The New Keynesian Term Structure of Interest Rate – work in progress –

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Abstract

This paper examines the term structure of interest rates and the key drivers of the risk premium, identifying the main factors that influence the premium investors require for bearing economic risks within a micro-founded New Keynesian model featuring Calvo pricing. We address two central questions: What are the key factors shaping the term premium and the inflation risk premium, and how can fundamental economic shocks be classified as 'good' (reducing the premium) or 'bad' (increasing it)? We present a novel global solution that improves our understanding of how investors price these risks in Treasury markets. By solving the representative investor's fundamental pricing equation and applying the Feynman-Kac formula, we compute moments of key macroeconomic variables for direct comparison with empirical data. Additionally, we connect our findings to affine term structure models and provide identifying restrictions for economic shocks in empirical work.

Keywords: Monetary economics, Calvo pricing, Term structure of interest rates
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1. Introduction

The recent surge in government debt, along with rising interest rates and high inflation, has increased economic uncertainty and renewed questions regarding how investors price risks. The term structure of interest rates, which illustrates the relationship between bond yields and their maturities, has long been a central focus of both macroeconomic theory and financial economics (see Gürkaynak and Wright, 2012; Borovička and Hansen, 2016). A key component is the term premium on nominal bonds, which compensates investors for inflation and consumption risks over the bond's lifetime. Additionally, the term structure is influenced by monetary and fiscal policy decisions: Monetary policy directly affects short-term interest rates, and expectations of future monetary policy through forward guidance, can shape the entire yield curve (Rudebusch and Wu, 2008).[1] Fiscal policy influences the term structure by altering government borrowing needs and shaping expectations for economic growth.[2] These policies are crucial in shaping investor expectations and savings decisions, thus affecting term premiums across different maturities. A substantial body of financial literature has argued that these risk premiums are both significant and time-varying (Campbell and Shiller, 1991; Cochrane and Piazzesi, 2005; Cochrane, 2007; Gürkaynak, Sack, and Wright, 2007), and that inflation uncertainty is an important component of bond risk premia (Wright, 2011, 2012).

Understanding the determinants of the term premium is essential for gaining insights into the pricing of economic risks in financial markets. Alignment of the macro model with the general equilibrium asset pricing kernel is crucial, as it ensures that the term structure accurately reflects the aggregate risks fully consistent with the consumption and inflation dynamics, and the covariance between the stochastic discount factor (SDF) and bond yields. By incorporating price rigidities (cf. Clarida, Galí, and Gertler, 2000; Woodford, 2001, 2003; Smets and Wouters, 2003), the New Keynesian (NK) framework facilitates the analysis of how aggregate and policy-induced shocks impact the risk premium of nominal bonds in Treasury markets. Against this backdrop, our central research questions are: What are the key factors shaping the term premium and the inflation risk premium, and which fundamental economic shocks can be considered "good," reducing the term premium, or "bad," increasing it? Despite significant progress, existing macro-finance models, while insightful, still rely on local approximations (Rudebusch and Swanson, 2008, 2012; Hördahl and Tristani, 2008; van Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez, 2012; Andreasen, Fernández-Villaverde, and Rubio-Ramírez, 2018). Because of such limitations, the literature often abstracts from a detailed analy-

[1] Gürkaynak, Sack, and Swanson (2005) and others have argued that the Fed has increasingly relied on unconventional policies to shape market perceptions of the expected path of short-term rates (cf. Backus and Wright, 2007; Christensen and Rudebusch, 2012).

[2] Bansal and Shaliastovich (2013) show that bond risk premia rise with uncertainty about expected inflation and fall with uncertainty about expected growth. Corhay, Kind, Kung, and Morales (2023) argue that fiscal theory allows sovereign debt to influence expected inflation, even in a frictionless economy.

sis of the premium investors demand to hedge against fundamental economic risks, and focuses on empirical measures such as the slope of the yield curve and excess returns (Rudebusch and Swanson, 2012; Christiano, Motto, and Rostagno, 2014). We require a global solution approach that goes beyond local dynamics, fully capturing the nonlinearities of the underlying model and their effects on financial markets, while establishing a clear conntection between the term premium and relevant empirical measures.[3]

This paper aims to identify the most relevant factors influencing the term premium, especially when key macroeconomic variables like consumption and inflation are endogenous. Building upon the seminal contributions of exogenous pricing kernels (Vasicek, 1977) and real production models (Cox, Ingersoll, and Ross, 1985a,b), we utilize macroeconomic theories that employ a general equilibrium framework with forward-looking investors who possess rational expectations, wherein consumption dynamics and inflation are integral components of the model (Rudebusch and Swanson, 2008; Bekaert, Cho, and Moreno, 2010; Kaltenbrunner and Lochstoer, 2010). We show the contribution of aggregate shocks and how their interaction with monetary and fiscal policies affects the term premium, thus addressing our core question concerning 'good' and 'bad' fundamental shocks. We obtain the inflation risk premium as the difference of the term premia of the nominal and real zero-coupon Treasury bond.[4] For instance, we show that monetary policy shocks, while generating 'good' inflation (i.e., reducing the inflation risk premium), actually *increase* the riskiness of nominal bonds by raising the term premium. This occurs because the term premium on long-term real bonds rises more than that on nominal bonds. In contrast, technology shocks (supply-side shocks, i.e., move output and inflation in opposite directions), preference shocks and shocks to the natural rate (both demand-side shocks) increase the risk premium for both types of assets, *and* the inflation risk premium. Our modular approach allows for scalability with the number of shocks, and our illustrative examples show that patterns remain remarkably robust, even when extending the model with additional shocks or a more complex correlation structure.

This work contributes to the literature in several ways. First, it develops a novel global solution for a macro-finance term structure model that incorporates recursive preferences (Kreps and Porteus, 1978; Duffie and Epstein, 1992) and is embedded in a NK framework with Calvo pricing (Calvo, 1983), overcoming the limitations of local approximations. Our analysis identifies the economic risks that either amplify or mitigate the premium demanded by investors in Treasury markets. By solving the fundamental pricing equation (PDE), we compute moments of key macro and financial variables using the Feynman-Kac formula, which allows for direct comparison with empirical data. Second, it emphasizes the role of endogenous consumption and inflation dynamics in determining

[3]If the underlying model exhibits economically significant nonlinearities, the prevalent Campbell-Shiller log-linearization can generate large numerical errors (cf. Pohl, Schmedders, and Wilms, 2018).

[4]Cieslak and Pflueger (2023) study the inflation risk premium in a reduced-form NK model and argue that 'good' inflation can be linked to demand shocks, and 'bad' inflation to cost-push shocks.

yield curve characteristics. We categorize economic shocks into those that create favorable and unfavorable shifts and compute the sensitivity of the implied bond price to the state variable, thereby obtaining their price elasticities. This helps to understand which shocks increase the term premium of Treasury bonds at a given maturity, and links our key insights on SDF dynamics to the development of next-generation affine term structure models (ATSM). Third, by focusing on the interplay between shocks and term premium, this study not only offers novel insights into the pricing of fundamental risks, but also provides identifying restrictions for empirical work.

Financial market data provides a powerful lens through which to focus macroeconomic modeling efforts. By restricting attention to shocks and preferences that are reflected in market data, we can better identify the risks and the premiums investors are willing to pay to avoid uncertainty in their consumption and portfolio decisions. Haubrich, Pennacchi, and Ritchken (2012) empirically decompose Treasury yields and analyze the term structures of real rates, expected inflation, and inflation risk premia. Building on exisiting macro-finance models, we consider an NK model with Calvo pring and recursive preferences, with an important role for long-run risks (Bansal and Yaron, 2004), a time-varying inflation objective (Gürkaynak, Sack, and Swanson, 2005; Rudebusch and Swanson, 2012), and temporary and persistent shocks to the natural rate (preferences), and study the implications for Treasury markets (see also Gürkaynak, Sack, and Wright, 2007; Gürkaynak and Wright, 2012). We think that the representative agent model is a useful benchmark, as it highlights how uncertainty about parameters, policy instruments, or other variables is encoded in financial data, but still allows for analytical results in the frictionless limit with closed-form expressions for the value function in the case of long-run risk and monetary policy shocks, or assuming unit elasticity of intertemporal substitution and/or non-recursive preferences. We focus on equilibria with active monetary policy, in which the enlarged set of policy instruments includes long-end target rates (the liquidity trap scenario is studied in Werning, 2012; Cochrane, 2017; Wieland, 2019). More research is required to extend the analysis to heterogeneous agent NK framework (HANK), where the asset pricing kernel is determined by the joint saving and portfolio decisions (Gabaix, Lasry, Lions, and Moll, 2016; Achdou, Han, Lasry, Lions, and Moll, 2021).

The rest of the paper is organized as follows. In Section 2 we study the nonlinear version of the NK model with shocks and the flexible-price scenario. Section 3 presents the results, discussing the effects of uncertainty, the categorization of shocks and the asset pricing implications for the term structure of interest rates. Section 4 concludes.

2. The Nonlinear New Keynesian Model with Shocks

We describe now the environment for our investigation. It is the continuous-time version of the standard NK model (cf. Woodford, 2003). We summarize the equilibrium dynamics,

show how to compute impulse responses, compute the effects of uncertainty, and how to solve the model in the policy function space. Throughout the paper, we keep the nonlinear structure of the model, which turns out to be quite relevant for non-normal times when considering large shocks and/or large deviations from the point of approximation.

2.1. The Model

The basic structure of the model is as follows. A representative household consumes, saves, and supplies labor. The final output is assembled by a final good producer, who uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent labor to manufacture their good, and face the constraint that they can only adjust the price following Calvo's pricing rule (Calvo, 1983). Finally, there is a monetary authority that fixes the short-term nominal interest rate through open market operations with public debt, and a fiscal authority that taxes and consumes. We introduce four stochastic shocks, one to preferences (which can be loosely interpreted as a shock to aggregate demand, temporarily affecting the real interest rate), one to technology (interpreted as a shock to aggregate supply), one to monetary policy, and one to fiscal policy. For simplicity, we do not explicitly model a shock to the inflation target, which is considered a policy instrument under the discretion of the central bank.

2.1.1. Households

Preferences. Consider an economy with a single consumer, interpreted as a representative of a large number of identical consumers. The representative consumer maximizes lifetime recursive utility (cf. Svensson, 1989; Duffie and Epstein, 1992),

$$U_0 \equiv \mathbb{E}_0 \int_0^\infty f(\mathbb{Z}_t, \mathbb{X}_t, U_t)\,\mathrm{d}t \tag{1}$$

in which $\mathbb{Z}_t \in \mathbb{U}_z$ denotes the n-vector of states, $\mathbb{X}_t \in \mathbb{U}_x$ denotes the m-vector of controls, and $f : \mathbb{U}_z \times \mathbb{U}_x \times \mathbb{R} \to \mathbb{R}$ is a normalized aggregator.

In what follows, we introduce preferences shocks d_t and variation in subjective discount factors r_t. Suppose that $\log d_t$ follows an Ornstein-Uhlenbeck (OU) process,

$$\mathrm{d}\log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}. \tag{2}$$

It allows analytical tractability and ensures that d_t cannot become negative. The process $B_{d,t}$ is a standard Brownian motion for $\log d_t$, or equivalently,

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}.$$

Below, for this shock and the other exogenous stochastic processes, we will use both the

formulation in level and in logs depending on the context and ease of notation. In addition to preference shocks, we allow for time-variation in discount rates r_t,

$$\mathrm{d}r_t = \rho_r(\rho - r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t} \tag{3}$$

with $\rho > 0$ denoting the subjective rate of time preference, with small but positive mean reversion ρ_r (near random walk) such that $r_{t+\Delta} \approx r_t + \sigma_r(B_{r,t+\Delta} - B_{r,t})$ has the ability to capture more persistent changes in the subjective time preference.

Suppose $\mathbb{X}_t = (c_t, l_t)^\top$ are the controls, and the *normalized aggregator* reads

$$f(\mathbb{Z}_t, \mathbb{X}_t, U_t) = d_t \left(\frac{(c_t/z_t)^{1-1/\eta}}{1-1/\eta} - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right) ((1-\gamma) r_t U_t)^{1-\frac{1-1/\eta}{1-\gamma}} - \frac{1-\gamma}{1-1/\eta} r_t U_t, \tag{4}$$

for $\gamma \neq 1$ and $\eta \neq 1$, where z_t captures a stochastic trend, and $\gamma > 0$ reflects risk aversion, and $\eta > 0$ is the elasticity of intertemporal substitution.[5] Our normalized aggregator in (4) preserves intertemporal consistency, but can relax the assumption of time-additivity (Kreps and Porteus, 1978; Duffie and Epstein, 1992; Rudebusch and Swanson, 2012).[6] Moreover, our specification in (4) follows additively separable utility kernel, in line with little empirical evidence of nonseparability between consumption and leisure, and consistent with balanced growth (cf. Rudebusch and Swanson, 2012). We assume that stationary consumption (instead of consumption levels) enters the utility kernel. Alternatively, one may introduce a trend for leisure as in to restore the balanced-growth

Let a_t denote real financial wealth, the household's real wealth evolves according to

$$\mathrm{d}a_t = \left(r_t^b a_t - c_t + w_t l_t + T_t + \digamma_t \right) \mathrm{d}t, \tag{5}$$

in which $r_t^b \equiv i_t - \pi_t$ where i_t is the nominal interest rate on government bonds, π_t the rate of inflation of the price level p_t (or price of the consumption good), w_t is the real wage, T_t is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy.

2.1.2. The final good producer

There is one final good produced using intermediate goods with the production function:

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} \,\mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{6}$$

[5] Note that the coefficient of relative risk aversion is different in a utility function that includes labor. This results extends to the case of recursive preferences and is used to define the coefficient of relative risk aversion in our setting (cf. Swanson, 2012, 2018).

[6] Versions with ordinal-equivalent aggregator functions can be found in Chacko and Viceira (2005), Wachter (2013), and more recently Parra-Alvarez, Posch, and Schrimpf (2022). This formulation better compares with standard time-additive preferences and allows to apply L'Hôpital's rule for $1/\eta \to 1$.

where ε is the elasticity of substitution.

Final good producers are perfectly competitive and maximize profits subject to the production function (6), taking as given all intermediate goods prices p_{it} and the final good price p_t. Hence, the input demand functions are $y_{it} = (p_{it}/p_t)^{-\varepsilon} y_t$ for all i and

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon} \mathrm{d}i \right)^{\frac{1}{1-\varepsilon}} \tag{7}$$

is the (aggregate) price level.

2.1.3. Intermediate good producers

Each intermediate firm produces differentiated goods out of labor using

$$y_{it} = A_t z_t l_{it}, \tag{8}$$

where l_{it} is the amount of the labor input rented by the firm and where A_t follows

$$\mathrm{d} \log A_t = -\rho_A \log A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t}, \tag{9}$$

and z_t, following

$$\begin{aligned} \mathrm{d}z_t &= x_t z_t \mathrm{d}t + \sigma_t z_t \mathrm{d}B_{z,t}, & (10) \\ \mathrm{d}x_t &= \rho_x(\mu - x_t)\mathrm{d}t + \sigma_t \sigma_x \mathrm{d}B_{x,t}, & (11) \\ \mathrm{d}\sigma_t &= \rho_\sigma(\sigma_z - \sigma_t)\mathrm{d}t + \sigma_\sigma \mathrm{d}B_{\sigma,t}, & (12) \end{aligned}$$

is a non-stationary component of technology, a geometric Brownian motion with drift x_t, which induces persistent changes in the growth rate of z_t, and captures long-run risk as in Andreasen and Jørgensen (2020), and $\sigma_t \geq 0$ for fluctuating economic uncertainty around σ_z, or stochastic volatility following Heston (1993).[7]

Most importantly, marginal costs of the intermediate good producers are the same across firms,

$$mc_t = w_t/(A_t z_t). \tag{13}$$

The monopolistic firms engage in infrequent price setting à la Calvo. We assume that intermediate good producers reoptimize their prices p_{it} only when a price-change signal is received. The probability (density) of receiving such a signal h periods from today is assumed to be independent of the last time the firm got the signal, and given by $\delta e^{-\delta h}$,

[7] For comparison with the contributions by Heston (1993), Cox, Ingersoll, and Ross (1985b), and Bansal and Yaron (2004), we assume $\sigma_t \geq 0$, but do not mechanically enforce the restriction as in (2) or (9).

$\delta > 0$. Thus $e^{-\delta(\tau-t)}$ denotes the probability of not having received a signal during $\tau - t$,

$$1 - \int_t^\tau \delta e^{-\delta(h-t)}\,\mathrm{d}h = 1 - \left(-e^{-\delta(\tau-t)} + 1\right) = e^{-\delta(\tau-t)}. \tag{14}$$

A fraction of firms will receive the price-change signal per unit of time. All other firms cannot reoptimize their price, but (partially) index their price to π_t^* with $\mathrm{d}\bar{P}_{it} = \chi\pi_t^*\bar{P}_{it}\,\mathrm{d}t$, where the degree of indexation is $\chi \in [0,1]$. This implies that if a firm cannot reoptimize its price at t for a period length of $\tau - t$, its price at τ is $p_{i\tau} = p_{it}e^{\int_t^\tau \chi\pi_s^*\,\mathrm{d}s}$. Note that the higher the parameter δ, the lower price rigidities, in the frictionless case $\delta \to \infty$. Hence, for $\delta < \infty$ prices are set to maximize the expected discounted profits,

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{m_\tau}{m_t} e^{-\delta(\tau-t)} \left(\frac{p_{i\tau}}{p_\tau} y_{i\tau} - mc_\tau y_{i\tau}\right) \mathrm{d}\tau \quad \text{s.t.} \quad y_{i\tau} = \left(\frac{p_{i\tau}}{p_\tau}\right)^{-\varepsilon} y_\tau,$$

where m_t denotes the equilibrium state-price density $m_\tau \equiv \exp(\int_t^\tau f_U(c_u, l_u, V)\,\mathrm{d}u)\lambda_\tau$.

After dropping constants, we may write the first-order condition as

$$p_{it}x_{1,t} = \frac{\varepsilon}{\varepsilon - 1} p_t x_{2,t} \quad \Rightarrow \quad \Pi_t^* = \frac{\varepsilon}{\varepsilon - 1}\frac{x_{2,t}}{x_{1,t}}, \tag{15}$$

in which $\Pi_t^* \equiv p_{it}/p_t$ is the ratio between the optimal new price (common across all firms that can reset their prices) and the price of the final good and where we define the auxiliary variables (interpreted as expected discounted marginal revenue and marginal costs),

$$x_{1,t} \equiv \mathbb{E}_t \int_t^\infty m_\tau e^{-\delta(\tau-t)} \left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon} e^{\int_t^\tau (1-\varepsilon)\chi\pi_s^*\,\mathrm{d}s} y_\tau \mathrm{d}\tau,$$
$$x_{2,t} \equiv \mathbb{E}_t \int_t^\infty m_\tau e^{-\delta(\tau-t)} mc_\tau \left(\frac{p_t}{p_\tau}\right)^{-\varepsilon} e^{-\int_t^\tau \varepsilon\chi\pi_s^*\,\mathrm{d}s} y_\tau \mathrm{d}\tau.$$

Both variables are forward looking (or jump variables) and determined in equilibrium.

Differentiating $x_{1,t}$ with respect to time gives

$$\mathrm{d}x_{1,t} = ((\delta - f_U + (1-\varepsilon)(\pi_t - \chi\pi_t^*))x_{1,t} - \lambda_t y_t)\,\mathrm{d}t, \tag{16}$$

in which we define inflation $\pi_t \equiv (1/\,\mathrm{d}t)\mathbb{E}_t(\mathrm{d}p_t/p_t)$. Similarly we obtain:

$$\mathrm{d}x_{2,t} = ((\delta - f_U - \varepsilon(\pi_t - \chi\pi_t^*))x_{2,t} - \lambda_t mc_t y_t)\,\mathrm{d}t. \tag{17}$$

Assuming that the price-change is stochastically independent across firms gives:

$$p_t^{1-\varepsilon} = \int_{-\infty}^t \delta e^{-\delta(t-\tau)} \left(p_{i\tau} e^{\int_\tau^t \chi\pi_s^*\,\mathrm{d}s}\right)^{1-\varepsilon} \mathrm{d}\tau,$$

making p_t a predetermined variable (Calvo's insight), because its level is given by past price quotations. Differentiating with respect to time and using $\mathrm{d}p_t^{1-\varepsilon} = (1-\varepsilon)\, p_t^{-\varepsilon}\mathrm{d}p_t$, we arrive at $(1-\varepsilon)\ \mathrm{d}p_t = \left(\delta p_{it}^{1-\varepsilon} p_t^{\varepsilon-1} - (\delta - (1-\varepsilon)\chi\pi_t^*)\right) p_t \mathrm{d}t$, which implies

$$\pi_t - \chi\pi_t^* = \frac{\delta}{1-\varepsilon}\left((\Pi_t^*)^{1-\varepsilon} - 1\right). \tag{18}$$

Differentiating (18) with respect to time, we obtain the inflation dynamics as

$$\mathrm{d}(\pi_t - \chi\pi_t^*) = -(\delta + (1-\varepsilon)(\pi_t - \chi\pi_t^*))\left(\pi_t - \chi\pi_t^* + (mc_t/x_{2,t} - 1/x_{1,t})m_t y_t\right)\ \mathrm{d}t, \tag{19}$$

which is the NK Phillips-curve.

2.1.4. The government problem

In normal times, we assume that the government sets the nominal interest rate i_t through open market operations according to two alternative setups, i.e., the feedback model

$$i_t - i_{ss} = \phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/(y_{ss}z_t) - 1), \quad \phi_\pi > 0,\ \phi_y \geq 0, \tag{20a}$$

or the partial adjustment model

$$\mathrm{d}i_t = \theta(\phi_\pi(\pi_t - \pi_t^*) + \phi_y(y_t/(y_{ss}z_t) - 1) - (i_t - i_{ss}))\mathrm{d}t + \sigma_i \mathrm{d}B_{i,t}, \quad \theta > 0, \tag{20b}$$

which includes a response to inflation and output, and a desire to smooth interest rates. The partial adjustment model in (20b) specifies a time lag between the inflation rate and the interest rate, and allows for an output response and monetary policy shocks.

We assume similar to Ireland (2007) that

$$\begin{aligned}\mathrm{d}\pi_t^* \;=\;& \rho_\pi(\pi_{ss} - \pi_t^*)\,\mathrm{d}t + \sigma_\pi\,\mathrm{d}B_{\pi,t} + \sigma_{\pi,r}\,\mathrm{d}B_{r,t} + \sigma_{\pi,i}\,\mathrm{d}B_{i,t} + \sigma_{\pi,d}\,\mathrm{d}B_{d,t} + \sigma_{\pi,A}\,\mathrm{d}B_{A,t} \\ & + \sigma_{\pi,z}\,\mathrm{d}B_{z,t} + \sigma_{\pi,x}\,\mathrm{d}B_{x,t} + \sigma_{\pi,\sigma}\,\mathrm{d}B_{\sigma,t} + \sigma_{\pi,g}\,\mathrm{d}B_{g,t}\end{aligned} \tag{21}$$

where the coefficients $\sigma_{\pi,r},\ \sigma_{\pi,i},\ \sigma_{\pi,d},\ \sigma_{\pi,A},\ \sigma_{\pi,z},\ \sigma_{\pi,x},\ \sigma_{\pi,\sigma},\ \sigma_{\pi,g}$ are preferences by the central bank which allows to systematically adjust its inflation target in response to either shock. Similar to (3), a small parameter ρ_π captures rather persistent changes in the inflation target. According the definition used in Ireland (2007), an adverse supply shock (corresponding to negative realizations of $B_{z,t}$ or $B_{A,t}$) tends to increase prices. To translate the short-run price pressures into more persistent movements in the inflation rate, e.g., $\sigma_{\pi,A} < 0$ helps us to formalize the stories of post-war monetary policy.

Following the traditional macro literature, we keep the fiscal block simple and assume zero net supply of government bonds such that $a_t = 0$ for all t. Transfers finance a given stream of government consumption g_t, expressed as a constant share of output, $s_g s_{g,t}$,

with a mean s_g and a stochastic component $s_{g,t}$ that follows

$$\mathrm{d}\log s_{g,t} = -\rho_g \log s_{g,t}\mathrm{d}t + \sigma_g \mathrm{d}B_{g,t}, \tag{22}$$

such that

$$g_t = s_g s_{g,t} y_t \equiv -T_t. \tag{23}$$

For a more elaborate fiscal block involving public debt and its transmission channels to inflation dynamics and the term structure, see Liemen and Posch (2023).

2.1.5. Aggregation

First, we derive an expression for aggregate demand:

$$y_t = c_t + g_t. \tag{24}$$

As in endowment economies, there is no possibility of transferring output intertemporally, meaning that the demand for each intermediate good producer is

$$y_{it} = (c_t + g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \qquad \forall i. \tag{25}$$

From (8), we aggregate the demand of all firms $A_t z_t l_{it} = (c_t + g_t)\,(p_{it}/p_t)^{-\varepsilon}$, and get $A_t z_t \int_0^1 l_{it}\mathrm{d}i = (c_t + g_t)\int_0^1 (p_{it}/p_t)^{-\varepsilon}\mathrm{d}i$, which can be written as

$$c_t + g_t = y_t = \frac{A_t z_t}{v_t} l_t, \tag{26}$$

by defining $v_t \equiv \int_0^1 (p_{it}/p_t)^{-\varepsilon}\mathrm{d}i$ as the aggregate loss of efficiency induced by price dispersion of the intermediate goods. Similar to the price level, v_t is a predetermined variable:

$$v_t = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} e^{-\varepsilon\int_\tau^t \chi\pi_s^* \,\mathrm{d}s}\mathrm{d}\tau. \tag{27}$$

Differentiating this expression with respect to time gives:

$$\mathrm{d}v_t \;=\; \left(\delta\left(\Pi_t^*\right)^{-\varepsilon} + \left(\varepsilon(\pi_t - \chi\pi_t^*) - \delta\right) v_t\right)\,\mathrm{d}t. \tag{28}$$

Similarly, we obtain an expression for aggregate profits (cf. Online Appendix I.1):

$$\digamma_t = (1 - mc_t v_t) y_t. \tag{29}$$

The profits of the firms in the economy are distributed to households as shown in (5).

2.2. Equilibria with price stickiness

For the case of $\delta < \infty$, we obtain the first-order conditions by defining the state space $\mathbb{U}_z \subseteq \mathbb{R}^{n_d}$ and the control region $\mathbb{U}_x \subseteq \mathbb{R}^m$, the *normalized aggregator* $f : \mathbb{U}_z \times \mathbb{U}_x \times \mathbb{R} \to \mathbb{R}$, the *drift function* $g : \mathbb{U}_z \times \mathbb{U}_x \to \mathbb{R}^{n_d}$, and the *diffusion function* $\sigma : \mathbb{U}_z \to \mathbb{R}^{n_d \times n_d}$. Given our description of the problem, we define the household's *value function*:

$$V(\mathbb{Z}_0; \mathbb{Y}_0) \equiv \max_{\{\mathbb{X}_t\}_{t=0}^{\infty}} U_0, \tag{30}$$

in which $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$ is a vector taken as parametric by the representative household and being determined in equilibrium being as functions of the state variables, subject to

$$\mathrm{d}\mathbb{Z}_t = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\,\mathrm{d}t + \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t, \tag{31}$$

where B_t is a given vector of independent standard Brownian motions. The instantaneous covariance matrix of $\mathbb{Z}_t$ is $\Sigma(\mathbb{Z}_t) \equiv \sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top$, which may be less than full rank n_d.

By choosing the control $\mathbb{X}_t \in \mathbb{R}^2_+$, the Hamilton-Jacobi-Bellman (HJB) equation reads:

$$0 = \max_{\mathbb{X}_t} \left\{ f(\mathbb{Z}_t, \mathbb{X}_t, V(\mathbb{Z}_t; \mathbb{Y}_t)) + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right) \right\}, \tag{32}$$

where $V_{\mathbb{Z}}$ is the n-vector of partial derivatives $\partial V / \partial \mathbb{Z}_t$, $V_{\mathbb{Z}\mathbb{Z}}$ is the $n_d \times n_d$ Hessian matrix, and $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. A neat result about our formulation is that the HJB equation (32) is a deterministic functional equation. In the discrete-time version, we would need to approximate expectations (or the n_d-dimensional integral).

From the discussion above, we define the *drift function* (with partial adjustment):

$$g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t) = \begin{bmatrix} (i_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t & (5) \\ \theta\phi_\pi(\pi_t - \pi_t^*) + \theta\phi_y(y_t/(y_{ss}z_t) - 1) - \theta(i_t - i_{ss}) & (20b) \\ \delta\left(1 + (1-\varepsilon)(\pi_t - \chi\pi_t^*)/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi_t - \chi\pi_t^*) - \delta)\,v_t & (28) \\ -(\rho_d \log d_t - \frac{1}{2}\sigma_d^2)d_t & (2) \\ -(\rho_A \log A_t - \frac{1}{2}\sigma_A^2)A_t & (9) \\ -(\rho_g \log s_{g,t} - \frac{1}{2}\sigma_g^2)s_{g,t} & (22) \\ x_t z_t & (10) \\ \rho_x(\mu - x_t) & (11) \\ \rho_\sigma(\sigma_z - \sigma_t) & (12) \\ \rho_r(\rho - r_t) & (3) \\ \rho_\pi(\pi_{ss} - \pi_t^*) & (21) \end{bmatrix},$$

and the *diffusion function* of the state transition equations:

$$\sigma(\mathbb{Z}_t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_d d_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_A A_t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_g s_{g,t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_t z_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_t \sigma_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_\sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_r & 0 \\ 0 & \sigma_{\pi,i} & 0 & \sigma_{\pi,d} & \sigma_{\pi,A} & \sigma_{\pi,g} & \sigma_{\pi,z} & \sigma_{\pi,x} & \sigma_{\pi,\sigma} & \sigma_{\pi,r} & \sigma_\pi \end{bmatrix} \begin{matrix} (5) \\ (20b) \\ (28) \\ (2) \\ (9) \\ (22) \\ (10) \\ (11) \\ (12) \\ (3) \\ (21) \end{matrix} .$$

The first-order conditions with respect to c_t and l_t for any interior solution are:

$$\begin{aligned} f_c(\mathbb{Z}_t, \mathbb{X}_t, V) &= V_a, \\ -f_l(\mathbb{Z}_t, \mathbb{X}_t, V) &= V_a w_t, \end{aligned}$$

which means for $\gamma \neq 1$

$$d_t c_t^{-1/\eta} z_t^{-(1-1/\eta)} \left((1-\gamma) r_t V\right)^{1-\frac{1-1/\eta}{1-\gamma}} = V_a, \tag{33}$$

$$d_t \psi l_t^{\vartheta} \left((1-\gamma) r_t V\right)^{1-\frac{1-1/\eta}{1-\gamma}} = V_a w_t, \tag{34}$$

and for $\gamma = 1/\eta = 1$

$$d_t (c_t/z_t)^{-1}/z_t = V_a, \tag{35}$$

$$d_t \psi l_t^{\vartheta} = V_a w_t, \tag{36}$$

or, by eliminating the costate variable we obtain $-f_l/f_c = \psi l_t^{\vartheta} c_t^{1/\eta} z_t^{1-1/\eta} = w_t$, which is the standard static optimality condition between labor and consumption.

Most notably, the first-order conditions (33) and (34) yield *optimal controls*:

$$\mathbb{X}_t = \mathbb{X}(\mathbb{Z}_t; V(\mathbb{Z}_t; \mathbb{Y}_t), V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t), \mathbb{Y}_t) \equiv \begin{bmatrix} c(\mathbb{Z}_t; V(\mathbb{Z}_t; \mathbb{Y}_t), V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t), \mathbb{Y}_t) \\ l(\mathbb{Z}_t; V(\mathbb{Z}_t; \mathbb{Y}_t), V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t), \mathbb{Y}_t) \end{bmatrix}$$

or

$$\mathbb{X}_t = \begin{bmatrix} \left((V_a(\mathbb{Z}_t; \mathbb{Y}_t))^{-1} d_t z^{-(1-1/\eta)} \left((1-\gamma) r_t V\right)^{1-\frac{1-1/\eta}{1-\gamma}} \right)^{\eta} \\ \left(\left(V_a(\mathbb{Z}_t; \mathbb{Y}_t) w_t\right)^{-1} d_t \psi \left((1-\gamma) r_t V\right)^{1-\frac{1-1/\eta}{1-\gamma}} \right)^{-1/\vartheta} \end{bmatrix}.$$

In particular, the first-order conditions (33) and (34) make the optimal controls functions

of the states, $c_t = c(\mathbb{Z}_t; \mathbb{Y}_t)$, $l_t = l(\mathbb{Z}_t; \mathbb{Y}_t)$. Hence, the concentrated HJB equation reads:

$$0 = f(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}_t), V(\mathbb{Z}_t; \mathbb{Y}_t)) + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}); \mathbb{Y}))^\top V_{\mathbb{Z}} + \tfrac{1}{2}\text{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right), \quad (37)$$

and

$$0 = (f_U + i_t - \pi_t)\lambda_t + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}); \mathbb{Y}))^\top \lambda_{\mathbb{Z}} + \tfrac{1}{2}\text{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right), \quad (38)$$

or

$$\begin{aligned} 0 &= f_U + i_t - \pi_t + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}); \mathbb{Y}))^\top \tilde{\lambda}_{\mathbb{Z}} + \tfrac{1}{2}\text{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}\right) \\ &= f_U + i_t - \pi_t + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}); \mathbb{Y}))^\top \tilde{\lambda}_{\mathbb{Z}} + \tfrac{1}{2}\text{tr}\left(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}\right) - \sigma_t \sigma_{\pi,z} \tilde{\lambda}_\pi + \sigma_t^2 \end{aligned}$$

for the costate equation by using the envelope theorem, where $\lambda_{\mathbb{Z}}$ is the n-vector of partial derivatives $\partial\lambda_t/\partial\mathbb{Z}_t$, and $\lambda_{\mathbb{Z}\mathbb{Z}}$ is the $n \times n$ Hessian matrix of the costate variable.[8]

Note that $V_a(\mathbb{Z}_t; \mathbb{Y}_t) \equiv \lambda_t$ in (33), (34), and (38) is readily interpreted as the marginal value of wealth or the current value of a unit of consumption in period t, and determines the asset pricing kernel in this economy. In what follows, we provide the asset pricing kernel or the stochastic discount factor (SDF) consistent with equilibrium dynamics of macro aggregates, which can be used to price any asset in the economy.

As we show in the appendix, the marginal value of wealth evolves according to:

$$\mathrm{d}\lambda_t = -(f_U + i_t - \pi_t)\lambda_t \mathrm{d}t + \lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t \quad (39)$$

and the value function follows:

$$\mathrm{d}V = -f(\mathbb{Z}_t, \mathbb{X}_t, V)\,\mathrm{d}t + V_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t. \quad (40)$$

The implicit Euler equation (39) has a simple interpretation: the change in the marginal value of wealth depends on the effective rate of time preference $-f_U$ minus the real interest rate and the shocks to the economy. Note that the effective rate of time preference $-f_U$ depends on the preference for the resolution of uncertainty

$$-f_U = \frac{1-\gamma}{1-1/\eta}\left(f_{UU}V + r_t\right), \quad \gamma \neq 1,\ 1/\eta \neq 1$$

It shows that for the case of $V < 0$, an early resolution of uncertainty, i.e., $f_{UU} > 0$, the convex normalized aggregator effectively decreases the discount factor r_t.

Definition 1. *The recursive-competitive equilibrium of the NK model is defined by the sequence $\{\mathbb{Z}_t, \mathbb{Y}_t, \lambda_t, V_t\}_{t=0}^\infty$, which satisfies the dynamic and static equations in (31), (13), (15), (16), (17), (18), (23), (24), (29), (33), (39), and (40).*

[8] Judd (1992) proposes to define (numerical) Euler equation errors based on (38) to gauge accuracy.

Following Duffie and Skiadas (1994), unlike in the case of time-additive utility, the case of recursive utility implies that the state-price density depends on the value function. In particular, we define (cf. Wachter, 2013, p.996)

$$m_s/m_t = \exp\left(\int_t^s f_U(\mathbb{Z}_\tau, \mathbb{X}_\tau, V)\,\mathrm{d}\tau\right)\frac{f_c(\mathbb{Z}_s, \mathbb{X}_s, V_s)}{f_c(\mathbb{Z}_t, \mathbb{X}_t, V_t)}, \tag{41}$$

where f_c, and f_U denote partial derivatives.[9] Using $\lambda_t = f_c$, we may also write

$$\begin{aligned}\mathrm{d}m_t &= f_U(\mathbb{Z}_t, \mathbb{X}_t, V)m_t\,\mathrm{d}t + m_t/\lambda_t\,\mathrm{d}\lambda_t \\ &= -\left(i_t - \pi_t\right)m_t\mathrm{d}t + m_t\lambda_{\mathbb{Z}}^\top/\lambda_t\sigma(\mathbb{Z}_t)\,\mathrm{d}B_t\end{aligned} \tag{42}$$

For the case of $1/\eta = \gamma$ simplifies to $f_U = -r_t$ and $f_c = d_t c_t^{-1/\eta} z_t^{-(1-1/\eta)}$ and the SDF is:

$$m_s/m_t = e^{-\int_t^s r_\tau\,\mathrm{d}\tau}\frac{V_a(\mathbb{Z}_s; \mathbb{Y}_s)}{V_a(\mathbb{Z}_t; \mathbb{Y}_t)}, \quad \text{and} \quad m_s \equiv e^{-\int_t^s r_\tau\,\mathrm{d}\tau}\lambda_s, \tag{43}$$

or, equivalently, the present value shadow price (cf. Hansen and Scheinkman, 2009).

2.2.1. Equilibrium dynamics

In what follows, we restrict our attention to solutions leading the economy towards a (stochastic) balanced growth path, in which $\pi_t \to \pi_t^*$. Because government bonds are in net zero supply, we arrive at a system of 6 endogenous processes, i.e., for the auxiliary variables $x_{1,t}$, $x_{2,t}$, price dispersion v_t, the Taylor rule i_t, the Euler equation λ_t, the value function V, and the 6 exogenous processes, which summarize equilibrium dynamics:

$$\mathrm{d}\lambda_t = -\left(f_U + i_t - \pi_t\right)\lambda_t\mathrm{d}t + \lambda_{\mathbb{Z}}^\top\sigma(\mathbb{Z}_t)\,\mathrm{d}B_t \tag{44a}$$

$$\mathrm{d}V = -f(\mathbb{Z}_t, \mathbb{X}_t, V(\mathbb{Z}_t; \mathbb{Y}_t))\,\mathrm{d}t + V_{\mathbb{Z}}^\top\sigma(\mathbb{Z}_t)\,\mathrm{d}B_t \tag{44b}$$

$$\mathrm{d}x_{1,t} = ((\delta - f_U + (1-\varepsilon)(\pi_t - \chi\pi_t^*))x_{1,t} - \lambda_t y_t)\,\mathrm{d}t \tag{44c}$$

$$\mathrm{d}x_{2,t} = ((\delta - f_U - \varepsilon(\pi_t - \chi\pi_t^*))x_{2,t} - \lambda_t m c_t y_t)\,\mathrm{d}t \tag{44d}$$

$$\mathrm{d}\mathbb{Z}_t = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\,\mathrm{d}t + \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t. \tag{44e}$$

In the NK model, the auxiliary variables $x_{2,t}/x_{1,t}$ determine the optimal relative price Π_t^* in (15) for the price setting firms, and thus inflation π_t in (18) or $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$, and

$$w_t = -\frac{f_l(\mathbb{Z}_t, \mathbb{X}_t, V)}{f_c(\mathbb{Z}_t, \mathbb{X}_t, V)} = \psi l_t^\vartheta c_t^{1/\eta} z_t^{1-1/\eta} \tag{45}$$

from the first-order conditions (33) and (34) pins down marginal costs in (13).

[9] Because the exponential term is (locally) deterministic, the covariances of the state-price density with fundamentals, are determined by the marginal value of wealth $f_c(c_t, V) = \lambda_t$ (Wachter, 2013, p.1022).

Proposition 2. *Defining stationary variables* $\hat{\mathbb{Z}}_t \equiv (\hat{a}_t, i_t, v_t, d_t, A_t, s_{g,t}, \hat{z}_t, x_t, \sigma_t, r_t, \pi_t^*)^\top$, $\hat{z}_t \equiv z_t/z_t = 1$, $\hat{a}_t \equiv a_t/z_t$, *aggregates* $\hat{\mathbb{Y}}_t \equiv (mc_t, \Pi_t^*, x_{1,t}, x_{2,t}, \pi_t, g_t, \hat{T}_t, \hat{y}_t, \hat{F}_t, \hat{w}_t)^\top$ *where* $\hat{T}_t \equiv T_t/z_t$, $\hat{y}_t \equiv y_t/z_t$, $\hat{F}_t \equiv F_t/z_t$, $\hat{w}_t \equiv w_t/z_t$, *controls* $\hat{\mathbb{X}}_t \equiv (\hat{c}_t, l_t)^\top$ *where* $\hat{c}_t \equiv c_t/z_t$, *and the costate* $\hat{\lambda}_t \equiv \lambda_t/z_t^{-1} = \lambda_t z_t$, *the value function (1) can be written as*

$$V(\mathbb{Z}_t; \mathbb{Y}_t) = V(\mathbb{Z}_t) \equiv V(\hat{\mathbb{Z}}_t; \hat{\mathbb{Y}}_t) = V(\hat{\mathbb{Z}}_t),$$

where $V_{\hat{a}}(\hat{\mathbb{Z}}_t) \equiv V_a z_t = \lambda_t z_t = \hat{\lambda}_t$, *and* $z_t \lambda_z/\lambda_t = -1$, *and* $V_z(\mathbb{Z}) = -\hat{a}_t V_a(\mathbb{Z}_t)$.

Proof. Online Appendix I.6 ■

Proposition 3. *By eliminating trends of all non-stationary variables using* $\hat{y}_t = y_t/z_t$, $\hat{c}_t = c_t/z_t$, $\hat{z}_t = 1$ *and for the costate* $\hat{\lambda}_t = \lambda_t z_t$,

$$\begin{aligned}
d\hat{\lambda}_t &= -(f_U + i_t - \pi_t - x_t + \sigma_t^2 - \sigma_t \tilde{\lambda}_\pi \sigma_{\pi,z})\hat{\lambda}_t dt + \hat{\lambda}_{\hat{\mathbb{Z}}}^\top \sigma(\hat{\mathbb{Z}}_t)\, dB_t && \text{(46a)}\\
dV &= -f(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t, V(\hat{\mathbb{Z}}_t; \hat{\mathbb{Y}}_t))\, dt + V_{\hat{\mathbb{Z}}}^\top \sigma(\hat{\mathbb{Z}}_t)\, dB_t && \text{(46b)}\\
dx_{1,t} &= ((\delta - f_U + (1-\varepsilon)(\pi_t - \chi\pi_t^*))x_{1,t} - \hat{\lambda}_t \hat{y}_t)\, dt && \text{(46c)}\\
dx_{2,t} &= ((\delta - f_U - \varepsilon(\pi_t - \chi\pi_t^*))x_{2,t} - mc_t \hat{\lambda}_t \hat{y}_t)\, dt && \text{(46d)}\\
d\hat{\mathbb{Z}}_t &= g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)\, dt + \sigma(\hat{\mathbb{Z}}_t)\, dB_t. && \text{(46e)}
\end{aligned}$$

summarizes a stationary system in which z_t *no longer appears as a state variable.*

Proof. Online Appendix I.7 ■

Note that a stationary system is not required for our numerical solution, we rather use the (necessary) condition that the system approaches a balanced growth path.

2.2.2. Numerical solution of the (conditional) deterministic system

In this section, we compute the solution to the HJB equation from a deterministic system of differential equations (the boundary value problem), which works because the HJB equation in continuous time is a deterministic equation (cf. Chang, 2004).[10] Nonetheless, we have to account appropriately for risk. So the novel idea below is to transform the system of SDEs into a system of PDEs that solves the HJB equation.

Consider $\lambda_t = \lambda(\mathbb{Z}_t)$ and $V = V(\mathbb{Z}_t)$, and use Itô's formula to arrive at

$$\begin{aligned}
\mathrm{d}\lambda_t &= \mathrm{d}\mathbb{Z}_t^\top \lambda_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right) \mathrm{d}t, && (47)\\
\mathrm{d}V &= \mathrm{d}\mathbb{Z}_t^\top V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right) \mathrm{d}t, && (48)
\end{aligned}$$

which leads us to the following proposition.

[10] In contrast, the discrete-time HJB equation requires the analyst needs to evaluate the state space not only at the current information set, but also at future expected values, so the continuous-time approach does not require to numerically compute expectations (a burdensome step in discrete-time models).

Proposition 4. *By subtracting the Itô second-order terms from the dynamics for the costate $d\lambda_t$ in (39) and the value function $dV(\mathbb{Z}_t;\mathbb{Y}_t)$ in (40), respectively,*

$$\begin{aligned} d\lambda_t - \tfrac{1}{2} tr\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right) dt &= d\mathbb{Z}_t^\top \lambda_{\mathbb{Z}}, \\ dV - \tfrac{1}{2} tr\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right) dt &= d\mathbb{Z}_t^\top V_{\mathbb{Z}}, \end{aligned}$$

and inserting $d\lambda_t$ in (39), dV in (40), and the state dynamics $d\mathbb{Z}_t$ from (31) we eliminate time (and shocks) obtain the costate and the HJB equations in (38) and (37).

Proof. Insert $\mathrm{d}\lambda_t$ from (39) into the state dynamics in (31) to obtain

$$\begin{aligned} -\left(f_U + i_t - \pi_t\right)\lambda_t \mathrm{d}t + \lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right)\,\mathrm{d}t = \\ g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \lambda_{\mathbb{Z}}\,\mathrm{d}t + \,\mathrm{d}B_t^\top \sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}} \end{aligned}$$

which, by collecting terms noting that $\lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t = \,\mathrm{d}B_t^\top \sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}}$, implies (38). Follow the same steps for the concentrated HJB in (37). ■

A system of PDEs which implies the same policy function is constructed using (39) and (40) and Proposition 4 by subtracting Itô terms from the costate variable and the value function (accounting for risk) and set shocks to zero (absence of shocks), such that

$$\begin{aligned} \mathbb{E}_t(\mathrm{d}\lambda_t) - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right)\,\mathrm{d}t &= \mathbb{E}_t\left(\mathrm{d}\mathbb{Z}_t^\top \lambda_{\mathbb{Z}}\right), \\ \mathbb{E}_t\left(\mathrm{d}V\right) - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right)\mathrm{d}t &= \mathbb{E}_t\left(\mathrm{d}\mathbb{Z}_t^\top V_{\mathbb{Z}}\right), \end{aligned}$$

solves (38) and (37). In fact, similar to a "risk-neutral representation" we adjust the drift of each process by its covariance with the states. Hence, we write a "risk-adjusted" dynamics for the states and neglect the shocks and solve for the unknown policy functions from a system of deterministic PDEs. Hence, we refer to the system

$$\begin{aligned} \mathrm{d}\lambda(\mathbb{Z}_t) &= -(f_U + i_t - \pi_t)\lambda_t \mathrm{d}t - \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}})\,\mathrm{d}t, && (49) \\ \mathrm{d}V(\mathbb{Z}_t) &= -f(\mathbb{Z}_t, \mathbb{X}_t, V)\,\mathrm{d}t - \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}})\mathrm{d}t, && (50) \\ \mathrm{d}x_{1,t} &= ((\delta - f_U - (\varepsilon - 1)(\pi_t - \chi\pi_t^*))x_{1,t} - \lambda_t y_t)\,\mathrm{d}t, \\ \mathrm{d}x_{2,t} &= ((\delta - f_U - \varepsilon(\pi_t - \chi\pi_t^*))\,x_{2,t} - mc_t\lambda_t y_t)\,\mathrm{d}t, \\ \mathbb{E}_t\left(\mathrm{d}\mathbb{Z}_t\right) &= g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\,\mathrm{d}t, && (51) \end{aligned}$$

as the *conditional* deterministic system, in which the inflation rate π_t is endogenously determined from (18) and (15), together with the auxiliary variables $x_{1,t}$ and $x_{2,t}$. Observe that the system coincides with the perfect-foresight solution if $\|\sigma(\mathbb{Z}_t)\| = 0$.

Alternatively, we use the stationary system (46) and subtract the Itô second-order

terms from the dynamics for the transformed costate $\mathrm{d}\hat{\lambda}_t$ and replace (49) and (50) by

$$\begin{aligned}\mathbb{E}_t(\mathrm{d}\hat{\lambda}_t) - \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\,\mathrm{d}t &= \mathbb{E}_t(\mathrm{d}\hat{\mathbb{Z}}_t^\top \hat{\lambda}_{\hat{\mathbb{Z}}}),\\ \mathbb{E}_t\,(\mathrm{d}V) - \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\mathrm{d}t &= \mathbb{E}_t(\mathrm{d}\hat{\mathbb{Z}}_t^\top V_{\hat{\mathbb{Z}}}),\end{aligned}$$

such that we may solve the system of PDEs,

$$\begin{aligned}\mathrm{d}\hat{\lambda}(\hat{\mathbb{Z}}_t) &= -(f_U + i_t - \pi_t - x_t + \sigma_t^2 - \sigma_t\tilde{\lambda}_\pi\sigma_{\pi,z})\hat{\lambda}_t\mathrm{d}t\\ &\quad -\tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\,\mathrm{d}t,\\ \mathrm{d}V(\hat{\mathbb{Z}}_t) &= -f(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t, V(\hat{\mathbb{Z}}_t; \hat{\mathbb{Y}}_t))\,\mathrm{d}t - \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})\mathrm{d}t,\\ \mathrm{d}x_{1,t} &= ((\delta - f_U - (\varepsilon-1)(\pi_t - \chi\pi_t^*))x_{1,t} - \hat{\lambda}_t\hat{y}_t)\mathrm{d}t,\\ \mathrm{d}x_{2,t} &= ((\delta - f_U - \varepsilon(\pi_t - \chi\pi_t^*))\,x_{2,t} - mc_t\hat{\lambda}_t\hat{y}_t)\mathrm{d}t,\\ \mathbb{E}_t(\,\mathrm{d}\hat{\mathbb{Z}}_t) &= g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)\,\mathrm{d}t.\end{aligned}$$

Hence, while $\lambda(\mathbb{Z}_t)$ approaches a balanced growth path, $V(\mathbb{Z}_t)$ and $\hat{\lambda}(\hat{\mathbb{Z}}_t)$ are stationary.

With either approach, the steady-state interest rate, which is computed from the conditional deterministic system reads (for details see the Online Appendix):

$$i_{ss} = -(f_U - \pi_{ss} - \mu + \sigma_z^2 - \sigma_z\tilde{\lambda}_\pi\sigma_{\pi,z}) - \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_{ss})\sigma(\hat{\mathbb{Z}}_{ss})^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}).$$

Intuitively, the conditional deterministic system summarizes the dynamics under the presence of uncertainty, that agents internalize into their consumption-saving decision, conditional on no further shocks (conditional on the current information set). Agents do not change their decision as long as they remain on the stable manifold of the dynamic system, which is idle at the (stochastic) steady state value.[11]

We solve the system of PDEs by a waveform relaxation algorithm (dynamic iteration). The trick is to regard the system of PDEs as a system of functional ODEs, separating the time dimension from the solution in the policy space (cf. Posch and Trimborn, 2013). This separation allows us to parallelize the computation for improved performance. Following this idea, we compute the unknown derivatives starting from the perfect-foresight solution, $\sigma(\mathbb{Z}_t) = 0$, then fix $\tilde{\lambda}_{\mathbb{Z}} \equiv \lambda_{\mathbb{Z}}/\lambda_t$, $\tilde{\lambda}_{\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}}/\hat{\lambda}_t$, and $\tilde{\lambda}_{\mathbb{Z}\mathbb{Z}} \equiv \lambda_{\mathbb{Z}\mathbb{Z}}/\lambda_t$, $\tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/\hat{\lambda}_t$, and solve the system of ODEs. The initial value for the control and/or jump variables is used to approximate the solution in the policy function space (using tensor products of univariate grids as initial values), then update the solution, and either iterate until convergence or use the two-step approximate solution as discussed below (cf. Table 1).

The transitional dynamics account for risk and the policy functions are being solved globally, thus may differ considerably along the transition towards the steady state. We

[11] Though there will be a steady-state distribution, we follow the convention in the literature and define the fix point of this system as the 'stochastic steady state', and thus use both terms interchangeably.

compare the approach to the certainty equivalent linear approximation (for example used for the sequence-space Jacobians as in Auclert, Bardóczy, Rognlie, and Straub, 2021), and the solution two-step approach where we fixed the policy functions at the certainty equivalent (nonlinear) solution and computed the risk effects from that approximation. Hence, our approach could be developed along those lines to balance computational time and accuracy or address potential problems due to the curse of dimensionality.

In the boundary value problem (BVP) we seek a function $x : [0, T] \mapsto \mathbb{R}^k$ that satisfies the (conditional) deterministic system for given initial conditions for the states $\mathbb{Z}_0$ and the TVC assuming that variables approach their (stochastic) balanced growth path. Because the time horizon is infinite, we may use a time transformation:

$$t = \frac{\tau}{\nu(1-\tau)} \quad \text{for} \quad \tau \in [0, 1),$$

where ν is a positive (nuisance) parameter, such that for $t \to \infty$ we have that $\tau \to 1$. Alternatively, we may set T sufficiently large but finite number.[12]

2.2.3. Sequence-space Jacobians and two-step approximate solutions

Note that the recursion as set out in Table 1 is only required if we are interested in the solution of the stochastic model where $\|\sigma(\mathbb{Z}_t)\| \neq 0$. In this section we elaborate on approximate solutions based on the perfect-foresight equilibrium mappings of aggregate variables around the steady state (cf. Auclert, Bardóczy, Rognlie, and Straub, 2021). Those ideas allow us to efficiently compute the transitional dynamics by rapid computation of sequence-space Jacobians.[13] Alternatively, we may also fix the derivatives $\lambda_{\mathbb{ZZ}}$ and $V_{\mathbb{ZZ}}$ at the steady state and compute the effects of risk for $\|\sigma(\mathbb{Z}_t)\| > 0$, either using the perfect-foresight solution (henceforth two-step approximate solution) or iteratively from the risk-adjusted solution. The latter approaches keep the nonlinear solution for aggregate state variables, but assume constant effects of uncertainty.[14] Another approach, closer to sequence-space Jacobians, is to use first-order perturbation, which keeps second-order effects *and* a linear model (cf. Parra-Alvarez, Polattimur, and Posch, 2021).

Below, we compare the different solutions based on the first-order approximation, the two-step approximate solution, and the iterative approximate solution to the full nonlinear solutions of the PDEs in (38) and (37). While in the first approach we solve a linear system, the other approaches use the nonlinear dynamics of aggregate states.

[12] Trimborn, Koch, and Steger (2008) introduced the relaxation algorithm to applications in economics. In contrast to their approach, we use projection methods to solve the boundary value problem, which turns out to be relatively efficient and (even for a few approximation points) highly accurate.

[13] We fix the first-order derivatives $\lambda_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ at the steady state, which then implies that $\lambda_{\mathbb{ZZ}} = V_{\mathbb{ZZ}} = 0$.

[14] It still reflects the solution of the stochastic model and hence accounts for precautionary savings.

2.2.4. Numerical solution in the policy function space

We may alternatively solve the HJB equation (37) directly by collocation based on the Matlab `CompEcon` toolbox (Judd, 1992; Miranda and Fackler, 2002).

Since the functional form of the solution is unknown, a strategy for solving the HJB equation is to directly approximate $V(\mathbb{Z}_t; \mathbb{Y}(\mathbb{Z}_t)) \approx \phi(\mathbb{Z}_t)v$, in which v is an n-vector of coefficients and ϕ is the $n \times n$ interpolation matrix, the tensor product of the univariate bases at n_d nodes for dimension $d = 1, ..., \dim(\mathbb{Z}_t)$. The computational burden is reduced when replacing the tensor product by sparse grids (Winschel and Krätzig, 2010). Starting from the HJB equation (37), we may approximation of the value function and/or control variables for given set of collocation nodes and basis functions $\phi(\mathbb{Z}_t)$:

$$0 = f(\mathbb{Z}_t, \mathbb{X}_t, \phi(\mathbb{Z}_t)v) + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}))^\top \phi_{\mathbb{Z}}(\mathbb{Z}_t)v + \tfrac{1}{2}\text{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z}_t)v\right),$$

where $\phi_{\mathbb{Z}}$ and $\phi_{\mathbb{Z}\mathbb{Z}}$ are the known derivatives (multidimensional arrays) of the interpolation matrix evaluated at the specified nodes. Although this approach typically requires a good initial guess, it is extremely useful to verify whether an implied solution, e.g., obtained from the conditional deterministic system indeed solves the HJB equation.

2.3. The flexible-price limit

An important benchmark solution to the NK model is the case where prices become more flexible. In the frictionless limit, $\delta \to \infty$, the household's *value function* still is defined as in (30), and the firms set prices to maximize profits

$$\max_{p_{it}} \left(\frac{p_{it}}{p_t} y_{it} - mc_t y_{it}\right) \quad \text{s.t.} \quad y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t.$$

After dropping constants, we may write the first-order condition as:

$$\frac{p_{it}}{p_t} = \frac{\varepsilon}{\varepsilon - 1} mc_t$$

Hence, in the flexible-price case, all firms set prices $p_{it} = p_t$ such that $v_t \equiv 1$. Given the value of marginal costs, we compute consumption and hours from (45) and (13):

$$c_t = (1 - s_g s_{g,t})^{\frac{1/\eta+\vartheta-1}{1/\eta+\vartheta}} \left((\varepsilon - 1)/(\psi\varepsilon)\right)^{\frac{1}{1/\eta+\vartheta}} A_t^{\frac{1+\vartheta}{1/\eta+\vartheta}} z_t \tag{52}$$

such that $l_t = \left((\varepsilon - 1)/((1 - s_g s_{g,t})\psi\varepsilon)\right)^{\frac{1}{1/\eta+\vartheta}} A_t^{\frac{1-1/\eta}{1/\eta+\vartheta}}$ and the (natural) output:

$$y_t^n = (1 - s_g s_{g,t})^{-\frac{1}{1/\eta+\vartheta}} A_t^{\frac{1+\vartheta}{1/\eta+\vartheta}} \left((\varepsilon - 1)/(\psi\varepsilon)\right)^{\frac{1}{1/\eta+\vartheta}} z_t. \tag{53}$$

Based on the definition, the output gap is readily available from $x_t \equiv y_t/y_t^n - 1$.

Proposition 5 (Value function). *In the frictionless limit, $\delta \to \infty$, the value function of the representative consumer with recursive preferences in (1) reads*

$$V(\mathbb{Z}_t) = \frac{1}{1-\gamma}\left(\frac{1-1/\eta}{1-\gamma} d_t r_t^{1-\frac{1-1/\eta}{1-\gamma}} (1-\gamma)(c_t/z_t)^{-1/\eta}/z_t a_t + \frac{1-1/\eta}{1-\gamma} h\right)^{\frac{1-\gamma}{1-1/\eta}}, \tag{54}$$

for $\gamma \neq 1$ and $\eta \neq 1$, which for $\gamma = 1/\eta$ reads $V(\mathbb{Z}_t) = d_t(c_t/z_t)^{-1/\eta}/z_t a_t + h/(1-\gamma)$, and

$$\lim_{1/\eta \to 1} V(\mathbb{Z}_t) = \frac{1}{1-\gamma} \exp\left(d_t(1-\gamma) r_t c_t^{-1} a_t + h(d_t, A_t, s_{g,t}, r_t)\right), \quad \gamma \neq 1, \tag{55}$$

or

$$\lim_{1/\eta \to 1} V(\mathbb{Z}_t) = d_t c_t^{-1} a_t + h(d_t, A_t, s_{g,t}, r_t), \quad \gamma = 1, \tag{56}$$

where c_t is given by (52) and $h = h(d_t, A_t, s_{g,t}, r_t)$ satisfies the PDE

$$\begin{aligned} 0 &= ((1-\gamma)V)^{\frac{1-1/\eta}{1-\gamma}} V^{-1} f(\mathbb{Z}_t, \mathbb{X}_t, V) + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}); \mathbb{Y}))^\top h_{\mathbb{Z}} + \tfrac{1}{2} tr\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top h_{\mathbb{Z}\mathbb{Z}}\right) \\ &\quad + \left(1 - \frac{1-1/\eta}{1-\gamma}\right) ((1-\gamma)V)^{-\frac{1-1/\eta}{1-\gamma}} \tfrac{1}{2} tr\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top h_{\mathbb{Z}} h_{\mathbb{Z}}^\top\right), \quad \gamma \neq 1 \end{aligned} \tag{57}$$

or $-f(\mathbb{Z}_t, \mathbb{X}_t, V) = g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t; \mathbb{Y}); \mathbb{Y}))^\top h_{\mathbb{Z}} + \frac{1}{2} tr\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top h_{\mathbb{Z}\mathbb{Z}}\right)$ for $\gamma = 1/\eta = 1$.

Proof. Online Appendix I.8 ■

3. Results

Our results confirm previous findings that the effects of uncertainty in the NK model on the policy functions and macroeconomic aggregates are small, and primarily a level shift. Both the deterministic and the stochastic steady states are very close. In what follows, we consider the implications of price rigidities and fundamental shocks for the term structure of interest rate and the premium investors require for bearing economic risks.

3.1. Effects of uncertainty and the natural rate of interest

After having obtained the effects of uncertainty (either approach) on the costate and value functions and/or the consumption function, we use Itô's formula

$$\begin{aligned} \mathrm{d}\lambda_t &= g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \lambda_{\mathbb{Z}} \,\mathrm{d}t + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right) \mathrm{d}t + \lambda_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t \\ &= g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}} \lambda_t \,\mathrm{d}t + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}\right) \lambda_t \mathrm{d}t + \tilde{\lambda}_{\mathbb{Z}}^\top \sigma(\mathbb{Z}_t)\lambda_t \,\mathrm{d}B_t, \end{aligned}$$

where the implicit Euler equation (39) pins down the equilibrium inflation rate as

$$\pi_t = f_U + i_t + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}), \tag{58}$$

or equivalently

$$\pi_t = f_U + i_t - x_t + \sigma_t^2 - \sigma_t \sigma_\pi \tilde{\lambda}_\pi \sigma_{\pi,z} + \tfrac{1}{2}\sigma_{\pi,z}^2 \tilde{\lambda}_{\pi\pi} + g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}}\, \mathrm{d}t + \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}})$$

to study the effects of uncertainty on the consumption-saving decision and interest rates, compute impulse response functions, and the implied term structure of interest rates.

In general equilibrium, the risk-free rate is defined as (cf. Posch, 2011)

$$-\frac{1}{\mathrm{d}t}\mathbb{E}_t\left[\frac{\mathrm{d}m_t}{m_t}\right] = -f_U(\mathbb{Z}_t, \mathbb{X}_t, V) - \frac{1}{\mathrm{d}t}\mathbb{E}_t\left[\frac{\mathrm{d}\lambda_t}{\lambda_t}\right] \equiv r_t^f.$$

Observe that the *implicit* risk premium in the economy is zero because the instantaneous real return to the bond is riskless. Even though the zero-coupon bond is default-free, it is still risky in the sense that its price can covary with the household's marginal utility of consumption (e.g., Rudebusch and Swanson, 2008, p.115).

In the short-run, the nominal policy rate, $i_t \equiv r_t^f + \pi_t$, is under the control of the monetary authority.[15] Our result in (58) can be used to show how the equilibrium *real* interest rate, r_t^f, is affected by the different state variables. Using (58) we conclude:

$$r_t^f = -f_U - g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}} - \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}). \tag{59}$$

Below we compare the effects of uncertainty for the different solutions. We start with the first-order approximation where $\frac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \lambda_{\mathbb{Z}\mathbb{Z}}\right) = 0$, and then discuss the nonlinear solution, when evaluated at the same reference point. By construction, there is no effect on the steady-state interest rate in the first-order approximation, such that

$$r_{ss}^f = \rho, \quad i_{ss} = \rho + \pi_{ss}$$

and neither for the macroeconomic aggregates (because of certainty equivalence). In the nonlinear solution, $g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}}$ is not necessarily zero at the (deterministic) steady state for $\sigma(\mathbb{Z}_t) \neq 0$, which also implies that $-\frac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}) \neq 0$. As a key result, the equilibrium real interest rate (or the natural rate of interest) is different, even when evaluated at the same reference point (the deterministic steady state).

A key insight is that the shape of the term structure crucially depends on the solution method. In what follows, we compute moments of the limiting distribution for the variables of interest and show how local approximations can be misleading in studying the model-implied term structure of interest rates and moments of the ergodic distribution. Most importantly, we are mainly interested in identifying which fundamental shocks create "unfavorable shifts" in the relevant state varibles. An unfavorable shift is a change to

[15] Having the nominal rate i_t under the control of the monetary authority (henceforth policy rate), is a modelling assumption and reflects the idea of an overnight facility.

a state typically with lower consumption (higher costate, SDF or "hunger"), i.e.,

$$\tilde{\lambda}_{\mathbb{Z}_j} > 0, \quad j = 1, \ldots, n_d \tag{60}$$

or the market price of risk associated with state j.

For examply, an increase in technology puts downward pressure on prices causing marginal costs and inflation to fall, and generates higher output and consumption, which in turn decreases the marginal value of wealth (households feel wealthier), $\tilde{\lambda}_A < 0$. In contrast, an adverse monetary policy shock, i.e., higher nominal interest rates, cause consumption to fall through intertemporal substitution, which in turn lowers inflation, but increases the marginal value of wealth, $\tilde{\lambda}_i > 0$. It implies a correlation between consumption and inflation that is exactly opposite (cf. Rudebusch and Swanson, 2012).

Following Barsky, Justiniano, and Melosi (2014), we define the natural rate as the real interest rate prevailing in an economy with flexible prices, or the second-best equilibrium (cf. Blanchard and Galí, 2007), i.e., defining λ_t^n as the frictionless limits:

$$r_t^n \;=\; -f_U - g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{\lambda}_{\mathbb{Z}}^n - \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{\lambda}_{\mathbb{Z}\mathbb{Z}}^n) \tag{61}$$

where f_U and $g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)$ are evaluated at the flexible price values.

3.1.1. Term structure of interest rates

Following Rudebusch and Swanson (2012), the term premium on long-term nominal bonds compensates investors for inflation and consumption risk over the lifetime of the bond. The term premium is defined by comparing the equilibrium prices under the physical and the risk-neutral probability measures. Consider a (zero-coupon) nominal bond with unity payoff at maturity N. Using the expectation approach, the equilibrium price reads:

$$P_t^{(N)} = \mathbb{E}_t \left(m_{t+N}/m_t e^{-\int_t^{t+N} \pi_s ds} \right), \tag{62}$$

which can be solved, by simulating for a given maturity N forward a few thousand times, and take the average. Similarly, we could obtain the hypothetical bond price under the risk-neutral probability measure, i.e., using a risk-neutral discount factor (or equivalently the risk-free rate) rather than the household's SDF:

$$\bar{P}_t^{(N)} = \mathbb{E}_t^{\mathbb{Q}} \left(m_{t+N}/m_t e^{-\int_t^{t+N} \pi_s ds} \right) = \mathbb{E}_t^{\mathbb{Q}} \left(e^{-\int_t^{t+N} i_s \, \mathrm{d}s} \right) \tag{63}$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the expectation under the risk-neutral probability measure and thus the risk-neutral evaluation of the bond price (Rudebusch and Swanson, 2008, 2012). Then, the yield is the (fictional) interest rate that justifies the quoted price, such that the log price $p_t^{(N)} \equiv \log P_t^{(N)}$ satisfies $y_t^{(N)} = -(1/N)p_t^{(N)}$, and the yield curve is a plot of the

yields as a function of their maturity. Hence, the expectation approach is easily adapted for the perfect-foresight solution (the term premium would be zero).[16]

An alternative solution offers the PDE approach (Cochrane, 2005, chap. 19.4), in which the basic pricing equation for the price $P_t^{(N)}$ reads

$$\mathbb{E}_t\left(\left(\mathrm{d}P_t^{(N)}\right)/P_t^{(N)}\right) - \left(1/P_t^{(N)}(\partial P_t^{(N)}/\partial N) + i_t\right)\mathrm{d}t = -\mathbb{E}_t\left(\left(\mathrm{d}P_t^{(N)}/P_t^{(N)}\right)\left(\mathrm{d}\lambda_t/\lambda_t\right)\right).$$

where $-P_N^{(N)}/P_t^{(N)}$ is the instantaneous forward rate, and for the nominal zero-coupon bond we have that $\lim_{N\to 0}(\partial P_t^{(N)}/\partial N)/P_t^{(N)} = -i_t$. The pricing equation is an application of Feynman-Kac to compute the conditional expectation by PDEs (cf. Appendix A.1).

Proposition 6. *From the PDE approach we obtain the covariance with the SDF as*

$$-\mathbb{E}_t\left(\left(\,dP_t^{(N)}/P_t^{(N)}\right)\left(\,d\lambda_t/\lambda_t\right)\right) \;=\; -\tilde{\lambda}_{\mathbb{Z}}^{\top}\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top}\tilde{P}_{\mathbb{Z}}^{(N)}\,dt,$$

where $\tilde{P}_{\mathbb{Z}}^{(N)} \equiv P_{\mathbb{Z}}^{(N)}/P_t^{(N)}$.

Proof. Online Appendix I.10 ■

Following the PDE approach and Proposition 6, the fundamental PDE reads

$$\begin{aligned} g(\mathbb{Z}_t,\mathbb{X}_t;\mathbb{Y}_t)^{\top}P_{\mathbb{Z}}^{(N)} + \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top}P_{\mathbb{Z}\mathbb{Z}}^{(N)}) - (\partial P_t^{(N)}/\partial N) - P_t^{(N)}i_t \\ = -\tilde{\lambda}_{\mathbb{Z}}^{\top}\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top}P_{\mathbb{Z}}^{(N)} \end{aligned} \tag{64}$$

where for the hypothetical price $\bar{P}_t^{(N)}$ the covariance terms on the RHS are zero. Focusing on the covariance has a deeper motivation: Understanding whether fundamental shocks increase or decrease either the term premium or inflation risk premium crucially depends on the sensitivity of the bond price, $\tilde{P}_{\mathbb{Z}}$, because $\tilde{\lambda}_{\mathbb{Z}}$ is endogenously given in the NK model, and both sensitivities are largely independent from either the size of the shocks or the correlation structure summarized in the covariance matrix $\Sigma(\mathbb{Z}) \equiv \sigma(\mathbb{Z})\sigma(\mathbb{Z})^{\top}$.

Proposition 7 (Affine Term Structure Model). *Observe that in the ATSM the prices* $\ln P_t^{(N)}$ *(and* $\ln \bar{P}_t^{(N)}$*) are linear functions of the state variables,*

$$\ln P_t^{(N)} \approx \frac{A(N)}{N} - \frac{B(N)^{\top}}{N}\mathbb{Z}_t, \quad \textit{or} \quad P_t^{(N)} \approx e^{\frac{A(N)}{N} - \frac{B(N)^{\top}}{N}\mathbb{Z}_t}$$

where using the denominator layout (vector of derivatives as column vector)

$$\partial(\ln P^{(N)})/\partial\mathbb{Z}_t = -\frac{B(N)}{N}, \quad \tilde{P}_{\mathbb{Z}}^{(N)} = -\frac{B(N)}{N}$$

[16] The expectation approach is used in macroeconomic models (Gürkaynak, Sack, and Swanson, 2005; Rudebusch and Swanson, 2008, 2012), but is challenging to compute the complete term structure.

such that $\tilde{P}_{\mathbb{ZZ}}$ depends entirely on the outer product of $B(N)/N$,

$$\tilde{P}_{\mathbb{Z}}^{(N)} = \partial(\ln P^{(N)})/\partial\mathbb{Z}_t, \quad \textit{and} \quad \tilde{P}_{\mathbb{ZZ}}^{(N)} = -\tilde{P}_{\mathbb{Z}}^{(N)}\frac{B(N)}{N}^{\top} = \frac{B(N)B(N)^{\top}}{N^2}.$$

Hence

$$\tfrac{1}{2}tr(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top}\tilde{P}_{\mathbb{ZZ}}^{(N)}) = \tfrac{1}{2}tr((\tilde{P}_{\mathbb{Z}}^{(N)})^{\top}\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top}\tilde{P}_{\mathbb{Z}}^{(N)})$$

where $\tilde{P}_{\mathbb{Z}}^{(N)} \equiv P_{\mathbb{Z}}^{(N)}/P_t^{(N)}$, and $\tilde{P}_{\mathbb{ZZ}}^{(N)} \equiv P_{\mathbb{ZZ}}^{(N)}/P_t^{(N)}$.

Proof. Online Appendix I.11 ■

Most importantly, the covariance term in the fundamental pricing equation for the ATSM is the same as in the nonlinear version (cf. Online Appendix I.11)

$$g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^{\top}P_{\mathbb{Z}}^{(N)} + \tfrac{1}{2}\mathrm{tr}((P_{\mathbb{Z}}^{(N)})^{\top}\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top}\tilde{P}_{\mathbb{Z}}^{(N)}) - (\partial P_t^{(N)}/\partial N) - P_t^{(N)}i_t = -\lambda_{\mathbb{Z}}^{\top}\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top}P_{\mathbb{Z}}^{(N)}, \tag{65}$$

such that we do not have to rely on second-order derivatives,

$$P_{\mathbb{ZZ}}^{(N)} = -P_{\mathbb{Z}}^{(N)}\frac{B(N)}{N}^{\top} = P_t^{(N)}\frac{B(N)B(N)^{\top}}{N^2} = P_{\mathbb{Z}}^{(N)}(P_{\mathbb{Z}}^{(N)})^{\top}/P_t^{(N)}$$

or

$$\tilde{P}_{\mathbb{ZZ}}^{(N)} = \tilde{P}_{\mathbb{Z}}^{(N)}(\tilde{P}_{\mathbb{Z}}^{(N)})^{\top} \tag{66}$$

In either case (ATSM or nonlinear), however, we use the same strategy as it allows us to solve a linear system based on the fundamental pricing equation. Our results confirm the link in (66). Note also that for the ATSM we need an approximate solution for the costate as an affine function of the state variables (such that $\lambda_{\mathbb{Z}}$ is constant).

Our strategy is to approximate $P^{(N)} \approx \Phi(N, \mathbb{Z})u$, where $\Phi = \Phi_0 \otimes \Phi_{n_d} \otimes \Phi_{n_d-1} \otimes \cdots \otimes \Phi_1$ is the $n \times n$ interpolation matrix, i.e., the tensor product of the univariate interpolation matrices Φ_j based on $1 \times n_j$ elementary vectors $\phi_j(Z_j)$ for $j = 0, ..., n_d$, each producing

$$\phi(N, \mathbb{Z}) \equiv \phi_N(N) \otimes \phi_{n_d}(\mathbb{Z}_{n_d}) \otimes \phi_{n_d-1}(\mathbb{Z}_{n_d-1}) \otimes \cdots \otimes \phi_1(\mathbb{Z}_1),$$

row vectors of $n \equiv \prod_{i=0}^{n_d} n_i$ columns, properly stacked, such that each row of Φ corresponds to one k of the n nodal values of $S \equiv \{N, \mathbb{Z}\}$, or $\phi(N_k, \mathbb{Z}_{1k}, \ldots, \mathbb{Z}_{n_dk})$. Hence, the $n_j \times n_j$ interpolation matrices Φ_j evaluate the $1 \times n_j$ elementary vectors at their univariate nodes, each row corresponding to one of the n_j nodal values of either N or $\mathbb{Z}_1, \ldots, \mathbb{Z}_{n_d}$.

Suppose that we construct a $n \times n_d + 1$ grid S from univariate nodes N and $\mathbb{Z}$,

$$S \equiv \{N, \mathbb{Z}\} = \begin{bmatrix} N_1 & \mathbb{Z}_{1,1} & \mathbb{Z}_{1,2} & \dots & \mathbb{Z}_{1,n_d} \\ N_2 & \mathbb{Z}_{1,1} & \mathbb{Z}_{1,2} & \dots & \mathbb{Z}_{1,n_d} \\ \vdots & & & & \\ N_{n_0} & \mathbb{Z}_{1,1} & \mathbb{Z}_{1,2} & \dots & \mathbb{Z}_{1,n_d} \\ N_1 & \mathbb{Z}_{2,1} & \mathbb{Z}_{1,2} & \dots & \mathbb{Z}_{1,n_d} \\ N_2 & \mathbb{Z}_{2,1} & \mathbb{Z}_{1,2} & \dots & \mathbb{Z}_{1,n_d} \\ \vdots & & & & \\ N_{n_0} & \mathbb{Z}_{2,1} & \mathbb{Z}_{1,2} & \dots & \mathbb{Z}_{1,n_d} \\ N_1 & \mathbb{Z}_{3,1} & \mathbb{Z}_{1,2} & \dots & \mathbb{Z}_{1,n_d} \\ \vdots & & & & \\ N_{n_0} & \mathbb{Z}_{n_1,1} & \mathbb{Z}_{n_2,2} & \dots & \mathbb{Z}_{n_{n_d},n_d} \end{bmatrix}$$

in which the lth row represents one nodal point in the $n_d + 1$ dimensional space $\mathbb{R}^{n_d+1}$.

Similarly, we require $P^{(N)}_{\mathbb{Z}_j} \approx \Phi_{\mathbb{Z}_j} u$, $\partial P^{(N)}/\partial N \approx \Phi_N u$, $P^{(N)}_{\mathbb{Z}_j}$, and $P^{(N)}_{\mathbb{Z}_i\mathbb{Z}_j} \approx \Phi_{\mathbb{Z}_i\mathbb{Z}_j} u$, to approximate the first- and second-order derivatives, respectively,

$$\sum_{j=1}^{n_d} \text{diag}(g_j(S))\Phi_{\mathbb{Z}_j} u + \tfrac{1}{2}\sum_{j=1}^{n_d} \text{diag}(\Sigma_{jj})\Phi_{\mathbb{Z}_j\mathbb{Z}_j} u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \text{diag}(\Sigma_{jk})\Phi_{\mathbb{Z}_j\mathbb{Z}_k} u$$
$$-\Phi_N u - \text{diag}(i)\Phi u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d} \text{diag}(\tilde{\lambda}_{\mathbb{Z}_j}(S))\text{diag}(\Sigma_{jk})\Phi_{\mathbb{Z}_k} u,$$

where

$$\Sigma_{jk}(\mathbb{Z}) \equiv \sum_{i=1}^{n_d} \sigma_{ji}(\mathbb{Z})\sigma_{ki}(\mathbb{Z}),$$

or by defining $\circ$ (element-wise operation) of an n-vector and an $n \times k$ matrix as multiplying corresponding elements, operating along the k columns of the matrix, such that

$$\sum_{j=1}^{n_d} g_j(S) \circ \Phi_{\mathbb{Z}_j} u + \tfrac{1}{2}\sum_{j=1}^{n_d} \Sigma_{jj} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_j} u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \Sigma_{jk}(\mathbb{Z}) \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_k} u$$
$$-\Phi_N u - i \circ \Phi u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d} (\tilde{\lambda}_{\mathbb{Z}_j} \circ \Sigma_{jk}) \circ \Phi_{\mathbb{Z}_k} u$$

where component functions $g_j(\mathbb{Z}, \mathbb{X}; \mathbb{Y})$ are evaluated at the $n \times 1$ vector of n nodal values, and i is the n-vector along the grid S. Each of the matrices of partial derivatives $\Phi_{\mathbb{Z}_k}$ and $\Phi_{\mathbb{Z}_k\mathbb{Z}_l}$ is a $n \times n$, and each Σ_{jk} is $n \times 1$ and $\tilde{\lambda}_{\mathbb{Z}_j}(S)$ is $n \times 1$.

Given the linearity of the Feynman-Kac equation and the boundary conditions, a collocation solution requires solving a linear matrix equation, $\mathbf{\Phi} u = 0_n$ together with the

boundary condition $\Phi(0,\mathbb{Z})u = 1_n$, where

$$\mathbf{\Phi} \equiv \sum_{j=1}^{n_d} g_j \circ \Phi_{\mathbb{Z}_j} + \tfrac{1}{2}\Sigma_{jj} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_j} + \sum_{k=1+j}^{n_d} \Sigma_{jk} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_k} + \sum_{k=1}^{n_d} (\tilde{\lambda}_{\mathbb{Z}_j} \circ \Sigma_{jk}) \circ \Phi_{\mathbb{Z}_k} - \Phi_N - i \circ \Phi.$$

So we may concatenate the matrices and solve the linear $2n \times n$ equation

$$\left[\begin{array}{c} \mathbf{\Phi} \\ \Phi(0,\mathbb{Z}) \end{array}\right] u \;=\; \left[\begin{array}{c} 0_n \\ 1_n \end{array}\right] \quad \Leftrightarrow \quad u = \left[\begin{array}{c} \mathbf{\Phi} \\ \Phi(0,\mathbb{Z}) \end{array}\right]^{-1} \left[\begin{array}{c} 0_n \\ 1_n \end{array}\right]. \tag{67}$$

The solution to (64), or its approximation (67), implies the term structure of interest rates for *any* given state vector and maturity of (nominal) zero-coupon Treasury bonds,

$$y_t^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) = -\log P^{(N)}(\mathbb{Z}_t)/N, \tag{68}$$

or its risk-neutral version

$$\bar{y}_t^{(N)} \equiv \bar{y}^{(N)}(\mathbb{Z}_t) = -\log \bar{P}^{(N)}(\mathbb{Z}_t)/N. \tag{69}$$

A similar approach with $\lim_{N\to 0}(\partial P_{R,t}^{(N)}/\partial N)/P_{R,t}^{(N)} = -r_t^f$ implies the term structure of interest rates $r^{(N)}(\mathbb{Z}_t)$ and its risk neutral version $\bar{r}^{(N)}(\mathbb{Z}_t)$ for Treasury inflation-protected securities (TIPS), or the price of the real zero-coupon Treasury bonds. Based on those general equilibrium prices, many asset pricing facts are simply computed.

3.1.2. Forward interest rates

The use of forward interest rates is quite common in central banks among other indicators to assess the market expectations about future monetary policy. Although they contain the same information as the standard yield curve, forward rates more easily allow a separation of expectations for the short, medium and long term (cf. Svensson, 1994).

For example, we may compute the *one-year* forward rate $f_t^{N\to N+1}$ as

$$f_t^{N\to N+1} \;=\; \log P_t^{(N)} - \log P_t^{(N+1)} \equiv p_t^{(N)} - p_t^{(N+1)}$$

which is a forward contract fixed today of investing a dollar in N years into a zero-coupon bond and holding that bond for the remaining time to maturity $N+1$.

We define the instantaneous forward rate $f_t^{(N)}$ as the rate for a forward contract with an infinitesimal investment period after maturity N. It can be approximated with an overnight forward rate, i.e., with maturity one day after settlement (cf. Svensson, 1994),

$$f_t^{(N)} = -\partial \log P_t^{(N)}/\partial N = -1/P_t^{(N)} \partial P_t^{(N)}/\partial N \tag{70}$$

such that the fundamental pricing equation has the following interpretation

$$\mathbb{E}_t\left(\left(\mathrm{d}P_t^{(N)}\right)/P_t^{(N)}\right) - \left(-f_t^{(N)} + i_t\right)\mathrm{d}t = -\mathbb{E}_t\left(\left(\mathrm{d}P_t^{(N)}/P_t^{(N)}\right)\left(\mathrm{d}\lambda_t/\lambda_t\right)\right),$$

where we may define the instantaneous holding period return (hpr),

$$\begin{aligned}\mathrm{hpr}_t^{(N)} &= \mathrm{d}P_t^{(N)}/P_t^{(N)} - 1/P_t^{(N)}(\partial P_t^{(N)}/\partial N)\mathrm{d}t \\ &= \mathrm{d}P_t^{(N)}/P_t^{(N)} + f_t^{(N)}\mathrm{d}t,\end{aligned}$$

such that the expected $\mathrm{hpr}_t^{(N)}$ of the zero-coupon bond with maturity N to equal the spot rate of the short-term bond plus the premium of the long-term bond, which compensates the investor that i_t does not evolve as expected (related to its riskiness)

$$\mathbb{E}_t(\mathrm{hpr}_t^{(N)}) = i_t - \mathbb{E}_t\left(\left(\mathrm{d}P_t^{(N)}/P_t^{(N)}\right)\left(\mathrm{d}\lambda_t/\lambda_t\right)\right).$$

From our solution

$$\mathbb{E}_t(\mathrm{hpr}_t^{(N)}) = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top \tilde{P}_{\mathbb{Z}}^{(N)} + \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \tilde{P}_{\mathbb{Z}\mathbb{Z}}^{(N)}) + f_t^{(N)}$$

where

$$f_t^{(N)} = -(\partial P_t^{(N)}/\partial N)/P_t^{(N)}.$$

For example, consider the benchmark long-term bond in the US data, which is 10-year Treasury note (40-quarter zero-coupon bond) in Rudebusch and Swanson (2012). Even though the nominal bond is default-free, it is still risky in the sense that its price co-varies with the household's marginal utility of consumption. In the model, the covariance with the SDF is negative in the sense that the price generally falls for adverse shocks, i.e., when the SDF increases. Hence, households regard holding the bond as being riskier than rolling over short-term bonds and demand a premium over the short rate $i_t^{(N)}$.

3.1.3. Term premium and inflation risk premium

We define the (level of the) term premium

$$TP_{N,t}^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) - \bar{y}^{(N)}(\mathbb{Z}_t), \tag{71}$$

and similarly

$$TP_{R,t}^{(N)} \equiv r^{(N)}(\mathbb{Z}_t) - \bar{r}^{(N)}(\mathbb{Z}_t), \tag{72}$$

which measures the 'riskiness' of the nominal and the real zero-coupon bonds, i.e., the compensation that investors require for bearing the risk that i_t or r_t^f does not evolve as expected, respectively. We may decompose nominal term premium into the term premium

of the real bond and the inflation risk premium, i.e., the premium that investors pay for bearing the risk that π_t does not evolve as expected. Hence

$$\begin{aligned} RP_{\pi,t}^{(N)} &= TP_{N,t}^{(N)} - TP_{R,t}^{(N)} \\ &= y^{(N)}(\mathbb{Z}_t) - r^{(N)}(\mathbb{Z}_t) - (\bar{y}^{(N)}(\mathbb{Z}_t) - \bar{r}^{(N)}(\mathbb{Z}_t)), \end{aligned} \tag{73}$$

which follows Haubrich, Pennacchi, and Ritchken (2012).

Defining the forward interest rate $\bar{f}_t^{(N)}$ under the risk-neutral probability measure as

$$\bar{f}_t^{(N)} = \partial \log \bar{P}_t^{(N)} / \partial N = -1/\bar{P}_t^{(N)} \partial P_t^{(N)} / \partial N$$

we may define the forward (nominal) term premium

$$TP_{f,t}^{(N)} \equiv f_t^{(N)} - \bar{f}_t^{(N)}, \tag{74}$$

which is another measure for the compensation for risk (the excess return), and refers to the covariance with the SDF in Proposition 6.[17] Note that we may approximate

$$\begin{aligned} TP_t^{(N)} &\approx \mathbb{E}_t(\text{hpr}_t^{(N)}) - i_t \\ &= -\tilde{\lambda}_{\mathbb{Z}}^{\top} \sigma(\mathbb{Z}_t) \sigma(\mathbb{Z}_t)^{\top} \tilde{P}_{\mathbb{Z}}^{(N)} \\ &= -\tilde{\lambda}_{\mathbb{Z}}^{\top} \Sigma(\mathbb{Z}_t) \tilde{P}_{\mathbb{Z}}^{(N)} \;\equiv\; TP_{f,t}^{(N)}, \end{aligned} \tag{75}$$

which is the covariance measure of the term premium (or forward term premium).

Moreover, we may use (75) to approximate the inflation risk premium (73) by

$$RP_{\pi,t}^{(N)} \approx -\tilde{\lambda}_{\mathbb{Z}}^{\top} \Sigma(\mathbb{Z}_t) (\tilde{P}_{\mathbb{Z},N}^{(N)} - \tilde{P}_{\mathbb{Z},R}^{(N)})$$

i.e., which shocks generate 'good' versus 'bad' inflation (cf. Cieslak and Pflueger, 2023). Any unfavorable shock $\lambda_{\mathbb{Z}_j} > 0$ with $j = 1, \ldots, n_d$ creates higher term premia for the Treasury bonds iff its covariance with the SDF is positive, i.e., the sensitivity of the bond price is negative, $P_{\mathbb{Z}_j}^{(N)} < 0$ (diagonal covariance matrix), or the total effect of the jth shock on the bond price is negative, $\Sigma_j(\mathbb{Z}) P_{\mathbb{Z}}^{(N)} < 0$ (full covariance matrix).

Although the term premium is the most accurate measure of the riskiness of long-term bonds, it is not directly observable. Rudebusch and Swanson (2008, 2012) suggest that, for example, the average slope of the term structure serves as a useful empirical proxy for the term premium. However, the effectiveness of these measures depends on the type of shock. While the average slope of the term structure is a reliable indicator in the case of technology shocks, it does not appear to be a valid proxy for preference shocks. Hence,

[17] As argued in Svensson (1994), there are three related but distinct term premia, the forward term premium, the holding period term premium, and the rollover term premium.

such measures are only applicable for specific types of shocks (see Tables 1 and 2). For illustration, we may compute the slope of the yield curve as

$$TS_t^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) - i_t \tag{76}$$

such that $\mathbb{E}_t(TS_{t+k}^{(N)}) = \mathbb{E}_t(y^{(N)}(\mathbb{Z}_{t+k})) - \mathbb{E}_t(i_{t+k})$ and

$$\mathbb{E}(TS_t^{(N)}) = \lim_{t\to\infty} \mathbb{E}_t(y^{(N)}(\mathbb{Z}_t)) - \mathbb{E}(i_t) = \mathbb{E}(y^{(N)}(\mathbb{Z})) - \mathbb{E}(i) \tag{77}$$

denotes the average slope of the term structure.

3.1.4. Sign restrictions in VARs

When deriving sign restrictions for shock identification in VARs, we need to study the sensitivities of nominal or real yields with respect to shocks, e.g., $TP_{\mathbb{Z}}^{(N)} = y_{\mathbb{Z}}^{(N)} - \bar{y}_{\mathbb{Z}}^{(N)}$. For example, technology shocks imply

$$\partial y_t^{(N)}/\partial A_t \equiv y_A^{(N)} = -\frac{P_A^{(N)}}{P^{(N)}}/N = -\tilde{P}_A^{(N)}/N$$

and

$$\partial \bar{y}_t^{(N)}/\partial A_t \equiv \bar{y}_A^{(N)} = -\frac{P_A^{(N)}}{P^{(N)}}/N = -\tilde{\bar{P}}_A^{(N)}/N$$

for the risk neutral evaluation. Hence, we obtain

$$\partial TP_t^{(N)}/\partial A_t = -(\tilde{P}_A^{(N)} - \tilde{\bar{P}}_A^{(N)})/N$$

Similar to Rudebusch and Swanson (2012), the NK model generates countercyclical term premium (technology shocks).[18]

3.1.5. Model-implied forecasts

In fact, the model-implied inflation forecast is a forward contract to inflation, which can be more informative than using forward rates (Gürkaynak, Sack, and Wright, 2007). Similar to the term structure of interest rate, we compute the π_{t+N} as a function of the current state variables $\mathbb{Z}_t$ and the forecasting horizon N. For the N-year ahead future expected inflation rate, we compute $\pi_t^{(N)}$ from (cf. Appendix A.1)

$$\partial \pi_t^{(N)}/\partial N \;\; = \;\; g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)^\top \pi_{\hat{\mathbb{Z}}}^{(N)} + \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top \pi_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^{(N)})$$

[18] In the flexible price limit, the sign flips and generate counterfactual responses, i.e., technology shocks would decrease the nominal term premium and generates a procyclical premium.

together with the solution $\hat{\mathbb{X}}_t = \hat{\mathbb{X}}(\hat{\mathbb{Z}}_t)$ and $\hat{\mathbb{Y}}_t = \hat{\mathbb{Y}}(\hat{\mathbb{Z}}_t)$ and by imposing the boundary condition $\pi_t^{(0)} = \pi_t = \pi(\hat{\mathbb{Z}}_t)$. Similar to the term structure of interest rates, the solution to the PDE implies the N-years ahead inflation expectations for given state variables as

$$\pi_t^{(N)} = \pi^{(N)}(\hat{\mathbb{Z}}_t). \tag{78}$$

Because the model time unit is years, we may relate the N-year ahead inflation forecast $\pi_t^{(N)}$ to the empirical NY1Y measure. Similarly, we obtain the model-implied forecast for any other variable, such as the expected prices/yields or the expected state variables, or even their limiting distribution and any moment $\mathbb{E}(\hat{\mathbb{Z}}_t^k)$ for $k \geq 0$.

Consider the N-year ahead forecast for the spot rate, i.e., the expected instantaneous forward rate in N years hence, $i_t^{(N)}$, which is a function of the forecasting horizon and the current states $\hat{\mathbb{Z}}_t$. We may solve Kolmogorov's backward equation, which reads

$$\partial i_t^{(N)}/\partial N \;\; = \;\; g(\hat{\mathbb{Z}}_t, \hat{\mathbb{X}}_t; \hat{\mathbb{Y}}_t)^\top i_{\hat{\mathbb{Z}}}^{(N)} + \tfrac{1}{2}\mathrm{tr}(\sigma(\hat{\mathbb{Z}}_t)\sigma(\hat{\mathbb{Z}}_t)^\top i_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}^{(N)})$$

with boundary condition $i_t^{(0)} = i_t$. Moreover, as shown above, the expected spot rate

$$i_t^{(N)} \equiv \mathbb{E}_t(\mathrm{hpr}_t^{(N)}) - TP_{f,t}^{(N)},$$

or, equivalently, the holding period return under the risk-neutral evaluation.

Our strategy allows us to compute the coefficients from a *linear* equation where we approximate $i_t^{(N)} \approx \Phi(N, \hat{\mathbb{Z}}_t)u$ in which u is an n-vector of coefficients, and Φ denotes the known $n \times n$ basis matrix, the tensor product of the univariate interpolation matrices, and $\Phi_{\mathbb{Z}}$ and $\Phi_{\mathbb{Z}\mathbb{Z}}$ are the known multidimensional arrays of derivatives of the interpolation matrix, evaluated at the specified nodes (cf. Section 3.1.1). For example,

$$\sum_{j=1}^{n_d} g_j(\hat{\mathbb{Z}}, \hat{\mathbb{X}}; \hat{\mathbb{Y}}) \circ \Phi_{\hat{\mathbb{Z}}_j} u + \tfrac{1}{2}\sum_{j=1}^{n_d} \Sigma_{jj} \circ \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_j} u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \Sigma_{jk}(\hat{\mathbb{Z}}) \circ \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k} u = \Phi_N u$$

or

$$\sum_{j=1}^{n_d} \mathrm{diag}(g_j(S))\Phi_{\mathbb{Z}_j} u + \tfrac{1}{2}\sum_{j=1}^{n_d} \mathrm{diag}(\Sigma_{jj})\Phi_{\mathbb{Z}_j\mathbb{Z}_j} u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \mathrm{diag}(\Sigma_{jk})\Phi_{\mathbb{Z}_j\mathbb{Z}_k} u = \Phi_N u,$$

or more compactly, we require that at each collocation node $l \in 1, \ldots, n$,

$$g(\mathbb{Z}_l, \mathbb{X}_l; \mathbb{Y}_l)^\top \Phi_{\mathbb{Z}} u + \tfrac{1}{2}\mathrm{tr}(\sigma(\mathbb{Z}_l)\sigma(\mathbb{Z}_l)^\top \Phi_{\mathbb{Z}\mathbb{Z}} u) - \Phi_N u = 0$$

together with the n boundary conditions

$$\Phi(0, \hat{\mathbb{Z}}_l) = i_l 1_n, \quad l \in 1, \ldots, n$$

such that the linear $2n \times n$ equation for the unknown coefficients, defining

$$\mathbf{\Phi} \equiv \sum_{j=1}^{n_d} g_j \circ \Phi_{\mathbb{Z}_j} + \tfrac{1}{2}\Sigma_{jj} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_j} + \sum_{k=1+j}^{n_d} \Sigma_{jk} \circ \Phi_{\mathbb{Z}_j\mathbb{Z}_k} - \Phi_N,$$

and

$$\begin{bmatrix} \mathbf{\Phi} \\ \Phi(0,\mathbb{Z}) \end{bmatrix} u = \begin{bmatrix} 0_n \\ i \end{bmatrix} \quad \Leftrightarrow \quad u = \begin{bmatrix} \mathbf{\Phi} \\ \Phi(0,\mathbb{Z}) \end{bmatrix}^{-1} \begin{bmatrix} 0_n \\ i \end{bmatrix}, \tag{79}$$

where i denotes the $n \times 1$ vector in $S \in \mathbb{R}^{n_d}$. Similar to the term-structure of interest rates, the solution to (79) the N-period-ahead conditional expectation (forecast) for the variables of interest, e.g., the nominal interest rate i_t, given a particular state $\mathbb{Z}_0$. Under certain regularity conditions, the forecast converges to its unconditional mean.

3.1.6. Term structure of inflation

The term structure of inflation shows how expected inflation depends on the forecasting horizon. How is the measure related to empirical measure of break-even inflation?

Having computed nominal and TIPS yields of zero-coupon Treasury bonds, we can solve for the rates of inflation compensation. Those are the inflation rates which, if realized, would leave an investor indifferent between holding a real and a nominal Treasury security (Gürkaynak, Sack, and Wright, 2010, p.76). For the continuously compounded inflation compensation rate at maturity N, the breakeven inflation rate is,

$$BE_t^{(N)} \equiv y^{(N)}(\mathbb{Z}_t) - r^{(N)}(\mathbb{Z}_t), \tag{80}$$

The breakeven inflation rate is also used for computing the inflation risk premium in (73) by subtracting the risk-neutral breakeven inflation rate. Hence, this measure shows the model-implied expected inflation (only if inflation risk premium is negligible).

Below we show the implementation for models in order to shed light on the effects of structural shocks on term premia and the inflation risk premium. The latter compensates the investors of nominal bonds for the risk that inflation does not evolve as expected. If nominal bonds tend to provide higher returns in bad economic states, inflation would be good to hedge against unfavorable shifts (and would generate a negative premium).

3.2. Data

Our data is from the Federal Reserve Bank of St. Louis Economic Dataset (FRED), and the Cleveland Fed's Center for Inflation Research from 1990 to 2025.[19] The dataset primarily consists of macroeconomic indicators, including monthly consumption (PCE), inflation based on consumer price index (CPI) and personal consumption expenditures (PCEPI). Additionally, it includes wages (compensation of employees), hours (average weekly hours), and the output gap derived from quarterly US Gross Domestic Product (cf. Figure 1).[20] The financial market data encompasses the US Effective Federal Funds Rate (Fed Funds), the N-Year Treasury Constant Maturity Rate (Govt), and the N-Year Treasury Inflation Protected Securities Rate (TIPS). It also includes fitted data from Gürkaynak, Sack, and Wright (2007, 2010), Laubach and Williams (2003), and Haubrich, Pennacchi, and Ritchken (2012).[21]

[19] For details see the FRED: https://fred.stlouisfed.org and the Cleveland Fed's Center for Inflation Research: https://www.clevelandfed.org/indicators-and-data/inflation-expectations

[20] Since our results are not sensitive to the choice of data, we simply use HP-filtered data as a proxy for the output gap (see Figure 2 for alternative measures, cf. Kamber, Morley, and Wong, 2018).

[21] Available at https://www.federalreserve.gov/data/yield-curve-models.htm for the nominal yield curve (feds200628.csv from Gürkaynak, Sack, and Wright, 2007), the TIPS yield curve and inflation compensation (feds200805.csv from Gürkaynak, Sack, and Wright, 2010), the three-factor nominal term structure model (feds200533.csv from Kim and Wright, 2005), the inflation expectations and real rates (expectations.xlsx from Haubrich, Pennacchi, and Ritchken, 2012), and estimates of the natural rate from Laubach and Williams (2003): https://www.newyorkfed.org/research/policy/rstar

Table 1: Empirical and Model-Based Unconditional Moments (Preference Shocks)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	ATSM	Model Two-Step	Nonlinear	Steady-states determ.	stoch.
state variables							
$\mathbb{E}(i)/100$		0.0850	0.0700	0.0678	0.0678	0.0700	0.0744
$SD(i)/100$		0.0244	0.0257	0.0269	0.0269		
$\mathbb{E}(d)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(d)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(r)/100$		0.0300	0.0300	0.0300	0.0300	0.0300	0.0300
$SD(r)/100$		0.0109	0.0109	0.0109	0.0109		
macro data							
$SD(c)$	1.464	0.000	1.304	1.529	1.529		
$SD(l)$	0.466	0.000	1.304	1.796	1.796		
$SD(w)$	1.592	0.000	3.527	4.541	4.542		
interest rates							
$\mathbb{E}(i)$	2.845	8.501	7.000	6.776	6.780	7.000	7.445
$SD(i)$	2.364	2.444	2.570	2.691	2.691		
$\mathbb{E}(i-\pi)$	0.677	5.126	5.000	4.942	4.946	5.000	5.445
$SD(i-\pi)$	2.235	3.241	2.286	2.286	2.285		
$\mathbb{E}(r^n)$	2.052	5.126	5.000	4.731	5.126		
$SD(r^n)$	0.854	3.241	1.352	3.747	3.241		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	8.492	7.009	6.785	6.789	7.087	7.323
$\mathbb{E}(y^{(5)})$	3.757	8.362	7.004	6.785	6.789	6.917	6.980
$\mathbb{E}(y^{(10)})$	4.250	8.179	6.976	6.765	6.769	6.834	6.865
$SD(y^{(10)})$	1.963	1.306	0.816	0.779	0.778		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.322	−0.024	−0.011	−0.011	−0.611	−0.579
$SD(y^{(10)}-i)$	1.315	1.349	1.860	2.034	2.034		
$\mathbb{E}(f^{(10)})$	5.570	7.864	6.936	6.736	6.741	6.740	6.740
$SD(f^{(10)})$	1.900	0.423	0.057	0.048	0.048		
$\mathbb{E}(TP^{(10)})$	0.836	−0.208	0.088	0.103	0.103	0.101	0.101
$SD(TP^{(10)})$	0.749	0.013	0.000	0.005	0.005		
inflation							
$\mathbb{E}(\pi)$	2.164	3.375	2.000	1.834	1.834	2.000	2.000
$SD(\pi)$	0.993	4.942	1.192	1.050	1.050		
$\mathbb{E}(BE^{(10)})$	2.136	2.838	1.971	1.816	1.816	1.820	1.806
$SD(BE^{(10)})$	0.394	0.804	0.029	0.029	0.029		
$\mathbb{E}(RP^{(10)})$	0.412	−0.583	0.023	0.028	0.028	0.027	0.028
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	8.502	7.000	6.776	6.780	7.043	7.149
$\mathbb{E}(\pi^{(1)})$	2.385	3.376	2.000	1.834	1.834	1.823	1.776

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\sigma_r = 0.01$, $\sigma_z = \mu = 0.02$, $\eta = 0.55$)

Table 2: Empirical and Model-Based Unconditional Moments (Technology Shocks)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0694	0.0700	0.0695	0.0695	0.0700	0.0696
$SD(i)/100$		0.0146	0.0070	0.0071	0.0071		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(r)/100$		0.0300	0.0300	0.0300	0.0300	0.0300	0.0300
$SD(r)/100$		0.0000	0.0000	0.0000	0.0000		
macro data							
$SD(c)$	1.464	0.773	0.701	0.703	0.703		
$SD(l)$	0.466	0.316	0.590	0.602	0.602		
$SD(w)$	1.592	1.046	1.112	1.134	1.134		
interest rates							
$\mathbb{E}(i)$	2.845	6.939	7.000	6.949	6.949	7.000	6.963
$SD(i)$	2.364	1.460	0.700	0.707	0.707		
$\mathbb{E}(i-\pi)$	0.677	4.952	5.000	4.952	4.952	5.000	4.963
$SD(i-\pi)$	2.235	0.593	0.924	0.926	0.926		
$\mathbb{E}(r^n)$	2.052	4.952	5.000	4.952	4.952		
$SD(r^n)$	0.854	0.593	0.322	0.593	0.593		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	6.937	7.003	6.952	6.952	6.981	6.960
$\mathbb{E}(y^{(5)})$	3.757	6.916	7.006	6.955	6.955	6.961	6.954
$\mathbb{E}(y^{(10)})$	4.250	6.890	7.007	6.955	6.955	6.958	6.955
$SD(y^{(10)})$	1.963	0.564	0.131	0.130	0.130		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.049	0.007	0.006	0.006	−0.005	−0.008
$SD(y^{(10)}-i)$	1.315	0.914	0.610	0.616	0.616		
$\mathbb{E}(f^{(10)})$	5.570	6.848	7.007	6.956	6.956	6.955	6.955
$SD(f^{(10)})$	1.900	0.139	0.009	0.009	0.009		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	0.011	0.011	0.011	0.011	0.011
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	1.987	2.000	1.996	1.996	2.000	2.000
$SD(\pi)$	0.993	1.656	0.329	0.327	0.327		
$\mathbb{E}(BE^{(10)})$	2.136	1.931	2.000	1.996	1.996	1.994	1.995
$SD(BE^{(10)})$	0.394	0.213	0.006	0.006	0.006		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.002	−0.002	−0.002	−0.002	−0.002
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	6.939	7.000	6.949	6.949	6.961	6.950
$\mathbb{E}(\pi^{(1)})$	2.385	1.987	2.000	1.996	1.996	1.989	1.992

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\sigma_i = 0.01$, $\sigma_A = 0.01$, $\sigma_z = 0.02$, $\mu = 0.02$, $\eta = 0.55$)

Table 3: Empirical and Model-Based Unconditional Moments (Stochastic Volatility)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	ATSM	Model Two-Step	Nonlinear	Steady-states determ.	stoch.
state variables							
$\mathbb{E}(i)/100$		0.0695	0.0700	0.0694	0.0694	0.0700	0.0695
$SD(i)/100$		0.0146	0.0070	0.0071	0.0071		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(X)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(X)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(S)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(S)$		0.0109	0.0109	0.0109	0.0109		
macro data							
$SD(c)$	1.464	0.773	0.701	0.704	0.703		
$SD(l)$	0.466	0.316	0.590	0.603	0.602		
$SD(w)$	1.592	1.046	1.112	1.136	1.133		
interest rates							
$\mathbb{E}(i)$	2.845	6.951	7.000	6.941	6.936	7.000	6.955
$SD(i)$	2.364	1.460	0.700	0.708	0.707		
$\mathbb{E}(i-\pi)$	0.677	4.940	5.000	4.944	4.940	5.000	4.955
$SD(i-\pi)$	2.235	0.595	0.924	0.926	0.926		
$\mathbb{E}(r^n)$	2.052	4.940	5.000	4.940	4.940		
$SD(r^n)$	0.854	0.595	0.323	0.595	0.595		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	6.949	7.003	6.944	6.939	6.977	6.951
$\mathbb{E}(y^{(5)})$	3.757	6.928	7.006	6.947	6.941	6.951	6.943
$\mathbb{E}(y^{(10)})$	4.250	6.902	7.007	6.947	6.942	6.947	6.943
$SD(y^{(10)})$	1.963	0.564	0.131	0.131	0.131		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.049	0.007	0.006	0.006	−0.008	−0.012
$SD(y^{(10)}-i)$	1.315	0.914	0.610	0.617	0.616		
$\mathbb{E}(f^{(10)})$	5.570	6.859	7.007	6.948	6.942	6.942	6.942
$SD(f^{(10)})$	1.900	0.139	0.009	0.009	0.009		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	0.011	0.011	0.011	0.011	0.011
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	2.011	2.000	1.996	1.995	2.000	2.000
$SD(\pi)$	0.993	1.658	0.329	0.327	0.328		
$\mathbb{E}(BE^{(10)})$	2.136	1.955	2.000	1.996	1.995	1.993	1.994
$SD(BE^{(10)})$	0.394	0.255	0.007	0.007	0.007		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.002	−0.002	−0.002	−0.002	−0.002
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	6.951	7.000	6.941	6.936	6.954	6.940
$\mathbb{E}(\pi^{(1)})$	2.385	2.011	2.000	1.996	1.995	1.986	1.991

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\sigma_i = 0.01$, $\sigma_A = 0.01$, $\sigma_x = 0.01$, $\sigma_z = 0.02$, $\mu = 0.02$, $\eta = 0.55$)

Table 4: Empirical and Model-Based Unconditional Moments (Long-Run Risk)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0694	0.0700	0.0694	0.0694	0.0700	0.0697
$SD(i)/100$		0.0160	0.0099	0.0104	0.0104		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(X)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(X)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\pi^*)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\pi^*)$		0.0109	0.0109	0.0109	0.0109		
macro data							
$SD(c)$	1.464	0.773	0.784	0.818	0.818		
$SD(l)$	0.466	0.316	0.687	0.771	0.771		
$SD(w)$	1.592	1.046	1.465	1.669	1.669		
interest rates							
$\mathbb{E}(i)$	2.845	6.939	7.000	6.938	6.938	7.000	6.971
$SD(i)$	2.364	1.604	0.989	1.041	1.041		
$\mathbb{E}(i-\pi)$	0.677	4.952	5.000	4.946	4.946	5.000	4.971
$SD(i-\pi)$	2.235	0.593	1.291	1.302	1.302		
$\mathbb{E}(r^n)$	2.052	4.952	5.000	4.952	4.952		
$SD(r^n)$	0.854	0.593	0.323	0.593	0.593		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	6.937	7.002	6.939	6.939	6.977	6.9608
$\mathbb{E}(y^{(5)})$	3.757	6.914	6.993	6.930	6.930	6.939	6.9337
$\mathbb{E}(y^{(10)})$	4.250	6.882	6.985	6.922	6.922	6.926	6.9237
$SD(y^{(10)})$	1.963	0.696	0.263	0.264	0.264		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.057	−0.015	−0.016	−0.016	−0.045	−0.0482
$SD(y^{(10)}-i)$	1.315	0.974	0.786	0.840	0.840		
$\mathbb{E}(f^{(10)})$	5.570	6.828	6.974	6.911	6.911	6.911	6.9113
$SD(f^{(10)})$	1.900	0.201	0.019	0.019	0.019		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	−0.002	−0.002	−0.002	−0.002	−0.0025
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	1.987	2.000	1.992	1.992	2.000	2.000
$SD(\pi)$	0.993	1.785	1.420	1.411	1.411		
$\mathbb{E}(BE^{(10)})$	2.136	1.923	1.973	1.964	1.964	1.962	1.963
$SD(BE^{(10)})$	0.394	0.263	0.082	0.082	0.082		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.025	−0.025	−0.025	−0.025	−0.025
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	6.939	7.000	6.938	6.938	6.954	6.945
$\mathbb{E}(\pi^{(1)})$	2.385	1.987	2.000	1.992	1.992	1.981	1.983

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\mu = 0.02$, $\sigma_z = 0.02$, $\sigma_A = 0.01$, $\sigma_i = 0.01$, $\sigma_x = 0.01$, $\sigma_\pi = 0.01$, $\eta = 0.55$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,z} = \sigma_{\pi,x} = 0$)

Table 5: Empirical and Model-Based Unconditional Moments (Long-Run Risk)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0694	0.0700	0.0689	0.0689	0.0700	0.0695
$SD(i)/100$		0.0208	0.0168	0.0181	0.0181		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(X)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(X)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\pi^*)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\pi^*)$		0.0244	0.0244	0.0244	0.0244		
macro data							
$SD(c)$	1.464	0.773	1.101	1.221	1.221		
$SD(l)$	0.466	0.316	0.646	1.009	1.009		
$SD(w)$	1.592	1.046	2.197	2.847	2.847		
interest rates							
$\mathbb{E}(i)$	2.845	6.939	7.000	6.890	6.890	7.000	6.946
$SD(i)$	2.364	2.083	1.677	1.814	1.814		
$\mathbb{E}(i-\pi)$	0.677	4.952	5.000	4.904	4.904	5.000	4.946
$SD(i-\pi)$	2.235	0.593	1.836	1.888	1.888		
$\mathbb{E}(r^n)$	2.052	4.952	5.000	4.952	4.952		
$SD(r^n)$	0.854	0.593	0.322	0.593	0.593		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	6.937	6.993	6.882	6.882	6.951	6.920
$\mathbb{E}(y^{(5)})$	3.757	6.906	6.940	6.826	6.826	6.844	6.835
$\mathbb{E}(y^{(10)})$	4.250	6.850	6.902	6.788	6.788	6.797	6.792
$SD(y^{(10)})$	1.963	1.072	0.487	0.487	0.487		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.088	−0.098	−0.101	−0.101	−0.149	−0.154
$SD(y^{(10)}-i)$	1.315	1.184	1.231	1.382	1.382		
$\mathbb{E}(f^{(10)})$	5.570	6.748	6.854	6.741	6.741	6.740	6.740
$SD(f^{(10)})$	1.900	0.349	0.034	0.033	0.033		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	−0.050	−0.050	−0.050	−0.050	−0.050
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	1.987	2.000	1.986	1.986	2.000	2.000
$SD(\pi)$	0.993	2.226	2.921	2.909	2.909		
$\mathbb{E}(BE^{(10)})$	2.136	1.891	1.872	1.859	1.859	1.855	1.857
$SD(BE^{(10)})$	0.394	0.406	0.180	0.179	0.179		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.108	−0.108	−0.108	−0.108	−0.108
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	6.939	7.000	6.890	6.890	6.921	6.906
$\mathbb{E}(\pi^{(1)})$	2.385	1.987	2.000	1.986	1.986	1.969	1.975

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\mu = 0.02$, $\sigma_z = 0.02$, $\sigma_A = 0.01$, $\sigma_i = 0.01$, $\sigma_x = 0.01$, $\sigma_\pi = 0.01$, $\eta = 0.55$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,z} = \sigma_{\pi,x} = 0.01$, $\eta = 0.55$)

Table 6: Empirical and Model-Based Unconditional Moments (Long-Run Risk)

Unconditional moment	US data 1990-2025	Benchmark Flex-Price	Model ATSM	Model Two-Step	Model Nonlinear	Steady-states determ.	Steady-states stoch.
state variables							
$\mathbb{E}(i)/100$		0.0694	0.0700	0.0690	0.0690	0.0700	0.0706
$SD(i)/100$		0.0208	0.0175	0.0188	0.0188		
$\mathbb{E}(A)$		1.0001	1.0000	1.0001	1.0001	1.0000	1.0001
$SD(A)$		0.0109	0.0109	0.0109	0.0109		
$\mathbb{E}(X)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(X)$		0.0002	0.0002	0.0002	0.0002		
$\mathbb{E}(\pi^*)$		0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
$SD(\pi^*)$		0.0244	0.0244	0.0244	0.0244		
macro data							
$SD(c)$	1.464	0.773	1.006	1.129	1.129		
$SD(l)$	0.466	0.316	1.234	1.465	1.465		
$SD(w)$	1.592	1.046	2.597	3.153	3.153		
interest rates							
$\mathbb{E}(i)$	2.845	6.939	7.000	6.900	6.900	7.000	6.959
$SD(i)$	2.364	2.083	1.748	1.881	1.881		
$\mathbb{E}(i-\pi)$	0.677	4.952	5.000	4.940	4.940	5.000	4.959
$SD(i-\pi)$	2.235	0.593	2.543	2.566	2.566		
$\mathbb{E}(r^n)$	2.052	4.952	5.000		4.952		
$SD(r^n)$	0.854	0.593	0.322		0.593		
term structure							
$\mathbb{E}(y^{(1)})$	2.979	6.937	6.999	6.899	6.899	6.975	7.004
$\mathbb{E}(y^{(5)})$	3.757	6.906	6.941	6.838	6.838	6.852	6.858
$\mathbb{E}(y^{(10)})$	4.250	6.850	6.894	6.791	6.791	6.797	6.800
$SD(y^{(10)})$	1.963	1.072	0.563	0.565	0.565		
$\mathbb{E}(y^{(10)}-i)$	1.406	−0.088	−0.106	−0.109	−0.109	−0.262	−0.259
$SD(y^{(10)}-i)$	1.315	1.184	1.301	1.447	1.447		
$\mathbb{E}(f^{(10)})$	5.570	6.748	6.834	6.731	6.731	6.730	6.730
$SD(f^{(10)})$	1.900	0.349	0.041	0.040	0.040		
$\mathbb{E}(TP^{(10)})$	0.836	−0.007	−0.057	−0.056	−0.056	−0.056	−0.056
$SD(TP^{(10)})$	0.749	0.000	0.000	0.000	0.000		
inflation							
$\mathbb{E}(\pi)$	2.164	1.987	2.000	1.960	1.960	2.000	2.000
$SD(\pi)$	0.993	2.226	3.282	3.256	3.256		
$\mathbb{E}(BE^{(10)})$	2.136	1.891	1.857	1.818	1.818	1.814	1.812
$SD(BE^{(10)})$	0.394	0.406	0.185	0.185	0.185		
$\mathbb{E}(RP^{(10)})$	0.412	−0.020	−0.132	−0.132	−0.132	−0.132	−0.132
forecasts							
$\mathbb{E}(i^{(1)})$	2.912	6.939	7.000	6.900	6.900	6.933	6.944
$\mathbb{E}(\pi^{(1)})$	2.385	1.987	2.000	1.960	1.960	1.932	1.924

Notes: All variables are in percent expressed at an annual rate. Consumption and consumption-output ratios are based on quarterly data. Standard deviations were computed for logarithmic deviations from a Hodrick-Prescott trend. The short-term nominal interest rate, i_t, is the end-of-month federal funds rate from the Federal Reserve Board, reported in annualized percentage points. The sample runs from January, 1990, through August, 2020. ($\mu = 0.02$, $\sigma_z = 0.02$, $\sigma_A = 0.01$, $\sigma_i = 0.01$, $\sigma_x = 0.01$, $\sigma_\pi = 0.01$, $\eta = 0.55$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,z} = \sigma_{\pi,x} = -0.01$, $\eta = 0.55$)

3.3. 'Good' and 'Bad' shocks in the NK Model

Assets with a negative covariance between its payoffs with consumption can be used as a vehicle to hedge against unfavorable shifts, because they tend to pay off when the SDF is high. The reason is that they tend to provide higher returns when marginal utility is higher (often coinciding with bad economic times, i.e., in adverse situations) when the investor most need it to. Hence, investors are usually willing to accept lower expected returns for assets that hedge against unfavorable shifts. In what follows, we categorize fundamental economic shocks generating a positive risk premium as 'bad' shocks.

Cieslak and Pflueger (2023) define 'good' and 'bad' inflation, in which 'bad' inflation increases bond risk premia, while 'good' inflation decreases bond risk premia. The authors define a demand shock as anything that can increase consumption and output at a given risk-free rate, whereas a supply (or cost-push) shock captures any disturbance to the relationship between the output gap and marginal costs of production due to monopolistic competition.[22] They conclude that demand shocks generally lead to inflation of the 'good' variety and result in a negative inflation risk premium, while supply shocks lower the real value of long-term nominal bonds, generate inflation of the 'bad' variety, inducing a positive inflation risk premium. Based on our definition in (73), we can categorize the shocks in supply and demand shocks and demonstrate that the conclusion is more nuanced. Specifically, we show that monetary policy shocks, while producing 'good' inflation, actually increase the riskiness of nominal bonds by raising the term premium. This is due to the fact that the term premium on long-term real bonds rises more than that on nominal bonds. We find that technology shocks (supply-side shocks, which move output and inflation in opposite directions), preference shocks, and shocks to the natural rate (both demand-side shocks, which move output and inflation in the same direction) increase the nominal and real term premium and the inflation risk premium.

We ask which structural shocks induce "unfavorable shifts" as in (60). For example, by raising interest rates, monetary policy shocks make it more attractive to save and less attractive to borrow, increasing the marginal value of wealth, $\tilde{\lambda}_i > 0$. Such shocks are inducing an adverse situation, typically associated with lower consumption. The same holds for preference shocks, $\tilde{\lambda}_d > 0$, discount rate shocks, $\tilde{\lambda}_r > 0$, and shocks to economic uncertainty, $\tilde{\lambda}_\sigma > 0$. In contrast, supply shocks (or so-called cost-push), however, decrease the marginal value of wealth by lowering the marginal costs of production, and we obtain for temporary shocks to technology $\tilde{\lambda}_A < 0$, permanent shocks to technology $\tilde{\lambda}_z < 0$, shocks to technology growth $\tilde{\lambda}_x < 0$, and shocks to the inflation target, $\tilde{\lambda}_\pi < 0$.

Having identified the sensitivities of the marginal value of wealth, $\lambda_{\mathbb{Z}_j}$, we are ready to categorize the fundamental shocks based on the sign of $\Sigma_j(\mathbb{Z})P_{\mathbb{Z}}^{(N)}$, i.e., the change of

[22] Cieslak and Pflueger (2023) define (reduced-form) demand shocks as anything that enters the Euler equation, whereas supply shocks as appearing in the Phillips curve equation.

the bond price with respect to states $P_{\mathbb{Z}}^{(N)}$ (or percentage terms $\tilde{P}_{\mathbb{Z}}^{(N)}$) weighted by the jth line of the covariance matrix, hence from (75) the jth shock (risk) generates

$$TP_j^{(N)} \approx -\tilde{\lambda}_{\mathbb{Z}_j}\Sigma_j(\mathbb{Z})\tilde{P}_{\mathbb{Z}}^{(N)} \tag{81}$$

The presence of risk associated with state j will increase the term premium as long as the effects on the discount factor and yields are qualitatively the same (same direction). This decomposition holds for the price of both the nominal and the real zero-coupon Treasury bond. 'Bad' shocks may generate 'good' inflation, if they generate a higher term premium for real bonds than for nominal bonds, generating a negative inflation risk premium

$$RP_j^{(N)} \approx -\tilde{\lambda}_{\mathbb{Z}_j}\Sigma_j(\mathbb{Z})(\tilde{P}_{\mathbb{Z},N}^{(N)} - \tilde{P}_{\mathbb{Z},R}^{(N)}) \tag{82}$$

It is important to note that the decomposition is helpful for identifying the theoretical contribution and importance of shocks to the *level* of term premia and risk premia. This is complementary to sign restrictions of impulse responses for shock identification used in the literature, which sheds light on the cyclicality or variance decomposition in VARs.

Summarizing, 'good' shocks are those who decrease the term premium, i.e., they tend to move yields and the discount factor in opposite directions. Fundamental shocks with unfavorable shifts in the state j, where $\tilde{\lambda}_{\mathbb{Z}_j} > 0$, thus can be 'good' as long as they also decrease yields. Moreover, whether or not those fundamental shocks generate 'good' inflation (decrease the inflation risk premium), provided they move nominal and real yields in the same direction, depends quantitatively on the price sensitivities.[23]

3.3.1. Technology shocks

For example, the model with technology shocks for $\delta < \infty$ has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 3$ relevant states. We use n_0 nodal points for N, $n_1, \ldots, n_3$ for the relevant states $\hat{\mathbb{Z}} = (i, v, A)$, such that $n \equiv n_0 \times n_1 \times n_2 \times n_3 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 1890$ ($n = 70 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi}u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular,

[23] Rudebusch and Swanson (2012) study the term premium of long-term nominal bonds. From the IRFs of bond price and consumption, we may categorize their shocks into fundamental shocks of the 'good' and 'bad' variety as follows: For the fundamental shocks to monetary policy, fiscal policy and technology, the discount factor is negatively related to consumption. Hence, the term premium is larger, the more negative the covariance between the discount factor and the price of the bond, i.e., for fundamental shocks that move both consumption and bond prices in the same direction ('bad' shocks).

the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_{ss})/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi(S_l) - \chi\pi_{ss}) - \delta)\, v_l \\ -(\rho_A \log A_l - \frac{1}{2}\sigma_A^2)A_l \end{bmatrix}$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_A A_l \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 \end{bmatrix}$$

such that

$$\Sigma_{jk}(\hat{\mathbb{Z}}_l) \equiv \sigma_{j1}(\hat{\mathbb{Z}}_l)\sigma_{k1}(\hat{\mathbb{Z}}_l) + \sigma_{j2}(\hat{\mathbb{Z}}_l)\sigma_{k2}(\hat{\mathbb{Z}}_l) + \sigma_{j3}(\hat{\mathbb{Z}}_l)\sigma_{k3}(\hat{\mathbb{Z}}),$$

denotes the lth element in the $n \times 1$ matrix Σ_{jk} for $j = 1, \ldots, 3$ and $k = 1, \ldots, 3$, i.e.,

$$\Sigma_{1k}(\hat{\mathbb{Z}}_l) = \sigma_{11}(\hat{\mathbb{Z}}_l)\sigma_{k1}(\hat{\mathbb{Z}}_l) + \sigma_{12}(\hat{\mathbb{Z}}_l)\sigma_{k2}(\hat{\mathbb{Z}}_l) + \sigma_{13}(\hat{\mathbb{Z}}_l)\sigma_{k3}(\hat{\mathbb{Z}}_l)$$

or $\Sigma_{11}(\hat{\mathbb{Z}}) = \sigma_i^2 \cdot 1_n$, $\Sigma_{12}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{13}(\hat{\mathbb{Z}}) = 0_n$,

$$\begin{aligned} \Sigma_{11}(\hat{\mathbb{Z}}_l) &= \sigma_{11}(\hat{\mathbb{Z}}_l)\sigma_{11}(\hat{\mathbb{Z}}_l) + \sigma_{12}(\hat{\mathbb{Z}}_l)\sigma_{12}(\hat{\mathbb{Z}}_l) + \sigma_{13}(\hat{\mathbb{Z}}_l)\sigma_{13}(\hat{\mathbb{Z}}_l) = \sigma_i^2, \\ \Sigma_{12}(\hat{\mathbb{Z}}_l) &= \sigma_{11}(\hat{\mathbb{Z}}_l)\sigma_{21}(\hat{\mathbb{Z}}_l) + \sigma_{12}(\hat{\mathbb{Z}}_l)\sigma_{22}(\hat{\mathbb{Z}}_l) + \sigma_{13}(\hat{\mathbb{Z}}_l)\sigma_{23}(\hat{\mathbb{Z}}_l) = 0 \\ \Sigma_{13}(\hat{\mathbb{Z}}_l) &= \sigma_{11}(\hat{\mathbb{Z}}_l)\sigma_{31}(\hat{\mathbb{Z}}_l) + \sigma_{12}(\hat{\mathbb{Z}}_l)\sigma_{32}(\hat{\mathbb{Z}}_l) + \sigma_{13}(\hat{\mathbb{Z}}_l)\sigma_{33}(\hat{\mathbb{Z}}_l) = 0, \end{aligned}$$

$$\Sigma_{2k}(\hat{\mathbb{Z}}_l) = \sigma_{21}(\hat{\mathbb{Z}}_l)\sigma_{k1}(\hat{\mathbb{Z}}_l) + \sigma_{22}(\hat{\mathbb{Z}}_l)\sigma_{k2}(\hat{\mathbb{Z}}_l) + \sigma_{23}(\hat{\mathbb{Z}}_l)\sigma_{k3}(\hat{\mathbb{Z}}_l)$$

or $\Sigma_{21}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{22}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{23}(\hat{\mathbb{Z}}) = 0_n$,

$$\begin{aligned} \Sigma_{21}(\hat{\mathbb{Z}}_l) &= \sigma_{21}(\hat{\mathbb{Z}}_l)\sigma_{11}(\hat{\mathbb{Z}}_l) + \sigma_{22}(\hat{\mathbb{Z}}_l)\sigma_{12}(\hat{\mathbb{Z}}_l) + \sigma_{23}(\hat{\mathbb{Z}}_l)\sigma_{13}(\hat{\mathbb{Z}}_l) = 0 \\ \Sigma_{22}(\hat{\mathbb{Z}}_l) &= \sigma_{21}(\hat{\mathbb{Z}}_l)\sigma_{21}(\hat{\mathbb{Z}}_l) + \sigma_{22}(\hat{\mathbb{Z}}_l)\sigma_{22}(\hat{\mathbb{Z}}_l) + \sigma_{23}(\hat{\mathbb{Z}}_l)\sigma_{23}(\hat{\mathbb{Z}}_l) = 0 \\ \Sigma_{23}(\hat{\mathbb{Z}}_l) &= \sigma_{21}(\hat{\mathbb{Z}}_l)\sigma_{31}(\hat{\mathbb{Z}}_l) + \sigma_{22}(\hat{\mathbb{Z}}_l)\sigma_{32}(\hat{\mathbb{Z}}_l) + \sigma_{23}(\hat{\mathbb{Z}}_l)\sigma_{33}(\hat{\mathbb{Z}}_l) = 0 \end{aligned}$$

or $\Sigma_{31}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{32}(\hat{\mathbb{Z}}) = 0_n$, $\Sigma_{33}(\hat{\mathbb{Z}}) = \sigma_A^2 A^2$,

$$\begin{aligned} \Sigma_{31}(\hat{\mathbb{Z}}_l) &= \sigma_{31}(\hat{\mathbb{Z}}_l)\sigma_{11}(\hat{\mathbb{Z}}_l) + \sigma_{32}(\hat{\mathbb{Z}}_l)\sigma_{12}(\hat{\mathbb{Z}}_l) + \sigma_{33}(\hat{\mathbb{Z}}_l)\sigma_{13}(\hat{\mathbb{Z}}_l) = 0 \\ \Sigma_{32}(\hat{\mathbb{Z}}_l) &= \sigma_{31}(\hat{\mathbb{Z}}_l)\sigma_{21}(\hat{\mathbb{Z}}_l) + \sigma_{32}(\hat{\mathbb{Z}}_l)\sigma_{22}(\hat{\mathbb{Z}}_l) + \sigma_{33}(\hat{\mathbb{Z}}_l)\sigma_{23}(\hat{\mathbb{Z}}_l) = 0 \\ \Sigma_{33}(\hat{\mathbb{Z}}_l) &= \sigma_{31}(\hat{\mathbb{Z}}_l)\sigma_{31}(\hat{\mathbb{Z}}_l) + \sigma_{32}(\hat{\mathbb{Z}}_l)\sigma_{32}(\hat{\mathbb{Z}}_l) + \sigma_{33}(\hat{\mathbb{Z}}_l)\sigma_{33}(\hat{\mathbb{Z}}_l) = \sigma_A^2 A_l^2 \end{aligned}$$

together with the solution of the HJB equation, which yields

$$\hat{\lambda} \equiv \hat{\lambda}(\hat{\mathbb{Z}}), \quad V \equiv V(\hat{\mathbb{Z}}), \quad \pi \equiv \pi(\hat{\mathbb{Z}}), \quad \hat{y} \equiv \hat{y}(\hat{\mathbb{Z}})$$

and derivatives

$$\hat{\lambda}_{\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad V_{\hat{\mathbb{Z}}} \equiv V_{\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}(\hat{\mathbb{Z}}), \quad V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}(\hat{\mathbb{Z}})$$

with

$$\lambda_{\hat{\mathbb{Z}}} \equiv \begin{bmatrix} \hat{\lambda}_i(\hat{\mathbb{Z}}) \\ \hat{\lambda}_v(\hat{\mathbb{Z}}) \\ \hat{\lambda}_A(\hat{\mathbb{Z}}) \end{bmatrix}, \quad V_{\hat{\mathbb{Z}}} \equiv \begin{bmatrix} V_i(\hat{\mathbb{Z}}) \\ V_v(\hat{\mathbb{Z}}) \\ V_A(\hat{\mathbb{Z}}) \end{bmatrix},$$

$$\hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \begin{bmatrix} \hat{\lambda}_{ii}(\hat{\mathbb{Z}}) & \hat{\lambda}_{iv}(\hat{\mathbb{Z}}) & \hat{\lambda}_{iA}(\hat{\mathbb{Z}}) \\ \hat{\lambda}_{vi}(\hat{\mathbb{Z}}) & \hat{\lambda}_{vv}(\hat{\mathbb{Z}}) & \hat{\lambda}_{vA}(\hat{\mathbb{Z}}) \\ \hat{\lambda}_{Ai}(\hat{\mathbb{Z}}) & \hat{\lambda}_{Av}(\hat{\mathbb{Z}}) & \hat{\lambda}_{AA}(\hat{\mathbb{Z}}) \end{bmatrix}, \quad V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \begin{bmatrix} V_{ii}(\hat{\mathbb{Z}}) & V_{iv}(\hat{\mathbb{Z}}) & V_{iA}(\hat{\mathbb{Z}}) \\ V_{vi}(\hat{\mathbb{Z}}) & V_{vv}(\hat{\mathbb{Z}}) & V_{vA}(\hat{\mathbb{Z}}) \\ V_{Ai}(\hat{\mathbb{Z}}) & V_{Av}(\hat{\mathbb{Z}}) & V_{AA}(\hat{\mathbb{Z}}) \end{bmatrix}$$

or

$$\tilde{\lambda}_{\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}}/\hat{\lambda}, \quad \tilde{V}_{\hat{\mathbb{Z}}} \equiv V_{\hat{\mathbb{Z}}}/V, \quad \tilde{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv \hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/\hat{\lambda}, \quad \tilde{V}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}} \equiv V_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}/V$$

such that

$$\begin{aligned} \mathrm{tr}\Big(\Sigma(S_l)\hat{\lambda}_{\hat{\mathbb{Z}}\hat{\mathbb{Z}}}\Big) &= \begin{bmatrix} \sigma_i^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 \end{bmatrix} \begin{bmatrix} \hat{\lambda}_{ii}(S_l) & \hat{\lambda}_{iv}(S_l) & \hat{\lambda}_{iA}(S_l) \\ \hat{\lambda}_{vi}(S_l) & \hat{\lambda}_{vv}(S_l) & \hat{\lambda}_{vA}(S_l) \\ \hat{\lambda}_{Ai}(S_l) & \hat{\lambda}_{Av}(S_l) & \hat{\lambda}_{AA}(S_l) \end{bmatrix} \\ &= \sigma_i^2 \hat{\lambda}_{ii} + \sigma_A^2 A^2 \hat{\lambda}_{AA} \end{aligned}$$

and $\Phi_{\hat{\mathbb{Z}}_j}$ and $\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}$ where

$$\begin{aligned} P_N^{(N)}(S_l) &\approx \Phi_N(N_l, \hat{\mathbb{Z}}_l)u = \Phi_N(S_l)u, \quad l = 1, \ldots, n, \\ P_j^{(N)}(S_l) &\approx \Phi_{\hat{\mathbb{Z}}_j}(N_l, \hat{\mathbb{Z}}_l)u = \Phi_{\hat{\mathbb{Z}}_j}(S_l)u, \quad l = 1, \ldots, n, \quad j = 1, \ldots, d, \\ P_{jk}^{(N)}(S_l) &\approx \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(N_l, \hat{\mathbb{Z}}_l)u = \Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)u, \quad l = 1, \ldots, n, \quad j, k = 1, \ldots, d, \end{aligned}$$

evaluates the lth row of the basis matrix $\Phi(S_l)$, respectively its derivatives $\Phi_N(S_l)$, $\Phi_{\hat{\mathbb{Z}}_j}(S_l)$, or $\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)$, such that at each collocation node $S_l \in \mathbb{R}^4$ (each lth row of the grid S)

$$\begin{aligned} &\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\hat{\mathbb{Z}}_j}(S_l)u + \tfrac{1}{2}\sum_{j=1}^{n_d} \Sigma_{jj}(S_l)\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_j}(S_l)u + \sum_{j=1}^{n_d}\sum_{k=1+j}^{n_d} \Sigma_{jk}(S_l)\Phi_{\hat{\mathbb{Z}}_j\hat{\mathbb{Z}}_k}(S_l)u \\ &-\Phi_N(S_l)u - i_l\Phi(S_l)u = -\sum_{j=1}^{n_d}\sum_{k=1}^{n_d} \tilde{\lambda}_{\mathbb{Z}_j}(S_l)\Sigma_{jk}(S_l)\Phi_{\mathbb{Z}_k}(S_l)u \end{aligned} \tag{83}$$

Hence, we may construct the collocation matrix $\mathbf{\Phi}$, whose typical lth elements are

$$\begin{aligned} &\sum_{j=1}^{n_d} g_j(S_l)\Phi_{\hat{\mathbb{Z}}_j}(S_l) + \tfrac{1}{2}\left(\Sigma_{11}(S_l)\Phi_{\hat{\mathbb{Z}}_1\hat{\mathbb{Z}}_1}(S_l) + \Sigma_{33}(S_l)\Phi_{\hat{\mathbb{Z}}_3\hat{\mathbb{Z}}_3}(S_l)\right) - \Phi_N(S_l) - i_l\Phi(S_l) \\ &+\tilde{\lambda}_{\hat{\mathbb{Z}}_1}(S_l)\Sigma_{11}(S_l)\Phi_{\hat{\mathbb{Z}}_1}(S_l) + \tilde{\lambda}_{\hat{\mathbb{Z}}_3}(S_l)\Sigma_{33}(S_l)\Phi_{\hat{\mathbb{Z}}_3}(S_l) \end{aligned}$$

or

$$g_1(S_l)\Phi_i(S_l) + g_2(S_l)\Phi_v(S_l) + g_3(S_l)\Phi_A(S_l) + \tfrac{1}{2}\left(\sigma_i^2\Phi_{ii}(S_l) + \sigma_A^2 A_l^2 \Phi_{AA}(S_l)\right)$$
$$-\Phi_N(S_l) - i_l\Phi(S_l) + \tilde{\lambda}_i\sigma_i^2\Phi_i(S_l) + \tilde{\lambda}_A A_l^2\sigma_A^2\Phi_A(S_l)$$

such that $\mathbf{\Phi}u = 0$ together with the n boundary conditions requiring $\Phi(0, \hat{\mathbb{Z}})u = 1$.

Table 4 confirms our presumption that $\tilde{\lambda}_A < 0$, and $\lambda_z < 0$ such that only monetary policy shocks create unfavorable shifts, $\tilde{\lambda}_i > 0$. Because prices of existing bonds tend to fall if interest rates rise (effects more sensitive for assets with longer duration), yields have a negative covariance with consumption and hence create positive term premium. Vice versa, prices of existing bonds tend to rise with positive (temporary) technology shocks and also create positive term premium.

3.3.2. Long-run risks

In the long-run risks (LRR) model, embedded in the NK model with time-varying inflation target for $\delta < \infty$, has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 5$ relevant states. We use n_0 nodal points for N, $n_1, \ldots, n_5$ for the relevant states $\hat{\mathbb{Z}} = (i, v, A, x, \pi^*)$, such that $n \equiv n_0 \times n_1 \times n_2 \times n_3 \times n_3 \times n_4 \times n_5 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 17010$ ($n = 70 \times 3 \times 3 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi}u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular, the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_l^*) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_l^*)/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi(S_l) - \chi\pi_l^*) - \delta)\, v_l \\ -(\rho_A \log A_l - \tfrac{1}{2}\sigma_A^2)A_l \\ \rho_x(\mu - x_l) \\ \rho_\pi(\pi_{ss} - \pi_l^*) \end{bmatrix}$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A A_l & 0 & 0 \\ 0 & 0 & 0 & \sigma_z\sigma_x & 0 \\ \sigma_{\pi,i} & 0 & \sigma_{\pi,A} & \sigma_{\pi,x} & \sigma_\pi + \sigma_{\pi,z} \end{bmatrix},$$

or

$$\sigma(S_l)\sigma(S_l)^\top = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 & \sigma_i\sigma_{\pi,i} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 & 0 & \sigma_A A_l \sigma_{\pi,A} \\ 0 & 0 & 0 & \sigma_z^2\sigma_x^2 & \sigma_z\sigma_x\sigma_{\pi,x} \\ \sigma_i\sigma_{\pi,i} & 0 & \sigma_A A_l \sigma_{\pi,A} & \sigma_z\sigma_x\sigma_{\pi,x} & \sigma_\pi^2 + \sigma_{\pi,i}^2 + \sigma_{\pi,A}^2 + \sigma_{\pi,x}^2 + \sigma_{\pi,z}^2 \end{bmatrix}$$

3.3.3. Preference shocks

In the model with preference shocks for $\delta < \infty$ has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 4$ relevant states. We use n_0 nodal points for N, $n_1, \ldots, n_4$ for the relevant states $\hat{\mathbb{Z}} = (i, v, d, r)$, such that $n \equiv n_0 \times n_1 \times n_2 \times n_3 \times n_4 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 5670$ ($n = 70 \times 3 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi} u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular, the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_{ss})/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi(S_l) - \chi\pi_{ss}) - \delta)\, v_l \\ -(\rho_d \log d_l - \frac{1}{2}\sigma_d^2) d_l \\ \rho_r(\rho - r_l) \end{bmatrix}$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_d d_l & 0 \\ 0 & 0 & 0 & \sigma_r \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_d^2 d_l^2 & 0 \\ 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}$$

3.3.4. Stochastic volatility

In the model with stochastic volatility for $\delta < \infty$ has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 5$ relevant states. We use n_0 nodal points for N, $n_1, \ldots, n_5$ for the relevant states $\hat{\mathbb{Z}} = (i, v, A, x, \sigma)$, such that $n \equiv n_0 \times n_1 \times \ldots \times n_5 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 17010$ ($n = 70 \times 3 \times 3 \times 3 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi} u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular,

the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ \delta\left(1 + (1-\varepsilon)(\pi(S_l) - \chi\pi_{ss})/\delta\right)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon(\pi(S_l) - \chi\pi_{ss}) - \delta)\, v_l \\ -(\rho_A \log A_l - \frac{1}{2}\sigma_A^2)A_l \\ \rho_x(\mu - x_l) \\ \rho_\sigma(\sigma_z - \sigma_l) \end{bmatrix}$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A A_l & 0 & 0 \\ 0 & 0 & 0 & \sigma_l \sigma_x & 0 \\ 0 & 0 & 0 & 0 & \sigma_\sigma \end{bmatrix}, \quad \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_A^2 A_l^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_l^2 \sigma_x^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\sigma^2 \end{bmatrix}$$

3.3.5. Flexible prices

The flexible-price limit of the NK model for $\delta \to \infty$ with $\pi_t^* \equiv \pi_{ss}$, $d_t \equiv 1$, $r_t \equiv \rho$, and $s_{g,t} \equiv 1$ has $n_d \equiv \dim(\hat{\mathbb{Z}}) = 2$ relevant states. We use n_0 nodal points for N, n_1, n_2 for $\hat{\mathbb{Z}} = (i, A)$, such that $n \equiv n_0 \times n_1 \times n_2 = \prod_{i=0}^{n_d} n_i$. We choose a Chebychev polynomial approximation and construct the grid of $n = 630$ ($n = 70 \times 3 \times 3$) collocation nodes.

We require that the approximant to satisfy the fundamental pricing equation at the n Chebychev nodes $\mathbf{\Phi} u = 0_n$ and the n boundary conditions $\Phi(0, \hat{\mathbb{Z}})u = 1_n$. In particular, the approximant requires system matrices at each collocation node S_l with $l = 1, \ldots, n$

$$g(S_l) = \begin{bmatrix} \theta\phi_\pi(\pi(S_l) - \pi_{ss}) + \theta\phi_y(\hat{y}(S_l))/y_{ss} - 1) - \theta(i_l - i_{ss}) \\ -(\rho_A \log A_l - \frac{1}{2}\sigma_A^2)A_l \end{bmatrix}$$

and

$$\sigma(S_l) = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_A A_l \end{bmatrix}, \quad \sigma(S_l)\sigma(S_l)^\top = \Sigma(S_l) = \begin{bmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_A^2 A_l^2 \end{bmatrix}$$

Table 7: 'good' and 'bad' shocks in the NK model

Variables (cyclical)	Shocks in the NK model						
	monet. policy i_t	techn. level A_t	techn. growth x_t	stoch. vol. σ_t	pref. d_t	natural rate r_t	infl. target π_t^*
output	−	+	+	−	+	+	+
marginal costs	−	−	+	−	+	+	+
inflation	−	−	+	−	+	+	+
value function	−	+	+	−	−	+	+
discount factor	+	−	−	+	+	+	−
yields (nominal)	+	−	+	−	+	+	+
yields (real)	+	−	+	−	+	+	−
term premium (nominal)	+	+	−	−	+	+	−
term premium (real)	+	+	−	−	+	+	+
inflation risk premium	−	+	−	−	+	+	−

Notes: All variables are computed for the parameterization in Table 2

Table 8: Sign restrictions of IRFs in the NK model (cyclicality)

Variables	Sign restrictions of IRFs in the NK model						
	monetary policy i_t	techn. level A_t	techn. growth x_t	stoch. vol. σ_t	pref. d_t	natural rate r_t	inflation target π_t^*
term premium (nominal)	−	−	+	−	−	−	+
term premium (real)	−	−	+	−	−	−	+
inflation risk premium	+	+	−	−	−	−	−

Notes: All variables are computed for the parameterization in Table 2

Table 9: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Preference Shocks)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.077	1.077
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.041	−1.041
$\tilde{\lambda}_d$	1.015	0.562	0.540	0.540
$\tilde{\lambda}_{dd}$	0.022	0.000	−0.718	−0.714
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
$\tilde{\lambda}_r$	5.658	1.837	1.635	1.635
$\tilde{\lambda}_{rr}$	−171.077	0.000	−96.946	−97.708
Nominal				
$\tilde{P}_i^{(10)}$	−3.661	−0.692	−0.761	−0.761
$\tilde{P}_{ii}^{(10)}$	14.174	0.477	3.412	3.411
$\tilde{P}_d^{(10)}$	0.816	−0.714	−0.672	−0.673
$\tilde{P}_{dd}^{(10)}$	−0.121	0.505	1.784	1.748
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_r^{(10)}$	0.298	−1.051	−0.832	−0.832
$\tilde{P}_{rr}^{(10)}$	−9.844	1.099	50.638	50.603
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.019	−1.071	−1.071
$\tilde{P}_{ii}^{(10)}$	0.000	1.037	3.263	3.262
$\tilde{P}_d^{(10)}$	−0.986	−0.532	−0.499	−0.500
$\tilde{P}_{dd}^{(10)}$	1.958	0.279	1.247	1.220
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_r^{(10)}$	−2.407	−0.019	0.035	0.035
$\tilde{P}_{rr}^{(10)}$	81.069	0.002	8.125	8.106

Notes: All variables are computed for the parameterization in Table 2 (standard deviation of 1 percent), $\sigma_z = \mu = 0.02$. In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 10: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Technology Shocks)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.019	1.019
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.013	−1.015
$\tilde{\lambda}_A$	−1.299	−0.714	−0.714	−0.714
$\tilde{\lambda}_{AA}$	2.987	0.000	0.446	0.443
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.659	−0.692	−0.690	−0.690
$\tilde{P}_{ii}^{(10)}$	14.490	0.485	3.501	3.501
$\tilde{P}_A^{(10)}$	−1.056	0.921	0.917	0.918
$\tilde{P}_{AA}^{(10)}$	2.219	0.851	1.075	1.075
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.019	−1.017	−1.017
$\tilde{P}_{ii}^{(10)}$	0.000	1.060	3.368	3.367
$\tilde{P}_A^{(10)}$	1.272	0.686	0.683	0.683
$\tilde{P}_{AA}^{(10)}$	0.349	0.481	0.673	0.673
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 (sd shocks of 1 percent $\sigma_z = 0.02$). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 11: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Stochastic Volatility)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.018	1.018
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.012	−1.013
$\tilde{\lambda}_A$	−1.299	−0.714	−0.715	−0.715
$\tilde{\lambda}_{AA}$	2.988	0.000	0.446	0.444
$\tilde{\lambda}_X$	0.000	−1.075	−1.074	−1.074
$\tilde{\lambda}_{XX}$	0.000	0.000	−1.540	−1.539
$\tilde{\lambda}_S$	0.000	0.000	0.043	0.043
$\tilde{\lambda}_{SS}$	0.000	0.000	1.617	1.617
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.659	−0.692	−0.689	−0.689
$\tilde{P}_{ii}^{(10)}$	14.491	0.478	3.502	3.500
$\tilde{P}_A^{(10)}$	−1.055	0.921	0.918	0.918
$\tilde{P}_{AA}^{(10)}$	2.217	0.849	1.074	1.075
$\tilde{P}_X^{(10)}$	1.939	−1.694	−1.689	−1.689
$\tilde{P}_{XX}^{(10)}$	3.933	2.870	6.801	6.804
$\tilde{P}_S^{(10)}$	−0.078	0.000	0.068	0.068
$\tilde{P}_{SS}^{(10)}$	−2.083	0.000	1.011	1.010
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	−0.000	−1.019	−1.016	−1.016
$\tilde{P}_{ii}^{(10)}$	0.000	1.038	3.370	3.368
$\tilde{P}_A^{(10)}$	1.271	0.687	0.684	0.684
$\tilde{P}_{AA}^{(10)}$	0.345	0.472	0.672	0.672
$\tilde{P}_X^{(10)}$	−2.338	−1.263	−1.258	−1.258
$\tilde{P}_{XX}^{(10)}$	5.466	1.595	4.631	4.634
$\tilde{P}_S^{(10)}$	0.094	0.000	0.050	0.050
$\tilde{P}_{SS}^{(10)}$	2.382	0.000	0.757	0.756
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$, $\sigma_x = 0.01$, $\sigma_z = 0.02$). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 12: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Long-Run Risk)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.018	1.018
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.013	−1.012
$\tilde{\lambda}_A$	−1.299	−0.714	−0.715	−0.715
$\tilde{\lambda}_{AA}$	2.988	0.000	0.446	0.446
$\tilde{\lambda}_X$	0.000	−1.075	−1.075	−1.075
$\tilde{\lambda}_{XX}$	0.000	0.000	−1.540	−1.540
$\tilde{\lambda}_\pi$	0.000	−1.075	−1.075	−1.075
$\tilde{\lambda}_{\pi\pi}$	0.000	0.000	−1.544	−1.521
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.659	−0.692	−0.689	−0.689
$\tilde{P}_{ii}^{(10)}$	14.485	0.485	3.500	3.500
$\tilde{P}_A^{(10)}$	−1.055	0.921	0.918	0.918
$\tilde{P}_{AA}^{(10)}$	2.217	0.852	1.065	1.064
$\tilde{P}_X^{(10)}$	1.939	−1.694	−1.689	−1.689
$\tilde{P}_{XX}^{(10)}$	3.931	2.879	6.771	6.770
$\tilde{P}_\pi^{(10)}$	1.938	−1.694	−1.687	−1.687
$\tilde{P}_{\pi\pi}^{(10)}$	3.930	2.878	6.797	6.798
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.019	−1.016	−1.016
$\tilde{P}_{ii}^{(10)}$	0.000	1.060	3.368	3.368
$\tilde{P}_A^{(10)}$	1.271	0.687	0.683	0.683
$\tilde{P}_{AA}^{(10)}$	0.345	0.481	0.664	0.664
$\tilde{P}_X^{(10)}$	−2.338	−1.262	−1.257	−1.257
$\tilde{P}_{XX}^{(10)}$	5.466	1.626	4.606	4.605
$\tilde{P}_\pi^{(10)}$	−0.000	1.075	1.079	1.079
$\tilde{P}_{\pi\pi}^{(10)}$	−0.000	1.188	4.219	4.219
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$ standard deviation of 1 percent). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection ($\sigma_z = 0.02$, $\sigma_i = 0.01$, $\sigma_A = 0.01$, $\sigma_x = 0.01$, $\sigma_\pi = 0.01$, $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,x} = \sigma_{\pi,z} = 0$).

Table 13: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Long-Run Risk)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.018	1.018
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.012	−1.012
$\tilde{\lambda}_A$	−1.299	−0.714	−0.715	−0.715
$\tilde{\lambda}_{AA}$	2.988	0.000	0.447	0.447
$\tilde{\lambda}_X$	0.000	−1.075	−1.074	−1.074
$\tilde{\lambda}_{XX}$	0.000	0.000	−1.539	−1.539
$\tilde{\lambda}_\pi$	0.000	−1.075	−1.074	−1.074
$\tilde{\lambda}_{\pi\pi}$	0.000	0.000	−1.544	−1.544
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.657	−0.692	−0.688	−0.688
$\tilde{P}_{ii}^{(10)}$	14.466	0.485	3.488	3.488
$\tilde{P}_A^{(10)}$	−1.054	0.921	0.918	0.918
$\tilde{P}_{AA}^{(10)}$	2.214	0.852	1.021	1.021
$\tilde{P}_X^{(10)}$	1.937	−1.694	−1.689	−1.689
$\tilde{P}_{XX}^{(10)}$	3.924	2.883	6.642	6.642
$\tilde{P}_\pi^{(10)}$	1.935	−1.692	−1.683	−1.683
$\tilde{P}_{\pi\pi}^{(10)}$	3.921	2.871	6.746	6.746
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	−0.000	−1.018	−1.015	−1.015
$\tilde{P}_{ii}^{(10)}$	−0.000	1.060	3.362	3.362
$\tilde{P}_A^{(10)}$	1.271	0.686	0.682	0.682
$\tilde{P}_{AA}^{(10)}$	0.345	0.481	0.631	0.631
$\tilde{P}_X^{(10)}$	−2.338	−1.262	−1.255	−1.255
$\tilde{P}_{XX}^{(10)}$	5.466	1.629	4.488	4.488
$\tilde{P}_\pi^{(10)}$	0.000	1.075	1.070	1.070
$\tilde{P}_{\pi\pi}^{(10)}$	−0.000	1.188	4.201	4.201
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$, sd of 1 percent, and correlations $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,x} = \sigma_{\pi,z} = 0.01$). In the flexible price scenario, we set $\phi_\pi = 0.5$.

Table 14: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Long-Run Risk)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.021	1.021
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.016	−1.016
$\tilde{\lambda}_A$	−1.299	−0.714	−0.714	−0.714
$\tilde{\lambda}_{AA}$	2.988	0.000	0.442	0.442
$\tilde{\lambda}_X$	0.000	−1.075	−1.077	−1.077
$\tilde{\lambda}_{XX}$	0.000	0.000	−1.565	−1.565
$\tilde{\lambda}_\pi$	0.000	−1.075	−1.078	−1.078
$\tilde{\lambda}_{\pi\pi}$	0.000	0.000	−1.562	−1.562
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.657	−0.692	−0.692	−0.692
$\tilde{P}_{ii}^{(10)}$	14.466	0.485	3.498	3.498
$\tilde{P}_A^{(10)}$	−1.054	0.921	0.914	0.914
$\tilde{P}_{AA}^{(10)}$	2.214	0.851	1.023	1.023
$\tilde{P}_X^{(10)}$	1.937	−1.694	−1.686	−1.686
$\tilde{P}_{XX}^{(10)}$	3.924	2.879	6.677	6.677
$\tilde{P}_\pi^{(10)}$	1.935	−1.692	−1.679	−1.680
$\tilde{P}_{\pi\pi}^{(10)}$	3.921	2.871	6.776	6.776
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	−0.000	−1.018	−1.019	−1.019
$\tilde{P}_{ii}^{(10)}$	−0.000	1.060	3.374	3.374
$\tilde{P}_A^{(10)}$	1.271	0.686	0.680	0.680
$\tilde{P}_{AA}^{(10)}$	0.345	0.480	0.614	0.614
$\tilde{P}_X^{(10)}$	−2.338	−1.262	−1.252	−1.252
$\tilde{P}_{XX}^{(10)}$	5.466	1.626	4.437	4.437
$\tilde{P}_\pi^{(10)}$	0.000	1.075	1.072	1.072
$\tilde{P}_{\pi\pi}^{(10)}$	−0.000	1.188	4.214	4.214
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$, sd of 1 percent, and correlations $\sigma_{\pi,A} = \sigma_{\pi,i} = \sigma_{\pi,x} = \sigma_{\pi,z} = -0.01$). In the flexible price scenario, we set $\phi_\pi = 0.5$.

4. Conclusion

This paper analyzes the determinants of the term premium on nominal bonds within a micro-founded New Keynesian (NK) framework, offering a novel global solution to better understand how investors price economic risks. It is the first to provide a full analytical investigation of uncertainty in a nonlinear NK model. Our key contribution is the classification of fundamental shocks as 'good' or 'bad' based on their impact on the term premium, which sheds light on the term structure of interest rates. While monetary policy shocks generate 'good' inflation by reducing the inflation risk premium, they actually raise the riskiness of nominal bonds, thereby increasing the term premium. In contrast, other demand shocks (related to real and nominal long-run risks) tend to lower both the inflation risk premium and the nominal term premium, whereas preference shocks, shocks to the natural rate, and supply-side technology shocks increase both, underscoring the importance of distinguishing between different shock types in bond pricing.

Furthermore, by employing a global solution method with recursive preferences and endogenous consumption and inflation dynamics, our model captures nonlinear effects that local approximations often miss. This provides deeper insights into yield curve dynamics and the associated risk premia demanded by investors. We find that temporary and persistent shocks to interest rates and inflation are required to consistently match key macroeconomic aggregates with the observed term structure. One interpretation is that monetary policy actions or other temporary shocks may trigger persistent variations in target rates, a perspective supported by empirical data (Uribe, 2017). Our model helps to resolve apparent term structure anomalies and confirms that uncertainty shocks are isomorphic to discount factor shocks (Barsky, Justiniano, and Melosi, 2014).

We believe that this paper is a starting point for several lines of research. Our analysis has important implications for both theoretical and empirical research. By establishing identifying restrictions for economic shocks in empirical models, we bridge the gap between theoretical asset pricing and macro-finance ATSM. Additionally, our framework provides a foundation for extending macro-finance term structure models to include heterogeneous agents (HANK) and idiosyncratic income shocks (cf. Kaplan, Moll, and Violante, 2018) or alternative policy scenarios such as fiscal dominance. Future research should explore the implications of incorporating heterogeneous beliefs, market segmentation, liquidity effects, and/or including a financial sector (e.g., Brunnermeier and Sannikov, 2014) to refine our understanding of the mechanisms driving risk and term premia. Though our identification of shocks provides a useful benchmark, eventually, the data should be used to jointly pin down the parameters together with the shocks. One key advantage of the continuous-time approach is that the model solution is consistent with different frequencies of macro and financial data (cf. Christensen, Posch, and van der Wel, 2016).

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A. Appendix

A.1. The Feynman-Kac formula

The Feynman-Kac formula is a useful generalization of Kolmogorov's backward equation (Øksendal, 2000, Theorems 8.2.1 and 8.1.1) and links the probability theory to the theory of partial differential equations (PDEs). Let X_t be the solution to the Itô process,

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

and let $h : \mathbb{R} \to \mathbb{R}$ be a C^2 function such that

$$f(t,x) = \mathbb{E}\left[h(X_t)|\mathcal{F}_0\right]$$

denotes the conditional expectation of $h(X_t)$ given the information set $\mathcal{F}_0$. Then $f(t,\cdot)$ for each t satisfies the PDE (Kolmogorov's backward equation, Theorem 8.1.1)

$$f_1'(t,x) = f_2'(t,x)\mu(x) + \tfrac{1}{2}f_{22}''(x)\sigma(x)^2 \tag{A.1}$$

with boundary condition $f(0,x) = h(x)$.

Moreover, let $q : \mathbb{R} \to \mathbb{R}$ be a C^1 function, and let

$$g(t,x) = \mathbb{E}\left[\exp\left(-\int_0^t q(X_s)\,\mathrm{d}s\right)h(X_t)|\mathcal{F}_0\right]$$

Then $g(t,\cdot)$ for each t satisfies the PDE (Feynman-Kac, Theorem 8.2.1)

$$g_1'(t,x) = g_2'(t,x)\mu(x) + \tfrac{1}{2}g_{22}''(t,x)\sigma(x)^2 - q(x)g(t,x) \tag{A.2}$$

with boundary condition $g(0,x) = h(x)$.

B. Tables and Figures

B.1. Tables

Table 1: Summary of the solution algorithm

Step 1	(Initialization)	Provide an initial guess for the unknown derivatives for a given set of collocation nodes and basis functions.
Step 2	(Solution)	Compute the optimal value of the controls for the set of nodal values for the state variables.
Step 3	(Update)	Update the consumption function derivatives.
Step 4	(Iteration)	Repeat Steps 2 and 3 until convergence.

Table 2: Parameterization

ϑ	1	Frisch labor supply elasticity
ρ	0.03	subjective rate of time preference, $\rho = -4 \log 0.9925$
ψ	1	preference for leisure
δ	0.65	Calvo parameter for probability of firms receiving signal, $\delta = -4 \log 0.85$
ε	25	elasticity of substitution intermediate goods
s_g	0	share of government consumption
μ	0	average growth rate technology
ρ_d	0.4214	autoregressive component preference shock, $\rho_d = -4 \log 0.9$
ρ_r	0.4214	autoregressive component discount rate shock, $\rho_g = -4 \log 0.9$
ρ_A	0.4214	autoregressive component technology shock, $\rho_A = -4 \log 0.9$
ρ_g	0.4214	autoregressive component government shock, $\rho_g = -4 \log 0.9$
ρ_x	0.4214	autoregressive component technology growth shock, $\rho_g = -4 \log 0.9$
ρ_σ	0.4214	autoregressive component technology volatility shock, $\rho_g = -4 \log 0.9$
ρ_π	0.4214	autoregressive component inflation target shock, $\rho_g = -4 \log 0.9$
σ_i	0.01	variance monetary policy shock
σ_d	0.01	variance preference shock
σ_A	0.01	variance technology shock
σ_g	0	variance government shock
σ_z	0.02	variance permanent technology shock
σ_x	0.01	variance technology growth shock
σ_r	0.01	variance discount rate shock
σ_σ	0.01	variance of stochastic volatility shock
σ_π	0.01	variance inflation target shock
ϕ_π	4	inflation response Taylor rule
ϕ_y	0	output response Taylor rule
θ	0.5	interest rate smoothing Taylor rule
π_{ss}	0.02	inflation target rate
χ	1	indexation at steady-state inflation rate, $\chi = 1$ is full indexation

Table 3: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Preference Shocks)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.035	1.035
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.019	−1.020
$\tilde{\lambda}_d$	1.015	0.562	0.559	0.559
$\tilde{\lambda}_{dd}$	0.022	0.000	−0.703	−0.699
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
$\tilde{\lambda}_r$	5.658	1.837	1.801	1.801
$\tilde{\lambda}_{rr}$	−171.077	0.000	−94.980	−95.741
Nominal				
$\tilde{P}_i^{(10)}$	−3.662	−0.692	−0.702	−0.704
$\tilde{P}_{ii}^{(10)}$	13.862	0.911	3.391	3.396
$\tilde{P}_d^{(10)}$	0.817	−0.713	−0.697	−0.697
$\tilde{P}_{dd}^{(10)}$	−0.130	−0.403	1.735	1.774
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_r^{(10)}$	0.298	−1.051	−0.985	−0.985
$\tilde{P}_{rr}^{(10)}$	−9.844	1.557	55.858	55.820
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.019	−1.027	−1.028
$\tilde{P}_{ii}^{(10)}$	0.000	1.360	3.259	3.264
$\tilde{P}_d^{(10)}$	−0.986	−0.531	−0.518	−0.518
$\tilde{P}_{dd}^{(10)}$	1.958	−0.402	1.204	1.232
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_r^{(10)}$	−2.407	−0.019	−0.008	−0.008
$\tilde{P}_{rr}^{(10)}$	81.069	−0.414	11.913	11.864

Notes: All variables are computed for the parameterization in Table 2 (standard deviation of 1 percent), $\sigma_z = \mu = 0$. In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 4: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Technology Shocks)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.019	1.019
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.013	−1.013
$\tilde{\lambda}_A$	−1.299	−0.714	−0.714	−0.714
$\tilde{\lambda}_{AA}$	2.987	0.000	0.444	0.444
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.660	−0.692	−0.690	−0.690
$\tilde{P}_{ii}^{(10)}$	14.475	0.485	3.502	3.502
$\tilde{P}_A^{(10)}$	−1.056	0.921	0.917	0.917
$\tilde{P}_{AA}^{(10)}$	2.218	0.851	1.075	1.075
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.019	−1.017	−1.017
$\tilde{P}_{ii}^{(10)}$	0.000	1.060	3.369	3.369
$\tilde{P}_A^{(10)}$	1.272	0.686	0.683	0.683
$\tilde{P}_{AA}^{(10)}$	0.349	0.481	0.673	0.673
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 (sd shocks of 1 percent $\sigma_z = 0$). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 5: Sensitivity of SDF and 10-year Zero-Coupon Bond Prices (Stochastic Volatility)

Derivatives	Benchmark	Model		
	Flex-Price	ATSM	Two-Step	Nonlinear
SDF				
$\tilde{\lambda}_i$	0.000	1.019	1.018	1.018
$\tilde{\lambda}_{ii}$	0.000	0.000	−1.013	−1.014
$\tilde{\lambda}_A$	−1.299	−0.714	−0.715	−0.715
$\tilde{\lambda}_{AA}$	2.988	0.000	0.446	0.443
$\tilde{\lambda}_X$	0.000	−1.075	−1.075	−1.074
$\tilde{\lambda}_{XX}$	0.000	0.000	−1.541	−1.543
$\tilde{\lambda}_S$	0.000	0.000	0.035	0.035
$\tilde{\lambda}_{SS}$	0.000	0.000	1.309	1.308
$\tilde{\lambda}_z$	−1.000	−1.000	−1.000	−1.000
$\tilde{\lambda}_{zz}$	2.000	0.000	2.000	2.000
Nominal				
$\tilde{P}_i^{(10)}$	−3.659	−0.692	−0.689	−0.689
$\tilde{P}_{ii}^{(10)}$	14.485	0.478	3.500	3.499
$\tilde{P}_A^{(10)}$	−1.055	0.921	0.918	0.918
$\tilde{P}_{AA}^{(10)}$	2.217	0.849	1.063	1.063
$\tilde{P}_X^{(10)}$	1.938	−1.694	−1.687	−1.688
$\tilde{P}_{XX}^{(10)}$	3.930	2.869	6.792	6.792
$\tilde{P}_S^{(10)}$	−0.044	0.000	0.169	0.169
$\tilde{P}_{SS}^{(10)}$	−1.086	0.000	3.865	3.866
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000
TIPS				
$\tilde{P}_i^{(10)}$	0.000	−1.019	−1.016	−1.016
$\tilde{P}_{ii}^{(10)}$	0.000	1.038	3.369	3.367
$\tilde{P}_A^{(10)}$	1.271	0.687	0.683	0.683
$\tilde{P}_{AA}^{(10)}$	0.345	0.472	0.662	0.662
$\tilde{P}_X^{(10)}$	−2.337	−1.263	−1.257	−1.257
$\tilde{P}_{XX}^{(10)}$	5.461	1.595	4.627	4.626
$\tilde{P}_S^{(10)}$	0.152	0.000	0.121	0.121
$\tilde{P}_{SS}^{(10)}$	3.966	0.000	2.745	2.746
$\tilde{P}_z^{(10)}$	0.000	0.000	0.000	0.000
$\tilde{P}_{zz}^{(10)}$	0.000	0.000	0.000	0.000

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$, $\sigma_x \sigma_z = 0.01$). In the flexible price scenario, we set $\phi_\pi = 0.5$ for unique equilibrium selection.

Table 6: 'good' and 'bad' shocks in the flexible price limit

Variables (cyclical)	Shocks in the flexible price limit						
	monet. policy i_t	techn. level A_t	techn. growth x_t	stoch. vol. σ_t	pref. d_t	natural rate r_t	infl. target π_t^*
output	0	+	0	0	0	0	0
marginal costs	0	0	0	0	0	0	0
inflation	+	+	0	0	−	−	+
value function	0	+	0	0	−	+	0
discount factor	0	−	0	0	+	+	0
yields (nominal)	+	+	−	+	−	−	−
yields (real)	0	−	+	−	+	+	0
term premium (nominal)	0	−	0	0	−	−	0
term premium (real)	0	+	0	0	+	+	0
inflation risk premium	0	−	0	0	−	−	0

Notes: All variables are computed for the parameterization in Table 2 ($\eta = 0.55$ standard deviation of 1 percent, and correlations $\sigma_{\pi,i} = 0.01$ and $\sigma_{\pi,A} = -0.01$)

Table 7: Sign restrictions of IRFs in the flexible price limit (cyclicality)

Variables	Sign restrictions of IRFs in the flexible price limit						
	monetary policy i_t	techn. level A_t	techn. growth x_t	stoch. vol. σ_t	pref. d_t	natural rate r_t	inflation target π_t^*
term premium (nominal)	0	+	0	0	+	+	0
term premium (real)	0	0	0	0	−	−	0
inflation risk premium	0	+	0	0	+	+	0

Notes: All variables are computed for the parameterization in Table 2 using the nonlinear solution.

B.2. Figures

Figure 1: US federal funds rate, 10-year treasury rate and inflation rate

In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the Consumer Price Index (Core CPI), seasonally adjusted, the 10-Year Treasury Inflation Protected Securities Rate (10Y TIPS), at the monthly frequency, and the Output gap (HP Filter) at the quarterly frequency. All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

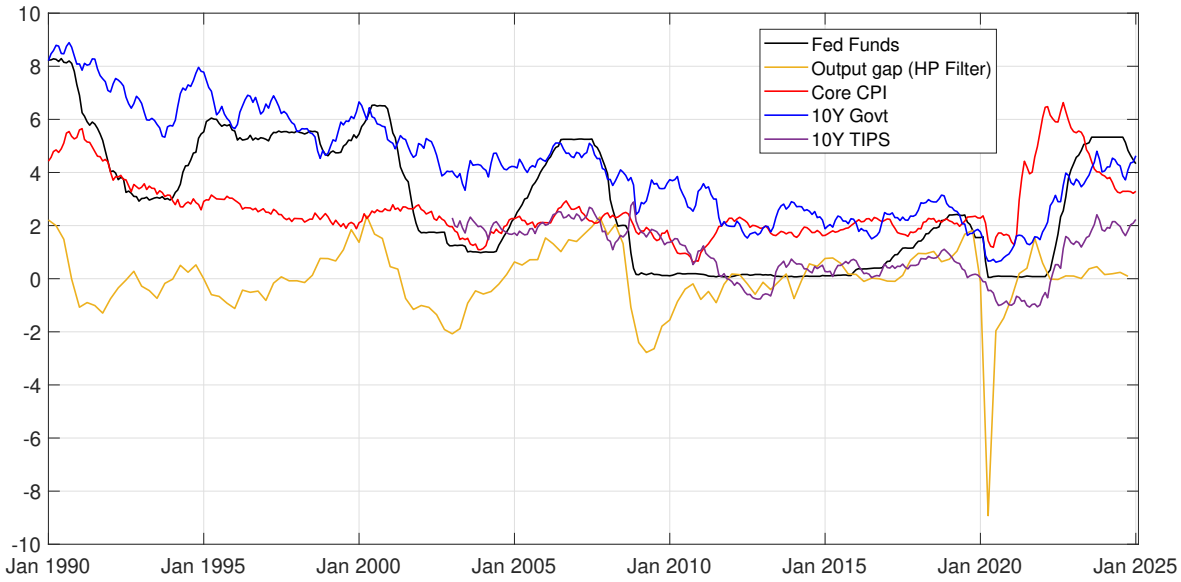


Figure 2: US federal funds rate, output gap, cyclical components

In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), and different estimates of the Output gap based on potential output from the Congressional Budget Office (CBO), the Hodrick-Prescott (HP) filter, and the Beveridge-Nelson (BN) trend-cycle decomposition, and the same filter with dynamic mean adjustment (DMA). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from 1990Q1, through 2024Q4.

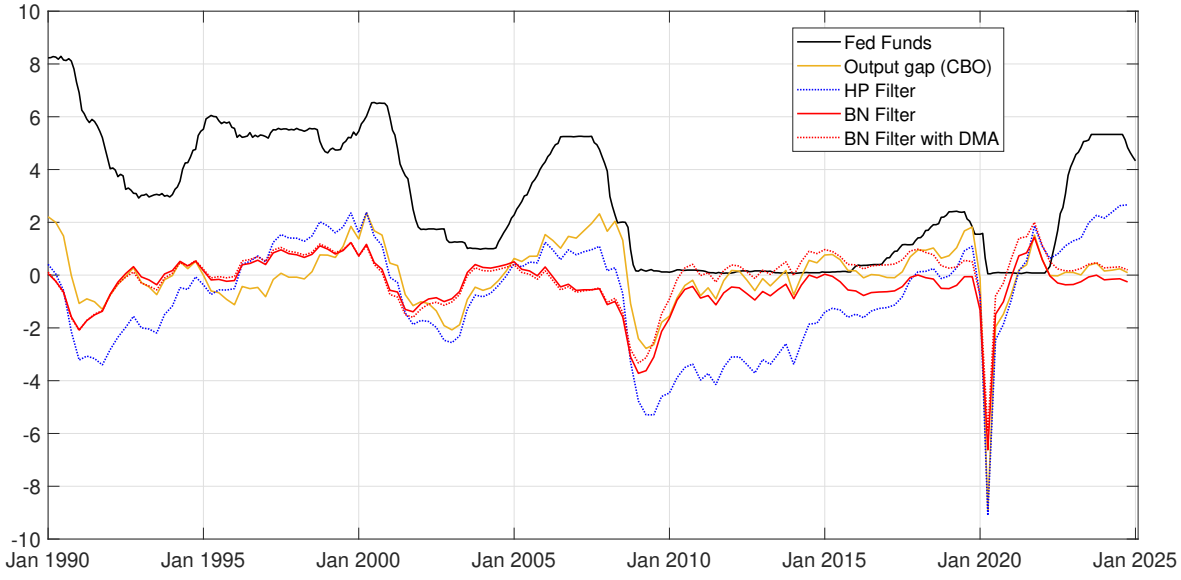


Figure 3: Macro data

In this figure we show time series plots of Real Personal Consumption Expenditures (Real Consumption), Real Compensation of Employees (Real Wages), Average Weekly Hours of Production and Nonsupervisory Employees (Hours). All series are HP filtered cyclical components, obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

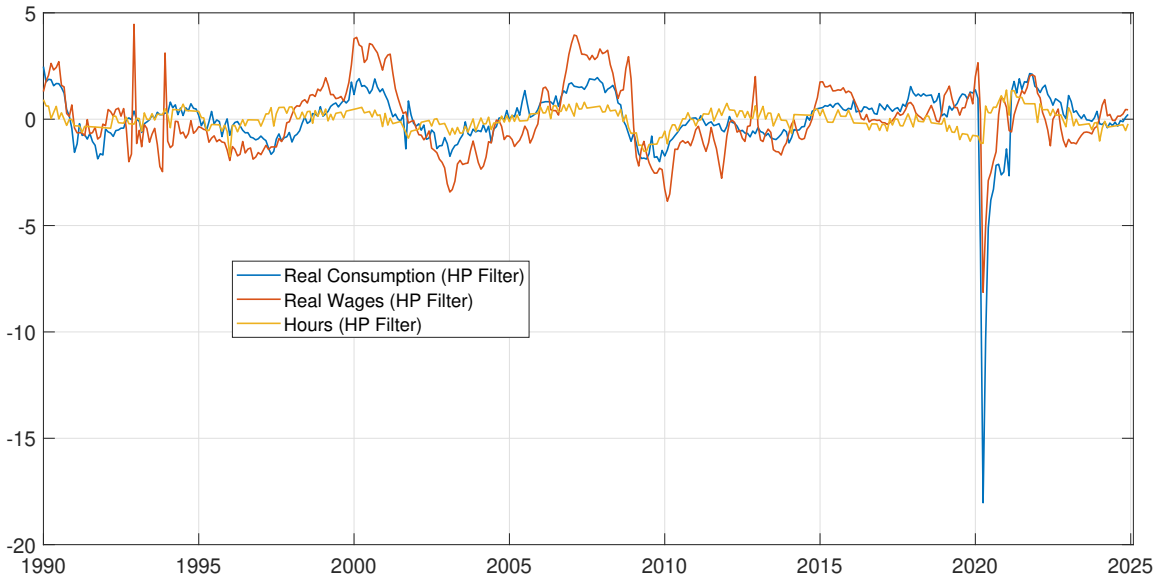


Figure 4: Interest Rates

In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), the Riskfree Rate (Real Rate), and the Estimate of the Natural Rate (Natural Rate) from Laubach and Williams (2003). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED) and the Federal Reserve Bank of New York. The sample runs from January 1990 to January 2025.

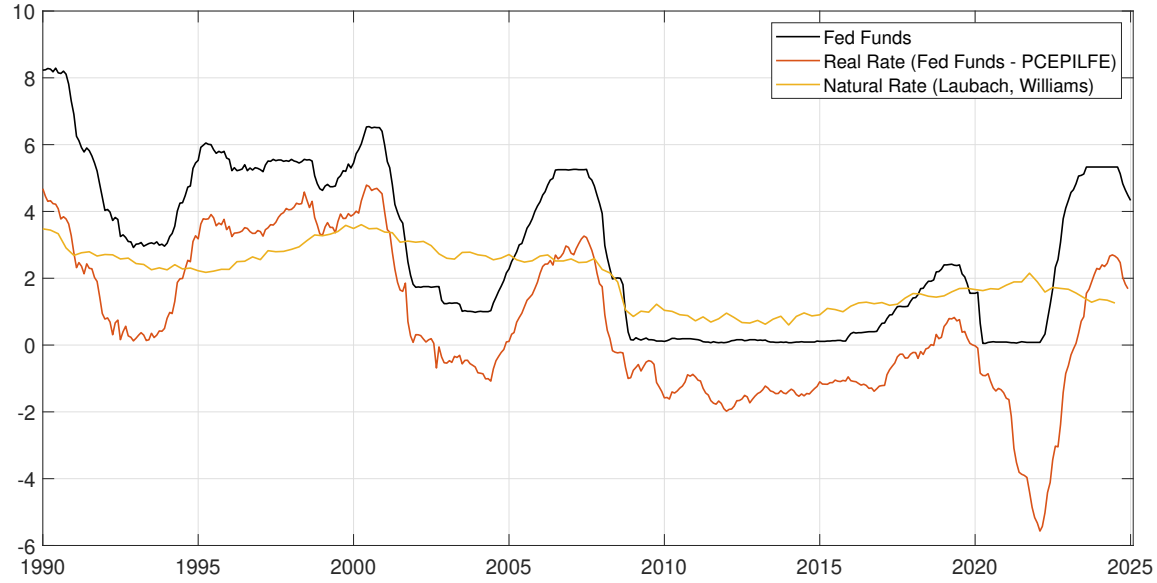


Figure 5: Term Structure of Interest Rates

In this figure we show time series plots of the Market Yield on U.S. Treasury Securities at X-Year Constant Maturity (1Y Govt, 5Y Govt, 10Y Govt), and the Spread of the 10 Year Yield over the Fed Funds Rate (Slope). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

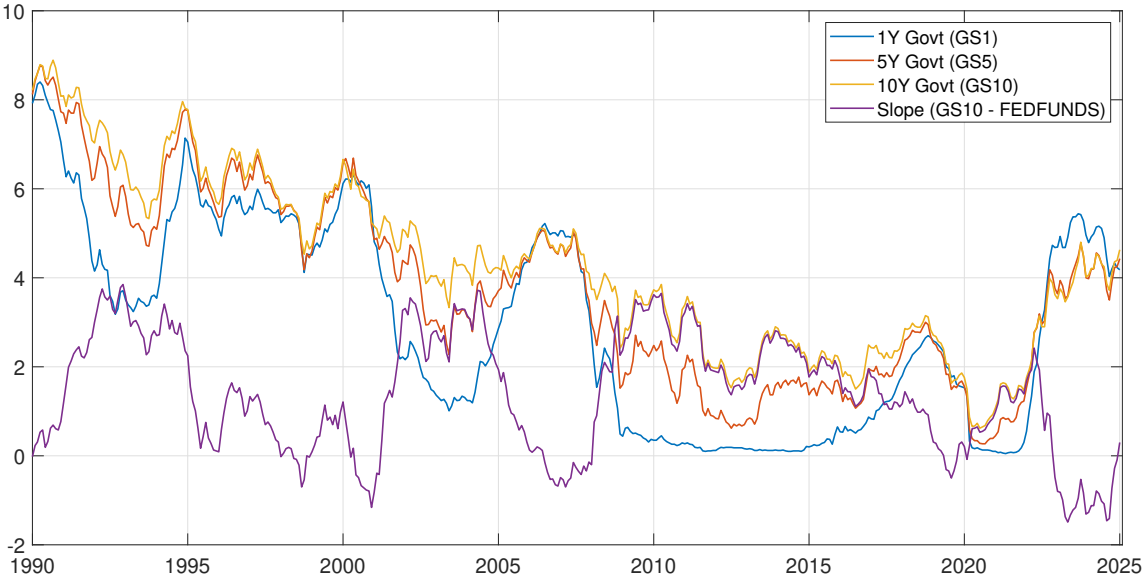


Figure 6: Forward Rates, Term Premium, and Inflation Risk Premium

In this figure we show time series plots of Fitted Instantaneous Forward Rate 10 Years Hence (Forward Rate 10Y), the Term Premium on an 10 Year Zero-Coupon Bond (Term Premium 10Y), the Instantaneous Forward Term Premium 10 Years Hence (Forward Term Premium 10Y), and the Fitted Inflation Risk Premium 10 Years (Inflation Risk Premium 10Y). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED), and from the Cleveland Fed's Center for Inflation Research. The sample runs from January 1990 to January 2025.

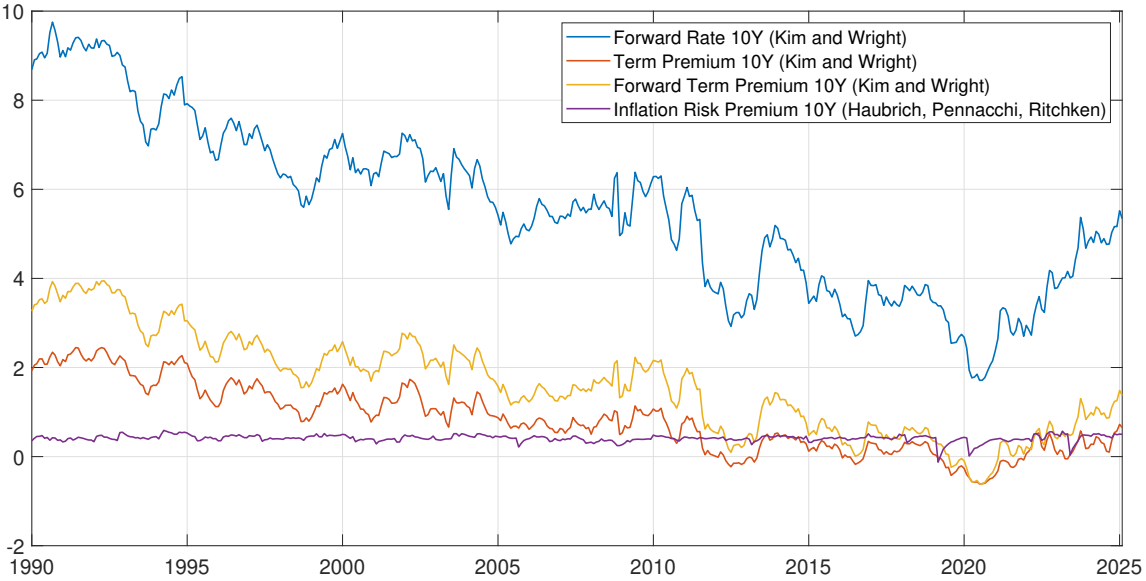


Figure 7: Inflation

In this figure we show time series plots of Percentage Changes from Year ago of the Consumer Price Index for All Urban Consumers: All Items (CPI Headline), Consumer Price Index for All Urban Consumers: All Items Less Food and Energy (CPI Core), Personal Consumption Expenditures: Chain-type Price Index (PCE Headline), and the Fed's primary measure of Personal Consumption Expenditures Excluding Food and Energy (PCE Core). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

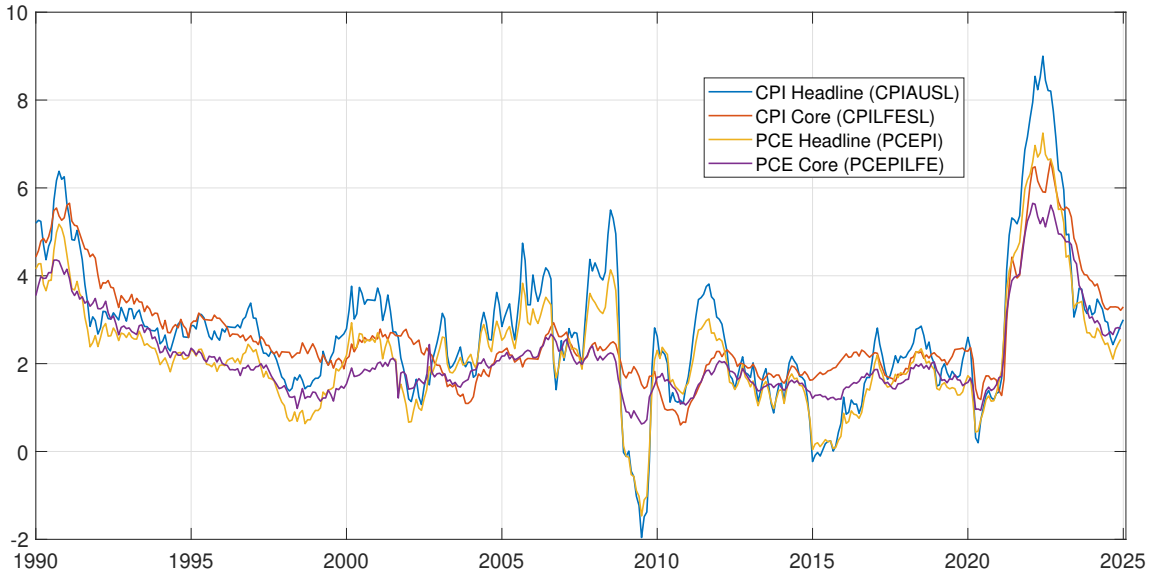


Figure 8: Forecasts

In this figure we show time series plots of Percentage Changes from Year ago of the Consumer Price Index for All Urban Consumers: All Items Less Food and Energy (Inflation), Fitted inflation expectations 1 Year (Inflation Expectation 1Y), the Fitted Instantaneous Forward Rate minus Instantaneous Forward Term Premium 1 year Hence (Expected Short Rate 1Y) based on Kim and Wright (2005), and the Inflation Compensation of the Zero-coupon Continuously Compounded 10 Years (Breakeven Inflation 10Y) from Gürkaynak, Sack, and Wright (2010). All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January 1990 to January 2025.

