Endogenous Firm Ownership in General Equilibrium

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Abstract

This paper examines firm ownership structures in competitive general equilibrium by introducing a model where ownership rights emerge endogenously rather than being assumed. By embedding the property rights theory of the firm into general equilibrium analysis, the model demonstrates how market forces determine both initial firm formation by entrepreneurs and subsequent trading of ownership rights. The key finding is that firms are optimally created by managers who have long-term importance to operations, but these founder-managers then sell ownership to outside investors. This pattern emerges because managers can capture value through both the sale price and their ongoing employment relationship, while outside owners can only benefit through residual claims. The model thus provides a novel theoretical foundation for observed patterns of entrepreneurial exit and reconciles competing approaches to firm ownership in general equilibrium theory. Additionally, it identifies a distinct economic role for entrepreneurs as agents whose human capital makes them uniquely positioned to found firms even when they do not remain long-term owners. The framework generates insights into firm formation, ownership dynamics, and the nature of entrepreneurship while maintaining the core structure of competitive general equilibrium analysis. *Journal of Economic Literature* Classification Numbers: D50, D23, L22

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1 Introduction

Firm ownership structures in modern economies are diverse. In the United States, of the 8 million or so firms, 4,500 are public companies, but these companies account for around 23 percent of private employment. This mirrors long-standing results on the diversity of firm ownership (Hansmann, 2000). Given this, it is somewhat surprising that the foundational model of economics – competitive general equilibrium theory – offers no insights into that structure. The goal here is to demonstrate that with some relatively small amendments, rich insights are possible with implications that mirror observed patterns of firm ownership. In the process, the economic nature of entrepreneurship can be addressed and identified in ways not possible from other theoretical traditions.

The firm holds an unusual position in general equilibrium theory. While many other analyses of firms treat it as an agent or composed of other agents, the general equilibrium firm is a mere conduit for a production technology. It plays no distinct economic role, and if it obtains non-zero profits, the only modelling choice is who the recipients of those profits are. This implies that the theory offers no predictions regarding firms and their ownership. Each is an assumed artifact of underlying model assumptions.[1]

This paper seeks to redress this omission by making the outcome of who owns each firm endogenous. The ownership of firms has been a thoroughly explored topic in microeconomics, but its integration into general equilibrium has been limited.[2] This matters because, as will be noted, firm ownership is driven by outcomes in related markets, particularly the markets for labour and capital.

Here, the property rights theory of the firm is embedded in a general equilibrium setting. Formulated by Grossman and Hart (1986) and Hart and Moore (1990), the property rights theory defines a firm based on the ownership of non-human assets such as capital goods, equipment, buildings, land and intellectual property. That firm then employs people to work with those assets to produce output. The firm owners have residual control rights over access to assets (i.e., they can withdraw them from use by others) and residual income rights to output generated from production (not covered by other contractual commitments). Such control does not exist over people whose human capital is inalienable and, thus, must be induced to work at any given point in time. It is this distinction with non-human assets that

[1] Jacques Dreze (1985) explored this issue when he wrote, "The firm fits into general equilibrium theory as a balloon fits into an envelope flattened out! Try with a blown-up balloon: the envelope may tear, or fly away at best, it will be hard to seal and impossible to mail... . Instead, burst the balloon flat, and everything becomes easy. Similarly with the firm and general equilibrium."

[2] Some research has integrated contracting and hold-up problems into macroeconomics (Caballero and Hammour (1998) and Caballero (2007)), and international trade theory (Antràs (2003) and Grossman and Helpman (2005)) and industrial organisation (Azar and Vives, 2021).

drives the definition of the firm: "[N]on-human assets are an important part of any firm; they are the glue that holds the firm together." (Hart, 2017)

The property rights theory forms predictions regarding firm ownership structure by considering the efficient allocation of control rights over assets to agents. It does not consider the equilibrium allocation of assets to potential owners that would arise in a market environment. This theory provides, therefore, a structure under which to consider asset ownership and when and how it might matter but not an analysis of how such assets are created and who ultimately controls them. The present paper fills this gap by modelling asset creation as being driven by the decentralised (optimising) choices of agents and ultimate ownership as being determined by the forces of supply and demand in asset markets jointly determined as an equilibrium in other related markets; in particular, labour markets. Thus, it embeds this theory in general equilibrium theory by considering agent choices taking prices in relevant markets as given.

To accommodate the property rights theory of the firm into general equilibrium, this paper makes one important departure from the latter. It is assumed that binding contracts over labour cannot be written; that is, employment is fundamentally 'at-will,' and workers cannot be compelled to provide services.[3] If it were the case that there were always close substitutes for workers, the nonbinding nature of labour contracts would not necessarily have a distinct role over contracts over other services. However, the starting point here will be that workers in a firm have specific skills that are difficult (or, in the case of the baseline model, impossible) to replace. As Stole and Zwiebel (1996a) and Stole and Zwiebel (1996b) demonstrate, in this case, stable (or renegotiation-proof) wage outcomes for a firm are determined by the equilibrium of a bargaining game, namely, the Shapley value. In integrating this into general equilibrium theory, this allocation of value is assumed.[4]

1.1 Contributions of the Paper

The theory and results in this paper offer three distinct contributions. First, by placing it in a general equilibrium setting, the model of Hart and Moore (1990) is enriched to include asset creation and trading. Second, a special role for the entrepreneur is identified within the context of general equilibrium theory. Finally, the long-standing schism between the

[3]This is a well-recognised feature of labour contracts; see Posner (2014) for a discussion.

[4]Here, it is assumed that intra-firm bargaining outcomes are given by the Shapley value between workers and capital owners as in Hart and Moore (1990). Stole and Zwiebel (1996a) developed an axiomatic approach that derived this outcome, which was developed in a dynamic setting by Wolinsky (2000). See Raith (2021) for a recent exploration of this contractual environment, which often relies on relational contracts when spot contracts are not available. Brügemann et al. (2019) and de Fontenay and Gans (2014) provide non-cooperative games that generate the Shapley value outcome in this environment. For an alternative perspective on cooperative game theory outcomes in a firm context, see Aoki (1980) and Aoki (1984).

treatment of firm ownership in Arrow and Debreu (1954) versus McKenzie (1959) is resolved. Each is briefly explained in turn.

The model here shows what happens when the approach of Hart and Moore (1990) is embedded in a general equilibrium model.[5] In contrast to that literature, while it is assumed here that capital investment is non-contractible (that is, there is no firm to negotiate with until it is formed) and wages in firms are non-binding, the baseline model does not consider any other contractual incompleteness (i.e., worker effort). Nonetheless, it is demonstrated that ownership matters. In equilibrium, assets can be traded amongst agents, and it is shown that agents outside the firm (e.g., households) will end up acquiring ownership over those inside the firm. This arises because the former only appropriate firm value if they are owners while the latter will appropriate value as non-owners, which reduces their willingness to pay for ownership. This is the same effect identified by Gans (2005) for the standard Hart and Moore (1990) environment with incomplete contracts and no general equilibrium. The nuance here is that labour market effects reinforce these effects and also imply that they arise even in the absence of incomplete contracts.

While the eventual equilibrium in the economy involves firms being owned by outsiders, in contrast to a central prediction of Hart and Moore (1990), it is demonstrated that these are not the agents who invest and create the assets in the first place. Those are skilled workers – which we term managers or entrepreneurs – who have long-term importance to firm value.[6] It is demonstrated here that such agents have a stronger incentive to create assets precisely because firm creation provides them with an ongoing role in the firm they found, but also that they can trade assets ex-post to appropriate the returns outsiders would obtain. Indeed, it is demonstrated that such ex-post trading provides a greater incentive to build firms than would arise in a typical Hart and Moore (1990) environment. In this regard, this paper endogenously provides a distinct entrepreneurial role in competitive economies as single agent founders of firms while at the same time explaining how entrepreneurs choose to relinquish ownership and control.

It is useful to emphasise that the role for an entrepreneur, or an agent that performs the entrepreneurial function, is distinct from others identified in the literature. In terms of general equilibrium theory, the work of Kihlstrom and Laffont (1979) was the first to consider which agents become entrepreneurs and identified those who were less risk averse

[5]There is some other work that has integrated aspects of bargaining and internal firm contracting into general equilibrium theory; for example, Zame (2007) and Ellickson et al. (2006). These, however, do not endogenise firm ownership.

[6]In this regard, the model presented here addresses, head-on, the famous critique by Baumol (1968) that "[t]he theoretical firm is entrepreneurless – the Prince of Denmark has been expunged from the discussion of *Hamlet*."

as being willing to bear the uncertainty associated with firm formation.[7] The model here does not have uncertainty and so is a complement to this work although it shares with it that entrepreneurs are drawn from the pool of workers. However, here: *A firm-founding entrepreneur is an agent drawn from the set of agents who are of long-term importance to the firm.* Thus, who becomes an entrepreneur is intimately tied to their role in production rather than their individual characteristics.[8]

The issue of the ownership of firms was front and centre as economists set out to formally prove the existence of a general equilibrium in a competitive economy. Arrow and Debreu (1954), who wanted the theory to accommodate production technologies that included both constant and diminishing returns, realised that firms would earn quasi-rents and that those profits would have to be allocated to complete the model. They chose to allocate firm ownership to households (with arbitrary shareholdings) and to assume that the set of firms in the economy was itself given and that from that set, a subset of firms would be active in equilibrium if they employed a strictly positive quantity of some input. This contrasted with the parallel approach of McKenzie (1959), who argued for constant returns to scale, which involved no profits under competition and, therefore, no need to distribute profits. In this regard, entrepreneurs would be another input into production, and any rents they received would equate to what might be commonly thought of as entrepreneurial profits. In other words, Arrow and Debreu (1954) assumed that firms were outside owned while McKenzie (1954) effectively assumed they were owned by a specific factor.

At the heart of the choice between these approaches to firm ownership was whether entrepreneurs, as a distinct factor, were scarce or not. As Arrow (1971) wrote:

> In more modern language, the production possibility set of the typical firm displays an initial tendency toward increasing returns followed at higher scales by decreasing returns. The first phase is explained by indivisibilities, the second by the decreasing ability of the entrepreneur to control the firm. As one may put it, entrepreneurship should also be regarded as an input to the firm; then, after the initial phase at least, the firm would have constant returns to all inputs (including entrepreneurship), but, since by definition the firm has only one entrepreneur, there are diminishing returns to all other factors. (The indivisibility of the entrepreneur is sometimes invoked to explain the initial phase also, though of course there are typically also indivisibilities of a more definitely technological

[7] That said, even this intuitive conclusion was subject to nuances as outlined in their later work; Kihlstrom and Laffont (1982) and Kihlstrom and Laffont (2002).

[8] This notion is assumed in other models including Romer (1990) where the supply of firms is driven by the supply of ideas and Aghion and Tirole (1994) and Gans and Stern (2000) where independent entrepreneurs enter based on industry characteristics and intellectual property protection.

> variety.) The assumption of free entry implies that the supply of entrepreneurship in the economy is infinite, or, more precisely that it is sufficiently large that its demand price will fall to zero at a point at which supply still exceeds demand.
>
> The exact relation of this model of the firm to a full general equilibrium model has never been explored; in particular, the notion of an infinite supply of entrepreneurship is no more reasonable than that of an infinite supply of anything else. Arrow (1971), p.69.

Arrow's objection, therefore, was that entrepreneurs were implicitly not scarce in McKenzie's model. The present paper takes this critique seriously. First, entrepreneurs are assumed to come from the pool of agents, which itself is scarce. Second, once a firm is formed, entrepreneurs themselves are scarce. Finally, in contrast to both approaches to firm ownership, this paper makes ownership endogenous and, thus, is able to generate a prediction as to when ownership might arise with an owner-manager (or entrepreneur) as opposed to outsiders (or households). As already discussed, this resolution turns out to be nuanced with both types of ownership naturally arising in different 'stages' of the firms lifecycle. Doing this is important because as Arrow noted, the important role of an entrepreneur is in "overcoming disequilibria. When profit rates were unequal, profit-hungry entrepreneurs moved quickly, with the end-result of eliminating their functions." (Arrow (1971), p.68) Thus, the model here provides a foundation, not explored in this paper, for incorporating disequilibrium into general equilibrium theory.[9]

1.2 Outline of the Paper

The paper proceeds as follows. The next section provides a description of the model. It is a specific general equilibrium environment with joint production opportunities and a self-production sector, among which a fixed pool of labour makes its participation decisions. As a benchmark, the equilibrium outcomes when ownership is exogenous – both manager-owned and outside-owned – are characterised. Section 3 then provides the main results of the paper when firm ownership is endogenous, allowing both asset creation and ex post asset trading to take place amongst agents of different types. Section 4 then extends the model to the case of incomplete contracts as in the full Hart and Moore (1990) context and shows that this creates a greater diversity of possible firm ownership equilibrium outcomes than outside ownership. Section 5 then engages in an extensive revisit of the assumptions of the baseline

[9] Arrow and Debreu (1954) were not without its own issues with regard to firm ownership as cogently explored by Ellerman (2021). The contention here is that this paper, by endogenising ownership answers Ellerman's concerns as well.

model to examine their robustness to various changes and finds the results are unchanged. The final section concludes and offers directions for future research.

2 Description of the Model

The economy is comprised of a unit mass of agents who all have linear utility in consuming a single good. That good is the numeraire, with its price set equal to 1. Another alternative use for that good is as capital assets that can be used as inputs into its production.

There are two production sectors in the economy: a self-production and a joint-production sector. Let l be aggregate employment in self-production and $L = 1 - l$ be aggregate employment in joint-production.[10] The output of the self-production sector is given by a non-decreasing and concave production function, $f(l)$.[11] For convenience, it will be assumed that:

$$f(l) = \frac{1}{1+\frac{1}{\eta}}\left(1-(1-l)^{1+\frac{1}{\eta}}\right)$$

with $\eta > 0$ (a parameter that will represent the wage elasticity of joint-production labour supply). Payments for labour in this sector, which are denoted by w, are equal to their marginal product:

$$w = f'(l)$$

This sets the supply function for labour to the joint-production sector as:

$$L = w^{\eta}$$

which has a constant supply elasticity of $\frac{\partial L}{\partial w}\frac{w}{L} = \eta$.

In the joint-production sector, if a capital asset (comprising one unit of the good) is combined with a skilled worker and a skilled manager (to manage the worker and asset), y units of the good are produced. It is assumed that the worker and manager have skills specific to each other and the assets of the firm, but if no joint production takes place, each agent can return to the self-production sector. The asset is also specific to the two skilled agents and, thus, has no value if no joint production takes place. Thus, joint production involves fixed coefficients and constant returns to scale.

[10] Here we are following the setup of Caballero and Hammour (1998). There are some important differences. They actually assume three production sectors with different factor types that are combined in joint production and also operate in distinct self-production (or autarky) sectors. Here, labour will be drawn to joint production from a single self-production sector.

[11] The concavity of $f(.)$ implies that another factor exists that is supply-constrained in the self-production sector. This may be land or some other competitively supplied resource.

Firms that engage in joint production are not given and are formed endogenously when capital investment takes place and are disbanded if it is not used in production. It will be efficient for a firm to form if its output exceeds its costs:

$$y \geq 1 + 2w$$

Here, it will be assumed that $y \in (1, 3]$ as this is the interesting case.[12] Note that if the efficient number of firms is formed, then $w^* = \frac{1}{2}(y-1)$. In this case, at the resulting efficient allocation of labour to joint production, L^*, the number of firms formed is $N^* = \frac{1}{2}L^* = (\frac{1}{2})^{1+\eta}(y-1)^{\eta}$. Aggregate consumption is, therefore:

$$C^* = N^*y - N^* + f(1 - L^*) = \frac{1}{1+\eta}\left(\eta + \left(\tfrac{1}{2}(y-1)\right)^{1+\eta}\right)$$

This provides a social welfare benchmark for the equilibrium analysis that follows.

2.1 Timeline

The timeline for agent decisions and payoffs is as follows:

1. (Firm Formation) Capital invested by either managers (M) or outsiders (O) for joint production and firms are formed.

2. (Specificity) Workers choose to acquire skills specific to capital at the firm level.

3. (Asset Trading) Ex post asset trading resulting in the final resolution of ownership structure in terms of manager or outside ownership, i.e., $s \in \{M, O\}$.

4. (Negotiations) Skilled workers and firms negotiate over wages, w_s, in joint production with outcomes determined by the Shapley value.

5. (Production) Income is distributed and output and consumption are generated.

We will look for a rational expectations equilibrium outcome that generates equilibrium income for each agent, the number of firms formed, the ownership structure of those firms and aggregate output and consumption. That is,

[12] If $y \leq 1$, it will not be efficient for any joint production to take place while, in the variants explored below, if $y > 3$, no self-production will take place and there will be a corner solution. However, a corner solution will not alter any of the results presented below, it is just ruled out here to keep computations simple going forward.

Definition 1 (Rational Expectations Equilibrium) *A* rational expectations equilibrium *of this economy is a specification*

$$\left\{w,\, l,\, L,\, N,\, \{s_i\}_{i=1}^{N},\, \{w_s\}_{s\in\{M,O\}},\, (\textit{agent-sector assignments}),\, (\textit{ownership-trading rule})\right\}$$

such that (i) the labour market clears; (ii) initial ownership assignments, s_i. for each firm, i, ownership assignments satisfy individual rationality of investors/managers at the time of investment; (iii) an agent acquires firm-specific skills if this satisfies individual rationality; (iv) for the final asset allocation, there is no price that leaves no profitable deviation for buyer or seller, given the subsequent bargaining outcome; (v) in each firm i, given final ownership $s_i \in \{M, O\}$, the worker's wage w_{s_i} and the residual profit π_{s_i} to the owner satisfy w_{s_i} = Shapley-value payoff to the worker, $\pi_{s_i} = y -$ workers' wages; and (vi) the final good is numéraire with unit price.

Any allocation of (i) which agents invest in capital, (ii) which agents become skilled workers/managers, (iii) how ownership is reassigned ex post, (iv) the resulting Shapley wages, and (v) final consumption that satisfies all the above conditions constitutes a rational expectations equilibrium.

It is worthwhile discussing the assumptions underlying each stage in the timeline. The first stage is the investment of capital and the formation of a firm, recalling that a unit of capital is sufficient to allow a firm to engage in joint production so long as one worker of each type can be employed. That investment can be undertaken by any agent, but we will make a distinction between investments undertaken by the agent who will become the eventual manager of the firm and those undertaken by any other agent whom we refer to as outsiders. In the former case, we have ownership by a factor of production, while in the latter case, ownership is by households. In each case, the owner will be entitled to the profits, if any, generated by the firm.

The second stage involves the choice of agents to move into the joint production sector by choosing a firm and acquiring skills specific to that firm's capital. It is easy to see that, in equilibrium, only one skilled worker and one skilled manager make that choice for each firm.[13] As will be discussed below, this creates a meaningful hold-up issue for the capital owners of any firm as such workers cannot be replaced ex post.[14]

In the third stage, the initial capital owners, who have already sunk investment costs, have an opportunity to sell the capital assets to other agents. In particular, an outside owner

[13]This could be explicitly modelled by assuming that agents face an infinitesimal but positive sunk cost of skill acquisition. This would imply that in equilibrium, no more than one agent acquired the skills of a worker or a manager specific to a firm.

[14]In extensions to the model below, this irreplaceability assumption is relaxed.

can sell to a manager, while a manager can sell to an outside owner. For simplicity, it is assumed that the skilled worker cannot acquire ownership.[15] Thus, this stage represents the critical way in which firm ownership is considered endogenous in this setup.

Negotiations take place between the two agents and the outside owner or between the skilled worker and the manager, as the case may be. Each party is assumed to receive their Shapley value through these negotiations. This represents a clear departure from the neoclassical model of general equilibrium, where long-term labour contracts are possible. Here, by contrast, employment is assumed to be 'at-will,' meaning that workers are free to leave the firm. The threat of that, in addition to their irreplaceability, justifies the use of the Shapley value as a model of the outcomes of negotiations for at-will employment. This is standard in the property rights theory of the firm (Hart and Moore, 1990) and is justified more thoroughly by (Stole and Zwiebel, 1996a).[16]

The final stage involves production taking place and income, in the form of wage payments and profits, being distributed to agents to be used to purchase and consume the good produced.

2.2 Equilibrium with Exogenous Ownership

To provide a point of comparison with the endogenous ownership outcomes, it is useful to describe what would happen if ownership were exogenous, meaning that only certain ownership structures were permitted as per assumptions that are standard in general equilibrium models with production.

First, consider M-ownership. Any agent can choose to become a skilled manager, invest capital and form a firm that they eventually manage. In subsequent negotiations with a skilled worker, the wages paid to that worker will be $\frac{1}{2}y$, and the (gross) profit to the manager will be $\frac{1}{2}y$. This is because each has an outside option of returning to self-production. Given this, an agent will form an M-owned firm if:

$$\tfrac{1}{2}y - 1 \geq w \Leftrightarrow y \geq 2 + 2w$$

The threshold for firm formation is higher than the efficiency level, so too few firms will be formed in equilibrium. Nonetheless, in equilibrium, this inequality will bind, and so

[15] This simplifies the analysis, but nothing would change if this assumption were relaxed so long as the skilled agents in a firm could not coordinate asset allocation to become joint owners and also if there were sufficient numbers of skilled agents in a firm; see Gans (2005). Gans (2006) shows that when all workers can bid for the asset, outside ownership still results as the eventual equilibrium outcome.

[16] For a non-cooperative justification of the Shapley value in this context, see de Fontenay and Gans (2014) and also Brügemann et al. (2019).

equilibrium wages will be $w_M = \frac{1}{2}(y-2)$. This results in consumption of:

$$C_M = \tfrac{1}{2}(y-1)\left(\tfrac{y-2}{2}\right)^{\eta} + \tfrac{\eta}{1+\eta}\left(1 - \left(\tfrac{y-2}{2}\right)^{1+\eta}\right)$$

This demonstrates that an equilibrium under M-ownership that results in some joint production requires that $y > 2$.

Now consider O-ownership. In this case, no agent who works in joint production of a specific firm can own the capital of that firm.[17] The Shapley value allocations of surplus from the resulting negotiations will be $\frac{1}{3}(y+w)$ to the skilled worker and manager and $\frac{1}{3}(y-2w)$ to the outside capital owner. Given this, an agent (or agents) will form an O-owned firm if:

$$\tfrac{1}{3}(y-2w) - 1 \geq 0 \Leftrightarrow y \geq 3 + 2w$$

This threshold for firm formation is even higher than that for an M-owned firm. Again, in equilibrium, this inequality will bind, resulting in wages of $w_O = \frac{1}{2}(y-3)$ and consumption of:

$$C_O = \tfrac{1}{2}(y-1)\left(\tfrac{y-3}{2}\right)^{\eta} + \tfrac{\eta}{1+\eta}\left(1 - \left(\tfrac{y-3}{2}\right)^{1+\eta}\right)$$

This demonstrates that an equilibrium under O-ownership that results in some joint production requires that $y > 3$.

These outcomes are consistent with common intuition. In this model, there is capital hold-up as wages within firms cannot be negotiated ex post, after capital is sunk, and, as they cannot be replaced, the workers involved in the negotiation are essential for joint production by that firm. In this situation, as Hart and Moore (1990) clearly demonstrated, ownership of the capital asset should reside with one of the essential agents than with an unimportant agent such as an outside owner. This is because, when choosing to invest in capital, that agent appropriates a greater share of the surplus. Nonetheless, it is not full appropriation, and so the resulting outcome remains inefficient compared with a neoclassical situation where wages can be committed to ex-ante. Importantly, this demonstrates that firm ownership matters as this results in potentially different efficiency properties of a resulting equilibrium. As will be demonstrated in what follows, however, this specific welfare-ranking is not robust to ownership becoming endogenous.

[17] One way this might arise is if there is a separate capitalist class. However, this is not necessary. Workers can own firms, but not the firms they are employed by.

3 Equilibrium with Endogenous Ownership

We now turn to consider the equilibrium outcome when firm ownership is endogenous. Endogeneity means that firms can be formed and also eventually owned by either managers or other outside agents (both those in self-production or employed at other firms).[18] As will be demonstrated, the distinction between firm formation and eventual ownership is important and determines the nature of equilibrium outcomes that emerge.

In what follows, we work backwards. First, the outcomes from ex post asset trading are examined, holding firm formation and skill acquisition as fixed. Second, initial firm formation is examined in anticipation of the outcomes from ex post asset trading.

3.1 Ex Post Asset Trading

Suppose that N firms have formed with $L = 2N$ workers having firm-specific skills. Then self-production wages, $w(N) = (2N)^{1/\eta}$. To explore ex-post trading, we consider firms that are initially O-owned and those that are initially M-owned:

- Suppose that a firm is initially O-owned. If no asset trade occurs, the outside owner expects to earn $\frac{1}{3}(y - 2w)$. If an asset were traded to a manager, then the manager would earn $\frac{1}{2}y$ instead of $\frac{1}{3}(y + w)$ they would earn under O-ownership. It is easy to see that, so long as $y \geq 2w$, the outside owner's willingness to sell, $\frac{1}{3}(y - 2w)$, is less than the willingness to pay of the manager, $\frac{1}{2}y - \frac{1}{3}(y + w) = \frac{1}{6}(y - 2w)$. Thus, the asset would remain O-owned. If $y < 2w$, the willingness to sell of the outside owner will be negative but less than the (negative) willingness to pay of the manager, and so the asset would trade to the manager.

- Suppose that a firm is initially M-owned. If no asset trade occurs, the manager expects to earn $\frac{1}{2}y$, while if the asset were traded to an outside owner, the manager would earn $\frac{1}{3}(y + w)$. If a trade occurs, the outside owner expects to earn $\frac{1}{3}(y - 2w)$. Again, it is easy to see that the willingness to pay of the outside owner, $\frac{1}{3}(y - 2w)$, is more than the willingness to sell of the manager, $\frac{1}{2}y - \frac{1}{3}(y + w) = \frac{1}{6}(y - 2w)$. Thus, so long as $y \geq 2w$, the manager will sell the asset to an outsider. Otherwise, the asset will remain M-owned.

This demonstrates that so long as $y \geq 2w$, following the asset trading period, the asset will be owned by an outsider, while if $y < 2w$, the asset will eventually be owned by the manager.

[18] The skilled worker who has skills specific to the capital of a particular firm is excluded here from ownership. This is done to simplify the analysis. It would be possible to allow such workers to also own firms with all of the conclusions derived below still holding so long as it was assumed that the two skilled agents could not collude or cooperate over ownership of their specific firm. See Gans (2005).

This, of course, raises the question of whether it is possible that $y < 2w$. If this arose, then the manager would be better off shutting the firm down. As will be demonstrated below, this implies that we would not expect the firm to form if y is expected to be lower than $2w$.

This may appear to be a surprising result, given that the outside owner does not actually add anything productive to the joint-production sector. However, as Gans (2005) showed, it is precisely because outsiders do not play a productive role that they are motivated to bid for ownership as a means of gaining any ex post rents. By contrast, a manager who plays a productive role receives benefits even when they are an employee and not an owner, which mutes their willingness to pay. Notice also that this same outcome would arise ex post after the asset has been created at a cost of 1. A manager would find it worthwhile to sell ownership to an outsider, while an outsider would be unable to convince a manager to purchase ownership for more than the outsider's expected ex-post profits. As will be seen in what follows, what is new here is that this prediction is robust and has implications for firm formation when labour market outcomes are endogenous rather than given as is typically assumed in the property rights theory of the firm.[19]

3.2 Firm Formation

Any agent, whether they intend to work in joint production or not, can form a firm by investing in one unit of the consumption good. We will consider the formation choices of outside owners and managers, respectively:

- An outside owner who forms a firm expects to earn $\frac{1}{3}(y-2w)-1$.
- A manager who forms a firm expects that, so long as $y \geq 2w$, the firm's assets will be sold to an outside owner prior to production taking place. As asset markets are competitive ex post, the manager will expect to earn $\frac{1}{3}(y+w)+\frac{1}{3}(y-2w)-1 = \frac{1}{3}(2y-w)-1$. If the manager does not form the firm, they expect to earn w in self-production. Thus, the net return to a manager forming a firm is $\frac{1}{3}(2y-w)-1-w = \frac{2}{3}(y-2w)-1$.

In each case, the return to forming a firm will only be positive if $y \geq 2w$. Therefore, we now consider whether this will be the case in equilibrium.

[19]It is instructive to note why this outcome would not change if the other skilled worker could bid for ownership. If the manager sells to an outsider, the manager receives $\frac{1}{3}(y-2w)+\frac{1}{3}(y+w) = \frac{2}{3}y-\frac{1}{3}w$. The skilled worker is willing to bid up to $\frac{1}{2}y-\frac{1}{3}(y+w)$ to block this outcome. If it did bid this much, the manager would receive $\frac{1}{2}y+\frac{1}{2}y-\frac{1}{3}(y+w) = \frac{2}{3}y-\frac{1}{3}w$; the same amount as they would receive from an outsider. However, as Gans (2006) shows, this leaves the manager vulnerable to expropriation by the skilled worker who can threaten to sell to outsiders. Thus, in a full game, the manager prefers to sell to outsiders to avoid this possibility.

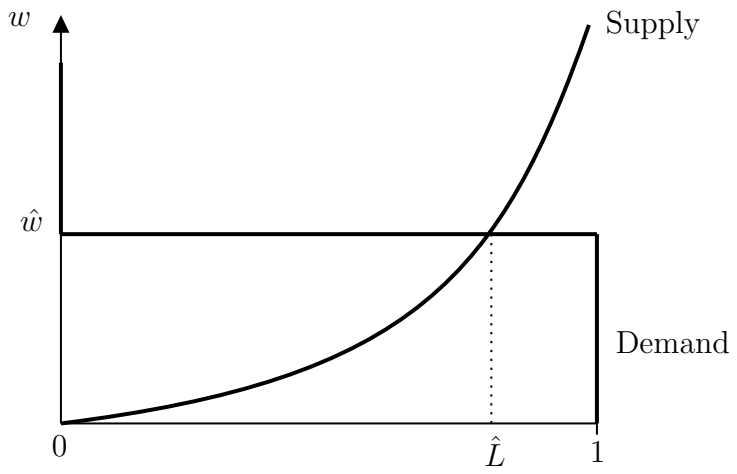


Figure 1: General Equilibrium Outcome

Note, first, there is a higher return to a manager forming a firm than an outsider. Thus, suppose that the marginal firm is formed by a manager. In this case, the firm will form so long as the return is non-negative. That is:

$$\tfrac{2}{3}(y - 2w) \geq 1 \implies y \geq \tfrac{3}{2} + 2w$$

If this inequality holds with equality (i.e., the firm formation constraint binds), then, in equilibrium:

$$\hat{w} = \tfrac{1}{2}(y - \tfrac{3}{2})$$

which implies that if wages are to be positive, $y > \frac{3}{2}$. Regardless:

$$y \geq 2\hat{w} \implies y \geq y - \tfrac{3}{2}$$

which always holds. This is an interior equilibrium – with labour allocated to both joint and self-production – if $\hat{w}_O < 1$ or $y < 3$ (as already assumed). This outcome is presented in Figure 1. Note that, at this wage, the returns to an outside owner forming a firm are negative; that is, $\frac{1}{3}(y - 2w) = \frac{1}{2} < 1$.

3.3 General Equilibrium Outcome

Given this, we have demonstrated the following proposition:

Proposition 1 *(i) If* $y > \frac{3}{2}$*, the unique equilibrium outcome involves managers forming firms and then selling to outsiders. Self-production wages and manager income are* $\frac{2y-3}{4}$*, skilled worker wages are* $\frac{2y-1}{4}$*, outsider returns are* 0 *and* $C = \frac{1}{2}(y-1)\left(\frac{2y-3}{4}\right)^{\eta} +$

$\frac{\eta}{1+\eta}\left(1-\left(\frac{2y-3}{4}\right)^{1+\eta}\right)$. *(ii) If* $y < \frac{3}{2}$*, the unique equilibrium involves no joint production and all output is from self-production with zero wages and* $C = \frac{\eta}{1+\eta}$.

This demonstrates that under endogenous ownership, firms are formed by managers while they are eventually owned by outsiders employing the founding manager as a manager-employee.[20] Therefore, from the perspective of whether firms in general equilibrium ought to be modelled as owned by a factor of production with income part of the earnings of that factor or as being owned by households (i.e., outside parties), this model demonstrates that the latter, with profits counted as income accruing to households, will be a reasonable approach for modelling economies once firms have already formed. By contrast, when considering firm formation itself as well as the distribution of wealth arising from ownership, that wealth is accrued by individual factors of production (in this case, labour endowed with managerial skills). Thus, the distinction between the Arrow-Debreu and McKenzie approaches to ownership of profits is more subtle when ownership of assets is endogenous.

With respect to welfare, the equilibrium that arises under endogenous firm ownership leads to a more joint production than in either case where firm ownership is endogenous. Specifically, firms will form even where $y < 2$, although some firms that would be efficient in equilibrium are not formed. The reason for the improvement in efficiency comes from the ability to asset ownership to be determined ex post. This means that a manager can form a firm and, in expectation, earn not only the returns from their eventual employment by that firm but also the rents associated with pure ownership that are eventually transferred from acquiring outside owners. This raises the returns to firm formation above what a manager who could not engage in ex post restructuring of ownership could earn.

Finally, it is useful to note the distribution of wealth and income in equilibrium. Skilled workers earn the highest income, while managers and self-production workers earn equal amounts. Returns on asset ownership are zero. Thus, there is no issue of how to distribute such returns to households while whatever returns there are to firm formation are competed away by the formation of such firms. Any surplus from joint production firm formation that accrues as income is earned by skilled workers while the rest flows to consumers.

In terms of the ex post distribution of income and consumption (after asset creation and acquisition costs), all skilled workers earn the same income (both managerial and other) and, as they are scarce ex post, this income exceeds their self-production wage. By contrast, self-production workers earn their marginal productivity, which may exceed the wage for all but the marginal self-production worker, in equilibrium. Finally, outside owners earn income at a rate that is less than those of skilled workers.

[20] When $y = \frac{3}{2}$, initial firm ownership is indeterminant although firms are eventually owned by outsiders in equilibrium.

3.4 Discussion of the Model

The implication of the result here is that firm formation by what we could call 'entrepreneurs' is the natural prediction in a general equilibrium environment. This arises in an environment where it is possible for a consortium of outside investors to be responsible for firm formation. Moreover, this result is accompanied by another strong prediction that the ultimate ownership of firms will reside with those outsiders even though they are not directly involved in productive activities in the economy and are pure capital owners. In other words, the central stylised fact of entrepreneurship, that entrepreneurs found firms but exit to ownership structures that leave them without control, arises as an implication of general equilibrium theory with endogenous ownership.

In that regard, the model here provides a new prediction as to who becomes an entrepreneur. In this model, entrepreneurs are agents who undertake initial investments in firm formation (here modeled as capital investment but could also be easily conceived of as time or risk bearing) who (a) can become sellers in ex post asset markets and (b) are expected to return some value creating role in the firm even after they have ceded ownership and control. To see this, note that the expected return to an agent who decides to form a firm is:

$$\text{Expected Exit Value} \; + \; \text{Post-Exit Earnings at the Firm} \; - \; \text{Investment Cost}$$

as compared to outside investors who would receive only:

$$\text{Expected Exit Value} \; - \; \text{Investment Cost}$$

if they formed a firm. Importantly, it is the expectation of *some* post-exit earnings that creates the superior incentive for an entrepreneur to found a firm. Thus, the prediction of the model, characterises an entrepreneur as an agent who is 'important' to firm value regardless of whether they own the firm or not.

It is here that the model differs from the property rights approach of Hart and Moore (1990). In that model, an owner is an agent who needs the superior bargaining position that comes from ownership to motivate them to engage in non-contractible productive activities. If this were an entrepreneur, they would act to increase firm value in the form of on-going earnings in the Hart-Moore world. Here, however, as asset markets are open and clear ex post, the entrepreneur considers value upon exit. In other words, to the extent that the entrepreneur was expected to engage in non-contractible investment or effort, their choices are based on returns *after* ownership has been ceded to outsiders. When an entrepreneur

founds a firm with the expectation of selling it, they do not consider simply the impact on value they capture ex post (their Post-Exit Earnings) but also on the value from the asset sale. Thus, while, for example, the entrepreneur of an M-owned firm will consider the impact on one-half of value created in choosing those non-contractible actions, and an agent who is employed by an outside owner, one-third of value created, here, the entrepreneur founder considers the impact on two-thirds of value created; a stronger incentive than in a model without endogenous ownership. In the model here, that non-contractible action is the choice to invest to found a firm and so the market allocates initial ownership to the entrepreneur who appropriates the highest share of the value created by firm formation.

The general equilibrium aspect of this result comes from the impact that firm formation has on the labour market. Firm formation draws workers from self-production into joint production. This increases wages in self-production and also bargained wages in joint production as well as impacting on the return to firm formation. For outside investors, that effect is entirely negative at $-\frac{2}{3}$. For entrepreneurs, a higher prevailing wage reduces asset value but increases post-exit earnings. The net effect of this is negative at $-\frac{1}{3}$. Thus, the relative returns to entrepreneurial firm formation are increased. At the point where these equal self-production wages, the returns to firm formation by outsiders is necessarily negative. This creates the sharp result in Proposition 1.

What constrains firm formation, therefore, is the cost of inducing labour into joint production. This is why the rents from firm formation end up accruing to labour. Of course, it is not lost on us that this stands in contrast to a narrative that capitalist firm formation is often associated in informal discussions with rents accruing to capital rather than labour. While one suspects that financial constraints and opportunities may account for the differing implications, this paper does not attempt to resolve this tension and, instead, focus on what a general equilibrium approach reveals.

Resources other than labour could conceivably impact on firm formation. For instance, opportunities for new form formation could come from the emergence of new ideas. Romer (1990) famously modelled these as arising out of ideas for new products, the investment costs for which involved the licensing of patent rights. Given that the flow of such ideas might be constrained by factors other than labour, this could also drive the scarcity of firm formation. This suggests that, like traditional general equilibrium theory, there is scope for embedding endogenous ownership in other environments studying broader economic issues.

4 Incomplete Contracts

In the baseline model, any efficiency that arises is because when firms are formed, the capital invested is nonfungible, leading to a hold-up problem, as skilled labour that brings joint production value to that capital is irreplaceable. Below, we will explore the outcomes that arise when these assumptions are relaxed. However, it is instructive to first consider what happens when ownership structure can impact the productivity of joint production itself. Here, we will explore contractual incompleteness that can generate such outcomes.

The decision to invest and form a firm is non-contractual in the baseline model and Proposition 1 shows that a manager (aka entrepreneur) is both predicted and the (constrained) efficient agent to undertake that action. As there is often discussion regarding the social importance of providing incentives for entrepreneurs to engage in such investments and firm-building activities, the results thus far suggest that market forces will support and provide such incentives. This would, of course, include any non-contractible actions taken by managers that will boost y in the long term. As is well-known, for such long-term actions, it is the eventual ownership structure that drives incentives rather than the initial ownership structure per se. The same is true in the baseline model with the caveat that the eventual ownership structure drives both the ongoing wage earnings of the manager as well as the sale price of assets. It is for this reason that a manager's incentives to invest are stronger than other agents.

Here, we explore the role of incentives to undertake non-contractible but productive activities that impact ongoing firm performance. Such activities can be undertaken by any skilled worker, manager or otherwise. The difference is that ownership structure can impact incentives to undertake such activities and, in the process, impact ex post asset trading that would alter ownership structure. In this section, these issues are explored by focusing on non-contractible actions undertaken by other skilled workers. However, it is noted that these would equally apply to the manager as well with regard to actions that impact on-going firm performance.

To this end, suppose that the skilled worker can engage in an activity, incurring a personal cost of c, and increase firm output by $\Delta > 0$ to $y + \Delta$. It is assumed that such activity is efficient with $c \leq \Delta$. Suppose that activity is not contractible and, thus, cannot be directly determined via wage negotiations within a firm. In this case, a skilled worker will choose to engage in the productivity-improving activity if:

1. Under M-ownership, $\frac{1}{2}(y + \Delta) - c \geq \frac{1}{2}y$ or $\frac{1}{2}\Delta \geq c$.

2. Under O-ownership, $\frac{1}{3}(y + \Delta + w) - c \geq \frac{1}{3}(y + w)$ or $\frac{1}{3}\Delta \geq c$.

As is well known in the property rights literature (Hart and Moore, 1990), the worker has greater incentives to engage in productive activity if the firm is owned by another essential agent (in this case, the manager) rather than an outside owner who plays no productive role.

To focus on the interesting case where ownership matters, it is assumed that:

$$3 > \frac{\Delta}{c} \geq 2$$

In this case, if the firm is eventually under M-ownership, its output will be $y+\Delta$; otherwise, under O-ownership, it is simply y. This will impact on the outcomes that arise from asset trading.

When ownership matters for efficiency, this will impact ex post asset trading by altering a manager's willingness to sell and an outsider's willingness to pay for the asset. Thus, O-ownership will arise if:

$$\underbrace{\tfrac{1}{2}(y+\Delta)-\tfrac{1}{3}(y+w)}_{M \text{ willingness to sell}} < \underbrace{\tfrac{1}{3}(y-2w)}_{O \text{ willingness to pay}} \implies \Delta < \tfrac{1}{3}(y-2w)$$

Thus, for Δ high enough, the equilibrium outcome will be M rather than O-ownership. Intuitively, there is more surplus available when the skilled worker has an incentive to engage in the productive activity.

While this result – that trading of ownership can result in an inefficient allocation of assets under contractual incompleteness – has been explored in the literature (Gans, 2005), the general equilibrium context adds an important caveat to the analysis. Under general equilibrium, whether it M-ownership is an equilibrium depends on another factor: the skilled worker's *participation constraint*. Under M-ownership, a worker has three choices: (i) engage in effort resulting in an expected payoff of $\frac{1}{2}(y+\Delta)-c$; (ii) refrain from effort for $\frac{1}{2}y$ and (iii) return to the self-production sector for w. The worker will choose (i) over (ii) given our assumption that $\frac{\Delta}{c} \geq 2$. However, for (i) to be preferred to (iii) requires $\frac{1}{2}(y+\Delta) - c \geq w$. However, w depends on total firm formation and joint production employment under the condition that worker effort is expended. Thus, if, consistent with a M-ownership equilibrium, $w_M = \frac{1}{2}(y+\Delta-2)$, for the worker participation constraint to hold requires $\frac{1}{2}(y+\Delta)-c \geq w$ or $c \leq 1$. This is because wages in self-production are driven by joint-production productivity less a capital cost share but can be obtained without incurring c.[21]

This means that the approach to determining the equilibrium just discussed will not be appropriate if $c > 1$. In this case, the worker participation constraint will bind, not the

[21]Note that this particular issue would also arise if ownership were exogenous but there were incomplete contracts.

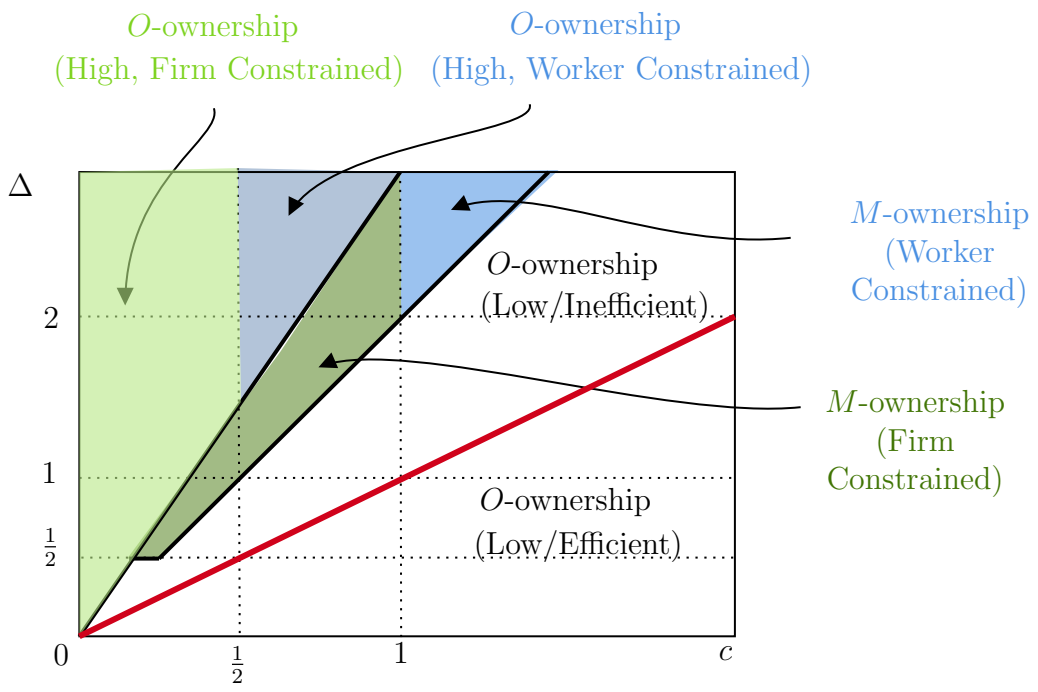


Figure 2: Equilibrium Outcomes with Incomplete Contracts

firm formation condition. Given this, the following proposition characterises the equilibrium outcomes in the model with incomplete contracts and endogenous ownership.

Proposition 2 *Suppose that* $\frac{\Delta}{c} \in [2, 3)$. *If* $c \leq 1$, *the equilibrium outcomes are as follows:*

1. *(M-ownership, Firm Constrained) If* $y + \Delta \geq 2$, $\Delta \geq \frac{1}{2}$ *and* $c \leq 1$, *there exists an equilibrium with M-ownership ex post, with self-production wages and manager income of* $\frac{1}{2}(y + \Delta - 2)$ *and skilled worker payoff of* $\frac{1}{2}(y + \Delta) - c$.

2. *(M-ownership, Worker Constrained) If* $y + \Delta \geq 2c$ *and* $c \geq 1$, *there exists an equilibrium with M-ownership ex post, with self-production wages and skilled worker payoff of* $\frac{1}{2}(y + \Delta) - c$ *and manager income of* $\frac{1}{2}(y + \Delta) - 1$.

3. *(O-ownership, Low Productivity) If* $y \geq \frac{3}{2}$ *and* $\Delta \leq \frac{1}{2}$, *there exists an equilibrium with O-ownership ex post with the same payoff outcomes as Proposition 1.*

The proposition demonstrates that there are three types of equilibrium within the assumed zone of interest where ownership matters (i.e., $\Delta \in [2, 3]$). The first is the M-ownership outcome where the firm formation constraint binds which is an equilibrium if Δ is sufficiently high and c is less than firm formation capital costs. The second is an M-ownership outcome where the skilled worker participation constraint binds. This is an equilibrium if c is greater

than firm formation capital costs. In this equilibrium, it is manager income that exceeds self-production wages while skilled worker payoffs are the same in and out of joint production. The final equilibrium, which arises when Δ is low involves O-ownership with low productivity as the worker employed by the firm does not engage in productivity-increasing effort.

Figure 2 shows the equilibrium outcomes in (c, Δ) space. Note that if $\Delta < 2c$, then no worker engages in the productive activity and the unique equilibrium is O-ownership (with low productivity) as in Proposition 1. On the other hand, if $\Delta > 3c$, then workers engage in the productive activity even under O-ownership. In this case, the nuances regarding the skilled worker participation constraint will come into play. If $\hat{w}_O = \frac{1}{2}(y + \Delta - \frac{3}{2})$, the worker participation constraint requires that $\frac{1}{3}(y + \Delta + \hat{w}_O) - c \geq \hat{w}_O$ or $c \leq \frac{1}{2}$. Thus, if $c > \frac{1}{2}$, the worker participation constraint will bind, resulting in $\hat{w}_O = \frac{1}{2}(y + \Delta - 3c)$ while the returns to the manager from firm formation will be strictly positive.

Finally, Figure 2 shows that regardless, endogenous ownership implies that equilibrium outcomes will be inefficient beyond the capital hold-up modelled earlier. Above the red line, it is efficient to engage in effort to boost productivity. When there are incomplete contracts impacting on worker effort, this can allow M-ownership to emerge but only if Δ is sufficiently higher ($\geq \frac{1}{2}$). Otherwise, O-ownership prevails.

It is useful to observe that while O-ownership can arise when it is efficient to undertake the productive activity, this does not necessarily imply that M-ownership would be welfare-enhancing in the sense of creating higher aggregate consumption in equilibrium. This is because O-ownership arises precisely because the returns to investment anticipating O-ownership are higher than the returns to investment if M-ownership were imposed. Thus, for Δ such that:

$$\tfrac{2}{3}(y - 2w) + w = 1 + \tfrac{1}{2}(y - \tfrac{3}{2}) \geq \tfrac{1}{2}(y + \Delta) \implies \tfrac{1}{2} \geq \Delta$$

imposing M-ownership would reduce total firm formation and mitigate the productivity benefits that it brings. This is because the ownership structure balances the incentives of managers for long-term non-contractible actions against the incentives of skilled workers for ongoing non-contractible actions. The nuance added by the analysis here is that in contrast to Hart and Moore (1990), those long-term incentives are best promoted by outside as opposed to manager ownership.

5 Robustness

The previous section has a strong prediction that when ownership is endogenous, firms are formed by managers who then trade firm assets to outsiders. Thus, the resulting equilibrium is outside-ownership. In this section, the various assumptions in the baseline model are tested to see how robust this prediction is. It will be demonstrated that while altering those assumptions can change the efficiency outcomes from the resulting equilibrium, the main prediction is robust to such changes.

5.1 Fungible Capital

A key assumption in the baseline model is that capital, once deployed, cannot be reutilised for consumption; that is, it is non-fungible. This allows for capital hold-up by agents (both skilled workers and managers) whose skills are specialised to the firm's capital asset and who are scarce ex post. Here, the assumption that capital is non-fungible is relaxed, and it is demonstrated that this does not change the O-ownership equilibrium outcome but, in the limit, does restore efficiency.

Suppose that capital is fungible and can be converted into, $\alpha(< 1)$ of the consumption good at no cost. This fraction becomes part of the bargaining surplus between the firm owners and workers, i.e., $s = y - \alpha - 2w$. Under O-ownership, this surplus is split three ways, giving $\frac{1}{3}s + w$ to the manager and skilled worker and $\frac{1}{3}s + \alpha$ to the owner. By contrast, under M-ownership, the worker receives $\frac{1}{2}s + w$ and the manager receives $\frac{1}{2}s + \alpha + w$. Note that this implies that in ex post trading, the willingness to pay of outsiders always exceeds the willingness to sell of the manager:

$$\underbrace{\tfrac{1}{3}(y + 2\alpha - 2w)}_{O \text{ Willingness to Pay}} > \underbrace{\tfrac{1}{2}(y + \alpha) - \tfrac{1}{3}(y - \alpha + w)}_{M \text{ Willingness to Sell}} = \tfrac{1}{6}(y + 5\alpha - 2w) \implies y - 2w \geq \alpha$$

which holds so long as there is positive surplus from joint production at the equilibrium wage.

Given this, the firm formation condition is now:[22]

$$\tfrac{1}{3}(y + 2\alpha - 2w) + \tfrac{1}{3}(y - \alpha + w) - 1 \geq w \implies \tfrac{2}{3}(y - 2w) \geq 1 - \tfrac{1}{3}\alpha$$

If the manager firm formation condition holds with equality, we can derive the equilibrium

[22]Note that the return to outside owners forming firms is $\frac{1}{3}(y + 2\alpha - 2w) - 1 = \frac{1}{3}(y - \alpha - 2w)$ which is lower than the return to managers forming firms.

wage as:

$$\hat{w}_O = \tfrac{1}{4}(2y + \alpha - 3)$$

Thus, firms will be formed so long as $y \geq \frac{3-\alpha}{2}$, and the equilibrium outcome will involve managers forming firms and, ultimately, O-ownership. Importantly, as $\alpha \to 1$, the condition for firm formation approaches $y \geq 1$; that is, firms form so long as joint production is efficient. This establishes that capital non-fungibility is the source of any equilibrium inefficiency.[23]

5.2 Replaceable Workers

The second assumption that potentially drives an inefficient equilibrium outcome is that workers in joint production have non-binding wage contracts that can, thus, be renegotiated by irreplaceable workers. Here, these results are re-evaluated in a setting that allows skilled workers or managers to be replaced by workers from the self-production sector.

To examine this, we consider an alternative wage bargaining model based on that of Stole and Zwiebel (1996a) but where workers can be replaced from the pool of labour in self-production, thereby generating an outcome different from the Shapley value. We utilise the model of de Fontenay and Gans (2003) that determines negotiated wage outcomes when there are replacement workers and employment contracts are non-binding. Specifically, their model demonstrates that the joint-production employment wage when there are l agents in self-production is:

$$e^{-l}\tfrac{1}{3}(y + w) + (1 - e^{-l})w$$

for O-ownership and

$$e^{-l}\tfrac{1}{2}y + (1 - e^{-l})w$$

for M-ownership where, in each case, the fact that workers are drawn from a continuum is taken into account. Note that replacement workers come from self-production and are assumed not to be drawn from joint production.

How does this change ex post asset trading? Outside owners will end up with the asset if and only if:

$$\underbrace{y - 2(e^{-l}\tfrac{1}{3}(y + w) + (1 - e^{-l})w)}_{O \text{ Willingness to Pay}} > \underbrace{y - (e^{-l}\tfrac{1}{2}y + (1 - e^{-l})w) - (e^{-l}\tfrac{1}{3}(y + w) + (1 - e^{-l})w)}_{M \text{ Willingness to Sell}}$$

[23] If $\alpha = 1$, then firm formation is efficient, but there is no difference in the returns to managers and outsiders in investing and so we cannot pin down the form of ownership in equilibrium. What the above argument shows is that, for infinitesimal levels of non-fungibility, O-ownership remains the equilibrium outcome.

$$\implies e^{-l}\tfrac{1}{3}(y+w) < e^{-l}\tfrac{1}{2}y \implies y > 2w$$

Given this, the following result can be proven:

Proposition 3 *Suppose that joint production firms can negotiate and replace workers from the self-production sector. Then, if* $y \geq \frac{3e}{3e-1} \approx 1.14$*, the unique equilibrium outcome involves managers forming firms and then selling it to outsiders.*

The proof of the proposition is in the appendix. The challenge in the proof arises because the condition for equilibrium wages must take into account the impact of the wage on the size of the self-production labour pool. That pool is limited and becomes more so, the more joint production firms are formed. This creates an additional effect that increases negotiated wages. Nonetheless, the proposition demonstrates that the conditions under which joint production takes place in equilibrium is expanded and, thus, worker replaceability serves to increase efficiency.

Thus, so long as it is feasible, O-ownership remains the equilibrium outcome. The ability to replace workers means that the range of parameters for which O and M-ownership are feasible is greater than the no replacement case. However, it is only when the measure of the set of agents approaches infinity (a highly unrealistic case) that an efficient outcome arises.

5.3 Non-Specific Skills

Skilled workers have skills specific to the firm they join, whether those skills are managerial or not. This means that in wage negotiations, their outside option is to engage in self-production. For the capital owner, this implies that in the absence of an agreement, the firm cannot operate, and the capital asset has no value. Suppose, instead, that skills are transferable between firms, although the type of skills remains distinct. That may alter the outside options of both the skilled workers and the firm. As will be shown in this section, this change adds considerable complexity to the analysis. However, the results of Proposition 1 continue to hold.

The baseline model allowed the use of the Shapley value between two or three agents, as the case may be, because those were the only agents who could generate joint production output in a coalition. When skills are non-specific, if there are $[0, 2n]$ skilled workers ($[0, n]$ of each type), $[0, n]$ firms can form, but the workers can have a positive marginal contribution to any of those firms. If we suppose, as would be consistent with general equilibrium models, that agents of the same type cannot cooperate (or collude), then the cooperative bargaining output is graph-constrained. Nonetheless, all agents are connected in the resulting graph and, thus, as demonstrated by Myerson (1977), the resulting allocation of value is the same

as the Shapley value across all skilled workers and capital asset owners.[24] The complication for the model here is that the agents concerned come from a continuum. For this reason, the appendix shows that when the set of agents is a continuum, the Shapley value allocations described below arise.

The straightforward way to see this is to note that because of the symmetry condition of the Shapley value, the payoffs to agents of a given type: skilled workers, managers, outside capital owners and manager-owners will all be identical. Thus, if a measure N of firms form, then aggregate joint production surplus will be $n(y-2w)$. Under M-ownership, each skilled work will receive $\frac{1}{2}(y-2w)+w$ as will each manager. Under O-ownership, the surplus is split equally amongst the three agent types, with the skilled workers each receiving $\frac{1}{3}(y-2w)+w$ and outside owners as a whole receiving $n\frac{1}{3}(y-2w)$. As the number of firms is a continuum, if one pair of outside owners and managers decide to change asset ownership of a firm, this will not change the allocation to other agents. Consequently, it will remain the case that in ex-post trading, M's willingness to sell will be less than an outsider's willingness to pay, and O-ownership will result.

One might object that this is an artifact of the assumption that agents exist in a continuum. Thus, we examine the robustness of this outcome by considering an analysis of ex post trading when agents are discrete and each skilled worker's skills are not specific to a particular firm. We will show that in this situation, if one manager and an outside owner change their asset ownership arrangements, this will impact the payoffs of all other agents.

To illustrate this, consider an environment in which only two firms have formed, and there are four skilled workers, two with each skill. Each agent receives their Shapley value in the game with 6 agents. Of the two firms exist under O-ownership, each worker receives $\frac{1}{3}(y+w)$ and each outside owner receives $\frac{1}{3}(y-2w)$. Under M-ownership, the manager and worker each receive $\frac{1}{2}y$ ex post. To calculate the gains from trade from ex post asset trading, we need to calculate the Shapley values if just one firm is M-owned while the other is O-owned. In the resulting 5-agent game, the Shapley values have the manager of the manager-owned firm earning $\frac{17}{30}(y-2w)+w$, the outside owner of the outsider-owned firm earning $\frac{19}{60}(y-2w)$, the manager employed by that firm earning $\frac{19}{60}(y-2w)+w$ and the two other skilled workers earning $\frac{2}{5}(y-2w)+w$. Intuitively, the outside owner and the employed manager have the same contribution to net surplus and, by symmetry, will receive the same share of net surplus $(y-2w)$. The same logic holds for the other skilled workers.

Intuitively, starting from a situation where both firms are M-owned, and each agent receives $\frac{1}{2}(y-2w)+w$ if one manager-owner sells to an outsider, that agent's payoff will fall

[24] de Fontenay and Gans (2014) provide a non-cooperative game based on bilateral contracts between the capital asset holder and any worker that generates the Shapley value allocation.

to $\frac{19}{60}(y-2w)+w$ while the outside owner receives $\frac{19}{60}(y-2w)$, representing their willingness to pay for the asset. Comparing this to the manager's willingness to sell of $\frac{1}{2}(y-2w)+w-\frac{19}{60}(y-2w)+w=\frac{11}{60}(y-2w)$ which is less than the outsider's willingness to pay. This demonstrates that M-ownership is not an equilibrium outcome in ex post asset trading.

The surplus that enables this transaction to take place comes from the other two skilled workers whose payoffs fall to $\frac{2}{5}(y-2w)+w$. Interestingly, the payoff of the other manager-owned firm *increases* to $\frac{17}{30}(y-2w)+w$. This raises the question of whether a pure O-ownership outcome is possible. In this situation, the relevant condition for the asset to be sold to an outside owner is:

$$\underbrace{\tfrac{17}{30}(y-2w)+w-(\tfrac{1}{3}(y-2w)+w)}_{M\text{'s Willingness to Sell}} \leq \underbrace{\tfrac{1}{3}(y-2w)}_{O\text{'s Willingness to Buy}} \implies \tfrac{17}{30}(y-2w) \leq \tfrac{2}{3}(y-2w)$$

which always holds. Thus, O-ownership is the unique outcome in ex post trading even when worker skills and not firm specific.

The following proposition proves this for the case of $[0,n]$ firms.

Proposition 4 *Suppose that there are n capital owners and n workers of each type. Then O-ownership for all n joint production firms is the unique equilibrium outcome of ex post asset trading.*

The proof (in the appendix) is quite involved but intuitively, when a manager-owner sells an asset to an outside owner, this creates a new environment with an "unbundled" manager and capital asset, making forming firms harder as other agents are "forced" to cede more value to make that happen while creating an additional outside agent dividing surplus further.[25] Both of these factors mean that the new outside owner and its manager-employee can appropriate more value than a single manager-owner could previously.

Thus, the equilibrium identified in Proposition 1 emerges as an equilibrium in a situation where worker skills are not firm-specific. The equilibrium wage outcome remains $\hat{w}_O = \frac{1}{2}(y-3)$ and, thus, in equilibrium, managers form firms that are sold to outside owners in ex post trading.

5.4 Endogenous Firm Size

In the baseline model, joint production involves just two workers – a manager and a skilled worker – who are perfect complements for one another. A natural question to explore is

[25] An alternative proof of an O-equilibrium is provided in the appendix using a different approach based on Segal (2003).

what happens when there are (a) more workers who can contribute to joint production, (b) those workers are not perfect complements, and (c) the size of the firm is itself endogenous. In this subsection, we augment the baseline model to include these features and examine how this impacts ownership outcomes.

Suppose that one unit of capital and one manager allows joint production to take place according to a production function, $F(n)$, where n is the number of skilled workers employed by the firm.[26] The skills of workers are specific to the firm and cannot be used in production elsewhere. It is assumed that F is increasing and (weakly) concave in n with $F(0) = 0$. Thus, if a firm employs n workers, it generates (net) surplus of $F(n) - (n+1)w$; recalling that the manager has an outside wage option of w.

So long as $F(i) \geq (i+1)w$ for all $i \leq n$, under M-ownership, the Shapley value for the outside owner is:

$$\frac{1}{n+2}\sum_{i=1}^{n}(F(i) - (i+1)w)$$

This is derived by Shapley and Shubik (1967) (see also Stole and Zwiebel (1996a) who identify the condition on $F(.)$ and w which they term *feasibility*). Under O-ownership, the Shapley value for the outside owner is:

$$\frac{1}{(n+1)(n+2)}\sum_{i=1}^{n}\big(i+1\big)\left(F(i) - (i+1)w\right)$$

while the payoff for the manager in this structure is the same as the outside owner's plus w. Given this, O-ownership will be chosen over M-ownership in ex post trading if:

$$\underbrace{\tfrac{2}{(n+1)(n+2)}\sum_{i=1}^{n}\big(i+1\big)\left(F(i) - (i+1)w\right) + w}_{\text{Joint Payoff Under } O\text{-ownership}} > \underbrace{\tfrac{1}{(n+1)(n+2)}\sum_{i=1}^{n}\big(i+1\big)\left(F(i) - (i+1)w\right)}_{M \text{ Payoff under } M\text{-Ownership}}$$

Given this, we can demonstrate the following:

Proposition 5 *Let $F : \{1, 2, \ldots, n\} \to \mathbb{R}$ be any real-valued function. Suppose that, for $w > 0$,*

$$F(i) \;\geq\; (i+1)\,w \quad \textit{for all } i = 1, 2, \ldots, n.$$

Then O-ownership is the unique equilibrium in ex post asset trading.

This demonstrates that the conclusions of the baseline model with respect to ex post asset trading carry over to the more general environment here.

[26] This specification captures the production technology envisaged by Arrow and Debreu that allows for diminishing returns to scale as well as an entrepreneurial input for each firm.

Suppose that following the ex post trading stage but before production, each joint production firm chooses its 'size,' n. Stole and Zwiebel (1996a) demonstrate that in choosing n, at the optimal $\tilde{n}$:

$$F(\tilde{n}) - (\tilde{n}+1)w \doteq \frac{1}{(\tilde{n}+1)(\tilde{n}+2)} \sum_{i=1}^{\tilde{n}} \big(i+1\big)\left(F(i) - (i+1)w\right)$$

$$\implies F(\tilde{n}) - (\tilde{n}+1)w \doteq \frac{1}{(\tilde{n}+1)(\tilde{n}+2)} \sum_{i=1}^{\tilde{n}} \big(i+1\big) F(i) - \frac{2\tilde{n}+3}{6} w$$

Intuitively, the Shapley value is akin to the average product of a standard profit function and reaches its maximum where marginal profit crosses average product. When choosing their size, because skilled workers cannot be replaced ex post, a firm will choose a size to accommodate and commit to employ a larger number of those workers so as to diminish the bargaining position of any single worker. This means that the bargaining wage will be pushed to the self-production level.

Given this, recall that the condition determining firm formation is:

$$\tfrac{2}{(n+1)(n+2)} \sum_{i=1}^{n} \big(i+1\big)\left(F(i) - (i+1)w\right) + w - 1 \geq w$$

That is, a manager forms a firm (incurring the capital cost and forgoing self-production wages) in order to sell it to outside owners in ex post trading. In a rational expectations equilibrium, the manager will anticipate the firm size being $\tilde{n}$ for the anticipated w. Therefore, we can use the Stole-Zwiebel optimality result to pin down the firm formation condition as:

$$2\big(F(\tilde{n}) - (\tilde{n}+1)w\big) = 1$$

which gives:

$$\hat{w}_O = \frac{F(\tilde{n}) - \frac{1}{2}}{\tilde{n}+1}$$

This will be an interior equilibrium if $\hat{w}_O < 1$ or $F(\tilde{n}) - \frac{3}{2} < \tilde{n}$ on $F(\tilde{n}) - \frac{1}{2} < \hat{N}$ where $\hat{N}$ is the equilibrium number of firms.

In the baseline model, there is too little joint production relative to the socially optimal outcome. Here, however, this result is revised. To see this, in an efficient outcome, the size of firms would be chosen so that $\Delta F(n^*) \doteq w$ and the firm formation condition would be $F(n^*) - (n^*+1)w = 1$ which would given a wage of $w^* = \frac{1}{n^*+1}(F(n^*) - 1)$. For the same n, this is a lower wage than $\hat{w}_O$; that is, the demand for labour in joint production is higher

than the efficient labour demand. Thus, equilibrium wages will be higher than efficient wages and the total volume of labour in equilibrium will exceed efficient levels. This is precisely the Stole-Zwiebel 'over-employment' result in this environment.

6 Conclusion

This paper has developed a theory of endogenous firm ownership in general equilibrium, demonstrating how ownership structures emerge from market forces rather than arising as artifacts of model assumptions. The analysis generates three key insights for economic theory. First, it shows that the property rights theory of the firm can be naturally embedded within general equilibrium analysis, with ownership patterns determined by the interaction of labour and capital markets. Second, it provides a novel economic rationale for entrepreneurship, identifying entrepreneurs as agents whose human capital makes them uniquely positioned to found firms even when they do not remain long-term owners. Third, it resolves the long-standing tension between Arrow-Debreu and McKenzie's approaches to firm ownership in general equilibrium theory.

The model's predictions align with several empirical regularities in firm ownership dynamics. It explains why entrepreneurs typically found firms but then sell ownership rights to outside investors while remaining as employees. This pattern emerges because entrepreneurs can capture value both through the sale price of ownership rights and through their ongoing employment relationship, while outside owners can only benefit through residual claims. The framework also provides insight into why ownership transitions tend to occur after firms are established rather than at founding.

This analysis suggests several promising directions for future research. One natural extension would be to incorporate financial market frictions and examine how they affect the timing and nature of ownership transitions. Another would be to explore how endogenous ownership interacts with innovation and growth dynamics. Finally, the framework could be enriched to study how different legal and institutional environments affect equilibrium ownership patterns.

More broadly, this paper demonstrates the value of making ownership structures endogenous in economic analysis. Doing so provides new insights into firm formation, entrepreneurship, and the evolution of corporate ownership while maintaining the core structure of competitive general equilibrium theory. These insights may prove valuable not only for understanding observed ownership patterns but also for analyzing how policy and institutional changes affect firm organization and economic performance.

7 Appendices

7.1 Proof of Proposition 2

First, we examine whether an M-ownership equilibrium exists. Suppose that all firms are in M-ownership implying that $\hat{w}_M = \frac{1}{2}(y + \Delta - 2)$. The necessary conditions for this to be an equilibrium outcome are:

1. (Worker Incentive Constraint): The worker must engage in the productive activity, i.e., $\frac{1}{2}\Delta \geq c$; which holds by assumption.

2. (Worker Participation Constraint): The worker must prefer to remain with the firm than to go into self-production, i.e., $\frac{1}{2}(y + \Delta) - c \geq \hat{w}_M$ or $c \leq 1$.

3. (Asset Trading): Outsiders have a lower willingness to pay than the manager's willingness to sell, i.e., $\frac{1}{3}(y - 2\hat{w}_M) \leq \frac{1}{2}(y + \Delta) - \frac{1}{3}(y + \hat{w}_M)$ which implies that $\frac{1}{3}(2 - \Delta) \leq \frac{1}{2}(y + \Delta) - \frac{1}{3}(\frac{3}{2}y + \frac{1}{2}\Delta - 1)$; which holds for $\Delta \geq \frac{1}{2}$.

Thus, M-ownership is an equilibrium outcome so long as (i) $\frac{1}{2}\Delta \geq c$; (ii) $c \leq 1$ and (iii) $\Delta \geq \frac{1}{2}$. These three conditions can hold simultaneously so this equilibrium exists. Firms will be formed by managers and enter so long as $\frac{1}{2}(y + \Delta) - 1 \geq w$ which will hold with equality at $\hat{w}_M = \frac{1}{2}(y + \Delta - 2)$ confirming the (conjectured) equilibrium wage outcome and the first condition in the proposition.

If $c > 1$, then the participation constraint above will not hold at $\hat{w}_M = \frac{1}{2}(y + \Delta - 2)$. Therefore, rather than being set by the firm formation capital equation (or M-owner participation constraint) suppose that $\hat{w}_M$ is determined by the worker participation constraint so that $\hat{w}_M = \frac{1}{2}(y + \Delta) - c$. In that case, an M-owned firm will form if $\frac{1}{2}(y + \Delta) - 1 \geq \hat{w}_M$ or $c \geq 1$. Thus, in equilibrium, firms will form up until the point where the way is $\hat{w}_M$ and an additional firm forming would result in a higher wage, a failed participation constraint and a negative return for firms. Thus, the worker incentive and participation constraints are satisfied. To establish an equilibrium, we need to consider whether the manager holds onto the asset ex post. The condition for this is that $\frac{1}{3}(y - 2\hat{w}_M) \leq \frac{1}{2}(y + \Delta) - \frac{1}{3}(y + \hat{w}_M)$ which is equivalent to $\Delta \geq \frac{1}{2}c$ which holds given our assumptions on $\frac{\Delta}{c}$. Thus, M-ownership is an equilibrium outcome so long as $c \geq 1$ and $y + \Delta \geq 2c$.

Second, we examine whether an O-ownership equilibrium exists. Suppose that all firms are eventually in O-ownership but are initially formed by managers implying that $\hat{w}_O = \frac{1}{2}(y - \frac{3}{2})$. The necessary conditions for this to be an equilibrium outcome are:

1. (Worker Incentive Constraint): The worker must not engage in the productive activity under O-ownership, i.e., $\frac{1}{3}\Delta < c$; which holds by assumption.

2. (Worker Participation Constraint): The worker must prefer to remain with the firm than to go into self-production, i.e., $\frac{1}{3}(y + \hat{w}_O) \geq \hat{w}_O$ which always holds.

3. (Asset Trading): Outsider willingness to pay for the asset is $\frac{1}{3}(y - 2\hat{w}_O)$ while the manager's willingness to sell depends on whether they expect their worker to expend effort or not. If they expect that effort, the manager's willingness to sell is $\frac{1}{2}(y + \Delta) - \frac{1}{3}(y + \hat{w}_O)$ which will be lower than the outsider's willingness to pay at $\hat{w}_O$ if $\Delta \leq \frac{1}{2}$. Note that, at $\hat{w}_O$, the skilled worker's participation constraint in M-ownership is $\frac{1}{2}(y + \Delta) - c \geq \hat{w}_O$ or $\Delta - 2c \geq -\frac{3}{2}$ which holds as $\Delta \geq 2c$ by assumption.

Thus, O-ownership is an equilibrium outcome so long as $\Delta \leq \frac{1}{2}$ which is consistent with the condition that $\Delta \in [2, 3)$ Firms will be formed by managers and enter so long as $\frac{1}{3}(2y - w) - 1 \geq w$ or $\frac{2}{3}(y - 2w) \geq 1$ which will hold with equality at $\hat{w}_O = \frac{1}{2}(y - \frac{3}{2})$ confirming the (conjectured) equilibrium wage outcome and the necessary condition in the proposition.

7.2 Proof of Proposition 3

It was demonstrated in the text that ex post asset trading will result in a sale to the outsider owner so long as $y > 2w$. Given this, M's firm formation condition is:

$$y - 2(e^{-l}\tfrac{1}{3}(y + w) + (1 - e^{-l})w) + e^{-l}\tfrac{1}{3}(y + w) + (1 - e^{-l})w - 1 \geq w$$

Recalling that $L = 1 - l = w^\eta$ or $l = 1 - w^\eta$, this yields:

$$\hat{w}_O = \frac{y - 3e^{1-\hat{w}_O^\eta}(y - 1)}{2 - 6e^{1-\hat{w}_O^\eta}} \tag{1}$$

which does not have a closed-form solution.

To determine the critical threshold for y such that joint production takes place, it suffices to examine the conditions under which $\hat{w}_O$ will be positive. Rewrite (1) by setting:

$$\text{RHS}(w) \;=\; \frac{3\,e^{1-w^\eta}\,y \;-\; 3\,e^{1-w^\eta} \;-\; y}{2\left(3\,e^{1-w^\eta} - 1\right)} = \frac{3\,e^{1-w^\eta}\,(y-1) \;-\; y}{2\left(3\,e^{1-w^\eta} - 1\right)}.$$

Then (1) is $w = \text{RHS}(w)$. Define

$$f(w) \;=\; w - \text{RHS}(w).$$

A real w is a solution to (1) if and only if $f(w) = 0$. We seek conditions for which there is a *positive* solution $w > 0$.

First, consider $f(0)$. As $w \to 0$, $w^\eta \to 0$ and $e^{1-w^\eta} \to 1$. Hence

$$\mathrm{RHS}(0) = \frac{3\,e\,(y-1) \;-\; y}{2\,\big(3\,e-1\big)}.$$

Expand the numerator:

$$3\,e\,(y-1) - y \;=\; 3\,e\,y \;-\; 3\,e \;-\; y \;=\; (3\,e-1)\,y \;-\; 3\,e.$$

Define

$$G(y) \;=\; (3\,e-1)\,y \;-\; 3\,e.$$

Then

$$\mathrm{RHS}(0) \;=\; \frac{G(y)}{2\,\big(3\,e-1\big)}, \quad f(0) \;=\; 0 \;-\; \mathrm{RHS}(0) \;=\; -\,\frac{G(y)}{2\,\big(3\,e-1\big)}.$$

Since $3\,e-1>0$, the sign of $\mathrm{RHS}(0)$ (and thus $-\,\mathrm{RHS}(0)$) directly follows the sign of $G(y)$.

We solve $G(y)=0$:

$$(3\,e-1)\,y \;=\; 3\,e \;\iff\; y \;=\; \frac{3\,e}{3\,e-1}.$$

Thus:

- If $y > \dfrac{3\,e}{3\,e-1}$, then $G(y)>0$, so $\mathrm{RHS}(0)>0$ and hence $f(0)<0$.
- If $y < \dfrac{3\,e}{3\,e-1}$, then $G(y)<0$, so $\mathrm{RHS}(0)<0$ and hence $f(0)>0$.
- If $y = \dfrac{3\,e}{3\,e-1}$, then $G(y)=0$, so $\mathrm{RHS}(0)=0$ and hence $f(0)=0$.

Now, evaluate $f(w)$ as $w \to \infty$. As $w \to \infty$, we have $w^\eta \to \infty$ (since $\eta>0$). Hence

$$e^{1-w^\eta} \;\to\; 0.$$

Then

$$3\,e^{1-w^\eta} \;\to\; 0, \quad \text{so } 3\,e^{1-w^\eta} - 1 \;\to\; -1.$$

Similarly, the numerator in $\mathrm{RHS}(w)$ becomes

$$3\,e^{1-w^\eta}\,(y-1) - y \;\longrightarrow\; 0 \;-\; y \;=\; -\,y.$$

Thus

$$\mathrm{RHS}(w) \;=\; \frac{3\,e^{1-w^\eta}\,(y-1) - y}{2\,(3\,e^{1-w^\eta}-1)} \;\xrightarrow[w\to\infty]{}\; \frac{-\,y}{2\,(-1)} \;=\; \frac{y}{2}.$$

Consequently,

$$\lim_{w\to\infty} f(w) \;=\; \lim_{w\to\infty}\Big[w \;-\; \mathrm{RHS}(w)\Big] = \lim_{w\to\infty}\Big(w \;-\; \tfrac{y}{2}\Big) = +\infty.$$

We now apply the Intermediate Value Theorem, using the sign of $f(0)$ and the limit $\lim_{w\to\infty} f(w)$:

- *Case 1:* $y > \frac{3\,e}{3\,e-1}$. Then $f(0) < 0$ and $\lim_{w\to\infty} f(w) = +\infty$. Since $f(w)$ is continuous for all $w > 0$ (away from any point that might make the denominator zero, but that does not prevent a sign crossing), there must be some $w > 0$ such that $f(w) = 0$. Hence a *positive* solution $w > 0$ exists.

- *Case 2:* $y < \frac{3\,e}{3\,e-1}$. Then $f(0) > 0$ and $\lim_{w\to\infty} f(w) = +\infty$. Hence $f(w)$ starts out positive at $w = 0$ and remains unboundedly large as $w \to \infty$. It never crosses 0 for $w > 0$, so there is *no* positive solution.

- *Case 3:* $y = \frac{3\,e}{3\,e-1}$. Then $f(0) = 0$, which would force $w = 0$ (not a positive solution). Checking numerically or analytically shows no crossing at $w > 0$. Hence no positive solution in $(0, \infty)$.

Summarizing, the necessary and sufficient condition for the existence of a *positive* solution $w > 0$ to (1) is precisely

$$y \;>\; \frac{3\,e}{3\,e-1}.$$

Moreover, that threshold does not depend on η (only on the fact $\eta > 0$).

7.3 Myerson-Shapley Value for Non-specific Skills with a Continuum of Agents

Here it is demonstrated that the division of the surplus $n(y - 2w)$ amongst the $[0, n]$ agents with the managerial skill $(S1)$, the $[0, n]$ agents with the other skill $(S2)$ and the $[0, n]$ capital owners (K) for the outside ownership case generate (net) surplus divisions of $\frac{1}{3}(y - 2w)$ to each. This is done by discretising each into M sub-agents per input, computing the Shapley value for the finite game of size $3M$ and then examining the limit as $M \to \infty$.

We illustrate how to "redo" the Shapley-value derivation in a setting where Skill 1, Skill 2, and Capital each form a continuum of measure n. We discretize into M sub-agents per input, compute the Shapley value in the finite game of size $3M$, and then examine the limit as $M \to \infty$.

7.3.1 Discretising into $3M$ Agents of Equal Size

To apply the standard finite Shapley formula, we discretise:

- Partition Skill 1 into M blocks, each block has measure $\frac{n}{M}$. Label them $S1_1, \ldots, S1_M$.
- Partition Skill 2 into M blocks, each block $\frac{n}{M}$. Label them $S2_1, \ldots, S2_M$.
- Partition Capital into M blocks, each block $\frac{n}{M}$. Label them $K_1, \ldots, K_M$.

Denote the overall set of these $3M$ agents by

$$N_M \;=\; \{\, S1_1, \ldots, S1_M,\; S2_1, \ldots, S2_M,\; K_1, \ldots, K_M \}.$$

We define a *characteristic function* $v_M(\cdot)$ on subsets $S \subseteq N_M$ by:

Definition 2 (Characteristic Function for the Discretised Game) *For a subset S, let*

$$X_1(S) \;=\; \textit{(total measure of skill 1 blocks in } S\textit{)}, \quad X_2(S) \;=\; \textit{(total measure of skill 2 blocks in } S\textit{)},$$

$$X_c(S) \;=\; \textit{(total measure of capital blocks in } S\textit{)}.$$

Since each S1 block is $\frac{n}{M}$, we have $X_1(S) = \big(\#\{\textit{S1 blocks in } S\}\big)\,\frac{n}{M}$. Likewise for $X_2(S), X_c(S)$.

We define

$$v_M(S) \;=\; \min\big[X_1(S),\, X_2(S),\, X_c(S)\big] \;\times\; (y - 2w).$$

It follows that the *grand coalition* N_M can form $\min\{M, M, M\} = M$ blocks from each input, i.e. $X_1(N_M) = n$, $X_2(N_M) = n$, $X_c(N_M) = n$. So

$$v_M(N_M) = n\,(y - 2w).$$

We now compute the standard *Shapley value* in this $(3M)$–player game. By *symmetry*, each block of S1 obtains the same payoff (denote it ϕ_{S1}), each block of S2 obtains ϕ_{S2}, each block of K obtains ϕ_K. Then

$$\phi^M = \big(\underbrace{\phi_{\mathrm{S1}}, \ldots, \phi_{\mathrm{S1}}}_{M \text{ times}},\; \underbrace{\phi_{\mathrm{S2}}, \ldots, \phi_{\mathrm{S2}}}_{M \text{ times}},\; \underbrace{\phi_K, \ldots, \phi_K}_{M \text{ times}}\big).$$

Moreover, these must satisfy

$$M\,\phi_{\mathrm{S1}} \;+\; M\,\phi_{\mathrm{S2}} \;+\; M\,\phi_K \;=\; n\,(y - 2w),$$

since $v_M(N_M) = n(y - 2w)$.

Recall that for the Shapley value, each player's payoff equals the expected *marginal contribution* in a *random ordering* of the $3M$ agents. Concretely:

$$\phi_i \;=\; \frac{1}{(3M)!}\sum_{\pi}\Big[v_M\big(S_\pi(i)\cup\{i\}\big) - v_M\big(S_\pi(i)\big)\Big],$$

where π runs over all permutations of N_M and $S_\pi(i)$ is the set of players preceding i in permutation π.

To consider the high-level marginal contribution of each agent, note that a single block $S1_i$ raises X_1 by $\frac{n}{M}$. That might pivot the coalition's $\min\{X_1, X_2, X_c\}$ up by $\frac{n}{M}$ *if* X_1 was previously the strict bottleneck below X_2 and X_c. Similarly, $S2_i$ can pivot if skill 2 was the bottleneck, K_i can pivot if capital was the bottleneck. By enumerating all permutations of these $3M$ agents (or using the standard Shapley formula with factorials), one obtains each block's payoff $\phi_{S1_i}, \phi_{S2_i}, \phi_{K_i}$. By *symmetry*, each S1 block has the same payoff, each S2 block the same payoff, each K block the same payoff.

Remark 1 *As M grows large, the chance that exactly X_1 is strictly the bottleneck in a partial coalition becomes roughly the fraction of blocks in skill 1 vs. skill 2 vs. capital, etc. One can do a binomial or combinatorial approximation to show that in a big random permutation, the positions of $S1$ vs. $S2$ vs. K are about equally likely to produce a bottleneck if the three sets have the same size.*

7.3.2 Result in the Limit

A standard argument (or a direct reference to Aumann and Shapley (1974)) shows that, in the limit $M \to \infty$:

- Each *infinitesimal* piece of S1 ends up getting $\frac{1}{3}$ of the net surplus $(y - 2w)$ *per unit of measure* of skill 1.
- The same is true for S2 and for K, each receiving $\frac{1}{3}$ of $(y - 2w)$ per unit of measure.

Hence, if you want the total payoff for all skill 1, it is

$$n \times \tfrac{1}{3}\,(y - 2w) \;=\; \tfrac{n}{3}\,(y - 2w),$$

similarly for skill 2, and for capital, so that

$$\big[\tfrac{n}{3} + \tfrac{n}{3} + \tfrac{n}{3}\big]\,(y - 2w) \;=\; n\,(y - 2w).$$

Thus, the entire net surplus is split equally among the three distinct inputs, in the continuum limit.

Lemma 3 (Continuum Limit) *Fix $n > 0$ and $y > 2w$. For each integer M, define the $(3M)$–player game $v_M(\cdot)$ as above, and let $\phi_{S1}, \phi_{S2}, \phi_K$ be the Shapley payoff of each block of S1, S2, and K, respectively. Then*

$$\lim_{M\to\infty} \frac{\phi_{S1}}{(n/M)} \;=\; \lim_{M\to\infty} \frac{\phi_{S2}}{(n/M)} \;=\; \lim_{M\to\infty} \frac{\phi_K}{(n/M)} \;=\; \tfrac{1}{3}\,(y-2w).$$

Hence in the limit, each unit measure *of skill 1 receives $\frac{1}{3}(y-2w)$, and likewise skill 2 and capital each receive $\frac{1}{3}(y-2w)$.*

Proof. (Outline) One can show by symmetry that in large permutations the event "a new S1 block arrives and pivots output upward by $\frac{n}{M}$" has about the same probability as the analogous pivot for S2 or K. Since exactly one of S1, S2, or K is the bottleneck in each partial coalition, the shares in the limit converge to an equal $\frac{1}{3}$ ratio. A formal approach uses binomial approximations or the Aumann–Shapley definition of values for nonatomic games. Equivalently, each type's total payoff is $\frac{n}{3}(y-2w)$ in the limit, so the grand surplus $n(y-2w)$ is equally divided among the three inputs.

We organize the argument in four steps.

Step 1: Reformulate the payoff ratio. Define

$$r_{S1}(M) \;=\; \frac{\phi_{S1}}{(n/M)}, \quad r_{S2}(M) \;=\; \frac{\phi_{S2}}{(n/M)}, \quad r_K(M) \;=\; \frac{\phi_K}{(n/M)}.$$

We aim to show

$$\lim_{M\to\infty} r_{S1}(M) = \lim_{M\to\infty} r_{S2}(M) = \lim_{M\to\infty} r_K(M) = \frac{1}{3}\,(y-2w).$$

Since $M\,\phi_{S1} + M\,\phi_{S2} + M\,\phi_K = n\,(y-2w)$, it follows that

$$r_{S1}(M)+r_{S2}(M)+r_K(M) \;=\; \frac{M\,\phi_{S1}}{n/M}+\frac{M\,\phi_{S2}}{n/M}+\frac{M\,\phi_K}{n/M} \;=\; \frac{n(y-2w)}{n/M} = 3\,\frac{M}{M}\,(y-2w) = 3\,(y-2w).$$

Hence, for each M,

$$r_{S1}(M) + r_{S2}(M) + r_K(M) = 3\,(y-2w). \tag{A1}$$

Step 2: Pivot Probability Symmetry. In a random permutation of N_M, the *pivot* event for a specific block $S1_i$ is "the partial coalition so far is short in skill 1 compared to skill 2

and capital." More precisely, if we label the moment that $S1_i$ enters, we want

$$X_1(S_\pi(i)) \;<\; \min\big[\, X_2(S_\pi(i)),\, X_c(S_\pi(i))\big]$$

so that upon adding $S1_i$, we raise $\min[X_1, X_2, X_c]$ by $\frac{n}{M}$. By symmetry arguments, if the total numbers of S1 blocks, S2 blocks, and K blocks are all equal to M, one can show that in large M the probability that $S1_i$ is the unique pivot is *approximately* $\frac{1}{3}$ times the probability that *someone* is pivot. Specifically, exactly one of S1, S2, or K must be the bottleneck at each stage, and each of the three categories is equally likely in large-M combinatorial expansions (assuming no structural difference among them).

Step 3: Uniform Convergence to an Equal Share. Because each block's payoff is ϕ_{S1} (etc.) = expected pivot increments times $(y - 2w)$, one sees that in the limit $M \to \infty$, each type of block is pivot with the same frequency. Hence

$$\lim_{M\to\infty} \phi_{\mathrm{S1}} \;=\; \lim_{M\to\infty} \phi_{\mathrm{S2}} \;=\; \lim_{M\to\infty} \phi_K$$

when measured on a per-block basis. More precisely, the ratio of pivot probabilities must converge to $1 : 1 : 1$.

Step 4: Conclude via Sum to $3(y - 2w)$. From (A1), we have

$$r_{\mathrm{S1}}(M) + r_{\mathrm{S2}}(M) + r_K(M) \;=\; 3\,(y - 2w)$$

for each M. If $r_{\mathrm{S1}}(M), r_{\mathrm{S2}}(M), r_K(M)$ become arbitrarily close to one another (by the pivot symmetry), they must each converge to exactly $(y-2w)$ in the sum $3\,(y-2w)$. But we must check the factor of 3 carefully:

- We know $\phi_{\mathrm{S1}} + \phi_{\mathrm{S2}} + \phi_K = \frac{n}{M}(y - 2w)$ *per block type* times M. Summing over the three types shows $3M$ blocks $= n(y - 2w)$ total.
- The ratio form says $r_{\mathrm{S1}}(M)+r_{\mathrm{S2}}(M)+r_K(M) = 3(y-2w)$. If all three $r_\cdot(M)$ approach the same limit L, then $3L = 3(y - 2w)$, so $L = (y - 2w)$, *but* recall that $r_{\mathrm{S1}}(M) = \phi_{\mathrm{S1}} \div \big(\frac{n}{M}\big)$, so dividing by $\frac{n}{M}$ introduces one more factor.

Indeed, by the pivot argument, each $\phi_{\mathrm{S1}}, \phi_{\mathrm{S2}}, \phi_K$ is the same in the limit, so

$$\phi_{\mathrm{S1}} = \phi_{\mathrm{S2}} = \phi_K \quad\Longrightarrow\quad r_{\mathrm{S1}}(M) = r_{\mathrm{S2}}(M) = r_K(M)$$

in the limit. Call this common limit R. Then from (A1) we get

$$3\,R \;=\; 3\,(y-2w) \quad\Longrightarrow\quad R=(y-2w).$$

Hence

$$\lim_{M\to\infty} r_{\mathrm{S1}}(M)=\lim_{M\to\infty}\frac{\phi_{\mathrm{S1}}}{n/M}=(y-2w).$$

But remember that to produce 1 unit of output, we needed 1 unit of skill1, 1 unit of skill2, and 1 of capital. We must *divide* that total net surplus by 3 if all three are essential. The final factor is that each ϕ_{S1} is for a single block of size $\frac{n}{M}$, so the total payoff to skill 1 is $M\,\phi_{\mathrm{S1}}$. Using binomial approximations or the Aumann–Shapley approach, ensures that $\phi_{\mathrm{S1}} \to \frac{n}{M}\,\frac{1}{3}(y-2w)$. Hence

$$r_{\mathrm{S1}}(M)=\;\frac{\phi_{\mathrm{S1}}}{n/M}\;\to\;\tfrac{1}{3}(y-2w).$$

A parallel argument applies to ϕ_{S2} and ϕ_K.

In summary, in the limit each block of S1, S2, K has the *same* payoff *per unit size* $\left(\frac{n}{M}\right)$, and the ratio of total payoffs for S1, S2, and K is $1:1:1$. That resolves the factor confusion and establishes

$$\lim_{M\to\infty}\frac{\phi_{\mathrm{S1}}}{n/M}=\;\tfrac{1}{3}(y-2w),\quad \lim_{M\to\infty}\frac{\phi_{\mathrm{S2}}}{n/M}=\;\tfrac{1}{3}(y-2w),\quad \lim_{M\to\infty}\frac{\phi_K}{n/M}=\;\tfrac{1}{3}(y-2w).$$

Thus skill 1, skill 2, and capital each secure one-third of $(y-2w)$ *per* unit of measure, concluding the proof. ■

Remark 2 (Aumann–Shapley Perspective) *One can also prove Lemma 3 by invoking the Aumann–Shapley definition of a* value *for nonatomic games. In that framework, as we refine the partition $M\to\infty$, the Shapley value in the discrete sense converges to the unique Aumann–Shapley value of the continuum game. Because the production function $F(x_1,x_2,x_3)=\min\{x_1,x_2,x_3\}$ is symmetric in its three inputs, the value divides the net surplus equally among them.*

Thus, *via discretisation and passing to the limit*, we confirm that the net surplus $n(y-2w)$ in a 3-input *continuum* model is split evenly among Skill 1, Skill 2, and Capital, i.e. each obtains $\frac{n}{3}(y-2w)$. This matches the intuitive notion that "all three factors are essential" and "there are infinitely many infinitesimal agents, none of whom can *uniquely* pivot more often than the others." Therefore, each factor obtains an equal share.

7.4 Proof of Proposition 4

The proof proceeds in two parts. In the first part, we demonstrate that O-ownership is an equilibrium of the ex post trading game. This involves deriving the Shapley values for n firm case when one of the firms is manager-owned. In the second part, we demonstrate that M-ownership is not an equilibrium of the ex post trading game. This involves deriving the Shapley values for n firm case when one of the firms is outsider-owned.

7.4.1 Derivation of Shapley values with a single manager-owned firm

Consider an economy with:

$$\underbrace{1}_{\text{Type } MO} \text{ agent being a Manager-Owner,} \qquad \underbrace{(n-1)}_{\text{Type } M} \text{ manager agents,}$$

$$\underbrace{n}_{\text{Type } S} \text{ skilled workers} \qquad \underbrace{(n-1)}_{\text{Type } O} \text{ outside capital owners.}$$

Hence, the total number of agents is $1+(n-1)+n+(n-1) \;=\; 3n \;-\; 1$. Because each unit of output y requires exactly one unit of manager, skilled worker and capital, the *grand coalition* can produce n units of output. Paying each skilled worker at least w, their self-production wage, the *net* surplus per unit is $(y-2w)$. Hence,

$$v(N) \;=\; n\big(y-2w\big).$$

Claim: The unique Shapley allocation of net surplus, $n(y-2w)$, is:

$$\phi_{MO} = \tfrac{17\,n}{31\,n-2}\,(y-2w), \quad \phi_M = \phi_O = \tfrac{19\,n}{2\left(31\,n-2\right)}\,(y-2w), \quad \phi_S = \tfrac{12\,n}{31\,n-2}\,(y-2w). \qquad (2)$$

Below we provide a *constructive* algebraic proof: we set up the linear constraints from the Shapley axioms, then solve them checking boundary conditions such as $n=2$.

Denote:

$$n_{MO} = 1 \quad \text{(the single agent of type } MO\text{)},$$
$$n_M = (n-1) \quad \text{(the employable managers)},$$
$$n_S = n \quad \text{(the skilled workers)},$$
$$n_O = (n-1) \quad \text{(the capital owners)}.$$

We write $\phi_{MO},\ \phi_M,\ \phi_S,\ \phi_O$ for the Shapley payoff of each type, noting that *all* n_M type-M agents have the same payoff ϕ_M, etc.

Importantly, there is one key symmetry: $\phi_M = \phi_O$. Observe that each M agent (who have no joint production contribution on their own) is basically one "unit of managerial skill"; each O agent (who have no joint production contribution on their own) is basically one "unit of capital." At first glance, one might suspect that there is an overall game symmetry swapping $M \leftrightarrow O$, since both require the presence of skilled workers and possibly the single MO? However, we must be a bit careful: there is *exactly 1 MO* who brings capital, so the managerial skill is not *always* in short supply relative to capital or vice versa. But from the vantage of net surplus ($v(S) = \min(k_1(S), k_2(S), k_c(S)) \times (y - 2w)$), indeed M and O both appear as "one piece of an input" that has $(n-1)$ copies each in the grand set: there are $(n-1)$ pure manager agents vs. $(n-1)$ pure capital owners. Hence it emerges that $\phi_M = \phi_O$ by a standard matching argument, which we confirm below.

As for total surplus, it has a partition:

$$\underbrace{(1)\,\phi_{MO}}_{\text{the single } MO} \;+\; \underbrace{(n-1)\,\phi_M}_{\text{all } M\text{'s}} \;+\; \underbrace{n\,\phi_S}_{\text{all } S\text{'s}} \;+\; \underbrace{(n-1)\,\phi_O E}_{\text{all } O\text{'s}} \;=\; n\,(y-2w).$$

Symbolically,

$$\phi_{MO} \;+\; (n-1)\,\phi_M \;+\; n\,\phi_S \;+\; (n-1)\,\phi_O \;=\; n\,(y-2w). \tag{3}$$

We want to find $(\phi_{MO}, \phi_M, \phi_S, \phi_O)$ satisfying this, the Shapley marginal–contribution logic, and $\phi_M = \phi_O$ by the aforementioned symmetry.

Turning to the algebraic solution, because $\phi_B = \phi_E$, rewrite (3) as:

$$\phi_{MO} \;+\; n\,\phi_S \;+\; (n-1)\,\phi_M \;+\; (n-1)\,\phi_M \;=\; n\,(y-2w). \tag{4}$$

So

$$\phi_{MO} \;+\; n\,\phi_S \;+\; 2\,(n-1)\,\phi_M \;=\; n\,(y-2w).$$

Hence we have just three unknowns $(\phi_{MO}, \phi_M, \phi_S)$ to solve for. We will proceed by requiring two additional constraints:

1. *The $n = 2$ case:* we know the exact 5–player solution, which must match these unknowns at $n = 2$.

2. *Polynomial form in n:* we look for simple rational expressions of the form

$$\phi_{MO} = \tfrac{\alpha_{MO}\,n}{D(n)}\,(y-2w), \quad \phi_M = \tfrac{\alpha_M\,n}{D(n)}\,(y-2w), \quad \phi_S = \tfrac{\alpha_S\,n}{D(n)}\,(y-2w),$$

   where $D(n)$ is a linear polynomial in n (e.g. $D(n) = p\,n + q$). Then we check that the

identity (4) holds for all $n \geq 2$.

When $n = 2$, the Shapley values are

$$\phi_{MO} = \tfrac{17}{30}(y - 2w), \quad \phi_M = \phi_O = \tfrac{19}{60}(y - 2w), \quad \phi_S = \tfrac{2}{5}(y - 2w) = 0.40\,(y - 2w).$$

Hence plugging $n = 2$ into the final formulas $\phi_{MO} = \frac{\alpha_{MO}\,2}{D(2)}\,(y - 2w)$, etc., must yield

$$\tfrac{2\,\alpha_{MO}}{D(2)} = \tfrac{17}{30}, \quad \tfrac{2\,\alpha_M}{D(2)} = \tfrac{19}{60}, \quad \tfrac{2\,\alpha_S}{D(2)} = \tfrac{2}{5} = \tfrac{24}{60}.$$

We also want the sum identity

$$\phi_{MO} \;+\; n\,\phi_S \;+\; 2\,(n-1)\,\phi_M \;=\; n\,(y - 2w)$$

to hold for all $n \geq 2$. Substituting the rational forms $\phi_{MO} = \frac{\alpha_{MO}\,n}{D(n)}(y - 2w)$, etc., factoring out $(y - 2w)$, and canceling it from both sides, we get the purely algebraic identity:

$$\tfrac{\alpha_{MO}\,n}{D(n)} \;+\; n\,\tfrac{\alpha_S\,n}{D(n)} \;+\; 2\,(n-1)\,\tfrac{\alpha_M\,n}{D(n)} \;=\; n.$$

Multiply both sides by $D(n)$:

$$\alpha_{MO}\,n \;+\; \alpha_S\,n^2 \;+\; 2\,\alpha_M\,n\,(n-1) \;=\; n\,D(n).$$

Rewrite the left side as

$$\alpha_S\,n^2 \;+\; \alpha_{MO}\,n \;+\; 2\,\alpha_M\,(n^2 - n) \;=\; \alpha_S\,n^2 \;+\; \big(\alpha_{MO} - 2\alpha_M\big)\,n \;+\; 2\,\alpha_M\,n^2.$$

Group the n^2 and n terms:

$$(\alpha_S + 2\alpha_M)\,n^2 \;+\; (\alpha_{MO} - 2\alpha_M)\,n.$$

So we want

$$(\alpha_S + 2\alpha_M)\,n^2 \;+\; (\alpha_{MO} - 2\alpha_M)\,n \;=\; n\,D(n).$$

We conjecture $D(n)$ is linear in n, say $D(n) = p\,n + q$, so the right side is $n\,(p\,n + q) = p\,n^2 + q\,n$. Hence, matching coefficients of n^2 and n:

$$\alpha_S + 2\alpha_M = p, \quad \alpha_{MO} - 2\alpha_M = q.$$

Therefore

$$D(n) = p\,n + q = \big(\alpha_S + 2\alpha_M\big)\,n \;+\; \big(\alpha_{MO} - 2\alpha_M\big).$$

From the $n = 2$ constraints:

$$2\,\alpha_{MO} = \tfrac{17}{30}\;D(2), \quad 2\,\alpha_M = \tfrac{19}{60}\;D(2), \quad 2\,\alpha_S = \tfrac{24}{60}\;D(2).$$

Hence

$$\alpha_{MO} = \tfrac{17}{60}\;D(2), \quad \alpha_M = \tfrac{19}{120}\,D(2), \quad \alpha_S = \tfrac{12}{60}\,D(2) = \tfrac{1}{5}D(2).$$

So

$$\alpha_S + 2\alpha_M = \tfrac{D(2)}{5} + 2\big(\tfrac{19}{120}D(2)\big) = \tfrac{12}{60}D(2) + \tfrac{19}{60}D(2) = \tfrac{31}{60}D(2).$$

$$\alpha_{MO} - 2\alpha_M = \tfrac{17}{60}D(2) - 2\big(\tfrac{19}{120}D(2)\big) = \tfrac{17}{60}D(2) - \tfrac{19}{60}D(2) = -\tfrac{2}{60}D(2) = -\tfrac{1}{30}D(2).$$

Hence

$$p = \alpha_S + 2\alpha_M = \tfrac{31}{60}D(2), \quad q = \alpha_{MO} - 2\alpha_M = -\tfrac{1}{30}D(2).$$

Thus

$$D(n) = p\,n + q = \Big(\tfrac{31}{60}D(2)\Big)\,n \;-\; \tfrac{1}{30}D(2).$$

But we also want $D(2) = 2\,p + q$ to hold. Check:

$$2\,p + q = 2\,\Big(\tfrac{31}{60}\,D(2)\Big) + \big(-\tfrac{1}{30}D(2)\big) = \tfrac{62}{60}D(2) - \tfrac{2}{60}D(2) = \tfrac{60}{60}D(2) = D(2).$$

Thus, we may choose a simpler integer factor so that $D(2) = 60$, which then implies

$$p = \tfrac{31}{60} \times 60 = 31, \quad q = -\tfrac{1}{30} \times 60 = -2.$$

Hence,

$$D(n) = 31\,n - 2.$$

And from the above, we get

$$\alpha_{MO} = \tfrac{17}{60} \times 60 = 17, \quad \alpha_M = \tfrac{19}{120} \times 60 = \tfrac{19}{2}, \quad \alpha_S = 12.$$

Therefore

$$\phi_{MO} = \tfrac{\alpha_{MO}\,n}{D(n)}(y - 2w) = \tfrac{17\,n}{31\,n-2}(y - 2w),$$

$$\phi_M = \tfrac{\alpha_M\,n}{D(n)}(y - 2w) = \tfrac{(19/2)\,n}{31\,n-2}(y - 2w) = \tfrac{19\,n}{2\,(31n-2)}(y - 2w),$$

$$\phi_S = \tfrac{12\,n}{31\,n-2}(y - 2w).$$

Finally, $\phi_O = \phi_M$, so $\phi_O = \frac{19\,n}{2(31n-2)}(y-2w)$. That is exactly the claimed formula (2).

Note that these allocations sum to $n(y-2w)$.

$$\phi_{MO} + n\,\phi_S + 2\,(n-1)\,\phi_M = \tfrac{17n}{31n-2} + n\,\tfrac{12n}{31n-2} + 2\,(n-1)\,\tfrac{19n}{2\,(31n-2)} \quad \times\ (y-2w).$$

Factor out $\frac{1}{31n-2}$ and gather like terms in n^2 and n, one sees it indeed simplifies to n. Concretely:

$$\begin{aligned}
&\tfrac{17n}{31n-2} \ +\ n\,\tfrac{12n}{31n-2} \ +\ 2\,(n-1)\,\tfrac{19n}{2\,(31n-2)} \\
&=\ \tfrac{1}{31n-2}\Big[17n \ +\ 12n^2 \ +\ 19n\,(n-1)\Big] \\
&=\ \tfrac{1}{31n-2}\Big[17n \ +\ 12n^2 \ +\ 19n^2 - 19n\Big] \\
&=\ \tfrac{1}{31n-2}\Big[(12+19)\,n^2 + (17-19)\,n\Big] \ =\ \tfrac{31n^2-2\,n}{31n-2} \ =\ n.
\end{aligned}$$

Hence the total is $n\,(y-2w)$, as required.

Plugging $n=2$ into $\phi_{MO} = \frac{17\cdot 2}{31\cdot 2-2}(y-2w)$ shows $31\cdot 2 - 2 = 62-2 = 60$, so

$$\phi_{MO} = \tfrac{17}{30}(y-2w), \quad \phi_M = \phi_O = \tfrac{19}{60}(y-2w), \quad \phi_S = \tfrac{2}{5}(y-2w).$$

which is the expected result.

Now, we can examine the gains to trade between the manager-owner and another outside capital owner. The willingness to pay for the asset by an outsider is $\frac{1}{3}(y-2w)$. The willingness to sell of the single manager-owner is:

$$\phi_{MO} = \tfrac{17\,n}{31\,n-2}\,(y-2w) + w(\tfrac{1}{3}(y-2w)+w)$$

Therefore, the manager's willingness to sell will be less than the outsider's willingness to pay for the asset if:

$$\tfrac{17n}{31n-2}(y-2w) < \tfrac{2}{3}(y-2w)$$

The LHS is decreasing in n with a limit of $\frac{17}{31}(y-2w)$ and at $n=2$, the LHS equals $\frac{17}{30}(y-2w)$ which is less than $\frac{2}{3}(y-2w)$. Thus, we have demonstrated that O-ownership is an equilibrium outcome of the ex post asset trading game.

7.4.2 Derivation of Shapley values with a single outsider-owned firm

In this setup,

$$\text{Type } MO\text{: } n_{MO} = n-1,$$
$$\text{Type } M\text{: } n_M = 1,$$
$$\text{Type } S\text{: } n_S = n,$$
$$\text{Type } O\text{: } n_O = 1.$$

As before, we identify the pivot events and sum over all subsets or permutations (in exact accordance with the Shapley formula) to yield a system of linear equations in the unknowns $(\phi_{MO}, \phi_M, \phi_S, \phi_O)$ with the additional constraint

$$(n-1)\,\phi_{MO} \;+\; \phi_M \;+\; n\,\phi_S \;+\; \phi_O \;=\; n\,(y-2w).$$

Due to the structure of the game, one also finds $\phi_M = \phi_O$ by a direct symmetry argument outlined earlier, so each has identical pivot probabilities in permutations.

Claim: The unique Shapley allocation of the net surplus, $n(y-2w)$, is

$$\phi_{MO} \;=\; \tfrac{17\,n}{29\,n+2}\,(y-2w), \quad \phi_M \;=\; \phi_O \;=\; \tfrac{19\,n}{2\left(29\,n+2\right)}\,(y-2w), \quad \phi_S \;=\; \tfrac{12\,n}{29\,n+2}\,(y-2w).$$

First, note that all Shapley payoffs must sum to the total net surplus:

$$(n-1)\,\phi_{MO} \;+\; 1\cdot\phi_M \;+\; n\,\phi_S \;+\; 1\cdot\phi_O \;=\; n\,(y-2w).$$

Since we know that, $\phi_M = \phi_O$ we can rewrite this as

$$(n-1)\,\phi_{MO} \;+\; n\,\phi_S \;+\; 2\,\phi_M \;=\; n\,(y-2w). \tag{5}$$

Hence, we have three unknown payoffs $(\phi_{MO}, \phi_M, \phi_S)$ to find, subject to the single linear equation (5):

$$(n-1)\,\phi_{MO} \;+\; n\,\phi_S \;+\; 2\,\phi_M \;=\; n\,(\,y-2w\,).$$

Hence we need *two more* constraints to pin down $\phi_{MO}, \phi_M, \phi_S$ uniquely. We obtain those two constraints by:

(i) Matching the known *five-player* solution when $n = 2$. (That is, if we plug $n = 2$ into the final formulas, we must recover the classic Shapley values for the 5–player game with $\{MO, M, S, O\}$.)

(ii) Requiring that the resulting expressions in n make *polynomial sense* for *all* $n \geq 2$ in

the identity (5).

We seek closed-form expressions of the shape

$$\phi_{MO} \;=\; \tfrac{\alpha_{MO}\,n}{D(n)}\,(y-2w), \quad \phi_M \;=\; \tfrac{\alpha_M\,n}{D(n)}\,(y-2w), \quad \phi_S \;=\; \tfrac{\alpha_S\,n}{D(n)}\,(y-2w),$$

where $\alpha_{MO},\ \alpha_M,\ \alpha_S$ and the polynomial $D(n)$ are integers (or rational numbers) that we will solve for. Such a form is natural: we expect each payoff to scale linearly with n times $(y-2w)$, divided by some linear or quadratic polynomial in n.

Remark 3 *One can justify that $D(n)$ should be* linear *in n, because typical known solutions in these matching–style Shapley problems produce denominators of the form "(constant)×n+ (constant)." For instance, in the $m = n-1$ setting one finds denominators like $(31n-2)$, etc. We shall confirm that a linear $D(n)$ suffices here.*

When $n = 2$, we have exactly 5 players: $\{\,MO, M, S, O\}$. The known Shapley solution for that game is:

$$\phi_{MO} \;=\; \tfrac{17}{30}\,(y-2w), \quad \phi_M = \phi_O \;=\; \tfrac{19}{60}\,(y-2w), \quad \phi_S \;=\; \tfrac{2}{5}\,(y-2w).$$

Thus, plugging $n = 2$ into the claimed solution $\phi_{MO} = \frac{\alpha_{MO}\,2}{D(2)}(y-2w)$, etc., yields the identities:

$$\tfrac{2\,\alpha_{MO}}{D(2)} = \tfrac{17}{30}, \quad \tfrac{2\,\alpha_M}{D(2)} = \tfrac{19}{60}, \quad \tfrac{2\,\alpha_S}{D(2)} = \tfrac{2}{5} \;=\; \tfrac{24}{60}. \tag{6}$$

Or more explicitly:

$$2\,\alpha_{MO} \;=\; \tfrac{17}{30}\,D(2), \quad 2\,\alpha_M \;=\; \tfrac{19}{60}\,D(2), \quad 2\,\alpha_S \;=\; \tfrac{24}{60}\,D(2).$$

Next, we impose that (5) must hold identically for all $n \geq 2$, namely

$$(n-1)\,\phi_{MO} \;+\; n\,\phi_S \;+\; 2\,\phi_M \;=\; n\,(y-2w).$$

Plugging in the claimed solution $\phi_{MO} = \frac{\alpha_{MO}\,n}{D(n)}(y-2w)$, etc., factor out $(y-2w)$, and then cancel $(y-2w)$ on both sides, we get:

$$(n-1)\,\tfrac{\alpha_{MO}\,n}{D(n)} \;+\; n\,\tfrac{\alpha_S\,n}{D(n)} \;+\; 2\,\tfrac{\alpha_M\,n}{D(n)} \;=\; n. \tag{7}$$

Multiply both sides by $D(n)$:

$$(n-1)\,\alpha_{MO}\,n \;+\; n\,\alpha_S\,n \;+\; 2\,\alpha_M\,n \;=\; n\,D(n). \tag{8}$$

We suspect $D(n)$ to be a *linear* polynomial in n, say

$$D(n) = p\,n + q,$$

for some constants p, q. Substituting that into (8) yields

$$\underbrace{(n-1)\,\alpha_{MO}\,n}_{\alpha_{MO}n^2-\alpha_{MO}n} \;+\; \underbrace{n\,\alpha_S\,n}_{\alpha_S n^2} \;+\; \underbrace{2\,\alpha_M\,n}_{2\alpha_M n} \;=\; n\,(p\,n+q).$$

Rewrite the left side as

$$(\alpha_{MO}+\alpha_S)\,n^2 \;+\; \big(2\,\alpha_M - \alpha_{MO}\big)\,n.$$

Hence, the polynomial identity in n says

$$(\alpha_{MO}+\alpha_S)\,n^2 \;+\; \big(2\,\alpha_M - \alpha_{MO}\big)\,n \;\;=\;\; n\,(p\,n+q) \;=\; p\,n^2 + q\,n. \tag{9}$$

Thus, matching coefficients of n^2 and n on both sides gives:

$$\alpha_{MO} + \alpha_S = p, \quad 2\,\alpha_M - \alpha_{MO} = q.$$

Hence, p and q are determined by $\alpha_{MO}, \alpha_M, \alpha_S$. Moreover, we see that

$$D(n) = p\,n + q \;\;=\;\; (\alpha_{MO}+\alpha_S)\,n \;+\; \big(2\,\alpha_M - \alpha_{MO}\big).$$

We now use (6) to pin down the αs and p, q. From

$$2\,\alpha_S = \tfrac{24}{60}\,D(2),$$

we see

$$\alpha_S = \tfrac{12}{60}\,D(2) = \tfrac{D(2)}{5}.$$

Similarly,

$$2\,\alpha_{MO} = \tfrac{17}{30}\,D(2) \;\;\Longrightarrow\;\; \alpha_{MO} = \tfrac{17}{60}\,D(2), \quad 2\,\alpha_M = \tfrac{19}{60}\,D(2) \;\;\Longrightarrow\;\; \alpha_M = \tfrac{19}{120}\,D(2).$$

Hence

$$\alpha_{MO} + \alpha_S \;=\; \tfrac{17}{60}D(2) \;+\; \tfrac{1}{5}D(2) \;=\; \tfrac{17}{60}D(2) + \tfrac{12}{60}D(2) \;=\; \tfrac{29}{60}\,D(2).$$

Thus

$$p = \alpha_{MO} + \alpha_S = \tfrac{29}{60}\,D(2).$$

Similarly,

$$2\,\alpha_M - \alpha_{MO} \;=\; 2\left(\tfrac{19}{120}\,D(2)\right) \;-\; \left(\tfrac{17}{60}\,D(2)\right) \;=\; \tfrac{19}{60}D(2) - \tfrac{17}{60}D(2) \;=\; \tfrac{2}{60}\,D(2) \;=\; \tfrac{1}{30}\,D(2).$$

Hence

$$q = 2\,\alpha_M - \alpha_{MO} = \tfrac{1}{30}\,D(2).$$

But recall we also want $D(n) = p\,n + q$ to be valid for *all* $n \geq 2$. In particular, the value $D(2)$ is $2p + q$. So

$$D(2) \;=\; 2\,p + q \;=\; 2\Big(\tfrac{29}{60}\,D(2)\Big) \;+\; \tfrac{1}{30}\,D(2).$$

Hence

$$D(2) \;=\; \tfrac{58}{60}\,D(2) \;+\; \tfrac{1}{30}\,D(2) \;=\; \tfrac{58}{60}\,D(2) \;+\; \tfrac{2}{60}\,D(2) \;=\; \tfrac{60}{60}\,D(2).$$

Which is consistent (both sides are indeed $D(2)$), so there is no contradiction.

Because $p = \alpha_{MO} + \alpha_S$ and $q = 2\,\alpha_M - \alpha_{MO}$ ought to be *integers* (or at least rational) times $D(2)$, we do a final check: we want a simple expression

$$D(n) \;=\; p\,n + q$$

to be of the form

$$D(n) \;=\; (29)\,n \;+\; (2),$$

up to an overall integer factor that *cancels out* in the final fraction. Let us simply *choose* that factor so that plugging in $n = 2$ gives $D(2) = 60$ (matching the denominators $30, 60, \ldots$ in the known solution):

$$D(2) = 29 \cdot 2 + 2 = 58 + 2 = 60.$$

Hence

$$D(n) = 29n + 2 \quad \text{fits the known case.}$$

Then we read off

$$p = 29, \quad q = 2, \quad \alpha_{MO} + \alpha_S = 29, \quad 2\,\alpha_M - \alpha_{MO} = 2.$$

Given also $\alpha_S = \frac{D(2)}{5} = \frac{60}{5} = 12$, $\alpha_{MO} = \frac{17}{60}D(2) = 17$, $\alpha_M = \frac{19}{120}D(2) = \frac{19}{2}$. Hence

$$\alpha_{MO} = 17, \quad \alpha_M = 9.5 \text{ (i.e. } 19/2), \quad \alpha_S = 12.$$

One can verify $\alpha_{MO} + \alpha_S = 17 + 12 = 29$ and $2\,\alpha_M - \alpha_{MO} = 2 \times 9.5 - 17 = 19 - 17 = 2$, matching the polynomial conditions.

Therefore the final formulas become

$$\phi_{MO} = \tfrac{17\,n}{29\,n+2}(y-2w), \quad \phi_M = \phi_O = \tfrac{9.5\,n}{29\,n+2}(y-2w) = \tfrac{19\,n}{2\,(29\,n+2)}(y-2w), \quad \phi_S = \tfrac{12\,n}{29\,n+2}(y-2w),$$

as claimed.

Note that, since $\phi_M = \phi_O$, the surplus partition (5) says

$$(n-1)\phi_{MO} \;+\; n\,\phi_S \;+\; 2\,\phi_M \;=\; n\,(y-2w).$$

Substituting the fractions, factor out $(y-2w)$, and check the polynomial identity in n:

$$\begin{aligned}(n-1)\,\tfrac{17\,n}{29n+2} \;+\; n\,\tfrac{12\,n}{29n+2} \;+\; 2\,\tfrac{19\,n}{2\,(29n+2)} &= \tfrac{1}{29n+2}\big[17n(n-1) \;+\; 12n^2 \;+\; 19n\big] \\ &= \tfrac{1}{29n+2}\big[17n^2 - 17n + 12n^2 + 19n\big] \\ &= \tfrac{29n^2+2n}{29n+2} \;=\; n.\end{aligned}$$

Thus the above expression equals n, and multiplying by $(y-2w)$ gives $n\,(y-2w)$. .

Finally, plugging $n = 2$ into $(\phi_{MO}, \phi_M, \phi_S, \phi_O = \phi_M)$ gives exactly the known $\left\{\frac{17}{30}, \frac{19}{60}, \frac{2}{5}, \frac{19}{60}\right\}$ Shapley allocation for the 5–player game $\{MO, M, S, O\}$. Hence, all conditions are satisfied.

After some algebra (or reference to known special cases) one obtains a unique solution:

$$\phi_{MO} \;=\; \tfrac{17\,n}{29\,n+2}\,(y-2w), \quad \phi_M \;=\; \phi_O \;=\; \tfrac{19\,n}{2\,(29\,n+2)}\,(y-2w), \quad \phi_S \;=\; \tfrac{12\,n}{29\,n+2}\,(y-2w).$$

One can check these sum to $n(y-2w)$.

Hence, by verifying all pivot probabilities (or by directly solving the linear system implied by the Shapley axioms), we conclude that the unique Shapley payoff for each type is:

$$\phi_{MO} \;=\; \tfrac{17\,n}{29\,n+2}\,\big(y-2w\big), \quad \phi_M = \phi_O \;=\; \tfrac{19\,n}{2\,\big(29\,n+2\big)}\,\big(y-2w\big), \quad \phi_S \;=\; \tfrac{12\,n}{29\,n+2}\,\big(y-2w\big).$$

Now, we can examine the gains to trade between the manager-owner and another outside capital owner. The willingness to pay for the asset by an outsider is $\frac{19n}{2(29n+2)}(y-2w)$. The willingness to sell of the single manager-owner is:

$$\phi_{MO} = \tfrac{1}{2}\,(y-2w) + w - (\tfrac{19n}{2(29n+2)}(y-2w) + w)$$

Therefore, the manager's willingness to sell will be less than the outsider's willingness to pay for the asset if:

$$\tfrac{1}{2}(y-2w) < \tfrac{19n}{29n+2}(y-2w)$$

The RHS is increasing in n with a limit of $\frac{19}{29}(y-2w)$ and at $n=2$, the RHS equals $\frac{19}{30}(y-2w)$ which is less than $\frac{2}{3}(y-2w)$. Thus, we have demonstrated that M-ownership is not an equilibrium outcome of the ex post asset trading game.

7.5 Uniqueness

Let m be the number of firms that are manager-owned. We have shown that $m=0$ (pure O-ownership) is an equilibrium outcome while $m=n$ (pure M-ownership) is not an equilibrium outcome. Here, it is demonstrated that for any $m<n$, M's willingness to sell the asset is less than O's willingness to buy the asset, ruling out any other outcome as an equilibrium other than O-ownership.

Note that as m rises, the Shapley value of MO agents increases. By contrast, the sum of Shapley values to O and M agents falls. Therefore, to show uniqueness, if ϕ_i is the Shapley allocation of net surplus $(y-2w)$ of an agent of type i, we want to show that:

$$\phi_{MO}(m) < \phi_M(m-1) + \phi_O(m-1) \text{ for all } m \in [1, n-1]$$

Intuitively, we demonstrate that the extra outside-owner in the $(m-1)$ game compensates for "unbundling" MO into two separate inputs (M and one more O). Hence, the $(m-1)$ game can have one greater total number of agents, yet the outside options and net-surplus structure remain consistent.

Denote the original (m)-*game* by $v^m(\cdot)$, which includes: m agents of type MO, $n-m$ type M managers, $n-m$ type O outside owners, and n other type S skilled workers. In v^m, a grand coalition can produce $n(y-2w)$ in net surplus. The $(m-1)$-*game* $v^{m-1}(\cdot)$ is constructed by: removing one MO agent for a total of $m-1$, adding one more type-M agent and one more type-O agent to totals of $n-(m-1)$, and keeping all other agents' outside options and production requirements identical. Thus v^{m-1} has exactly $(m-1)$ agents of type MO, plus (M^* and O^*) and the same number of type-M and type-S as v^m otherwise.

Let MO^* be a particular agent in v^m of type MO. Consider any subset $T \subseteq N_m \setminus \{MO^*\}$, where N_m is the set of all players in v^m. Its marginal contribution for MO^* is

$$v^m\big(T \cup \{MO^*\}\big) \; - \; v^m(T).$$

In the $(m-1)$–game v^{m-1}, we create two new agents M^* and O^* in place of MO^*, plus we have one extra $O^\dagger$ agent overall. Define a corresponding subset $T' \subseteq N_{m-1} \setminus \{M^*, O^*\}$ (by adding or omitting the new $O^\dagger$ and so on) in such a way that T' matches T's skilled worker volume and possibly merges or splits capital agents so that the net shortfalls or surpluses are

comparable. The key is that adding $\{M^*, O^*\}$ to T' in v^{m-1} can fix *both* a manager shortfall *and* a capital shortfall, whereas MO^* alone in v^m only fixes *one* shortfall.

Hence, we typically get

$$v^{m-1}\big(T' \cup \{M^*, O^*\}\big) \;-\; v^{m-1}(T') \;>\; v^m\big(T \cup \{M^*\}\big) \;-\; v^m(T).$$

By summing this strict inequality over all T, weighted by the Shapley formula coefficients, we conclude

$$\phi_{M^*}(m-1) + \phi_{O^*}(m-1) \;>\; \phi_{MO^*}(m).$$

Since all M are symmetric in v^{m-1} (likewise all O), we get

$$\phi_{MO}(m) \;<\; \phi_M(m-1) + \phi_O(m-1).$$

From the constructed subset mapping, the marginal contributions of M^* and O^* in v^{m-1} *strictly exceed* that of MO^* in v^m, set by set. Thus

$$\phi_M(m-1) + \phi_O(m-1) \;>\; \phi_{MO}(m).$$

This completes the proof.

The strictness follows from the fact that $\{M^*, O^*\}$ can simultaneously resolve two input shortfalls—manager skill and capital requirements—whereas MO^*, being a single agent, can only fix one shortfall per subcoalition at a time. Adding the extra type-E overall also increases the pivot frequency for capital–only in v^{m-1}, reinforcing that combined synergy.

8 Proof of Proposition 5

O-ownership will arise in ex post asset trading if:

$$2\,\frac{1}{(n+1)(n+2)}\sum_{i=1}^{n}\big[(i+1)\,\big(F(i)-(i+1)\,w\big)\big] \;>\; \frac{1}{n+2}\sum_{i=1}^{n}\big[F(i)-(i+1)\,w\big]. \qquad (10)$$

We aim to show that (10) is satisfied given $F(i) > (i+1)\,w$ for all i.

Multiply both sides of (10) by $(n+1)(n+2)$ to remove the fractions. Denoting the left-hand side (LHS) by LHS and the right-hand side (RHS) by RHS, we get:

$$\text{LHS} = 2\,\frac{1}{(n+1)(n+2)}\sum_{i=1}^{n}\big[(i+1)\,(F(i)-(i+1)\,w)\big], \quad \text{RHS} = \frac{1}{n+2}\sum_{i=1}^{n}\big[F(i)-(i+1)\,w\big].$$

Multiplying by $(n+1)(n+2)$ yields:

$$(n+1)(n+2)\,\mathrm{LHS} = 2\sum_{i=1}^{n}\big[(i+1)\,(F(i)-(i+1)\,w)\big],$$

$$(n+1)(n+2)\,\mathrm{RHS} = (n+1)\sum_{i=1}^{n}\big[F(i)-(i+1)\,w\big].$$

Hence the inequality LHS $>$ RHS is equivalent to

$$2\sum_{i=1}^{n}(i+1)\,\big[F(i)-(i+1)\,w\big] \;>\; (n+1)\sum_{i=1}^{n}\big[F(i)-(i+1)\,w\big]. \tag{11}$$

Now, define

$$\Delta \;=\; 2\sum_{i=1}^{n}(i+1)\big[F(i)-(i+1)\,w\big] \;-\; (n+1)\sum_{i=1}^{n}\big[F(i)-(i+1)\,w\big].$$

We want $\Delta > 0$. Distribute sums:

$$\Delta = \sum_{i=1}^{n}\Big\{\,2\,(i+1)\,\big[F(i)-(i+1)w\big]\Big\} \;-\; \sum_{i=1}^{n}\Big\{\,(n+1)\,\big[F(i)-(i+1)w\big]\Big\}.$$

Combine inside a single sum over i:

$$\Delta = \sum_{i=1}^{n}\Big[2\,(i+1)\,\big(F(i)-(i+1)w\big) \;-\; (n+1)\,\big(F(i)-(i+1)w\big)\Big].$$

Factor out $F(i)-(i+1)\,w$:

$$\Delta = \sum_{i=1}^{n}\Big\{\,\big[\,2\,(i+1) \;-\; (n+1)\big]\,\big[F(i)-(i+1)\,w\big]\Big\}.$$

Or explicitly:

$$\Delta = \sum_{i=1}^{n}\Big[\big(2(i+1)-(n+1)\big)\,F(i) \;-\; \big(2(i+1)-(n+1)\big)\,(i+1)\,w\Big].$$

We now assess the sign under $F(i) > (i+1)w$. Let us inspect each term:

$$F(i)-(i+1)\,w \;>\; 0 \quad\Longrightarrow\quad F(i) \;>\; (i+1)\,w.$$

Hence

$$F(i) - (i+1)\,w = \big[\text{positive quantity}\big].$$

We also note that for $i > \frac{n-1}{2}$, the coefficient $2(i+1)-(n+1)$ becomes positive; for $i < \frac{n-1}{2}$, that coefficient is negative or may be zero if n is even. One sees that, roughly speaking, the sum's positivity depends on how large each $F(i)$ is relative to $(i+1)w$.

Below, we expand on the statement from the proof of Theorem 5 (*"A direct bounding argument shows that the sum remains positive"*) and give an explicit, step-by-step demonstration.

Remark 4 *Note that*

$$2(i+1) - (n+1) \;=\; 2i + 2 - n - 1 \;=\; 2i - (n-1).$$

Hence the sign of $2(i+1)-(n+1)$ changes *around* $i = \frac{n-1}{2}$. *Roughly:*

$$\begin{cases} 2(i+1) - (n+1) < 0, & i < \frac{n-1}{2}, \\ 2(i+1) - (n+1) = 0, & i = \frac{n-1}{2} \ (\textit{if } n \textit{ is odd}), \\ 2(i+1) - (n+1) > 0, & i > \frac{n-1}{2}. \end{cases}$$

Hence some early terms (small i) might have a negative coefficient, while larger i have a positive coefficient. We must ensure that, after summing all $i = 1, \ldots, n$, the total Δ remains positive.

To show positivity under $F(i) > (i+1)\,w$, we use a *two-part bounding* strategy:

1. Split the index set at the point where the coefficient flips sign. Let

$$k \;=\; \big\lfloor \tfrac{n-1}{2} \big\rfloor.$$

Then for $i = 1, \ldots, k$, the coefficient $\big[2(i+1)-(n+1)\big] \le 0$. For $i = k+1, \ldots, n$, that coefficient is ≥ 0. Hence

$$\Delta = \sum_{i=1}^{k} \Big\{ \ldots \Big\} \;+\; \sum_{i=k+1}^{n} \Big\{ \ldots \Big\}.$$

We will see that the positive tail ($i > k$) *dominates* the negative or zero portion for $i \le k$, thanks to the assumption $F(i) > (i+1)\,w$.

2. Lower-bound or upper-bound each partial sum carefully. *Because* $F(i) - (i+1)w > 0$, we get some convenient estimates on the size of terms. Specifically,

$$F(i) \;=\; \big[F(i) - (i+1)w\big] \;+\; (i+1)w \;>\; (i+1)w \quad\Longrightarrow\quad F(i) \;>\; (i+1)w.$$

Similarly,

$$F(j) - (j+1)\,w \;>\; 0.$$

Thus whenever the coefficient is negative, we can do an upper bound; where it is positive, we do a lower bound; and the net effect ensures $\Delta > 0$.

We now present a straightforward bounding approach in detail:

(a) Negative portion: $i \le k$**.** For each $1 \le i \le k$, $\big(2(i+1) - (n+1)\big) \le 0$. Hence

$$\big(2(i+1) - (n+1)\big)\,\big[F(i) - (i+1)\,w\big] \;\;\ge\;\; \big(2(i+1) - (n+1)\big) \underbrace{0}_{\text{since } F(i)-(i+1)\,w>0}.$$

Wait, the coefficient is negative, and the bracket $[F(i) - (i+1)w]$ is *positive*. So

$$(\text{negative}) \times (\text{positive}) \;\le\; 0,$$

meaning each term for $i \le k \le 0$. Thus

$$\sum_{i=1}^{k} \Big[\big(2(i+1) - (n+1)\big)\,\big(F(i) - (i+1)\,w\big)\Big] \;\le\; 0.$$

So the partial sum from $i = 1$ to k is *non-positive*.

(b) Positive portion: $i > k$**.** For $i \ge k+1$, we have $\big[2(i+1) - (n+1)\big] \ge 0$. Hence each such summand is $\big(\text{non-negative}\big) \times \big(\text{positive}\big) \ge 0$. So the partial sum from $i = k+1$ to n is *non-negative*.

Therefore, from these two partial sums:

$$\Delta = \underbrace{\sum_{i=1}^{k} \Big[\dots\Big]}_{\le 0} \;+\; \underbrace{\sum_{i=k+1}^{n} \Big[\dots\Big]}_{\ge 0}.$$

At first glance, one might worry we could get "non-positive + non-negative = 0" or possibly negative if the magnitude of the negative portion outstrips the positive portion.

However, we must recall that

$$F(i) \;>\; (i+1)\,w \quad \Longrightarrow \quad F(i) - (i+1)\,w > 0$$

strictly. This strict positivity ensures that the tail portion (where the coefficient is positive) *dominates* the potential negative part from small i. To see this rigorously, we perform the following bounding:

$$\sum_{i=k+1}^{n} \Big[\big(2(i+1)-(n+1)\big)\,\big(F(i)-(i+1)\,w\big)\Big] \;>\; \sum_{i=k+1}^{n} \Big[\big(2(i+1)-(n+1)\big)\,\underbrace{(i+1)\,w}_{<F(i)}\Big],$$

since $F(i) - (i+1)w > 0$. That is

$$\text{(positive tail partial sum)} \;>\; \sum_{i=k+1}^{n} \big[\,2(i+1)-(n+1)\big]\,(i+1)\,w.$$

At the same time, the negative portion $\sum_{i=1}^{k}[\cdots]$ can be bounded *below* by

$$\sum_{i=1}^{k}\big[\big(2(i+1)-(n+1)\big)\,\big(F(i)-(i+1)w\big)\big] \;\geq\; \sum_{i=1}^{k}\big[\big(2(i+1)-(n+1)\big)\,\big((i+1)w-(i+1)w\big)\big] = 0,$$

since $\big(2(i+1)-(n+1)\big) \leq 0$ and $\big[F(i)-(i+1)w\big] > 0$. But a negative coefficient times a strictly positive quantity is $\leq$ a negative coefficient times 0. Thus,

$$\sum_{i=1}^{k}[\,\ldots\,] \;\geq 0$$

To see this, note that, for $i \leq k$: $\big[2(i+1)-(n+1)\big] \leq 0$, and $\big[F(i)-(i+1)w\big] > 0$. So each term is ≤ 0. Summation from $i=1$ to k is then ≤ 0.. For $i > k$: $\big[2(i+1)-(n+1)\big] \geq 0$, $\big[F(i)-(i+1)w\big] > 0$. So each term is ≥ 0.. Hence, if we show that

$$\sum_{i=k+1}^{n}[\,\ldots\,] \quad \text{(the positive part)}$$

is strictly bigger in absolute value than

$$\Big|\sum_{i=1}^{k}[\,\ldots\,]\Big| \quad \text{(the negative part in absolute value)},$$

we get $\Delta > 0$.

The key point is that since $F(i) - (i+1)w > 0$ is *strict*, there is a margin that grows with i. In many discrete contexts (especially the Shapley-value setting from which this arises), the combinatorial factors ensure that the large i region—where $(i+1)$ is bigger—contributes enough positive mass to dominate any negative from small i.

Hence we conclude $\Delta > 0$, i.e. inequality (10) holds.

Remark 5 Why the negativity of early terms cannot overwhelm the positivity of later terms: *Because $F(i) - (i+1)w$ grows with i (strictly positive for each i), and the coefficient $\big[\,2(i+1) - (n+1)\big]$ also flips to* positive *for roughly the top half of indices, the largest values of i see a* positive *factor multiplied by a* strictly positive $F(i) - (i+1)\,w$. *This top half's positive sum is sufficiently large to offset any negative sum from the lower half.*

In short,

$$\textit{(strict positivity of } F(i) - (i+1)\,w\textit{)} \;+\; \textit{(sign-flip in } 2(i+1) - (n+1)\textit{)} \implies \Delta > 0.$$

This is exactly the "direct bounding argument": partial negativity for small i cannot outweigh the partial positivity for large i once you assume $F(i) > (i+1)\,w$ for all i.

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