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GENERALIZED PROSPECT THEORY

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Prospect theory (PT) and its cumulative form (CPT) successfully capture behavioral deviations from expected utility theory (EUT), like loss aversion and probability weighting. However, PT faces challenges in dynamic settings, largely due to its descriptive nature and exogenously imposed parameters. This paper introduces a dynamic model, building on Neural Differential Utility (NDU), where loss aversion and a generalized form of probability weighting emerge endogenously from a resource-constrained agent's optimal behavior. The agent balances the benefits of aligning their preferences with the environment against the cognitive and affective costs of preference adjustment. Crucially, CPT is shown to be a special, limiting case of this more general framework. The model offers testable predictions, distinguishing it from CPT, and provides a psychologically grounded microfoundation for reference-dependent preferences. This approach provides steps toward the Generalized Prospect Theory (GPT).

KEYWORDS: Neural Differential Utility, Prospect Theory, Rational Expectations, Endogenous Preferences, Deep Learning Theory, AI-economics.

"*Prospect theory is unlikely to provide an accurate description of decisions made in the wealth frame.*" (Kahneman, 2002, p. 464)

1. INTRODUCTION

Behavioral economics has documented systematic deviations from expected utility theory (EUT), with loss aversion (losses looming larger than equivalent gains) and probability weighting (non-linear transformation of probabilities) being promi-

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nent examples (Kahneman and Tversky (1979), Tversky and Kahneman (1992)). Prospect Theory (PT, Kahneman and Tversky (1979)) and its cumulative form (CPT, Tversky and Kahneman (1992)) successfully capture these deviations. However, while successful in explaining behavioral anomalies in static settings, PT faces challenges in dynamic and complex economic environments (e.g., Novemsky and Kahneman (2005), Camerer (2005)). These challenges stem primarily from PT's descriptive nature, relying on features like a reference point, diminishing sensitivity, and non-linear probability weighting without explaining their underlying mechanisms.

This paper addresses this gap by providing a dynamic model, building on the axiomatic framework of Neural Differential Utility (NDU, Qi (2024)), where both loss aversion and an emergent, generalized probability weighting arise endogenously from a resource-constrained agent's optimal behavior[1] in the face of cognitive and affective costs. This work represents a crucial step towards a **Generalized Prospect Theory (GPT)**. Endogenizing preferences, particularly features like loss aversion, is crucial because:

1. It provides deeper insight into why these behaviors arise (moving beyond mere description).
2. It connects seemingly disparate phenomena (risk aversion, loss aversion, probability weighting) through a common underlying mechanism within a unified axiomatic framework.
3. It leads to more accurate predictions, especially in dynamic settings where preferences may evolve, overcoming limitations of static preferences or exogenously imposed changes.

Our approach, grounded in NDU, is based on the premise that cognitive and affective processes are costly (e.g., Simon (1955), Camerer et al. (2005), Kahneman (2011)). Agents have limited resources for cognitive control (processing information about consumption relative to a reference point) and emotional regulation (managing the affective responses to gains and losses), modeled through a cost function

[1] Qi (2024) provide a formal discussion of the resource-constrained agent. For further details on the foundational concept of resource constraints, please also refer to (e.g., Gershman et al. (2015), Lieder and Griffiths (2020), Bhui et al. (2021)).

that penalizes deviations from a baseline preference setting. The agent actively modulates their response to consumption, balancing the benefits of aligning their preferences with their environment against these costs (Qi (2024)). This connects to a growing literature exploring the cognitive and neural underpinnings of decision-making (e.g., Rangel et al. (2008), Glimcher (2013)) and the impact of emotions on the decision process (e.g., Loewenstein (2001)). Neuroscientific studies provide supporting evidence, revealing asymmetric brain processing of gains and losses, with heightened amygdala activity and increased autonomic arousal during losses (e.g., Kuhnen and Knutson (2005), Yacubian et al. (2006), De Martino et al. (2006), Tom et al. (2007), Sokol-Hessner et al. (2009), Kandel et al. (2013), Frydman et al. (2014)). These findings motivate our incorporation of affective costs. (See Table I for a mapping of NDU to neural functioning.)

TABLE I

NDU and Two Dimensions of Neural Functioning

	Cognitive	**Affective**
Controlled Serial Effortful	I. Deliberate preference formation (e.g., planning, reasoning) Parameters optimization (θ_t)	II. Emotional regulation (e.g., resisting impulses) Parameters optimization (θ_t)
Automatic Parallel Effortless	III. Heuristic judgments, biases (e.g., framing effects) Preference state (u_t) dynamics	IV. Immediate emotional responses (e.g., fear, excitement) Preference state (u_t) dynamics

Note: This table provides a two-dimensional framework (also see Qi (2024)) for understanding the psychological factors that influence preferences, informed by research such as Camerer et al. (2005) and Kahneman (2011). **Controlled** processes are conscious, deliberate, and resource-consuming. **Automatic** processes are fast, intuitive, and operate largely outside of conscious awareness. **Cognitive** influences relate to beliefs and knowledge, **affective** influences relate to emotions and moods. Together, these quadrants play a role in shaping preferences and form the conceptual basis for the NDU. Specifically, controlled processes influence the parameters θ_t of the ResNets, while automatic processes govern the dynamics of the preference state u_t.

1.1. *Generalized Prospect Theory (GPT) and Neural Differential Utility (NDU)*

The main contribution of this paper is to utilize NDU (see Section 2) as a step toward a Generalized Prospect Theory (GPT). We formally define GPT as a dynamic extension of Prospect Theory incorporating:

- **Dynamic Reference Dependence:** The reference point, determined by factors like past consumption and social comparisons, is modeled here using rational expectations for future consumption, emphasizing the agent's rationality.
- **Endogenous Loss Aversion:** Loss aversion arises from the agent's optimal response to affective costs, providing a microfoundation, contrasting with models that assume it exogenously (e.g., Kahneman and Tversky (1979), Barberis et al. (2001), Kőszegi and Rabin (2006), Barberis and Xiong (2012), Pagel (2017)).
- **Generalized Probability Weighting:** The NDU model exhibits behavior qualitatively resembling CPT. The value function's concavity in gains and convexity in losses effectively transforms objective probabilities into subjective decision weights, mirroring CPT's probability weighting. However, the agent uses objective probabilities, $\mathbb{P}$. The distortion arises from the optimal response to preference adjustment costs, reflected in the value function's shape. Proposition 6 confirms that the model does not violate first-order stochastic dominance.
- **CPT as a Special Case:** As shown in Proposition 7, CPT can be viewed as a limiting case of the NDU model. As the cost parameter for losses, λ_a, approaches infinity (with fixed λ_c), the value function converges pointwise to a function with a kink at the reference point, similar to CPT's value function. This provides a theoretical link between GPT and CPT.
- **Role of Cognitive and Affective Costs:** Parameters λ_c and λ_a capture individual differences in risk preferences and responses to gains and losses, allowing for richer heterogeneity than CPT. GPT explicitly incorporates cognitive and affective processes as drivers of preference formation and adjustment, modeled through cost functions influencing the agent's choice of a control variable representing cognitive and affective effort.
- **Rational Expectations:** The model maintains rational expectations. Agents correctly anticipate future outcome distributions and use these objective probabilities. The perceived distortions in probability weighting arise endogenously, not from biased probability perceptions.
- **Testable Predictions:** We derive testable predictions differentiating our model from CPT, including the effects of cognitive load and emotional regulation on risk preferences.

The NDU model provides a specific instantiation of GPT. NDU uses continuous-time Residual Networks (ResNets) to model the dynamics of a latent preference state, u_t, which evolves according to a controlled stochastic differential equation (Forward Propagation), driven by the consumption process and the agent's control, θ_t (representing cognitive/affective effort). The agent's continuation utility, V_s, is governed by a backward stochastic differential equation (Backward Propagation), incorporating an aggregator function, f_{θ_s}, and a cost function, $\mathcal{L}(u_s, \theta_s)$, representing cognitive and affective costs of preference adjustment. The cognitive and affective costs in NDU are informed by neuroeconomic evidence (see Table I).

The key innovation of NDU, in the context of GPT, is that it provides a mechanism through which loss aversion and probability weighting emerge endogenously. The asymmetry in the cost function (Equation 14), specifically affective costs only in the loss domain, leads to a value function exhibiting loss aversion (steeper in the loss domain than the gain domain, as shown in Figure 3). The curvature of this endogenously determined value function then gives rise to emergent probability weighting (Definition 3 and Proposition 4).

Remark 1—Key Departures from CPT: The NDU's critical distinctions, in constructing this GPT, relative to CPT, are:

1. **Endogeneity:** Loss aversion and probability weighting are endogenously determined.
2. **Dynamics:** Preferences change over time, responding to consumption history and the agent's cognitive/affective state. The model allows for dynamic probability weighting (e.g., Gonzalez and Wu (1999)), absent in standard CPT, arising from the interplay of cognitive and affective costs. Section 4 discusses this extensively.
3. **Mechanism:** The model provides a specific mechanism-cognitive and affective costs-explaining why loss aversion and probability weighting arise.
4. **Continuous Time:** Provides more modeling flexibility.

In summary, NDU provides an axiomatic framework for a potential GPT. It offers a dynamic, continuous-time model where key features of prospect theory emerge endogenously from optimal responses to cognitive and affective costs. This

opens new avenues for research, bridging behavioral economics, the neoclassical paradigm, and potentially, AI agent design.

1.2. *Limitations of Prospect Theory and Existing Alternatives*

Traditional models with static preferences, including PT, often fail to capture the complexities of human behavior in dynamic settings. Key critiques of PT, which motivate our approach, include:

- **Descriptive Focus, Normative Ambiguity:** PT's descriptive nature, unlike EUT's axiomatic foundation, creates ambiguity in dynamic contexts, offering limited guidance on optimal dynamic decision-making (Kahneman (2011)). The potential for manipulation of reference points (through framing effects, defaults, etc.-see Tversky and Kahneman (1981), Johnson and Goldstein (2003), Hardisty et al. (2010)) further complicates welfare analysis.[2]
- **Exogenous Reference Point:** PT treats the reference point as exogenous, lacking a universally accepted formation mechanism (e.g., Kőszegi and Rabin (2006, 2007)). This is restrictive in dynamic settings where reference points plausibly evolve endogenously (see Novemsky and Kahneman (2005)).
- **Parameter Instability:** PT parameters (loss aversion, probability weighting) exhibit context dependence (Starmer (2000)), with substantial variation in estimated coefficients (Bruhin et al. (2010), Barberis (2013)), limiting generalizability. Theoretical analysis and empirical evidence also suggest that probability weighting may not always follow the inverse-S shape assumed by CPT (Prelec (1998), Bruhin et al. (2010)).
- **Limited Scope:** PT primarily focuses on single-period decisions, restricting its application to dynamic and strategic environments.[3] It cannot directly address

[2] Prelec (1998)'s probability weighting function, while a significant improvement over earlier formulations, doesn't fully resolve the dynamic issues of PT. It provides a more precise description of how probabilities are distorted at a given point in time, but it doesn't address the fundamental problems of framing, reference point dynamics, and time inconsistency that arise in dynamic contexts. The axioms underlying Prelec's function are primarily focused on static choices.

[3] While research has explored dynamic extensions (e.g., Barberis (2000), Kőszegi and Rabin (2006), Barberis and Xiong (2012), Ingersoll and Jin (2013), Pagel (2017) for theoretical analysis, and Imas (2016) for empirical tests), these often face challenges in maintaining tractability and capturing intertemporal

issues like consumption smoothing or investment under uncertainty. Even in static settings, Kahneman (2002) acknowledge PT's limitations within the wealth frame,[4] a consideration that becomes even more complex dynamically.

- **Cognitive Process "Black Box":** PT describes observed choices but lacks explicit modeling of the underlying cognitive and affective mechanisms. The functional forms are derived from behavior rather than models of cognitive processes like attention or memory (e.g., Busemeyer and Townsend (1993)), restricting extrapolation to contexts with altered cognitive load, time pressure, or novel information (e.g., Simon (1955), Payne (1993), Gabaix (2014)).
- **Alternative Models:** While various alternative models exist, including regret theory (Bell (1982), Loomes and Sugden (1982)), disappointment aversion (Gul (1991)), rank-dependent utility theory (Quiggin (1982)), and salience theory (Bordalo et al. (2012)), they often capture only specific aspects of non-EUT behavior and do not provide a unified framework that endogenizes both loss aversion and generalized probability weighting.

These critiques highlight the need for dynamic preference models that capture the evolving nature of preferences and provide a microfoundation for observed deviations from conventional EUT.

TABLE II

COMPARISON WITH EXISTING LITERATURE

	Reference Point	Loss Aversion	Methodology
Kahneman and Tversky (1979)	Exogenous	Exogenous	Experimental
Barberis et al. (2001)	Return	Exogenous	Recursive Utility
Kőszegi and Rabin (2006)	Expectations	Exogenous	Gain-Loss Utility
Barberis and Xiong (2012)	Wealth	Exogenous	Realization Utility
Pagel (2017)	Expectations	Exogenous	News-Utility
This Model	Expectations	Endogenous	NDU

decision-making under uncertainty. PT's non-expected utility nature also complicates dynamic programming.

[4] Qi (2025) demonstrate GPT's application in consumption-based asset pricing, highlighting its advantage in analyzing decisions within the wealth framework.

Table II compares our model to related existing literature, highlighting our key contribution of endogenizing loss aversion within a dynamic framework. Our work is related to, but distinct from, habit formation models (e.g., Campbell and Cochrane (1999), Constantinides (1990)) and recursive utility (e.g., Kreps and Porteus (1978), Epstein and Zin (1989), Duffie and Epstein (1992)). Habit formation models don't explicitly model cognitive/affective costs, and while recursive utility allows for non-expected utility preferences, it doesn't typically endogenize loss aversion or link to neuroeconomic mechanisms in the same way. Our model can be seen as a generalized form of stochastic differential utility (Duffie and Epstein (1992)), but with a focus on neuroeconomic grounding and the endogenous emergence of behavioral biases.

The paper is organized as follows. Section 2 introduces the NDU. Section 3 introduces the model setup, derives the agent's decision problem, and characterizes the cost function. Section 4 presents the solution to the model, including the value function and optimal effort, and establishes the connection to CPT. Section 5 quantitatively analyzes the model solutions. Section 6 concludes. Proofs are in Appendix. Additional information can be found in the Supplement.

2. PRELIMINARIES

This section introduces the notation, definitions, and assumptions necessary for the Neural Differential Utility (NDU, see Qi (2024)) model.

Notations. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions: the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, with $T > 0$ as the finite time horizon. An agent chooses a consumption path $(c_t)_{t\in[0,T]}$, where $c_t \in \mathcal{C} \subset \mathbb{R}^n$ is the consumption bundle at time t, measured in units of a numeraire good. The agent's utility is represented by a utility function $V : \mathcal{C} \times [0, T] \to \mathbb{R}$, which we aim to approximate using deep learning. We consider a specific class of ResNets architectures with L layers and element-wise activation function $\eta(\cdot)$. Let $\Theta \subset \mathbb{R}^d$ be a compact parameter space. The hypothesis space $\mathcal{H}$ consists of functions parameterized by these ResNets:

DEFINITION 1—Continuous-time ResNets: A continuous-time ResNets is a stochastic process u_t taking values in $\mathbb{R}^q$, driven by an adapted stochastic process $c_t \in \mathcal{C} \subset \mathbb{R}^n$, and parameterized by a measurable, right-continuous, and uniformly bounded parameter function $\theta_t \in \Theta \subset \mathbb{R}^d$. The evolution of u_t is governed by a controlled stochastic differential equation (SDE):

$$du_t = h_{\theta_t}(u_t, c_t)dt + \sigma^h_{\theta_t}(u_t, c_t)d\mathcal{Z}^h_t, \quad u_0 = \phi(c_0), \tag{1}$$

where:

- $h_{\theta_t} : \mathbb{R}^q \times \mathcal{C} \to \mathbb{R}^q$ is a function that specifies the architecture of the network. It is parameterized by θ_t. Economically, h_{θ_t} governs the deterministic evolution of the preference state, capturing how preferences change based on current consumption and the existing preference state.
- $\sigma^h_{\theta_t} : \mathbb{R}^q \times \mathcal{C} \to \mathbb{R}^{q\times m}$ is a progressively measurable and uniformly bounded volatility matrix, with dimension m depending on the dimension of the martingale $\mathcal{Z}^h$. This term captures random shocks on preference evolution.
- $\mathcal{Z}^h_t$ is an m-dimensional martingale process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, representing the source of randomness in preference dynamics.
- $\phi : \mathcal{C} \to \mathbb{R}^q$ determines the initial value of the process u_t based on the initial consumption c_0. This function maps the initial consumption experience to an initial preference state.

The continuous-time ResNets, represented by the flow map Φ_T, models the evolution of an agent's preference state over time. Each infinitesimal transformation of the preference state is defined by the residual block:

$$h_{\theta_t}(u_t, c_t) = A_t\eta(B_t u_t + \Gamma_t c_t + b_t), \tag{2}$$

where $\theta_t = (A_t, B_t, \Gamma_t, b_t)$ with $A_t \in \mathbb{R}^{q\times m}$, $B_t \in \mathbb{R}^{m\times q}$, $b_t \in \mathbb{R}^m$, and $\eta : \mathbb{R}^m \to \mathbb{R}^m$ is an element-wise activation function. This block can be interpreted as a continuous-time transformation of the preference state through m neurons. The matrix $\Gamma_t \in \mathbb{R}^{m\times n}$ introduces a direct dependence of the preference state dynamics on the consumption path. See Qi (2024) for discussions of the activation function.

Definition 2—Neural Differential Utility: Consider an agent with a finite time horizon $T > 0$ operating within a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. The consumption process $c = \{c_t\}_{t\in[0,T]}$, where $c_t \in \mathcal{C}_t \subset \mathbb{R}^n$ and $\mathcal{C}_t$ is a closed, convex, and time-dependent set. We assume that c is progressively measurable with respect to $\{\mathcal{F}_t\}$ and that $\mathbb{E}[\int_0^T \|c_t\|^2 dt] < \infty$. The agent also chooses an $\mathcal{F}_t$-adapted control process $\theta = \{\theta_t\}_{t\in[0,T]}$, where $\theta_t \in \Theta \subset \mathbb{R}^{d_\theta}$ and Θ is a convex, compact set. Let $\mathcal{A}$ denote the space of admissible control processes, i.e., progressively measurable processes with values in Θ, such that $\mathbb{E}\left[\int_0^T \|\theta_t\|^2 dt\right] < \infty$. The agent's continuation utility at time s, denoted by $V_s \in \mathbb{R}$, and her dynamic preference state, $u_t \in \mathbb{R}^q$, are governed by the following FBSDE system:

$$du_t = h_{\theta_t}(u_t, c_t)dt + \sigma^h_{\theta_t}(u_t, c_t)d\mathcal{Z}^h_t, \quad u_0 = \phi(c_0), \qquad \text{(Forward Propagation)}$$

$$dV_s = -[f_{\theta_s}(c_s, u_s, V_s) - \lambda\mathcal{L}(u_s, \theta_s)]ds + \sigma_s d\mathcal{Z}_s, \quad V_T = 0, \qquad \text{(Backward Propagation)}$$

where $\theta_t = (A_t, B_t, \Gamma_t, b_t)$ with $A_t \in \mathbb{R}^{q\times m}$, $B_t \in \mathbb{R}^{m\times q}$, $b_t \in \mathbb{R}^m$, and $\Gamma \in \mathbb{R}^{m\times n}$. The function $h_{\theta_t}(u_t, c_t)$ is defined in (2). $\mathcal{Z}_t$ and $\mathcal{Z}^h_t$ are square-integrable $\{\mathcal{F}_t\}$-martingales, σ_t and $\sigma^h_{\theta_t}$ are adapted processes. The other components are:

- $f_{\theta_t}(c_t, u_t, V_t)$: The aggregator function, parameterized by ResNets. We assume f is Lipschitz continuous in (c, u, V) for each fixed θ and continuous in θ for fixed (c, u, V).
- $\phi : \mathcal{C}_0 \to \mathbb{R}^q$: A Lipschitz continuous function mapping the initial consumption c_0 to the initial preference state u_0.
- $\lambda \geq 0$: A scaling parameter representing the weight of cognitive and affective preference adjustment costs.
- $\mathcal{L}(u_t, \theta_t)$: The cognitive and affective preference adjustment costs function. We primarily focus on a quadratic cost function: $\mathcal{L}(u_t, \theta_t) = \|\theta_t - \theta_0\|^2$, where θ_0 represents a baseline or default control strategy.[5]

[5]This baseline reflects the agent's preference evolution in the absence of active intervention. The quadratic cost function penalizes deviations from this baseline, with larger deviations incurring higher costs. This form captures the concept of increasing marginal cost while maintaining analytical tractability.

Optimization Problem. The agent's optimization problem at time t is given by:

$$V_t(c;\theta) = \sup_{\theta \in \mathcal{A}_T} \mathbb{E}\left[\int_t^T \left(f_{\theta_s}(c_s, u_s, V_s) - \lambda \mathcal{L}(u_s, \theta_s)\right) ds \middle| \mathcal{F}_t\right], \tag{3}$$

subject to (Forward Propagation), (Backward Propagation).

Key Insights. For illustrative purposes only (see Qi (2024)), we now consider a highly simplified, linear version of the model. This simplification does not apply to the general results presented later.

- **Forward Propagation: Forming Current Preference State u_t.**

$$u_t = \theta_t c_t. \qquad \text{(Linear Forward Propagation)}$$

- **Backward Propagation: Preference Adjustment and Expected Utility.**

$$dV_s = -\left[\theta_s c_s - \beta V(c_s) - \frac{\lambda c_s \theta_s^2}{2}\right] ds + \sigma_s d\mathcal{Z}_s, \qquad \text{(Linear Backward Propagation)}$$

Proposition 1—Endogenous Risk Aversion: *Consider the NDU framework as defined by equations* (Linear Forward Propagation) *and* (Linear Backward Propagation). *An agent maximizes expected lifetime utility by dynamically choosing the preference parameter sequence $\{\theta_s\}_{s\in[t,\infty)}$ from time t onward. The consumption (data generating) process, c_t, follows the geometric Brownian motion:*

$$dc_t = \mu c_t dt + \sigma c_t d\mathcal{Z}_t, \quad c_0 > 0, \tag{DGP}$$

where $\mu \in \mathbb{R}$ denotes the growth rate, $\sigma > 0$ is the volatility, and $\mathcal{Z}_t$ is a standard Brownian motion. The agent's objective is to find the optimal policy $\{\theta_s^*\}_{s\in[t,\infty)}$ to maximize the objective function starting from time t:

$$\sup_{\{\theta_s\}_{s\in[t,\infty)}} \mathbb{E}_t\left[\int_t^\infty e^{-\beta(s-t)}\left(\theta_s c_s - \frac{1}{2}\lambda c_s \theta_s^2\right) ds\right], \tag{4}$$

where $\beta > 0$ is the discount rate, $\lambda > 0$ is the preference adjustment costs parameter. Assume the value function $V(c)$ satisfies $V(1) = 0$ to uniquely identify the solution and the transversality condition $\lim_{t\to\infty} \mathbb{E}_t[e^{-\beta(s-t)}V(c_s)] = 0$. The Hamilton-Jacobi-Bellman (HJB) equation is given by

$$0 = \sup_{\theta} \left\{ \theta c - \frac{1}{2}\lambda c\theta^2 - \beta V(c) + \mu c V'(c) + \frac{1}{2}\sigma^2 c^2 V''(c) \right\}. \tag{5}$$

Then, under the condition $\beta > \mu$, the optimal preference parameter is given by $\theta^*(c) = \frac{1}{\lambda}$, and the value function is:

$$V(c) = \frac{c - c^\gamma}{2\lambda(\beta - \mu)}, \quad \textit{with} \quad \gamma = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\beta}}{\sigma^2} < 0. \tag{6}$$

The coefficient of relative risk aversion, $\mathcal{R}(c) = -\frac{cV''(c)}{V'(c)}$, approaches $1 - \gamma > 1$ as $c \to 0$, indicating significant risk aversion, and $\mathcal{R}(c) \to 0$ as $c \to \infty$. To better understand the value function in equation (6), we define $u(c) = V'(c)$, where $u(c) = \frac{1-\gamma c^{\gamma-1}}{2\lambda(\beta-\mu)}$. Analyzing $u(c)$'s derivatives, we obtain a local measure of relative risk aversion: $\rho = -c \cdot \frac{u''(c)}{u'(c)} = 2 - \gamma$. This suggests a linkage to Constant Relative Risk Aversion (CRRA) utility.

PROOF: See Proof of Proposition 1 in Qi (2024). *Q.E.D.*

3. ENDOGENOUS LOSS AVERSION

3.1. *Model Setup with Normalized Consumption*

The model setup is a direct extension of Proposition 1, with two key differences: (i) the agent is concerned about expected consumption as a reference point, and (ii) she incurs an additional affective preference adjustment costs when consumption falls below this reference point, as described below. Recall the consumption process c_t at time t in Equation (DGP):

$$dc_t = \mu c_t dt + \sigma c_t d\mathcal{Z}_t, \quad c_0 > 0, \tag{7}$$

The reference point, $\bar{c}_t$, is the rational expectation of consumption at time t, conditional on information available at time 0:

$$\bar{c}_t = \mathbb{E}[c_t|\mathcal{F}_0] = c_0 e^{\mu t}. \tag{8}$$

This specification links the reference point to agent's forward-looking rational expectations about future consumption, aligning with the rational expectations paradigm prevalent in macroeconomic modeling (e.g., Lucas Jr (1972)) and observable trends such as those in the U.S. Bureau of Economic Analysis (BEA) data on personal consumption expenditures.[6]

Following Kőszegi and Rabin (2006), Pagel (2017), we define the normalized consumption process, x_t, as the percentage deviation of consumption from the reference point (Barberis and Xiong (2012)):[7]

$$x_t = \frac{c_t}{\bar{c}_t} = \frac{c_t}{c_0 e^{\mu t}}. \tag{9}$$

x_t represents the percentage deviation of consumption from its expected trajectory. $x_t > 1$ is a consumption "gain," $x_t < 1$ a "loss," and $x_t = 1$ means consumption is at the expected level. This captures diminishing sensitivity: a fixed absolute deviation is a larger percentage deviation when expected consumption is lower, aligning with Weber's Law (Fechner (1860)) and the concavity of standard utility functions. The

[6] Rational expectations are a natural choice in this setting as individuals are assumed to form expectations about future consumption based on available information and the underlying economic model. This contrasts with adaptive expectations formulations, where the reference point would be a backward-looking weighted average of past consumption (e.g., Friedman and Becker (1957)), potentially failing to capture the forward-looking nature of consumption decisions in a growing economy where individuals anticipate future income and consumption growth. Using forward-looking expectations also ensures internal consistency within the model.

[7] **Alternative Reference Points:** While we focus on rational expectations, we acknowledge that alternative reference point specifications could lead to different results. Social comparisons (Abel (1990)) might introduce strategic interactions between agents, while past consumption peaks (Constantinides (1990)) could lead to habit formation effects. However, the core mechanism of endogenous loss aversion, driven by the asymmetry in cognitive and affective costs, would likely persist even with these alternative specifications, though the precise shape of the value function and the degree of loss aversion might change. Such extensions might lead to interesting interactions between individual preferences and aggregate dynamics. However, we focus on rational expectations for clarity and consistency with standard macroeconomic models.

proportional normalization is intuitive, consistent with empirical findings (Heath et al. (1999)), and offers a tractable analytical framework, leading to a stationary normalized consumption process.[8]

3.2. *Proportional Normalization*

Applying Itô's Lemma to the normalized consumption process, we find that its dynamics are governed by:

$$dx_t = \sigma x_t d\mathcal{Z}_t, \quad x_t > 0. \tag{10}$$

The zero drift reflects that x_t is a martingale under rational expectations. The volatility term, σx_t, implies that the volatility of normalized consumption is proportional to its level. This result is important because it simplifies the analysis, allowing us to focus on a time-homogeneous problem. The lack of mean reversion in x_t is crucial. Deviations from expected consumption can be persistent, leading to sustained periods of perceived gains or losses. This persistence is essential for generating endogenous loss aversion, as past experiences have a lasting impact. This contrasts with models with habit formation, where consumption tends to revert to a habit level.[9]

[8] The proportional normalization is chosen for its intuitive interpretation and its consistency with empirical findings suggesting that individuals evaluate outcomes relative to a reference level (see, e.g., Heath et al. (1999)). It also offers a more tractable analytical framework compared to alternative specifications like a simple difference ($c_t - \bar{c}_t$). The advantage of proportional normalization lies in the fact that the percentage deviation provides a scale-invariant measure of the consumption gap, making it suitable for comparisons across different levels of expected consumption. Moreover, it leads to a stationary normalized consumption process, which simplifies the analysis. While other normalizations are possible, they might introduce additional complexities. For example, a simple difference normalization would lead to a non-stationary process with a time-varying drift, making it more difficult to derive closed-form solutions.

[9] The advantage of this formulation is its straightforward interpretation in terms of proportional deviations (see Barberis and Xiong (2012) and Ingersoll and Jin (2013)) from the reference point.

3.3. *Agent's Decision Problem*

The agent's preference state is represented by a one-layer ResNets with a linear activation function, $u_t = \theta_t x_t$, where θ_t is a control variable reflecting cognitive and affective effort.[10] The agent faces cognitive and affective costs, $\mathcal{L}(u_t, \theta_t)$. The value function, $V(x)$, is:

$$V(x) = \sup_{\theta \in \mathcal{A}} \mathbb{E}\left[\int_t^\infty e^{-\beta(s-t)} \left(\theta_s x_s - \mathcal{L}(\theta_s, x_s) \right) ds \middle| x_t = x \right], \tag{11}$$

where $\beta > 0$ is the subjective discount rate. $\mathcal{L}(\theta_s, x_s)$ represents the cognitive and affective costs (detailed below). θ_t is a progressively measurable control process taking values in $\mathbb{R}$ and belonging to the admissible set $\mathcal{A}$, defined as:

$$\mathcal{A} = \{\theta : [0, \infty) \to \mathbb{R} \mid \theta \text{ is } \mathcal{G}_t\text{-progressively measurable}\},$$

and satisfying $\mathbb{E}\left[\int_0^T |\theta_s|^2 ds\right] < \infty$ for all $T > 0$. The parameter space for β, σ, λ_c, and λ_a is $\beta, \sigma, \lambda_c, \lambda_a \in (0, \infty)$. The value function evolves according to the following BSDE:

$$dV(x_t) = -\left(\theta_t x_t - \mathcal{L}(\theta_t, x_t) - \beta V(x_t) \right) dt + \xi_V(t) d\mathcal{Z}_t, \tag{12}$$

where $\xi_V(t) = \sigma x_t V_x(x_t)$. This leads to the HJB equation:

$$\beta V(x) = \sup_\theta \left\{ \theta x - \mathcal{L}(\theta, x) + \frac{1}{2}\sigma^2 x^2 V_{xx}(x) \right\}. \tag{13}$$

3.4. *Economic Interpretation of θ_t and Effort Costs*

The control variable θ_t represents the agent's cognitive and affective effort at time t, modulating the utility derived from normalized consumption, x_t. A higher

[10] This specification can be interpreted as a simplified representation of a more complex neural network that maps consumption deviations to utility. While a one-layer network with a linear activation function is a simplification, it captures the essential feature of preference adjustments being driven by deviations from the reference point.

θ_t amplifies the experienced utility. We interpret θ_t as encompassing two related aspects,[11] drawing upon dual-process theories (e.g., Kahneman (2011)):

- *Cognitive Effort:* Mental resources devoted to processing information about consumption relative to the reference point. This corresponds to "System 2" processes (Kahneman (2011)). Higher effort involves closer attention, more careful comparison, and more deliberate planning. Increased cognitive effort intensifies the experienced utility by drawing more attention to the consumption experience. This can be interpreted as an attention cost, where paying more attention to consumption requires scarce mental resources (e.g., Sims (2003)).
- *Affective Effort (Emotion Regulation):* Effort exerted to regulate the emotional response to consumption outcomes (Gross and John, 2003, Gross et al., 2014). This involves actively savoring gains (increasing θ_t when $x_t > 1$) or mitigating the pain of losses (adjusting θ_t when $x_t < 1$). This represents the effort of managing the feeling associated with the consumption outcome. This can be linked to the concept of self-control, where effort is exerted to align immediate experiences with long-term goals (e.g., Bratslavsky et al. (1998)).

Remark 2: The agent faces a fundamental trade-off: increasing θ_t enhances the utility derived from consumption, but it also incurs cognitive and affective costs. The optimal level of effort balances these costs and benefits. This is a key departure from standard models, where utility is a direct, immutable function of consumption. Here, utility is actively constructed by the agent through cognitive and affective effort. This trade-off is central to generating endogenous loss aversion.

[11] It is important to acknowledge that the interpretation of θ_t as a "meta-level" control is a simplification. Real cognitive and affective processes are likely more complex and involve multiple interacting mechanisms. However, this abstraction allows us to capture the essential feature of endogenous preference modulation within a tractable mathematical framework. The agent's choice of θ_t can also be interpreted in terms of self-control or willpower, where effort is exerted to align immediate experiences with long-term goals.

3.5. *Asymmetric Cognitive and Affective Costs*

We introduce a cost function that captures both cognitive and affective components associated with adjusting preferences:

$$\mathcal{L}(\theta_t, x_t) = \underbrace{\frac{\lambda_c x_t}{2}\theta_t^2}_{\text{Cognitive Costs}} + \underbrace{\frac{\lambda_a \max(1-x_t, 0)}{2}\theta_t^2}_{\text{Affective Costs}}, \tag{14}$$

where $\lambda_c > 0$ and $\lambda_a > 0$.

- *Cognitive Costs:* Proportional to normalized consumption, x_t, and quadratic in effort, θ_t^2. λ_c captures intensity. The x_t term reflects that processing larger consumption levels is more demanding, consistent with mental accounting and findings in numerical cognition suggesting processing larger differences requires more effort (Moyer and Landauer (1967)). The quadratic form, θ_t^2, implies increasing marginal costs, reflecting limited cognitive resources and diminishing returns.[12]
- *Affective Costs:* Incurred only when $x_t < 1$ and proportional to the shortfall, $1 - x_t$. λ_a captures intensity. Losses trigger emotional responses requiring effort to manage. The quadratic form, θ_t^2, implies increasing marginal costs.[13] $\max(1 - x_t, 0)$ ensures one-sided costs,[14] reflecting the asymmetry between gains and losses, supported by neuroeconomic findings showing stronger affective responses to losses than equivalent gains (Yacubian et al. (2006), De Martino et al. (2006), Tom et al. (2007), Sokol-Hessner et al. (2009)).[15]

[12] This is consistent with findings in numerical cognition suggesting that processing larger numerical differences requires more cognitive effort (Moyer and Landauer (1967)). The quadratic form, θ_t^2, implies increasing marginal costs of effort. This is consistent with the idea that cognitive resources are limited and subject to diminishing returns and fatigue. It becomes increasingly difficult to sustain high levels of cognitive effort.

[13] The quadratic form, θ_t^2, again implies increasing marginal costs of effort, suggesting that it becomes increasingly difficult to regulate intense emotional responses.

[14] The term $\max(1 - x_t, 0)$ can also be written as $\frac{(1-x+|1-x|)}{2}$. While we choose a linear dependence on $1 - x$, we acknowledge alternative possibilities like a concave (diminishing sensitivity) or convex form, but opt for linearity for analytical tractability.

[15] This asymmetry is motivated by neuroeconomic findings suggesting that losses elicit stronger affective responses than equivalent gains, as reflected in increased amygdala activity and autonomic

REMARK 3—Justification for Functional Form and Parameter Identification: The quadratic form for both cost components is a standard assumption representing diminishing returns to effort and increasing marginal costs. This reflects the likely limitations of both cognitive and affective resources. The proportionality of cognitive costs to x_t and affective costs to $\max(1 - x_t, 0)$ are key features: cognitive effort scales with the magnitude of consumption (relative to the reference point), while affective effort is tied specifically to losses.[16] Regarding identification, λ_c and λ_a govern the intensities of cognitive and affective costs, respectively. A higher λ_a/λ_c implies greater loss aversion.

The structure of the cognitive and affective costs, $\mathcal{L}(\theta, x)$, is central to the agent's optimal choice of effort, θ, and ultimately, to the emergence of endogenous loss aversion. Figure 1 provides a three-dimensional visualization of these costs. Panel A illustrates the cognitive cost component, $\mathcal{L}_c(\theta, x) = \frac{\lambda_c x}{2}\theta^2$, which is quadratic in effort, θ, and scales linearly with normalized consumption, x. This reflects the increasing marginal cost of cognitive effort and the greater difficulty of processing larger consumption levels. Panel B depicts the total cost, $\mathcal{L}(\theta, x) = \mathcal{L}_c(\theta, x) + \mathcal{L}_a(\theta, x)$, which incorporates the affective cost component, $\mathcal{L}_a(\theta, x) = \frac{\lambda_a \max(1-x,0)}{2}\theta^2$. The affective cost is only incurred when consumption falls below the reference point ($x < 1$), capturing the effort required to regulate the emotional response to losses. The asymmetry in the total cost function, driven by the presence of $\mathcal{L}_a(\theta, x)$ only for $x < 1$, is a key driver of the model's results.

Figure 2 provides two-dimensional cross-sections of the cost functions, further clarifying their properties. Panel A shows both the total cost and cognitive cost for several fixed values of normalized consumption ($x = 0.5, 1.0, 1.5$) as a function of

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arousal (e.g., Yacubian et al. (2006), De Martino et al. (2006), Tom et al. (2007), Sokol-Hessner et al. (2009). While alternative functional forms (e.g., linear, exponential) could be considered, the quadratic form is chosen for its analytical tractability and consistency with general notions of diminishing returns to effort. The specific form of the asymmetry is chosen for simplicity, but the model's qualitative results are robust to alternative specifications that capture the greater salience of losses.

[16] While other functional forms are possible, the quadratic form offers a balance between tractability and realism. It allows for a nuanced response (unlike, say, linear costs) and captures diminishing returns (unlike exponential costs which might lead to unrealistic corner solutions). The chosen form captures key stylized facts: gain-loss asymmetry, sensitivity to $|x_t|$, and cognitive-affective interaction.

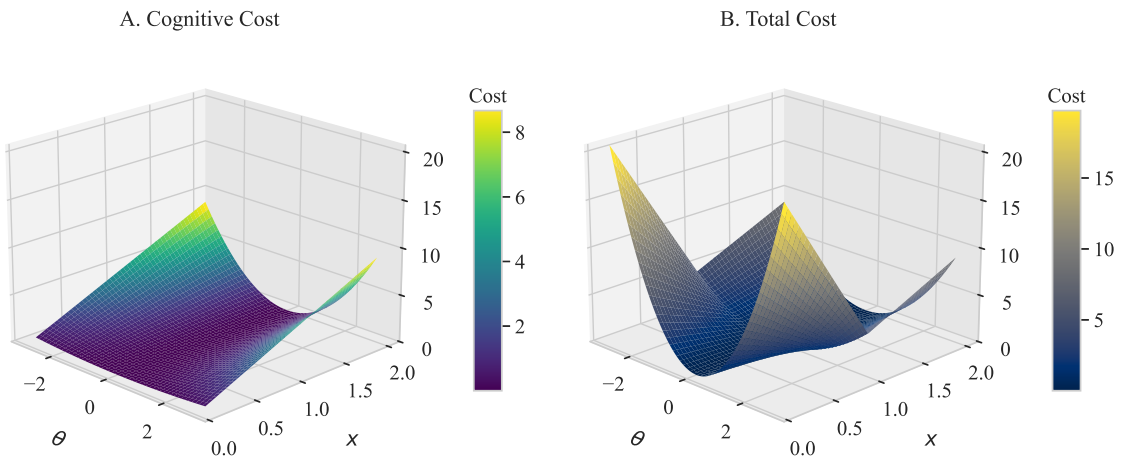


FIGURE 1.—Cognitive and Total Cost Functions.

Note: **Panel A** depicts the cognitive cost function, $\mathcal{L}_c(\theta, x) = \frac{\lambda_c x}{2}\theta^2$, as a function of the cognitive/affective effort, θ, and normalized consumption, x. **Panel B** shows the total cost function, $\mathcal{L}(\theta, x) = \frac{\lambda_c x}{2}\theta^2 + \frac{\lambda_a \max(1-x,0)}{2}\theta^2$, which includes both cognitive and affective costs. The affective cost component is only active when $x < 1$, representing the cost of regulating emotions associated with losses. Parameter values: $\lambda_c = 1$, $\lambda_a = 5$.

effort, θ. The difference between the total cost and cognitive cost curves for $x = 0.5$ highlights the contribution of the affective cost in the loss domain. Panel B illustrates the costs for a fixed level of effort ($\theta = 1$) as a function of normalized consumption, x. This asymmetry in costs, combined with the agent's desire to maximize utility net of these costs, leads to the endogenous emergence of loss aversion, which will be demonstrated in the subsequent analysis of the optimal effort and implied utility function (see Figure 3).

4. GENERALIZED PROSPECT THEORY

4.1. *Model Solution*

Because the problem is time-homogeneous with an infinite horizon, we seek a stationary solution: $V(x_t) = V(x)$. The value function solves the HJB equation (13). The transversality condition is:

$$\lim_{t\to\infty} e^{-\beta t}\mathbb{E}[|V(x_t)|] = 0, \tag{15}$$

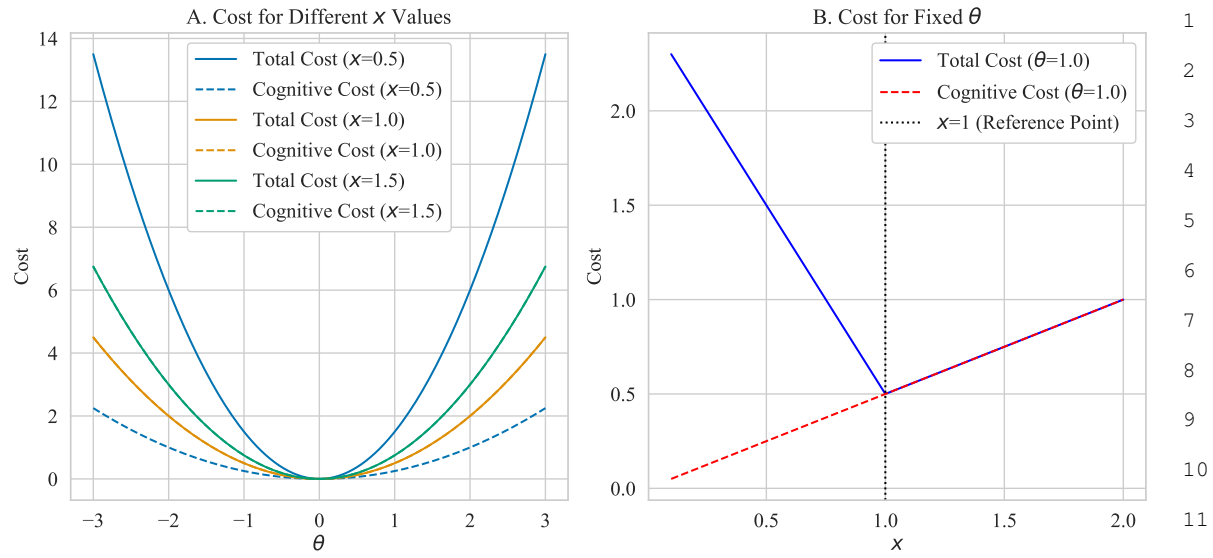


FIGURE 2.—Cost Function Cross-Sections.

Note: **Panel A** displays the total cost, $\mathcal{L}(\theta, x)$, and cognitive cost, $\mathcal{L}_c(\theta, x)$, for different fixed levels of normalized consumption ($x = 0.5, 1.0, 1.5$) as a function of effort, θ. **Panel B** shows the total and cognitive costs for a fixed level of effort ($\theta = 1$) as a function of normalized consumption, x. The discontinuity in the total cost at $x = 1$ (indicated by the vertical dotted line) reflects the activation of the affective cost component for losses ($x < 1$). Parameter values: $\lambda_c = 1$, $\lambda_a = 5$.

utility does not explode as time goes to infinity. Economically, it means that the subjective discount rate must be sufficiently high relative to the volatility of consumption.

PROPOSITION 2: *Consider the HJB equation* (13) *with* $x > 0$. *Assume* $\beta > \sigma^2/2$, $\sigma > 0$, $\lambda_c > 0$, *and* $\lambda_a > 0$. *Define* $V_0 := V(1)$. *We define* $V(x)$ *piecewise, with* $V_+(x)$ *for* $x \geq 1$ *(gains) and* $V_-(x)$ *for* $0 < x \leq 1$ *(losses). We impose value matching*[17] $(V_+(1) = V_-(1))$, *smooth pasting*[18] $(V'_+(1) = V'_-(1))$, *and the transversality condition* (15). *The solution is:*

[17] $V(x)$ is continuous at $x = 1$, with $V_0 = V(1)$. This condition ensures that the value function is well-behaved at the reference point, reflecting the idea that there is no abrupt change in value when transitioning from losses to gains.

[18] The first derivative of $V(x)$ is continuous at $x = 1$. This condition ensures a smooth transition in the agent's valuation and is economically motivated by the absence of arbitrage opportunities at the reference point (see Dixit (2013)).

(i) For $x \geq 1$ (Gains Region):

$$V_+(x) = \left(V_0 - \frac{1}{2\lambda_c\beta}\right) x^{\gamma_2} + \frac{x}{2\lambda_c\beta}, \quad \theta_+^* = \frac{1}{\lambda_c}. \tag{16}$$

(ii) For $0 < x \leq 1$ (Losses Region, $\lambda_c = \lambda_a =: \lambda$):

$$V_-(x) = \left(V_0 - \frac{1}{2\lambda(\beta - \sigma^2)}\right) x^{\gamma_1} + \frac{x^2}{2\lambda(\beta - \sigma^2)}, \quad \theta_-^* = \frac{x}{\lambda}. \tag{17}$$

where

$$\gamma_1 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\beta}{\sigma^2}} > 1, \quad \gamma_2 = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{2\beta}{\sigma^2}} < 0,$$

$$V_0 = \frac{1}{2\lambda(\gamma_2 - \gamma_1)} \left(\frac{\gamma_2 - 1}{\beta} - \frac{\gamma_1 - 2}{\beta - \sigma^2}\right).$$

(iii) For $0 < x \leq 1$ (Losses Region, $\lambda_c \neq \lambda_a$): Assume $0 < \lambda_c < 2\lambda_a$. The solution is:

$$V_-(x) = B_1 x^{\gamma_1} + V_p^*(x), \quad \theta_-^* = \frac{x}{\lambda_c x + \lambda_a(1 - x)}, \tag{18}$$

where

$$V_p^*(x) = \frac{1}{\sigma^2\lambda_a(\gamma_2 - \gamma_1)} \left[\frac{x^2}{2 - \gamma_1} {}_2F_1(1, 2 - \gamma_1; 3 - \gamma_1; -kx) \right.$$
$$\left. - \frac{x^2}{2 - \gamma_2} {}_2F_1(1, 2 - \gamma_2; 3 - \gamma_2; -kx)\right], \tag{19}$$

and

$$\gamma_1 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\beta}{\sigma^2}} > 1, \quad \gamma_2 = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{2\beta}{\sigma^2}} < 0, \quad k = \frac{\lambda_c - \lambda_a}{\lambda_a}.$$

${}_2F_1(a,b;c;z)$ denotes the hypergeometric function.[19] *B_1 and V_0 are determined by solving the following system numerically:*

$$V_0 = B_1 + V_p^*(1),$$
$$\left(V_0 - \frac{1}{2\lambda_c\beta}\right)\gamma_2 + \frac{1}{2\lambda_c\beta} = B_1\gamma_1 + \left.\frac{dV_p^*(x)}{dx}\right|_{x=1},$$

where

$$\left.\frac{dV_p^*(x)}{dx}\right|_{x=1} = \frac{1}{\sigma^2\lambda_a(\gamma_2-\gamma_1)}\left[\frac{2}{2-\gamma_1}{}_2F_1(1,2-\gamma_1;3-\gamma_1;-k)\right.$$
$$-\frac{k}{3-\gamma_1}{}_2F_1(2,3-\gamma_1;4-\gamma_1;-k) - \frac{2}{2-\gamma_2}{}_2F_1(1,2-\gamma_2;3-\gamma_2;-k)$$
$$\left.+\frac{k}{3-\gamma_2}{}_2F_1(2,3-\gamma_2;4-\gamma_2;-k)\right].$$

The hypergeometric series converges for $|kx|<1$. Since $0<x\leq 1$, this requires $|\frac{\lambda_c-\lambda_a}{\lambda_a}|<1$, or $0<\lambda_c<2\lambda_a$.

Proof: See Appendix A. *Q.E.D.*

The solution reveals several key features. First, the value function is continuous and differentiable everywhere, including at the reference point ($x=1$), due to the value matching and smooth pasting conditions. Second, the optimal effort in the gains region (θ_+^*) is constant, while the optimal effort in the losses region (θ_-^*) is increasing in x. Third, the value function exhibits a kink at $x=1$ in the limiting case when λ_a approaches infinity, as shown in Proposition 7. This kink is a hallmark of loss aversion.

[19] ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function (e.g., Olver et al. (2010)), defined by the series

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(q)_n$ is the Pochhammer symbol (rising factorial), defined by $(q)_0=1$ and $(q)_n = q(q+1)(q+2)\cdots(q+n-1)$ for $n\geq 1$. The series converges for $|z|<1$. In our case, $z=-kx$, where $k=(\lambda_c-\lambda_a)/\lambda_a$ and $0<x\leq 1$. Convergence requires $|-kx|<1$, or $|k|<1$ since $x>0$. This is equivalent to $0<\lambda_c<2\lambda_a$.

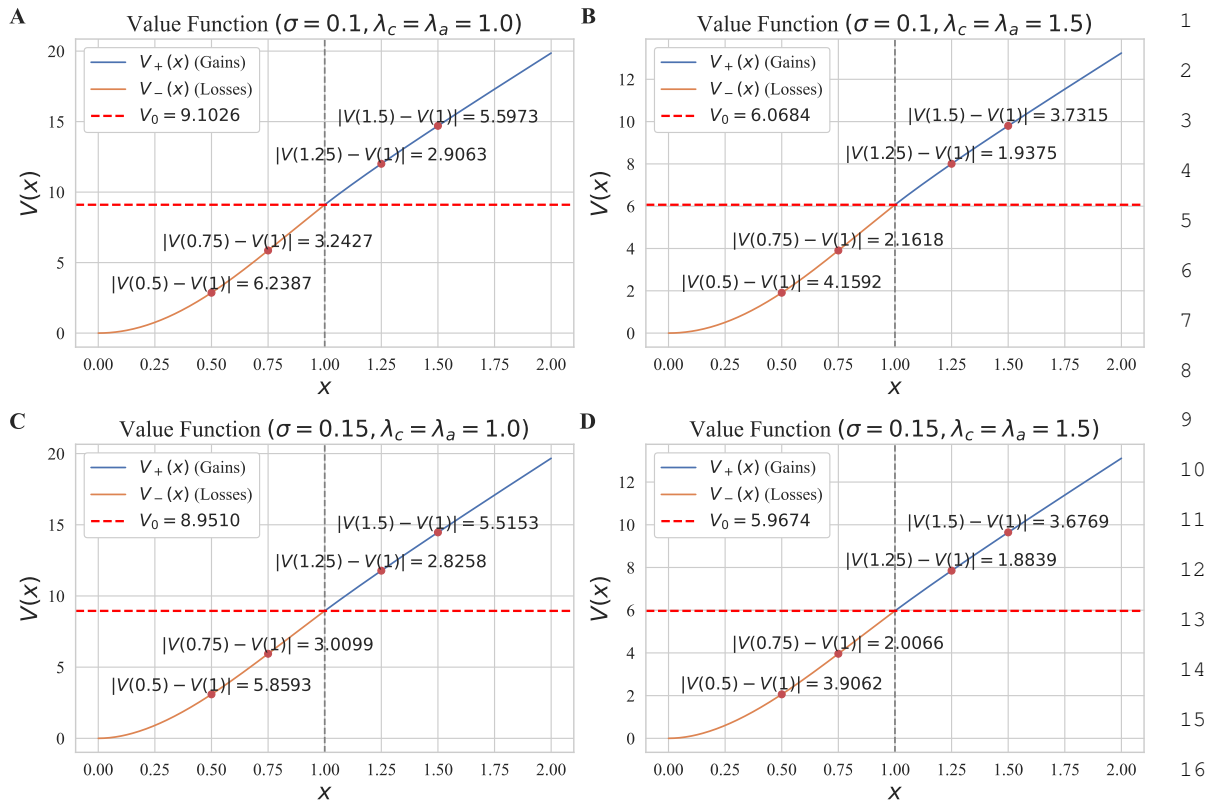


FIGURE 3.—Value Function with Endogenous Loss Aversion.

Note: The figure illustrates the value function. Panels A/B: $\sigma = 0.1$; Panels C/D: $\sigma = 0.15$. Panels A/C: $\lambda_c = \lambda_a = 1.0$; Panels B/D: $\lambda_c = \lambda_a = 1.5$. Blue: $V_+(x)$ (gains); Orange: $V_-(x)$ (losses); Red dashed: V_0. The figure demonstrates: (1) Smooth pasting at $x = 1$. (2) Loss aversion: $V_-(x)$ is steeper than $V_+(x)$ near $x = 1$. (3) Greater loss aversion with larger λ_c and λ_a. (4) Higher σ leads to lower V_0. Numerical values confirm asymmetry: $|V(0.75) - V(1)| > |V(1.25) - V(1)|$ and $|V(0.5) - V(1)| > |V(1.5) - V(1)|$ for all parameter combinations. Parameters: $\sigma = 0.1$ (Panels A/B), $\sigma = 0.15$ (Panels C/D). $\lambda_c = \lambda_a = 1.0$ (Panels A/C), $\lambda_c = \lambda_a = 1.5$ (Panels B/D), and $\beta = 0.05$ (all panels). See Section 5.1 for the discussion of the parameter choice.

Figure 3 illustrates the value function for different parameter values. The figure shows that the value function is steeper for losses than for gains, indicating loss aversion. It also shows that loss aversion increases with λ_a and λ_c, and that the value function is smooth at the reference point.

THEOREM 3—Verification Theorem: *Let $V(x)$ be a C^2 solution to (13) for $x \neq 1$ and C^1 for all $x > 0$, satisfying (15). Suppose $\exists K > 0$ and $\alpha \in (0,1)$ such that $|V(x)| \leq K(1 + x^\alpha)$*

for all $x > 0$. Then $V(x)$ is the unique value function, and the corresponding optimal control θ^* is the optimal control for (11).[20]

PROOF: See Appendix A.1. *Q.E.D.*

4.2. *Connection to Cumulative Prospect Theory (CPT)*

With $dx_t = \sigma x_t d\mathcal{Z}_t$, this model leads to emergent properties similar to CPT.

1. Key Features: x_t is log-normally distributed, non-negative, and Markovian. $V(x)$ represents expected discounted utility under the optimal policy, θ^*. The agent weights probabilities according to the rational expectations measure, $\mathbb{P}$. The agent does not subjectively distort probabilities.

2. Connection to CPT: We define emergent probability weighting:

DEFINITION 3—Emergent Probability Weighting: Consider a decision maker (DM) with a value function $V(x)$ defined over normalized consumption outcomes $x \in \mathbb{R}^+$, where $x = 1$ represents a reference point. Let $\mathcal{L}$ be the set of all simple lotteries over outcomes in $\mathbb{R}^+$. A lottery $L \in \mathcal{L}$ is represented as $L = (x_1, p_1; \ldots; x_n, p_n)$, where x_i are outcomes and p_i are the corresponding objective probabilities ($\sum_{i=1}^n p_i = 1$). The DM's certainty equivalent for lottery L under the value function V, denoted $CE_V(L)$, is defined implicitly by:

$$V(CE_V(L)) = \sum_{i=1}^{n} p_i V(x_i). \tag{20}$$

We say that the DM's preferences, represented by V, exhibit *emergent probability weighting* if there exists a strictly increasing, continuous function $\pi : [0,1] \to [0,1]$ with $\pi(0) = 0$ and $\pi(1) = 1$, and a strictly increasing, continuous representation value function $v : \mathbb{R}^+ \to \mathbb{R}$, such that for all lotteries $L \in \mathcal{L}$, there exists a certainty

[20] The growth condition $|V(x)| \leq K(1 + x^\alpha)$ for $\alpha \in (0,1)$ is crucial for ensuring the model's economic reasonableness and mathematical consistency. The restriction $\alpha \in (0,1)$ ensures that the value function does not grow too rapidly with consumption. Values of $\alpha \geq 1$ are economically unreasonable as they would imply that marginal utility is increasing with consumption, violating the fundamental principle of diminishing marginal utility. This restriction is essential for the transversality condition to hold and for the verification theorem to apply.

equivalent $CE_{\pi,v}(L)$ satisfying:

$$v(CE_{\pi,v}(L)) = \sum_{i=1}^{n} \pi(p_i)v(x_i), \tag{21}$$

and $CE_V(L) = CE_{\pi,v}(L)$. In other words, the DM's choices can be represented as if they are evaluating lotteries using the probability weighting function $\pi(\cdot)$ and the representation value function $v(\cdot)$.

The model exhibits emergent probability weighting from the curvature of the endogenously determined value function. Higher λ_c leads to a more concave value function in gains. Higher λ_a leads to a lower value function in losses. This curvature effectively transforms objective probabilities into subjective decision weights resembling CPT's probability weighting. The concavity in gains and convexity in losses (see Figure 3) imply the agent behaves as if overweighting small probabilities and underweighting large probabilities. The agent, while using objective probabilities, acts as if distorting them due to the value function's shape.

Proposition 4—Emergent Probability Weighting in the NDU Model: *The NDU model, with value function $V(x)$ as defined in Proposition 2, exhibits emergent probability weighting. Specifically, for any simple lottery $L = (x_1, p_1; \ldots; x_n, p_n)$, there exist functions $\pi(p)$ and $v(x)$, satisfying the conditions in Definition 3, such that $CE_V(L) = CE_{\pi,v}(L)$.*

Proof: See Appendix C. *Q.E.D.*

Proposition 5—Endogenous Loss Aversion: *The value function in Proposition 2 exhibits stable loss aversion when $\lambda_c = \lambda_a$. This is manifested in two key ways:*

(i) ***Asymmetric Value Differences:*** *For any sufficiently small $\Delta x > 0$:*

$$|V(1 - \Delta x) - V(1)| > |V(1 + \Delta x) - V(1)|. \tag{22}$$

(ii) ***Greater Curvature for Losses:*** *The value function is generally more concave in the losses region near $x = 1$ than in the gains region. That is, $|\lim_{x\to 1^-} V''_-(x)| > |\lim_{x\to 1^+} V''_+(x)|$ for most parameter combinations.*

PROOF: See Appendix A.2. *Q.E.D.*

REMARK 4—Reference Point: The behavior at the reference point follows the pairs (x, V_0), with $x = 1$ distinguishing gains ($x \geq 1$) from losses ($0 < x \leq 1$), and $V_0 = V(1)$ represents the endogenous valuation at the reference point to denote the boundary between losses region and gains region.

3. The Free Boundary V_0:

- **Determination:** V_0 is endogenously determined by simultaneously solving the value matching condition $V_+(1) = V_-(1)$ and the smooth pasting condition $V'_+(1) = V'_-(1)$. This ensures that the value function is continuously differentiable at the reference point, reflecting the idea that the agent's marginal valuation should be consistent when transitioning from gains to losses. The smooth pasting condition is economically motivated by the absence of arbitrage opportunities at the reference point.
- **Significance:** V_0 represents the agent's value at the reference point, effectively separating the gains and losses regions. It is a critical level that influences the agent's optimal behavior. For example, an agent with a higher V_0 might be willing to take on more risk in the gains region, as the value at the reference point is already high. This is consistent with the findings of Novemsky and Kahneman (2005) on the impact of boundaries on choices, where the location of a boundary can significantly affect decision-making.
- **Dependence:** V_0 depends on the model parameters $(\lambda_c, \lambda_a, \sigma, \beta)$, reflecting the agent's preferences and the characteristics of the stochastic environment. For example, an increase in loss aversion (higher λ_a) would typically lead to a lower V_0, as the agent is more sensitive to losses and thus values the reference point less. An increase in uncertainty (higher σ) might also affect V_0, depending on the interaction with other parameters.
- **CPT Interpretation:** V_0 can be seen as the subjective value assigned to the status quo or the reference point itself. In CPT, the value function is often assumed to be zero at the reference point. However, in this model, V_0 can be positive or negative, reflecting the dynamic nature of the reference point (Camerer (2005)). A positive V_0 suggests that the agent derives some intrinsic

value from being at the reference point ($x = 1$), while a negative V_0 could indicate a "status quo bias" where the agent is averse to moving away from the reference point even if it is not intrinsically valuable. The magnitude and sign of V_0 will depend on the interplay between the agent's risk aversion, loss aversion, and the stochastic environment.[21]

PROPOSITION 6—No Violation of Stochastic Dominance: *The model does not predict violations of first-order stochastic dominance.*

PROOF: See Appendix A.3 *Q.E.D.*

PROPOSITION 7—CPT as a Limiting Case: *As $\lambda_a \to \infty$, keeping λ_c fixed, the value function (Proposition 2) converges pointwise to a value function $V^*(x)$ with a kink at $x = 1$, and the decision-making behavior of the agent converges to that implied by a CPT model. Specifically, let $CE(\lambda_a)$ be the certainty equivalent of a lottery under the NDU model with parameter λ_a, and let CE_{CPT} be the certainty equivalent of the same lottery under a CPT model with value function $v(x) = x^\alpha$ for $x \geq 1$ and $v(x) = -L(-x)^\alpha$ for $x < 1$ (where $\alpha \in (0,1)$ and $L > 1$ are constants) and the Tversky-Kahneman (Tversky and Kahneman (1992)) probability weighting function $\pi(p)$. Then, $\lim_{\lambda_a \to \infty} CE(\lambda_a) = CE_{CPT}$, where $\alpha = \gamma_1$ and L is determined by the limit of the ratio of the left and right derivatives of the value function at $x = 1$.*

PROOF: See Appendix B. *Q.E.D.*

This proposition establishes a formal connection between our model and CPT. It shows that CPT, with its characteristic kinked value function and probability weighting, can be viewed as a special case of our model where the cost of affective

[21] Empirical evidence for a non-zero value at the reference point is somewhat limited. However, some studies have found that individuals exhibit a "status quo bias," where they tend to stick with their current situation even when presented with objectively better alternatives (Kahneman et al. (1991)). This could be interpreted as evidence for a positive V_0. Other studies have found that individuals are more likely to accept gambles that are framed as avoiding losses rather than acquiring gains, even when the gambles are objectively identical (Tversky and Kahneman (1981)). This could be interpreted as evidence for a negative V_0. More research is needed to directly measure V_0 and understand its determinants. One possible approach would be to elicit individuals' willingness to pay to avoid small deviations from their expected consumption level.

effort becomes infinitely large. This is a significant result because it provides a micro-foundation for CPT, which is typically taken as a descriptive model without a deeper theoretical justification.

5. QUANTITATIVE ANALYSIS

This section explores the model's quantitative behavior and the sensitivity of its predictions to changes in the key parameters: λ_c (cognitive cost intensity), λ_a (affective cost intensity), β (discount factor), and σ (volatility). We focus on how these parameters, individually and jointly, influence the degree of endogenous loss aversion and the potential for dynamic emergent probability weighting. This analysis directly informs the identification strategies discussed in Section A.

5.1. *Parameter Choice*

Our baseline calibration targets aggregate consumption behavior.[22] We calibrate the consumption volatility parameter, σ, within the range $[0.01, 0.2]$. This value is consistent with estimates from U.S. National Income and Product Accounts (NIPA) data on real personal consumption expenditures (PCE); and from micro-level panel data, such as the Panel Study of Income Dynamics (PSID) and the Consumer Expenditure Survey (CEX), which typically report much higher levels of consumption volatility than aggregate data. Specifically, calculating the standard deviation of the annualized quarterly growth rate of real PCE over extended periods (e.g., 1947-2024) typically yields values within this range.[23] Table III summarizes the typical ranges of consumption volatility found in different data sources and corresponding literature.

The discount factor, β, is set to 0.05 in the baseline calibration. This value satisfies the constraint $\beta > \sigma^2/2$, which is necessary for the existence of a well-defined

[22] We focus on matching key empirical features of consumption data, particularly its volatility. A crucial distinction is made between calibrating to aggregate consumption dynamics versus individual household consumption.

[23] We acknowledge that aggregate data may underestimate the true volatility of consumption due to smoothing and aggregation biases inherent in national accounts, as well as the potential for time aggregation bias (e.g., Blinder and Deaton (1985), Campbell and Deaton (1989)). Individual household consumption is likely to be significantly more volatile.

TABLE III

CONSUMPTION VOLATILITY ESTIMATES FROM DIFFERENT DATA SOURCES

Data Source	Range (Annualized σ)	Key Literature
NIPA (Aggregate)	0.01 - 0.03	Stock and Watson (2002)
PSID/CEX (Individual)	0.10 - 0.30	Blundell et al. (2008)
DSGE Models	0.005 - 0.02	Smets and Wouters (2007)

Note: This table summarizes estimates of annualized consumption volatility (σ) from various sources. NIPA (National Income and Product Accounts, (e.g., Davis and Kahn (2008), Blinder and Deaton (1985))) data typically yield lower estimates due to aggregation and smoothing. PSID (Panel Study of Income Dynamics) and CEX (Consumer Expenditure Survey) data provide individual-level estimates (e.g., Storesletten et al. (2004), Hall and Mishkin (1982)), which are considerably higher, but are subject to measurement error. DSGE (Dynamic Stochastic General Equilibrium, (e.g., Kydland and Prescott (1982))) models are calibrated or estimated using a variety of data sources and methodologies, leading to a range of volatility parameters.

solution and ensures that the agent's utility does not explode. We also examine the implications of alternative parameter values through extensive sensitivity analysis, as reported in subsequent sections. To demonstrate the robustness of our findings, we perform sensitivity analysis across a range of σ values. For aggregate consumption, we explore the interval $[0.01, 0.03]$. We also present results for higher values (with a maximum 0.2) of σ, consistent with individual-level consumption volatility, to illustrate the model's behavior under increased uncertainty. The parameters λ_c and λ_a are chosen to match some stylized facts.

5.2. *Preferences Instability*

This section investigates the sensitivity of the agent's preferences to changes in the key parameters. We examine how these parameters influence the value function, $V(x)$, and the optimal control (effort), $\theta^*(x)$, through comparative statics. Crucially, this analysis helps quantify the degree of endogenous loss aversion and identify conditions for non-inverse-S shape probability weighting.

Figure 4 illustrates the distinct effects of λ_c and λ_a. In Panel A ($\lambda_c = 1.0$, varying λ_a), increasing λ_a primarily depresses the value function in the loss region ($x < 1$). A higher λ_a signifies a greater cost to regulating emotions associated with losses, leading to a lower overall value. Notably, the value function in the gains region ($x \geq 1$) is also affected by λ_a due to the influence of the affective cost component at the reference point, highlighting the interconnectedness of the two domains. Panel B shows the corresponding optimal control. As λ_a increases, $\theta^*(x)$ decreases

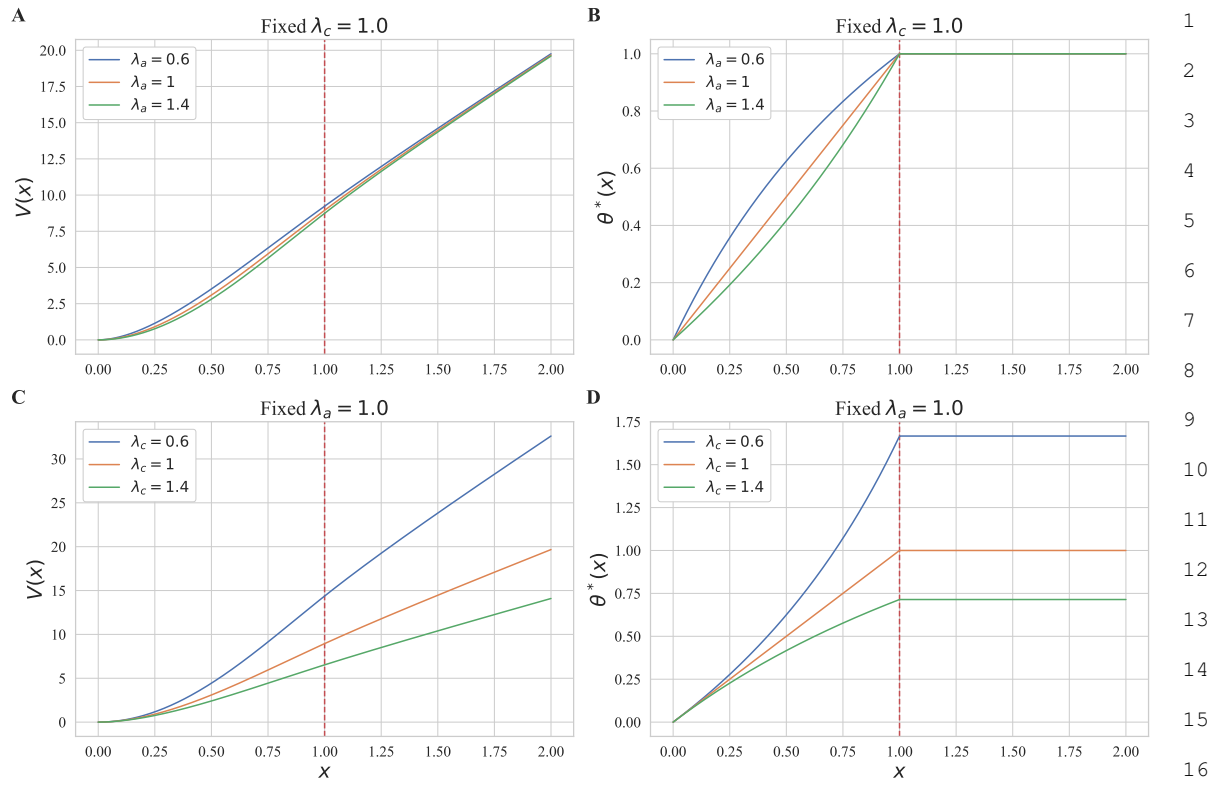


FIGURE 4.—Comparative Statics of Value Function and Optimal Control with Varying λ_c and λ_a. **Panel A**: Value function, $V(x)$, with fixed $\lambda_c = 1.0$ and varying λ_a. Increasing λ_a primarily depresses the value function in the loss region ($x < 1$), reflecting the higher cost of regulating emotions associated with losses. **Panel B**: Optimal control, $\theta^*(x)$, with fixed $\lambda_c = 1.0$ and varying λ_a. As λ_a increases, $\theta^*(x)$ decreases in the loss region and transitions from concave to convex. **Panel C**: Value function, $V(x)$, with fixed $\lambda_a = 1.0$ and varying λ_c. Increasing λ_c lowers $V(x)$ for both gains and losses and makes the value function more concave. **Panel D**: Optimal control, $\theta^*(x)$, with fixed $\lambda_a = 1.0$ and varying λ_c. Increasing λ_c decreases $\theta^*(x)$ for all x, reflecting the higher cost of cognitive effort. The vertical dashed red line in each panel indicates the reference point, $x = 1$. Baseline parameters: $\beta = 0.05$ and $\sigma = 0.15$.

in the loss region and transitions from concave to convex, reflecting the reduced incentive to exert costly affective effort. In the gains region, $\theta^*(x)$ remains constant at $1/\lambda_c = 1$, as established in Proposition 2.

Panel C ($\lambda_a = 1.0$, varying λ_c) demonstrates that changes in λ_c affect the value function across the entire domain. Increasing λ_c lowers $V(x)$ for both gains and losses, and makes the value function more concave. Higher cognitive costs reduce the agent's willingness to amplify utility, leading to a lower overall value. Panel D shows that increasing λ_c decreases $\theta^*(x)$ for all x. The inverse relationship in the

gains region ($\theta^*(x) = 1/\lambda_c$) is evident. In the loss region, the decrease is non-linear, reflecting the interplay between cognitive and affective costs. The differences in $\theta^*(x)$ across different λ_c values are less pronounced in the loss region compared to the gains region.

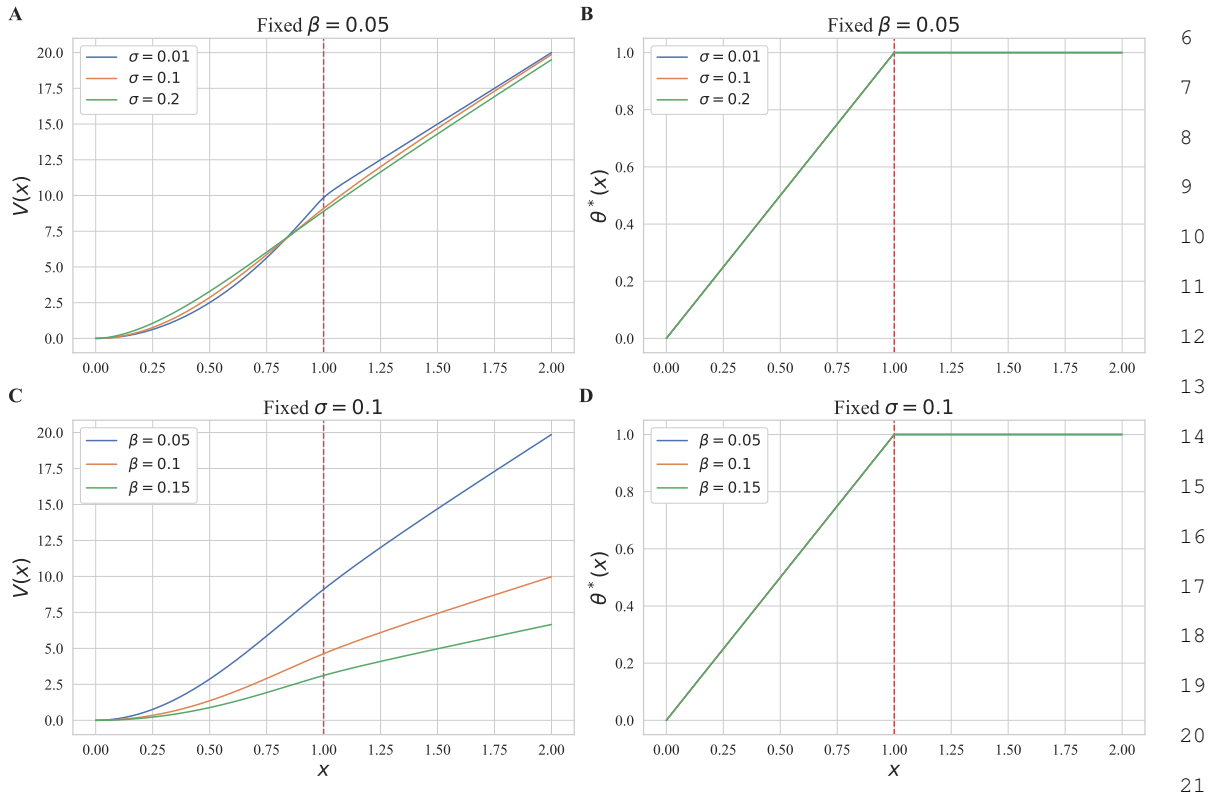


FIGURE 5.—Comparative Statics of Value Function and Optimal Control with Varying β and σ. **Panel A**: Value function, $V(x)$, with fixed $\beta = 0.05$ and varying σ. As σ increases, the value function generally decreases in the gains region. In the loss region, the value function initially decreases and then relatively increases with σ, subject to the constraint $\beta > \sigma^2/2$. **Panel B**: Optimal control, $\theta^*(x)$, with fixed $\beta = 0.05$ and varying σ. Changes in σ do not affect $\theta^*(x)$, as σ influences the consumption process dynamics but not the cost function directly. **Panel C**: Value function, $V(x)$, with fixed $\sigma = 0.1$ and varying β. Increasing β lowers the value function across the entire domain. **Panel D**: Optimal control, $\theta^*(x)$, with fixed $\sigma = 0.1$ and varying β. Changes in β do not affect $\theta^*(x)$ compared to the cost parameters. The vertical dashed red line in each panel indicates the reference point, $x = 1$. $\lambda_c = \lambda_a = 1.0$ for all panels.

Figure 5 explores the impact of β and σ. Panel A (fixed $\beta = 0.05$, varying σ) reveals a non-monotonic relationship between volatility and the value function, particularly in the loss region. As σ increases, the value function generally decreases in the gains region. In the loss region, the value function initially (close to $x = 1$)

decreases and then relatively increases with σ, subject to the constraint $\beta > \sigma^2/2$. Higher volatility increases the uncertainty of future consumption, reducing overall welfare. Panel B shows that changes in σ do not affect $\theta^*(x)$, as σ influences the consumption process dynamics but not the cost function directly. Panel C (fixed $\sigma = 0.1$, varying β) shows that increasing β lowers the value function. A higher discount factor implies the agent values future utility less, leading to a lower overall value. Panel D demonstrates that, like σ, changes in β do not affect $\theta^*(x)$ compared to changes in the cost parameters, reinforcing the importance of the cost structure in determining optimal effort.

5.3. *Endogenous Loss Aversion*

To quantify the degree of endogenous loss aversion, we calculate the ratio of value differences around the reference point:

$$\text{Loss Aversion Ratio} = \frac{|V(1-\Delta x) - V(1)|}{|V(1+\Delta x) - V(1)|}, \tag{23}$$

where Δx represents a small deviation from the reference point ($x = 1$). This ratio measures the relative steepness of the value function in the loss domain compared to the gain domain. A ratio greater than 1 indicates loss aversion: a loss of Δx has a greater impact on value than a gain of the same magnitude.

Figure 6 illustrates the loss aversion ratio as a function of λ_a for different values of λ_c, holding β and σ constant. The loss aversion ratio is calculated using the solutions derived in Proposition 2, specifically utilizing numerical methods to solve the system of equations defined by value matching and smooth pasting conditions, as detailed in the proposition.

Remark 5—Quantitative Range: The model can generate a range of loss aversion ratios consistent with empirical estimates. Typical estimates of loss aversion in the experimental literature range from 1.8 to 2.1 (e.g., Brown et al. (2024)). Figure 6 shows that this range can be achieved with plausible parameter combinations. This demonstrates that significant loss aversion can emerge when λ_a is substantially larger than λ_c.

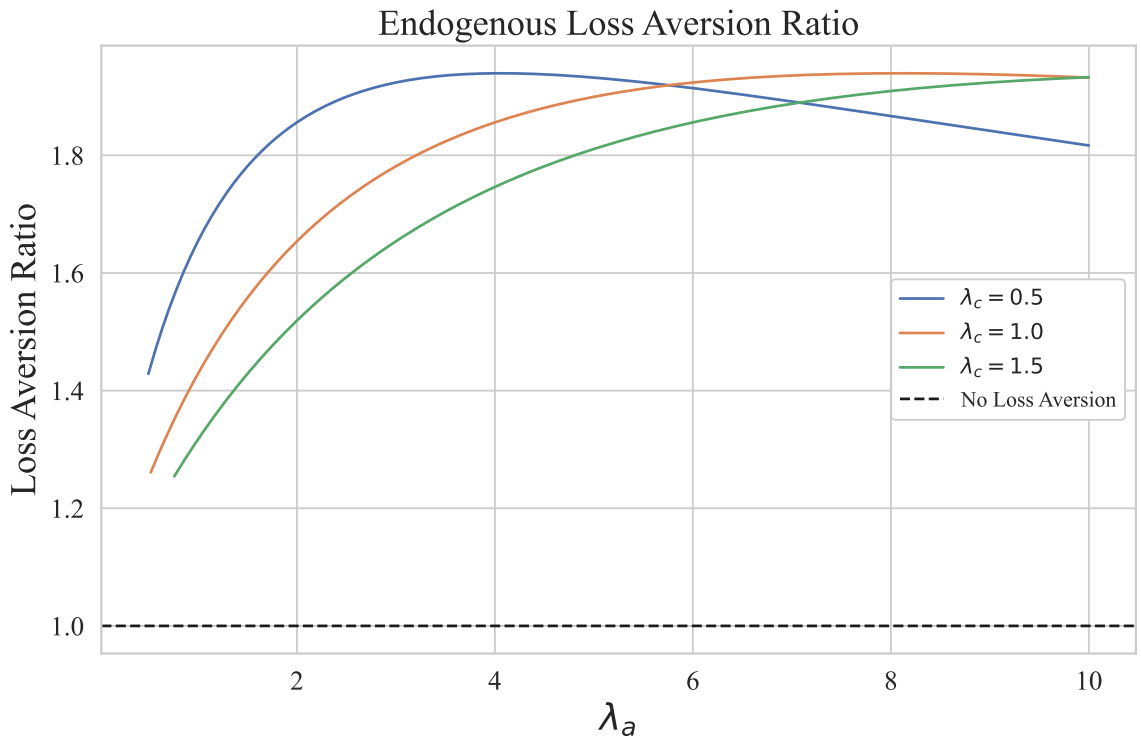


FIGURE 6.—Endogenous Loss Aversion Ratio. The figure plots the loss aversion ratio, $|V(1-\Delta x) - V(1)|/|V(1+\Delta x) - V(1)|$, as a function of λ_a, for different values of λ_c. We set $\Delta x = 0.1$ (a 5% deviation from the reference point). The horizontal dashed line at 1.0 indicates the absence of loss aversion. Values above 1.0 indicate loss aversion. Baseline parameters: $\beta = 0.1$ and $\sigma = 0.02$.

Figure 7 shows the sensitivity of the loss aversion ratio to the choice of Δx. While the magnitude of the ratio changes with Δx, the qualitative result that the ratio is greater than 1 (indicating loss aversion) holds for a range of Δx values. The ratio is relatively stable for small values of Δx (e.g., between 0.01 and 0.10). We use $\Delta x = 0.1$ as a representative value throughout the analysis.

These findings, generated through the computational implementation of the model, have important implications for empirical work, as discussed in detail in Section A. The non-linearity and joint determination of λ_c and λ_a highlight the necessity of a structural model for estimation. The numerical results confirm and extend the theoretical predictions, providing a quantitative understanding of the model's behavior and the emergence of loss aversion.

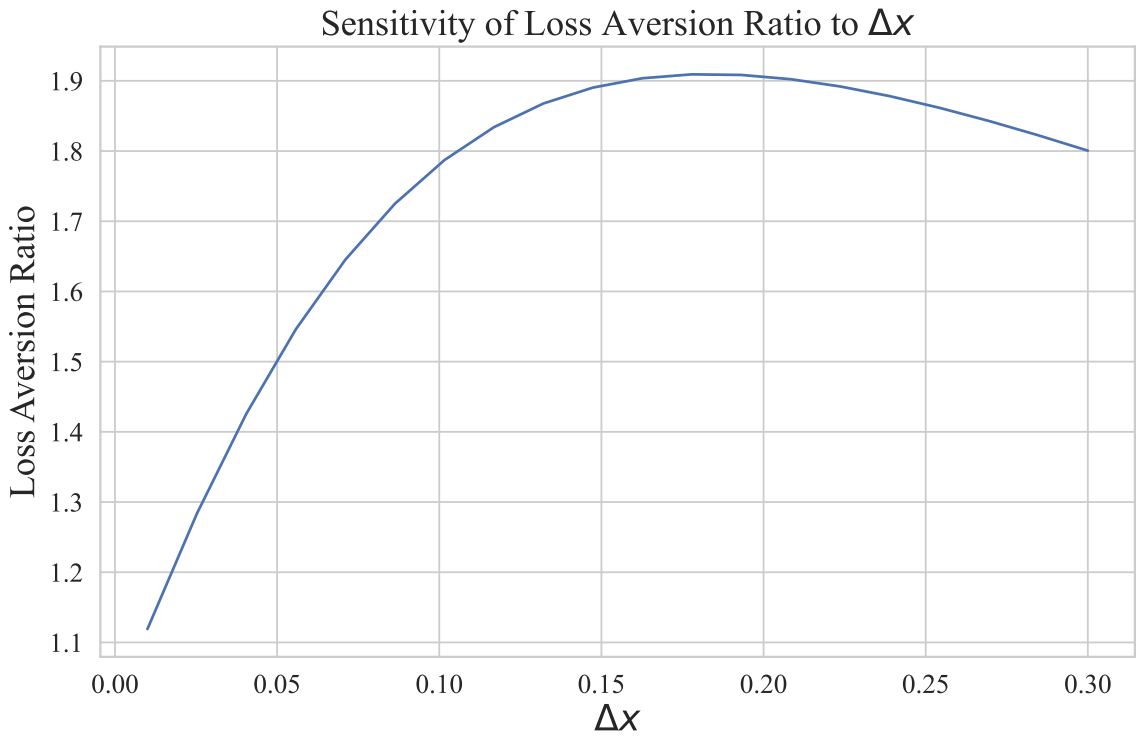


FIGURE 7.—Sensitivity of Loss Aversion Ratio to Δx. The figure plots the loss aversion ratio, $|V(1-\Delta x)-V(1)|/|V(1+\Delta x)-V(1)|$, for different values of Δx, ranging from 0.01 to 0.30. Parameters: $\lambda_c = 1.0$, $\lambda_a = 3$, $\beta = 0.1$, and $\sigma = 0.02$.

5.4. *Analysis of Emergent Probability Weighting*

A key distinction between our model and standard Cumulative Prospect Theory (CPT) lies in the shape of the emergent probability weighting function, $\pi(p)$. While standard CPT typically assumes a monotonically increasing (though often inverse-S shaped) $\pi(p)$, our model allows for non-inverse-S shape. This arises from the interplay between the curvature of the value function in the gains and losses regions and the endogenous determination of effort, θ^*. The shape of $\pi(p)$ is a direct consequence of the shape of the value function, $V(x)$. To illustrate this, consider a binary lottery with outcomes $x_1 < 1$ (a loss) and $x_2 > 1$ (a gain), and corresponding probabilities p and $1-p$. The NDU model's certainty equivalent, CE, satisfies:

$$V(CE) = pV(x_1) + (1-p)V(x_2). \tag{24}$$

The emergent probability weighting function, $\pi(p)$, is implicitly defined by relating the objective probabilities in a lottery to the certainty equivalent generated by the NDU model (see Proposition 4). We can represent the certainty equivalent as if it were calculated using a non-linear probability weighting:

$$v(CE) = \pi(p)v(x_1) + (1 - \pi(p))v(x_2), \tag{25}$$

where $v(x)$ is a representation value function. For simplicity, we can use a linear representation value function with a loss aversion coefficient of 1: $v(x) = x$ for $x \geq 1$ and $v(x) = -(-x)$ for $x < 1$. This allows us to isolate the effect of the emergent probability weighting. Then, $\pi(p)$ can be calculated as:

$$\pi(p) = \begin{cases} \frac{CE - x_2}{x_1 - x_2}, & \text{if } CE \geq 1 \\ \frac{-(-CE) - x_2}{x_1 - x_2}, & \text{if } CE < 1 \end{cases} \tag{26}$$

If the value function is sufficiently more concave in the gains region than in the losses region, the agent's behavior may be represented as if they are underweighting probabilities of gains more as the probability of the gain increases (within a certain range). This leads to the potential for non-inverse-S shape emergent probability weighting. The conditions for non-inverse-S shape depend on the interplay of all four parameters (λ_c, λ_a, β, and σ). A relatively high λ_c compared to λ_a makes the value function more concave for gains, which is a key driver for non-inverse-S shape.

Figure 8 illustrates the emergent probability weighting function, $\pi(p)$, generated by the model for two different parameter sets. Panel A shows a case where $\pi(p)$ is monotonically increasing, but exhibits the characteristic inverse-S shape often associated with CPT: overweighting of small probabilities and underweighting of large probabilities. This occurs when λ_a is sufficiently larger than λ_c. Panel B, however, demonstrates the possibility of non-inverse-S probability weighting. This non-inverse-S arises because the relatively high cognitive cost parameter, λ_c, makes the value function significantly more concave in the gains region.

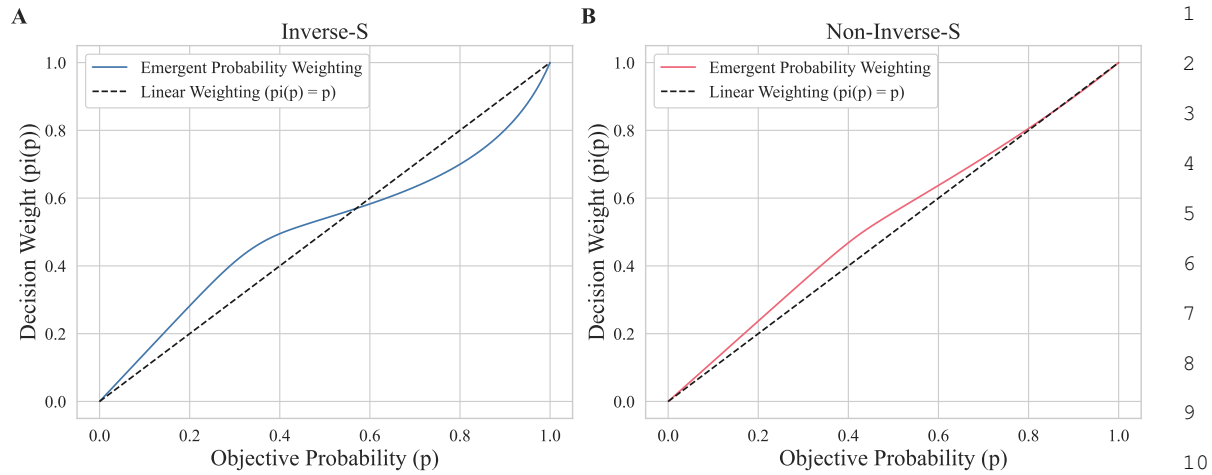


FIGURE 8.—Emergent Probability Weighting Functions.

Note: **Panel A** depicts a monotonically increasing emergent probability weighting function, exhibiting the typical inverse-S shape. Parameter values: $\beta = 0.1$, $\sigma = 0.02$, $\lambda_c = 1.0$, $\lambda_a = 5.0$, $x_1 = 0.5$, $x_2 = 1.5$. **Panel B** shows a non-inverse-S emergent probability weighting function. Parameter values: $\beta = 0.1$, $\sigma = 0.02$, $\lambda_c = 1.0$, $\lambda_a = 0.6$, $x_1 = 0.5$, $x_2 = 1.5$. The dashed line in both panels represents linear probability weighting ($\pi(p) = p$).

This potential for non-inverse-S probability weighting is a crucial testable implication that distinguishes our model from standard CPT. Observing such non-inverse-S shape in experimental data (e.g., Gonzalez and Wu (1999)) would provide strong support for the endogenous effort mechanism. It would suggest that individuals are not simply distorting probabilities in a fixed way, but rather that their weighting of probabilities is a consequence of a dynamic optimization process involving cognitive and affective costs.

6. CONCLUDING REMARKS

This paper presents a model of decision-making under risk where loss aversion and apparent probability weighting emerge endogenously from cognitive and affective costs. The model generates a framework that encompasses CPT as a limiting case and allows for richer behaviors. The key findings are:

- *Endogenous Loss Aversion:* Loss aversion is not assumed but arises from the agent's optimal response to affective costs associated with losses. This provides a process-based explanation for why losses loom larger than gains.

- *Endogenous Probability Weighting:* The model generates apparent probability distortions endogenously. These distortions arise from the endogenous shape of the value function, providing a unified explanation for both loss aversion and the appearance of probability weighting, without any separate probability weighting function.
- *CPT as a Special Case:* As the affective cost parameter λ_a approaches infinity, the model's predictions converge to those of CPT, providing a microfoundation for this important behavioral model.
- *Role of Cognitive and Affective Costs:* The parameters λ_c and λ_a provide a nuanced way to capture individual differences in risk preferences and responses to gains and losses.

Our model provides microfoundations for key aspects of behavioral economics, deriving loss aversion and probability weighting (Kahneman and Tversky (1979), Tversky and Kahneman (1992)) from underlying cognitive and affective cost functions. It extends reference-dependent preference models (e.g., Kőszegi and Rabin (2006)) by focusing on the process of responding to reference point deviations, rather than solely on the reference point (here, the rational expectation). The model integrates dual-process theories (e.g., Camerer et al. (2005), Kahneman (2011)) by incorporating both cognitive effort (System 2) and affective effort (emotion regulation), resonating with research on emotions in decision-making (Loewenstein (2001), Caplin and Leahy (2001). Finally, the cognitive and affective cost constructs offer potential connections to neuroeconomic investigations of decision processes (Rangel et al. (2008), Glimcher (2013)).

APPENDIX A: Proof of Proposition 2

Proof:

Case (i): $x \geq 1$ **(Gains Region):**

1. **Substituting the Cost Function:** With $\lambda_c > 0$, the cost function for $x \geq 1$ is:

$$\mathcal{L}(\theta, x) = \frac{\lambda_c x}{2}\theta^2.$$

The HJB equation (13) becomes:

$$\beta V_+(x) = \sup_\theta \left\{ \theta x - \frac{\lambda_c x}{2}\theta^2 + \frac{1}{2}\sigma^2 x^2 V_+''(x) \right\}.$$

2. **Finding Optimal θ_+:** Taking the derivative with respect to θ and setting it to zero:

$$\frac{\partial}{\partial \theta}\left(\theta x - \frac{\lambda_c x}{2}\theta^2\right) = x - \lambda_c x\theta = 0 \implies \theta_+^* = \frac{1}{\lambda_c}.$$

3. **Admissibility of θ_+^*:** We need to show $\mathbb{E}\left[\int_0^T |\theta_+^*(s)|^2 ds\right] < \infty$ for all $T > 0$. Since $\theta_+^* = 1/\lambda_c$ is a constant:

$$\mathbb{E}\left[\int_0^T |\theta_+^*(s)|^2 ds\right] = \int_0^T (1/\lambda_c)^2 ds = (1/\lambda_c)^2 T < \infty.$$

Thus, θ_+^* is admissible.

4. **Solving the ODE:** Substituting θ_+^* into the HJB equation:

$$\beta V_+(x) = \frac{x}{2\lambda_c} + \frac{1}{2}\sigma^2 x^2 V_+''(x).$$

This is a second-order linear non-homogeneous ODE. We guess a solution of the form $V_+(x) = Ax^\gamma + Bx$. Substituting this into the ODE:

$$\beta(Ax^\gamma + Bx) = \frac{x}{2\lambda_c} + \frac{1}{2}\sigma^2 x^2 (A\gamma(\gamma-1)x^{\gamma-2})$$

$$\beta A x^\gamma + \beta B x = \frac{x}{2\lambda_c} + \frac{1}{2}\sigma^2 A\gamma(\gamma-1)x^\gamma.$$

Matching coefficients:

- For the x^γ term: $\beta A = \frac{1}{2}\sigma^2 A\gamma(\gamma-1) \implies \gamma(\gamma-1) = \frac{2\beta}{\sigma^2}$.
- For the x term: $\beta B = \frac{1}{2\lambda_c} \implies B = \frac{1}{2\lambda_c \beta}$.

The characteristic equation for γ is $\gamma^2 - \gamma - \frac{2\beta}{\sigma^2} = 0$. The roots are:

$$\gamma_{1,2} = \frac{1 \pm \sqrt{1 + \frac{8\beta}{\sigma^2}}}{2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2\beta}{\sigma^2}}.$$

We denote the positive root as γ_1 and the negative root as γ_2. To satisfy the transversality condition and the growth condition in the verification theorem, we choose the negative root, γ_2. Thus, the general solution for $x \geq 1$ is:

$$V_+(x) = Ax^{\gamma_2} + \frac{x}{2\lambda_c \beta}.$$

Case (ii): $0 < x \leq 1$ **(Losses Region,** $\lambda_c = \lambda_a = \lambda$**):**

1. **Substituting the Cost Function:** With $\lambda_c = \lambda_a = \lambda$, the cost function for $0 < x \leq 1$ is:

$$\mathcal{L}(\theta, x) = \frac{\lambda}{2}\theta^2.$$

The HJB equation becomes:

$$\beta V_-(x) = \sup_\theta \left\{ \theta x - \frac{\lambda}{2}\theta^2 + \frac{1}{2}\sigma^2 x^2 V_-''(x) \right\}.$$

2. **Finding Optimal** θ_-**:** Taking the derivative with respect to θ and setting it to zero:

$$\frac{\partial}{\partial \theta}\left(\theta x - \frac{\lambda}{2}\theta^2\right) = x - \lambda\theta = 0 \implies \theta_-^* = \frac{x}{\lambda}.$$

3. **Admissibility of** θ_-^***:** We need to show $\mathbb{E}\left[\int_0^T |\theta_-^*(s)|^2 ds\right] < \infty$ for all $T > 0$. Since $\theta_-^* = x/\lambda$:

$$\mathbb{E}\left[\int_0^T |\theta_-^*(s)|^2 ds\right] = \mathbb{E}\left[\int_0^T (x_s/\lambda)^2 ds\right] = (1/\lambda^2)\int_0^T \mathbb{E}[x_s^2] ds.$$

Since x_s is a geometric Brownian motion with zero drift, given x_0, $\mathbb{E}[x_s^2] = x_0^2 \exp(\sigma^2 s)$. Thus,

$$\mathbb{E}\left[\int_0^T |\theta_-^*(s)|^2 ds\right] = (x_0^2/\lambda^2)\int_0^T \exp(\sigma^2 s) ds = (x_0^2/\lambda^2)\frac{e^{\sigma^2 T}-1}{\sigma^2} < \infty.$$

Thus, θ_-^* is admissible.

4. **Solving the ODE:** Substituting θ_-^* into the HJB equation:

$$\beta V_-(x) = \frac{x^2}{2\lambda} + \frac{1}{2}\sigma^2 x^2 V_-''(x).$$

We guess a solution of the form $V_-(x) = Cx^\gamma + Dx^2$. Substituting:

$$\beta(Cx^\gamma + Dx^2) = \frac{x^2}{2\lambda} + \frac{1}{2}\sigma^2 x^2 (C\gamma(\gamma-1)x^{\gamma-2})$$

$$\beta C x^\gamma + \beta D x^2 = \frac{x^2}{2\lambda} + \frac{1}{2}\sigma^2 C\gamma(\gamma-1)x^\gamma.$$

Matching coefficients:

- For x^γ: $\beta C = \frac{1}{2}\sigma^2 C\gamma(\gamma-1) \implies \gamma(\gamma-1) = \frac{2\beta}{\sigma^2}$ (same as before).
- For x^2: $\beta D = \frac{1}{2\lambda} + \sigma^2 D \implies D(\beta - \sigma^2) = \frac{1}{2\lambda} \implies D = \frac{1}{2\lambda(\beta-\sigma^2)}$.

We get the same characteristic equation for γ as before, so $\gamma = \gamma_1$ or $\gamma = \gamma_2$. Since $\gamma_1 > 1$, to satisfy the condition that $\lim_{x\to 0} V_-(x)$ exists (and is finite), we choose γ_1. Thus, the general solution for $0 < x \leq 1$ is:

$$V_-(x) = Cx^{\gamma_1} + \frac{x^2}{2\lambda(\beta-\sigma^2)}.$$

Combining the Solutions and Applying Boundary Conditions:

1. **Value Matching:** $V_+(1) = V_-(1)$. This gives:

$$A + \frac{1}{2\lambda_c \beta} = C + \frac{1}{2\lambda(\beta-\sigma^2)} = V_0.$$

2. **Smooth Pasting:** $V'_+(1) = V'_-(1)$. We have:

$$V'_+(x) = A\gamma_2 x^{\gamma_2 - 1} + \frac{1}{2\lambda_c \beta}, \quad V'_-(x) = C\gamma_1 x^{\gamma_1 - 1} + \frac{x}{\lambda(\beta - \sigma^2)}.$$

Evaluating at $x = 1$:

$$A\gamma_2 + \frac{1}{2\lambda_c \beta} = C\gamma_1 + \frac{1}{\lambda(\beta - \sigma^2)}.$$

3. **Solving for V_0, A, and C:** From value matching, $A = V_0 - \frac{1}{2\lambda_c \beta}$ and $C = V_0 - \frac{1}{2\lambda(\beta - \sigma^2)}$. Substituting into the smooth pasting condition (and noting $\lambda_c = \lambda$ in this case):

$$\left(V_0 - \frac{1}{2\lambda\beta}\right)\gamma_2 + \frac{1}{2\lambda\beta} = \left(V_0 - \frac{1}{2\lambda(\beta - \sigma^2)}\right)\gamma_1 + \frac{1}{\lambda(\beta - \sigma^2)}.$$

Solving for V_0:

$$V_0(\gamma_2 - \gamma_1) = \frac{\gamma_2 - 1}{2\lambda\beta} - \frac{\gamma_1 - 2}{2\lambda(\beta - \sigma^2)} \implies V_0 = \frac{1}{2\lambda(\gamma_2 - \gamma_1)}\left(\frac{\gamma_2 - 1}{\beta} - \frac{\gamma_1 - 2}{\beta - \sigma^2}\right).$$

We then find A and C using the value matching conditions:

$$A = V_0 - \frac{1}{2\lambda\beta}, \qquad C = V_0 - \frac{1}{2\lambda(\beta - \sigma^2)}.$$

Case (iii): $0 < x \leq 1$ **(Losses Region,** $\lambda_c \neq \lambda_a$**):**

1. **Substituting the Cost Function:** The HJB equation becomes:

   - For $x \geq 1$, $\beta V = \sup_\theta \{\theta x - \frac{\lambda_c x}{2}\theta^2 + \frac{1}{2}\sigma^2 x^2 V_{xx}\}$.
   - For $0 < x \leq 1$, $\beta V = \sup_\theta \{\theta x - \frac{\lambda_c x + \lambda_a(1-x)}{2}\theta^2 + \frac{1}{2}\sigma^2 x^2 V_{xx}\}$.

2. **Finding Optimal θ:**

   - For $x \geq 1$: $\theta^*_+ = \frac{1}{\lambda_c}$ (same as before).

- For $0 < x \leq 1$: Taking the derivative with respect to θ and setting it to zero:

$$\frac{\partial}{\partial \theta}\left(\theta x - \frac{\lambda_c x + \lambda_a(1-x)}{2}\theta^2\right) = x - (\lambda_c x + \lambda_a(1-x))\theta = 0,$$

which gives

$$\theta_-^* = \frac{x}{\lambda_c x + \lambda_a(1-x)}.$$

3. **Admissibility of θ_-^***: We need to show that $\mathbb{E}\left[\int_0^T |\theta_-^*(s)|^2 ds\right] < \infty$ for all $T > 0$. Since $0 < x \leq 1$ and $\lambda_c, \lambda_a > 0$, we have:

$$0 < \theta_-^*(s) = \frac{x_s}{\lambda_c x_s + \lambda_a(1-x_s)} \leq \frac{x_s}{\lambda_a(1-x_s)}.$$

We also observe that

$$0 < \theta_-^*(s) \leq \frac{1}{\min\{\lambda_a, \lambda_c\}}.$$

Since x_s follows a geometric Brownian motion and $\theta_-^*(s)$ is bounded, we have: $\mathbb{E}\left[\int_0^T |\theta_-^*(s)|^2 ds\right] \leq \frac{1}{(\min\{\lambda_c,\lambda_a\})^2}\int_0^T \mathbb{E}[1] ds = \frac{T}{(\min\{\lambda_c,\lambda_a\})^2} < \infty$ for all $T > 0$. Thus, θ_-^* is admissible.

4. **Solving the ODE:**

- For $x \geq 1$: The solution is the same as in Case (i):

$$V_+(x) = \left(V_0 - \frac{1}{2\lambda_c \beta}\right) x^{\gamma_2} + \frac{x}{2\lambda_c \beta}.$$

- For $0 < x \leq 1$: Substituting θ_-^* into the HJB equation yields:

$$\beta V_-(x) = \frac{x^2}{2(\lambda_c x + \lambda_a(1-x))} + \frac{1}{2}\sigma^2 x^2 V_-''(x). \tag{27}$$

This is a second-order linear non-homogeneous ODE. We can rewrite it in standard form:

$$V_-''(x) - \frac{2\beta}{\sigma^2 x^2} V_-(x) = -\frac{1}{\sigma^2(\lambda_c x + \lambda_a(1-x))}.$$

The homogeneous solution to this equation is of the form $V_h(x) = C_1 x^{\gamma_1} + C_2 x^{\gamma_2}$, where γ_1 and γ_2 are the same roots as before, determined by the characteristic equation $\gamma(\gamma-1) = \frac{2\beta}{\sigma^2}$.

- *Derivation of the Particular Solution (using Variation of Parameters):* We seek a particular solution of the form $V_p(x) = u_1(x)x^{\gamma_1} + u_2(x)x^{\gamma_2}$, where $u_1(x)$ and $u_2(x)$ are functions to be determined. The method of variation of parameters gives us the following system of equations:

$$u_1'(x)x^{\gamma_1} + u_2'(x)x^{\gamma_2} = 0,$$
$$u_1'(x)\gamma_1 x^{\gamma_1-1} + u_2'(x)\gamma_2 x^{\gamma_2-1} = -\frac{1}{\sigma^2(\lambda_c x + \lambda_a(1-x))}.$$

This is a system of linear equations in $u_1'(x)$ and $u_2'(x)$. Multiplying the first equation by $\gamma_1 x^{-1}$ and subtracting it from the second equation, we get:

$$u_2'(x)(\gamma_2 - \gamma_1)x^{\gamma_2-1} = -\frac{1}{\sigma^2(\lambda_c x + \lambda_a(1-x))},$$

which gives

$$u_2'(x) = \frac{x^{1-\gamma_2}}{\sigma^2(\gamma_1 - \gamma_2)(\lambda_c x + \lambda_a(1-x))}.$$

Substituting this back into the first equation, we get:

$$u_1'(x) = -u_2'(x)x^{\gamma_2-\gamma_1} = -\frac{x^{1-\gamma_1}}{\sigma^2(\gamma_1 - \gamma_2)(\lambda_c x + \lambda_a(1-x))}.$$

Letting $k = \frac{\lambda_c - \lambda_a}{\lambda_a}$, we can rewrite these as:

$$u_1'(x) = -\frac{1}{\lambda_a \sigma^2 (\gamma_1 - \gamma_2)} \frac{x^{1-\gamma_1}}{1+kx},$$
$$u_2'(x) = \frac{1}{\lambda_a \sigma^2 (\gamma_1 - \gamma_2)} \frac{x^{1-\gamma_2}}{1+kx}.$$

Now we integrate to find $u_1(x)$ and $u_2(x)$. Using the integral formula involving the hypergeometric function:

$$\int \frac{x^{p-1}}{1+kx} dx = \frac{x^p}{p} {}_2F_1(1, p; p+1; -kx) + C,$$

where the integral converges for $|kx| < 1$, we obtain:

$$u_1(x) = -\frac{1}{\lambda_a \sigma^2 (\gamma_1 - \gamma_2)} \frac{x^{2-\gamma_1}}{2-\gamma_1} {}_2F_1(1, 2-\gamma_1; 3-\gamma_1; -kx) + C_1,$$
$$u_2(x) = \frac{1}{\lambda_a \sigma^2 (\gamma_1 - \gamma_2)} \frac{x^{2-\gamma_2}}{2-\gamma_2} {}_2F_1(1, 2-\gamma_2; 3-\gamma_2; -kx) + C_2.$$

We can set the constants of integration $C_1 = C_2 = 0$ to find a particular solution. Thus:

$$\begin{aligned} V_p^*(x) &= u_1(x) x^{\gamma_1} + u_2(x) x^{\gamma_2} \\ &= \frac{1}{\sigma^2 \lambda_a (\gamma_2 - \gamma_1)} \left[\frac{x^2}{2-\gamma_2} {}_2F_1(1, 2-\gamma_2; 3-\gamma_2; -kx) \right. \\ &\quad \left. - \frac{x^2}{2-\gamma_1} {}_2F_1(1, 2-\gamma_1; 3-\gamma_1; -kx) \right]. \end{aligned}$$

The general solution is then $V_-(x) = B_1 x^{\gamma_1} + V_p^*(x)$, where B_1 is a constant determined by the boundary conditions, and we choose the positive root γ_1 to ensure the solution remains bounded as x approaches 0.

5. **Value Matching and Smooth Pasting:**

- **Value Matching:** $V_+(1) = V_-(1)$. This gives:

$$V_0 = B_1 + V_p^*(1).$$

- **Smooth Pasting:** $V_+'(1) = V_-'(1)$. This gives:

$$\left(V_0 - \frac{1}{2\lambda_c\beta}\right)\gamma_2 + \frac{1}{2\lambda_c\beta} = B_1\gamma_1 + \left.\frac{dV_p^*(x)}{dx}\right|_{x=1},$$

where

$$\begin{aligned}\frac{dV_p^*(x)}{dx} &= \frac{d}{dx}\left[u_1(x)x^{\gamma_1} + u_2(x)x^{\gamma_2}\right] \\ &= u_1'(x)x^{\gamma_1} + \gamma_1 u_1(x)x^{\gamma_1-1} + u_2'(x)x^{\gamma_2} + \gamma_2 u_2(x)x^{\gamma_2-1}.\end{aligned}$$

Substituting the expressions for $u_1'(x)$, $u_2'(x)$, $u_1(x)$, and $u_2(x)$ and evaluating at $x = 1$, we obtain the expression for $\left.\frac{dV_p^*(x)}{dx}\right|_{x=1}$ given in the proposition statement.

These two equations (value matching and smooth pasting) form a system of two equations in two unknowns (V_0 and B_1). This system is solved numerically.

6. **Verification of Convergence of the Hypergeometric Series:** The hypergeometric series converges for $|-kx| < 1$. Since $0 < x \leq 1$, this requires $|k| < 1$, which is equivalent to $|(\lambda_c - \lambda_a)/\lambda_a| < 1$. This simplifies to $0 < \lambda_c < 2\lambda_a$.

Q.E.D.

A.1. Proof of Theorem 3

PROOF: Let $v(t,x)$ be an arbitrary admissible control, and define $J(t,x;v)$ as the objective function in (11) with control v:

$$J(t,x;v) = \mathbb{E}\left[\int_t^\infty e^{-\beta(s-t)}\left(v_s x_s - \mathcal{L}(v_s,x_s)\right)ds \middle| x_t = x\right].$$

Let $\tau_n = \inf\{t \geq 0 : x_t \geq n \text{ or } x_t \leq 1/n\}$. Since x_t is a continuous process, $\tau_n \to \infty$ almost surely as $n \to \infty$. Applying Itô's Lemma to $e^{-\beta(s-t)}V(x_s)$ from t to $\tau_n \wedge T$ (where $T > t$):

1. **Apply Itô's Lemma:**

$$e^{-\beta(\tau_n \wedge T - t)}V(x_{\tau_n \wedge T}) - V(x_t)$$
$$= \int_t^{\tau_n \wedge T} e^{-\beta(s-t)} \left(-\beta V(x_s) + \frac{1}{2}\sigma^2 x_s^2 V_{xx}(x_s) \right) ds + \int_t^{\tau_n \wedge T} e^{-\beta(s-t)} \sigma x_s V_x(x_s) d\mathcal{Z}_s.$$

2. **Take Expectations:**

$$\mathbb{E}\left[e^{-\beta(\tau_n \wedge T - t)}V(x_{\tau_n \wedge T}) \right] - V(x_t)$$
$$= \mathbb{E}\left[\int_t^{\tau_n \wedge T} e^{-\beta(s-t)} \left(-\beta V(x_s) + \frac{1}{2}\sigma^2 x_s^2 V_{xx}(x_s) \right) ds \right].$$

   The expectation of the stochastic integral is zero because $V_x(x)$ satisfies appropriate integrability conditions due to the growth condition on $V(x)$ and the properties of geometric Brownian motion.

3. **Use HJB Inequality:** Since V solves the HJB equation, for any admissible control v_s:

$$\beta V(x_s) \geq v_s x_s - \mathcal{L}(v_s, x_s) + \frac{1}{2}\sigma^2 x_s^2 V_{xx}(x_s).$$

4. **Substitute and Rearrange:** Substituting this inequality into the previous equation:

$$\mathbb{E}\left[e^{-\beta(\tau_n \wedge T - t)}V(x_{\tau_n \wedge T}) \right] - V(x_t) \leq \mathbb{E}\left[\int_t^{\tau_n \wedge T} e^{-\beta(s-t)} \left(-v_s x_s + \mathcal{L}(v_s, x_s) \right) ds \right].$$

   Rearranging:

$$V(x_t) \geq \mathbb{E}\left[\int_t^{\tau_n \wedge T} e^{-\beta(s-t)} \left(v_s x_s - \mathcal{L}(v_s, x_s) \right) ds \right] - \mathbb{E}\left[e^{-\beta(\tau_n \wedge T - t)}V(x_{\tau_n \wedge T}) \right].$$

5. **Take Limits:** Since $|V(x)| \leq K(1+x^{\alpha})$ and x_t is a geometric Brownian motion, and $\tau_n \to \infty$ as $n \to \infty$, we have $\lim_{n\to\infty} \mathbb{E}\left[e^{-\beta(\tau_n \wedge T - t)} V(x_{\tau_n \wedge T})\right] = \mathbb{E}\left[e^{-\beta(T-t)} V(x_T)\right]$. Also, by the Monotone Convergence Theorem:

$$\lim_{n\to\infty} \mathbb{E}\left[\int_t^{\tau_n \wedge T} e^{-\beta(s-t)} \left(v_s x_s - \mathcal{L}(v_s, x_s)\right) ds\right] = \mathbb{E}\left[\int_t^{T} e^{-\beta(s-t)} \left(v_s x_s - \mathcal{L}(v_s, x_s)\right) ds\right].$$

Taking the limit as $n \to \infty$:

$$V(x_t) \geq \mathbb{E}\left[\int_t^{T} e^{-\beta(s-t)} \left(v_s x_s - \mathcal{L}(v_s, x_s)\right) ds\right] - \mathbb{E}\left[e^{-\beta(T-t)} V(x_T)\right].$$

Taking the limit as $T \to \infty$, and using the transversality condition, the second term vanishes:

$$V(x_t) \geq \mathbb{E}\left[\int_t^{\infty} e^{-\beta(s-t)} \left(v_s x_s - \mathcal{L}(v_s, x_s)\right) ds\right] = J(t, x; v).$$

Since v is arbitrary, $V(x_t) \geq \sup_{v\in\mathcal{A}} J(t, x; v)$.

6. **Optimal Control:** Consider the optimal control θ^* from the maximization in the HJB equation. Applying Itô's Lemma with θ^*, we have equality, and thus $V(x_t) = J(t, x; \theta^*)$. This implies $V(x_t) = \sup_{v\in\mathcal{A}} J(t, x; v)$.

Q.E.D.

A.2. Proof of Proposition 5

PROOF:

(i) **Asymmetric Value Differences:** This follows directly from the different functional forms of $V_+(x)$ and $V_-(x)$, resulting from the asymmetric cost function. While the smooth pasting condition ensures $V_+'(1) = V_-'(1)$, the overall shape of the value function is steeper for losses than for gains in a neighborhood of $x = 1$. This is demonstrated numerically in Figure 3 and its caption. A formal analytic proof for all parameter values and all Δx is cumbersome due to the complexity of $V_-(x)$ in the general case ($\lambda_c \neq \lambda_a$). However, the numerical

evidence and the structure of the solutions (different effort levels) strongly support this.

(ii) **Greater Curvature for Losses:** We examine the second derivatives of $V_+(x)$ and $V_-(x)$:

- **For $x \geq 1$ (Gains):**

$$V_+(x) = \left(V_0 - \frac{1}{2\lambda_c\beta}\right) x^{\gamma_2} + \frac{x}{2\lambda_c\beta}$$
$$V'_+(x) = \left(V_0 - \frac{1}{2\lambda_c\beta}\right) \gamma_2 x^{\gamma_2-1} + \frac{1}{2\lambda_c\beta}$$
$$V''_+(x) = \left(V_0 - \frac{1}{2\lambda_c\beta}\right) \gamma_2(\gamma_2-1) x^{\gamma_2-2}$$
$$\lim_{x\to 1^+} V''_+(x) = \left(V_0 - \frac{1}{2\lambda_c\beta}\right) \gamma_2(\gamma_2-1),$$

- **For $0 < x \leq 1$ and $\lambda_c = \lambda_a = \lambda$ (Losses - Simplified Case):**

$$V_-(x) = \left(V_0 - \frac{1}{2\lambda(\beta-\sigma^2)}\right) x^{\gamma_1} + \frac{x^2}{2\lambda(\beta-\sigma^2)}$$
$$V'_-(x) = \left(V_0 - \frac{1}{2\lambda(\beta-\sigma^2)}\right) \gamma_1 x^{\gamma_1-1} + \frac{x}{\lambda(\beta-\sigma^2)}$$
$$V''_-(x) = \left(V_0 - \frac{1}{2\lambda(\beta-\sigma^2)}\right) \gamma_1(\gamma_1-1) x^{\gamma_1-2} + \frac{1}{\lambda(\beta-\sigma^2)}$$
$$\lim_{x\to 1^-} V''_-(x) = \left(V_0 - \frac{1}{2\lambda(\beta-\sigma^2)}\right) \gamma_1(\gamma_1-1) + \frac{1}{\lambda(\beta-\sigma^2)},$$

- **For $0 < x \leq 1$ and $\lambda_c \neq \lambda_a$ (Losses - General Case):** This case is more complex due to the presence of the hypergeometric function in $V_p^*(x)$. We have:

$$V_-(x) = B_1 x^{\gamma_1} + V_p^*(x),$$

where $V_p^*(x)$ is given by equation (19). We need to compute the second derivative of $V_p^*(x)$, which is:

$$\begin{aligned}\frac{dV_p^*(x)}{dx} = \frac{1}{\sigma^2\lambda_a(\gamma_2-\gamma_1)}\Bigg[&\frac{2x}{2-\gamma_1}{}_2F_1(1,2-\gamma_1;3-\gamma_1;-kx)\\&+\frac{kx^2}{3-\gamma_1}{}_2F_1(2,3-\gamma_1;4-\gamma_1;-kx)\\&-\frac{2x}{2-\gamma_2}{}_2F_1(1,2-\gamma_2;3-\gamma_2;-kx)\\&-\frac{kx^2}{3-\gamma_2}{}_2F_1(2,3-\gamma_2;4-\gamma_2;-kx)\Bigg].\end{aligned}$$

Taking derivative once again, we have

$$\begin{aligned}\frac{d^2V_p^*(x)}{dx^2} = \frac{1}{\sigma^2\lambda_a(\gamma_2-\gamma_1)}\Bigg[&\frac{2}{2-\gamma_1}{}_2F_1(1,2-\gamma_1;3-\gamma_1;-kx)\\&+\frac{4kx}{3-\gamma_1}{}_2F_1(2,3-\gamma_1;4-\gamma_1;-kx)\\&+\frac{2k^2x^2}{(3-\gamma_1)(4-\gamma_1)}{}_2F_1(3,4-\gamma_1;5-\gamma_1;-kx)\\&-\frac{2}{2-\gamma_2}{}_2F_1(1,2-\gamma_2;3-\gamma_2;-kx)\\&-\frac{4kx}{3-\gamma_2}{}_2F_1(2,3-\gamma_2;4-\gamma_2;-kx)\\&-\frac{2k^2x^2}{(3-\gamma_2)(4-\gamma_2)}{}_2F_1(3,4-\gamma_2;5-\gamma_2;-kx)\Bigg].\end{aligned}$$

Therefore,

$$V_-''(x) = B_1\gamma_1(\gamma_1-1)x^{\gamma_1-2} + \frac{d^2V_p^*(x)}{dx^2}.$$

The limit is therefore

$$\begin{aligned}\lim_{x\to 1^-} V_-''(x) = B_1\gamma_1(\gamma_1-1) \\ &+ \frac{1}{\sigma^2\lambda_a(\gamma_2-\gamma_1)}\left[\frac{2}{2-\gamma_1}{}_2F_1(1,2-\gamma_1;3-\gamma_1;-k)\right. \\ &+ \frac{4k}{3-\gamma_1}{}_2F_1(2,3-\gamma_1;4-\gamma_1;-k) \\ &+ \frac{2k^2}{(3-\gamma_1)(4-\gamma_1)}{}_2F_1(3,4-\gamma_1;5-\gamma_1;-k) \\ &- \frac{2}{2-\gamma_2}{}_2F_1(1,2-\gamma_2;3-\gamma_2;-k) \\ &- \frac{4k}{3-\gamma_2}{}_2F_1(2,3-\gamma_2;4-\gamma_2;-k) \\ &\left.- \frac{2k^2}{(3-\gamma_2)(4-\gamma_2)}{}_2F_1(3,4-\gamma_2;5-\gamma_2;-k)\right].\end{aligned}$$

Comparing $\lim_{x\to 1^+} V_+''(x)$ and $\lim_{x\to 1^-} V_-''(x)$ analytically is difficult in the general case. However, we can make the following observations:

- Since $\gamma_2 < 0$ and $\gamma_2 - 1 < 0$, the term $\gamma_2(\gamma_2-1)$ is positive. Therefore, $V_+''(x)$ is always positive, indicating concavity in the gains region.
- In the simplified case ($\lambda_c = \lambda_a$), since $\gamma_1 > 1$ and $\gamma_1 - 1 > 0$, the term $\left(V_0 - \frac{1}{2\lambda(\beta-\sigma^2)}\right)\gamma_1(\gamma_1-1)$ can be positive or negative depending on the value of V_0. However, the term $\frac{1}{\lambda(\beta-\sigma^2)}$ is always positive. The overall sign of $V_-''(x)$ is ambiguous without further analysis, but it will generally have a larger magnitude than $V_+''(1)$ due to the asymmetry introduced by the optimal effort.
- In the general case ($\lambda_c \neq \lambda_a$), numerical evaluation is necessary to definitively compare the magnitudes of the second derivatives. However, the presence of λ_a in the cost function for losses, and the resulting different functional form for $V_-(x)$, leads to greater curvature (more negative second derivative) for losses compared to gains in a neighborhood of

$x = 1$ for a wide range of parameter values. The different curvature is the manifestation of loss aversion.

Q.E.D.

A.3. Proof of Proposition 6

PROOF: Let X and Y be two random variables representing normalized consumption, with cumulative distribution functions (CDFs) $F_X(x)$ and $F_Y(x)$, respectively. First-order stochastic dominance states that if $F_X(x) \leq F_Y(x)$ for all x, and $F_X(x) < F_Y(x)$ for some x, then X is preferred to Y. In our model, this means $\mathbb{E}[V(X)] \geq \mathbb{E}[V(Y)]$. Since $V(x)$ is a strictly increasing function of x (because $V'(x) > 0$ for all $x > 0$, which we have established in the derivation of the solution), this condition holds. To show this formally, we can use integration by parts. Let $v(x) = V(x)$. Then:

$$\begin{aligned}\mathbb{E}[v(X)] &= \int_0^\infty v(x) dF_X(x) = \int_0^\infty v(x) f_X(x) dx \\ &= v(x)F_X(x)\Big|_0^\infty - \int_0^\infty F_X(x)v'(x)dx \\ &= -\int_0^\infty F_X(x)v'(x)dx,\end{aligned}$$

where the boundary $v(x)F_X(x)\Big|_0^\infty$ vanishes because $F_X(0) = 0$ and $\lim_{x\to\infty} F_X(x)v(x) = 0$ (due to $F_X(\infty) = 1$ and the growth condition on $V(x)$ in the verification theorem). Similarly, $\mathbb{E}[v(Y)] = -\int_0^\infty F_Y(x)v'(x)dx$. Therefore:

$$\mathbb{E}[v(X)] - \mathbb{E}[v(Y)] = -\int_0^\infty [F_X(x) - F_Y(x)]v'(x)dx.$$

If X stochastically dominates Y, then $F_X(x) \leq F_Y(x)$ for all x, so $F_X(x) - F_Y(x) \leq 0$. Since $v'(x) = V'(x) > 0$ for all x, the integrand is non-positive for all x. Thus, the integral is non-negative:

$$\mathbb{E}[V(X)] - \mathbb{E}[V(Y)] \geq 0 \implies \mathbb{E}[V(X)] \geq \mathbb{E}[V(Y)].$$

This shows that the model does not violate first-order stochastic dominance. *Q.E.D.*

APPENDIX B: Proof of Proposition 7

Proof: We want to show that as $\lambda_a \to \infty$, the value function of the NDU model converges to a value function with a kink at $x = 1$, and the agent's decision-making behavior converges to that implied by a CPT model. We proceed in several steps:

1. **Rescaling and Parameterization:** Let $\epsilon = 1/\lambda_a$. As $\lambda_a \to \infty$, $\epsilon \to 0$. Rewrite the HJB equation (Equation 13) for $0 < x \leq 1$ in terms of ϵ:

$$\beta V(x) = \sup_{\theta} \left\{ \theta x - \frac{\lambda_c x + (1-x)/\epsilon}{2} \theta^2 + \frac{1}{2}\sigma^2 x^2 V_{xx}(x) \right\}. \tag{28}$$

   This is a singular perturbation problem because the cost term contains $1/\epsilon$, which becomes unbounded as $\epsilon \to 0$, while the coefficient of the second derivative, V_{xx}, remains independent of ϵ.

2. **Outer Solution (x Bounded Away from 1):** For a fixed $x < 1$, as $\epsilon \to 0$, the term $(1-x)/\epsilon$ dominates the cost function. The optimal control in the losses region (from Proposition 2) is:

$$\theta_-^*(x, \epsilon) = \frac{x}{\lambda_c x + (1-x)/\epsilon}.$$

   Taking the limit as $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \theta_-^*(x, \epsilon) = 0.$$

   Substituting this into the HJB equation (28), we obtain the outer equation:

$$\beta V_-(x) = \frac{1}{2}\sigma^2 x^2 V_-''(x). \tag{29}$$

   The general solution to this equation is $V_-(x) = A_1 x^{\gamma_1} + A_2 x^{\gamma_2}$, where γ_1 and γ_2 are the roots of the characteristic equation $\gamma(\gamma - 1) = 2\beta/\sigma^2$, as defined in Proposition 2. Since $\gamma_2 < 0$ and we require the solution to be bounded as

$x \to 0$, we must have $A_2 = 0$. Thus, the outer solution is:

$$V_-(x) = A_1 x^{\gamma_1}, \quad \text{for } 0 < x < 1. \tag{30}$$

3. **Inner Solution (Near $x = 1$):** To analyze the behavior near $x = 1$, we introduce a stretched variable $y = (1 - x)/\epsilon$. This rescales the region near $x = 1$. As $\epsilon \to 0$, this region shrinks, but the stretched variable y allows us to examine the behavior in detail. Let $\tilde{V}(y) = V(1 - \epsilon y)$. Then:

$$V_x = \frac{dV}{dx} = \frac{d\tilde{V}}{dy}\frac{dy}{dx} = -\epsilon^{-1}\tilde{V}'(y),$$

$$V_{xx} = \frac{d^2V}{dx^2} = \epsilon^{-2}\tilde{V}''(y).$$

Substituting into the HJB equation (28) for $0 < x < 1$, we get:

$$\beta\tilde{V}(y) = \sup_\theta \left\{ \theta(1 - \epsilon y) - \frac{\lambda_c(1 - \epsilon y) + y}{2}\theta^2 + \frac{1}{2}\sigma^2(1 - \epsilon y)^2\epsilon^{-2}\tilde{V}''(y) \right\}. \tag{31}$$

We argue that the term $\epsilon^{-2}\tilde{V}''(y)$ vanishes as $\epsilon \to 0$. This is the most delicate part of the proof. We proceed by contradiction, similar in spirit to viscosity solution arguments, but adapted to this specific problem.

Claim 8: $\lim_{\epsilon\to 0} \epsilon^{-2}\tilde{V}''(y) = 0$ *uniformly on compact subsets of* $(0, \infty)$.

Proof of Claim: Suppose, for the sake of contradiction, that the claim is false. Then there exists a compact set $K \subset (0, \infty)$, a sequence $\epsilon_n \to 0$, and a sequence $y_n \in K$ such that either $\epsilon_n^{-2}\tilde{V}''(y_n; \epsilon_n) \to +\infty$ or $\epsilon_n^{-2}\tilde{V}''(y_n; \epsilon_n) \to -\infty$ as $n \to \infty$. (Here, we explicitly write $\tilde{V}(y; \epsilon)$ to emphasize the dependence on ϵ).

- *Case 1:* $\epsilon_n^{-2}\tilde{V}''(y_n; \epsilon_n) \to +\infty$. This implies that $\tilde{V}''(y_n; \epsilon_n)$ grows faster than ϵ_n^2. Since $(1 - \epsilon_n y_n)^2$ is bounded (because y_n is in a compact set), the entire term $\frac{1}{2}\sigma^2(1 - \epsilon_n y_n)^2\epsilon_n^{-2}\tilde{V}''(y_n; \epsilon_n)$ goes to $+\infty$. This would imply that $\beta\tilde{V}(y_n; \epsilon_n) \to +\infty$, contradicting the growth condition on the value function (Theorem 3).

- *Case 2:* $\epsilon_n^{-2}\tilde{V}''(y_n;\epsilon_n) \to -\infty$. In this case, the HJB equation (31) would become, in the limit,

$$\beta\tilde{V}(y) = -\infty + \sup_{\theta}\left\{\theta(1-\epsilon y) - \frac{\lambda_c(1-\epsilon y)+y}{2}\theta^2\right\}.$$

  This is also a contradiction, as the value function must be finite.

Therefore, our assumption that $\epsilon_n^{-2}\tilde{V}''(y_n;\epsilon_n)$ does not converge to 0 must be false. This establishes the claim. *Q.E.D.*

With the second derivative term vanishing, the HJB equation (31) simplifies to:

$$\beta\tilde{V}(y) = \sup_{\theta}\left\{\theta - \frac{\lambda_c+y}{2}\theta^2\right\}. \tag{32}$$

The optimal control is $\tilde{\theta}^* = 1/(\lambda_c+y)$. Substituting this back into (32) gives:

$$\beta\tilde{V}(y) = \frac{1}{2(\lambda_c+y)}.$$

The solution to this is:

$$\tilde{V}(y) = \frac{1}{2\beta(\lambda_c+y)} + C, \tag{33}$$

where C is a constant of integration.

4. **Matching:** We need to match the inner solution (33) with the outer solution (30) and the solution for $x \geq 1$ (the gains region). We use an intermediate variable approach. Let $\eta = (1-x)/\epsilon^\alpha$, where $0 < \alpha < 1$. As $\epsilon \to 0$, $\eta \to \infty$ if x is fixed and less than 1. The matching condition requires that:

$$\lim_{\epsilon\to 0} V_-(1-\epsilon^\alpha\eta) = \lim_{\epsilon\to 0}\tilde{V}(\epsilon^{\alpha-1}\eta). \tag{34}$$

For the outer solution, $V_-(1-\epsilon^\alpha\eta) = A_1(1-\epsilon^\alpha\eta)^{\gamma_1}$. As $\epsilon \to 0$, this approaches A_1. For the inner solution, $\tilde{V}(\epsilon^{\alpha-1}\eta) = \frac{1}{2\beta(\lambda_c+\epsilon^{\alpha-1}\eta)} + C$. As $\epsilon \to 0$, since $\alpha - 1 < 0$, this approaches C. Therefore, the matching condition requires $A_1 = C$. We

also need to match the inner solution with the gains region solution, $V_+(x)$. As $\epsilon \to 0$, $V_+(1) = V_-(1) = V_0$, and $V'_+(1) = V'_-(1)$. Recall,

$$V_+(x) = \left(V_0 - \frac{1}{2\lambda_c\beta}\right) x^{\gamma_2} + \frac{x}{2\lambda_c\beta}, \quad \theta^*_+ = \frac{1}{\lambda_c}.$$

5. **Asymptotic Value Function:** As $\epsilon \to 0$ (i.e., $\lambda_a \to \infty$), the value function converges to:

$$V^*(x) = \begin{cases} A_1 x^{\gamma_1}, & 0 < x < 1 \\ \left(V_0 - \frac{1}{2\lambda_c\beta}\right) x^{\gamma_2} + \frac{x}{2\lambda_c\beta}, & x \geq 1 \end{cases} \tag{35}$$

The smooth pasting condition no longer holds in the limit.

6. **The Kink and Loss Aversion Coefficient:** The limiting value function $V^*(x)$ has a kink at $x = 1$. The left-hand derivative at $x = 1$ is:

$$\lim_{x \to 1^-} \frac{dV^*}{dx} = A_1\gamma_1.$$

The right-hand derivative at $x = 1$ is:

$$\lim_{x \to 1^+} \frac{dV^*}{dx} = \left(V_0 - \frac{1}{2\lambda_c\beta}\right)\gamma_2 + \frac{1}{2\lambda_c\beta}.$$

From smooth pasting condition, we know that

$$V'_+(1) = \left(V_0 - \frac{1}{2\lambda_c\beta}\right)\gamma_2 + \frac{1}{2\lambda_c\beta} = A_1\gamma_1.$$

The loss aversion coefficient, L, in the CPT representation can be related to the ratio of these derivatives as follows. To relate $V^*(x)$ to a CPT value function, consider the form $v(x) = x^\alpha$ for $x \geq 1$ and $v(x) = -L(-x)^\alpha$ for $x < 1$. The derivatives are:

$$v'(x) = \begin{cases} \alpha x^{\alpha-1}, & x \geq 1 \\ L\alpha(-x)^{\alpha-1}, & x < 1 \end{cases}$$

We match the exponents to the exponents in our limiting value function, so $\alpha = \gamma_1$. The CPT loss aversion parameter L can be approximated as the ratio of the left and right hand-side derivative:

$$L = \frac{A_1\gamma_1}{\left(V_0 - \frac{1}{2\lambda_c\beta}\right)\gamma_2 + \frac{1}{2\lambda_c\beta}}$$

Because $A_1\gamma_1 = \left(V_0 - \frac{1}{2\lambda_c\beta}\right)\gamma_2 + \frac{1}{2\lambda_c\beta}$, so $L = 1$.

7. **Convergence of Certainty Equivalents:** Let $CE(\epsilon)$ be the certainty equivalent of a lottery under the NDU model with parameter $\epsilon = 1/\lambda_a$. Let CE_{CPT} be the certainty equivalent of the same lottery under a CPT model with the value function described above and a probability weighting function $\pi(p)$. We want to show that $\lim_{\epsilon\to 0} CE(\epsilon) = CE_{CPT}$. Consider a simple lottery $(x_1, p; x_2, 1-p)$ with $x_1 < 1 < x_2$. The NDU certainty equivalent is defined by:

$$V(CE(\epsilon);\epsilon) = pV(x_1;\epsilon) + (1-p)V(x_2;\epsilon),$$

where we write $V(x;\epsilon)$ to emphasize the dependence on ϵ. The CPT certainty equivalent is defined by:

$$v(CE_{CPT}) = \pi(p)v(x_1) + (1-\pi(p))v(x_2),$$

where $v(x) = x^{\gamma_1}$ for $x \geq 1$ and $v(x) = -(-x)^{\gamma_1}$ for $x < 1$. Because $V(x;\epsilon)$ converges pointwise to $V^*(x)$ as $\epsilon \to 0$, and because the probability of x_t landing exactly at $x = 1$ is zero, the certainty equivalent under the NDU model will converge to that of the CPT model.

Q.E.D.

APPENDIX C: Proof of Proposition 4 (Emergent Probability Weighting)

Proof: The proof proceeds by construction. We will demonstrate how to derive a probability weighting function, $\pi(p)$, and a representation value function, $v(x)$,

that satisfy Definition 3, such that the certainty equivalents generated by the NDU model's value function, $V(x)$, are replicated by the representation.

- **Step 1: Setup and Notation**

  - Consider the NDU model with the value function $V(x)$ as defined in Proposition 2. This value function is strictly increasing and continuous (except possibly for a kink at $x = 1$ in the limiting case where $\lambda_a \to \infty$).
  - Let $\mathcal{L}$ be the set of all simple lotteries over normalized consumption outcomes $x \in \mathbb{R}^+$. A lottery $L \in \mathcal{L}$ is represented as $L = (x_1, p_1; \ldots; x_n, p_n)$, where x_i are the outcomes and p_i are the corresponding objective probabilities (with $\sum_{i=1}^{n} p_i = 1$).
  - The NDU model's certainty equivalent for lottery L, denoted $CE_V(L)$, is defined implicitly by:

$$V(CE_V(L)) = \sum_{i=1}^{n} p_i V(x_i). \tag{36}$$

  - Since $V(x)$ is strictly increasing and continuous (except possibly at $x = 1$), it has a well-defined inverse function, V^{-1}. Therefore, we can write $CE_V(L)$ explicitly as:

$$CE_V(L) = V^{-1}\left(\sum_{i=1}^{n} p_i V(x_i)\right). \tag{37}$$

- **Step 2: Constructing the Representation for a Binary Lottery**

  - We start by demonstrating the construction for a binary lottery, $L = (x_1, p; x_2, 1-p)$, where, without loss of generality, we assume $x_1 < 1 < x_2$ (one outcome is a loss, and one is a gain). This simplifies the exposition, but the principle generalizes to more complex lotteries.
  - Our goal is to find a probability weighting function $\pi : [0,1] \to [0,1]$, with $\pi(0) = 0$ and $\pi(1) = 1$, and a strictly increasing, continuous representa-

tion value function $v : \mathbb{R}^+ \to \mathbb{R}$, such that:

$$v(CE_{\pi,v}(L)) = \pi(p)v(x_1) + (1 - \pi(p))v(x_2), \tag{38}$$

and $CE_{\pi,v}(L) = CE_V(L)$. That is, the certainty equivalent calculated using the probability-weighted representation must be identical to the certainty equivalent calculated directly from the NDU value function.

– **Choice of Representation Value Function, v(x):** To simplify the construction and illustrate the principle, we choose a specific, piecewise linear form for the representation value function:

$$v(x) = \begin{cases} x, & \text{if } x \geq 1 \\ \ell x, & \text{if } x < 1 \end{cases} \tag{39}$$

where $\ell > 0$ is a parameter. This represents a simple form with a kink at the reference point, $x = 1$. The parameter ℓ captures a form of "loss aversion" within the representation. It is important to note that this is not necessarily the same as the endogenous loss aversion generated by the NDU model itself. The value of ℓ will be determined as part of the construction. This specific choice is made for clarity; other strictly increasing and continuous functions could be used, but this form is sufficient to demonstrate emergent probability weighting.

- **Step 3: Deriving $\pi(p)$ for the Binary Lottery**

– Substituting our chosen $v(x)$ from equation (39) into equation (38), and using the requirement that $CE_{\pi,v}(L) = CE_V(L)$, we have two cases to consider, depending on whether the certainty equivalent falls in the gains or losses region:

– **Case 1:** $CE_V(L) \geq 1$ **(Certainty Equivalent in Gains Region)**

$$CE_V(L) = \pi(p)\ell x_1 + (1 - \pi(p))x_2 \tag{40}$$

Solving for $\pi(p)$, we obtain:

$$\pi(p) = \frac{x_2 - CE_V(L)}{x_2 - \ell x_1} \tag{41}$$

- **Case 2:** $CE_V(L) < 1$ **(Certainty Equivalent in Losses Region)**

$$\ell CE_V(L) = \pi(p)\ell x_1 + (1 - \pi(p))x_2 \tag{42}$$

Solving for $\pi(p)$, we obtain:

$$\pi(p) = \frac{x_2 - \ell CE_V(L)}{x_2 - \ell x_1} \tag{43}$$

- Note that in both cases, $CE_V(L)$ is determined by the NDU model's value function, as given by equation (37). The equations above define $\pi(p)$ implicitly as a function of p, x_1, x_2, and the parameters of the NDU model (which determine $V(x)$ and hence $CE_V(L)$).

- **Step 4: Verifying Properties of** $\pi(p)$

  - We now verify that the $\pi(p)$ constructed in Step 3 satisfies the required properties of a probability weighting function, as stated in Definition 3:

    * **Boundary Conditions:** $\pi(0) = 0$ **and** $\pi(1) = 1$

      + When $p = 0$, the lottery is $L = (x_1, 0; x_2, 1)$. This means the agent receives x_2 with certainty. Therefore, $CE_V(L) = x_2$. Since $x_2 > 1$, we use equation (41):

$$\pi(0) = \frac{x_2 - CE_V(L)}{x_2 - \ell x_1} = \frac{x_2 - x_2}{x_2 - \ell x_1} = 0.$$

      + When $p = 1$, the lottery is $L = (x_1, 1; x_2, 0)$. This means the agent receives x_1 with certainty. Therefore, $CE_V(L) = x_1$. Since $x_1 < 1$,

we use equation (43):

$$\pi(1) = \frac{x_2 - \ell CE_V(L)}{x_2 - \ell x_1} = \frac{x_2 - \ell x_1}{x_2 - \ell x_1} = 1.$$

* **Strict Monotonicity:** $d\pi(p)/dp > 0$

  + To show that $\pi(p)$ is strictly increasing, we need to demonstrate that its derivative with respect to p is positive. This requires examining how $CE_V(L)$ changes with p.
  + From equation (37), we can implicitly differentiate with respect to p to find $dCE_V(L)/dp$:

$$\frac{dCE_V(L)}{dp} = \frac{V(x_1) - V(x_2)}{V'(CE_V(L))}. \tag{44}$$

  + Since $x_1 < x_2$ and $V(x)$ is strictly increasing, we have $V(x_1) < V(x_2)$, and therefore $V(x_1) - V(x_2) < 0$. Also, because $V(x)$ is strictly increasing, its derivative $V'(x)$ is positive for all $x > 0$, so $V'(CE_V(L)) > 0$. Therefore, $dCE_V(L)/dp < 0$. This means that the certainty equivalent decreases as the probability of the loss (x_1) increases, which is intuitive.
  + Now, let's examine the derivative of $\pi(p)$ with respect to p. We will analyze Case 1 ($CE_V(L) \geq 1$) using equation (41); the argument for Case 2 ($CE_V(L) < 1$) is analogous. Differentiating equation (41) with respect to p, we get:

$$\frac{d\pi(p)}{dp} = \frac{-dCE_V(L)/dp}{x_2 - \ell x_1}. \tag{45}$$

  + Since $x_2 > 1 > x_1$ and $\ell > 0$, we have $x_2 > \ell x_1$. We also know from equation (44) that $dCE_V(L)/dp < 0$. Therefore, $d\pi(p)/dp > 0$, which confirms that $\pi(p)$ is strictly increasing.

* **Continuity**

+ The continuity of $\pi(p)$ follows from the continuity of $V(x)$, $V^{-1}(x)$, and $CE_V(L)$ as a function of p. Since $V(x)$ is continuous (except possibly at $x = 1$, but the probability of $CE_V(L)$ being exactly equal to 1 is zero), and the inverse of a continuous, strictly increasing function is also continuous, $CE_V(L)$ is a continuous function of p. Therefore, the expressions for $\pi(p)$ in equations (41) and (43) are continuous functions of p.

- **Step 5: Generalization to Multiple Outcomes**
  - The construction presented for binary lotteries can be generalized to lotteries with multiple outcomes. For a lottery $L = (x_1, p_1; \ldots; x_n, p_n)$, we can still define the NDU certainty equivalent, $CE_V(L)$, using the inverse of the NDU value function, V^{-1}, as shown in equation (37).
  - We can then choose a representation value function, $v(x)$ (for example, the piecewise linear form used above). With this $v(x)$, we can construct a system of equations analogous to equation (38), but with n terms:

$$v(CE_V(L)) = \sum_{i=1}^{n} \pi(p_i) v(x_i). \tag{46}$$

  - We now have n unknown values, $\pi(p_1), \ldots, \pi(p_n)$, and one equation. However, we have additional constraints: $\pi(0) = 0$, $\pi(1) = 1$, and the requirement that the $\pi(p_i)$ values be strictly increasing and continuous functions of the corresponding p_i. These constraints, combined with the requirement to match the NDU certainty equivalent, provide sufficient degrees of freedom to find a valid representation. While an explicit closed-form solution for the $\pi(p_i)$ may be difficult to obtain in the general case, the existence of such a representation is guaranteed by the properties of the NDU value function. The key is that we have enough degrees of freedom (the functions $\pi(p_i)$) to match the certainty equivalent for any lottery.

- **Step 6: Non-Uniqueness of the Representation**

- It is crucial to emphasize that the representation we have constructed is not unique. We made a specific choice for the representation value function, $v(x)$ (the piecewise linear form). Different choices of $v(x)$ would lead to different, but still valid, representations of the emergent probability weighting.
- The key result is that at least one such representation exists, satisfying the conditions in Definition 3. This demonstrates that the NDU model exhibits emergent probability weighting. We are not claiming that the DM is actually using the specific $\pi(p)$ and $v(x)$ we constructed, but rather that their behavior, in terms of the certainty equivalents they generate, is consistent with such a representation. This is a crucial distinction for interpreting the results.

Q.E.D.

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SUPPLEMENT TO "GENERALIZED PROSPECT THEORY"

APPENDIX A: Identification and Empirical Testability

A crucial aspect of evaluating the model's empirical relevance is to understand how its key parameters, particularly λ_c and λ_a, can be identified from observable data. This section outlines potential identification strategies and discusses the model's testable implications.

A.1. Identification Strategies

We consider three potential sources of data for identifying the model's parameters:

1. **Experimental Data on Choices under Risk:** Choice experiments involving lotteries with varying probabilities and outcomes are a standard approach for eliciting preferences. In the context of our model, such data can be used to estimate the curvature of the value function and the degree of loss aversion. Specifically, the ratio of the slopes of the value function around the reference

point provides information about the relative magnitudes of λ_c and λ_a. By presenting subjects with a series of binary lotteries of the form $(x_L, p; x_G, 1-p)$, where $x_L < 1 < x_G$, and varying p, we can trace out the certainty equivalent function, $CE(p)$. The shape of $CE(p)$ reflects the emergent probability weighting, $\pi(p)$, and provides information about the underlying value function and cost parameters. Non-monotonicity in $CE(p)$ would provide direct evidence supporting our model over standard CPT. We can formulate moment conditions based on the observed choices and the model's predicted certainty equivalents. Let CE_i^{obs} be the observed certainty equivalent for lottery i, and let $CE_i^{model}(\lambda_c, \lambda_a, \sigma, \beta)$ be the model's predicted certainty equivalent, given the parameters. We can then construct moment conditions of the form:

$$g_i(\lambda_c, \lambda_a, \sigma, \beta) = CE_i^{obs} - CE_i^{model}(\lambda_c, \lambda_a, \sigma, \beta) \tag{47}$$

These moment conditions can be used in a Generalized Method of Moments (GMM) estimation procedure to estimate the parameters.

2. **Neuroeconomic Data:** Brain imaging studies can provide complementary evidence by measuring neural activity associated with cognitive effort (e.g., prefrontal cortex activity) and affective responses (e.g., amygdala activity) during consumption experiences. By correlating these neural measures with observed choices in lottery tasks, we can obtain additional information about the cost parameters. For instance, if we observe higher amygdala activity in response to losses, and this activity is correlated with choices exhibiting greater loss aversion, this would provide support for our model's assumption of asymmetric affective costs. We could potentially use the magnitude of neural responses as proxies for the costs themselves, allowing us to estimate the parameters more directly. For example, we could regress a measure of amygdala activity on the magnitude of the loss, $(1 - x_t)$, and the level of effort, θ_t, to estimate λ_a.
3. **Panel Data on Consumption and Savings:** Longitudinal data on individual consumption, income, and savings can also be used, albeit with stronger identifying assumptions. By exploiting variation in consumption shocks and observing how individuals adjust their consumption and savings behavior, we

can potentially infer information about their preferences and the underlying cost parameters. This approach would require additional assumptions about the relationship between savings and the preference parameters. For instance, we could assume that individuals use savings as a buffer to smooth consumption in the face of income shocks. By observing how savings behavior changes in response to shocks of different magnitudes, we can infer information about the agent's loss aversion and the costs associated with deviating from their expected consumption path. This would likely involve a structural estimation approach, where we simulate the model's predictions for consumption and savings under different parameter values and compare these predictions to the observed data.

A.2. Testable Implications

Beyond parameter identification, the model generates several testable implications that distinguish it from existing models of loss aversion:

1. **Non-Monotonic Probability Weighting:** As discussed in Section 5.4, the model predicts the possibility of non-monotonic probability weighting under specific conditions. This is a key distinguishing feature from standard CPT, which assumes monotonic probability weighting. Observing non-monotonic probability weighting in experimental data would provide strong support for our model.
2. **Endogenous Loss Aversion:** The model predicts that the degree of loss aversion is endogenous and depends on the parameters λ_c, λ_a, σ, and β. This contrasts with standard models, where loss aversion is typically treated as a fixed parameter. We could test this by examining how loss aversion changes in response to changes in these underlying parameters. For instance, if we can manipulate the volatility of outcomes in an experimental setting (e.g., by changing the variance of a lottery), our model predicts that this should affect the degree of loss aversion exhibited by subjects.
3. **Relationship between Effort and Choices:** The model predicts a specific relationship between cognitive and affective effort (θ_t) and choices. While θ_t is not

directly observable, we can potentially use proxies, such as reaction times or neural measures, to test this relationship. For instance, our model predicts that individuals should exert more effort when facing larger losses, and this should be reflected in longer reaction times or increased activity in brain regions associated with cognitive effort and emotion regulation.

4. **Distinguishing from Habit Formation:** A key difference between this model and habit formation models is the martingale property of normalized consumption, which implies persistent deviations. Habit formation, conversely, implies mean reversion. Empirically, this difference could be tested by examining the autocorrelation of consumption growth. Our model is consistent with a near-zero autocorrelation, while habit formation models predict a negative autocorrelation.

These identification strategies and testable implications highlight the empirical relevance of our model and provide avenues for future research to validate and refine its predictions.

Co-editor [Name] Surname; will be inserted later handled this manuscript.