} & \mu(D_j) = \nu(\{y_j\}) = S_j \quad \text{Mass conservation} \\ & \bigcup_{j=1}^M D_j = \Omega \text{ and } \mu(D_i \cap D_j) = \emptyset, \text{ Partitioning of firms} \end{cases} \tag{34}$$

The optimal partition formulation of the multi-pollutant optimal transport problem provides a clear understanding of the following proposition.

Proposition 4.1 *Let us assume that the matching function takes the form of an inner product:*

$$V(x,y_j) = \sum_{k=1}^{N} x_i y_{jk}.$$

Then, there exists a multivariate convex function ϕ and an optimal partition $\Omega = \bigcup_{j=1}^M D_j$ of firms such that

$$\begin{cases} T(x) = \nabla\Phi(x) = y_j \ \text{ for } x \in D_j, & \text{Permit optimally assigned to a firm} \\ \mu(D_j) = S_j, \ j = 1,\ldots,M & \text{Aggregate consistency} \end{cases} \tag{35}$$

[13] Refer to the discussions by Wolansky (2015) regarding the interplay between optimal transport techniques and optimal partitioning.

Proof. This result is a direct application of Brenier's theorem Brenier (1987, 1991), a foundational result in optimal transport theory. When the matching function is an inner product, the optimal assignment of permits to firms can be characterized as follows. There exists a convex function ϕ whose gradient $\nabla\phi$ defines the optimal assignment $T(x) = \nabla\phi(x)$. This gradient maps each firm x to the permit y_j that maximizes the matching value $V(x, y_j)$. The firms are divided into groups D_j, where each group D_j corresponds to a specific permit type y_j. The size of each group D_j matches the number of available permits S_j, ensuring that the allocation is feasible. The optimal assignment maximizes the total social welfare, given by the integral $\sum_{j=1}^{M} \int_{D_j} V(x, y_j) f(x)\, dx$, ensuring that the allocation is both efficient and consistent with the available permits. The system of equations in (35) guarantees that the assignment is optimal and aggregate constraints are satisfied. The gradient $\nabla\phi(x)$ acts as a mechanism to align firms' characteristics with the attributes of the permits, ensuring an efficient allocation.[14]

As shown in (35), the existence and uniqueness of the optimal assignment of multi-pollutants to heterogeneous firms are crucial for designing efficient multi-pollutant cap-and-trade markets. The complexity arises from the multidimensional characteristics of firms and their varying preferences. The inner product matching function captures the interplay between multi-pollutant permits and firms' attributes, while the optimal transport map $T(x) = \nabla\phi(x)$ orchestrates the allocation process. This gradient-driven strategy aligns firms' characteristics with permit attributes, ensuring an efficient and harmonized allocation. Further details about the function ϕ will be provided later in section 4.2.

Proposition 4.2 *Optimal multi-pollutant permit assignment with a general matching function*

Let $V(x, y)$ be a general matching function that is regular and bounded from above. Then, there exists an optimal partition $\Omega = \bigcup_{j=1}^{M} D_j$ of firms and a function ϕ such that:

$$\begin{cases} T(x) = e_x(\nabla\Phi(x)) = y_j, \quad \text{for } x \in D_j, \quad \textit{Permit optimally assigned to a firm} \\ \\ \mu(D_j) = S_j,\ j = 1, \ldots, M, \qquad\qquad \textit{Aggregate consistency} \end{cases} \tag{36}$$

where $e_x(.)$ is the inverse of the mapping $y_j \to \nabla_x V(x, y_j)$.

Proof. This result generalizes Brenier's theorem Brenier (1987) and builds on work by Carlier (2003); Gangbo and McCann (1995, 1996); Abdellaoui (1998); Abdellaoui and Heinich (1999). When the matching function $V(x, y)$ is not an inner product, the optimal assignment of permits to firms is determined by the gradient of a function ϕ, transformed by the operator $e_x(.)$. This operator ensures that the assignment $T(x) = e_x(\nabla\phi(x))$ aligns firms' characteristics x with permit attributes y_j in a way that maximizes social welfare while satisfying aggregate constraints. The proposition extends the optimal transport framework to cases where the matching function $V(x, y)$

[14] Let us mention that the system of equations (35) resembles the Oliker and Prussner (1989) numerical characterization of the profoundly nonlinear Monge-AmpÃ¨re equations. This connection offers a valuable linkage between the realms of optimal transport theory and the theory of fully nonlinear elliptic equations.

is more general than an inner product. Here, $V(x,y)$ captures the "value" of matching a firm with characteristics x to a permit with attributes y. The operator $e_x(.)$ acts as a bridge, translating the gradient of ϕ (which encodes optimality) into the permit space. This ensures that firms are assigned permits in a way that maximizes overall welfare while respecting the availability of permits.[15]

Remark 3 *Brenier (1991) theorem establishes a fundamental connection between optimal transport theory and convex analysis by showing that an optimal transport map must be the gradient of a convex function. In the context of the proposition, this implies that the optimal assignment of multi-pollutants to firms in the cap-and-trade market is guided by an underlying convex structure. The fact that a convex function is a natural generalization of increasing functions in dimensions greater than one - monotone rearrangement to the multidimensional setting - implies for two distinct firms x and x′ that do not belong to the same partition,* $\langle \nabla\phi(x) - \nabla\phi(x'), x - x' \rangle > 0$. *Alternatively, this can be expressed as* $\langle y_i - y_j, x - x' \rangle > 0$. *This suggests that firms with more significant pollution characteristics* ($x > x'$) *will be assigned multi-pollutant permits of greater size* ($T(x) = y_i > T(x') = y_j$).[16] *In essence, the convex structure of the assignment map influences the allocation of permits based on firms' pollution attributes, leading to a nuanced and efficient distribution of permits in the multi-pollutant market. Leveraging multidimensional rearrangement properties of Brenier's map enables a more robust, efficient, and equitable design of the multi-pollutant market, facilitating effective policy interventions to achieve economic and environmental goals simultaneously.*

4.2 Dual formulation of the multi-pollutant optimal transport problem

The multi-pollutant optimal transport problem can be elegantly reformulated as a shadow price problem, involving diverse polluting firms with distinct characteristics, using the dual formulation of the Kantarovich optimization (Kantorovich, 1942; Gangbo and McCann, 1996; Aurenhammer et al., 1998).

This dual approach provides invaluable insights into the interactions between firms with varying pollution attributes within the shadow price system. Each firm endeavors to maximize its net payoff by carefully selecting the multipollutant permit that aligns with its specific characteristics. The power of the dual Kantarovich approach lies in its ability to analyze and design multi-pollutant cap-and-trade systems, enabling policymakers to navigate the complexities of firm interactions, pollution attributes, and shadow price mechanisms. By embracing this framework, we equip ourselves to tackle the challenges of multi-pollutant management with a profound understanding of the corresponding shadow price conditions, thus unlocking new possibilities for employing market mechanisms in reaching impactful policy interventions.

[15]This generalization introduces computational challenges due to the nonlinearity and multidimensionality of the problem. Future research could focus on developing efficient algorithms to compute the inverse mapping $e_x(.)$, enabling practical applications of this framework.

[16]A more general property, called p-cyclically monotone is used in separable Banach spaces by Abdellaoui (1998); Abdellaoui and Heinich (1999) to dicuss the optimal transport map.

$$\underbrace{\Phi(\mu,\nu,V)}_{\text{Social Value}} = \begin{cases} \inf\limits_{\phi(x),\psi(y_j)} & \left[\int_\Omega \phi(x) f(x)\,dx + \sum_{j=1}^M S_j \psi(y_j)\right] \quad \text{Dual Value Function Minimization} \\ \text{s.t.} & \psi(y_j) + \phi(x) \geq V(x,y_j) \quad \forall x \in \Omega,\ y_j,\ j = 1,\ldots,M \quad \text{Shadow Prices Constraints} \\ & d\mu(x) = f(x)\,dx \quad \text{Distribution of Firms over } \Omega \\ & \nu(y) = \sum_{j=1}^M S_j \delta_{y=y_j} \quad \text{Distribution of Multi-pollutant Permits} \end{cases} \tag{37}$$

In the dual formulation of the multi-pollutant optimal transport problem, Lagrange multipliers $\phi(x)$ and $\psi(y_j)$ are used to address constraints from the initial setup, providing significant economic interpretations:

The multiplier $\phi(x)$ represents the net payoff (profit) of a firm exhibiting polluting characteristics x within the distribution μ of firms. The valuation that a firm x has for the multi-pollutant permit y_j is denoted $V(x,y_j)$. If a firm of type x purchases the permit y_j at the price $\psi(y_j)$, its net payoff is calculated as its valuation for the permit minus the cost: $V(x,y_j) - \psi(y_j)$. Given a set of prices $\{\psi(y_j)\}_{j=1,\ldots,M}$, the firm x will select the permit y_j that maximizes the quantity $V(x,y_j) - \psi(y_j)$.

The multiplier $\psi(y_j)$ serves as the shadow price for the emission permit y_j within the target distribution ν. It reflects the permit's market value or cost, indicating its demand level. A high $\psi(y_j)$ suggests that the permit is highly sought after, and firms are willing to pay a premium for it. The constraint $\phi(x) + \psi(y_j) \geq V(x,y_j)$ ensures that the total value a firm gains from using the permit y_j corresponds to the sum of the permit's cost and the firm's net payoff.

These multipliers, $\phi(x)$ and $\psi(y_j)$, are crucial in ensuring that the constraints of the multi-pollutant optimal transport problem are satisfied, enabling the optimal allocation of permits among firms. They create a pricing mechanism that balances the demand and supply of emission permits, aiming to maximize overall social welfare. Moreover, these multipliers act as marginal values, indicating the effect of incremental changes in the distribution of firms and permits. The optimal values of these multipliers guide both the firms' permit selections and the pricing of permits, facilitating an efficient and socially beneficial distribution of permits in the multi-pollutant cap-and-trade system.

Proposition 4.3 ***Optimal Assignment and Shadow Prices*** ψ
Given a distribution $d\mu(x) = f(x)dx$ of polluting firms and a portfolio of multi-pollutant permits $Y = \{y_1,\ldots,y_M\}$, the optimal assignment of multi-pollutant permits to polluting firms can be reduced to finding the shadow prices (multipliers) $\phi(x)$ and $\psi(y_j)$ that optimize the dual formulation of the social optimal problem.

Let $G : \mathbb{R}^M \to \mathbb{R}^M$ denote the mapping $\psi \to (\mu(D_1(Y,\psi)),\ldots,\mu(D_M(Y,\psi)))$, where $\mu(D_j(Y,\psi))$ represents the measure of firms assigned to permit y_j. Let $S = (S_1,\ldots,S_M)$ denote the vector of proportions of the multi-pollutant permit portfolio $\nu = \sum_{j=1}^M S_j \delta_{y_j}$. The task of finding an optimal map in the multi-pollutant optimal transport problem, involving the heterogeneous distribution of

firms $d\mu = f(x)dx$ and the permit portfolio ν, reduces to solving the pricing equation:

$$G(\psi) = S.$$

This transformation simplifies the problem from an infinite-dimensional to a finite-dimensional framework, yielding a system of nonlinear equations. The solution to this system provides the shadow prices ψ that balance the demand for permits with their supply, ensuring an efficient allocation of multi-pollutant permits across firms. The finite-dimensional nature of the problem arises from the discrete portfolio of permits Y, which allows the optimization to be reduced to solving for M shadow prices ψ_j, where M is the number of permits.

Proof. The Monge-Kantorovich theorem (Kantorovich, 1942; Aurenhammer et al., 1998; Gangbo and McCann, 1996; Carlier, 2003) establishes a duality relationship between the primal and dual problems of optimal transport.[17] It states that the value of the primal problem, which maximizes the matching value of transporting one measure to another, equals the value of the dual problem, which minimizes a functional subject to constraints on shadow prices.[18]

To derive the optimal shadow prices in the dual formulation (37), we first establish the relationship between the shadow prices $\psi(y_j)$ (denoted as ψ_j) and a firm with polluting characteristics x. For a given shadow price vector $\psi = \{\psi_1, \ldots, \psi_M\}$, the firm's maximized net payoff value $\phi(x)$ is given by:

$$\phi(x, \psi) = \sup_{y_j, j=1,\ldots,M} \left[V(x, y_j) - \psi_j \right]. \tag{38}$$

This equation represents the firm's optimal choice of pollution permits y_j that maximizes its net payoff, given the shadow prices ψ_j.

Using the Kantorovich dual formulation from (37) and the relationship in (38), we can rewrite the dual problem with a finite set of variables $\psi = \{\psi_1, \ldots, \psi_M\}$. The dual problem now becomes:

$$\inf_{(\psi_j)_{j=1,\ldots,M}} F(\psi), \tag{39}$$

[17]The duality property is crucial in understanding and solving optimal transport problems. The proof relies on results from Kantorovich (1942); Gangbo and McCann (1996); Aurenhammer et al. (1998); Villani (2009); Carlier (2003), applied to multi-pollutant optimal transport problems.

[18]The Monge-Kantorovich theorem ensures that the optimal assignment of multi-pollutants to firms can be decentralized in a market equilibrium where supply and demand for permits are balanced through shadow prices.

where $F(\psi)$ is defined as:

$$\begin{aligned} F(\psi) &= \int_{\Omega} \phi(x,\psi) f(x)\, dx + \sum_{j=1}^{M} S_j \psi_j \\ &= \int_{\Omega} \left[\max_{j=1,\ldots,M} \left\{ V(x,y_j) - \psi_j \right\} \right] f(x) dx + \sum_{j=1}^{M} S_j \psi_j \qquad (40) \\ &= \sum_{j=1}^{M} \left[\int_{D_j} \left(V(x,y_j) - \psi_j \right) f(x) dx + S_j \psi_j \right] \\ &= \sum_{j=1}^{M} \int_{D_j} V(x,y_j) f(x) dx + \sum_{j=1}^{M} \left(S_j - \mu(D_j) \right) \psi_j, \qquad (41) \end{aligned}$$

Here, Ω is partitioned into regions $D_j(Y,\psi)$, defined as:

$$\Omega = \bigcup_{j=1}^{M} D_j(Y,\psi) \quad \text{and} \quad D_j(Y,\psi) = \left\{ x \in \Omega : V(x,y_j) - \psi_j = \max_{k=1,\ldots,M} \left\{ V(x,y_k) - \psi_k \right\} \right\}.$$

Each region $D_j(Y,\psi)$ represents the set of firms x that optimally choose the pollution permit y_j given the shadow prices ψ. The term $\mu(D_j)$ denotes the measure of firms in D_j, which is derived from the distribution $f(x)$.

The social planner's objective is to find the optimal vector of shadow prices $\psi = \{\psi_1, \ldots, \psi_M\}$ that minimizes the unconstrained concave function $F(\psi)$. This function can be expressed as:

$$F(\psi) = W(\psi) + \sum_{j=1}^{M} S_j \psi_j, \qquad (42)$$

where:

$$W(\psi) = \int_{\Omega} \left[\max_{j=1,\ldots,M} \left\{ V(x,y_j) - \psi_j \right\} \right] f(x) dx.$$

The term $W(\psi)$ represents the aggregate net payoff of all firms, while $\sum_{j=1}^{M} S_j \psi_j$ accounts for the total cost of the pollution permits at the given shadow prices. The optimal shadow prices ψ balance these two components, ensuring that the social value $\Phi(\mu,\nu,V)$ is optimized. Note that at the optimum, $F(\psi) = \Phi(\mu,\nu,V)$.[19]

To find the optimal ψ, we compute the first-order condition of the convex function $F(\psi)$:

$$\frac{\partial F(\psi)}{\partial \psi_j} = 0, \qquad (43)$$

[19]The constraints on the shadow prices ensure that the net payoff $\phi(x)$ for each firm x is at least as large as the value $V(x,y_j)$ minus the shadow price ψ_j. At the optimum, the constraint becomes binding, i.e., $\psi_j + \phi(x) = V(x,y_j)$. This binding condition is crucial for determining the optimal allocation of permits and the corresponding shadow prices.

which, using (42), implies:

$$\frac{\partial W(\psi)}{\partial \psi_j} + S_j = 0, \quad j = 1, \ldots, M. \tag{44}$$

As shown in Appendix C, the convexity of $F(\psi)$ guarantees the existence of the shadow prices ψ. Notably, the system of shadow price equations (44) is identical to the system of competitive market price equations (20). Both systems can be expressed concisely as:

$$G(\psi) = S,$$

where $G(\psi) = \left(-\frac{\partial W(\psi)}{\partial \psi_1}, \ldots, -\frac{\partial W(\psi)}{\partial \psi_M}\right)$ represents the vector demand mapping of multi-pollutant permits, and $S = (S_1, \ldots, S_M)$ denotes the weights of the multi-pollutant portfolio. The existence of shadow prices for the social planner problem ensures the existence of equilibrium market prices for the multi-pollutant cap-and-trade market.

Therefore, the shadow prices ψ used by the social planner for the optimal multi-pollutant allocation correspond exactly to the system of competitive prices p that equate demand and supply for the portfolio of multi-pollutants, as shown in (20). This implies that the allocations resulting from the social planner's problem and the competitive equilibrium are consistent with the two fundamental theorems of welfare economics in multi-pollutant cap-and-trade markets, demonstrating the existence of an optimal partition of firms supported by both competitive market incentives and the social planner's maximization goal.

This proof demonstrates how the optimal multi-pollutant allocation problem can be effectively solved by finding the appropriate shadow prices ψ, which determine the permit prices that balance market demand and supply. These shadow prices play a critical role in maximizing social welfare and ensuring efficient multi-pollutant cap-and-trade systems.

5 Decentralization of Optimal Multi-pollutant Allocation

Proposition 5.1 ***Decentralization via Market Prices***
The shadow values $(\psi_1, \ldots, \psi_M)$ used by the social planner for the optimal allocation of multi-pollutant permits to heterogeneous firms correspond exactly to the system of competitive prices $(p_1, \ldots, p_M)$ that equate the demand and supply for the portfolio of multi-pollutants.

The social planner's allocation of multi-pollutant permits can be decentralized through a system of market prices $(p_j)_{j=1,\ldots,M}$ such that:

$$\underbrace{T(x)}_{\text{Permit assigned to firm } x} = \arg\max_{j=1,\ldots,M} \Big[V(x, y_j) - p_j\Big].$$

Given an optimal vector of prices $p = (p_1, \ldots, p_M)$, the unique optimal transport map that allocates heterogeneous polluting firms to the targeted distribution of multi-pollutant permits can

be expressed as:

$$T = \sum_{j=1}^{M} y_j \mathbb{1}_{D_j(Y,p)}, \tag{45}$$

where $\mathbb{1}_{D_j(Y,p)}$ is the indicator function of the demand set $D_j(Y,p)$.[20]

If the matching function is the inner product $V(x,y_j) = xy_j = \sum_{k=1}^{d} x_k y_{jk}$ in the Euclidean multi-pollutant space, the optimal mapping T can be represented as the gradient of a convex function:

$$T = \nabla\phi, \tag{46}$$

where the convex function ϕ is explicitly given by:

$$\phi(x,p) = \max_{j=1,\ldots,M} \left\{ xy_j - p_j \right\}. \tag{47}$$

The twist condition-that the mapping $y_j \to D_x V(x,y_j)$ is injective-ensures that the optimal assignment results in a partition of firms, except for a negligible set $\mu(D_j \cap D_l) = 0$. This condition also guarantees the uniqueness of the optimal assignment (transport map) of firms to multi-pollutant permits.

6 Stability of Multi-Pollutant Cap-and-Trade Markets Under Changes in Permit Portfolios

Understanding how multi-pollutant cap-and-trade markets react to changes in the portfolio of permits is crucial for ensuring market stability and effectiveness. Changes in the portfolio may involve adjustments to the composition of permits (e.g., altering the mix of pollutants they cover) or changes in the total mass of permits available (e.g., increasing or decreasing the mass supply of some permits). Regulators may need to make such adjustments over time to address public health concerns, environmental goals, or evolving policy priorities. The impact of these changes on the system of prices can be understood through the hedonic nature of permit pricing. As established earlier, the shadow prices derived from the dual formulation of the optimal transport problem are identical to the decentralized market prices. From a hedonic perspective, the price of a permit is determined by its unique combination of pollutants, the distribution of polluting firms, and the matching function that connects firms to permits. Each permit is valued based on how well its characteristics align with the needs and constraints of firms. This framework allows us to analyze how changes in the permit portfolio affect market prices and stability.

6.1 Regularity Conditions for Stability

To ensure the stability of multi-pollutant cap-and-trade markets, certain regularity conditions must be satisfied. These conditions provide the mathematical foundation for analyzing how changes in

[20] This means that $T(x)$ assigns the permit y_j to the firm x when x belongs to the demand set $D_j(Y,\psi)$. The mapping $T(x)$ effectively partitions the firms into regions $D_j(Y,\psi)$, each corresponding to a specific permit y_j.

the permit portfolio influence market prices and firm behavior. Below, we outline the key conditions:

- **Twice Continuously Differentiable Matching Function**: The function $V(x, y_j)$, which describes the value of permit y_j to a firm with characteristics x, must have continuous second-order derivatives. This ensures smoothness and regularity in the relationship between firms' emission profiles and their permit choices.

- **Twist Condition**: The gradient vectors $\nabla_x V(x, y_j)$ and $\nabla_x V(x, y_k)$ must be distinct for all x in the domain Ω and for all distinct pairs of permit types (y_j, y_k). This condition ensures that firms can differentiate between permits based on their pollutant characteristics, reflecting the unique value of each permit type.

- **Connectedness of Firm Distribution**: The distribution of firms μ on Ω must satisfy the Poincaré-Wirtinger inequality. This ensures that the support of μ is well-connected and free of isolated regions, allowing for meaningful comparisons and averages across the domain. In practical terms, this condition guarantees that the distribution of firms is sufficiently smooth and interconnected, enabling stable market dynamics.

- **Smooth Differentiability and Strong Monotonicity of Demand**: The demand function $G(\psi) = \big(\mu(D_1(Y, \psi)), \ldots, \mu(D_M(Y, \psi))\big)$, which represents the demand for permits as a function of prices, must be smoothly differentiable and strongly monotonic. This ensures that the demand for permits responds predictably to changes in prices, maintaining market stability. The strong monotonicity of $G(\cdot)$ relies on the connectedness of the firm distribution and the geometric properties of the matching function.

These conditions collectively ensure that the multi-pollutant cap-and-trade market remains stable and responsive to changes in the permit portfolio. They provide the necessary mathematical framework for analyzing how small adjustments to the portfolio—whether in composition or mass—affect market prices and firm behavior.

6.2 Stability of Multi-Pollutant Permit Prices

The stability of permit prices under changes in the portfolio is a concern for regulators. The following proposition formalizes this stability, showing that small changes in the permit portfolio lead to proportionally small changes in market prices:

Proposition 6.1 ***Stability of Multi-Pollutant Permit Prices*** *Consider a market with heterogeneous polluting firms distributed according to a continuous measure $d\mu(x) = f(x)dx$. Let $\nu = \sum_{j=1}^{M} S_j \delta_{y_j}$ and $\tilde{\nu} = \sum_{j=1}^{M} \tilde{S}_j \delta_{y_j}$ be two portfolios of multi-pollutant permits, where:*

- $(y_j)_{j=1,\ldots,M}$ *represents the set of multi-pollutant permit types in the two portfolios,*

- $S = (S_j)_{j=1,\ldots,M}$ *and* $\tilde{S} = (\tilde{S}_j)_{j=1,\ldots,M}$ *are the respective shares or weights of the multi-pollutant permits in the two portfolios.*

Let $p = (p_j)_{j=1,\dots,M}$ and $\tilde{p} = (\tilde{p}_j)_{j=1,\dots,M}$ denote the systems of market prices associated with the portfolios ν and $\tilde{\nu}$, respectively. Then, there exists a positive constant $K_0(\Omega,\mu,V,S,\tilde{S}) > 0$,[21] such that the following stability inequality holds:

$$\underbrace{\|p-\tilde{p}\|}_{\text{Distance between permit prices}} < K_0(\Omega,\mu,V,S,\tilde{S}) \underbrace{\|S-\tilde{S}\|}_{\text{Distance between portfolios' weights}} . \tag{48}$$

Proof. This proof applies the mathematical findings of Bansil and Kitagawa (2022, Theorem 1.10) to the multi-pollutant optimal transport problem.[22] The key idea is to leverage the regularity and invertibility of the multivariate demand mapping $G(p) = \big(\mu(D_1(Y,p)),\dots,\mu(D_M(Y,p))\big)$, which ensures that market demand adjusts smoothly to changes in the supply of multi-pollutant permits. The proof relies on the Poincaré-Wirtinger inequality, which guarantees that the distribution of polluting firms is sufficiently smooth and well-connected. This condition ensures that the demand mapping $G(p)$ is well-behaved and strongly monotonic, meaning it responds predictably to changes in permit prices. Intuitively, this prevents abrupt shifts in demand and prices when the permit portfolio is adjusted. The stability constant $K_0(\Omega,\mu,V,S,\tilde{S})$ quantifies how sensitive permit prices are to changes in the portfolio. It depends on several factors: the diversity of firms in the market (Ω), the distribution of their pollution characteristics (μ), the sensitivity of the matching function V, and the composition of the permit allocations (S and $\tilde{S}$). Together, these factors determine how smoothly permit prices adjust to policy changes. In essence, the result shows that permit prices adjust smoothly to changes in the portfolio, provided the distribution of firms is well-behaved and the matching function is sufficiently regular. This ensures that regulators can adjust permit allocations without causing destabilizing price swings in the market.

This proposition says that there exists a positive constant $K_0(\Omega,\mu,V,S,\tilde{S})$ such that a small change in the allocation of weights (i.e., reallocation of permits) between the two portfolios will not result in a dramatic change in the associated system of prices of multi-pollutant permits. In other words, the distance between the two sets of shadow prices, represented by $\|p-\tilde{p}\|$, will be bounded by $K_0(\Omega,\mu,V,S,\tilde{S})$ times the distance between weights of the two portfolios, $\|S-\tilde{S}\|$. This stability inequality ensures that slight changes in the distribution of permits across different emission types will have only a limited impact on the corresponding prices of multi-pollutant permits. The proposition provides valuable insights into the robustness of the multi-pollutant cap-and-trade system and suggests that the permit prices remain relatively stable even with small shifts

[21] The coefficient $K_0(\Omega,\mu,V,S,\tilde{S})$ depends on several key factors influencing permit price dynamics in response to changes in the government's multi-pollutant permit portfolio. The set of firms in the market (Ω) and the statistical properties of their distribution (μ) play a central role, as they determine the diversity of pollution sources and the market's capacity to absorb regulatory adjustments. A distribution μ satisfying the Poincaré-Wirtinger inequality ensures firms are sufficiently dispersed, preventing sharp price fluctuations during permit reallocations. Additionally, the sensitivity of the matching function (V) to changes in firm characteristics affects price stability; if small shifts in firm attributes lead to large changes in permit assignments, prices will become more volatile. Finally, the composition of the initial and adjusted permit allocations (S and $\tilde{S}$) influences price adjustments, as larger shifts in portfolio composition can amplify market responses. Together, these factors determine whether permit prices adjust smoothly or experience significant swings in response to policy changes.

[22] We thank Professor Jun Kitagawa for valuable discussions regarding their work.

in permit allocations.

Proposition 6.2 ***Stability of Geometric Partitions in the Multi-Pollutant Cap-and-Trade Markets*** *Consider a market with heterogeneous polluting firms distributed according to a continuous measure $d\mu(x) = f(x)dx$. Let $\nu = \sum_{j=1}^{M} S_j \delta_{y_j}$ and $\tilde{\nu} = \sum_{j=1}^{M} \tilde{S}_j \delta_{y_j}$ be two portfolios of multi-pollutant permits, where:*

- $(y_j)_{j=1,\dots,M}$ *represents the set of multi-pollutant permit types in the two portfolios,*
- $S = (S_j)_{j=1,\dots,M}$ *and* $\tilde{S} = (\tilde{S}_j)_{j=1,\dots,M}$ *are the respective shares or weights of the multi-pollutant permits in the two portfolios.*

If the weighting gap $\|S - \tilde{S}\|$ is sufficiently small, then there exists a positive constant $K_1(\Omega, M, N, \mu, S, \tilde{S}) > 0$ such that the following stability inequality holds:

$$\underbrace{\left[\mathbf{d}_H(D_j, \widetilde{D}_j)\right]^N}_{\text{Difference in demand sets}} < K_1(\Omega, M, N, \mu, S, \tilde{S}) \underbrace{\|S - \tilde{S}\|}_{\text{Distance between the weights}}, \tag{49}$$

for $j = 1, \dots, M$. Here:

- $\mathbf{d}_H$ *denotes the Hausdorff distance, which quantifies the difference between two sets,*
- D_j *and* $\widetilde{D}_j$ *represent the demand sets for pollutant j under the portfolios ν and $\tilde{\nu}$, respectively.*

Proof. This provides a compelling application of the mathematical findings from Bansil and Kitagawa (2022, Theorem 1.10) to the multi-pollutant optimal transport problem. An important remark to mention is that if the Hausdorff distance between two demand sets $\mathbf{d}_H(D_j, \widetilde{D}_j)$ is zero, it implies that the two sets D_j and $\widetilde{D}_j$ are equal, as per their construction in the mathematical proof.

This proposition examines the stability of the geometric partitions in the multi-pollutant cap-and-trade market when there are small changes in the allocation of weights (i.e., reallocation of permits) between the two portfolios. If the weighting gap, represented by $\|S - \tilde{S}\|$, is sufficiently small, there exists a constant $K_1(\Omega, M, N, \mu, S, \tilde{S})$ such that the Hausdorff distance, denoted by $\mathbf{d}_H(D_j, \widetilde{D}_j)$, between the two sets of associated optimal partitions $(D_j)_{j=1,\dots,M}$ and $(\tilde{D}_j)_{j=1,\dots,M}$ is bounded by $K_1(\Omega, M, N, \mu, S, \tilde{S})\|S - \tilde{S}\|$. The proposition demonstrates that small changes in the distribution of permits across different emission types will not lead to significant variations in the geometric partitions of the market. The stability of the market partitions implies that the overall structure of the multi-pollutant cap-and-trade system remains relatively consistent, providing policymakers and market participants with valuable insights into the system's robustness.

Now, let's delve into a result that grants substantial flexibility for stabilizing market decisions by permitting slight alterations in both the composition of multi-pollutant permits themselves, denoted as $\{y_j\}_{j=1}^{M}$ and $\{\tilde{y}_j\}_{j=1}^{M}$, as well as their respective weights, represented as $\{\tilde{S}_j\}_{j=1}^{M}$ and $\{S_j\}_{j=1}^{M}$.

Proposition 6.3 ***Stability of Permit Assignments to Firms Under Variations in Multi-Pollutant Permit Composition or Shares***

Consider a market with heterogeneous polluting firms distributed according to a continuous measure $d\mu(x) = f(x)dx$. Suppose there are two portfolios of multi-pollutants, represented by the measures:

$$\nu = \sum_{j=1}^{M} S_j \delta_{y_j} \quad \text{and} \quad \tilde{\nu} = \sum_{j=1}^{M} \tilde{S}_j \delta_{\tilde{y}_j},$$

where:

- $Y = \{y_1, \ldots, y_M\}$ is the set of pollutant types in portfolio ν,
- $\tilde{Y} = \{\tilde{y}_1, \ldots, \tilde{y}_M\}$ is the set of pollutant types in portfolio $\tilde{\nu}$,
- S_j and $\tilde{S}_j$ represent the respective quantities of pollutants in the two portfolios.

If the matching function governing firm-pollutant interactions is given by

$$V(x, y_j) = \sum_{k=1}^{M} x_k y_{jk},$$

then there exists a positive constant $K_3(\Omega, S, \tilde{S}, Y, \tilde{Y}) > 0$ such that the deviation in optimal firm-to-pollutant assignments between the two portfolios satisfies the inequality:

$$\underbrace{\|T - \tilde{T}\|_{L^2}}_{\text{Difference in optimal assignments}} \leq K_3(\Omega, S, \tilde{S}, Y, \tilde{Y}) \underbrace{d_2(\nu, \tilde{\nu})}_{\text{Distance between the two portfolios}}. \tag{50}$$

Here:

- T represents the optimal assignment of firms to multi-pollutants under the portfolio ν,
- $\tilde{T}$ represents the optimal assignment of firms to multi-pollutants under the portfolio $\tilde{\nu}$, and
- $d_2(\nu, \tilde{\nu})$ measures the difference between the two portfolios in terms of both the types of pollutants (y_j vs. $\tilde{y}_j$) and their quantities (S_j vs. $\tilde{S}_j$).[23]

Proof. This proposition directly applies the mathematical result by Mérigot et al. (2020)[Theorem 3.1] to the multi-pollutant cap-and-trade economic framework. This proposition holds significant importance as it demonstrates the stability of the multi-pollutant cap-and-trade market against minor variations in the set of pollutants and their respective weights. Such adaptability underscores the market's robustness, establishing it as a reliable instrument for environmental regulation and pollution control.

[23]The term $d_2(\nu, \tilde{\nu})$ is defined as $d_2(\nu, \tilde{\nu}) = W_2(\nu, \tilde{\nu})^{2/5}$, where $W_2(\nu, \tilde{\nu})$ is the Wasserstein distance between ν and $\tilde{\nu}$. The Wasserstein distance measures the dissimilarity between the two portfolio, considering both the composition and quantities of pollutants. For a technical exposition of the Wasserstein distance, see Villani (2009). If $d_2(\nu, \tilde{\nu}) = 0$, the two portfolios are identical in both composition and quantity, which implies, based on the inequality (50), that the optimal assignments T and $\tilde{T}$ will also be identical.

Proposition 6.4 ***Stability of Firm Switching Behavior in Multi-Pollutant Permit Markets*** *Consider a market with heterogeneous polluting firms distributed according to a continuous measure $d\mu(x) = f(x)dx$. Let there be two scenarios of multi-pollutant permit allocations:*

$$\nu = \sum_{j=1}^{M} S_j \delta_{y_j} \quad \textit{and} \quad \tilde{\nu} = \sum_{j=1}^{M} \tilde{S}_j \delta_{y_j},$$

where S_j and $\tilde{S}_j$ represent the proportions of permits y_j in the initial and modified portfolios, respectively. Let the associated shadow prices for the permits be $p = (p_1, \ldots, p_M)$ and $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_M)$ for the two scenarios.

The following stability inequality holds:

$$\sum_{j=1}^{N} \Delta_\mu(D_j(Y,p), D_j(Y,\tilde{p})) \leq 4N\|S - \tilde{S}\|_1,$$

where:

- *$\Delta_\mu(D_j(Y,p), D_j(Y,\tilde{p}))$ measures the difference in the number of firms assigned to permit j between the two scenarios. This includes firms that:*
  - *Enter the new demand set for permit j under $\tilde{p}$ but were not in the original demand set under p, or*
  - *Leave the original demand set for permit j under p to join another permit set under $\tilde{p}$.*
- *$\|S - \tilde{S}\|_1$ is the norm of the difference between the initial and modified mass permit allocations.*

This proposition establishes that the stability of firm switching behavior in multi-pollutant permit markets is directly tied to the magnitude of policy changes. Specifically, smaller differences in permit allocations ($\|S - \tilde{S}\|_1$) lead to fewer firms switching demand sets, either entering or leaving. Firms that switch are precisely those not in the intersection of the initial and final demand sets for a given permit. This result implies that minor regulatory changes cause minimal disruptions in how firms interact with permits, ensuring market stability and predictability.

Proof. This is an application of the mathematical findings from Bansil and Kitagawa (2022, Theorem 1.3) to the multi-pollutant optimal transport problem.

6.3 What are the policy implications of the stability results in the multi-pollutant optimal transport problem?

The stability results can enhance the real-world implementations of multi-pollutant markets for at least three reasons Firstly, the stability results assure policymakers that small perturbations in the weights of the multi-pollutant permit portfolio will lead to only minor changes in the optimal

permit allocation. This provides confidence that the chosen permit allocation strategy will remain effective and efficient in achieving pollution reduction targets even when faced with varying scenarios. Secondly, the findings suggest that the optimal permit allocation can lead to improved market performance and reduced volatility in the prices of multi-pollutant permits. A stable allocation minimizes abrupt fluctuations in permit prices, promoting market predictability and reducing uncertainty for firms and investors operating within the cap-and-trade system. Thirdly, the stability results facilitate easier updates and revisions to permit allocation policies. Policymakers can adjust the allocation in response to new information or data without compromising the overall effectiveness of the cap-and-trade system. This adaptability allows the policy to remain robust and responsive to changing environmental conditions and evolving market dynamics.

Remark 4 *In a multi-pollutant optimal transport problem, "border indifference" refers to polluting firms located at the boundaries of two demand sets in the partition as they show no strong preference for choosing one multi-pollutant pollutant over the other. This implies that firms at the border are equally willing to choose these two multiple pollutants, and the net payoffs associated with choosing any of these two multi-pollutant permits are equal.*[24] *More formally, if a firm of type x is indifferent between purchasing the multi-pollutant permit y_j at price p_j and acquiring the multi-pollutant permit y_k at price p_k, it signifies that the price difference between permit y_j and permit y_k precisely offsets the variance in the firm values attributed to these two permits. Mathematically, this can be expressed as $p_j - p_k = V(x,y_j) - V(x,y_k)$. Interestingly, this equality also suggests that $y_j > y_k$ leads to $p_j > p_k$, indicating that a permit y_j containing a larger proportion of pollutant rights than permit y_k will command a higher price.*

Remark 5 *The standard cap-and-trade system with one pollutant represents a simplified and specialized approach compared to the more general multi-pollutant cap-and-trade market using optimal transport theory. While the one-pollutant system focuses solely on allocating permits for a single pollutant, the multi-pollutant problem takes into account the intricate interactions between multiple pollutants and their respective permits, considering the distribution of polluting firms in a multi-dimensional space. By extending the market mechanism to encompass multiple pollutants and firms, the multi-pollutant optimal transport problem offers a comprehensive and realistic framework to address the complexities and interdependencies among various pollutants, thus becoming a powerful tool for designing efficient and equitable cap-and-trade markets. In summary, the standard cap-and-trade with one pollutant can be viewed as a special case of the optimal transport multi-pollutant problem, as it deals exclusively with a single pollutant and overlooks the broader considerations present in the more general multi-pollutant setting.*

Remark 6 *The design of multi-pollutant cap-and-trade markets with optimal transport theory presents a novel and powerful approach to address equitable or inequality concerns among polluting firms. This methodology allows for a more precise allocation of emission permits, considering the heterogeneity of pollution characteristics of firms. By optimizing the transportation of*

[24] For a firm of type x that is not indifferent, the firm value to choose the permit y_j over the permit y_k is strictly larger than the difference in prices. That is, $V(x,y_j) - V(x,y_k) > p_j - p_k$.

pollution from sources to targets and tailoring permit allocations based on firms' specific needs, the approach promotes fairness and efficiency in compliance. Moreover, the use of cutting-edge optimization techniques enables dynamic adjustments of permits over time, accommodating the evolving challenges faced by different firms. Overall, the integration of optimal transport theory offers a sophisticated and equitable solution to tackle pollution disparities and foster sustainability in achieving environmental objectives.

7 Numerical approach and algorithmic for tackling multi-pollutant cap-and-trade markets

In this section, we delve into a numerical demonstration concerning permit prices and the optimal distribution of demands within a multi-pollutant cap-and-trade market. This approach introduces a visual representation that serves to complement the theoretical aspects, offering a tangible grasp of key insights. This is particularly important given the inherent complexity of the multi-dimensional nature of the multi-pollutant cap-and-trade problem.

Our approach combines computational optimal transport methods with geometry techniques, leading to a visual presentation of how multi-dimensional permits are distributed across a continuous range of polluting firms. To tackle the underlying numerical challenges in solving optimal transport problems, we draw from stochastic gradient descent techniques (Peyré et al., 2019; Genevay et al., 2016; Bourne et al., 2020; Aurenhammer et al., 1998; Bourne et al., 2022; Galichon, 2016). Through iterative adaptations of these techniques, we converge towards equilibrium permit prices. Additionally, we harness computational geometry methods to determine the optimal demand sets for multi-pollutant permits. For a comprehensive understanding of the algorithm's intricacies, please refer to Appendix D.

7.1 Convergence to equilibrium in multi-Pollutant markets

The numerical approach for computing the shadow prices $\psi_j,\ j = 1,...,M$ in multi-pollutant markets are similar to a tatonnement process, comparable to a dynamic price adjustment mechanism, whose goal is to align the demands for multi-pollutant permits and the supplies. As presented in Appendix D, the algorithmic equation that guides the updating of shadow prices reads:

$$\psi_j^{(k+1)} = \psi_j^{(k)} - \eta \left(S_j - \mu(D_j^{(sample)}) \right)$$

Here, $\psi_j^{(k+1)}$ represents the updated shadow price for permit i at iteration $k+1$, while $\psi_j^{(k)}$ is the current shadow price at iteration t. The right-hand side of the equation involves a critical interplay between the supply, denoted as S_j, and the demand size $\mu(D_j^{(sample)})$ arising from polluting firms associated with permit j.

The equation reflects a continuous process of equilibrium exploration. The difference $S_j - \mu(D_j^{(sample)})$ captures the disparity between supply and demand. If supply exceeds demand, i.e., $S_j > \mu(D_j^{(sample)})$, the updated shadow price $\psi_j^{(k+1)}$ is expected to decrease. Conversely, if demand

outstrips supply, implying $S_j < \mu(D_j^{(sample)})$, the shadow price $\psi_j^{(k+1)}$ is anticipated to rise. The repeated application of this process, iteratively guided by the learning rate η, gradually steers the shadow prices toward equilibrium. The algorithm is adept at sensing and responding to imbalances between supply and demand, aligning the prices such that they harmonize with the overall market dynamics. By iteratively updating shadow prices based on the interplay of supply and demand, the algorithm not only drives equilibrium but also provides insights into the sensitivity of permit prices to shifts in market conditions, thereby aiding the understanding of permit pricing dynamics in complex environmental economics scenarios.

7.2 Example of simulation of a multi-pollutant cap-and-trade market

In a multi-pollutant cap-and-trade market, we consider a pollution space denoted as Ω, which is a bounded subset of the two-dimensional Euclidean space $\mathbb{R}^2$. A two-dimensional vector represents each firm's pollution characteristics in this space, with components labeled as x_1 and x_2. Let us assume that distribution of polluting firms in this space is described by a mixture of three Gaussians, where each Gaussian component is characterized by its weight (w_i), means in both dimensions ($\mu_{x_{1i}}$ and $\mu_{x_{2i}}$), and standard deviation (σ_i).[25]

$$f(x_1, x_2) = \sum_{i=1}^{3} w_i \frac{1}{2\pi\sigma_i^2} \exp\left(-\frac{1}{2}\left(\frac{(x_1-\mu_{x_{1i}})^2}{\sigma_i^2} + \frac{(x_2-\mu_{x_{2i}})^2}{\sigma_i^2}\right)\right)$$

where w_i is the weight of the i-th Gaussian component, $\mu_{x_{1i}}$ and $\mu_{x_{2i}}$ are the means of the i-th Gaussian in the x_1 and x_2 dimensions respectively, and σ_i is the standard deviation (variance) of the i-th Gaussian. Now, let's assume that there are two pollutants of interest, CO_2 and SO_2, and visualize the distribution of polluting firms in a bounded domain using the Gaussian mixture graph.

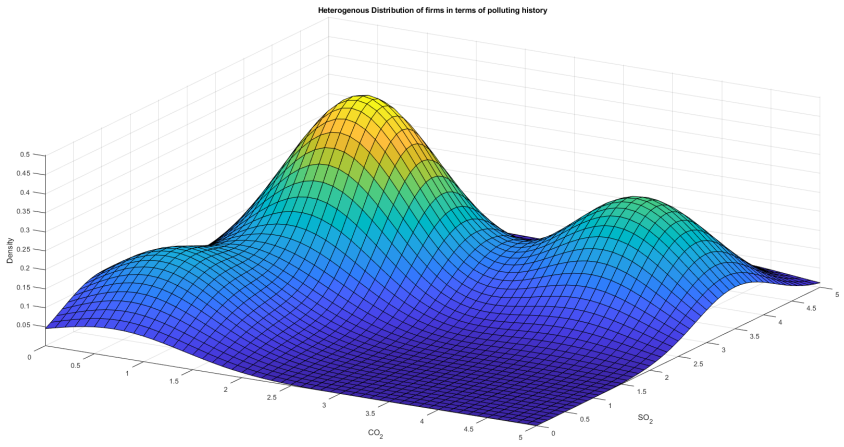
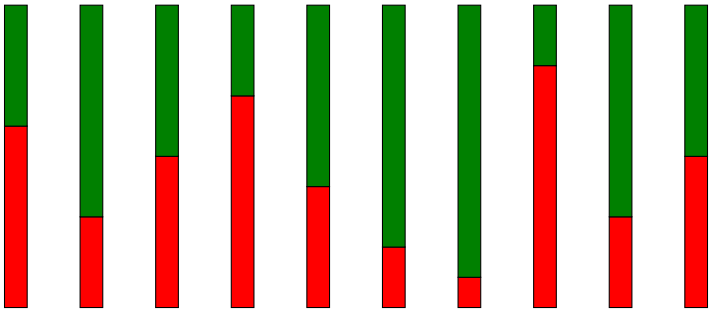


Figure 1: Distribution of polluting firms in a multi-pollutant cap-and-trade market

[25]This mixture model allows us to capture the complexities of pollution patterns more effectively than a single Gaussian distribution.

The government agency issues a multi-pollutant portfolio, consisting of 10 different permit types, each denoted by y_j.

$$\nu = \sum_{i=1}^{10} S_j \delta_{y_j} \tag{51}$$



A portfolio of 10 multi-pollutant permits

The matching function represented by $V(x, y_j) = \sum_{k=1}^{2} x_k y_{jk}$ allows us to measure the compatibility between a firm with x and the permit type y_j.

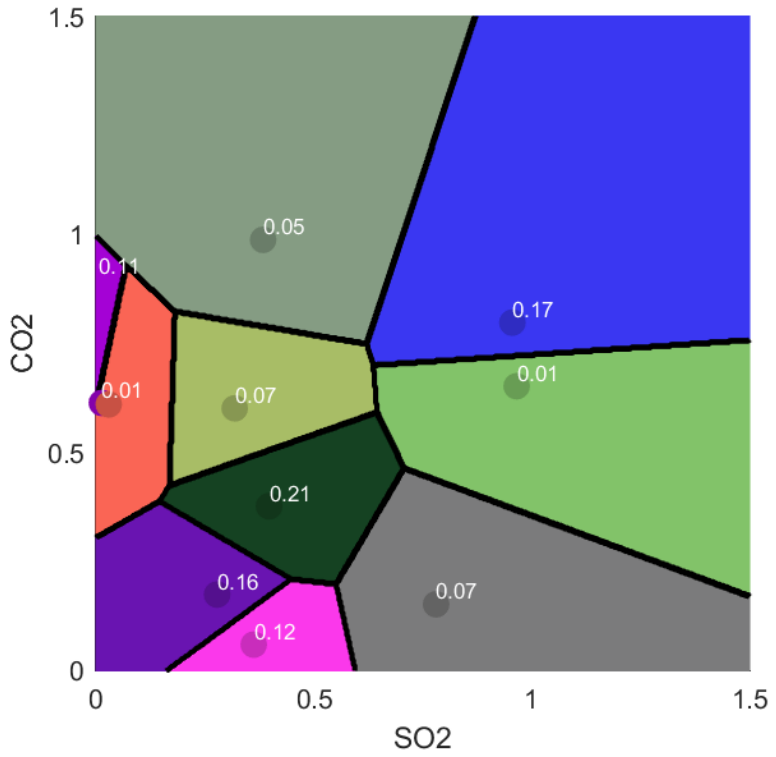


Figure 2: Illustration of the application of a combined approach using stochastic gradient descent and computational geometry numerical techniques to determine the optimal demand partition in a multi-pollutant cap-and-trade market with a portfolio of $M = 10$ distinct types of permits. The small circles represent the multi-pollutant permits $\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$ in the pollution space Ω. The computed shadow prices are shown next to each permit. Each permit is associated with a relative weight proportion in the following set: $\{S_1 = 10\%, S_2 = 10\%, S_3 = 10\%, S_4 = 10\%, S_5 = 10\%, S_6 = 10\%, S_7 = 10\%, S_8 = 10\%, S_9 = 10\%, S_{10} = 10\%\}$. The polygons represent the computed optimal demand partition, where each polygon represents the set of firms demanding the permit located within it. Numerical convergence is achieved after 60 iterations.

Overall, the algorithm combines stochastic gradient descent with computational optima transport to dynamically adjust the demand partition cells, ensuring an efficient allocation of permits to match the distribution of polluting firms while respecting market clearing. In order for the numerical scheme to converge to the equilibrium state within the multi-pollutant cap-and-trade system, the algorithm iteratively adjusts the shadow prices of permits. More specifically, it raises the shadow price of permits that are associated with a demand size larger than the supply size of the corresponding multi-pollutant permit. On the other hand, it decreases the shadow price of permits that correspond to a demand size smaller than the supply size of the respective multi-pollutant permit. This iterative adjustment process is instrumental in harmonizing the demand and supply dynamics, ultimately driving the system towards its equilibrium state.

Finally, it's important to acknowledge that while our simulation has been conducted in a two-dimensional space for illustrative purposes, the insights gained from this process hold promise for tackling real-world challenges involving a substantial number of multi-pollutants. The power

of computational optimal transport lies in its potential to navigate complex multi-dimensional spaces, even though this might introduce dimensionality challenges, warranting careful consideration of computational resources. Fortunately, the substantial advancements in computational optimal transport techniques witnessed over the past decade offer a promising signal that policymakers can confidently address complex, high-dimensional multi-pollutant issues.

8 Is There Equivalence Between Primary Markets for Multi-Pollutant Permits and Clustered Secondary Markets for Individual Pollutant Rights?

A multi-pollutant cap-and-trade system assigns permits to firms in a primary market, where each permit accounts for multiple pollutants simultaneously. This approach reflects the joint environmental impact of pollutants, such as greenhouse gases and particulate matter, which often interact in complex ways. A question for policymakers is: *Can a secondary market that trades individual pollutant rights separately fully replicate the primary market for multi-pollutant permits?* To answer this, we analyze whether decomposing multi-pollutant permits into individual pollutant rights preserves the economic and environmental integrity of the original market design.

8.1 Implicit Prices and Market Dimensionality

The implicit prices λ_i for individual pollutants are determined by the system of equations:

$$\underbrace{p_j(\mu,\nu,V)}_{\text{Price of permit } j} = \underbrace{\sum_{i=1}^{N} \lambda_i y_{ji}}_{\text{Weighted sum of pollutant contributions and their implicit prices}}, \quad j = 1,\ldots,M, \tag{52}$$

where p_j is the price of permit j, and y_{ji} represents the contribution of pollutant i to permit j. In matrix notation, this system can be expressed as:

$$\underbrace{\begin{bmatrix} p_1(\mu,\nu,V) \\ \vdots \\ p_M(\mu,\nu,V) \end{bmatrix}}_{\text{Permit prices } \mathbf{p}} = \underbrace{\begin{bmatrix} y_{11} & \cdots & y_{1N} \\ \vdots & \ddots & \vdots \\ y_{M1} & \cdots & y_{MN} \end{bmatrix}}_{\text{Multi-pollutant permits } \mathbf{Y}} \underbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix}}_{\text{Implicit pollutant prices } \lambda}. \tag{53}$$

When the number of permits M equals the number of pollutants N and the matrix $\mathbf{Y}$ is invertible, the implicit prices can be directly computed as:

$$\lambda = \mathbf{Y}^{-1}\mathbf{p}(\mu,\nu,V). \tag{54}$$

However, in practical settings, it is expected that $M > N$, meaning there are more permits than pollutants. This reflects the reality that regulators design multi-pollutant permits to capture a wide range of environmental and economic scenarios, leading to a high-dimensional primary market. In such cases, the system is overdetermined, and approximation methods like least squares estimation

are required to estimate λ:

$$\lambda = \left(\mathbf{Y}^\top \mathbf{Y}\right)^{-1} \mathbf{Y}^\top \mathbf{p}(\mu, \nu, V).$$

This equation acts as a "dimensionality reduction machine," projecting the high-dimensional permit prices $\mathbf{p}$ into a lower-dimensional space of implicit pollutant prices λ. Importantly, this projection is not exact, and there is no strict equivalence between the primary and secondary markets. The approximation introduces some loss of information, particularly regarding the complex interactions between pollutants encoded in the primary market.

8.2 Practical Implications for Policymakers

For policymakers, the key takeaway is that the transition from a primary multi-pollutant market to a secondary market for individual pollutant rights involves trade-offs. While the secondary market simplifies trading by focusing on individual pollutants, it cannot fully replicate the nuanced interactions captured in the primary market. This is especially true when $M > N$, as the approximation process inherently loses some information about how pollutants interact. The equation $\lambda = \left(\mathbf{Y}^\top \mathbf{Y}\right)^{-1} \mathbf{Y}^\top \mathbf{p}(\mu, \nu, V)$ provides a practical tool for estimating implicit pollutant prices, but it is not a perfect solution. Policymakers should be aware that dimensionality reduction introduces approximation errors. The projection from M permits to N pollutants is lossy, meaning some information about pollutant interactions is lost. This can lead to inefficiencies or unintended environmental outcomes, such as reducing emissions of one pollutant while inadvertently increasing emissions of another. There is no strict equivalence between markets. The primary market operates in a high-dimensional space that captures complex pollutant interactions, while the secondary market operates in a lower-dimensional space that simplifies trading but sacrifices some of this complexity.

8.3 Secondary Clustered Markets: A Practical Alternative

Once implicit pollutant prices are established, firms with surplus allowances for specific pollutants can sell them back to the regulator or trade them among themselves. This trading is more efficient among firms that initially purchased the same permit in the primary market, leading to a secondary clustered market for individual pollution rights. Firms within the same cluster are more similar in the multidimensional pollution space, making trades within the cluster more efficient and predictable. However, even with clustering, the secondary market cannot fully replicate the primary market. Policymakers must carefully weigh the benefits of simplified trading against the risks of information loss and potential environmental trade-offs.

8.4 Information Loss and Dimensionality Challenges in Secondary Markets for Individual Pollutant Rights

Replacing the primary multi-pollutant market with a secondary market for individual pollutants introduces significant challenges related to information loss and dimensionality. The primary market operates in a high-dimensional space, where M multi-pollutant permits account for N individual pollutants, often with $M > N$. This high dimensionality captures the complex interactions between pollutants, which are encoded in the matrix $\mathbf{Y}$ of multi-pollutant permit effects. However, transitioning to a secondary market that trades individual pollutant rights reduces this complexity, potentially losing critical information about pollutant interactions.

The implicit prices λ in the secondary market are estimated as:

$$\lambda = \left(\mathbf{Y}^\top \mathbf{Y}\right)^{-1} \mathbf{Y}^\top \mathbf{p}(\mu, \nu, V), \tag{55}$$

where $\mathbf{p}(\mu, \nu, V)$ represents the system of prices in the primary market, which depends on the distribution of firms μ, the permit portfolio ν, and the matching function V. This estimation provides a lower-dimensional projection of the permit prices. However, this projection is inherently lossy, as λ is only an approximation and does not fully replicate the value of the primary market prices. The residual error:

$$\Delta = \frac{\|\mathbf{p}(\mu, \nu, V) - \mathbf{Y}\lambda\|_F}{\|\mathbf{p}(\mu, \nu, V)\|_F}, \tag{56}$$

quantifies the discrepancy between the high-dimensional primary market and the lower-dimensional secondary market. A small Δ indicates a close approximation, while a large Δ signals significant information loss, particularly regarding pollutant interactions.

This dimensionality reduction mirrors challenges in statistical representation, where high-dimensional systems are often approximated in lower dimensions at the cost of losing nuanced relationships. In the context of pollution markets, this loss tells us that separate markets for individual pollutants —using the approximate prices λ—are not equivalent to the primary market with multi-pollutant permits. The primary market accounts for several factors, including the distribution of firms, the composition of permits, and the multidimensional welfare function, which are not fully captured in the secondary market. As a result, policies designed for individual pollutants may fail to account for critical interactions between pollutants, leading to inefficient or counterproductive outcomes. In summary, while secondary markets for individual pollutants offer simplicity, they may come at the cost of losing critical information about pollutant interactions. Policymakers must carefully consider these trade-offs when designing market-based mechanisms to ensure that environmental goals are met without unintended consequences.

9 Concluding Remarks

In this paper, we have developed a novel framework for designing multi-pollutant cap-and-trade systems, addressing one of the most pressing challenges in environmental policy: managing si-

multaneous exposure to multiple pollutants. Unlike traditional approaches that treat pollutants in isolation, our framework accounts for the complex interactions between pollutants, offering a more comprehensive and effective solution for improving air quality and mitigating climate change. By integrating tools from optimal transport theory and environmental economics, we provide policymakers with a robust mechanism to allocate multi-pollutant permits efficiently, ensuring that environmental and health outcomes are better protected while minimizing economic disruption.

A central contribution of our work is the recognition that pollutants do not exist in isolation—they mix, interact, and amplify their harmful effects. Traditional single-pollutant markets often overlook these interactions, leading to inefficiencies and unintended consequences. For example, reducing emissions of one pollutant might inadvertently increase emissions of another, undermining the overall environmental benefits. Our multi-pollutant framework addresses this issue by designing permits that account for the combined effects of pollutants, ensuring that policies are both effective and equitable. This approach is particularly relevant for policymakers seeking to tackle air quality and climate change challenges in a holistic manner.

Another key insight from our analysis is the importance of balancing trade-offs between primary and secondary markets. Primary markets for multi-pollutant permits operate in a high-dimensional space, capturing the intricate relationships between pollutants and firms. While secondary markets for individual pollutant rights simplify trading, they risk losing critical information about these interactions. Policymakers must carefully navigate these trade-offs, ensuring that secondary markets complement rather than replace primary markets. Clustering firms with similar multi-pollutant profiles can help mitigate some of these challenges, but it is not a complete solution. Secondary markets should be viewed as a complementary tool, enhancing flexibility without sacrificing the nuanced interactions captured in primary markets.

The implications of our framework extend beyond local air quality management. By addressing multidimensional heterogeneity in pollution sources, regulatory requirements, and environmental impacts, our approach is adaptable to a wide range of policy contexts. For instance, it can be extended to a multi-regional context, enabling the analysis of cross-regional spillover effects, inter-regional permit trading, and the harmonization of environmental policies across jurisdictions. Such a multi-regional, multi-pollutant approach enhances the scalability of cap-and-trade systems, facilitating effective management of transboundary pollution and fostering cooperation among regions to achieve shared environmental goals.

Looking ahead, fostering interdisciplinary collaboration among economists, scientists, and policymakers will be essential for advancing multi-pollutant market-based mechanisms. A longer-term objective should be adopting a "one environment" approach that considers the health impacts of multiple pollutants across air, water, and agriculture to ensure the well-being of society and the environment. By integrating insights from diverse fields, we can develop innovative solutions that address the complex challenges of pollution management in a holistic and sustainable manner.

In conclusion, this work underscores the importance of addressing multi-pollutant interactions in environmental policy design. By leveraging the tools of economics, mathematics, and environmental science, we can develop more effective and comprehensive strategies for managing pollution and mitigating climate change. The challenges of transitioning between primary and

secondary markets highlight the need for careful policy design and interdisciplinary collaboration. As we move forward, adopting a holistic and multidimensional approach will be key to achieving sustainable environmental outcomes and safeguarding public health. Policymakers have a critical role to play in advancing this agenda. By embracing innovative frameworks like the one presented here, they can design policies that are not only effective but also adaptable to the complexities of real-world pollution challenges. The time to act is now—by working together, we can create a cleaner, healthier, and more sustainable future for all.

Appendix

A First order partial derivatives of the multivariate function $W(p)$

Given the statistical distribution of polluting firms $d\mu = f(x)dx$, and with the price of permits p set, let's initiate our exploration by expressing the total payoff function $W(p)$ for polluting firms.

$$W(p) = \int_{\Omega} \pi(x,p) f(x) dx \tag{57}$$

$$\frac{\partial \pi(x,p)}{\partial p_j} = \begin{cases} -1 & \text{if } x \in D_j(Y,p) \\ 0 & \text{if } x \text{ satisfies } V(x,y_j) - p_j < \max\limits_{j'=1,..M} \left[V(x,y_{j'}) - p_{j'} \right] \\ \text{Not defined if} & x \in D_j(Y,p) \bigcap D_{j'}(Y,p) \text{ for some } j \neq j'. \end{cases} \tag{58}$$

where

$$D_j(Y,p) = \left\{ x \in \Omega;\ V(x,y_j) - p_j = \max_{j'=1,..M} \left\{ V(x,y_k) - p_{j'} \right\} \right\}.$$

Since the measure $d\mu(x) = f(x)dx$ is absolutely continuous, it is important to note that the last scenario corresponds to a negligible set since

$$\mu(D_i(Y,p) \bigcap D_j(Y,p)) = \mu(\{x \in \Omega; V(x,y_j) - p_j = V(x,y_{j'}) - p_{j'}) = 0.$$

This leads to compute the partial derivative of the welfare function W with respect to p_j as follows:

$$\begin{aligned} \frac{\partial W}{\partial p_j} &= \frac{\partial}{\partial p_j} \int_{\Omega} \pi(x,p) f(x) dx \\ &= \int_{\Omega} \frac{\partial \pi(x,p)}{\partial p_j} f(x) dx \\ &= \int_{D_j(Y,p)} (-1) f(x) dx \\ &= -\mu(D_j(Y,p)) \leq 0 \end{aligned} \tag{59}$$

In this equation, $D_j(Y,p)$ signifies the demand for permit y_j, indicating the collection of firms choosing y_j as the optimal selection. This observation reveals that the derivative of the total payoff function of firms with respect to the price of permit p_j is either negative or zero, implying a strategic interaction among firms within the market.

B Second-Order Partial Derivatives of the Multivariate Function $W(p)$

Starting from (59), recall that the first partial derivative of $W(p)$ with respect to p_j is given by:

$$\frac{\partial W}{\partial p_j} = \int_{D_j(Y,p)} (-1) f(x)\, dx. \tag{60}$$

Our objective is to compute the second partial derivative with respect to p_i:

$$\begin{aligned} \frac{\partial^2 W}{\partial p_i \partial p_j}(p) &= \frac{\partial}{\partial p_i}\left[\frac{\partial W}{\partial p_j}\right] && (61) \\ &= \frac{\partial}{\partial p_i}\left[\int_{D_j(Y,p)} (-1) f(x)\, dx\right]. && (62) \end{aligned}$$

A key observation is that when the price p_i changes marginally, the demand set $D_j(Y,p)$ also changes. This occurs because firms adjust their permit choices in response to price changes, leading to shifts in the boundaries of the demand sets. To account for these changes, we apply the Reynolds Transport Theorem, which allows us to compute derivatives of integrals over domains that vary with respect to a parameter (in this case, prices).[26]

Applying the Reynolds Transport Theorem, the second partial derivative can be expressed as:

$$\frac{\partial^2 W}{\partial p_i \partial p_j}(p) = \int_{D_i(Y,p)\cap \partial D_j(Y,p)} (-1)(\delta m \cdot \mathbf{n})\, f(x)\, d\mathcal{H}(x),$$

where:

- $\partial D_j(Y,p)$ represents the boundary of the demand set $D_j(Y,p)$,
- The outward normal vector to the boundary is given by: $\mathbf{n}(x) = \frac{\nabla_x V(x,y_i) - \nabla_x V(x,y_j)}{\left\|\nabla_x V(x,y_i) - \nabla_x V(x,y_j)\right\|}$, where $\nabla_x V(x,y)$ denotes the gradient of V with respect to x.[27] This vector $\mathbf{n}(x)$ indicates the direction in which firm x is most likely to switch permits.
- $\delta m = \frac{dx}{dp_j}$ captures the rate at which firms move across the boundary in response to price changes, and
- $d\mathcal{H}(x)$ is the surface measure on the boundary.

[26] When prices change, firms adjust their demand for permits, which alters the boundaries of the demand sets $D_j(Y,p)$. The Reynolds Transport Theorem helps us understand how these changes affect the integral of firm distributions over these sets. Intuitively, the theorem tells us that the change in the integral (which represents the total demand for permits) depends on how firms move across the boundaries of the demand sets in response to price changes. Specifically, the change is proportional to the rate at which firms switch from one permit to another, weighted by the density of firms at the boundary. This switching behavior is driven by the relative benefits firms derive from different permits, as captured by the gradients of their value functions $V_x(x,y_i)$ and $V_x(x,y_j)$.

[27] The gradient $\nabla_x V(x,y)$ is the vector of partial derivatives of V with respect to each component of $x = (x_1,\ldots,x_N)$. In economic terms, it represents how the value function V changes as the firm's characteristics x vary. The difference $\nabla_x V(x,y_i) - \nabla_x V(x,y_j)$ captures how firms' preferences for permits y_i and y_j differ.

The term $\delta m \cdot \mathbf{n}$ represents the rate at which firms cross the boundary of the demand set $D_j(Y,p)$ in response to a change in price p_j. This rate depends on the relative benefits firms derive from permits y_i and y_j, as captured by the gradients of their value functions $\nabla_x V(x,y_i)$ and $\nabla_x V(x,y_j)$. Intuitively, firms switch permits when the marginal benefit of one permit exceeds that of another, and the rate of switching is proportional to the difference in these marginal benefits.

This result shows that the second derivative depends on how firms switch between permits at the boundaries of the demand sets, weighted by the density of firms at those boundaries. This switching behavior is driven by the relative benefits firms derive from different permits, as captured by the gradients of their value functions $V(x,y_i)$ and $V(x,y_j)$.

These techniques are used by De Gournay et al. (2019) and Kitagawa et al. (2019) to compute the second-order partial derivatives of functions with moving domains in an optimal transport problem between a continuous measure and weighted sums of Dirac measures. By applying these results to the multi-pollutant cap-and-trade market problem,[28] the second-order partial derivatives of $W(p)$ are obtained as follows:

$$\frac{\partial^2 W}{\partial p_i \partial p_j}(p) = -\int_{D_i(Y,p_i)\cap D_j(Y,p_j)} \frac{f(x)}{\left\|\nabla_x V(x,y_i) - \nabla_x V(x,y_j)\right\|} d\mathcal{H}(x) \leq 0 \quad (i \neq j), \tag{63}$$

and

$$\frac{\partial^2 W}{\partial p_j^2}(p) = -\sum_{l \neq j} \frac{\partial^2 W}{\partial p_l \partial p_j}(p) \geq 0. \tag{64}$$

Here, $d\mathcal{H}(x)$ is the $N-1$-dimensional Hausdorff measure on the boundary of the demand sets,[29] and $f(x)$ is the density function associated with the mass measure of polluting firms μ.

The Hessian matrix of $W(p)$ is symmetric. Moreover, since the mixed partial derivatives of W are negative while all main diagonal entries are positive, the Hessian matrix of W is positive semi-definite, implying that $W(p)$ is a convex multivariate function in p.

Upon differentiation of (12) with respect to p_i, it becomes evident that

$$\frac{\partial \mu(D_j(Y,p))}{\partial p_i} = -\frac{\partial^2 W}{\partial p_i \partial p_j}(p) \geq 0. \tag{65}$$

This result shows that an increase in the price of permit p_i leads to an increase in the measure of firms assigned to permit p_j, reflecting the substitution effect between permits.

[28] When applying the findings from De Gournay et al. (2019) to our multi-pollutant problem, we leverage the principle that seeking the maximum value of a payoff function is equivalent to seeking the minimum value of the negation of that payoff function. This concept is often employed to convert problems centered around max functions into equivalent problems involving min functions, and vice versa.

[29] The intersection of two demand sets, $D_i(Y,p)\cap D_j(Y,p)$, in a two-dimensional pollution space ($N=2$) is typically a line segment or an empty set, depending on the geometric arrangement of the demand sets. In higher-dimensional pollution spaces ($N>2$), the intersection can form more complex geometric shapes, determined by the matching function and the distribution of heterogeneous firms within the multidimensional space.

Similarly, taking derivatives with respect to p_j on both sides of (12) leads to

$$\frac{\partial^2 W}{\partial p_j^2}(p) = -\frac{\partial \mu(D_j(Y,p))}{\partial p_j}. \tag{66}$$

Given that the left-hand side of (66) is non-negative (as per (64)), we deduce that

$$\frac{\partial \mu(D_j(Y,p))}{\partial p_j} \leq 0. \tag{67}$$

This implies that an increase in the price of permit p_j reduces the measure of firms assigned to that permit, consistent with the law of demand.

Finally, utilizing the symmetry relationship in the second partial derivatives:

$$\frac{\partial^2 W}{\partial p_i \partial p_j}(p) = \frac{\partial^2 W}{\partial p_j \partial p_i}(p), \tag{68}$$

we infer from (59) that:

$$\frac{\partial \mu(D_j(Y,p))}{\partial p_i} = \frac{\partial \mu(D_i(Y,p))}{\partial p_j}. \tag{69}$$

This symmetry reflects the reciprocal nature of substitution effects between permits in the multi-pollutant cap-and-trade market.

C Proof of the Convexity of the Objective Function $F(\psi)$ and Existence of Optimal Shadow Prices in the Dual Formulation of the Social Planner Problem

The dual objective function is given by:

$$F(\psi) = \sum_{j=1}^{M} \int_{D_j} V(x,y_j) f(x)\, dx + \sum_{j=1}^{M} \left(S_j - \mu(D_j)\right) \psi_j, \tag{70}$$

where the demand set $D_j(Y,\psi)$ is defined as:

$$D_j(Y,\psi) = \left\{ x \in \Omega : V(x,y_j) - \psi_j = \max_{k=1,\ldots,M} \{V(x,y_k) - \psi_k\} \right\},$$

and $\Omega = \bigcup_{j=1}^{M} D_j(Y,\psi)$.

To analyze the convexity of $F(\psi)$, we introduce a function $\tilde{W}$ defined as:

$$\tilde{W}(A,\psi) = \int_{\Omega} \left(V(x,y_{A(x)}) - \psi_{A(x)}\right) d\mu(x), \tag{71}$$

where $A : \Omega \to Y = \{y_1,\ldots,y_M\}$ is a surjective assignment mapping each firm $x \in \Omega$ to a permit y_j. Let $A^{-1}(\{j\}) = \{x \in \Omega : A(x) = j\}$ denote the set of firms assigned to permit y_j. Using this

notation, $\tilde{W}$ can be rewritten as:

$$\begin{aligned}\tilde{W}(A,\psi) &= \sum_j \int_{A^{-1}(\{j\})} \big(V(x,y_j) - \psi_j\big)\, d\mu(x) \\ &= \sum_j \int_{A^{-1}(\{j\})} V(x,y_j)\, d\mu(x) - \sum_j \psi_j \mu\left(A^{-1}(\{j\})\right). \end{aligned} \tag{72}$$

Here, the first term is independent of ψ, while the second term is linear in ψ. Thus, for a fixed assignment A, the function $\psi \mapsto \tilde{W}(A,\psi)$ is affine (linear plus a constant).

Next, we define the optimal assignment A^*_ψ as the assignment that maximizes the integrand $V(x,y_j) - \psi_j$ for each $x \in \Omega$. Formally:

$$A^*_\psi(x) = \arg\max_{j=1,\dots,M} \left[V(x,y_j) - \psi_j\right]. \tag{73}$$

By construction, A^*_ψ satisfies:

$$V(x,y_{A^*_\psi(x)}) - \psi_{A^*_\psi(x)}) \geq V(x,y_{A(x)}) - \psi_{A(x)}, \tag{74}$$

for any assignment A. Integrating this inequality over Ω yields:

$$\int_\Omega \left[V(x,y_{A^*_\psi(x)}) - \psi_{A^*_\psi(x)})\right] f(x)\, dx \geq \int_\Omega \left[V(x,y_{A(x)}) - \psi_{A(x)})\right] f(x)\, dx. \tag{75}$$

This implies:

$$\tilde{W}(A^*_\psi,\psi) \geq \tilde{W}(A,\psi), \tag{76}$$

for all assignments A. Therefore, $\tilde{W}(A^*_\psi,\psi)$ is the supremum of $\tilde{W}(A,\psi)$ over all possible assignments A:

$$\tilde{W}(A^*_\psi,\psi) = \sup_A \tilde{W}(A,\psi).$$

Since $\tilde{W}(A,\psi)$ is affine in ψ for each fixed A, the function $\psi \mapsto \tilde{W}(A^*_\psi,\psi)$ is the upper envelope of a family of affine functions. By a standard result in convex analysis, the upper envelope of affine functions is convex. Thus, $\psi \mapsto \tilde{W}(A^*_\psi,\psi)$ is convex.

Now, observe that the demand set $D_j(Y,\psi)$ is precisely the set of firms assigned to permit y_j under the optimal assignment A^*_ψ:

$$D_j(Y,\psi) = {A^*_\psi}^{-1}(j). \tag{77}$$

This allows us to express $W(\psi)$, the first term in the dual objective $F(\psi)$, as:

$$W(\psi) = \sum_{j=1}^M \int_{D_j} V(x,y_j) f(x)\, dx = \tilde{W}(A^*_\psi,\psi).$$

Since $\tilde{W}(A^*_\psi,\psi)$ is convex in ψ, and the second term in $F(\psi)$ is linear in ψ, the dual objective $F(\psi)$

is the sum of a convex function and a linear function:

$$F(\psi) = W(\psi) + \sum_{j=1}^{M} S_j \psi_j = \tilde{W}(A^*_\psi, \psi) + \sum_j S_j \psi_j. \tag{78}$$

As the sum of a convex function and a linear function is convex, $F(\psi)$ is convex.

Finally, the existence of a solution to the dual Kantorovich problem follows from the convexity of $F(\psi)$ and the existence of shadow prices $\psi = \{\psi_1, \ldots, \psi_M\}$ that minimize $F(\psi)$. This completes the proof.

D Stochastic Gradient Descent Algorithm for computing prices and demand partition in multi-pollutant cap-and-trade market

Algorithm 1 Algorithm for multi-pollutant cap-and-trade markets

Require: Ω represents the multi-pollutant space, $f(x)$ is the density distribution of polluting firms. Define weights $S_1,\ldots,S_M$ such that $S_j > 0$ and $\sum_{j=1}^{M} S_j = 1 = \int_\Omega f(x)dx$, associated with the multi-pollutant permits $y_1,\ldots,y_M$ in Ω. We also consider the convergence tolerance ε, the total number of iterations K that required to satisfy an error tolerance, and a learning rate η.

Ensure: Prices $(\psi_1,\ldots,\psi_M)$ for the demand sets $\{D_j\}_{j=1}^{M}$ such that each demand set D_i has volume size S_i with up to ε relative error, i.e., $\frac{|\mu(D_i)-S_j|}{S_j} < \varepsilon$ for all $j \in \{1,\ldots,M\}$.

1: **Initialisation:** The M multi-pollutant permits $y_1,\ldots,y_M$ in Ω are fixed. Set shadow prices to zero: $\psi^{(0)} = (\psi_1^{(0)},\ldots,\psi_M^{(0)}) = 0$.

2: **for** $k = 1,\ldots,K$ **do**

3: **Optimization using SGD method:**

4: Use a numerical optimization method to find $\psi^{(k)} = (\psi_1^{(k)},...,\psi_M^{(k)})$ that minmizes the dual objective functional shown in (40):

$$\tilde{F}_k(\psi_1,...,\psi_M) = \sum_{i=1}^{M}(S_j - \mu(D_j))\psi_j + \sum_{j=1}^{M}\int_{D_j} V(x,y_j)f(x)dx$$

where D_j is the demand set generated by the multi-pollutant points $(y_j)_{j=1}^{M}$ and current shadow prices $(\psi^{(k-1)} = (\psi_1^{(k-1)},...,\psi_M^{(k-1)})$.

5: The gradient of $\tilde{F}$ w.r.t ψ_j is:

$$\frac{\partial \tilde{F}_k}{\partial \psi_j} = S_i - \mu(D_j) \quad j = 1,...,M$$

Denote this gradient as $\nabla_{\psi_i}\tilde{F}_k$.

6: In SGD, instead of using all the demand sets for gradient computation, randomly select a single demand set $D_j^{(sample)}$. Thus, $\nabla_{\psi_j}\tilde{F}_k^{(sample)} = S_j - \mu(D_j^{(sample)}$ is the gradient of $\tilde{F}_k$ w.r.t ψ_j using this randomly selected demand set from the sample.

7: Update shadow values for permit y_j using:

$$\psi_j^{(k+1)} = \psi_j^{(k)} - \eta\nabla_{\psi_j}\tilde{F}_k^{(sample)}$$

or

$$\psi_j^{(k+1)} = \psi_j^{(k)} - \eta\left(S_j - \mu(D_j^{(sample)}\right)$$

where $D_j^{(sample)}$ is the demand set randomly selected from the sample, and η is the learning rate.

8: Continue until stopping the error tolerance $\frac{|\mu(D_j)-S_j|}{S_j} < \varepsilon$ for all $j \in \{1,\ldots,M\}$ is met.

9: **end for**

10: **Return:** Output the computed shadow prices $\{\psi_j\}_{j=1}^{M} = \{\psi_j^{(K)}\}_{j=1}^{M}$.

The algorithm employs a stochastic gradient descent method to iteratively update the shadow prices of permits, utilizing the stochastic gradient descent method (SGD). The algorithm dynamically

adapts the demand partition for multi-pollutant permits. This iterative process persists until the shadow prices converge, a condition determined by the termination threshold ε. The final demand sets computed during the last iteration signify the optimal partition of polluting firms.

References

ABDELLAOUI, T. (1998): "Optimal solution of a Monge-Kantorovitch transportation problem," *Journal of Computational and Applied Mathematics*, 96, 149–161.

ABDELLAOUI, T. AND H. HEINICH (1999): "Caractérisation d'une solution optimale au problème de Monge-Kantorovitch," *Bulletin de la Société Mathématique de France*, 127, 429–443.

AMBEC, S. AND J. CORIA (2018): "Policy Spillovers in the Regulation of Multiple Pollutants," *Journal of Environmental Economics and Management*, 87, 114–134.

AURENHAMMER, F., F. HOFFMANN, AND B. ARONOV (1998): "Minkowski-type theorems and least-squares clustering," *Algorithmica*, 20, 61–76.

BANSIL, M. AND J. KITAGAWA (2022): "Quantitative Stability in the Geometry of Semi-discrete Optimal Transport," *International Mathematics Research Notices*, 2022, 7354–7389.

BILLIONNET, C., D. SHERRILL, I. ANNESI-MAESANO, AND G. STUDY (2012): "Estimating the Health Effects of Exposure to Multi-Pollutant Mixture," *Annals of Epidemiology*, 22, 126–141.

BOURNE, D., C. EGAN, B. PELLONI, AND M. WILKINSON (2022): "Semi-discrete optimal transport methods for the semi-geostrophic equations," *Calculus of Variations and Partial Differential Equations*, 61, 39.

BOURNE, D., P. KOK, S. ROPER, AND W. SPANJER (2020): "Laguerre tessellations and polycrystalline microstructures: a fast algorithm for generating grains of given volumes," *Philosophical Magazine*, 100, 2677–2707.

BRENIER, Y. (1987): "Décomposition polaire et réarrangement monotone des champs de vecteurs," *C.R. Acad. Sci. Paris, Sér I Math.*, 305, 805–808.

——— (1991): "Polar factorization and monotone rearrangement of vector-valued functions," *Communications on Pure and Applied Mathematics*, 44, 375–417.

BURTRAW, D., A. KRUPNICK, K. PALMER, A. PAUL, M. TOMAN, AND C. BLOYD (2003): "Ancillary benefits of reduced air pollution in the US from moderate greenhouse gas mitigation policies in the electricity sector," *Journal of Environmental Economics and Management*, 45, 650–673.

CAREY, M. (1977): "Integrability and Mathematical Programming Models: A Survey and a Parametric Approach," *Econometrica*, 45, 1957–1976.

CARLIER, G. (2003): *Duality and existence for a class of mass transportation problems and economic applications*, Tokyo: Springer Japan, 1–21.

CHIAPPORI, P., R. MCCANN, AND L. NESHEIM (2010): "Hedonic price equilibria, stable matching, and optimal transport: Equivalence, topology, and uniqueness," *Economic Theory*, 42, 317–354.

COHAN, D. S., J. W. BOYLAN, A. MARMUR, AND M. N. KHAN (2007): "An integrated framework for multipollutant air quality management and its application in Georgia," *Environmental Management*, 40, 545–554.

COUNCIL, N. R. (2004): *Air Quality Management in the United States*, National Academies Press.

DALES, J. (1974): *Pollution, property & prices.*, Canadian university paperbooks, University of Toronto Press.

DE GOURNAY, F., J. KAHN, AND L. LEBRAT (2019): "Differentiation and regularity of semi-discrete optimal transport with respect to the parameters of the discrete measure," *Numerische Mathematik*, 141, 429–453.

DOMINICI, F., R. PENG, C. BARR, , AND M. BELL (2010): "Protecting human health from air pollution: shifting from a single-pollutant to a multipollutant approach," *Epidemiology*, 21, 187–194.

EKELAND, I. (2010): "Notes on Optimal Transportation," *Economic Theory*, 42, 437–459.

EKELAND, I., J. J. HECKMAN, AND L. NESHEIM (2004): "Identification and Estimation of Hedonic Models," *Journal of Political Economy*, 112, S60–S109.

EPPLE, D. (1987): "Hedonic prices and implicit markets: Estimating demand and supply functions for differentiated products," *Journal of Political Economy*, 95, 59–80.

FANN, N., H. ROMAN, C. FULCHER, M. GENTILE, B. HUBBELL, K. WESSON, AND J. LEVY (2011): "Maximizing health benefits and minimizing inequality: incorporating local-scale data in the design and evaluation of air quality policies," *Risk analysis: an official publication of the Society for Risk Analysis*, 31, 908–922.

FIGALLI, A., R. KIM, AND R. MCCANN (2011): "When is multidimensional screening a convex program?" *Journal of Economic Theory*, 146, 454–478.

FISCHER, C. AND G. JACOBSEN (2021): "The Green New Deal and the Future of Carbon Pricing," *Journal of Policy Analysis and Management*, 40, 988–995.

GALICHON, A. (2016): *Optimal Transport Methods in Economics*, Princeton University Press.

——— (2021a): "A survey of some recent applications of optimal transport methods to econometrics," .

——— (2021b): "The unreasonable effectiveness of optimal transport in economics," Papers, arXiv.org.

GANGBO, W. AND R. MCCANN (1996): "The Geometry of Optimal Transportation," *Acta Math.*, 177, 113–161.

GANGBO, W. AND R. J. MCCANN (1995): "Optimal Maps in Monge's Mass Transport Problem," *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 321, 1653–1658.

GENEVAY, A., M. CUTURI, G. PEYRÉ, AND F. BACH (2016): "Stochastic optimization for large-scale optimal transport," in *Advances in Neural Information Processing Systems*, 3440–3448.

GODZINSKI, A. AND M. S. CASTILLO (2021): "Disentangling the Effects of Air Pollutants with Many Instruments," *Journal of Environmental Economics and Management*, 109, 102489.

GREENBAUM, D. AND R. SHAIKH (2010): "First steps toward multipollutant science for air quality decisions," *Epidemiology*, 21, 195–197.

HIDY, G. AND W. PENNELL (2010): "Multipollutant air quality management," *Journal of the Air & Waste Management Association*, 60.

HOTELLING, H. (1938): "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates," *Econometrica*, 6, 242–269.

HU, J., Q. YING, Y. WANG, AND H. ZHANG (2015): "Characterizing multi-pollutant air pollution in China: Comparison of three air quality indices," *Environment International*, 84, 17–25.

HUANG, W., Y. ZHOU, X. CHEN, X. ZENG, L. KNIBBS, Y. ZHANG, B. JALALUDIN, S. DHARMAGE, L. MORAWSKA, Y. GUO, X. YANG, L. ZHANG, A. SHAN, J. CHEN, T. WANG, J. HEINRICH, M. GAO, L. LIN, X. XIAO, P. ZHOU, Y. YU, N. TANG, AND G. DONG (2023): "Individual and joint associations of long-term exposure to air pollutants and cardiopulmonary mortality: a 22-year cohort study in Northern China," *The Lancet Regional Health - Western Pacific*, 36, 100776.

KANTOROVICH, L. (1942): "On the translocation of masses," *C.R. (Doklady) Acad. Sci. URSS (N.S.)*, 37, 199–201.

KITAGAWA, J., Q. MÉRIGOT, AND B. THIBERT (2019): "Convergence of a Newton algorithm for semi-discrete optimal transport," *Journal of the European Mathematical Society*, 21, 2603–2651.

KORTENKAMP, A. AND M. FAUST (2018): "Regulate to reduce chemical mixture risk: Regulatory systems must better provide for risks from exposure to multiple chemicals," *Science*, 361, 224.

LEIRIÃO, L., M. DE OLIVEIRA, T. MARTINS, AND S. MIRAGLIA (2023): "A Multi-Pollutant and Meteorological Analysis of Cardiorespiratory Mortality among the Elderly in São Paulo, Brazil - An Artificial Neural Networks Approach," *International Journal of Environmental Research and Public Health*, 20.

MASIOL, M., C. AGOSTINELLI, G. FORMENTON, E. TARABOTTI, AND B. PAVONI (2014): "Thirteen years of air pollution hourly monitoring in a large city: Potential sources, trends, cycles and effects of car-free days," *Science of the Total Environment*, 494-495, 84–96.

MCFADDEN, D. (1974): "Conditional logit analysis of qualitative choice behavior," *Frontiers in econometrics*, 105, 105.

MÉRIGOT, Q., A. DELALANDE, AND F. CHAZAL (2020): "Quantitative stability of optimal transport maps and linearization of the 2-Wasserstein space," in *Proceedings of the Twenty Third International Conference on Artificial Intelligence and Statistics*, ed. by S. Chiappa and R. Calandra, PMLR, vol. 108 of *Proceedings of Machine Learning Research*, 3186–3196.

MONGE, G. (1781): "Mémoire sur la théorie des déblais et des remblais," *Histoire de l'Académie Royale des Sciences de Paris, avec les Mémoires de Mathématique et de Physique pour la même année*, 666–704.

MONTERO, J. (2001): "Multipollutant Markets," *The RAND Journal of Economics*, 32, 762–774.

MOULIN, H. AND Y. SPRUMONT (2007): "Air Allocation of Production Externalities: Recent Results," *Revue d'économie politique*, 117, 7–36.

NEWELL, R., W. A. PIZER, AND D. RAIMI (2014): "Carbon Markets: Past, Present, and Future," *Annual Review of Resource Economics*, 6, 191–215.

OLIKER, I. AND D. PRUSSNER (1989): "On the numerical solution of the equation $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = f$ and its discretizations, I," *Numerische Mathematik*, 54, 271–293.

OSTROM, E. (2011): "Reflections on "Some Unsettled Problems of Irrigation"," *American Economic Review*, 101, 49–63.

PEYRÉ, G., M. CUTURI, ET AL. (2019): "Computational optimal transport," *Foundations and Trends® in Machine Learning*, 11, 355–607.

PHANEUF, D. (2013): "Heterogeneity in Environmental Demand," *Annual Review of Resource Economics*, 5, 227–244.

ROCHET, J.-C. AND P. CHONÉ (1998): "Ironing, sweeping, and multidimensional screening," *Econometrica*, 66, 783–826.

ROSEN, S. (1974): "Hedonic prices and implicit markets: Product differentiation in pure competition," *Journal of Political Economy*, 82, 34–55.

SAMUELSON, P. (1950): "The Problem of Integrability in Utility Theory," *Economica*, 17, 355–385.

SILBERBERG, E. (1972): "Duality and the Many Consumer's Surpluses." *The American Economic Review*, 62, 942–952.

STAFOGGIA, M., S. BREITNER, H. R., AND X. BASAGAÑA (2017): "Statistical Approaches to Address Multi-Pollutant Mixtures and Multiple Exposures: The State of the Science," *Current Environmental Health Reports*, 4, 481–490.

STAVINS, R. N. (2001): *Experience with Market-Based Environmental Policy Instruments*, Amsterdam: North-Holland/Elsevier Science.

SUN, J., J. SCHREIFELS, J. WANG, J. FU, AND S. WANG (2014): "Cost Estimate of multi-pollutant abatement from the power sector in the Yangtze River Delta region of China," *Energy Policy*, 69, 478–488.

U.S. ENVIRONMENTAL PROTECTION AGENCY (2024): "Interim Framework for Advancing Consideration of Cumulative Impacts," Fact Sheet.

VILLANI, C. (2009): *Optimal transport: old and new*, Springer.

WINQUIST, A., E. KIRRANE, M. KLEIN, M. STRICKLAND, L. DARROW, S. SARNAT, K. GASS, J. MULHOLLAND, A. RUSSELL, AND P. TOLBERT (2014): "Joint Effects of Ambient Air Pollutants on Pediatric Asthma Emergency Department Visits in Atlanta, 1998-2004," *Epidemiology*, 25, 666–673.

WOLANSKY, G. (2015): "On semi-discrete Monge–Kantorovich and generalized partitions," *Journal of Optimization Theory and Applications*, 165, 359–384.

WU, J., S. LI, J. DUAN, Y. LI, J. WANG, P. DENG, C. MENG, W. WANG, H. YUAN, Y. LU, M. SHEN, AND Q. ZHAO (2023): "Association of joint exposure to various ambient air pollutants during adolescence with blood pressure in young adulthood," *Journal of Clinical Hypertension*.

YU, L., W. LIU, X. WANG, Z. YE, Q. TAN, W. QIU, X. NIE, M. LI, B. WANG, AND W. CHEN (2022): "A review of practical statistical methods used in epidemiological studies to estimate the health effects of multi-pollutant mixture," *Environmental Pollution*, 306.

ZHU, J., L. CHEN, AND H. LIAO (2022): "Multi-pollutant air pollution and associated health risks in China from 2014 to 2020," *Atmospheric Environment*, 268.