Stochastic Dominance and Financial Pricing in Peer-to-Peer Insurance*

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Abstract: This paper studies the economic benefits of peer-to-peer insurance for pools of dependent and heterogeneous risks. We characterize the form of insurance contracts with proportional surplus sharing that is consistent with a second-order stochastic dominance requirement and financial pricing in an incomplete market setting. Moreover, conditions under which the proposed contract design generates utility benefits for all risk-averse policyholders are derived. In particular, we identify necessary second-moment conditions on the risks being pooled and establish sufficient conditions related to the policyholder's background risk. The results provide new insights into the design and benefits of simple risk pooling solutions that require only limited information about the joint loss distribution and the policyholders' preferences.

Keywords: Peer-to-Peer Insurance, Mutual Insurance, Financial Pricing, Risk Sharing, Stochastic Dominance, Heterogeneous Risks

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1 Introduction

New products in retail markets and risk-sharing solutions for large losses, often referred to as peer-to-peer (P2P) insurance, have led to renewed interest in mutual insurance arrangements.[1] Economically, P2P insurance pools risks for a given premium and distributes potential profits or losses back to policyholders rather than to the owners of an insurance company. The resulting pools are typically heterogeneous and may exhibit stochastic dependencies.[2] In contrast to the case of homogeneous risks, where diversification benefits and utility gains are well documented for all risk-averse policyholders[3], the economic benefits of P2P insurance for policyholders are less clear when heterogeneous risks are pooled. Individual benefits are then determined by the nature of the risks in the pool and the way in which heterogeneity is reflected in P2P insurance contracts, i.e., in the pricing and profit-sharing rules applied by the pool.

Following the seminal work of Borch (1962), a comprehensive literature shows how to design optimal risk exchanges given the individual preferences of the pool participants (Wilson 1968, Bühlmann and Jewell 1979). In contrast, more recent studies on P2P insurance propose contracts that do not require policyholder-specific preference information, for example, by using axiomatic techniques (Denuit et al. 2022, Jiao et al. 2023) or arguments based on network structures (Abdikerimova and Feng 2022). Complementing this recent work on preference-free contract designs, our paper develops an economic approach that combines ideas from financial pricing with the requirement that the contract can potentially generate utility gains for all risk-averse agents. Based on this approach, we are able to characterize a unique form of P2P contracts with linear surplus-sharing if the risks in the pool are not entirely unsystematic and we establish conditions on the relevant risks which are sufficient for the occurrence of utility gains.

Our formal analysis relies on the following assumptions: We investigate P2P insurance in a one-period model with n agents facing heterogeneous insurable risks and independent background risks. A P2P insurer offers full coverage of the insurable loss for a premium paid at time $t = 0$. At time $t = 1$, the policyholders of the P2P insurer receive the indemnity payment as well

[1] See Feng (2023) for a comprehensive review of recent innovations in P2P or, more generally, decentralized insurance. The close economic relation between modern P2P and traditional mutual insurance is, for example, highlighted by NAIC (2024) stating that P2P is "as much a return to the old roots of insurance as a leap forward." We use the term P2P insurance in a broad economic sense to refer to mutual insurance agreements, regardless of their technical implementation.

[2] Note that the pooling of heterogeneous risks is also relevant for traditional insurers. Although these companies classify risks and thus select (approximately) homogeneous exposures in individual portfolios, they pool them in one legal entity. Consequently, their overall underwriting portfolio also consists of heterogeneous risks.

[3] Cf. Samuelson (1967) for an early analysis of diversification benefits with homogeneous risks.

as a profit and loss participation. The participation fully redistributes the P2P insurer's total surplus to its policyholders based on percentage shares that are fixed at time $t = 0$ as part of the P2P insurance contract. To determine market-consistent premiums of the P2P contracts, we apply an arbitrary Stochastic Discount Factor (SDF). We do not assume that markets are complete and, therefore, rely on a utility-based criterion to evaluate P2P insurance solutions from the policyholders' perspective.[4] In particular, second-order stochastic dominance (SSD) relations are used to derive general statements about the policyholders' benefits from P2P insurance.

Within this model framework, we first characterize a P2P insurance contract that satisfies two conditions: it is consistent with (i) SSD of P2P insurance and (ii) no-arbitrage pricing building on a given SDF. If the losses in the pool are not fully unsystematic, these two conditions are sufficient to characterize the policyholder's participation and the price of the contract up to the total safety loading. Each policyholder's participation is then based on her relative contribution to the systematic risk of the pool as measured by the covariance of the total claim amount and the given SDF. If all claims in the pool are uncorrelated with the SDF, the design of the policyholders' profit-and-loss participation is not constrained by the above-mentioned conditions. In both cases, the premium corresponds to the expected claim and a share of the total safety loading, which is the same as the policyholder's share of the pool's surplus.

After describing the properties of the proposed P2P insurance contract, the following conditions for the occurrence of utility gains from P2P insurance are established. We first derive necessary second-moment conditions imposing constraints on the relationship between the systematic and the total risk of the P2P insurance pool that must be satisfied for SSD of P2P insurance. Then, we show that P2P insurance based on the proposed contract is always beneficial if the second-moment conditions hold and the policyholder's background risk is large enough.[5] Complementing this general SSD result, we provide minimum-variance bounds on the background risk that hold under a fourth-order Taylor expansion of the policyholders' utility functions.

We then investigate a special case where the systematic risk exposure of all claims in the P2P insurer's portfolio is identical for a given SDF, while still allowing for differences in the idiosyncratic risk of the pooled claims. In this case, the participation rule of the P2P contract simplifies to an

[4] See Bauer et al. (2013) for a comprehensive review on previous studies that apply ideas from financial pricing in insurance. Note that, given the incompleteness of the market, the SDF and the corresponding martingale measure are not unique. We avoid restricting our analysis to a specific measure (such as the variance-optimal measure or the minimal entropy measure) in order to derive results that are as generally applicable as possible.

[5] In our Internet Appendix, we also derive an alternative sufficient condition that restricts the joint distribution of the risks in the pool.

equal distribution of the insurer's profit or loss, and the safety loading included in the premium is the same for all policyholders. Accordingly, the choice of a specific SDF becomes irrelevant. Furthermore, simpler conditions for utility gains can be provided in the case of homogeneous systematic risk exposures.

Finally, we consider three extensions of our baseline analysis: First, we discuss how operating expenses affect our results. Second, we demonstrate that solvency constraints can be used to determine the overall magnitude of the safety loading in the P2P insurance premiums without affecting the policyholders' utility. Third, we describe the impact of investment risk on the previously characterized contract form by assuming that both individual policyholders and the P2P insurer invest their funds in a risky portfolio.

Our analysis contributes to the following strands of literature:

First, the results presented in this study contribute to the growing literature on contract designs for P2P insurance (see, e.g., Denuit et al. 2022, Abdikerimova and Feng 2022, Feng et al. 2023) by proposing a new P2P insurance contract based on a combination of financial pricing and an SSD requirement. Second, our analysis is related to the vast literature on optimal risk exchanges building on the work of Borch (1962) and Wilson (1968). More specifically, our work complements risk sharing analyses that incorporate financial fairness constraints (Bühlmann and Jewell 1979, Schumacher 2018) and recent work investigating the effects of increasing the size of the risk pool (Anthropelos et al. 2024). Whereas the traditional approach to risk sharing requires information on the preferences of each pool participant or the choice of a specific risk measure (see, e.g., Embrechts et al. 2018), recent contributions have proposed axioms to derive and evaluate basically preference-free sharing rules (Denuit et al. 2022, Jiao et al. 2023). Adding to this recent strand of the literature, we propose a new contract design that builds on ideas from financial pricing instead of an axiomatic approach. Third, our results contribute to the extensive literature on the effects of diversification in insurance (Cummins 1974) and, in particular, to recent studies examining the benefits of risk pooling from the policyholders' point of view (Schmeiser and Orozco-Garcia 2021, Huggenberger and Albrecht 2022). Whereas the above-mentioned studies mainly focus on homogeneous risk exposures, we derive SSD results for individual pooling benefits from P2P insurance with heterogeneous risks. Fourth, our paper complements the economic literature that studies the effect of independent background risk on insurance demand and, more generally, risk-taking behavior (Doherty and Schlesinger 1983, Kimball 1993, Gollier and Pratt 1996). In particular, we exploit a recent result by Pomatto et al. (2020) to show that P2P insurance can generate benefits for all-risk averse

policyholders if the risk pool is selected such that it satisfies the derived second-moment conditions and policyholders are facing sufficiently large independent background risks.

The structure of this paper is as follows: Section 2 presents our model framework and the assumptions underlying our analysis. Section 3 summarizes our baseline results on the form of the P2P contract and its benefits for risk-averse policyholders. In Section 4, we discuss the effects of including operating expenses, solvency constraints, and investment risk. Finally, Section 5 provides our concluding remarks. Proofs of our formal results are provided in the Appendix. Additional results and examples can be found in an Internet Appendix.

2 Model Assumptions

2.1 Individual Risks

We investigate a one-period model with n heterogeneous agents, who hold a non-random wealth $w_{0,i}$, $i = 1, \dots, n$ at time $t = 0$ and who are facing two types of risks at time $t = 1$:

Assumption B1 *Agent i is exposed to a random insurable loss X_i and an uninsurable risk Y_i, which are both realized at time $t = 1$. Furthermore, it holds that $\mathbb{E}\big[X_i^2\big] < \infty$ and Y_i is independent of $X_1, \dots, X_n$ for all $i = 1, \dots, n$.*

Importantly, Assumption B1 does not impose a homogeneity restriction on the insurable risks $X_1, \dots, X_n$. It allows for risks that differ in their moments as well as in the type of distributions.[6] The marginal distributions of the insurable risks are only required to have finite variances, which has the following two important implications: First, it distinguishes our analysis from previous studies that comprehensively document the limitations of risk pooling and diversification for extremely fat-tailed risks (see, for example, Fama 1965, Ibragimov 2009). Second, the $\mathcal{L}^2$-constraint allows us to apply financial pricing and to measure systematic risk exposures using covariances. Y_i captures all kinds of background risks that cannot be insured and are independent of the insurable risks.

In our baseline analysis, we exclude investment risk by making the following assumption.[7]

Assumption B2 *All agents invest their available wealth between $t = 0$ and $t = 1$ at the risk-free rate $r_f \geq 0$.*

[6] We do not need impose the common restriction $X_i \geq 0$, but Assumption B1 also covers this important special case.

[7] Investment risk will be considered as an extension of our setting in Section 4.3.

Under the Assumptions B1 and B2, the wealth of agent i at time $t = 1$ without insurance is given by

$$W_i^{\text{no}} = (1 + r_f)\, w_{0,i} - X_i + Y_i. \tag{1}$$

Note that the sign of Y_i is just a convention and that we can rewrite W_i^{no} in terms of a zero-mean background risk and a modified initial wealth if $\mathbb{E}[|Y_i|] < \infty$ and risk-free lending between $t = 0$ and $t = 1$ is possible.[8] Furthermore, equation (1) shows that we do not assume limited liability of the policyholders, which could reduce the benefits of insurance (Gollier et al. 1997).

2.2 Peer-to-Peer Insurance

We investigate P2P insurance of the losses $X_1, \ldots, X_n$ or a subset of these exposures. To obtain insurance, agent i pays a premium π_i at time $t = 0$. At time $t = 1$, she receives a cash flow that consists of two components: her indemnity payment I_i based on the realization of her individual insurable loss X_i and a participation P_i in the total profit or loss of the pool.[9]

Formally, we can write the wealth of agent i participating in a P2P insurance pool as

$$W_i^{\text{p2p}}(X_1, \ldots, X_n) = (1 + r_f)\,(w_{0,i} - \pi_i) - X_i + Y_i + I_i + P_i(X_1, \ldots, X_n). \tag{2}$$

We consider a pool that fully indemnifies the agents' losses. Accordingly, we assume:

Assumption B3 *The P2P insurer provides full coverage of the individual losses X_i, i.e., $I_i = X_i$, for all $i = 1, \ldots, n$.*

We furthermore focus on underwriting risk and neglect operating expenses as well as investment risk in our baseline analysis.[10] The total surplus of a P2P insurer pooling the risky exposures $X_1, \ldots, X_n$ for the premiums $\pi_1, \ldots, \pi_n$ can then be written as

$$S(X_1, \ldots, X_n) = \sum_{j=1}^{n} (1 + r_f)\, \pi_j - \sum_{j=1}^{n} X_j, \tag{3}$$

where we assume that the premiums collected at time $t = 0$ are invested at the risk-free rate.

For this surplus, we impose the following type of sharing rule:

[8] In particular, we could choose $Y_i' = Y_i - \mathbb{E}[Y_i]$ and add the discounted value of $\mathbb{E}[Y_i]$ to $w_{0,i}' = w_{0,i} + \mathbb{E}[Y_i]\,(1+r_f)^{-1}$.

[9] Accordingly, the position in a P2P insurance pool can be understood as the combination of a policyholder and an equity holder stake, just as the position in a traditional mutual insurance company (Cummins et al. 1999).

[10] Operating expenses and investment risk will be added in Sections 4.1 and 4.3.

Assumption B4 *The surplus* $S(X_1, \ldots, X_n)$ *defined in equation* (3) *is fully shared among the pool participants with* $\theta_{i,n}$ *denoting the percentage participation of policyholder* i *determined at time* $t = 0$*. It thus holds that*

$$P_i(X_1, \ldots, X_n) = \theta_{i,n} \cdot S(X_1, \ldots, X_n), \tag{4}$$

for $i = 1, \ldots, n$ *with* $\sum_{i=1}^{n} \theta_{i,n} = 1$*.*

According to Assumption B4, a predetermined percentage share of the total surplus is assigned to each policyholder. The assumption ensures that profits are fully redistributed within the pool (if $S(X_1, \ldots, X_n) > 0$) and that losses are also fully covered by the pool members (if $S(X_1, \ldots, X_n) < 0$). Moreover, Assumption B4 implies that all pool members are able to provide additional payments at $t = 1$ if high losses occur and it rules out upper limits on these loss participations, e.g., upper bounds that depend on the magnitude of the individual premiums.[11]

Under the Assumptions B3 and B4, the agent's wealth with P2P insurance from equation (2) can be written as

$$W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) = (1 + r_f)\,(w_{0,i} - \pi_i) + Y_i + \theta_{i,n} \cdot S(X_1, \ldots, X_n). \tag{5}$$

A comparison of equations (1) and (5) shows that "no insurance" can be considered as a special case of P2P insurance with $n = 1$.

2.3 Individual Risk Evaluation and Insurance Pricing

We assume that the risks $X_1, \ldots, X_n$ affecting the agents' wealth cannot be perfectly replicated using financial assets, which is a common assumption for individual insurance events (Bauer et al. 2013, p. 629f).

On the policyholders' side, we therefore rely on a utility-based criterion for ranking different risk pooling arrangements. To obtain general statements on the benefits of P2P insurance which do not depend on specific preference assumptions, we derive second-order stochastic dominance (SSD) results. W_1 weakly dominates W_2 by SSD if $\mathbb{E}[u(W_1)] \geq \mathbb{E}[u(W_2)]$ for all increasing and concave functions provided that the expectations exist.[12] In this case, we write $W_1 \succeq_{ssd} W_2$. From

[11] While such upper bounds are common in practice, the relevance of these bounds for our results seems to be limited because the probability of additional payments to the pool at time $t = 1$ can be made arbitrarily small by including an adequate total safety loading at time $t = 0$, whose choice will not affect our baseline results on the benefits of P2P insurance as will be shown in Section 3.

[12] Therefore, SSD is also referred to as increasing concave order. Cf. Shaked and Shanthikumar (2007) for a general exposition to increasing concave and convex order and Levy (1992) for applications in economics and finance.

this definition, it follows that *every* risk-averse expected utility maximizer weakly prefers W_1 over W_2 if $W_1 \succeq_{ssd} W_2$. The criterion thus allows us to make general statements for all risk-averse agents within the expected utility framework. Furthermore, it is consistent with extensions of the expected utility paradigm such as rank-dependent expected utility with a concave probability weighting function (Chew et al. 1987).

Since we are interested in obtaining general preference-free results on the benefits of P2P insurance, we will apply the following criteria:

Property SSD1 *All risk-averse agents prefer P2P insurance in a pool consisting of the risks $X_1, \dots, X_n$ to no insurance, that means,*

$$W_i^{p2p}(X_1, \dots, X_n) \succeq_{ssd} W_i^{no} \qquad \text{for } i = 1, \dots, n. \tag{6}$$

Property SSD1 captures that all risk-averse agents benefit from P2P insurance. Note that this property is closely related to the so-called "willingness-to-join" property in the axiomatic framework proposed by Denuit et al. (2022).[13] We further investigate conditions ensuring that the existing policyholders benefit from an increase in the size of a P2P insurance pool.

Property SSD2 *All risk-averse policyholders, $i = 1, \dots, n-1$, pooling the risks $X_1, \dots, X_{n-1}$ prefer to admit a new policyholder with the risk X_n to the P2P insurance pool, that means,*

$$W_i^{p2p}(X_1, \dots, X_n) \succeq_{ssd} W_i^{p2p}(X_1, \dots, X_{n-1}) \qquad \text{for } i = 1, \dots, n-1. \tag{7}$$

Compared to Property SSD1, the Property SSD2 imposes a higher benchmark from the perspective of the current policyholders.[14] If Property SSD2 is always satisfied when admitting new policyholders to the pool, i.e., $W_i^{\mathrm{p2p}}(X_1, \dots, X_m) \succeq_{ssd} W_i^{\mathrm{p2p}}(X_1, \dots, X_{m-1})$ for all $m = 2, \dots, n$, and new members only join if it is beneficial for them, i.e., $W_m^{\mathrm{p2p}}(X_1, \dots, X_m) \succeq_{ssd} W_m^{\mathrm{no}}$, then the P2P insurance pool also satisfies Property SSD1. Property SSD2 is a reasonable criterion if the P2P insurance company is organized such that it acts in the best interest of current pool members – avoiding utility losses for them.[15]

[13] See Definition 3.33 and Remark 3.34 in Denuit et al. (2022).

[14] Anthropelos et al. (2024) refer to a closely related property in their analysis based on the entropic risk measure as "strong consensus", whereas Property SSD1 is comparable to their notion of "weak consensus".

[15] Note that this criterion is related to regulatory rules protecting existing policyholders in participating life insurance from adverse effects to their positions when new risks are included in a portfolio.

The proposed criteria can help to assess the benefits of P2P insurance from the policyholders' perspective – taking the current composition of the risk pool $(X_1, \ldots, X_n)$ or $(X_1, \ldots, X_{n-1})$ as given. However, the properties do not rule out that higher pooling benefits could be achieved in different pools $\tilde{X}_1, \ldots, \tilde{X}_m$ or with alternative risk sharing schemes that exploit information about the policyholders' specific utility functions. Note that SSD relations similar to the Properties SSD1 and SSD2 cannot be used to compare P2P insurance to policies offered by stock insurers with a premium exceeding the expected indemnity because the choice between such products depends on the policyholder's risk-aversion level.

For the pricing of P2P insurance, we will apply a financial approach building on the following assumption:

Assumption P1 *There are no arbitrage opportunities.*

The absence of arbitrage ensures the existence of a non-negative Stochastic Discount Factor (SDF) (Hansen and Richard 1987). However, since we assume that individual insurance claims cannot be replicated, we allow the market to be incomplete. Accordingly, the SDF does not have to be unique in our setting. We denote the set of non-negative SDFs by $\mathcal{M}$. For a given $M \in \mathcal{M}$, the definition of the SDF implies that the corresponding P2P insurance premium for policyholder i satisfies

$$\pi_{i,n}(X_1, \ldots, X_n) = \mathbb{E}[M \cdot (X_i + P_i(X_1, \ldots, X_n))]\,. \tag{8}$$

By adopting this approach, we avoid choosing a specific premium principle which can limit the generality of our findings and we are able to account for the profit and loss participation received by the agents joining a P2P insurance pool. Moreover, the market-based approach ensures that systematic risk exposure[16] is consistently priced and that P2P insurance prices are in line with competitive prices charged by stock insurers for full coverage of the insurable risks. Note that the premium in equation (8) can alternatively be written as a discounted payoff under a risk-neutral measure.[17] Furthermore, it can be understood as a premium principle in sense of Bühlmann and Jewell (1979) applied to the total payoff $X_i + P_i(X_1, \ldots, X_n)$ with the SDF M as loading factor.

[16] We say that the risk X_i is systematic if $\operatorname{cov}[X_i, M] \neq 0$.

[17] In particular, it holds that $\pi_{i,n} = (1 + r_f)^{-1}\,\mathbb{E}^Q[X_i + P_i(X_1, \ldots, X_n)]$, where the risk-neutral measure Q is obtained from the Radon-Nikodym derivative $Z = (1 + r_f)\,M$.

3 Baseline Analysis

In this section, we first present our main results for the general setting introduced in the previous section. Then we focus on the special case of risks that are homogeneous with respect to their systematic risk exposure and finally we discuss the relation of the proposed P2P contract to results from the risk sharing literature.

3.1 Heterogeneous Systematic Risk

3.1.1 Contract Form

We first characterize the set of premiums and profit sharing rules for P2P insurance that are consistent with (i) arbitrage-free pricing and can (ii) potentially generate benefits for all risk-averse agents as stated in Property SSD1.

Proposition 1 *Suppose that Assumptions B1-B4 as well as Assumption P1 hold and that the expectation of Y_i is finite for $i = 1, \dots, n$. Furthermore, assume that the P2P insurer applies the SDF $M \in \mathcal{M}$ for pricing the risks $X_1, \dots, X_n$. Then arbitrage-free premiums that are consistent with Property SSD1 must be of the form*

$$\pi^*_{i,n} = (1 + r_f)^{-1} \left[\mathbb{E}[X_i] + \theta^*_{i,n} \left(\sum_{j=1}^{n} (1 + r_f)\, \pi^*_{j,n} - \mathbb{E}[X_j] \right) \right] \quad \text{for } i = 1, \dots, n \tag{9}$$

and we distinguish the following cases for the corresponding profit and loss sharing rule:

i) If $\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] \neq 0$*, no-arbitrage pricing implies*

$$\theta^*_{i,n} = \frac{\operatorname{cov}[M, X_i]}{\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right]} \qquad \text{for } i = 1, \dots, n. \tag{10}$$

ii) If $\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] = 0$ *and* $\operatorname{cov}[M, X_{i_0}] \neq 0$ *for* $i_0 \in \{1, \dots, n\}$*, an arbitrage-free profit sharing rule that is consistent with Property SSD1 does not exist.*

iii) If $\operatorname{cov}[M, X_i] = 0$ *for all* $i = 1, \dots, n$*, then no arbitrage does not impose constraints on the profit and loss sharing rule* $\theta^*_{1,n}, \dots, \theta^*_{n,n}$*.*

Proof: See the Appendix.

According to equation (9), the premium paid for participation in the P2P insurance pool corresponds to the expected payoff at the end of the period for each policyholder discounted at the risk-free rate. Under the assumptions used in Proposition 1, the premium is only characterized up to the total safety loading $\sum_{j=1}^{n}(1+r_f)\,\pi_{j,n}^{*} - \mathbb{E}[X_j]$. For a given level of the total loading, equation (9) describes how to split this loading between the members of the pool. A simple solution corresponds to the actuarial fair premium $\pi_{i,n}^{*} = (1+r_f)^{-1} \cdot \mathbb{E}[X_i]$ for all $i = 1, \ldots, n$.

In general, equation (9) implies

$$W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) = (1+r_f)\, w_{0,i} + Y_i - \mathbb{E}[X_i] + \theta_{i,n}^{*} \cdot \sum_{j=1}^{n}(\mathbb{E}[X_j] - X_j) \tag{11}$$

for the agent's wealth with P2P insurance at time $t = 1$. Consequently, the choice of the safety loading is not relevant for the agent's final wealth, which is due to the redistribution of the loading back to the pool members through their profit and loss participation combined with the assumption that there is no default risk regarding the policyholder's participation.[18] In addition, equation (11) implies that the magnitudes of $w_{0,i}$ and r_f are irrelevant for the Properties SSD1 and SSD2.

Equation (11) also shows that buying P2P insurance for the premium $\pi_{i,n}^{*}$ does not change the agent's expected wealth, that means,

$$\mathbb{E}\Big[W_i^{\mathrm{p2p}}(X_1, \ldots, X_n)\Big] = \mathbb{E}\Big[W_i^{\mathrm{p2p}}(X_1, \ldots, X_{n-1})\Big] = \mathbb{E}[W_i^{\mathrm{no}}(X_i)]\,. \tag{12}$$

Interestingly, case i) in Proposition 1 states that there is a unique arbitrage-free profit and loss sharing rule that can make P2P insurance attractive for all risk-averse agents if the total risk of the pool is not fully unsystematic, that is, $\operatorname{cov}\Big[M, \sum_{j=1}^{n} X_j\Big] \neq 0$. The shares in the P2P insurer's surplus must then be given by the individual contributions to the total systematic risk of the P2P insurance pool.[19] If $\mathbb{E}\big[Y_i^2\big] < \infty$, this allocation implies

$$\operatorname{cov}\Big[M, W_i^{\mathrm{p2p}}(X_1, \ldots, X_n)\Big] = \operatorname{cov}\Big[M, W_i^{\mathrm{p2p}}(X_1, \ldots, X_{n-1})\Big] = \operatorname{cov}\left[M, W_i^{\mathrm{no}}\right]. \tag{13}$$

Accordingly, the systematic risk of each agent's position is not affected by the P2P risk sharing. As shown in the proof of Proposition 1, this is achieved by setting the profit and loss par-

[18] This observation is similar to the finding that the magnitude of the premium does not matter for the policyholder's utility in a pure mutual with homogeneous risks (Schmeiser and Orozco-Garcia 2021).

[19] A discussion of the relation between this sharing rule and risk allocations considered in the previous literature will be provided in Section 3.3.

ticipations such that the risk premia for the original claim and the participation are identical. This is not possible if there is a systematic component in some of the individual claims, i.e., $\operatorname{cov}[M, X_i] > 0$ (or $\operatorname{cov}[M, X_i] < 0$), but the systematic risk is fully diversified at the pool level, i.e., $\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] = 0$. In this case, a premium according to equation (9) would be too low (or too high), explaining the result in part ii) of Proposition 1. The result in iii) follows from a similar argument because $\operatorname{cov}[M, X_i + P_i(X_1, \ldots, X_n)]$ is always zero if all risks in the P2P pool are uncorrelated with the SDF.

Next, we present a simple example that illustrates the form of the contract given in Proposition 1.

Example 1 (Contract Form) *Suppose that* $w_{0,i} = 100$, $r_f = 0$, $Y_i = 0$, $\mathbb{E}[X_i] = 10$, $\operatorname{var}[X_i] = 4$, $i = 1, 2$, *and that the P2P insurance contract from Proposition 1 is applied. Then, it holds that* $\mathbb{E}[W_i^{no}] = \mathbb{E}\left[W_i^{p2p}\right] = 90$ *and the variance of wealth without insurance is given by* $\operatorname{var}[W_i^{no}] = 4$, $i = 1, 2$. *Now, let us compare the following two cases:*

i) $\operatorname{cov}[M, X_1] = \operatorname{cov}[M, X_2] = 1$ *and* $\operatorname{cov}[X_1, X_2] = 2$,

ii) $\operatorname{cov}[M, X_1] = 1$, $\operatorname{cov}[M, X_2] = 0$ *and* $\operatorname{cov}[X_1, X_2] = 0$.

In case i), we obtain $\theta_{1,2}^* = \theta_{2,2}^* = \frac{1}{2}$ *for the participation factors and* $\operatorname{var}[X_1 + X_2] = 12$. *This implies* $\operatorname{var}\left[W_1^{p2p}\right] = \operatorname{var}\left[W_2^{p2p}\right] = 3$. *In case ii), it holds that* $\theta_{1,2}^* = 1$, $\theta_{2,2}^* = 0$ *and* $\operatorname{var}[X_1 + X_2] = 8$, *which implies* $\operatorname{var}\left[W_1^{p2p}\right] = 8$ *and* $\operatorname{var}\left[W_2^{p2p}\right] = 0$. *If we assume mean-variance preferences, the P2P insurance contract would be beneficial to both agents in case i), whereas the first agent would not join the pool in case ii).*

3.1.2 Necessary Conditions

The previous example illustrates that the contract form derived in Proposition 1 is not sufficient to ensure the occurrence of utility gains in the sense of Property SSD1. We next present necessary conditions on the risks in the pool that must be satisfied if the Properties SSD1 and SSD2 hold.

Proposition 2 *Suppose that the assumptions used in Proposition 1 hold and that the corresponding pricing and profit sharing rules for* $\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] \neq 0$ *are applied. Furthermore, assume that* $\operatorname{var}[Y_i] < \infty$ *for* $i = 1, \ldots, n$.

i) Then,

$$\left|\frac{\operatorname{cov}[M, X_i]}{\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right]}\right| \leq \frac{\sigma[X_i]}{\sigma[\sum_{j=1}^{n} X_j]} \qquad \text{for all } i = 1, \ldots, n, \tag{14}$$

is a necessary condition for Property SSD1,

ii) and

$$\left|\frac{\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right]}{\operatorname{cov}\left[M, \sum_{j=1}^{n-1} X_j\right]}\right| \geq \frac{\sigma[\sum_{j=1}^{n} X_j]}{\sigma[\sum_{j=1}^{n-1} X_j]} \tag{15}$$

is a necessary condition for Property SSD2.

According to part i), joining a P2P insurance pool can only be beneficial to all risk-averse policyholders if the ratio of the policyholder's own systematic risk and the total systematic risk of the pool is not larger than the ratio of the policyholder's claim standard deviation and the standard deviation of the pool's total claim amount. Part ii) of Proposition 2 states that all risk-averse policyholders can only benefit from admitting a new agent to a P2P insurance pool if the resulting percentage increase in the total claim's standard deviation is not larger than the increase in the (absolute) magnitude of the pool's systematic risk exposure, as measured by its covariance with the SDF M. Both second-moment conditions are derived by exploiting that a higher variance is inconsistent with a SSD preference. Economically, these results capture that the individual agents also care about high levels of unsystematic risk, which is redistributed through P2P risk sharing but not compensated under financial no-arbitrage pricing.

We illustrate the conditions given in Proposition 2 with the following example:[20]

Example 2 (Normally-Distributed Risks) *We consider a simple stylized model for a pool of jointly normally distributed risks* $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $i = 1, \ldots, n$, *and assume* $Y_i = 0$, $i = 1, \ldots, n$. *Consistent with a standard linear factor model, the following structure on the moments of the individual claims is imposed:* $\mu_i = a_i + b_i \cdot \mathbb{E}[M]$, $\sigma_i^2 = b_i^2 \cdot \operatorname{var}[M] + \sigma_{\varepsilon,i}^2$, $\operatorname{cov}[X_i, M] = b_i \cdot \operatorname{var}[M]$ *and* $\operatorname{cov}[X_i, X_j] = b_i \cdot b_j \cdot \operatorname{var}[M]$, $i \neq j$, *with* $b_i \in \mathbb{R}$, $\sigma_{\varepsilon,i} > 0$ *for all* $i = 1, \ldots, n$. *We investigate the benefits of P2P insurance from the perspective of agent 1 assuming that her initial wealth is given by* $w_{1,0} = 10$ *and that her preferences are captured by an exponential utility function with the risk aversion parameter* $\gamma_1 = 0.5$, *that is,* $u_1(w) = 1 - \exp(-\gamma_1 \cdot w)$. *Moreover, we use* $r_f = 0.02$ *for the risk-free rate and assume* $\operatorname{var}[M] = 0.5^2$.

[20] Derivations of the results shown in this example are summarized in Section B of the Internet Appendix.

Under the given assumptions, $\sum_{j=1}^{n} X_j$ is also normally distributed, and we can rely on the moment-generating function of the normal distribution to compute the agent's expected utility. We investigate two cases with the following parameter values:

1) $a_i = 1$, $b_i = 4 + i$ and $\sigma_{\varepsilon,i} = 2$ for $i = 1, \ldots, 40$,

2) $a_i = 1$, $b_i = 4$ and $\sigma_{\varepsilon,i} = 2 + i$ for $i = 1, \ldots, 40$.

Accordingly, we analyze P2P insurance pools consisting of heterogeneous claims, whose variances are increasing in i—with the differences driven by systematic (unsystematic) risk in the first (second) case. We apply the P2P insurance contract derived in Proposition 1 and build on equation (11) *to calculate the agent's expected utility from P2P insurance. Panels A and B in Figure 1 show how the corresponding certainty equivalent varies when successively adding the risks $X_2, \ldots, X_{40}$ to the pool.*

Whereas the certainty equivalent turns out to be monotonically increasing in case 1, it is first increasing and then decreasing in case 2, which reflects that the variance differences in the latter case are associated with higher levels of uncompensated idiosyncratic risk. Panels C and D illustrate when the necessary conditions of Proposition 2 are satisfied by plotting the difference between the left- and right-hand sides of the inequalities in equations (14) *and* (15)*. They show that the utility losses in the second case are caused by violations of these necessary conditions.*

Although we were able to distinguish between cases with utility gains and losses in the previous example based on the second-moment conditions given in Proposition 2, these conditions are *not* sufficient for utility gains in general.[21] A relatively simple example illustrating this point can be obtained by using a fourth-order Taylor expansion[22] of the policyholders' expected utility

$$\mathbb{E}[u_i(W)] \approx T_4[u_i(W)] := u_i(\bar{w}) + \frac{u_i^{(2)}(\bar{w})}{2} M_2^*[W] + \frac{u_i^{(3)}(\bar{w})}{6} M_3^*[W] + \frac{u_i^{(4)}(\bar{w})}{24} M_4^*[W] \qquad (16)$$

with $\bar{w} := \mathbb{E}[W]$, $M_i^*[W] = \mathbb{E}\big[(W - \mathbb{E}[W])^i\big]$ as the i-th central moment of W and $u_i^{(k)} = \frac{du_i^k}{d^k w}$ as the k-th derivative of u_i.[23]

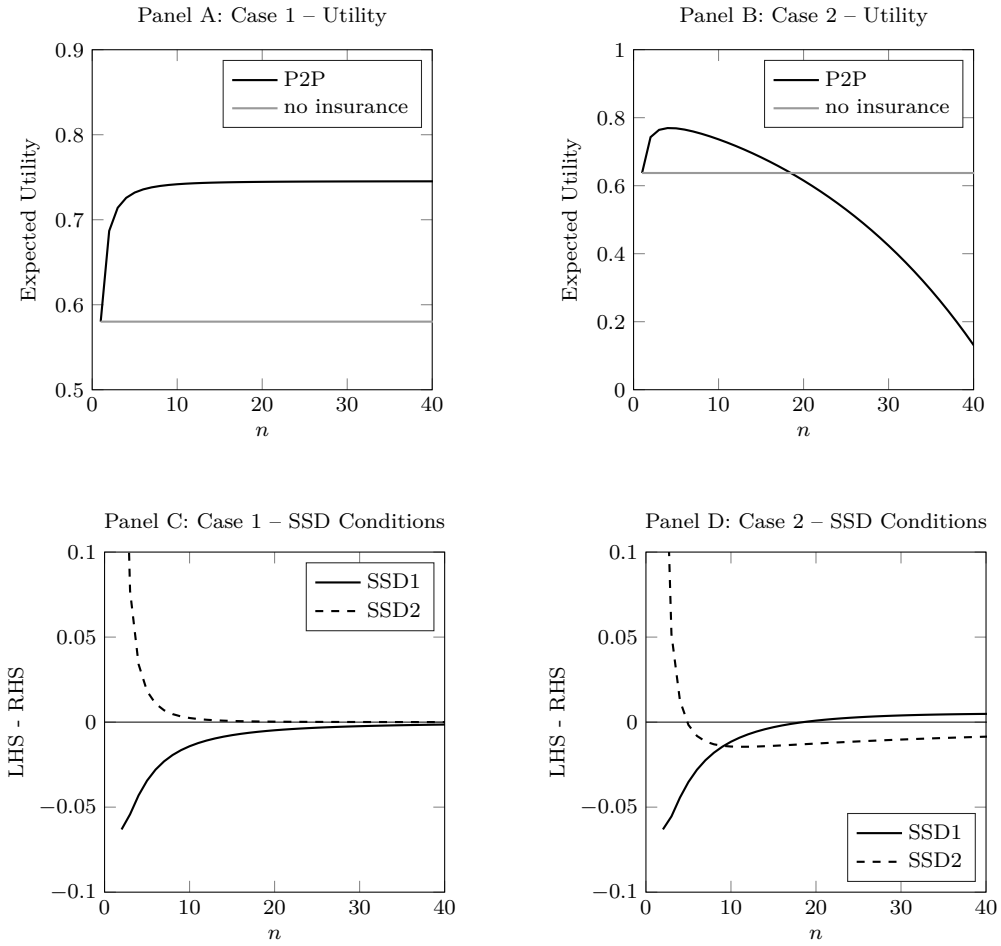
Example 3 (Non-Normal Risks) *As in the previous example, we investigate again the benefits of the P2P insurance contract derived in Proposition 1 from the perspective of the first policyholder*

[21] This observation mirrors the well-known differences in the theory of portfolio selection between mean-variance optimization and expected utility maximization (Levy 1974, Levy and Markowitz 1979).

[22] Cf. Tsiang (1972, p. 356) for a discussion of such approximation.

[23] Details on the required calculations for this example can be found in Section B of the Internet Appendix.

Figure 1: Normally-Distributed Risks



Note: This figure shows how the policyholder's utility changes when including additional risks in a P2P insurance pool that applies the contract derived in Proposition 1 under the assumptions described in Example 2. Moreover, it illustrates how changes in the policyholder's utility are related to the conditions derived in Proposition 2. We take the perspective of agent 1 and use an exponential utility function with $\gamma_1 = 0.5$. Furthermore, we assume $w_{0,1} = 10$, $r_f = 0.02$ and $Y_1 = 0$. The risks $X_1, \ldots, X_{40}$ follow a multivariate normal distribution with the variance increasing in i. We consider two cases for the model parameters: In case 1 (2), the increase in the variance is driven by a higher level of systematic (unsystematic) risk. See Example 2 for a full description of the model parameters. Panels A and B show the expected utility from buying P2P insurance as a function of the pool size in both cases. Panels C and D illustrate when the necessary conditions derived in Proposition 2 are satisfied from the first policyholder's perspective. They show the difference between the left- and right-hand sides of the inequalities in equations (14) and (15) for each pool size n. The necessary condition for Property SSD1 (SSD2) is satisfied when the solid (dashed) line is not larger (smaller) than zero.

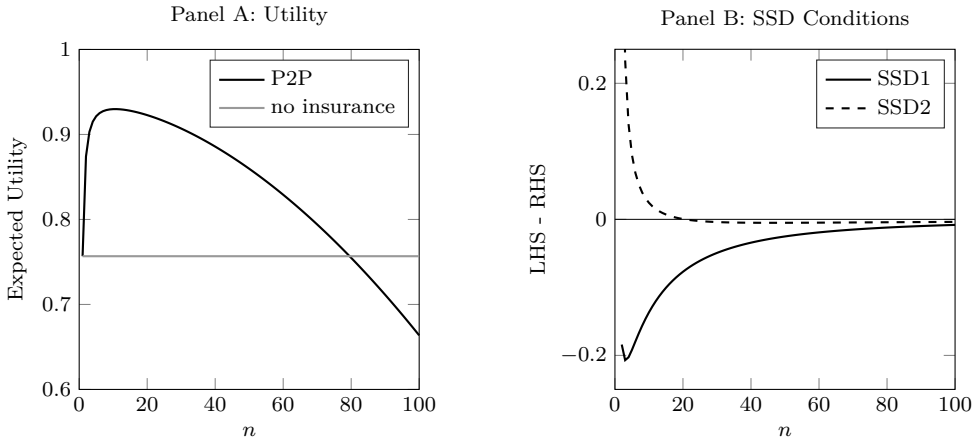
with initial wealth $w_{i,0} = 10$ *and* $Y_1 = 0$. *We now apply the Taylor expansion shown in equation* (16) *for an exponential utility function with* $\gamma_1 = 0.5$ *to evaluate the first policyholder's utility. Furthermore, we again use* $r_f = 0.02$ *and* $\text{var}[M] = 0.5^2$. *In contrast to the previous example, we now assume that the insurable risks* $X_1, \ldots, X_n$ *follow a mixture distribution with a common state indicator* S *(Wang 1998). We use two states with* $\mathbb{P}[S = L] = 0.99$ *and* $\mathbb{P}[S = H] = 0.01$. *Generalizing Example 2, we assume a multivariate normal distribution for* $X_1, \ldots, X_n$ *conditional on* $S = s$, $s \in \{L, H\}$, *with* $(X_1, \ldots, X_n)'|S = s \sim \mathcal{N}(\boldsymbol{\mu}_s, \boldsymbol{\Sigma}_s)$, *where* $(\boldsymbol{\mu}_s)_i = a_{i,s} + b_i \cdot \mathbb{E}[M]$, $(\boldsymbol{\Sigma}_s)_{ii} = b_i^2 \cdot \text{var}[M] + \sigma^2_{\varepsilon,i,s}$ *for* $i = 1, \ldots, n$ *and* $(\boldsymbol{\Sigma}_s)_{ij} = b_i \cdot b_j \cdot \text{var}[M]$ *for* $i \neq j$. *Furthermore, we suppose that the systematic risk exposure is not state-specific. In particular, we assume* $\text{cov}[X_i, M|S = L] = \text{cov}[X_i, M|S = H] = b_i \cdot \text{var}[M]$ *and* $\mathbb{E}[M \mid S = L] = \mathbb{E}[M \mid S = H] = \mathbb{E}[M]$ *for* $i = 1, \ldots, n$. *We apply* $a_{i,L} = 2$, $a_{i,H} = 2 + 3 \cdot \sqrt{i}$, $b_i = 2$ *and* $\sigma_{\varepsilon,i,L} = \sigma_{\varepsilon,i,H} = 4$ *for all* $i = 1, \ldots, n$. *Figure 2 summarizes the results. Panel A shows the first agent's expected utility from buying P2P insurance with the risks* $(X_1, \ldots, X_n)$ *measured by* $T_4[u_1(X_1, \ldots, X_n)]$ *for* $n = 2, \ldots, 100$. *The first policyholder's utility is nonmonotonic in the given example. It initially increases when including additional risks in the P2P pool but, subsequently, it decreases. For* $n \geq 80$, *the first policyholder's utility falls below the level of the no-insurance alternative although the necessary condition for Property SSD1 given in Proposition 2 is always satisfied for* $n \in [2, 100]$ *as shown in Panel B of Figure 2. Furthermore, the decrease in the utility of the first policyholder already starts for* $n = 12$ *while the corresponding necessary condition for Property SSD2 still holds, which can also be seen from Panel B.*

3.1.3 Sufficient Conditions

Building on recent findings in the theory of stochastic dominance (Pomatto et al. 2020), we next show that the proposed P2P insurance contract is always beneficial if the second-moment conditions given in Proposition 2 are satisfied and the policyholders' background risk is large enough. For this result, we will rely on the following definition: We say that a random variable Z' is noisier than another random variable Z if there exists an independent random variable W with $Z' = Z + W$ (Pomatto et al. 2020, p. 1880).

Using this notion and building on Theorem 2 in Pomatto et al. (2020), we are able to derive the following result:

Figure 2: Non-Normally-Distributed Risks



Note: This figure shows how the policyholder's utility changes when including additional risks in a P2P insurance pool that applies the contract derived in Proposition 1 under the assumptions described in Example 3. Moreover, it illustrates how changes in the policyholder's utility are related to the conditions derived in Proposition 2. We apply the Taylor expansion introduced in equation (16) to evaluate the first policyholder's expected utility assuming an exponential utility function with $\gamma_1 = 0.5$. Furthermore, we again assume $w_{0,1} = 10$, $r_f = 0.02$ and $Y_1 = 0$. The risks $X_1, \ldots, X_{100}$ follow a multivariate mixture of normals with two states and a common state indicator as described in Example 3. Panel A shows the expected utility of buying P2P insurance as a function of the pool size, and Panel B illustrates when the necessary conditions from Proposition 2 are satisfied. It shows the difference between the left- and right-hand sides of the inequalities in equations (14) and (15) for each pool size n. The necessary condition for Property SSD1 (SSD2) is satisfied when the solid (dashed) line is not larger (smaller) than zero.

Proposition 3 *Suppose that Assumptions B1-B4 and P1 are satisfied. Furthermore, assume that the P2P pool applies the arbitrage-free pricing and profit sharing rules from equations* (9) *and* (10) *in Proposition 1. For each collection of insurable losses* $(X_1, \ldots, X_n)$*, there exist random variables* $Y_{i,n}^*$*,* $i = 1, \ldots, n$*, and* Y_n^{**} *that are independent of* $X_1, \ldots, X_n$ *and have the following properties:*

i) If a strict inequality holds in equation (14) *and the background risk* Y_i *is noisier than* $Y_{i,n}^*$ *for* $i = 1, \ldots, n$*, then Property SSD1 is satisfied.*

ii) If a strict inequality holds in equation (15) *and the background risk* Y_i *is noisier than* Y_n^{**} *for* $i = 1, \ldots, n$*, then Property SSD2 is satisfied.*

Proof: See the Appendix.

Proposition 3 states that P2P insurance is always beneficial and a larger pool is preferable if the variance conditions derived in Proposition 2 are strictly satisfied and the policyholder's background risk Y_i is noisier than $Y_{i,n}^*$ and Y_n^{**}, $i = 1, \ldots, n$. The existence of the random variables $Y_{i,n}^*$ and Y_n^{**}, which define the relevant minimum background risk, follows from Theorem 2 in Pomatto et al.

(2020). As explained by Pomatto et al. (2020), $Y_{i,n}^*$ and Y_n^{**} must exhibit a certain level of tail risk in the sense that their moment-generating functions are not finite.[24] The relevance of this result likely depends on the lines of business that are analyzed or, more generally, on the relative magnitudes of the risk transferred to the pool and the policyholders' background risk.

In addition to the general background risk result formalized in Proposition 3, we next examine the special case in which the policyholder's expected utility can be approximated using the fourth-order Taylor expansion introduced in equation (16). We furthermore impose $u_i^{(4)} < 0$, which is a common preference assumption often referred to as temperance.[25] This specific setting will allow us to derive explicit lower bounds on the minimum variance of the background under which the proposed P2P contract is beneficial and larger pools are preferable.

Proposition 4 *Suppose that Assumptions B1-B4 as well as Assumption P1 are satisfied and that the P2P pool applies the arbitrage-free pricing and profit sharing rules from equations* (9) *and* (10) *in Proposition 1. Additionally, assume that* $\mathbb{E}\big[X_i^4\big] < \infty$, $\mathbb{E}\big[Y_i^4\big] < \infty$, $\mathbb{E}[Y_i] = 0$ *and* $u_i^{(4)} < 0$ *for all* $i = 1, \dots, n$. *Then there exist variance thresholds* $v_{i,n}^*$ *and* $v_{i,n}^{**}$ *for all* $i = 1, \dots, n$, $n \in \mathbb{N}$, $n \geq 2$ *that satisfy the following properties:*

i) If a strict inequality holds in equation (14) *and* $\operatorname{var}[Y_i] \geq v_{i,n}^*$, *then* $T_4[u_i(W_i^{p2p}(X_1, \dots, X_n))] \geq T_4[u_i(W_i^{no})]$ *for all* $i = 1, \dots, n$.

ii) If a strict inequality holds in equation (15) *and* $\operatorname{var}[Y_i] \geq v_{i,n}^{**}$, *then* $T_4[u_i(W_i^{p2p}(X_1, \dots, X_n))] \geq T_4[u_i(W_i^{p2p}(X_1, \dots, X_{n-1}))]$ *for all* $i = 1, \dots, n$.

Proof: See the Appendix.

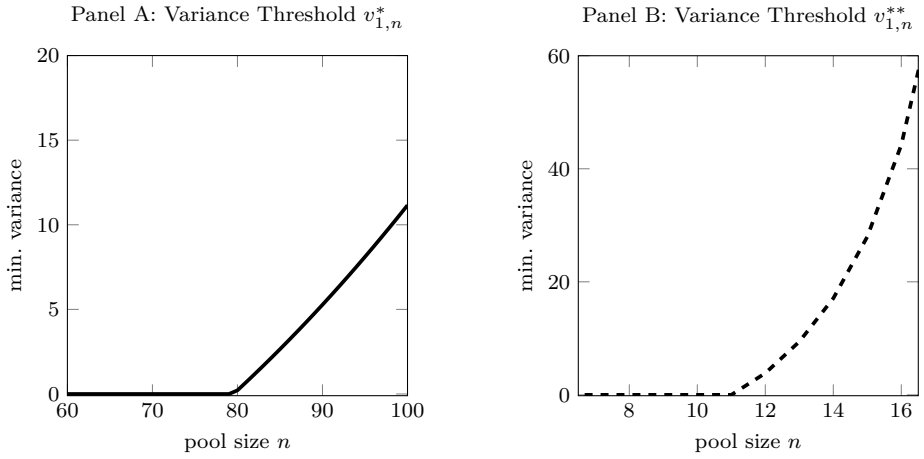
Under the assumptions in Proposition 4, we can characterize the magnitude of the minimum background risk based on the variance thresholds $v_{i,n}^*$ and $v_{i,n}^{**}$. These thresholds can be computed in closed form based on the derivatives of the policyholder's utility function and the moments of the relevant wealth positions, as shown in the proof of Proposition 4. The following example illustrates these results.

Example 4 (Minimum Background Risk) *We reexamine the setting from Example 3 and calculate the minimum variance thresholds* $v_{1,n}^*$ *and* $v_{1,n}^{**}$ *for the background risk derived in Proposition 4 from the perspective of the first policyholder. Panel A of Figure 3 shows the threshold* $v_{1,n}^*$

[24] This can be seen from the fact that independent background risk would *not* affect the preference ranking of the two positions under an exponential utility function. See Section V.A in Pomatto et al. (2020).

[25] Cf. Eeckhoudt and Schlesinger (2006) for this terminology and a discussion of this property.

Figure 3: Minimum Background Risk



Note: This figure illustrates the minimum background risk thresholds established in Proposition 4 under the assumptions described in Example 4. We again take the perspective of policyholder 1 and use the Taylor approximation introduced in equation (16) for an exponential utility function with $\gamma = 0.5$, $w_{0,1} = 10$ and $r_f = 0.02$. The risks $X_1, \ldots, X_{100}$ follow a multivariate mixture of normals with two states and a common state indicator as described in Example 3. Panel A shows the variance threshold $v^*_{1,n}$ for Property SSD1 around the pool size $n = 80$, where the expected utility of P2P insurance without background risk drops below the level of the uninsured position. Panel B shows the variance threshold $v^{**}_{1,n}$ for Property SSD2 around the pool size $n = 12$, where the first policyholder's utility starts decreasing although the necessary condition for Property SSD2 is still satisfied.

around $n = 80$, where the first policyholder's utility from P2P insurance without background risk drops below the level of no insurance, as shown in Panel A of Figure 2. Panel B shows the threshold $v^{**}_{1,n}$ around $n = 12$, where the benefits of larger pools become negative for the first policyholder.

We finally note that similar to well-known results from portfolio theory on the equivalence of mean-variance decision rules and the expected utility criterion (Meyer 1987, Sinn 1989), the second-moment conditions derived in Proposition 2 can be shown to be sufficient if the shape of the total loss distribution does not change when adding risks to the P2P insurance pool. We formally establish the corresponding result as Proposition A.1 in the Internet Appendix by imposing appropriate restrictions on the joint loss distribution.

3.2 Homogeneous Systematic Risk

We next focus on insurable risks satisfying the following homogeneity assumption on their systematic risk exposure.

Assumption P2 *It holds that* $\operatorname{cov}[M, X_i] = c_M$ *for* $i = 1, \ldots, n$ *and* $M \in \mathcal{M}$.

We thus investigate claims with identical systematic risk as measured by their covariance with an SDF M. For this case, we obtain the following result from Proposition 1:

Corollary 1 *Suppose that Assumptions B1-B4, P1 and P2 are satisfied. Then arbitrage-free premiums $\pi^*_{1,n}, \ldots, \pi^*_{n,n}$ consistent with Property SSD1 must satisfy*

$$\pi^*_{i,n} = (1 + r_f)^{-1} \left(\mathbb{E}[X_i] + l\right), \qquad i = 1, \ldots, n, \tag{17}$$

for $l \in \mathbb{R}$ and, if $\operatorname{cov}[M, X_i] \neq 0$, the unique profit and loss sharing rule is given by

$$\theta^*_{i,n} = \frac{1}{n} \qquad i = 1, \ldots, n. \tag{18}$$

Proof: See the Appendix.

Moreover, the profit sharing and the premium do not depend on the choice of a specific SDF for all $\tilde{M} \in \mathcal{M}$ that satisfy $\operatorname{cov}\left[\tilde{M}, X_i\right] = \operatorname{cov}\left[\tilde{M}, X_j\right]$.[26] In this case, the level of $\operatorname{cov}\left[\tilde{M}, X_i\right]$ is irrelevant for the premium and the participation. Interestingly, this result is specific to P2P insurance. In competitive markets, the premium charged by a stock insurer without profit and loss participation of the policyholders would depend on the choice of the SDF $M \in \mathcal{M}$ if $\operatorname{cov}[M, X_i] \neq 0$. Furthermore, by additionally imposing Assumption P2, we obtain a fully explicit characterization of the premium, which corresponds to the actuarial fair premium plus an arbitrary loading l that must be identical for all policies in the pool. Note that the following results also apply to the case $\operatorname{cov}[M, X_i] = 0$ with the limitation that the profit sharing is not uniquely identified in this case.

Under Assumption P2, we obtain the following simplified versions of the conditions from Proposition 2:

Corollary 2 *Suppose that the assumptions used in Corollary 1 hold and that the corresponding pricing and surplus rules from equations* (17) *and* (18) *are used.*

i) Then,

$$\operatorname{var}[X_i] \geq \operatorname{var}\left[\frac{1}{n}\sum_{j=1}^{n} X_j\right] \qquad \textit{for all } i = 1, \ldots, n, \tag{19}$$

is a necessary condition for Property SSD1,

[26] Note, however, that the choice between two SDFs M and M' matters if $\operatorname{cov}[M, X_i] = \operatorname{cov}[M, X_j]$ but $\operatorname{cov}[M', X_i] \neq \operatorname{cov}[M', X_j]$.

ii) and

$$\text{var}\left[\frac{1}{n}\sum_{j=1}^{n} X_j\right] \leq \text{var}\left[\frac{1}{n-1}\sum_{j=1}^{n-1} X_j\right] \tag{20}$$

is a necessary condition for Property SSD2.

Proof: See the Appendix.

Part i) shows that all risk-averse policyholders can only benefit from joining the pool if the variances of their individual insurable risks are at least as large as the variance of the average risk in the pool. According to part ii) of the corollary, an increase in the pool size can only be beneficial for all risk-averse pool members if adding the risk reduces the variance of the average claim in the pool. These conditions show that the order in which risks are added to the pool can affect the acceptability of a risk and the corresponding pooling benefits. If the risks are uncorrelated and ordered such that $\sigma_1 \geq \sigma_2 > \ldots > \sigma_n$, the condition in equation (20) is always satisfied.[27] In this case, it must still be verified that diversification benefits in the pool are sufficiently large such that agents', whose risks exhibit relatively small variances, are willing to buy P2P insurance as formalized by the condition in equation (19).

According to these results, a high degree of heterogeneity in the volatility of claims can limit the attractiveness of P2P insurance as a simple risk-sharing solution. If high-risk individuals are sufficiently risk-averse, they would still be willing to join the P2P insurance pool under less favorable conditions, such as higher premiums. However, the magnitude of such payments and their acceptability for the current policyholders can only be assessed under specific preference assumptions and is thus beyond the scope of our analysis focusing on SSD.

It is straightforward to see that a modified version of Proposition 3 can be derived under Assumption P2 by replacing the second-moment conditions from Proposition 2 with the simpler variance conditions from Corollary 1. This result is included as Corollary A.1 in the Internet Appendix. Furthermore, a rather strong result on the benefits of P2P insurance can be derived for losses that can be written as $X_i = \mu_i + Z_i$ with $\mu_i \in \mathbb{R}$, $i = 1, \ldots, n$, and exchangeable random variables $Z_1, \ldots, Z_n$.[28] For such losses that only differ in their expected claims, Proposition A.2 in the Internet Appendix shows that i) new policyholders are always willing to buy P2P insurance

[27] This follows from the Schur convexity of the function $g(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sigma_i^2 x_{[i]}^2$ for $x_i \geq 0$, $i = 1, \ldots, n$ and $\sigma_1 \geq \sigma_2 > \ldots > \sigma_n$, where $x_{[i]}$ is the i-th largest element of $(x_1, \ldots, x_n)$. Cf. Ibragimov (2009, Appendix C).

[28] Exchangeability requires that the joint distribution of $Z_1, \ldots, Z_n$ is invariant under permutations of the indices, i.e., $(Z_1, \ldots, Z_n) \overset{\text{d}}{=} (Z_{i_1}, \ldots, Z_{i_n})$ for all permutations $i_1, \ldots, i_n$ of $1, \ldots, n$ (McNeil et al. 2015, Definition 7.16).

and ii) larger P2P insurance pools are always weakly preferred to smaller pools by all current pool members without additional shape restrictions.

3.3 Relation to the Risk Sharing Literature

In this section, we discuss relations between the findings of our baseline analysis and previous results from the risk sharing and P2P insurance literature.[29]

If the agents face no credit risk and a loan is used to pay the premium at time $t = 0$, the value of the premium at time $t = 1$ reduced by the surplus participation of policyholder i based on the contract described in Proposition 1 corresponds to

$$\eta_i(X_1, \ldots, X_n) = (1 + r_f) \cdot \pi^*_{i,n} - \theta^*_{i,n} \cdot S(X_1, \ldots, X_n) \tag{21}$$

$$= \mathbb{E}[X_i] + \theta^*_{i,n} \left(\sum_{j=1}^{n} X_j - \mathbb{E}[X_j] \right). \tag{22}$$

Accordingly, the general P2P contract can be seen as a risk-sharing agreement in which the individual loss X_i is exchanged for the payment of $\eta_i(X_1, \ldots, X_n)$, which corresponds to the policyholder's expected loss and a participation in the difference between the realized total claim amount and its expectation. From the pricing result in equation (8) and the definition in equation (21), it follows with $\mathbb{E}[M] = (1 + r_f)^{-1}$ that $\mathbb{E}[M\, \eta_i(X_1, \ldots, X_n)] = \mathbb{E}[M\, X_i]$ for $i = 1, \ldots, n$. The risk exchange implied by the proposed P2P insurance contract is thus financially fair in the sense of Bühlmann and Jewell (1979, p. 247) for a premium principle with the specific loading factor $G = M$. If $\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] \neq 0$, Proposition 1 implies

$$\eta_i(X_1, \ldots, X_n) = \mathbb{E}[X_i] + \frac{\operatorname{cov}[M, X_i]}{\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right]} \left(\sum_{j=1}^{n} X_j - \mathbb{E}[X_j] \right). \tag{23}$$

In this case, our results identify a unique linear risk-sharing rule that is consistent with a given SDF M and can generate utility gains for all risk-averse policyholders. Interestingly, the uniqueness of the solution shown in equation (23) implies that alternative participation factors $\theta_{i,n}$ must violate some of our assumptions used when deriving Proposition 1. That means alternative factors $\theta_{i,n}$, $i = 1, \ldots, n$, would (i) either violate no-arbitrage pricing or (ii) they cannot create utility gains for *all* risk-averse agents irrespective of their specific preferences. Of course, alternative linear sharing

[29] Cf., for example, Denuit et al. (2022) and Feng et al. (2023) for recent reviews of this literature.

rules can still be beneficial or even preferable (and free of arbitrage) for specific utility functions as discussed in the literature on optimal risk sharing, in general, and on financially fair linear risk sharing (Schumacher 2018), in particular.

Note that, despite some structural similarity, the participation rule shown in equation (23) differs from the well-known covariance principle used in risk sharing and capital allocation (see, e.g., Bauer and Zanjani 2015 and Jiao et al. 2023). Whereas the P2P insurance contract derived under our baseline assumptions defines each agent's profit and loss participation based on her individual claim's contribution to P2P pool's *systematic* risk, the covariance principle relies on contributions to the *total* risk of the pool, as measured by the variance of the total claim amount. Accordingly, it relies on participation factors of the form

$$\theta_{i,n}^{\mathrm{cov}} = \frac{\mathrm{cov}\left[X_i, \sum_{j=1}^n X_j\right]}{\mathrm{var}\left[\sum_{j=1}^n X_j\right]}, \qquad \text{for } i = 1, \ldots, n. \tag{24}$$

Moreover, it is interesting to highlight the relation between the participation factors $\theta_{i,n}^*$ used in equation (23) and underwriting betas used in factor models for insurance pricing (Fairley 1979).[30] These betas use the covariance of a claim with a systematic factor to measure the claim's systematic risk.[31] However, for such betas, the covariance is standardized by the factor's variance instead of the P2P pool's systematic risk.

Finally, we note that the axiomatic techniques developed in the more recent literature on risk sharing and P2P insurance (Denuit et al. 2022, Jiao et al. 2023) can provide valuable insights for the choice of $\theta_{i,n}$ if the risks in the pool are completely unsystematic, i.e., if $\mathrm{cov}\,[X_i, M] = 0$ for all $i = 1, \ldots, n$. This application requires a linear sharing rule. For the much-discussed conditional mean risk-sharing rule (Denuit and Dhaene 2012), this requirement is for example satisfied under additional distributional restrictions.[32]

4 Extensions

In this section, we first investigate the robustness of our results with respect to introducing operating expenses and solvency constraints. Then we analyze the effect of investment risk on our findings.

[30] Note, moreover, that the sign of the risk exposures differs in models that measure systematic

[31] Note that the signs of such betas depend on how a claim's risk is measured (returns vs. losses) and on the choice of systematic component (market return vs. SDF itself).

[32] Cf. Denuit and Dhaene (2012, Example 4.7) for the case of normally distributed losses.

4.1 Operating Expenses

To understand the effect of operating expenses on the policyholder's wealth, we assume that each agent is charged an individual expense loading $e_{i,n}$ that can depend on the size of the insurance portfolio n. The loadings are used to cover operating expenses or transaction cost with a value of c_n at time $t = 1$.

If we assume that the loadings exactly cover these expenses, that is $\sum_{j=1}^{n}(1 + r_f)\, e_{j,n} = c_n$, they do not affect the surplus participation of the policyholders because we have

$$\sum_{j=1}^{n}(\pi^*_{j,n} + e_{j,n})\,(1 + r_f) - \sum_{j=1}^{n} X_j - c_n = S(X_1, \ldots, X_n). \tag{25}$$

In this case, a policyholder's wealth from buying P2P insurance with operating expenses can be calculated as $W_i^{\mathrm{p2p,e}}(X_1, \ldots, X_n) = W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) - e_{i,n}$, where W_i^{p2p} corresponds to the wealth level without an expense loading that we analyzed before. In combination with equation (12), this relation implies that the expected wealth with P2P insurance is smaller than the expected wealth of the uninsured position. The decision for P2P insurance thus depends on the level of $e_{i,n}$ and the policyholder's specific preferences, such that the general Property SSD1 cannot be analyzed in this setting. In contrast, our previous results concerning Property SSD2 can be extended to the case with operating expenses if the expense loading $e_{i,n}$ is assumed to be weakly decreasing in n (for example, due to decreasing fixed cost per policyholder). Then cost benefits strengthen a preference for larger P2P pools and our previous conditions for Property SSD2 will also be sufficient in the case with operating expenses under the given assumptions.

4.2 Solvency Constraints

Since the loading is not directly relevant to the policyholder's wealth under the assumptions used in Proposition 1, as shown in equation (11), it can be determined based on solvency considerations related to potential frictions, such as bankruptcy costs.

In our setting without equity holders, insolvency concerns could arise for policyholders if additional payments at $t = 1$ must be made due to high aggregate losses. Such payments can be avoided in most cases by including sufficiently large safety loadings as part of the premiums paid at time $t = 0$. In particular, the total loading $l_n := \sum_{j=1}^{n} \pi^*_{i,n} - \mathbb{E}[X_j]$ could be chosen such that the probability of additional contributions by the policyholders at time $t = 1$ becomes arbitrarily

small, i.e.,

$$\mathbb{P}\left[\sum_{j=1}^{n} X_j > \sum_{j=1}^{n}(1+r_f)\,\pi^*_{j,n}\right] \leq \alpha \tag{26}$$

for $\alpha \in (0,1)$. This condition can be translated into a standard Value-at-Risk (VaR_α) constraint

$$l_n = \sum_{j=1}^{n}(1+r_f)\,\pi^*_{j,n} - \mathbb{E}[X_j] \geq \text{VaR}_\alpha\left[\sum_{j=1}^{n}(X_j - \mathbb{E}[X_j])\right] \tag{27}$$

with the VaR_α defined as lower the $(1-\alpha)$-quantile of the loss distribution, i.e., $\text{VaR}_\alpha[L] = \inf\{l \in \mathbb{R} : \mathbb{P}[L \leq l] \geq 1-\alpha\}$.

If limiting the probability of additional contributions at time $t = 1$ is not a major concern for policyholders or a potential regulator, the P2P insurer could also choose solutions with a zero loading or even negative loadings – postponing payments to $t = 1$.

4.3 Investment Risk

Finally, we consider the effect of investment risk on our previous results by replacing Assumption B2 with the following assumption:

Assumption I1 *All agents invest their available wealth between $t = 0$ and $t = 1$ in a risky portfolio with the random return R_A.*

Under this assumptions, the wealth of policyholder i without insurance is given by $W_i^{\text{A,no}} = (1+R_A)\cdot w_{0,i} + Y_i - X_i$. Furthermore, we assume that the P2P insurance company also invests at the risky rate R_A, so that its surplus can be computed as follows:

Assumption I2 *The total surplus of the P2P insurance company is given by*

$$S^A(X_1,\ldots,X_n) = \left(\sum_{j=1}^{n}\pi_{j,n}\right)\cdot(1+R_A) - \sum_{j=1}^{n} X_j, \tag{28}$$

where R_A refers to the random asset return over the period $[0,1]$.

These assumptions effectively allow us to avoid changes in the agents' investment risk through P2P insurance, which would require a preference-specific compensation. Under these assumptions, we are able to prove the following generalization of the proposed SSD-consistent contract design to the case of risky investment portfolios.

Proposition 5 *Suppose that Assumptions B1, B3, B4, P1 as well as Assumptions I1, I2 hold and that the P2P insurer applies the SDF $M \in \mathcal{M}$. Then arbitrage-free premiums that are consistent with Property SSD1 must be of the form*

$$\pi_{i,n}^* = (1 + \mathbb{E}[R_A])^{-1} \cdot \left[\mathbb{E}[X_i] + \theta_{i,n}^* \cdot \left(\sum_{j=1}^{n} (1 + \mathbb{E}[R_A])\, \pi_{j,n}^* - \mathbb{E}[X_j] \right) \right] \tag{29}$$

for $i = 1, \ldots, n$. We distinguish the following cases for the corresponding profit and loss sharing rule:

i) If $\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] - \operatorname{cov}[M, R_A] \left(\sum_{j=1}^{n} \pi_{j,n}^*\right) \neq 0$, *no-arbitrage pricing implies*

$$\theta_{i,n}^* = \frac{\operatorname{cov}[M, X_i] - \operatorname{cov}[M, R_A]\, \pi_{i,n}^*}{\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] - \operatorname{cov}[M, R_A] \left(\sum_{j=1}^{n} \pi_{j,n}^*\right)} \qquad \text{for } i = 1, \ldots, n. \tag{30}$$

ii) If $\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] - \operatorname{cov}[M, R_A] \left(\sum_{j=1}^{n} \pi_{j,n}^*\right) = 0$ *and* $\operatorname{cov}[M, X_{i_0}] - \operatorname{cov}[M, R_A]\, \pi_{i_0,n}^* \neq 0$ *for an index $i_0 \in \{1, \ldots, n\}$, an arbitrage-free profit sharing rule that is consistent with Property SSD1 does not exist.*

iii) If $\operatorname{cov}[M, X_i] - \operatorname{cov}[M, R_A]\, \pi_{i,n}^* = 0$ *for all $i = 1, \ldots, n$, then no-arbitrage does not impose constraints on the profit and loss sharing rule $\theta_{1,n}^*, \ldots, \theta_{n,n}^*$.*

Proof: See the Appendix.

Similar to the case without investment risk, the premium of policyholder i can be written as the discounted expectation of her payoff at time $t = 1$. However, with investment risk, the discount factor is based on the expectation of the random return R_A instead of the risk-free rate as shown in equation (29). The cases in i) - iii) that describe the profit and loss participation are also similar to the corresponding results in Proposition 1. An important difference is that the profit and loss participation now depends on both, the systematic risk contribution of individual policyholders and the investment risk arising from premium payments. With $l_n := \sum_{j=1}^{n} (1 + \mathbb{E}[R_A]) \cdot \pi_{j,n}^* - \mathbb{E}[X_j]$ for the total safety loading of the pool, we can write the individual premium as $\pi_{i,n}^* = (1 + \mathbb{E}[R_A])^{-1} \cdot [\mathbb{E}[X_i] + \theta_{i,n} \cdot l_n]$ and it holds that $\sum_{j=1}^{n} \pi_j = (1 + \mathbb{E}[R_A])^{-1} \cdot \left[\sum_{j=1}^{n} \mathbb{E}[X_i] - l_n\right]$. By substituting these results in equation (30), we can show that the participation factor of policyholder i can also

be represented as

$$\theta_{i,n}^{*} = \frac{\operatorname{cov}\left[M, X_i\right] - \operatorname{cov}\left[M, R_A\right]\,\left(1+\mathbb{E}[R_A]\right)^{-1}\mathbb{E}[X_i]}{\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right] - \operatorname{cov}\left[M, R_A\right]\,\left(1+\mathbb{E}[R_A]\right)^{-1}\sum_{j=1}^{n}\mathbb{E}[X_j]}. \tag{31}$$

Accordingly, the participation factors do not depend on the magnitude of the safety loading, but they account for differences in the policyholders' expected losses. In contrast to the case without investment risk, one can still obtain a unique profit allocation when $\operatorname{cov}\left[M, X_i\right] = 0$ for all $i = 1, \ldots, n$, but $\operatorname{cov}\left[M, R_A\right] \neq 0$. In this case, the results in equations (30) and (31) imply

$$\theta_{i,n}^{*} = \frac{\pi_{i,n}^{*}}{\sum_{j=1}^{n}\pi_{j,n}^{*}} = \frac{\mathbb{E}[X_i]}{\sum_{j=1}^{n}\mathbb{E}[X_j]} \qquad \text{for } i = 1, \ldots, n. \tag{32}$$

For the policyholder's wealth from buying the P2P insurance contract described in Proposition 5, we obtain

$$\begin{aligned} &W_i^{\mathrm{A,p2p}}(X_1, \ldots, X_n) \\ &\qquad = (1+R_A)\,w_{0,i} + Y_i - \frac{1+R_A}{1+\mathbb{E}[R_A]}\,\mathbb{E}[X_i] + \theta_{i,n}^{*}\cdot\sum_{j=1}^{n}\left(\frac{1+R_A}{1+\mathbb{E}[R_A]}\,\mathbb{E}[X_j] - X_j\right). \end{aligned} \tag{33}$$

Note that this expression is structurally similar to the case without investment risk. For a risk-free investment, that is, $R_A = r_f$, the wealth shown in equation (33) simplifies to the expression in equation (11). Moreover, it again follows that P2P insurance does not affect the policyholder's expected wealth. In particular, it holds that

$$\mathbb{E}\Big[W_i^{\mathrm{A,p2p}}(X_1, \ldots, X_n)\Big] = (1+\mathbb{E}[R_A])\cdot w_0 + \mathbb{E}[Y_i] - \mathbb{E}[X_i] = \mathbb{E}\Big[W_i^{\mathrm{A,no}}\Big] \tag{34}$$

for $i = 1, \ldots, n$ and $n \in \mathbb{N}$. Equation (33) can be used to derive necessary second-moment conditions for the Properties SSD1 and SSD2 generalizing the results from Proposition 2 to the case with investment risk. Similar to the result in equation (22), the contract presented in Proposition 5 can also be translated into a risk sharing agreement, in which the individual loss X_i is exchanged for the payment of

$$\eta_i^{inv}(X_1, \ldots, X_n) = \frac{1+R_A}{1+\mathbb{E}[R_A]}\,\mathbb{E}[X_i] - \theta_{i,n}^{*}\left(\sum_{j=1}^{n}\frac{1+R_A}{1+\mathbb{E}[R_A]}\,\mathbb{E}[X_j] - \sum_{j=1}^{n} X_j\right). \tag{35}$$

Accordingly, the resulting risk exchange is again linear in $\sum_{j=1}^{n} X_j$ but with a random intercept determined by the asset return R_A. Moreover, the participation factors $\theta^*_{i,n}$, $i = 1, \ldots, n$, also account for the investment risk as shown in equation (31).

5 Conclusion

In this paper, we focus on P2P insurance with heterogeneous risks and analyze potential benefits for risk-averse policyholders. In contrast to the case of pooling homogeneous exposures, for which diversification results are well known, the occurrence of benefits from pooling heterogeneous exposures depends on the types of risks in the pool and on the form of the contract defining each policyholders' payoffs.

In a first step, we characterize the form of arbitrage-free premiums (while assuming an incomplete market setting) and profit-sharing rules that can generate utility gains from P2P insurance with heterogeneous risks for all risk-averse policyholders without requiring information about their specific preferences. Building on this form of the P2P insurance contract, we then establish necessary second-moment conditions for the occurrence of pooling benefits that apply to existing and potential new policyholders. These conditions restrict the relationship between the systematic and the total risk of the P2P pool and the corresponding quantities of the individual policyholders. Finally, we show that the proposed contract design and the second-moment conditions become sufficient for SSD of P2P insurance and a preference for including additional risks if the policyholders' background risks are large enough.

Given the renewed interest in P2P insurance driven by fintech innovation, our results can provide valuable insights for the design of new risk-sharing solutions with heterogeneous exposures that can generate benefits for all risk-averse policyholders. In particular, we derive a simple contract form that requires limited information on the moments of the risks in the pool and almost no information on the policyholder's preferences. Furthermore, the SSD conditions that we establish indicate how differences in second moments and systematic risk should be taken into account for the selection of P2P portfolios and that this insurance design might be especially valuable for exposure types that are not too large in comparison to the policyholders' background risk. In addition, our findings demonstrate the usefulness of financial pricing techniques for the characterization of preference-free risk pooling solutions, which might be an interesting direction for further research.

Appendix

Proof of Proposition 1: To simplify the notation, we set $P_{i,n} := P_i(X_1, \ldots, X_n)$. From the standard expansion of the pricing equation in terms of the covariance and $\mathbb{E}[M] = (1 + r_f)^{-1}$, it follows that

$$\pi^*_{i,n} = \mathbb{E}[M \cdot (X_i + P_{i,n})] = (1 + r_f)^{-1} (\mathbb{E}[X_i] + \mathbb{E}[P_{i,n}]) + \operatorname{cov}[M, X_i + P_{i,n}]. \tag{36}$$

Next, we note that Property SSD1 implies $\mathbb{E}\Big[W_i^{\mathrm{p2p}}(X_1, \ldots, X_n)\Big] \geq \mathbb{E}[W_i^{\mathrm{no}}]$ for all $i = 1, \ldots, n$. Using equations (1) and (5), we can rewrite this condition as

$$(1 + r_f)\, \pi^*_{i,n} \leq \mathbb{E}[X_i + P_{i,n}] = \mathbb{E}\left[X_i + \theta_{i,n} \left(\sum_{j=1}^{n} (1 + r_f)\, \pi^*_{j,n} - X_j \right) \right] \tag{37}$$

for all $i = 1, \ldots, n$. If there exists an index i_0 with $(1+r_f)\, \pi^*_{i_0,n} < \mathbb{E}[X_{i_0} + P_{i_0,n}]$, then equation (37) implies

$$\sum_{i=1}^{n} (1 + r_f)\, \pi^*_{i,n} < \sum_{i=1}^{n} \mathbb{E}\left[X_i + \theta_{i,n} \left(\sum_{j=1}^{n} (1 + r_f)\, \pi^*_{j,n} - X_j \right) \right]. \tag{38}$$

With Assumption B4, specifically $\sum_{i=1}^{n} \theta_{i,n} = 1$, we obtain the contradiction

$$\sum_{i=1}^{n} (1 + r_f)\, \pi^*_{i,n} - \mathbb{E}[X_i] < \sum_{j=1}^{n} (1 + r_f) \cdot \pi^*_{j,n} - \mathbb{E}[X_j]. \tag{39}$$

Therefore, we conclude

$$\pi^*_{i,n} = (1 + r_f)^{-1} \left[\mathbb{E}[X_i] + \theta_{i,n} \left(\sum_{j=1}^{n} (1 + r_f)\, \pi^*_{j,n} - \mathbb{E}[X_j] \right) \right] \qquad \text{for all } i = 1, \ldots, n, \tag{40}$$

which confirms equation (9). Comparing equations (36) and (40), we have $\operatorname{cov}[M, X_i + P_{i,n}] = 0$ for all $i = 1, \ldots, n$. This implies

$$\operatorname{cov}[M, X_i] - \theta_{i,n} \cdot \sum_{j=1}^{n} \operatorname{cov}[M, X_j] = 0 \qquad \text{for all } i = 1, \ldots, n. \tag{41}$$

If $\sum_{j=1}^{n} \operatorname{cov}[M, X_j] \neq 0$, equation (41) uniquely identifies the participation factors $\theta_{1,n}^*, \ldots, \theta_{n,n}^*$ given in equation (10), which confirms case i) in Proposition 1. The cases ii) and iii) also follow from equation (41): If $\operatorname{cov}[X_i, M] = 0$ for all $i = 1, \ldots, n$ and thus $\sum_{j=1}^{n} \operatorname{cov}[X_j, M] = 0$, equation (41) is always satisfied. If $\operatorname{cov}\left[\sum_{j=1}^{n} X_j, M\right] = 0$ but there exists an $i_0 \in \{1, \ldots, n\}$ with $\operatorname{cov}[X_{i_0}, M] \neq 0$, then equation (41) cannot be solved for i_0 — implying that an arbitrage-free P2P insurance solution consistent with Property SSD1 does not exist. ■

Proof of Proposition 2: From equations (10) and (11), it follows that

$$\operatorname{var}\left[W_i^{\mathrm{p2p}}(X_1, \ldots, X_n)\right] = \operatorname{var}[Y_i] + \frac{(\operatorname{cov}[M, X_i])^2}{(\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right])^2} \cdot \operatorname{var}\left[\sum_{j=1}^{n} X_j\right]. \tag{42}$$

Furthermore, $W_1 \succeq_{ssd} W_2$ and $\mathbb{E}[W_1] = \mathbb{E}[W_2]$ imply $W_1 \preceq_{cx} W_2$ and thus $\operatorname{var}[W_1] \leq \operatorname{var}[W_2]$ if $\operatorname{var}[W_2] < \infty$ (Shaked and Shanthikumar 2007, Theorem 4.A.35 (b) and Eq. (3.A.4)). To verify the statement in part i) of Proposition 2, we note that Property SSD1, i.e., $W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) \succeq_{ssd} W_i^{\mathrm{no}}$, implies $\operatorname{var}\left[W_i^{\mathrm{p2p}}(X_1, \ldots, X_n)\right] \leq \operatorname{var}[W_i^{\mathrm{no}}]$. Using equation (42) and the independence of the background risk Y_i, we obtain

$$\frac{(\operatorname{cov}[M, X_i])^2}{(\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right])^2} \cdot \operatorname{var}\left[\sum_{j=1}^{n} X_j\right] \leq \operatorname{var}[X_i]. \tag{43}$$

To show part ii), we apply the same reasoning. Property SSD2, i.e., $W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) \succeq_{ssd} W_i^{\mathrm{p2p}}(X_1, \ldots, X_{n-1})$ implies $\operatorname{var}\left[W_i^{\mathrm{p2p}}(X_1, \ldots, X_n)\right] \leq \operatorname{var}\left[W_i^{\mathrm{p2p}}(X_1, \ldots, X_{n-1})\right]$. With the variance representation from equation (42), this inequality can be rewritten as

$$\frac{\operatorname{var}\left[\sum_{j=1}^{n} X_j\right]}{\operatorname{var}\left[\sum_{j=1}^{n-1} X_j\right]} \leq \frac{(\operatorname{cov}[M, \sum_{k=1}^{n} X_k])^2}{\left(\operatorname{cov}\left[M, \sum_{k=1}^{n-1} X_k\right]\right)^2}. \qquad \blacksquare \tag{44}$$

Proof of Proposition 3: Building on the representation of W_i^{p2p} in equation (11), we introduce

$$\widetilde{W}_i^{\mathrm{p2p}}(X_1, \ldots, X_n) = (1 + r_f)\, w_{0,i} - \mathbb{E}[X_i] + \theta_{i,n}^* \cdot \left(\sum_{j=1}^{n} (\mathbb{E}[X_j] - X_j)\right). \tag{45}$$

It directly follows from this definition that $\mathbb{E}\left[\widetilde{W}_i^{\mathrm{p2p}}(X_1, \ldots, X_n)\right] = \mathbb{E}\left[\widetilde{W}_i^{\mathrm{p2p}}(X_1, \ldots, X_{n-1})\right]$ and $\mathbb{E}\left[\widetilde{W}_i^{\mathrm{p2p}}(X_1, \ldots, X_n)\right] = (1 + r_f)\, w_{0,i} - \mathbb{E}[X_i]$. Part i) can be shown by exploiting that a strict

inequality in equation (14) implies $\text{var}\Big[\widetilde{W}_n^{\text{p2p}}(X_1,\ldots,X_n)\Big] < \text{var}[X_i]$ for all $i = 1,\ldots,n$. Accordingly, Theorem 2 in Pomatto et al. (2020) ensures the existence of an independent random variable $Y_{i,n}^*$ such that

$$W_i^{\text{p2p}}(X_1,\ldots,X_n) = \widetilde{W}_i^{\text{p2p}}(X_1,\ldots,X_n) + Y_i \succeq_{ssd} (1+r_f)\, w_{0,i} - X_i + Y_i = W_i^{\text{no}} \tag{46}$$

if Y_i is noisier than $Y_{i,n}^*$.

Furthermore, a strict inequality in Proposition 2 ii) implies by construction $\text{var}\Big[\widetilde{W}_i^{\text{p2p}}(X_1,\ldots,X_n)\Big] < \text{var}\Big[\widetilde{W}_i^{\text{p2p}}(X_1,\ldots,X_{n-1})\Big]$. Therefore, Theorem 2 in Pomatto et al. (2020) also guarantees the existence of a random variable Y_n^{**} with

$$\widetilde{W}_i^{\text{p2p}}(X_1,\ldots,X_n) + Y_i \succeq_{ssd} \widetilde{W}_i^{\text{p2p}}(X_1,\ldots,X_{n-1}) + Y_i \tag{47}$$

if Y_i is noisier than Y_n^{**}, which confirms part ii) of Proposition 3 because of $W_i^{\text{p2p}}(X_1,\ldots,X_n) = \widetilde{W}_i^{\text{p2p}}(X_1,\ldots,X_n) + Y_i$. ■

Proof of Lemma 4: Consider two positions with random wealth W_1 and W_2 satisfying $\mathbb{E}[W_1] = \mathbb{E}[W_2] = \bar{w}$. Then, the definition of the Taylor expansion in equation (16) implies that $T_4[u_i(W_1)] \geq T_4[u_i(W_2)]$ is equivalent to

$$\frac{u_i^{(2)}(\bar{w})}{2}\,\Delta M_2^*[W_1,W_2] + \frac{u_i^{(3)}(\bar{w})}{6}\,\Delta M_3^*[W_1,W_2] + \frac{u_i^{(4)}(\bar{w})}{24}\,\Delta M_4^*[W_1,W_2] \geq 0, \tag{48}$$

where $\Delta M_k^*[W_1,W_2]$ denotes the difference in the k-th central moment, that is, $\Delta M_k^*[W_1,W_2] := M_k^*[W_1] - M_k^*[W_2]$ for $k = 1,2,3$. When adding independent background risk with $\mathbb{E}[Y] = 0$, we have $\mathbb{E}[W_j + Y] = \mathbb{E}[W_j] = \bar{w}$ and $M_2^*[W_j + Y] = \text{var}[W_j + Y] = \text{var}[W_j] + \text{var}[Y]$ for $j = 1,2$. To calculate the third and fourth central moment, we rely on $M_k^*[W_j + Y] = \mathbb{E}\big[(W_j - \mathbb{E}[W_j] + Y)^k\big]$, $j = 1,2$. Using $\mathbb{E}[Y] = 0$ and $\mathbb{E}[W_j - \mathbb{E}[W_j]] = 0$ as well as the independence of Y and W_j, it follows that

$$M_3^*[W_j + Y] = M_3^*[W_j] + \mathbb{E}\big[Y^3\big], \tag{49}$$

$$M_4^*[W_j + Y] = M_4^*[W_j] + 6 \cdot \text{var}[W_j] \cdot \text{var}[Y] + \mathbb{E}\big[Y^4\big] \tag{50}$$

for $j = 1,2$. Accordingly, the difference in the second and third central moments of W_1 and W_2 is

not affected by the background risk, that is, $\Delta M_k^*[W_1+Y, W_2+Y] = \Delta M_k^*[W_1, W_2]$ for $k = 2, 3$. In contrast, we obtain for the difference in the fourth central moments

$$\Delta M_4^*[W_1+Y, W_2+Y] = \Delta M_4^*[W_1, W_2] + 6 \cdot \Delta M_2^*[W_1, W_2] \cdot \mathrm{var}[Y]\,. \tag{51}$$

Using these results, we conclude that $T_4[u_i(W_1+Y)] \geq T_4[u_i(W_2+Y)]$ if and only if

$$\begin{aligned}\frac{u_i^{(2)}(\bar{w})}{2}\,\Delta M_2^*[W_1, W_2] + \frac{u_i^{(3)}(\bar{w})}{6}\,\Delta M_3^*[W_1, W_2] \\ + \frac{u_i^{(4)}(\bar{w})}{24}\,(\Delta M_4^*[W_1, W_2] + 6 \cdot \Delta M_2^*[W_1, W_2] \cdot \mathrm{var}[Y]) \geq 0.\end{aligned} \tag{52}$$

If $u^{(4)}(\bar{w}) < 0$ and $\Delta M_2^*[W_1, W_2] < 0$, that means, $\mathrm{var}[W_1] < \mathrm{var}[W_2]$, the inequality in equation (52) is satisfied if $\mathrm{var}[Y] \geq v_{min}$ with

$$v_{min} := \max\left\{-\frac{1}{6}\frac{\Delta M_4^*[W_1, W_2]}{\Delta M_2^*[W_1, W_2]} - \frac{2}{3}\frac{u_i^{(3)}(\bar{w})}{u_i^{(4)}(\bar{w})}\frac{\Delta M_3^*[W_1, W_2]}{\Delta M_2^*[W_1, W_2]} - 2\frac{u_i^{(2)}(\bar{w})}{u_i^{(4)}(\bar{w})};\, 0\right\}. \tag{53}$$

The results of Lemma 4 follow from applying this reasoning to $W_1 = W_i^{\mathrm{p2p}}(X_1, \ldots, X_n)$ and $W_2 = W_i^{\mathrm{no}}$ (part i) or $W_2 = W_i^{\mathrm{p2p}}(X_1, \ldots, X_{n-1})$ (part ii). In both cases, we exploit that the expected wealth is not changed when buying P2P insurance and that the required variance inequalities are satisfied if the conditions derived in Proposition 2 hold. ■

Proof of Corollary 1: Under Assumption P2 with $\mathrm{cov}\,[M, X_i] \neq 0$, it holds that $\mathrm{cov}\left[\sum_{j=1}^n X_j, M\right] \neq 0$ such that case i) from Proposition 1 applies and the surplus sharing according to equation (10) simplifies to

$$\theta_{i,n}^* = \frac{\mathrm{cov}\,[X_i, M]}{\mathrm{cov}\left[\sum_{j=1}^n X_j, M\right]} = \frac{c_M}{n \cdot c_M} = \frac{1}{n}. \tag{54}$$

Moreover, equation (9) implies that a set of arbitrage-free premiums $\pi_{1,n}^*, \ldots, \pi_{n,n}^*$ must solve the equation system

$$(1+r_f) \cdot \pi_{i,n}^* - \mathbb{E}[X_i] = \frac{1}{n}\sum_{j=1}^n (1+r_f)\,\pi_{j,n}^* - \mathbb{E}[X_j] \qquad i = 1, \ldots, n. \tag{55}$$

With $l_i := (1+r_f)\,\pi_{i,n}^* - \mathbb{E}[X_i]$, $i = 1, \ldots, n$, we can rewrite equation (55) as $l_i = \frac{1}{n}\sum_{j=1}^n l_j$ for

$i = 1, \ldots, n$. Therefore, it must hold that $l_i = l$ for $l \in \mathbb{R}$ and the premiums are thus given by $\pi^*_{i,n} = (1 + r_f)^{-1} \, (\mathbb{E}[X_i] + l)$, $i = 1, \ldots, n$.

Proof of Corollary 2: The conditions given in equations (19) and (20) are obtained by substituting $\operatorname{cov}[M, X_j] = c_M$ and $\operatorname{cov}\left[M, \sum_{j=1}^n X_j\right] = n \cdot c_M$ into equations (14) and (15). ■

Proof of Proposition 5: Let $P^A_{i,n} = \theta_{i,n} \cdot S^A(X_1, \ldots, X_n)$ denote the participation of policyholder i for a pool with the risks $(X_1, \ldots, X_n)$. We first use that SSD of the P2P contract requires $\mathbb{E}\Big[W_i^{\mathrm{A,p2p}}(X_1, \ldots, X_n)\Big] \geq \mathbb{E}\Big[W_i^{\mathrm{A,no}}\Big]$ for $i = 1, \ldots, n$, which implies $\mathbb{E}\Big[(1 + R_A)\pi^*_{i,n}\Big] \leq \mathbb{E}\Big[X_i + P^A_{i,n}\Big]$ for all $i = 1, \ldots, n$. As in the proof of Proposition 1, the assumption of a strict inequality for an index $i_0 \in \{1, \ldots, n\}$ can be shown to be inconsistent with $\sum_{i=1}^n \theta_{i,n} = 1$. Consequently, it must hold that

$$\mathbb{E}\big[(1 + R_A)\, \pi^*_{i,n}\big] = \mathbb{E}\big[X_i + P^A_{i,n}\big] \qquad \text{for all } i = 1, \ldots, n. \tag{56}$$

By solving for $\pi^*_{i,n}$ and using $\mathbb{E}\Big[P^A_{i,n}\Big] = \theta_{i,n} \left(\sum_{j=1}^n (1 + \mathbb{E}[R_A])\, \pi^*_{j,n} - \mathbb{E}[X_j]\right)$, we obtain equation (29). Next, we substitute the result from (56) into the standard pricing equation (36) and conclude

$$\pi^*_{i,n} = \frac{1 + \mathbb{E}[R_A]}{1 + r_f} \cdot \pi^*_{i,n} + \operatorname{cov}\left[M, X_i + P^A_{i,n}\right]. \tag{57}$$

Furthermore, we can exploit that $\mathbb{E}[M(1 + R_A)] = 1$ and $\mathbb{E}[M] = (1 + r_f)^{-1}$ imply $(1 + \mathbb{E}[R_A])(1 + r_f)^{-1} = (1 - \operatorname{cov}[M, R_A])$. Therefore, equation (57) can be rewritten as $\operatorname{cov}[M, R_A]\, \pi^*_{i,n} = \operatorname{cov}\Big[M, X_i + P^A_{i,n}\Big]$. With the definition of $P^A_{i,n}$, it follows that

$$\operatorname{cov}[M, R_A]\, \pi^*_{i,n} - \operatorname{cov}[M, X_i] = \theta_{i,n} \cdot \operatorname{cov}\left[M, \sum_{j=1}^n (1 + R_A)\, \pi^*_{j,n} - \sum_{j=1}^n X_j\right] \tag{58}$$

for all $i = 1, \ldots, n$. If $\operatorname{cov}\left[M, \sum_{j=1}^n X_j\right] - \operatorname{cov}[M, R_A] \left(\sum_{j=1}^n \pi^*_{j,n}\right) \neq 0$, this condition implies

$$\theta^*_{i,n} = \frac{\operatorname{cov}[M, X_i] - \operatorname{cov}[M, R_A]\, \pi^*_{i,n}}{\operatorname{cov}\left[M, \sum_{j=1}^n X_j\right] - \operatorname{cov}[M, R_A] \left(\sum_{j=1}^n \pi^*_{j,n}\right)} \tag{59}$$

for the fair participation, $i = 1, \ldots, n$. Otherwise, if $\operatorname{cov}[M, R_A]\, \pi^*_{i,n} - \operatorname{cov}[M, X_i] = 0$ for all $i = 1, \ldots, n$, equation (58) is satisfied for all $\theta_{i,n}$. In contrast, if $\operatorname{cov}[M, R_A]\, \pi^*_{i_0,n} - \operatorname{cov}[M, X_{i_0}] \neq 0$

for $i_0 \in \{1, \ldots, n\}$ but $\operatorname{cov}\left[M, \sum_{j=1}^n X_j\right] - \operatorname{cov}\left[M, R_A\right] \left(\sum_{j=1}^n \pi_{j,n}^*\right) = 0$, then equation (58) does not have a solution. ■

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Internet Appendix

Stochastic Dominance and Financial Pricing in Peer-to-Peer Insurance

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This Internet Appendix consists of two sections. Section A presents additional theoretical results. Section B provides details on the calculations used in the Examples 2-4.

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A Additional Results

Our alternative sufficient condition for utility gains from buying P2P insurance relies on the following restriction on the joint distribution of the risks in the pool:

Assumption H1 *$X_i = a_i + b_i Z_i$ with $a_i, b_i \in \mathbb{R}$, $b_i > 0$, $Z_i \overset{d}{=} Z$ for all $i = 1, \ldots, n$. It holds that $\sum_{j=1}^{n} X_j \overset{d}{=} c_n + d_n Z$ with $c_n, d_n \in \mathbb{R}$, $d_n > 0$ for all n.*

This assumption is for example satisfied for jointly normally distributed losses, but also for other types of risks with a joint elliptical distribution such as fat-tailed risks from a multivariate t-distribution.[1] For this class of risks, the combination of fair pricing and the conditions on the second moments given in Proposition 2 will be sufficient to establish benefits of P2P insurance for heterogeneous agents as formalized in the following result:

Proposition A.1 *Suppose that Assumptions B1-B4, P1 and H1 are satisfied and that* $\operatorname{cov}[M, X_i] \cdot \operatorname{cov}[M, X_j] \geq 0$ *for all $i, j = 1, \ldots, n$. Furthermore, suppose that the pricing and profit sharing results from equations* (9) *and* (10) *in Proposition 1 are used.*

i) If the moment condition in equation (14) *is satisfied, then Property SSD1 holds.*

ii) If the moment condition in equation (15) *is satisfied, then Property SSD2 holds.*

Proof: We first introduce $Z^* = \frac{Z - \mathbb{E}[Z]}{\sigma[Z]}$ and set $d_n^* = d_n \, \sigma[Z]$ as well as $c_n^* = c_n + d_n \, \mathbb{E}[Z]$. Then, it holds by construction that $\mathbb{E}[Z^*] = 0$, $\operatorname{var}[Z^*] = 1$ and

$$c_n^* + d_n^* \, Z^* = c_n + d_n \, Z = \sum_{j=1}^{n} X_j \tag{IA.1}$$

according to Assumption H1. By computing the first two moments in equation (IA.1), it furthermore follows that $c_n^* = \mathbb{E}\left[\sum_{j=1}^{n} X_j\right]$ and $d_n^* = \sqrt{\operatorname{var}\left[\sum_{j=1}^{n} X_j\right]}$. From combining this result with equation (11), we obtain

$$W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) \overset{\mathrm{d}}{=} (1 + r_f) \, w_{0,i} + Y_i - \mathbb{E}[X_i] + \theta_{i,n}^* \cdot \left[\sum_{j=1}^{n} \mathbb{E}[X_j] - (c_n^* + d_n^* \cdot Z^*) \right] \tag{IA.2}$$

$$= (1 + r_f) \, w_{0,i} + Y_i - \mathbb{E}[X_i] - \frac{\operatorname{cov}[M, X_i]}{\operatorname{cov}\left[M, \sum_{j=1}^{n} X_j\right]} \cdot d_n^* \cdot Z^*. \tag{IA.3}$$

[1] This follows from the stability of elliptical distributions under linear transformations (McNeil et al. 2015, p. 202).

Using

$$\tilde{d}_n = \frac{\operatorname{cov}[M, X_i]}{\operatorname{cov}\left[M, \sum_{j=1}^n X_j\right]} \cdot d_n^*, \tag{IA.4}$$

implies

$$W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) \overset{\mathrm{d}}{=} (1 + r_f)\, w_{0,i} + Y_i - \mathbb{E}[X_i] - \tilde{d}_n \cdot Z^*. \tag{IA.5}$$

To prove part i) of Proposition A.1, we exploit that the wealth without insurance can be written as

$$W_i^{\mathrm{no}} = (1 + r_f)\, w_{0,i} + Y_i - \mathbb{E}[X_i] - \sigma[X_i] \cdot Z^*. \tag{IA.6}$$

Given that Y_i is independent of $X_1, \ldots, X_n$ and thus of Z^*, it follows from the properties of SSD under linear transformations (Hadar and Russell 1971, Theorem 5) that

$$-\tilde{d}_n \cdot Z^* \succeq_{ssd} -\sigma[X_i] \cdot Z^* \qquad \Rightarrow \qquad W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) \succeq_{ssd} W_i^{\mathrm{no}}. \tag{IA.7}$$

To show that $-\tilde{d}_n \cdot Z^* \succeq_{ssd} -\sigma[X_i] \cdot Z^*$, we first note that equation (14) and $\frac{\operatorname{cov}[M,X_i]}{\operatorname{cov}\left[M,\sum_{j=1}^n X_j\right]} \geq 0$ imply $\tilde{d}_n \leq \sigma[X_i]$. Furthermore, according Shaked and Shanthikumar (2007, Theorem 4.A.3), $\tilde{d}_n\,(-Z^*) \succeq_{ssd} \sigma[X_i]\,(-Z^*)$ holds if and only if

$$\int_p^\infty Q_u[\tilde{d}_n \cdot (-Z^*)]\, du \leq \int_p^\infty Q_u[\sigma[X_i] \cdot (-Z^*)]\, du \qquad \text{for all } p \in (0,1). \tag{IA.8}$$

This condition is satisfied for $\tilde{d}_n \leq \sigma[X_i]$ due to $Q_\alpha[a\, Z^*] = a\, Q_\alpha[Z^*]$ for $a > 0$.

For part ii), the same reasoning can be applied. In particular, we exploit that equation (15) and $\frac{\operatorname{cov}[M,X_i]}{\operatorname{cov}\left[M,\sum_{j=1}^n X_j\right]} \geq 0$ imply $\tilde{d}_n \leq \tilde{d}_{n-1}$. Furthermore, we can again argue that

$$\tilde{d}_n \leq \tilde{d}_{n-1} \qquad \Rightarrow \qquad W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) \succeq_{ssd} W_i^{\mathrm{p2p}}(X_1, \ldots, X_{n-1}). \qquad \blacksquare \tag{IA.9}$$

Proposition A.1 shows that the second-moment conditions from the equations (14) and (15) imply pooling benefits for all risk-averse agents if the distributional restriction from Assumption H1 is satisfied and the sign of the systematic risk as measured by the covariance with the SDF is the same for all risks in the pool. Note that the latter sign condition can be dropped if the distribution of Z given in Assumption H1 is symmetric, in the sense that $Z \overset{\mathrm{d}}{=} -Z$. In general, this result is

related to the location-scale family or linear-class argument about the equivalence of mean-variance decision rules and the expected utility criterion (Meyer 1987, Sinn 1989).[2] Finally, we note that a simpler version of the result can be derived for the case of homogeneous systematic risk exposures by replacing the moment conditions in Proposition A.1 with the corresponding conditions from Corollary 2.

In the case of identical systematic risk exposures, we furthermore obtain the following simplified version of the general background risk result given in Proposition 3.

Corollary A.1 *Suppose that Assumptions B1-B4, P1 and P2 are satisfied. Furthermore, assume that the P2P pool applies the arbitrage-free pricing and profit sharing rules derived in Corollary 1. For each collection of insurable losses* $(X_1, \ldots, X_n)$*, there exist random variables* $Y_{i,n}^*$*,* $i = 1, \ldots, n$*, and* Y_n^{**} *that are independent of* $X_1, \ldots, X_n$ *and have the following properties:*

i) If a strict inequality holds in equation (19) *and the background risk* Y_i *is noisier than* $Y_{i,n}^*$ *for* $i = 1, \ldots, n$*, then Property SSD1 is satisfied.*

ii) If a strict inequality holds in equation (20) *and the background risk* Y_i *is noisier than* Y_n^{**} *for* $i = 1, \ldots, n$*, then Property SSD2 is satisfied.*

Corollary A.1 replaces the covariance-based conditions derived in Proposition 2 with the variance conditions presented in Corollary 2.

For the case of identical systematic risk exposures, we can finally derive a strong result on the benefits of P2P insurance without background risk if the joint distribution of the losses in the pool satisfies the following assumption:

Assumption H2 *Suppose that* $X_i = \mu_i + Z_i$ *with* $\mu_i \in \mathbb{R}$*,* $i = 1, \ldots, n$*, and an exchangeable sequence* $Z_1, \ldots, Z_n$ *satisfying* $\operatorname{var}\left[Z_i^2\right] < \infty$ *for* $i = 1, \ldots, n$*.*

For such risks that only differ in their expected claims, P2P insurance turns out to beneficial without additional shape restrictions.

[2] In an insurance setting, similar distributional restrictions have for example been applied by Dhaene et al. (2008, Theorem 5) or Huggenberger and Albrecht (2022, Proposition 6).

Proposition A.2 *Suppose that Assumptions B1-B4, P1, P2 and H2 are satisfied. Furthermore, assume that the P2P pool applies the arbitrage-free pricing and profit sharing rules from equations* (17) *and* (18) *in Corollary 1.*

i) Property SSD1 holds, that means, every risk-averse agent will benefit from obtaining P2P insurance.

ii) Property SSD2 holds, that means, every risk-averse policyholder, $i = 1, \ldots, n-1$, will benefit from accepting new policyholders to the P2P insurance pool.

Proof: Under Assumption H2 and $\mathbb{E}[Z_i] = \mu_Z$, it holds that $\mathbb{E}[X_j] = \mu_j + \mu_Z$. We thus obtain from equation (11)

$$W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) = (1 + r_f)\, w_{0,i} + Y_i - (\mu_i + \mu_Z) + \frac{1}{n} \left[\sum_{j=1}^{n} \mu_Z - Z_j \right], \tag{IA.10}$$

$$= (1 + r_f)\, w_{0,i} + Y_i - \mu_i - \frac{1}{n} \left[\sum_{j=1}^{n} Z_j \right]. \tag{IA.11}$$

Given that $Z_1, \ldots, Z_n$ is a sequence of exchangeable random variables, we can use the stochastic dominance result (Albrecht and Huggenberger 2017, Lemma 3.1)

$$-\frac{1}{n} \sum_{j=1}^{n} Z_j \succeq_{ssd} -\frac{1}{n-1} \sum_{j=1}^{n-1} Z_j \tag{IA.12}$$

in combination with the stability of SSD under the increasing linear transformation $g_i(x) = (1 + r_f)\, w_{0,i} - \mu_i + x$ to establish

$$(1 + r_f)\, w_{0,i} - \mu_i - \frac{1}{n} \left[\sum_{j=1}^{n} Z_j \right] \succeq_{ssd} (1 + r_f)\, w_{0,i} - \mu_i - \frac{1}{n-1} \left[\sum_{j=1}^{n-1} Z_j \right]. \tag{IA.13}$$

Since Y_i is independent of $Z_1, \ldots, Z_n$, it follows from the stability properties of SSD (Hadar and Russell 1971, Theorem 5) that $W_i^{\mathrm{p2p}}(X_1, \ldots, X_n) \succeq_{ssd} W_i^{\mathrm{p2p}}(X_1, \ldots, X_{n-1})$, which confirms part ii) of Proposition A.2. Part i) can be obtained along the same lines using $-\frac{1}{n} \sum_{j=1}^{n} Z_j \succeq_{ssd} -Z_i$. ■

This result extends the main findings of the mutual insurance analysis in Albrecht and Huggenberger (2017) to the case of heterogeneous risks under the additional assumption that the premiums

charged reflect differences in expected claims.

B Derivations Examples

Example 2

Example 2 builds on the assumption that the insurable losses follow a multivariate normal distribution with $(X_1, X_2, \ldots, X_n)' \sim \mathcal{N}_n(\boldsymbol{\mu}_n, \boldsymbol{\Sigma}_n)$, where $\boldsymbol{\mu}_n = (\mu_i)$ is a real-valued $n \times 1$ vector and $\boldsymbol{\Sigma}_n = (\sigma_{ij})$ is a positive definite $n \times n$ matrix. In this case, the closure of the normal distribution under linear transformations implies $\sum_{i=1}^n X_i \sim \mathcal{N}_1(\mu_{T,n}, \sigma^2_{T,n})$ for the total claim amount with

$$\mu_{T,n} = \sum_{i=1}^n \mu_i \qquad \text{and} \qquad \sigma^2_{T,n} = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}. \tag{IA.14}$$

From equations (11), (12) and $\mathbb{E}[Y_i] = 0$, we obtain

$$\mathbb{E}\Big[W_i^{\text{p2p}}(X_1, \ldots, X_n)\Big] = (1 + r_f)\, w_{0,i} - \mu_i, \tag{IA.15}$$

$$\text{var}\Big[W_i^{\text{p2p}}(X_1, \ldots, X_n)\Big] = \theta^2_{i,n} \text{var}\left[\sum_{j=1}^n X_j\right] = \theta^2_{i,n}\, \sigma^2_{T,n} \tag{IA.16}$$

for the first two moments of the policyholder's wealth when buying P2P insurance. Exploiting $\mu_i = \mathbb{E}[X_i]$ and $\sigma_{ij} = \text{cov}\,[X_i, X_j]$, we can easily compute these two moments. Finally, the policyholder's expected utility can be calculated in closed-form using the moment-generating function of the normal distribution:[3]

$$\begin{aligned}
&\mathbb{E}\Big[u_i(W_i^{\text{p2p}}(X_1, \ldots, X_n))\Big] \\
&\quad = 1 - \exp\left(-\gamma_i \cdot \mathbb{E}\Big[W_i^{\text{p2p}}(X_1, \ldots, X_n)\Big] + \frac{1}{2} \cdot \gamma_i^2 \cdot \text{var}\Big[W_i^{\text{p2p}}(X_1, \ldots, X_n)\Big]\right) &\text{(IA.17)} \\
&\quad = 1 - \exp\left(-\gamma_i \cdot ((1 + r_f)\, w_{0,i} - \mu_i) + \frac{1}{2} \cdot \gamma_i^2 \cdot \theta^2_{i,n} \cdot \sigma^2_{T,n}\right). &\text{(IA.18)}
\end{aligned}$$

[3]The moment-generating function of a normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ is given by

$$m_X(t) := \mathbb{E}[\exp(tX)] = \exp(t\mu + \frac{1}{2}t^2\sigma^2).$$

Examples 3 and 4

The Examples 3 and 4 rely on conditional instead of unconditional normal distributions. We introduce a state variable S that indicates whether a low loss state ($S = L$) or a high loss state ($S = H$) occurs. In these two states, we assume

$$(X_1, X_2, \ldots, X_n)'|S = L \sim \mathcal{N}_n(\boldsymbol{\mu}_{n,L}, \boldsymbol{\Sigma}_{n,L}), \tag{IA.19}$$

$$(X_1, X_2, \ldots, X_n)'|S = H \sim \mathcal{N}_n(\boldsymbol{\mu}_{n,H}, \boldsymbol{\Sigma}_{n,H}) \tag{IA.20}$$

for the multivariate distribution of the insurable losses with $\boldsymbol{\mu}_{n,L} = (\mu_{i,L})$, $\boldsymbol{\mu}_{n,H} = (\mu_{i,H})$ as $n \times 1$ vectors and $\boldsymbol{\Sigma}_{n,L} = (\sigma_{ij,L})$, $\boldsymbol{\Sigma}_{n,H} = (\sigma_{ij,H})$ as positive definite $n \times n$ matrices. The conditional expectations and the conditional variances of $\sum_{j=1}^n X_j$ are then given by

$$\mu_{T,n,s} = \mathbb{E}\left[\sum_{j=1}^n X_j \mid S = s\right] = \sum_{i=1}^n \mu_{i,s} \quad \text{and} \quad \sigma^2_{T,n,s} = \operatorname{var}\left[\sum_{j=1}^n X_j | S = s\right] = \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij,s} \tag{IA.21}$$

for $s \in \{L, H\}$. Building on these moments and the representation in equation (11) with $Y_i = 0$, we can compute the conditional moments of the policyholder's wealth for $s \in \{L, H\}$ as follows

$$\mathbb{E}\Big[W_i^{\text{p2p}}(X_1, \ldots, X_n) \mid S = s\Big] = (1 + r_f)\, w_{0,i} - \mathbb{E}[X_i] + \theta^*_{i,n} \cdot \left[\left(\sum_{j=1}^n \mathbb{E}[X_j]\right) - \mu_{T,n,s}\right], \tag{IA.22}$$

$$\operatorname{var}\Big[W_i^{\text{p2p}}(X_1, \ldots, X_n)|S = s\Big] = (\theta^*_{i,n})^2 \cdot \operatorname{var}\left[\sum_{j=1}^n X_j | S = s\right] = (\theta^*_{i,n})^2 \cdot \sigma^2_{T,n,s}. \tag{IA.23}$$

Note that the first expression involves the unconditional expectation of the policyholders' claims, which satisfy $\mathbb{E}[X_i] = \mathbb{P}[S = L] \cdot \mu_{i,L} + \mathbb{P}[S = H] \cdot \mu_{i,H}$, $i = 1, \ldots, n$.

Based on the state-specific moments of the wealth distribution given in equations (IA.22) and (IA.23), we can calculate the unconditional moments required for the Taylor expansion of the policyholder's utility using the following result (Frühwirth-Schnatter 2006, p. 11): For a two-state mixture distribution with $Z|S = L \sim \mathcal{N}(\mu_L, \sigma^2_L)$, $Z|S = H \sim \mathcal{N}_1(\mu_H, \sigma^2_H)$, $\mathbb{P}[S = L] = p_L$

and $\mathbb{P}[S = H] = p_H$, it holds that $\mu := \mathbb{E}[Z] = p_L \cdot \mu_L + p_H \cdot \mu_H$ and

$$\operatorname{var}[Z] = \left(\sum_{s \in \{L,H\}} p_s \cdot (\sigma_s^2 + \mu_s^2) \right) - \mu^2, \tag{IA.24}$$

$$\mathbb{E}\big[(Z-\mu)^3\big] = \sum_{s \in \{L,H\}} p_s \cdot (\mu_s - \mu)^3 + 3\,(\mu_s - \mu)\sigma_s^2, \tag{IA.25}$$

$$\mathbb{E}\big[(Z-\mu)^4\big] = \sum_{s \in \{L,H\}} p_s \cdot (\mu_s - \mu)^4 + 6\,(\mu_s - \mu)^2\sigma_s^2 + 3\,\sigma_s^4. \tag{IA.26}$$

We furthermore rely on the following derivatives of the exponential utility function, $u_i(w) = 1 - \exp(-\gamma \cdot w)$, to implement the Taylor expansion: $u_i^{(2)}(w) = -\gamma^2 \exp(-\gamma w)$, $u_i^{(3)}(w) = \gamma^3 \exp(-\gamma w)$, and $u_i^{(4)}(w) = -\gamma^4 \exp(-\gamma w)$.

To determine the participation factors $\theta_{i,n}^*$, $i = 1, \ldots, n$, we calculate the unconditional covariances $\operatorname{cov}[X_i, M]$, $i = 1, \ldots, n$ based on the given state-specific covariances. For this purpose, we note that $\operatorname{cov}[X_i, M] = \mathbb{E}[X_i \cdot M] - \mathbb{E}[X_i] \cdot \mathbb{E}[M]$ and

$$\mathbb{E}[X_i \cdot M] = \mathbb{P}[S = L] \cdot \mathbb{E}[X_i \cdot M \mid S = L] + \mathbb{P}[H = L] \cdot \mathbb{E}[X_i \cdot M \mid S = H]\,. \tag{IA.27}$$

The required state-specific expectations can be written as

$$\mathbb{E}[X_i \cdot M \mid S = s] = \operatorname{cov}[X_i, M | S = s] + \mathbb{E}[X_i \mid S = s] \cdot \mathbb{E}[M] \tag{IA.28}$$

for $s \in \{L, H\}$. If $\operatorname{cov}[X_i, M|S = L] = \operatorname{cov}[X_i, M|S = H] = c_i$, it follows that the unconditional covariance is equal to the conditional covariances, i.e.,

$$\begin{aligned} \operatorname{cov}[X_i, M] =& \mathbb{P}[S = L] \cdot (c_i + \mathbb{E}[X_i \mid S = L] \cdot \mathbb{E}[M]) \\ &+ \mathbb{P}[S = H] \cdot (c_i + \mathbb{E}[X_i \mid S = H] \cdot \mathbb{E}[M]) - \mathbb{E}[X_i] \cdot \mathbb{E}[M] = c_i. \end{aligned} \tag{IA.29}$$

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