R-squares in the Cross-section and the Time-series

PIERLUIGI BALDUZZI* TOMMASO BONAFINI† ROBERTO SAVONA‡

[VERY PRELIMINARY, PLEASE DO NOT QUOTE]

This draft: February 12, 2025

*Boston College. Corresponding author: Pierluigi Balduzzi, Carroll School of Management, Boston College, 140 Commonwealth Avenue, Chestnut Hill, Massachusetts, 02467; Tel: (617) 552-3976; Fax: (617) 552-0431; email:balduzzp@bc.edu.

†Department of Economics and Management, University of Brescia, Italy.

‡Department of Economics and Management, University of Brescia, Italy.

R-squares in the Cross-section and the Time-series

ABSTRACT

The R-square of a theoretical cross-sectional (CS) second-pass regression of test asset risk premia on factor loadings is factorized into two components. The first component captures the average time-series (TS) fit of the factor model across test assets. The second component reflects the relative risk-return trade-off of the model's optimal factor portfolio (OFP). Our factorization explains why weak factors, although not systematic, can work in the cross section because they generate near-arbitrage opportunities. For example, the improvement in pricing performance going from the CAPM to the FF3, FF5, and FF6 models is mainly due an increase in the relative risk-return trade-off of the model's OFP, not to an improvement in average TS fit.

JEL # G12

Introduction

Five decades of research on asset pricing models have not fully explained why cross-sectional regression methods connect theories to the data imprecisely. PB: What do you mean by "imprecisely"? Do you mean that the TS and CS approaches to inference lead to different results? MacKinlay (1995) makes a point on this, arguing that nonzero intercepts arise not only from missing risk factors but also from specific firm characteristics – this argument is in *time-series perspective*. Kozak and Nagel (2024) make another fundamental point, formalizing the importance in capturing the stock return covariances in factor construction – this argument is in *cross-sectional perspective*. Along the line connecting these two points, an impressive number of papers examined almost all individual component of asset pricing models, recommending: how to proceed with a rigorous diagnosis of the model (Jagannathan and Wang (1998); Kan et al. (2013); Kan and Zhang (1999a,b)); how to select (Pukthuanthong et al. (2019); Feng et al. (2020)), test (Harvey et al. (2016); Tian (2021)) and assemble (Stambaugh and Yuan (2017); Kozak et al. (2020)) factors in the zoo; how to compare and economically interpret asset pricing models (Barillas and Shanken (2017, 2018); Lewellen et al. (2010); Barillas et al. (2020)). However, we still do not know why results in time-series differ from those in cross-section (Ahmed et al. (2019)). Fama (2015) makes us aware that the time-series and cross-section regression approaches to test asset pricing models seem to be competitors, as "*they are complements, with lots of back and forth between them.*" And Harvey and Liu (2021) are straight to the point when they admit that the endless quest to identify factors in the cross-section is because there is no consensus about the factors' structure.

This paper makes an important progress to this question formalizing the relationship between the cross-sectional (CS) and the time-series (TS) fit of an asset pricing model. It

may seem surprising but, in fact, the existing literature has never formalized the connection between the two measures of fit. In doing this, we benefit from some key studies we build upon in connecting several cornerstones of the asset pricing modeling literature.

Cochrane (2011) first notes that asset pricing models such as the CAPM, C-CAPM, and I-CAPM propose factors that, while correlated to asset returns, need not be pervasive. On the other hand, according to these same models, pervasive factors, such as the returns on industry-sorted stock portfolios need not be priced; i.e., do not need to command non-zero risk premia. The factors identified in the literature (HML, SMB, etc.) are the excess returns on characteristic-sorted, long-short portfolios. If the characteristics quantify exposure to important priced risks that models such as the CAPM ignore, then we would expect the corresponding factors to be somewhat pervasive. On the other hand, if the factors are just the returns on near-arbitrage strategies with little liquidity, then there is no reason to expect them to be pervasive. Interestingly, a similar point is already in MacKinlay (1995), who notes that the "missing factors" of asset pricing models explain little of the common variability in stock returns. Even additional factors could explain little of common return variability – i.e., the factors are not pervasive – but nevertheless they are useful to enhance the cross-sectional fit. In this regards, Kozak et al. (2020) and Lettau and Pelger (2020) make an important distinction between factors explaining the covariance matrix of asset returns and those that have a high price of risk. But it is Kozak and Nagel (2024) who come closest to clarifying the link between the TS and CS dimensions when they ask about the conditions required to span the stochastic discount factor through the cross-sectional asset pricing factors. They prove that GLS cross-sectional regression slope factors are theoretically optimal as they perfectly span the multifactor SDF, whereas characteristic-based factors do span if the covariances of stock returns are as close as possible to characteristics. While Kozak

and Nagel (2024) conceptualize the sources of systematic risk and derive the conditions under which cross-sectional asset pricing factors span the SDF, we definitively explain why factors in cross-section may help improve the CS fit when they explain little in time-series. The reason passes through the link between the TS and the CS R-squares. To make clear the importance of this connection, consider the case of a linear asset pricing model with a single traded factor where test assets' returns are orthonormal, as in the case of the principal components (PCs) of the original returns, which is one of the cases scrutinized in our paper. What we show is that the CS R-square (R^2_{CS}) of a theoretical second-pass regression of N test assets' ex-ante risk premia on the corresponding population factor loadings can be represented as the product of two terms:

$$R^2_{CS} = \bar{R}^2_{TS} \times \frac{N(\hat{\lambda}^2/\sigma^2_f)}{SR^2_T}, \tag{1}$$

where an upper bar denotes a CS average. $\hat{\lambda}$ is the population value of the second-pass CS estimate of the factor risk premium and σ^2_f is the variance of the factor; SR_T is the Sharpe ratio on the tangency portfolio of the test assets and the term $\bar{R}^2_{TS}$ measures the average TS fit of the factor model. The term $\frac{N(\hat{\lambda}^2/\sigma^2_f)}{SR^2_T}$ measures the risk-return trade-off of the model's factor relative to the risk-return trade-off of the tangency portfolio of the test assets. In the case of a multi-factor model, we replace the single factor by the excess return of the Optimal Factor Portfolio (OFP), the tangency portfolio of the factor assets. While helping to simplify our factorization, this replacement is indeed consistent with a pricing kernel representation. In short, we display how the cross-sectional fit of an asset pricing model is depending on: i) the pervasiveness of the associated factor, and ii) the relative attractiveness of the risk-return trade-off of the factor.

To give a sense to the TS-CS R-square link, in Figure 1 we plot $\hat{\lambda}^2/\sigma^2_f$ as a function of Ave(R^2_{TSn}), for varying levels of $\bar{R}^2_{CS}$, the *iso-R^2 curves*, employing the Generalized Least

Squares (GLS) to regress the 25 Fama-French (FF) size–BM + the 30 FF industry portfolios onto the single and combined (tangency portfolios of) 6 FF factors.

The figure illustrates that individual factors do not seem to explain much of the time-series variability, and they have a limited explanatory power regarding the CS of test assets. However, combinations of the factors show an increase in the R^2_{CS} that is due mainly to an increase in the risk-return trade-off of the OFP. Note, in particular, how the most significant improvement is occuring when the Momentum factor is added (FF6).

Our results document a tension between the CAPM and the multi-factor models which lies in the nature of factors and the choice of the test assets. The CAPM delivers high CS R^2 because the Market is strongly systematic with high TS fit across test assets. Multi-factor models increase the CS fit because of the profitability of the factors. This happens for traded, non-traded factors, and principal components (PCs), as well as when moving from OLS to WLS, and GLS. In this case, we obtain lower R^2_{CS} values as a result of the relative (in)efficiency of the index portfolio sample mean-variance location.[1] The CS R-square factorization offers novel insights also on the properties of the test statistics used in the literature to assess whether the factors are priced ($\hat{\lambda} \neq 0$), useless ($\beta = 0$), and price expected excess returns exactly ($\hat{\alpha} = 0$). By running bootstrap-based inference on model parameters we provide new evidence on traded, not-traded and rotated asset-based factors. Overall, factors are priced/unpriced, useless/useful and able to price exactly depending on the test asset we use and whether the factors are included in the test assets. Bearing in mind that when comparing OLS vs. GLS we actually looking results from different test assets, since GLS can be seen as OLS performed on rotated data set, our main findings are as follows: first, factors are mostly priced with OLS, and all priced with GLS (when including

[1] As ponted out in Kandel and Stambaugh (1995), the goodness-of-fit measure for the GLS cross-sectional regression is the squared relative efficiency of the index portfolio.

factors in the test assets); second, we cannot reject that factors are useless in the GLS case whilst we do in OLS for the CAPM, Momentum, and multi-factor models; third, multi-factor models are rejected with GLS (α is statistically different from zero) but we cannot do with OLS.

Using our factorization we explain why this happens: Market is systematic and characteristic-based factors are not, and when combined together as optimal tangency portfolio, factors can be viewed as near-arbitrage opportunities with low systematic risk, priced but useless (GLS), and unable to price expected excess returns exactly (GLS). The near-arbitrage qualification of factors matters also for non-traded factors and PCs. While we cannot exclude that single non-traded factors are useless and unpriced (moving from OLS to GLS and vice-versa), their combination as optimal tangency portfolio into conditional CAPM models (Jagannathan and Wang (1996) and Lettau and Ludvigson (2001)) gives back factors that are priced and not useless, but unable to price exactly the test assets. As for PCs, we document a "tension" between the first PC and multi-factor version of the PCs similar to that of CAPM vs. multi-factor models, with the first PC priced and possibly useless, and multiple PCs combined as optimal portfolio priced and possibly useless alike because qualified as near-arbitrage opportunity (GLS). Again, we rule out the null hypotheses that PCs are exact.

In a 20-year moving window analysis, we document a substantial time variability of the R^2_{CS} and its components, with factors exhibiting a decreasing trend in their risk-return trade-off component. The tension also appears in this context, as the Market factor is showing a steady increase, while multi-factor model including Momentum peaking around 2004 before a gradual decline (GLS).

While profitability (the relative risk-return trade-off component of the cross-sectional R^2_{CS}) explains why factors in the cross-section do a good job even when in the time-series

they are weakly or not at all systematic, the choice of test assets can change the nature of factors. This is the case, for example, of the Momentum factor, which plays a key role in the improvement of the cross-sectional fit through the high contribution of the relative risk-return trade-off component. When adding the momentum-portfolio returns to the test-asset returns, the average TS fit increases substantially, changing the nature of Momentum into systematic factor. In other terms, profitability converts into pervasiveness, i.e., alphas becomes betas, because of the changed test-asset sample. In this respect, we complement the findings of Giglio et al. (2024). These authors deduce that factors are strong or weak depending on the set of test assets used in the estimation. Our results go one step further, as we show that while factors can be either pervasive or profitable based on the selected test assets – which echoes the strong-weak property of factors in Giglio et al. (2024) – we explain why weak factors, although not pervasive, can work in the cross section as they appear as near-arbitrage opportunities.

This paper is organized as follows. Section I summarizes the related literature. Section II derives the factorization of the cross-sectional R-square. In Section III we describe the bootstrap exercises we run estimate asset pricing models. Section IV describes the data while ection V presents the results. Section VI concludes.

I Related literature

Our paper connects several strands of the literature on asset pricing modeling. The 2–stage cross sectional regression (CSR), introduced in Black et al. (1972) and Fama and MacBeth (1973), has a central position in our general theoretical framework, which is articulated along robustness and misspecification in the 2–stage CSR, CS and TS asset pricing tests, and inference methods for model estimation.

The misspecification and robustness problems in the 2–stage CSR were already disucssed in Fama and MacBeth (1973). Subsequent papers explored the topic in more depth. Jagannathan and Wang (1998) highlight that the beta-pricing model is often misspecified and examine the implications of this on CSR estimators. Similarly, Kan and Zhang (1999b) investigate the properties of the two-pass CSR when incorporating a "useless factor" – a factor uncorrelated with the test asset returns. They show that the t-value of such a factor tends to converge to a high value in CSR, potentially misleading interpretations. They also show that the CS R-squared in a model with a useless factor do not converge in distribution to zero, but to a random variable. They propose testing whether the betas of the assets with respect to a particular factor are significantly different from zero in the first-pass time series regression before running the second-pass cross-sectional regression. This is equivalent to examining the distribution of the time-series fit in our decomposition. Furthermore, Kan and Zhang (1999a) show that even Generalized Method of Moments (GMM) techniques are vulnerable to useless factors when the model is misspecified. Gospodinov et al. (2014) find that conventional inference methods tend to erroneously conclude that useless factors are priced. To overcome this problem, they propose a procedure that eliminates useless factors and factors with zero risk premia.[2] Gospodinov et al. (2019) expose the problem of possibly misspecified and unidentified linear asset pricing models in a maximum likelihood context. Building on the insights in the previous papers, Bryzgalova et al. (2023) propose a Bayesian solution to address these challenges. Moreover, Kleibergen (2009), Burnside (2011), and Bryzgalova (2015), extend this discussion to factors that are not necessarily useless but weak. Kleibergen (2009) shows that the risk premia estimates of the two-pass procedure are spurious whenever the betas are relatively small and this problem is aggravated when the number of test assets is large. Burnside (2011) argues that sampling uncertainty in betas

[2]In doing this, the authors find that the statistical evidence of many non-traded factors is rather weak.

cannot be overlooked when inferring about factor risk premia. He also observes that excluding a constant from the cross-sectional regression leads to significantly smaller R^2 statistics, compared to models that include a constant. Bryzgalova (2015) proposes a new estimator that can be used as a model diagnostic and estimation technique for the risk premia. This estimator eliminates the impact of factors that are weakly correlated with asset returns and at the same time restores the identification of the strong factors in the model. In the Carhart (1997) model, this estimator find that the momentum factor is irrelevant when the test assets are 25 portfolios sorted by size and book-to-market ratio. At the same time, this estimator finds that nearly all the specifications with non-traded factors considered in the paper suffer from identification failures, even when mimicking portfolios for these non-traded factors are used.

In the two-pass estimation method, an important decision revolves around the choice of the weighting matrix used in the second-pass regression, namely, whether to employ an Ordinary Least Squares (OLS) or a Generalized Least Squares (GLS) regression. This choice has significant implications for the model's interpretability and accuracy. Kandel and Stambaugh (1995) discuss the implications of this choice by examining the mean-variance location of an index portfolio. They find that the GLS cross-sectional regression R^2 is a measure of relative efficiency determined by a portfolio's mean-variance location, unaffected by the repackaging of individual assets. Conversely, they observe that the OLS cross-sectional regression R^2 has no relation to the mean-variance location of an inefficient portfolio. Lewellen et al. (2010) criticize the use of OLS in the second-pass regression, noting that a model can show a good fit even if the sensitivities of the observed factors are small or zero. They also critique the low benchmark commonly adopted by the literature for declaring success in evaluating asset pricing models. This critique is particularly focused on the high cross-sectional R^2

values often celebrated when explaining average returns on Fama and French's (Fama and French (1993)) 25 size-B/M portfolios. Lewellen et al. (2010) emphasize that these portfolios inherently have a strong factor structure, where FF's factors alone explain a significant portion of the variation in returns – more than 90% of the time-variation and over 80% of the cross-sectional variation in average returns. Consequently, achieving a high cross-sectional R^2 with other models becomes less challenging, as loadings on almost any proposed factor are likely to align with expected returns. Essentially, a factor only needs to be weakly correlated with SMB or HML, and not with the tiny idiosyncratic three-factor residuals of the size-B/M portfolios. Thus, if test asset portfolios possess a strong inherent factor structure, then even factor portfolios with zero mean returns can produce high values in unrestricted cross-sectional regressions. They analyze various models and find that after expanding the universe of test assets, using GLS and using confidence intervals for the true CS R2, none of the models performs very well.

Kleibergen and Zhan (2015) delve deeper into the issues surrounding the use of weak factors, deriving expressions for the large sample distributions of both OLS and GLS R^2 when the betas of observed factors are small and possibly zero. They highlight the problem of unexplained factor structures in the first-pass residuals leading to large estimation errors in the estimated betas. These errors are often spanned by the unobserved true factors' betas when there's minimal correlation between the observed proxy factors and the true factors. This correlation produces large OLS R^2 values in the second-pass regression, misleadingly suggesting effectiveness. Kan et al. (2013) contribute to this discussion by comparing OLS and GLS from different perspectives. They suggest that OLS might be more relevant when focusing on expected returns for a specific set of assets or test portfolios, whereas GLS R^2 might be more pertinent from an investment perspective. The core of their paper, however,

lies in an econometric analysis model comparison based on R^2 in an excess returns specification with both no constraints on the zero-beta rate and the zero-beta constrained to equal the risk-free rate. Their work includes deriving asymptotic tests for multiple model comparisons, evaluating the joint hypothesis that a given model outperforms a set of alternatives in terms of cross-sectional R^2. Furthermore, Kan et al. (2013) address another important question – does a particular factor in a multifactor model make an incremental contribution to the model's overall explanatory power, considering the presence of other factors? They argue that one cannot answer this question merely by examining the usual risk premium coefficients on the multiple regression betas. Instead, they propose an approach that considers the cross-sectional relation with simple regression betas (or equivalently, asset covariances with the factors) as the explanatory variables. This approach aims to determine whether the corresponding coefficient differs significantly from zero.

The previous papers analyze the problem of misspecification and the choice of weighting matrices in factor models. However, they are agnostic on the rationale for using multiple factors in addition to the market factor. MacKinlay (1995) differentiates the violations of the CAPM in two categories – risk-based and nonrisk-based alternatives. For the risk-based category the deviation from the CAPM (*i.e.* the intercepts of the CAPM are statistically different form zero) is either missing risk factors or the misidentification of the market factor, so this is based on the assumption of investor rationality and perfect capital markets. The non-risk based alternative includes biases introduced in the empirical methodology, the existence of market frictions, or explanations arising from the presence of irrational investors. This differentiation has implication on the maximum squared Sharpe ratio of the tangency portfolio of all assets. For the risk-based alternatives the maximum squared SR is bounded, and for the nonrisk-based alternatives the maximum squared SR is a less useful

construct and can, in principle, be unbounded. The main points of MacKinlay (1995) are two. First, in the context of risk-based asset pricing models, it is difficult to differentiate between competing models through the zero-intercept test, because the null and alternative distribution have substantial overlap. Second, if the goal is to compare a risk-based pricing model with a nonrisk-based alternative, the zero-intercept test can be very useful since the distributions of the test statistic for these alternatives have little overlap. In his analysis, MacKinlay (1995) ends up suggesting that deviation from the CAPM are more likely to be related to nonrisk-based alternatives than risk-based ones, and that models like the Fama and French three-factors model will tend to create a portfolio like the optimal orthogonal portfolio, independently on the explanation for the CAPM deviations.

In the last decades a plethora of factors, the "factor zoo", have demonstrated effectiveness in explaining cross-sectional patterns in expected stock returns. While traditional perspectives held that a limited set of risk factors should be adequate to account for these patterns, contemporary literature challenges this notion. Kozak et al. (2020) assert that characteristics-sparse Stochastic Discount Factors (SDFs) fall short in providing a comprehensive summary of the cross-section of expected stock returns. Instead, they argue that an SDF formed by a substantial number of characteristics-based factors is essential for effective pricing. Kozak and Nagel (2024) further contribute by highlighting that heuristic factor construction methods often neglect the covariance matrix of individual stock returns. Consequently, these heuristic factors are only effective in spanning the SDF that accurately prices individual stocks under specific covariance matrix conditions. These conditions are more likely to be met when the econometrician employs a larger set of characteristics, even if these factors are not directly related to expected returns. The key is that they assist in capturing sources of stock return covariances. Kozak and Nagel (2024) also emphasize the

importance of hedging against unpriced risks associated with heuristic factors. The inefficiency of these factors often arises from their contamination with unpriced risks, which do not offer compensatory higher mean returns. Didisheim et al. (2024) stress the value of complexity in SDFs, noting that factor pricing models tend to exhibit improved out-of-sample performance as the number of factors increases. This interpretation challenges the traditional Arbitrage Pricing Theory (APT) perspective, which suggests that a small number of risk factors should suffice to capture the risk-return tradeoff among assets. Additionally, Bryzgalova et al. (2023) introduce a comprehensive Bayesian approach encompassing both tradable and nontradable factors. Their approach incorporates tradable restrictions on the price of risk when necessary. Notably, they find that the true latent SDF is densely represented within the space of observable factors.

None of the previous papers directly express the relation between the time-series fit in the first pass of the Fama-MacBeth regression and the cross-sectional fit in the second pass. The second pass focuses on the relation between factors and mean returns, the first pass on factors and second moment of returns. Pukthuanthong et al. (2019) propose a protocol for identifying genuine risk factors. They point out that a genuine factor needs to be related to covariance matrix returns, priced in the cross-section of returns and should yield a Sharpe ratio reasonable enough to be consistent with risk pricing. Related to the first condition of a genuine factor, Moskowitz (2003) examines the link between well-known factors and the covariance structure of returns. In particular he finds that while the size factor has a strong relation to covariance risk, value and momentum have respectively a weak and negligible relation with it.

Lettau and Pelger (2020) propose an alternative estimator, the risk-premium PCA (RP-PCA), to identify factors that simultaneously explain the time-series and cross-section of

stock returns. This estimator is an alternative to standard PCA and integrates the first and second moments information of the data. Essentially, it is a PCA with an additional penalty term that corrects for cross-sectional pricing errors. The goal is to find an estimator that can explain both the comovements in asset returns and the cross-section of expected returns. Lettau and Pelger (2020) argue that RP-PCA can identify both "weak" and "strong" factors. "Weak" factors are those that exhibit high Sharpe Ratios but are harder to detect with standard PCA since they affect only a subset of assets. In contrast, "strong" factors, such as the market factor, influence a broader range of assets.

This methodology exploits the importance of both the cross-sectional (CS) and time-series (TS) dimensions of an asset pricing model. However, to our knowledge, our research is the first to establish a direct link between the CS and TS explanatory power of a factor model. We specifically address the lack of TS explanatory power in existing factor models, arguing that this deficiency is a key reason for their poor performance and subsequent statistical rejection.

II The link between CS and TS fit

II.A Preliminaries

Consider a vector of returns in *excess* of the risk-free rate on N test assets, $\tilde{r}_t$, with mean $\tilde{\mu}$ and covariance matrix $\tilde{\Sigma}$. To allow for different weighting schemes in the second-pass CS regression, we introduce the matrix $W = H^\top H$, a positive definite symmetric square matrix that defines the "metric" of the fit of the model. We denote with r_t the "rotated" excess returns, $r_t \equiv H\tilde{r}_t$.

Consider now a single factor f_t, with corresponding $N \times 1$ vector of theoretical factor

betas β from the single-index model:

$$r_t = a + \beta f_t + \epsilon_t. \tag{2}$$

The factor can be either traded or non-traded. If it is traded, it can be included in the set of test asset returns. Note that the single factor f_t captures *all* the pricing properties of the asset pricing model. Indeed, we can construct a single-factor model that captures the pricing implications of several factors at once; see Section II.D below.

We estimate the premium on the factor and we evaluate the performance of the asset pricing model by projecting the N test asset population risk premia, $\mu_n \equiv E(r_{nt})$, onto the population factor betas β_n. We have

$$\mu_n = \beta_n \lambda + \alpha_n, \tag{3}$$

and

$$\hat{\lambda} = (\beta^\top \beta)^{-1} \beta^\top \mu \tag{4}$$

$$\alpha = \mu - \beta \hat{\lambda}. \tag{5}$$

The quantity

$$R^2_{CS} \equiv 1 - \frac{\hat{\alpha}^\top \hat{\alpha}}{\mu^\top \mu} \tag{6}$$

is the CS R-square. Note that the CS regression above can include a constant, in which case, the CS regression model becomes

$$\mu_n = \lambda_0 + \beta_n \lambda + \alpha_n, \tag{7}$$

where λ_0 is the differential between the zero-beta rate and the observed risk-free rate.

Also note that

$$\begin{aligned}\hat{\alpha} &= \mu - \beta\hat{\lambda} \\ &= \mu - \beta\hat{E}(f_t) + \beta\hat{E}(f_t) - \beta\hat{\lambda} \\ &= \alpha_{TS} + \beta[\hat{E}(f_t) - \hat{\lambda}] \end{aligned} \tag{8}$$

where $\alpha_{TS} \equiv \mu - \beta\hat{E}(f_t)$ and

$$\alpha_{TS} = \hat{\alpha} - \beta[\hat{E}(f_t) - \hat{\lambda}]. \tag{9}$$

Hence, we have ($\hat{\alpha}^\top\beta = \beta^\top\hat{\alpha} = 0$)

$$\hat{\alpha}_{TS}^\top\hat{\alpha}_{TS} = \hat{\alpha}^\top\hat{\alpha} + \beta^\top\beta[\hat{E}(f_t) - \hat{\lambda}]^2 \tag{10}$$

and

$$\begin{aligned} R^2_{Tot} &\equiv 1 - \frac{\hat{\alpha}_{TS}^\top\hat{\alpha}_{TS}}{\mu^\top\mu} \\ &= R^2_{CS} - \frac{\beta^\top\beta[\hat{E}(f_t) - \hat{\lambda}]^2}{\mu^\top\mu}. \end{aligned} \tag{11}$$

The overall fit of the model depends on its ability to price both test assets and factor. If test assets returns are orthonormal, as in the case of the PCs, R^2_{Tot} coincides with the GRS statistic.

II.B CS R2

We focus on the R-square statistic because it is an intuitive and easily available statistic measuring the performance of an asset pricing model. Whereas any model, being an approximation, is going to be rejected if subject to close enough scrutiny, the R-square can tell us whether a model, while formally rejected, is still useful. Moreover, the cross-sectional R-square has a direct link to other well-known statistics, such as the Gibbons-Ross-Shanken

F-statistic (Gibbons et al. (1989), GRS) and the Hansen-Jagannathan distance (Hansen and Jagannathan (1997), HJD); see discussion below.

Notice, however, that the point estimates for this measure of fit depend on the choice of metric in which the residuals of the CS regression are evaluated; i.e., they depend on the choice of *rotation* of the original raw test asset returns $\tilde{r}_t$ via the H matrix. We focus on three cases:

1. $W = I_N$: this corresponds to an OLS CS projection of the original expected excess returns on the betas.

2. $W = S^{-1}$, where $S = \text{diag}(\tilde{\Sigma})$ is a diagonal matrix with the same diagonal terms as $\tilde{\Sigma}$: this corresponds to a weighted least squares (WLS) CS projection of the original expected excess returns on the betas.

3. $W = \tilde{\Sigma}^{-1}$: this corresponds to a GLS CS projection of the original expected excess returns on the betas. In the empirical analysis, we consider the eigendecomposition of PCA.[3]

Using the eigendecomposition of $\tilde{\Sigma}^{-1}$, our *rotated* returns are $r_t = H\tilde{r}_t = \Lambda^{-\frac{1}{2}} A^\top \tilde{r}_t$ where A is the orthonormal matrix with the normalized eigenvectors of $\tilde{\Sigma}$ as columns and $\Lambda^{-\frac{1}{2}}$ is the diagonal matrix with the square roots of the reciprocals of the eigenvalues Λ of $\tilde{\Sigma}$ on the diagonal. Therefore, r_t are the standardized principal components of $\tilde{r}_t$.

Also note that the GLS case where $W = \hat{\Sigma}_{\epsilon\epsilon}^{-1}$ leads to the same estimate of λ as in the case where $W = \Sigma^{-1}$, but the CS R^2 equals

$$R^2_{CS} = 1 - \frac{\tilde{\alpha}^\top \Sigma^{-1}_{\tilde{\epsilon}\tilde{\epsilon}} \tilde{\alpha}}{\tilde{\mu}^\top \Sigma^{-1}_{\tilde{\epsilon}\tilde{\epsilon}} \tilde{\mu}}, \tag{12}$$

[3]Results of the factorization for a Cholesky decomposition of Σ are identical, even though the returns on the test assets – in this case, the Cholesky factors – differ from the PCs of asset returns.

where $\tilde{\alpha}^\top \Sigma_{\tilde{\epsilon}\tilde{\epsilon}}^{-1} \tilde{\alpha}$ is the W_u term in the F-statistic of Gibbons et al. (1989).

The estimate $\hat{\lambda}$ corresponds to the premium on a minimum-norm, unit-beta factor mimicking portfolio; i.e., a portfolio that minimizes $w^\top w$, subject to the constraint $w^\top \tilde{\beta} = 1$, where $w = \beta(\beta^\top \beta)^{-1}$ denotes the $N \times 1$ vector of portfolio weights. In the case of a traded factor, the factor may not be priced exactly — i.e., $E(f)$ may differ from $\hat{\lambda}$ — but the unit-beta mimicking portfolio is priced correctly by construction; i.e., $w^\top \alpha = (\beta^\top \beta)^{-1} \beta^\top \alpha = 0$, as the vector α is the vector of residuals of a regression of the test assets' risk premia on the factor betas (see above).

Also note that in the last case, the rotated excess returns are orthonormal; i.e., $\Sigma = I_N$. Hence, if we consider the eigenvalue decomposition of $\tilde{\Sigma}$, our exercise amounts to testing an asset pricing model on the standardized principal components of the returns on the test assets. Finally, in this case, $\mu^\top \mu$ equals the squared SR of the tangency portfolio of the test assets, whereas

$$\hat{\alpha}^\top \hat{\alpha} = HJD^2, \tag{13}$$

where HJD denotes the Hansen and Jagannathan (1997) distance measure. Hence, the CS R-square equals

$$R^2_{CS} = 1 - \frac{HJD^2}{SR^2_T}, \tag{14}$$

where SR_T is the Sharpe ratio of the tangency portfolio of the test assets.

II.C CS and TS fit

We now proceed to derive an explicit link between CS and TS explanatory power of the factor model. The derivation follows the analysis of MacKinlay (1995), with the difference that our focus is on the explanatory power of the candidate model, whereas MacKinlay (1995)

focuses on the explanatory power of the missing factor(s), as captured by the returns on the "optimal orthogonal portfolio."

Proposition II.1 *We can write* R^2_{CS} *as*

$$\begin{aligned} R^2_{CS} &= \frac{Ave(\sigma_n^2) - Ave(\sigma_{\epsilon n}^2)}{Ave(\mu_n^2)} \frac{\hat{\lambda}_f^2}{\sigma_f^2} \\ &= \left[1 - \frac{Ave(\sigma_{\epsilon n}^2)}{Ave(\sigma_n^2)}\right] \frac{\hat{\lambda}^2/\sigma_f^2}{Ave(\mu_n^2)/Ave(\sigma_n^2)}. \end{aligned} \tag{15}$$

Proof Let $\hat{\mu} = \beta\hat{\lambda}$ denote the "predicted" component of test asset risk premia as per the asset pricing model. We have:

$$\hat{\mu}\hat{\mu}^\top = \beta\beta^\top\hat{\lambda}^2. \tag{16}$$

From the factor model in equation (2) we also have:

$$\Sigma = \beta\beta^\top\sigma_f^2 + \Sigma_{\epsilon\epsilon}. \tag{17}$$

Combining equations (16) and (17) above, we have:

$$\hat{\mu}\hat{\mu}^\top = (\Sigma - \Sigma_{\epsilon\epsilon})\frac{\hat{\lambda}^2}{\sigma_f^2}. \tag{18}$$

Taking the trace of both sides of equation (18) above and dividing by $\mu^\top\mu$, we have the result in (15).

Equation (15) above is the main analytical result of this paper: a set of factor betas explains the CS of test asset risk premia if the factor is systematic, as measured by the term

$$1 - \frac{\text{Ave}(\sigma_{\epsilon n}^2)}{\text{Ave}(\sigma_n^2)} \tag{19}$$

and if it is priced relative to the test assets, as measured by the term

$$\frac{\hat{\lambda}^2/\sigma_f^2}{\text{Ave}(\mu_n^2)/\text{Ave}(\sigma_n^2)}. \tag{20}$$

Proposition II.2 *Note that (19) corresponds to the pseudo-R^2 in Kleibergen and Zhan (2015):*

$$\textit{pseudo-}R^2 = 1 - \frac{\sum_{n=1}^{N} \theta_{\epsilon n}}{\sum_{n=1}^{N} \theta_n} \tag{21}$$

where θ_n are the eigenvalues of Σ and $\theta_{\epsilon n}$ are the eigenvalues of $\Sigma_{\epsilon\epsilon}$.

Proof The eigendecompositions of the covariance matrices of the test assets returns and residuals of the single-factor models are $\Sigma = Q\Theta Q^\top$ and $\Sigma_{\epsilon\epsilon} = Q_\epsilon \Theta_\epsilon Q_\epsilon^\top$, respectively. Here, Q and Q_ϵ represent the matrices whose columns are the eigenvectors of Σ and $\Sigma_{\epsilon\epsilon}$, while Θ and Θ_ϵ are the diagonal matrices containing the corresponding eigenvalues, ordered in descending magnitude. The trace of $(\Sigma - \Sigma_{\epsilon\epsilon})$ is

$$\begin{aligned} \mathrm{Tr}(\Sigma - \Sigma_{\epsilon\epsilon}) &= \mathrm{Tr}(Q\Theta Q^\top) - \mathrm{Tr}(Q_\epsilon \Theta_\epsilon Q_\epsilon^\top) = \mathrm{Tr}(\Theta) - \mathrm{Tr}(\Theta_\epsilon) \\ &= \sum_{n=1}^{N} \theta_n - \sum_{n=1}^{N} \theta_{\epsilon n} \end{aligned} \tag{22}$$

Thus, by taking the traces of both sides of equation (18) and dividing by $\mu^\top \mu$, we have

$$\begin{aligned} R^2_{CS} &= \left[1 - \frac{\sum_{n=1}^{N} \theta_{\epsilon n}}{\sum_{n=1}^{N} \theta_n}\right] \frac{\hat{\lambda}^2/\sigma_f^2}{\sum_{n=1}^{N} \mu_n^2 / \sum_{n=1}^{N} \theta_n} \\ &= \left[1 - \frac{\mathrm{Ave}(\theta_{\epsilon n})}{\mathrm{Ave}(\theta_n)}\right] \frac{\hat{\lambda}^2/\sigma_f^2}{\mathrm{Ave}(\mu_n^2)/\mathrm{Ave}(\theta_n)} \\ &= \text{pseudo-}R^2 \times \frac{\hat{\lambda}^2/\sigma_f^2}{\mathrm{Ave}(\mu_n^2)/\mathrm{Ave}(\theta_n)} \end{aligned} \tag{23}$$

where $\mathrm{Ave}(\theta_n) = \mathrm{Ave}(\sigma_n^2)$ and $\mathrm{Ave}(\theta_{\epsilon n}) = \mathrm{Ave}(\sigma_{\epsilon n}^2)$.

Below, we specialize the result.

Proposition II.3 *In the WLS case, we have ($\sigma_n^2 = 1, \forall n$):*

$$R^2_{CS} = \textit{Ave}(R^2_{TSn}) \frac{\hat{\lambda}^2/\sigma_f^2}{\textit{Ave}(\mu_n^2)}, \tag{24}$$

where R^2_{TSn} is the TS R2 of the single-index model for the n-th excess return.

Proposition II.4 *In the GLS case, we have:*

$$R^2_{CS} = Ave(R^2_{TSn})\frac{N(\hat{\lambda}^2/\sigma^2_f)}{SR^2_{max}}. \tag{25}$$

In the case where the factor is a PC of test-asset returns, the average TS R2s is equal to $1/N$ where N is the number of test assets. The TS R2 for the multi-factor model is equal to K/N where K is the number of factors.

Proposition II.5 *In the GLS case when the factors are included in the test assets, we have*

$$1 - \frac{Ave(\sigma^2_{\epsilon n})}{Ave(\sigma^2_n)} = \frac{K}{N+K}. \tag{26}$$

Proof Let's consider a multi-factor model where the dependent variables are the PCs of the $N+K$ test assets returns and factors, $z_t = H\tilde{z}_t$ where $\tilde{z}_t = [\tilde{r}_t, f_t]^\top$ and H is a $(N+K)\times(N+K)$ rotation matrix. We can represent the factors f_t as a linear combination of $\tilde{z}_t$, $f_t = W\tilde{z}_t$ where W is a $K\times(N+K)$ matrix of coefficients. Therefore,

$$z_t = a + \beta W\tilde{z}_t + \epsilon_t \tag{27}$$

where $\beta = cov(H\tilde{z}_t, W\tilde{z}_t)(W\tilde{\Sigma}W^\top)^{-1} = H\tilde{\Sigma}W^\top(W\tilde{\Sigma}W^\top)^{-1}$. Here $\tilde{\Sigma}$ is the covariance matrix of $\tilde{z}_t$. The explained variance of the model is

$$\Sigma - \Sigma_{\epsilon\epsilon} = \beta W\tilde{\Sigma}W^\top\beta^\top. \tag{28}$$

Taking traces of both sides and remembering that $\Sigma = H\tilde{\Sigma}H^\top$ we have

$$tr(H\tilde{\Sigma}H^\top) - tr(\Sigma_{\epsilon\epsilon}) = tr(\beta W\tilde{\Sigma}W^\top\beta^\top). \tag{29}$$

Using the cyclic property of traces, plugging in the expression of β in the right side, and remembering that $\tilde{\Sigma}^{-1} = H^\top H$ in the GLS case, the expression simplifies to

$$(N + K) - tr(\Sigma_{\epsilon\epsilon}) = K. \tag{30}$$

Finally, dividing both sides by $(N + K)$ and remembering that $Ave(\sigma_n^2) = 1$, we obtain the result.

Proposition II.6 *In the GLS case when the factors are a linear transformation of the test assets returns, we have*

$$1 - \frac{Ave(\sigma_{\epsilon n}^2)}{Ave(\sigma_n^2)} = \frac{K}{N}. \tag{31}$$

Proof This proof follows the same reasoning as the previous one. However, in this case, the dependent variables are the principal components (PCs) of the N test assets returns, $r_t = H\tilde{r}_t$, where H is an $N \times N$ matrix, and the factors are a linear combination of the raw returns $f_t = W\tilde{r}_t$, where W is a $K \times N$ coefficient matrix. Therefore, in the last step, we substitute $(N + K)$ with N

Proposition II.6 includes all the cases where the factors are a linear combination of the returns. For example, this is true when the factors are the principal components (PCs) of the returns. It also applies when we use mimicking portfolios of non-traded factors as factors, since these portfolios are the variable component of a linear projection of the non-traded factors on the excess returns Balduzzi and Robotti (2008).

Proposition II.7 *In the GLS case, given K factors, the maximum TS R^2 attainable is*

$$1 - \frac{Ave(\sigma_{\epsilon n}^2)}{Ave(\sigma_n^2)} = \frac{K}{N}. \tag{32}$$

Proof In the GLS case, the dependent variables are the principal components (PCs) of the test asset returns. Since PCs are uncorrelated with each other, if we have K factors, the best we can do is to perfectly explain the time-series variability of K PCs. The K factors that maximize the time-series fit are a linear combination of K PCs, and therefore, they are uncorrelated with the other $N - K$ PCs. This implies that the maximum time-series R^2 attainable is K/N.

II.D Non-traded factors and multiple factors

If the factor is not traded, it can be replaced with (the variable component of) its projection onto the augmented span of excess returns. In this case, the factor is simply a linear combination of the test assets' excess returns. Hence, the factor is effectively included in the test assets.

Proposition II.8 *If there are K traded factors, then the multiple factors can be replaced with the single excess return on the tangency portfolio of the factor assets, the optimal factor portfolio (OFP).*[4]

$$f_t^\star = f_t^\top \Sigma_{ff}^{-1} \hat{\lambda}, \tag{33}$$

where $\hat{\lambda} = (\beta^\top \beta)^{-1} \beta^\top \hat{\mu}$ is now a $K \times 1$ vector of risk premia on unit-beta factor-mimicking portfolios.

The betas w.r.t. the OFP returns are given by:

$$\beta^\star = \Sigma_{rf^\star} \frac{1}{\sigma_{f^\star}^2}. \tag{34}$$

[4]In the case of non-traded factors, first we project the factors onto the test assets' excess returns, then we construct the tangency portfolio.

We have

$$\Sigma_{rf^\star} = \Sigma_{rf}\Sigma_{ff}^{-1}\hat{\lambda} \tag{35}$$

and

$$\sigma_{f^\star}^2 = \hat{\lambda}^\top \Sigma_{ff}^{-1}\hat{\lambda}, \tag{36}$$

which leads to

$$\beta^\star = \frac{1}{\hat{\lambda}^\top \Sigma_{ff}^{-1}\hat{\lambda}}\Sigma_{rf}\Sigma_{ff}^{-1}\hat{\lambda} = \frac{1}{\hat{\lambda}^\top \Sigma_{ff}^{-1}\hat{\lambda}}\beta\hat{\lambda}. \tag{37}$$

The CS estimate of the factor risk premium is given by

$$\begin{aligned}
\hat{\lambda}^\star &= [(\beta^\star)^\top\beta^\star]^{-1}(\beta^\star)^\top\mu \\
&= (\hat{\lambda}^\top\Sigma_{ff}^{-1}\hat{\lambda})(\hat{\lambda}^\top\beta^\top\beta\hat{\lambda})^{-1}\hat{\lambda}^\top\beta^\top\mu \\
&= (\hat{\lambda}^\top\Sigma_{ff}^{-1}\hat{\lambda})(\hat{\lambda}^\top\beta^\top\beta\hat{\lambda})^{-1}\hat{\lambda}^\top\beta^\top\beta\hat{\lambda} \\
&= \hat{\lambda}^\top\Sigma_{ff}^{-1}\hat{\lambda} \\
&= \sigma_{f^\star}^2.
\end{aligned} \tag{38}$$

The component of risk premia explained by the single-factor model is $\beta^\star\hat{\lambda}^\star$, which, based on equation (38) and (37), equals $\beta\hat{\lambda}$. This is the same component of the risk premia *jointly* explained by the original set of factors f_t.

In the next section, we provide an economic justification for our focus on the OFP returns in terms of the properties of a candidate pricing kernel.

II.E Relation to pricing kernel representation

Given the *normalized* candidate pricing kernel q_t ($E(q_t) = 1$), we have:

$$\alpha = E(q_t r_t) \tag{39}$$

where

$$q_t = 1 - b(f_t - \mu_f). \tag{40}$$

Hence, we have

$$\alpha = E(qr_t) = \mu - \Sigma_{rf} b \tag{41}$$

and

$$\begin{aligned} \hat{b} &= (\Sigma_{fr}\Sigma_{rf})^{-1}\Sigma_{fr}\mu \\ &= \frac{\hat{\lambda}}{\sigma_f^2}, \end{aligned} \tag{42}$$

where $\hat{b}$ is the coefficient of a CS regression of the test asset risk premia μ on the covariances between asset returns and the factor. In the case of multiple factors, we have:

$$\hat{b} = \Sigma_{ff}^{-1}\hat{\lambda} \tag{43}$$

and

$$q_t = 1 + \hat{b}^\top \mu_f - f_t^\star. \tag{44}$$

The same factorization of CS R2s arises in this context as in the case of beta models. We have:

$$\hat{\mu}\hat{\mu}^\top = \Sigma_{rf}\Sigma_{fr} b^2. \tag{45}$$

From the factor model

$$r_t = a + \beta_q q_t + \epsilon_t \tag{46}$$

where $\beta_q = -\Sigma_{rf}\frac{1}{\sigma_f^2}$, we have

$$\Sigma = \Sigma_{rf}\Sigma_{fr}\frac{1}{\sigma_f^2} + \Sigma_{\epsilon\epsilon}. \tag{47}$$

Combining equations (45) and (47) above, we have:

$$\hat{\mu}\hat{\mu}^{\top} = (\Sigma - \Sigma_{\epsilon\epsilon})\frac{\hat{\lambda}^2}{\sigma_f^2}, \tag{48}$$

which is the same as (18). Also note that the TS fit of a single-index model with the excess return on the OFP as the factor is the same as the fit of a TS regression of excess returns on the estimated pricing kernel.

III Bootstrap

As documented by Lewellen et al. (2010) and Kan et al. (2013), even very high CS R-squares may still lead to the rejection of an asset pricing model. Hence, we conduct a bootstrap exercise to implement our inference.

We consider bootstrap inference under three nulls and under the alternative. In all cases, we bootstrap blocks of monthly observations to account for serial correlation and ARCH effects. We generate bootstrap samples of raw excess returns using the estimates that correspond to a given choice of W; i.e., we use the OLS estimates to generate bootstrap samples on which we perform the estimation for case 1; we use the WLS estimates to generate bootstrap samples on which we perform the estimation for case 2; we use the GLS estimates to generate bootstrap samples on which we perform the estimation for case 3.

We consider the following nulls:

- **Null 1, unpriced factor** ($\lambda = 0$): The factor is not priced.

  Under this null hypothesis, to generate bootstrap samples for inference, we first demean the raw returns and the factors and add $\tilde{\alpha} = \hat{H}^{-1}\hat{\alpha}$ to the raw returns. Adjusted returns are denoted as $\tilde{r}^a = \{\tilde{r}_1^a, \ldots, \tilde{r}_T^a\}$, where $\tilde{r}_t^a = \tilde{r}_t - \mathbb{E}(\tilde{r}_t) + \tilde{\alpha}$ is a $N \times 1$ vector.

Similarly, adjusted factors are denoted as $f^a = \{f_1^a, \ldots, f_T^a\}$, where $f_t^a = f_t - \mathbb{E}(f_t)$ is a $K \times 1$ vector. Bootstrap samples $\tilde{r}^{a,b,*}$ and $f^{a,b,*}$ are then drawn with replacement blocks of m observations from $\tilde{r}^a$ and f^a, respectively, where $b = 1, \ldots, B$ and B is the number of bootstrap samples. The returns and factors series are drawn jointly for each sample (note that $\hat{H}$ and $\hat{\alpha}$ are held constant across all bootstrap samples).

- **Null 2, the model holds** ($\alpha = 0$): the factor prices expected excess returns exactly.

  Under this null hypothesis, to generate bootstrap samples for inference, we first demean the raw returns and the factors and add $\hat{\tilde{\beta}}\hat{\lambda} = \hat{H}^{-1}\hat{\beta}\hat{\lambda}$ to the raw returns and $\hat{\lambda}$ to the factors. Adjusted returns are denoted as $\tilde{r}^a = \{\tilde{r}_1^a, \ldots, \tilde{r}_T^a\}$, where $\tilde{r}_t^a = \tilde{r}_t - \mathbb{E}(\tilde{r}_t) + \hat{\tilde{\beta}}\hat{\lambda}$ is a $N \times 1$ vector. Similarly, adjusted factors are denoted as $f^a = \{f_1^a, \ldots, f_T^a\}$, where $f_t^a = f_t - \mathbb{E}(f_t) + \hat{\lambda}$ is a $K \times 1$ vector. Bootstrap samples $\tilde{r}^{a,b,*}$ and $f^{a,b,*}$ are then drawn with replacement blocks of m observations from $\tilde{r}^a$ and f^a, respectively, where $b = 1, \ldots, B$ and B is the number of bootstrap samples. The returns and factors series are drawn jointly for each sample (note that $\hat{H}$, $\hat{\beta}$, and $\hat{\lambda}$ are constant across bootstrap samples).

- **Null 3, useless factor** ($\beta = 0$): the factor is priced but does not explain excess returns.

  Bootstrap samples $\tilde{r}^{b,*}$ and $f^{b,\circ}$ are drawn with replacement blocks of m observations from $\tilde{r} = \{\tilde{r}_1, \ldots, \tilde{r}_T\}$ and $f = \{f_1, \ldots, f_T\}$, respectively. The returns and factors series are drawn independently for each sample.

- **Alternative**: the factor is priced, but it does not price excess returns exactly.

  Bootstrap samples $\tilde{r}^{b,*}$ and $f^{b,*}$ are drawn with replacement blocks of m observations from $\tilde{r} = \{\tilde{r}_1, \ldots, \tilde{r}_T\}$ and $f = \{f_1, \ldots, f_T\}$, respectively. The returns and factors

series are drawn jointly for each sample.

Given a vector of bootstrap raw returns $\tilde{r}_t^{b,*}$, we then recover the bootstrap returns $r_t^{b,*}$ by performing the rotation $r_t^{b,*} = \hat{H}^{b,*}\tilde{r}_t^{b,*}$. (Note that $\hat{H}^{b,*}$ is estimated based on the bootstrap sample of raw returns and changes with the bootstrap sample.)

IV Data

In this study, we use as test assets the well-documented 25 size-B/M portfolios along with 30 industry portfolios, sourced from the Kenneth R. French data library. Lewellen et al. (2010) point out that considering only the 25 size-B/M portfolios as test assets is a low hurdle to meet in claiming success given the strong factor structure of these portfolios. For this reason, following their advice, we include the 30 industry portfolios in the test assets.[5] We also extend the study by incorporating additional test assets: the 25 portfolios formed monthly on Size and Momentum, 25 portfolios formed on Size and Operating Profitability, and 25 portfolios formed on Size and Investment, also sourced from French data library. The factors that we use in this analysis are the Fama and French (2015) five factors—the excess return of the Market (Mkt.Rf), Size (SMB: small-minus-big), Value (HML: high-minus-low), Profitability (RMW: robust-minus-weak), and Investment (CMA: conservative-minus-aggressive)—along with Momentum (UMD: up-minus-down). Our sample period spans from July 1963, marking the beginning of the available data for the six factors in the Kenneth R. French database, through December 2022, providing nearly six decades of financial data.

We also analyze the case of non-traded factors, focusing on the conditional CAPM (C-LAB) by Jagannathan and Wang (1996), the unconditional consumption model (CG), and

[5]Readers interested in the descriptive statistics of these test assets can find the expected values and standard deviations of the monthly returns in the Appendix that is available on request.

the conditional version of the consumption CAPM by Lettau and Ludvigson (2001) (CC-CAY). Growth rate in per capita labor income (LAB) is from the Andriollo et al. (2024) database, the lagged yield spread between Baa- and Aaa-rated corporate bonds (Baa–Aaa), and the growth rate in real per capita nondurable consumption (seasonally adjusted at annual rates) (CG) from the Federal Reserve Economic Data. The consumption-wealth ratio factor is from Lettau's site (CAY). The data spans from July 1963 through September 2019, the last available data for the *cay* factor.[6] Following Balduzzi and Robotti (2008), we decide to work with the mimicking portfolios of the non-traded factors.

V Results

We apply our factorization and decomposition to the OLS, GLS, and WLS cases, using both traded and non-traded factors, as well as the first six principal components (PCs) of test assets returns as factors. When using the traded factors, we also inspect how asset pricing model's estimates change when factors and characteristic-based portfolio returns are included in the test assets. In doing this, we follow the prescription suggested in Lewellen et al. (2010), upon which a proposed factor, when it is a traded portfolio, should be included as one of the test assets. The issue is relevant for GLS regressions, as this case forces the regression to price the asset perfectly (the estimated risk premium matches the expected value of the factor); see Lewellen et al. (2010) and Kan et al. (2013).

To put our factorization into perspective, we display results through *iso-*R^2 *curves*, we

[6]Following Vissing-Jørgensen and Attanasio (2003), Kan et al. (2013) and Andriollo et al. (2024), we linearly interpolate the quarterly values of *cay* and the growth rate in real per capita nondurable consumption to permit analysis at the monthly frequency.

realize by fixing $R^2_{CS} = \bar{R}^2_{CS}$, solving for $\hat{\lambda}^2/\sigma_f^2$, then obtaining:

$$\hat{\lambda}^2/\sigma_f^2 = \frac{[\text{Ave}(\mu_n^2)/\text{Ave}(\sigma_n^2)] \times \bar{R}^2_{CS}}{\left[1 - \frac{\text{Ave}_{(\sigma^2_{\epsilon n})}}{\text{Ave}_{(\sigma_n^2)}}\right]}. \tag{49}$$

Analytically, we plot $\hat{\lambda}^2/\sigma_f^2$ as a function of $\left[1 - \frac{\text{Ave}_{(\sigma^2_{\epsilon n})}}{\text{Ave}_{(\sigma_n^2)}}\right]$, for different values of $\bar{R}^2_{CS}$, for both traded and non-traded factors in the OLS and GLS cases (Figures 1, 2, 3, and 4.[7]. In this way, we graphically depict the trade-off between pervasiveness and the relative risk-return trade-off of the factors for different values of the cross-sectional R^2_{CS}.

We finally analyze the behavior of R^2_{CS} over time using a moving window approach with a length of 20 years (240 months). We chose a 20-year window due to the large number of test assets (55), which necessitates a sufficiently long estimation period. The first data available with this moving window length is June 1983.

The following sections comment on OLS and GLS estimations, as the WLS results closely follow those in the OLS case with a few differences – in the case of the first PCs of returns, the time-series fit is not monotonically decreasing for the PCs, and not only the first PC (which was already high in the OLS case), but also some of the other PCs show a high ratio of squared Sharpe ratios (SRs), leading to relatively high R^2_{CS} values compared to the OLS case.[8]

V.A Traded factors

V.A.1 OLS

R^2_{CS} factorization

Table 1 reports results of cross-sectional R-square estimation and bootstrap inference

[7]Computationally, the iso-R^2 curves are constructed by setting the quantity $\text{Ave}(\mu_n^2)/\text{Ave}(\sigma_n^2)$ to the corresponding value computed on the test asset sample.

[8]In the Appendix we show the results and plot the isocurves for the WLS case.

under the three nulls $\alpha = 0$, $\beta = 0$, $\lambda = 0$, while in Table 2 and 3 we show the two component of our factorization. The root mean squared error (RMSE), computed through our bootstrap procedure under the alternative hypothesis, denotes variability in the R^2_{CS} estimates, with some factors showing R^2_{CS} estimated with greater precision, while others do not. For example, the R^2_{CS} of Market factor is 0.9158 with an RMSE of 0.1232; instead, for Value the R^2_{CS} is 0.0671 and the RMSE 0.0998. This variability in the estimation accuracy is tied to the accuracy of the pricing component of our factorization, as the TS Fit is generally estimated with greater precision across all factors compared to the pricing component.

The R^2_{CS} estimates are high for almost all factors (except for Value) and for the multi-factor models. This is consistent with the findings of Lewellen et al. (2010) and Kleibergen and Zhan (2015), who note that OLS R^2_{CS} can be high even when the factors are not systematic in the first pass of the Fama-MacBeth regression, or in the terminology of our paper, when the TS Fit is close to zero. This is especially true for non-traded factors, we discuss in the next section, but even for traded factors we observe low pervasiveness-high profitability pairing; see Figure 2. For instance, with the Momentum factor, although the TS Fit is close to zero (0.0375 with an RMSE of 0.0199), the factor still exhibits a relatively high R^2_{CS} (0.8218 with an RMSE of 0.1755) because of the high pricing component of the factor (the squared-SR ratio is 21.8900).

When including the factor in the test assets, Momentum becomes slightly more systematic (the TS Fit is 0.0458 with an RMSE of 0.0202), and the risk premium (reported in Table 4) moves closer to the expected value (though the gap remains substantial, at -1.7302 vs 0.6393), resulting in a lower squared-SR ratio (12.5572). Overall, the decrease in the pricing component has a more significant impact than the increase in the TS fit, leading the R^2_{CS} to decline.

The same pattern is observed for all other factors, except for the Market. In this case, including the factor in the test assets does not change the results. Market has in fact a different nature, being systematic (the TS Fit is 0.6317) with the lowest value of the pricing component (the squared-SR ratio is 1.4486) and showing low mispricing (the difference between the risk premium and the expected value is not substantial; see Table 4).

Moving to multiple factors, the R^2_{CS} improves relative to the Market. Our factorization gives a clinical understanding on the reasons underlying such an improvement. Multiple factors, combined together as optimal factor portfolios (OFPs), are less systematic and exhibit higher pricing components. While the FF3 model still shows a high TS Fit (this is because of the contribution of Market), the FF5 and FF6 models display lower TS Fit with substantial squared-SR ratios, because of the marginal contribution of Momentum, which exhibits the higher SR component among all single factors. Interestingly, note that OFPs tend to be correctly priced as the risk premiums are closely aligned with the expected values of the OFPs. This is the first tension we document between the CAPM and the multi-factor models, we attribute to the nature of factors. In more depth, the R^2_{CS} can be high because the factors are pervasive, as is the case for Market, or because they show high pricing component (squared-SR ratios), as is the case for Momentum. Also, multi-factor models in the form of OFPs, while consistent with the kernel representation, reflect on extreme factor weights[9], which lead factors to work well in the cross section notwithstanding the low pervasiveness again because they act together as near-arbitrage opportunities (high squared-SR ratios); this is indeed the case of FF6.

Parameter inference

[9]Green and Hollifield (1992) show that this is due to the dominance of a single factor in the covariance structure of returns.

We now consider the bootstrap inference results under the three nulls $\alpha = 0$, $\beta = 0$, $\lambda = 0$. Tables 1, 2 and 3 denote whether the factors are priced ($\hat{\lambda} \neq 0$), useless ($\beta = 0$), and price expected excess returns exactly ($\hat{\alpha} = 0$) at 5% level[10]. Based on the R^2_{CS} distributions, we reject the null hypothesis that factors are unpriced ($\lambda = 0$) with the exception of Value. However, we only reject the null that the factor is useless ($\beta = 0$) for Market, Momentum, and multiple-factor models. For single and multi-factor models we cannot reject that model holds ($\alpha = 0$), differently from the GLS case we discuss later in which we do reject the null for multiple-factor models. Under the null of useless factors, the confidence intervals for $\hat{\lambda}$ (and $\hat{\lambda}/\sigma_f$; see Appendix) are wider than under the other hypotheses, implying higher values for the top percentiles of the squared-SR ratio under the useless factors null. These high values are explained by the assumption that the factors are useless, making the risk premiums in the second pass of the Fama-MacBeth regression undefined. This is consistent with the findings of Kan and Zhang (1999b), which show that as the estimated betas approach zero, the absolute value of the estimated risk premiums must increase toward infinity, rather than decrease to zero, in order to explain the cross-sectional differences in expected returns. Note that under the nulls $H_0 : \alpha = 0$ and $H_0 : \lambda = 0$ the distribution of the TS Fit component is the same.[11]

[10] More results are in Appendix, in which we present the critical percentiles of the distributions of the cross-sectional R-squares and their components.

[11] Under $H_0 : \hat{\alpha} = 0$ and $H_0 : \hat{\lambda} = 0$ the distribution of the TS measure for each single factor is the same. This is due on how we have built our bootstraps. Under $H_0 : \hat{\alpha} = 0$ our bootstrapped returns are $r^b_t = [r_t - \mathbb{E}[r_t] + \beta\hat{\lambda}]_b$ and the factors $f^b_t = [f_t - \mathbb{E}[f_t] + \hat{\lambda}]_b$, while under $H_0 : \hat{\lambda} = 0$ our bootstrapped returns are $r^b_t = [r_t - \mathbb{E}[r_t] + \hat{\alpha}]_b$ and the factors $f^b_t = [f_t - \mathbb{E}[f_t]]_b$. In all these cases $var(r^b_t)$, $var(f^b_t)$ and $cov(r^b_t, f^b_t)$ depend on the non-constant part of the value that is draw from the original sample and this is the same for both hypothesis, since for both hypothesis the draws have the same seed. As a result the bootstrapped betas β^b under the two hypothesis are the same and also the variance-covariance matrix of the residuals is the same. This is not true for the OFP because in this case the OFP $f^{b*}_t = f^{b\top}_t \Sigma^{-1}_{f^b f^b} \hat{\lambda}^b$ depends on the bootstrapped factors and the bootstrapped lambdas that changes with the hypothesis.

V.A.2 GLS

R^2_{CS} Factorization

In the GLS case, the R^2_{CS} estimates are low and less precise compared to the OLS case, ranging from 0.0060 (the RMSE is 0.0132) for Robust-Minus-Weak (RMW), to 0.0568 (the RMSE is 0.0285) for Market. Note that when using the optimal factor portfolios (OFPs) the R^2_{CS} estimates improve. For instance, the FF6, which includes the five Fama-French (FF) factors and Momentum, raises the R^2_{CS} to 0.3206, with an RMSE of 0.1051. This is consistent with many studies (see, e.g., Lewellen et al. (2010), Kan et al. (2013), and Kleibergen and Zhan (2015)) documenting low values of CS Fit when using GLS.

As outlined in Proposition II.7, the value of the TS Fit is upper bounded in the GLS case. Table 2 and Figure 1 show a value of 0.0181 for Market, which is just below the theoretical maximum of $K/N = 1/55 = 0.0182$, with K the number of factors and N the number of test assets.[12] We also recall that when the factors are included in the test assets, Proposition 26 asserts that the model TS Fit equals $K/(N+K)$. This is exactly what we observe in Table 2.

It is clear that, in the GLS case, the CS Fit mainly depends on the ratio of squared Sharpe ratios. As discussed, the ratio reflects the risk-return trade-off of the model factor relative to that of the test assets tangency portfolio (see Equation 1). Therefore, the GLS R^2_{CS} reflects the factor proximity to the minimum-variance boundary, as noted by Kandel and Stambaugh (1995)[13] and emphasized by Lewellen et al. (2010). The Market has the highest R^2_{CS}, followed by Momentum, which shows the highest ratio of squared Sharpe ratios, while

[12] Note that the upper bounding applies also in the case of the OFPs, for which we are actually dealing with a single factor, the tangency portfolio of factors.

[13] The authors argue that the goodness-of-fit measure for the GLS cross-sectional regression is the squared relative efficiency of the index portfolio.

the Market ranks just after.

When factors are included in the test assets, the R^2_{CS} and the ratio of squared Sharpe ratios become interchangeable measures of relative efficiency, in line with the prescription of Lewellen et al. (2010) about the inclusion of traded factors within the test assets sample. When we do this, Momentum has the highest ratio of squared Sharpe ratios and, in turn, the highest R^2_{CS}, followed by Investment and Profitability; Market factor is ranked only fourth. Multi-factor models (OFPs) improve significantly the R^2_{CS} of single factors as a result of the extreme portfolio weights of the tangency portfolio (Green and Hollifield (1992)) which amplify the nature of near-arbitrage opportunity of the factors (higher ratios of squared Sharpe ratios).

Figure 1 offers a graphical representation of our results. The three factors of the Fama-French 3-factor model (FF3) share similar spanning and pricing properties. They explain, on average, less than 2% of the TS variability of test assets' returns and deliver a squared Sharpe ratio that is less than 0.02. As a result, their ability to explain the CS of test assets' risk premia is rather limited: the CS R^2 is less than 6% for Market and even lower for Value (HML) and Size (SMB). When these three factors are combined together (OFP), the cross sectional fit improves to about 12%, keeping the TS Fit essentially unchanged. Investment (CMA) and Profitability (RMW) factors explain less of the TS variability of test assets' returns – slightly above 1% – than the three FF factors and offer similar risk-return trade-offs. Therefore, the R^2_{CS} is less than that of the original three FF factors.

Moving to the five FF factors, the improvement in the CS R^2 is substantial, as the coefficient is roughly 17%. Again, the reason of such improvement is mainly due to the increased relative Sharpe ratio.

While Market is the pervasive factor by definition, and proved by our R-square factor-

ization, Momentum embodies the essence of a near-arbitrage opportunity. Researchers have given to Momentum the status of an independent factor. Ehsani and Linnainmaa (2022), by referring to Fama and French (2016), observe that models including it tend to explain anything more than just Momentum, while models without it are unable to explain such a factor. Our factorization leads to novel insights which help explain the source of the tension between CAPM (Market) and Multi-factor models (FF6). Specifically, when taken individually, Momentum is the least systematic among the FF factors (the TS Fit component is 0.0068), although the cross sectional fit is near 4%, slightly less than the Market; moreover, when including Momentum within the test assets sample, the CS R^2 for Momentum is the highest. In other terms, while less systematic, this factor can explain the cross-section of the test assets because of its high squared SR, especially when including it in the test assets. In this case, Momentum becomes the best individual factor model among all six FF factors.

Finally, when combined with the other five factors, Momentum contributes to raise the CS explanatory power over the 32%. The cross sectional fit improvement driven by the increase in the squared Sharpe ratio through Momentum is thus more substantial in the case of the OFP, which benefits from the extreme weighting induced by the tangency portfolio. In this case, the balancing between TS Fit and relative SR leans even more towards the pricing component, at the expense of pervasiveness of the OFP return; and indeed, the average TS Fit of FF6 is halved compared to that of FF5.

Parameter inference

Bootstrap inference results are rather interesting and quite different relative to OLS. Factors are all/almost all priced ($\lambda \neq 0$) when included/not included in the assets (see Table 1), while we cannot reject the null they are useless ($\beta = 0$) except for FF6. The reason is

attributable to the distribution of the relative SR (Table 3), for which we cannot reject the hypothesis that single and multi-factor models are useless, except for FF5. The combined effect of the two CS R^2 component distributions impact on the statistical significance of the β parameters. A plausible explanation of this result connect to Kandel and Stambaugh (1995). The authors provides the link between the index portfolio's mean-variance location and the properties of the fitted GLS mean-beta relation, showing that a slope coefficient of zero occurs only when the mean return on the index is equal to that of the global minimum-variance portfolio. We can therefore presume that the non-significantly different from zero betas for almost all single and multi-factor models are because the same factors are close to characterized themselves as global minimum-variance portfolios. Such a characterization may also impact on the $\alpha = 0$ null, we cannot reject for FF3, FF5 and FF6. These are mean-variance optimal portfolios and as such, the variance is minimized by construction, causing strong rejection of the null.

V.A.3 R^2_{CS} decomposition

The R^2_{CS} decomposition through Equation 11 offers a further cross sectional fit anatomy of the pricing models, investigating the impact of the factor mispricing. Specifically, Equation 11 asserts that the cross-sectional fit can be high because of the difference between the estimated risk premia and the expected value of factors, as measured by $[\hat{E}(f_t) - \hat{\lambda}]^2$. Note that this term is scaled by $\frac{\beta^\top \beta}{\mu^\top \mu}$ which measures the cross variability of betas in the test assets. The results in Table 4 show that the high CS Fit values for the Market model and the multiple-factor models are close to the R^2_{Tot} values (indicating low time-series intercepts in the first-pass regression) and minimal factor mispricing (small differences between the risk premium and the expected value of the factor or OFP). For the OFPs, although $\frac{\beta^\top \beta}{\mu^\top \mu}$ is relatively high, the difference between the risk premia and expected values is nearly zero,

resulting in an R^2_{Tot} close to R^2_{CS}.[14] This is not the case for the other single factors. For these, the R^2_{Tot} values are low or even negative, and the CS Fit is largely explained by the models' inability to price the factors correctly. For example, in the case of Momentum, the estimated risk premium is -2.2774, or -1.7302 when the factor is included on the left-hand side of the regression, compared to the factor's expected value of 0.6393.

In the GLS case (Table 5), R^2_{Tot} is nearly equal to the CS fit, since the risk premium estimates are close to the expected value of the factors. When the factors are included in the test assets, these estimates coincide exactly. This occurs because, when a factor portfolio is included as a left-hand-side asset, GLS forces the regression to price the asset perfectly: the risk premium on the factor equals the factor's average excess return exactly (Lewellen et al. (2010)).

V.B Non-traded factors and Principal Components

V.B.1 Non-traded factors

Tables 6, 7, and 8 present the results of our factorization for non-traded factors (Panel A), while Figure 4 plots the corresponding isocurves. The results of the TS Fit confirm the Proposition II.6, which states that factors that are linear combinations of the returns have a TS Fit equal to K/N, where K is the number of factors and N is the number of test assets. Hence, the individual factors with the highest CS Fit are those with the highest ratio of squared Sharpe ratios (SRs). In particular, *cay* is the best non-traded factor in terms of CS Fit because of a relatively high ratio of squared-SR.

[14] The high values of $\frac{\beta^\top \beta}{\mu^\top \mu}$ in Equation 11 for the Optimal Factor Portfolios arise because these portfolios are constructed as the tangency portfolio of the factors, which minimizes portfolio variance. This approach leads to an independent variable with low variability in the first-pass regression, prompting the model to "overcompensate" by assigning large coefficients (high β values) to better match the more variable test asset returns.

The bootstrap-based inference results show single non-traded factors useless/useful priced/unpriced depending on the OLS vs. GLS estimation technique. This is the case for Market and consumption growth (CG), useless with GLS, consumption-wealth ratio (CAY) and per capita labor income growth (LAB), useless and unpriced with OLS. However, their multivariate version in the form of conditional CAPM models of Jagannathan and Wang (1996) (C-LAB) and Lettau and Ludvigson (2001) (C-CAY), whose factors are combined as tangency portfolios, are priced and not useless, but unable to price exactly the test assets (αs are significantly different from 0).

An in-depth analysis of the R^2_{Tot} for the non-traded factors in the GLS case denotes that they are equal to the R^2_{CS} because, in such a case, the risk premia and the expected values of the factors are identical. The exception is C-LAB, which includes the Market factor; this explains why the corresponding OFP has a different risk premium relative to the expected value of the factors, and why R^2_{Tot} is slightly different from R^2_{CS}.

V.B.2 Principal Components

The same tables 6, 7, and 8 presents the decomposition using the first six principal components (PCs) of returns as factors (Panel B). We also construct the OFP with 3, 5, and 6 PCs to compare with the FF3, FF5, and FF6 models. The choice to use the first PCs as factors is motivated by the fact that a factor model with a small number of principal components performs like popular reduced-form factor models in the cross-section (Kozak et al. (2018)).

Since we use PCs, the time-series fit of the PCs decreases monotonically, as the first PC captures the most variation in returns, followed by the second, and so on. When we consider the OFPs of these PCs, the time-series fit is lower than that of the first PC. Notably, the highest ratio of squared Sharpe ratios for a single factor is also associated with the first PC,

resulting in the highest R^2_{CS}. The pricing components of the other PCs are relatively small, as well as for their time-series fits, then causing a lower R^2_{CS}. Instead, when using PCs as OFPs, the R^2_{CS} improves substantially, due to the combined effect of the time-series fit and the high ratio of squared SRs.

The first PC is, to a reasonable approximation, a *level* factor that places equal weight to all test asset returns (Kozak et al. (2018)). This explains why its factorization distills the almost the same components that derive by factorizing the Market factor. Moreover, the results from the OFPs with 3, 5, and 6 PCs closely resemble those from the FF3, FF5, and FF6 models.

The results from bootstrap inference denote a first PC priced and possibly useless (GLS), and multiple PCs combined as optimal portfolio priced and possibly useless alike because characterized as near-arbitrage opportunities (high relative SRs). Moreover, in the GLS case we reject the hypothesis that multi-PCs price exactly the test asset returns (α is statistically different from 0).

Finally, we have no difference between R^2_{Tot} and R^2_{CS}, as the estimated risk premia of the PCs are equal to the expected values of their returns.

V.C Discussion

As a whole, the results from traded, non-traded and PC factors, tell us that popular empirical single- and multi-factor asset pricing models only capture a relatively small fraction of the TS variability of test assets' returns. The improvement in the cross-sectional fit is driven by the risk-return trade-off of OFPs. Indeed, the relative Sharpe ratio always increases with the number of factors, while reducing the pervasiveness. The nature of near-arbitrage opportunity of the OFPs reflects the extreme portfolio weighting induced by the tangency

portfolio mechanism. The point connects with Green and Hollifield (1992), who show how the extreme (long/short) weights in efficient portfolios are caused by the dominance of a single factor in the covariance structure of returns. The effect is to obtain a diversified portfolio with no systematic risk (zero beta) – as well as eliminating residual risk – and high near-arbitrage opportunity, which grows as we add to the Market/first PC (dominant and systematic factor) the other factors/PCs, which are less systematic but profitable (high relative SR). This is what plausibly occurs with the OFPs, which move toward a global minimum-variance portfolio characterization (Kandel and Stambaugh (1995)) thereby causing betas to be indistinguishable from zero (we cannot reject the null that the factors and principal components combined together are useless).

A key finding of our factorization is that factors that explain little in time-series because they are less systematic can work well in the cross-section because of their pricing component (the relative risk-return trade-off component of the cross-sectional R^2). Moreover, the nature of factors (pervasive or near-arbitrage) could change depending on the choice of the test assets. This is what seems to be happening with Momentum, which significantly reduces its pricing component while increasing the CS R^2 when included in the test assets. Note that Momentum is recomputed on a monthly basis, then including the most recently available data, whereas SMB, HML, RMW, and CMA are updated annually using data of at least six months old, except for Size. In other terms, the tests we used do not include Momentum-portfolios and results could be affected because we regressed characteristic and industry based portfolios, recomputed at least on a semi-annual basis, onto a factor which is recomputed on a monthly basis also exhibiting high volatility, being based on the cumulative average returns from t-12 to t-2 months (then using the 11 months in between). Therefore, to better understand the impact of test assets on the factor characterization (pervasive vs. priced), we

now move on including characteristic-based portfolios on the left hand side of the regression equation.

V.D Test assets and factors characterization

To inspect how the choice of the test assets impacts on the nature of factors, we re-run our factorization by including in the tests: (1) the 25 Size-Momentum Portfolios; (2) the 25 Size-Operating Profitability Portfolios; (3) the 25 Size-Investment Portfolios. This exercise is executed for each of the three set of portfolios, next comparing the TS Fit and the relative SR components with and without the characteristic-based portfolios one set at the time. Once we obtain all estimates (see the Appendix), we focus on the log differences of the CS R^2 and the corresponding TS Fit and relative SR; in this way, we assess the variations in the two CS R^2 components occurred when changing the original 55 tests. Specifically, the log differences are computed as,

$$\ln\left(\frac{R^2_{CS}}{R^2_{CS^b}}\right) = \ln\left(\frac{TSFit}{TSFit^b}\right) + \ln\left(\frac{RSR}{RSR^b}\right),$$

where $R^2_{CS^b}$, $TSFit^b$, RSR^b are the CS R^2, the TS Fit and the relative Sharpe ratio estimated using the starting 55 tests with the FF6, when analyzing the Size-Momentum portfolios, and the FF5, for Size-Operating Profitability and Size-Investment portfolios. The results are reported in Table 9. The highest impact is for Size-Momentum portfolios: when the 25 portfolios are added to the tests, the TS Fit increases substantially together with a large decrease in the relative SR. With OLS the variations in the two components are almost the same in magnitude, while with GLS the reduction in the relative SR is much more substantial, and thus the CS R^2 deteriorates sizably. For Size-Operating Profitability and Size-Investment portfolios, the CS R^2 deteriorates as well with the GLS, but not in the same

magnitude of Size-Momentum portfolios, because of the reduction in the TS Fit. The results differ from the OLS estimation: both sets denote a rebalancing of the two components, with Size-Operating Profitability increasing the pervasiveness of FF5 (the log variations are negative for RSR and positive for TSF), and Size-Investment placing more weight on RSR. As a whole, these results confirm the changing nature of factors, from pervasive to near-arbitrage or vice-versa depending on the choice of the test assets. Note, also, that since the GLS can be viewed as an OLS run on the rotated test assets, the different results between OLS and GLS are because we are analyzing different test assets, then reinforcing our conclusions: factors can change their nature depending on the tests we use. In that, we complement Giglio et al. (2024), who argue that the tests used in the estimation of asset pricing models qualify the factors. By connecting factors with test assets, they conclude that the strength or weakness of a factor should be viewed as a property of the tests. When exemplifying the case of a possible factor, they write that liquidity "... *may be weak in a cross section of portfolios sorted by, say, size and value, but may be strong in a cross section of assets sorted by characteristics that well capture exposure to liquidity*". What we add to Giglio et al. (2024) is the reason why factors depend on tests and can work in the cross-section when they are weak. This is because of the two components of our CS R^2 factorization. On the one hand, their combination attributes the nature of systematic factor or near-arbitrage opportunity; on the other, factors that have low power in time series, can still increase the cross sectional fit through their pricing component.

V.E Moving window evidence

V.E.1 OLS

Traded factors Figures 5, 6, and 7 display the behavior of R^2_{CS} and its two components for the traded factors and the OFPs formed from these factors in the OLS case (factors not included in the test assets). Figures 5 and 6 show the factorization for the five Fama-French factors and the Momentum factor. Apart from the Market factor, whose CS Fit and components remain relatively stable over time, the other factors exhibit variability in both the CS Fit and its components.

The Market factor shows a high R^2_{CS} over time, primarily driven by the TS Fit, as indicated by a relatively stable squared-SR ratio, which varies only between 1 and 1.5 throughout the analysis period. The Market factor also maintains the highest TS Fit among the factors over this period.

For the Size factor, the CS Fit is high in the earlier part of the period but decreases around the year 2000. This decline in CS Fit is due to a gradual decrease in the TS Fit, starting around 1983, and a drop in the squared-SR ratio at the turn of the century. After 2000, the TS Fit for the Size factor remains stable, with an increase only after 2020. Meanwhile, the pricing component begins to rise after 2000, contributing to a subsequent increase in the CS Fit.

The Value factor exhibits an increasing CS Fit prior to 2000, primarily driven by a rise in the TS Fit, despite a declining squared-SR ratio during this period. After 2000, both the TS Fit and the pricing component begin to decline, reaching their lowest points in the second decade of the 21st century. However, after 2020, both components in the factorization start to increase, with a particular rise in the squared-SR ratio, resulting in a notable jump in

R^2_{CS}.

The Profitability factor shows a low R^2_{CS} before the year 2000, despite periods of relatively high squared-SR ratios. This is primarily due to the low TS Fit of the factor during that time. After 2000, the TS Fit begins to rise, resulting in an increase in the CS Fit. It's worth noting that, post-2000, the pricing component stabilizes compared to previous years. Although the TS Fit starts to decline after 2020, this does not impact the CS Fit, as it is offset by an increase in the squared-SR ratio.

The Investment factor shows a relatively stable CS Fit before the year 2000, with both components remaining stable. After 2000, the TS Fit declines, leading to a decrease in the CS Fit. Although the squared-SR ratio remains stable after 2000, it begins to exhibit significant fluctuations after 2020; however, these fluctuations do not impact the CS Fit, given the extremely low TS Fit.

The Momentum factor exhibits an unstable CS Fit before the year 2000, largely due to a highly variable pricing component during this period. After 2000, the CS Fit becomes more stable and starts to increase. Notably, the TS Fit has been rising since 1993, reaching a peak in 2021, while the squared-SR ratio has been declining since 1993, reaching low levels in the last decade. Over time, the factor has become more systematic, with its pricing component decreasing to levels comparable to those of other factors. As a result, the CS Fit, initially driven by the pricing component, has increasingly come to reflect the influence of the TS Fit. It is worth noting that this factor was first formalized in academic literature by Jegadeesh and Titman (1993) in 1993, coinciding with the initial rise in TS Fit and the decline in the squared-SR ratio. This aligns with the findings of Cho (2020), which suggest that asset pricing anomalies may, at least in part, represent "alphas" transformed into "betas" by the actions of arbitrageurs. The OFPs exhibit stable R^2_{CS} values. Figure 7 shows that as the

number of factors in the OFP increases, the TS Fit decreases, while the squared-SR ratio rises. This pattern confirms our previous observations that, for the OFP, the CS Fit is primarily driven by the pricing component rather than the TS Fit.

Figure 8 illustrates the behavior of the risk premia and expected values of the factors over time. The figure confirms the findings from Section V.A.1, showing that for the Market and OFPs, the difference between risk premia and expected values is nearly zero. In contrast, the other factors display varying differences between risk premia and expected values over time, with the Momentum factor exhibiting the largest and most variable difference.

Non-traded factors Figures 9 and 10 show the behavior of R^2_{CS} and its two components for the consumption growth, CAY and their interaction, the growth rate in per capita labor income, and the lagged yield spread between Baa- and Aaa-rated corporate bonds. Figure 11, on the other hand, presents the results for the OFPs related to the conditional version of the consumption CAPM by Lettau and Ludvigson (2001) (CC-CAY) and the conditional CAPM (C-LAB) by Jagannathan and Wang (1996).

Confirming findings from the literature regarding the "weakness" of non-traded factors, the graphs show that both the individual factors and the OFPs, with the exception of C-LAB, exhibit particularly low TS Fit. The higher TS Fit of the OFP associated with the conditional CAPM by Jagannathan and Wang (1996) can be attributed to the inclusion in this model of the Market factor, which is both traded and systematic.

The ratio of squared-SR for the non-traded factors is, on average, higher than that of the traded factors. For the OFPs, the ratio for CC-CAY is slightly higher than that of the OFPs related to the traded factors, while the ratio for C-LAB is of a comparable magnitude.

Another difference between traded and non-traded factors is that non-traded factors tend

to exhibit greater fluctuations in the ratio of squared-SR, resulting in R^2_{CS} values that are less stable over time. Consumption Growth is the only individual non-traded factor that shows more stability, while the components of the OFPs are also more stable than those of the individual factors.

Figure 12 displays the behavior of the risk premia and expected values of the factors and OFPs over time. The difference between the expected values of the factors and the risk premia is less pronounced compared to that of the traded factors (note that the scale in this graph differs from the one used for the traded factors).

V.E.2 GLS

Traded factors In the GLS case, the primary driver of the CS Fit is the squared-SR ratio, as illustrated in the moving window analysis in Figures 13, 14, and 15. As noted previously, the TS Fit in the GLS case is bounded above, which is evident in the figures. We also observe that the original three Fama-French factors and the OFP constructed from these factors maintain an almost constant TS Fit over the entire period, whereas the Profitability, Investment, and Momentum factors display a more variable TS Fit.

Examining the pricing component, we observe that all factors except for the Market and Momentum show a decreasing trend in the squared-SR ratio—and consequently in R^2_{CS}—over time. The Momentum factor had a squared-SR ratio close to zero before 1993, which then increased, peaking around 2004 before declining over the past fifteen years. The Market factor is unique in that its squared-SR ratio, and therefore its R^2_{CS}, increased throughout the analysis period.

The OFPs with 3 and 5 factors display a decreasing trend in the squared-SR ratio (and thus in R^2_{CS}) over the analysis period. In contrast, the OFP that includes the Momentum

factor exhibits a pattern similar to that of the Momentum factor itself, with a peak around 2004 followed by a subsequent decline.

Figure 16 illustrates the behavior of risk premia and expected values of the factors in the GLS case. Here, all factors show risk premia that are nearly equal to their expected values, with the exception of the Momentum factor, which still exhibits a noticeable difference between the two.

Non-traded factors Figures 17, 18, and 19 show the factorization of R^2_{CS} over time for the non-traded factors in the GLS case. As noted in Proposition II.6, the TS Fit in the GLS case is a constant, equal to K/N. Consequently, the R^2_{CS} and the ratio of squared-SR convey the same information, albeit on a different scale, which is evident from the graphs.

Figure 20 displays the moving window analysis of the risk premia and expected values of the factors, which are identical in this case (with the exception of C-LAB, which includes the Market factor, although the difference is not noticeable in the graph).

VI Conclusions

In this paper, we study the link between the pervasiveness of an asset pricing factor and the ability of the corresponding betas to explain the cross-section of risk premia. We show that the link goes through the profitability of the factor. Commonly used models do not explain the cross-section of premia mainly because the corresponding portfolio returns are not sufficiently pervasive. The momentum factor, on the other hand, improves pricing performance because of its profitability, not its pervasiveness. Our results have implications for the interpretation of asset pricing tests and for the choice of empirical factors.

References

Ahmed, S., Bu, Z., and Tsvetanov, D. (2019). Best of the Best: A Comparison of Factor Models. *Journal of Financial and Quantitative Analysis*, 54(4):1713–1758.

Andriollo, A., Robotti, C., and Zhang, X. (2024). Misspecification and Weak Identification in the Nontraded Factor Zoo.

Balduzzi, P. and Robotti, C. (2008). Mimicking Portfolios, Economic Risk Premia, and Tests of Multi-Beta Models. *Journal of Business & Economic Statistics*, 26(3):354–368.

Barillas, F., Kan, R., Robotti, C., and Shanken, J. (2020). Model Comparison with Sharpe Ratios. *Journal of Financial and Quantitative Analysis*, 55(6):1840–1874.

Barillas, F. and Shanken, J. (2017). Which Alpha? *The Review of Financial Studies*, 30(4):1316–1338.

Barillas, F. and Shanken, J. (2018). Comparing Asset Pricing Models. *The Journal of Finance*, 73(2):715–754.

Black, F., Jensen, M. C., and Scholes, M. (1972). The capital asset pricing model: Some empirical tests. In Jensen, M. C., editor, *Studies in the Theory of Capital Markets*, pages 79–124. Praeger, New York.

Bryzgalova, S. (2015). Spurious Factors in Linear Asset Pricing Models.

Bryzgalova, S., Huang, J., and Julliard, C. (2023). Bayesian Solutions for the Factor Zoo: We Just Ran Two Quadrillion Models. *The Journal of Finance*, 78(1):487–557.

Burnside, C. (2011). The Cross Section of Foreign Currency Risk Premia and Consumption Growth Risk: Comment. *American Economic Review*, 101(7):3456–3476.

Carhart, M. M. (1997). On Persistence in Mutual Fund Performance. *The Journal of Finance*, 52(1):57–82.

Cho, T. (2020). Turning alphas into betas: Arbitrage and endogenous risk. *Journal of Financial Economics*, 137(2):550–570.

Cochrane, J. H. (2011). Presidential Address: Discount Rates. *The Journal of Finance*, 66(4):1047–1108.

Didisheim, A., Ke, S. B., Kelly, B., and Malamud, S. (2024). APT or "AIPT"? The Surprising Dominance of Large Factor Models.

Ehsani, S. and Linnainmaa, J. T. (2022). Factor Momentum and the Momentum Factor. *The Journal of Finance*, 77(3):1877–1919.

Fama, E. F. (2015). Cross-Section Versus Time-Series Tests of Asset Pricing Models. *SSRN Electronic Journal*.

Fama, E. F. and French, K. R. (1993). Common Risk Factors in the Returns on Stocks and Bonds. *Journal of Financial Economics*, 33(1):3–56.

Fama, E. F. and French, K. R. (2015). A five-factor asset pricing model. *Journal of Financial Economics*, 116(1):1–22.

Fama, E. F. and French, K. R. (2016). Dissecting Anomalies with a Five-Factor Model. *Review of Financial Studies*, 29(1):69–103.

Fama, E. F. and MacBeth, J. D. (1973). Risk, Return, and Equilibrium: Empirical Tests. *Journal of Political Economy*, 81(3):607–636.

Feng, G., Giglio, S., and Xiu, D. (2020). Taming the Factor Zoo: A Test of New Factors. *The Journal of Finance*, 75(3):1327–1370.

Gibbons, M. R., Ross, S. A., and Shanken, J. (1989). A Test of the Efficiency of a Given Portfolio. *Econometrica*, 57(5):1121.

Giglio, S., Xiu, D., and Zhang, D. (2024). Test Assets and Weak Factors. *The Journal of Finance*, page jofi.13415.

Gospodinov, N., Kan, R., and Robotti, C. (2014). Misspecification-Robust Inference in Linear Asset-Pricing Models with Irrelevant Risk Factors. *Review of Financial Studies*, 27(7):2139–2170.

Gospodinov, N., Kan, R., and Robotti, C. (2019). Too Good to Be True? Fallacies in Evaluating Risk Factor Models. *Journal of Financial Economics*, 132(2):451–471.

Green, R. C. and Hollifield, B. (1992). When Will Mean-Variance Efficient Portfolios Be Well Diversified? *The Journal of Finance*, 47(5):1785–1809.

Hansen, L. P. and Jagannathan, R. (1997). Assessing Specification Errors in Stochastic Discount Factor Models. *The Journal of Finance*, 52(2):557–590.

Harvey, C. R. and Liu, Y. (2021). Lucky factors. *Journal of Financial Economics*, 141(2):413–435.

Harvey, C. R., Liu, Y., and Zhu, H. (2016). ... and the Cross-Section of Expected Returns. *Review of Financial Studies*, 29(1):5–68.

Jagannathan, R. and Wang, Z. (1996). The Conditional CAPM and the Cross-Section of Expected Returns. *The Journal of Finance*, 51(1):3–53.

Jagannathan, R. and Wang, Z. (1998). An Asymptotic Theory for Estimating Beta-Pricing Models Using Cross-Sectional Regression. *The Journal of Finance*, 53(4):1285–1309.

Jegadeesh, N. and Titman, S. (1993). Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency. *The Journal of Finance*, 48(1):65–91.

Kan, R., Robotti, C., and Shanken, J. (2013). Pricing Model Performance and the Two-Pass Cross-Sectional Regression Methodology. *The Journal of Finance*, 68(6):2617–2649.

Kan, R. and Zhang, C. (1999a). GMM Tests of Stochastic Discount Factor Models with Useless Factors. *Journal of Financial Economics*, 54(1):103–127.

Kan, R. and Zhang, C. (1999b). Two-Pass Tests of Asset Pricing Models with Useless Factors. *The Journal of Finance*, 54(1):203–235.

Kandel, S. and Stambaugh, R. F. (1995). Portfolio Inefficiency and the Cross-section of Expected Returns. *The Journal of Finance*, 50(1):157–184.

Kleibergen, F. (2009). Tests of Risk Premia in Linear Factor Models. *Journal of Econometrics*, 149(2):149–173.

Kleibergen, F. and Zhan, Z. (2015). Unexplained Factors and Their Effects on Second Pass R -squared's. *Journal of Econometrics*, 189(1):101–116.

Kozak, S. and Nagel, S. (2024). When Do Cross-sectional Asset Pricing Factors Span the Stochastic Discount Factor? *SSRN Electronic Journal*.

Kozak, S., Nagel, S., and Santosh, S. (2018). Interpreting Factor Models. *The Journal of Finance*, 73(3):1183–1223.

Kozak, S., Nagel, S., and Santosh, S. (2020). Shrinking the Cross-section. *Journal of Financial Economics*, 135(2):271–292.

Lettau, M. and Ludvigson, S. (2001). Consumption, Aggregate Wealth, and Expected Stock Returns. *The Journal of Finance*, 56(3):815–849.

Lettau, M. and Pelger, M. (2020). Factors That Fit the Time Series and Cross-Section of Stock Returns. *The Review of Financial Studies*, 33(5):2274–2325.

Lewellen, J., Nagel, S., and Shanken, J. (2010). A Skeptical Appraisal of Asset Pricing Tests. *Journal of Financial Economics*, 96(2):175–194.

MacKinlay, A. (1995). Multifactor Models do not Explain Deviations from the CAPM. *Journal of Financial Economics*, 38(1):3–28.

Moskowitz, T. J. (2003). An Analysis of Covariance Risk and Pricing Anomalies. *The Review of Financial Studies*, 16(2):417–457.

Pukthuanthong, K., Roll, R., and Subrahmanyam, A. (2019). A Protocol for Factor Identification. *The Review of Financial Studies*, 32(4):1573–1607.

Stambaugh, R. F. and Yuan, Y. (2017). Mispricing Factors. *The Review of Financial Studies*, 30(4):1270–1315.

Tian, M. (2021). Firm Characteristics and Empirical Factor Models: A Model Mining Experiment. *The Review of Financial Studies*, 34(12):6087–6125.

Vissing-Jørgensen, A. and Attanasio, O. P. (2003). Stock-Market Participation, Intertemporal Substitution, and Risk-Aversion. *American Economic Review*, 93(2):383–391.

Table 1: **Cross-sectional R^2: traded factors**.
The test assets are the 25 Portfolios Formed on Size and Book-to-Market and 30 Industry Portfolios from Kenneth French's site. The factors used are the Fama and French (2015) five factors—the excess return of the Market (Mkt.Rf), Size (SMB: small-minus-big), Value (HML: high-minus-low), Profitability (RMW: robust-minus-weak), and Investment (CMA: conservative-minus-aggressive)—along with Momentum (UMD: up-minus-down). The Optimal Factor Portfolios analyzed are the tangency portfolio of Mkt.Rf, SMB and HML (FF3), the tangency portfolio of Mkt.Rf, SMB, HML, RMW and CMA (FF5), and the tangency portfolio of all 6 factors (FF6). The data cover the period from July 1963 to December 2022. The Root Mean Square Error of the alternative hypothesis and the percentiles of the null hypotheses are determined using a block-bootstrap approach with a block dimension of 3 and 10,000 resamples. The null hypotheses tested are as follows: the factor prices expected excess returns exactly ($H_0 : \hat{\alpha} = 0$); the factor is not priced ($H_0 : \hat{\lambda} = 0$); and the factor is priced but does not explain excess returns ($H_0 : \beta = 0$). The cells are colored when the null that the factor prices expected excess returns exactly is not rejected, and when the nulls that the factor is not priced and the factor is priced but does not explain excess returns are rejected. The level of significance is 5%.

	Estimate (RMSE)		$H_0 : \alpha = 0$		$H_0 : \beta = 0$		$H_0 : \lambda = 0$	
	OLS	GLS	OLS	GLS	OLS	GLS	OLS	GLS
Panel A: Factors not included in the test assets								
Mkt.RF	0.9151	0.056755						
	(0.1237)	(0.0285)						
SMB	0.7533	0.015646						
	(0.1320)	(0.0141)						
HML	0.0883	0.035926						
	(0.1114)	(0.0222)						
RMW	0.4617	0.005975						
	(0.1574)	(0.0132)						
CMA	0.6305	0.021989						
	(0.1421)	(0.0181)						
UMD	0.8218	0.038372						
	(0.1755)	(0.0261)						
FF3	0.9351	0.119521						
	(0.0962)	(0.0447)						
FF5	0.9475	0.16677						
	(0.0802)	(0.0509)						
FF6	0.9581	0.320557						
	(0.0726)	(0.1051)						
Panel B: Factors included in the test assets								
Mkt.RF	0.9158	0.048373						
	(0.1232)	(0.0255)						
SMB	0.7454	0.018163						
	(0.1323)	(0.0150)						
HML	0.0671	0.036835						
	(0.0998)	(0.0223)						
RMW	0.4175	0.04988						
	(0.1511)	(0.0262)						
CMA	0.5954	0.06755						
	(0.1422)	(0.0303)						
UMD	0.5753	0.075189						
	(0.1801)	(0.0343)						
FF3	0.9355	0.110259						
	(0.0946)	(0.0420)						
FF5	0.9467	0.266505						
	(0.0788)	(0.0685)						
FF6	0.9569	0.346712						
	(0.0680)	(0.0852)						

Table 2: **Time-series Fit ($\mathbf{1 - Ave(\sigma^2_{\epsilon n})/Ave(\sigma^2_n)}$): traded factors**.
The test assets are the 25 Portfolios Formed on Size and Book-to-Market and 30 Industry Portfolios from Kenneth French's site. The factors used are the Fama and French (2015) five factors—the excess return of the Market (Mkt.Rf), Size (SMB: small-minus-big), Value (HML: high-minus-low), Profitability (RMW: robust-minus-weak), and Investment (CMA: conservative-minus-aggressive)—along with Momentum (UMD: up-minus-down). The Optimal Factor Portfolios analyzed are the tangency portfolio of Mkt.Rf, SMB and HML (FF3), the tangency portfolio of Mkt.Rf, SMB, HML, RMW and CMA (FF5), and the tangency portfolio of all 6 factors (FF6). The data cover the period from July 1963 to December 2022. The Root Mean Square Error of the alternative hypothesis and the percentiles of the null hypotheses are determined using a block-bootstrap approach with a block dimension of 3 and 10,000 resamples. The null hypothesis tested is that the factor is priced but does not explain excess returns ($H_0 : \beta = 0$). The cells are colored when the null is rejected. The level of significance is 5%. In the GLS case, when the factors are included in the test assets, the Time-series fit is equal to 1 divided by the number of test assets (*).

	Estimate (RMSE)		$H_0 : \beta = 0$	
	OLS	GLS	OLS	GLS
Panel A: Factors not included in the test assets				
Mkt.RF	0.6317	0.0181		
	(0.0218)	(0.0000)		
SMB	0.1864	0.0179		
	(0.0271)	(0.0000)		
HML	0.0338	0.0178		
	(0.0103)	(0.0000)		
RMW	0.0406	0.0116		
	(0.0125)	(0.0011)		
CMA	0.0710	0.0120		
	(0.0163)	(0.0009)		
UMD	0.0375	0.0068		
	(0.0199)	(0.0016)		
FF3	0.4842	0.0180		
	(0.1480)	(0.0001)		
FF5	0.2023	0.0135		
	(0.1114)	(0.0020)		
FF6	0.0999	0.0079		
	(0.0803)	(0.0026)		
Panel B: Factors included in the test assets				
Mkt.RF	0.6354	0.0179		*
	(0.0217)	(0.0000)		
SMB	0.1900	0.0179		*
	(0.0269)	(0.0000)		
HML	0.0380	0.0179		*
	(0.0103)	(0.0000)		
RMW	0.0430	0.0179		*
	(0.0126)	(0.0000)		
CMA	0.0730	0.0179		*
	(0.0163)	(0.0000)		
UMD	0.0458	0.0179		*
	(0.0202)	(0.0000)		
FF3	0.4696	0.0172		*
	(0.1451)	(0.0000)		
FF5	0.1934	0.0167		*
	(0.0986)	(0.0000)		
FF6	0.1332	0.0164		*
	(0.0621)	(0.0000)		

Table 3: **Ratio of squared Sharpe Ratios $\frac{\hat{\lambda}^2/\sigma_f^2}{\mathbf{Ave}(\mu_n^2)/\mathbf{Ave}(\sigma_n^2)}$: traded factors**.

The test assets are the 25 Portfolios Formed on Size and Book-to-Market and 30 Industry Portfolios from Kenneth French's site. The factors used are the Fama and French (2015) five factors—the excess return of the Market (Mkt.Rf), Size (SMB: small-minus-big), Value (HML: high-minus-low), Profitability (RMW: robust-minus-weak), and Investment (CMA: conservative-minus-aggressive)—along with Momentum (UMD: up-minus-down). The Optimal Factor Portfolios analyzed are the tangency portfolio of Mkt.Rf, SMB and HML (FF3), the tangency portfolio of Mkt.Rf, SMB, HML, RMW and CMA (FF5), and the tangency portfolio of all 6 factors (FF6). The data cover the period from July 1963 to December 2022. The Root Mean Square Error of the alternative hypothesis and the percentiles of the null hypotheses are determined using a block-bootstrap approach with a block dimension of 3 and 10,000 resamples. The null hypotheses tested are as follows: the factor prices expected excess returns exactly ($H_0 : \hat{\alpha} = 0$); the factor is not priced ($H_0 : \hat{\lambda} = 0$); and the factor is priced but does not explain excess returns ($H_0 : \beta = 0$). The cells are colored when the null that the factor prices expected excess returns exactly is not rejected, and when the nulls that the factor is not priced and the factor is priced but does not explain excess returns are rejected. The level of significance is 5%.

	Estimate (RMSE)		$H_0 : \alpha = 0$		$H_0 : \beta = 0$		$H_0 : \lambda = 0$	
	OLS	GLS	OLS	GLS	OLS	GLS	OLS	GLS
Panel A: Factors not included in the test assets								
Mkt.RF	1.4486	3.1387						
	(0.2061)	(1.5779)						
SMB	4.0417	0.8734						
	(0.7696)	(0.7885)						
HML	2.6151	2.0152						
	(2.2552)	(1.2429)						
RMW	11.3650	0.5167						
	(4.4388)	(1.1313)						
CMA	8.8751	1.8303						
	(2.4509)	(1.4283)						
UMD	21.8900	5.6312						
	(10.2358)	(3.8052)						
FF3	1.9314	6.6558						
	(1.1825)	(2.4891)						
FF5	4.6830	12.3361						
	(5.6545)	(4.0935)						
FF6	9.5944	40.4911						
	(11.4394)	(17.9051)						
Panel B: Factors included in the test assets								
Mkt.RF	1.4414	2.7089						
	(0.2040)	(1.4276)						
SMB	3.9226	1.0171						
	(0.7436)	(0.8397)						
HML	1.7662	2.0628						
	(1.8491)	(1.2508)						
RMW	9.7176	2.7933						
	(3.8524)	(1.4668)						
CMA	8.1579	3.7828						
	(2.2746)	(1.6956)						
UMD	12.5572	4.2106						
	(3.7385)	(1.9194)						
FF3	1.9919	6.3950						
	(1.1868)	(2.4368)						
FF5	4.8950	15.9903						
	(4.4835)	(4.1100)						
FF6	7.1834	21.1494						
	(6.1518)	(5.1960)						

Table 4: **Decomposition of the cross-sectional R-square: OLS**.
The test assets, downloaded from Kenneth French's site, are the 25 Portfolios Formed on Size and Book-to-Market, and the 30 Industry Portfolios. The factors used are the Fama and French (2015) five factors—the excess return of the Market (Mkt.Rf), Size (SMB: small-minus-big), Value (HML: high-minus-low), Profitability (RMW: robust-minus-weak), and Investment (CMA: conservative-minus-aggressive)—along with Momentum (UMD: up-minus-down). The data cover the period from July 1963 to December 2022. The symbol **§** indicates that the factors are included in the test assets.

R^2_{Tot}	**Mkt.RF**	**SMB**	**HML**	**RMW**	**CMA**	**UMD**	**FF3**	**FF5**	**FF6**
Panel A: factors not included in the test assets									
R^2_{CS}	0.9151	0.7533	0.0883	0.4617	0.6305	0.8218	0.9351	0.9475	0.9581
R^2_{Tot}	0.9003	0.4009	−0.1259	−0.3467	−0.6454	−0.5261	0.9244	0.9127	0.9500
$\Delta_{R^2} = R^2_{CS} - R^2_{Tot}$	0.0148	0.3523	0.2142	0.8084	1.2760	1.3479	0.0106	0.0348	0.0080
$\frac{\beta^\top \beta}{\mu^\top \mu}$	2.3089	1.5122	0.2826	0.6080	1.2430	0.1584	1371.2642	236.3332	56.9333
$\mathrm{E}(f^*_t)$	0.5495	0.2231	0.3116	0.2817	0.3009	0.6393	0.0289	0.0755	0.1178
$\hat{\lambda}$	0.6296	0.7058	−0.5589	−0.8714	−0.7122	−2.2774	0.0261	0.0633	0.1297
$[\mathrm{E}(f^*_t) - \hat{\lambda}]^2$	0.0064	0.2330	0.7578	1.3296	1.0265	8.5069	0.0000	0.0001	0.0001
Panel B: factors included in the test assets									
$R^2_{CS,\S}$	0.9158	0.7454	0.0671	0.4175	0.5954	0.5753	0.9355	0.9467	0.9569
$R^2_{Tot,\S}$	0.9014	0.4020	−0.1219	−0.3428	−0.6400	−0.5036	0.9258	0.9142	0.9401
$\Delta_{R^2_\S} = R^2_{CS,\S} - R^2_{Tot,\S}$	0.0144	0.3434	0.1889	0.7603	1.2354	1.0789	0.0097	0.0325	0.0168
$\frac{\beta_\S^\top \beta_\S}{\mu_\S^\top \mu_\S}$	2.3199	1.5460	0.3181	0.6427	1.2754	0.1922	1296.0001	216.4683	100.3712
$\mathrm{E}(f^*_t)$	0.5495	0.2231	0.3116	0.2817	0.3009	0.6393	0.0296	0.0784	0.1106
$\hat{\lambda}_\S$	0.6283	0.6944	−0.4592	−0.8060	−0.6833	−1.7302	0.0269	0.0661	0.0976
$[\mathrm{E}(f^*_t) - \hat{\lambda}_\S]^2$	0.0062	0.2221	0.5940	1.1830	0.9686	5.6147	0.0000	0.0002	0.0002
Panel C: Differential impact									
$\ln(\Delta_{R^2_\S}/\Delta_{R^2})$	−0.0267	−0.0257	−0.1253	−0.0613	−0.0323	−0.2226	−0.0938	−0.0680	0.7367
$\ln\left(\frac{\beta_\S^\top \beta_\S}{\mu_\S^\top \mu_\S} / \frac{\beta^\top \beta}{\mu^\top \mu}\right)$	0.0047	0.0221	0.1182	0.0555	0.0257	0.1929	−0.0565	−0.0878	0.5670
$\ln([\mathrm{E}(f^*_t) - \hat{\lambda}_\S]^2 / [\mathrm{E}(f^*_t) - \hat{\lambda}]^2)$	−0.0314	−0.0478	−0.2435	−0.1169	−0.0580	−0.4155	−0.0373	0.0198	0.1697

Table 5: **Decomposition of the cross-sectional R-square: GLS**.
The test assets, downloaded from Kenneth French's site, are the 25 Portfolios Formed on Size and Book-to-Market, and the 30 Industry Portfolios. The factors used are the Fama and French (2015) five factors—the excess return of the Market (Mkt.Rf), Size (SMB: small-minus-big), Value (HML: high-minus-low), Profitability (RMW: robust-minus-weak), and Investment (CMA: conservative-minus-aggressive)—along with Momentum (UMD: up-minus-down). The data cover the period from July 1963 to December 2022.

R^2_{Tot}	**Mkt.RF**	**SMB**	**HML**	**RMW**	**CMA**	**UMD**	**FF3**	**FF5**	**FF6**
R^2_{CS}	0.0568	0.0156	0.0359	0.0060	0.0220	0.0384	0.1195	0.1668	0.3206
R^2_{Tot}	0.0565	0.0156	0.0359	−0.0056	0.0171	0.0378	0.1194	0.1617	0.2889
$\Delta_{R^2} = R^2_{CS} - R^2_{Tot}$	0.0002	0.0001	0.0000	0.0116	0.0049	0.0006	0.0001	0.0050	0.0316
$\frac{\beta^\top \beta}{\mu^\top \mu}$	0.1649	0.3627	0.3724	0.4318	0.5245	0.0718	91.8903	37.3247	6.6592
$\mathrm{E}(f^*_t)$	0.5495	0.2231	0.3116	0.2817	0.3009	0.6393	0.0350	0.0784	0.1505
$\hat{\lambda}$	0.5866	0.2077	0.3106	0.1176	0.2048	0.7312	0.0361	0.0668	0.2194
$[\mathrm{E}(f^*_t) - \hat{\lambda}]^2$	0.0014	0.0002	0.0000	0.0269	0.0093	0.0085	0.0000	0.0001	0.0048

Table 6: **Cross-sectional R^2: non-traded factors and principal components of returns**.
The test assets are the 25 Portfolios Formed on Size and Book-to-Market and 30 Industry Portfolios from Kenneth French's site. In **Panel A**, the factors used are the excess return of the Market (Mkt.Rf) downloaded from French's site, and the mimicking portfolios of the following non-traded factors: the growth rate in real per capita nondurable consumption (CG), Lettau and Ludvigson (2001)'s consumption-wealth ratio (CAY), the interaction between CG and CAY (CG×CAY), the growth rate in per capita labor income (LAB), and the lagged yield spread between Baa- and Aaa-rated corporate bonds from FRED (Baa–Aaa). The Optimal Factor Portfolios analyzed are the tangency portfolio of CG, CAY, and CG×CAY (CC–CAY), and the tangency portfolio of Mkt.Rf, LAB and Baa–Aaa (C–LAB). For non-traded factors, the data cover the period from July 1963 to September 2019. In **Panel B**, the factors used are the first 6 Principal components of the test assets returns. We also display the results for the Optimal Factor Portfolios formed with the first three, five and six PCs (PC1–3, PC1–5 and PC1–6). The data cover the period from July 1963 to December 2022. The Root Mean Square Error of the alternative hypothesis and the percentiles of the null hypotheses are determined using a block-bootstrap approach with a block dimension of 3 and 10,000 resamples. The null hypotheses tested are as follows: the factor prices expected excess returns exactly ($H_0 : \hat{\alpha} = 0$); the factor is not priced ($H_0 : \hat{\lambda} = 0$); and the factor is priced but does not explain excess returns ($H_0 : \beta = 0$). The cells are colored when the null that the factor prices expected excess returns exactly is not rejected, and when the nulls that the factor is not priced and the factor is priced but does not explain excess returns are rejected. The level of significance is 5%.

	Estimate (RMSE)		$H_0 : \alpha = 0$		$H_0 : \beta = 0$		$H_0 : \lambda = 0$	
	OLS	GLS	OLS	GLS	OLS	GLS	OLS	GLS
Panel A: Non-traded factors								
Mkt.RF	0.9052	0.0514						
	(0.1380)	(0.0269)						
CG	0.8818	0.0028						
	(0.1390)	(0.0099)						
CAY	0.1199	0.1557						
	(0.1662)	(0.0553)						
CG×CAY	0.6698	0.0145						
	(0.1562)	(0.0146)						
LAB	0.1501	0.1050						
	(0.1818)	(0.0406)						
Baa–Aaa	0.7191	0.0021						
	(0.1969)	(0.0088)						
CC–CAY	0.9128	0.1709						
	(0.1244)	(0.0550)						
C–LAB	0.9054	0.1891						
	(0.1055)	(0.0576)						
Panel B: Principal Components of test assets returns								
PC1	0.917100	0.060750						
	0.123963	0.030399						
PC2	0.000099	0.000088						
	0.044432	0.007754						
PC3	0.014087	0.014496						
	0.070379	0.014393						
PC4	0.000360	0.000605						
	0.110928	0.007218						
PC5	0.018776	0.039458						
	0.113621	0.021761						
PC6	0.006025	0.015219						
	0.106800	0.013657						
PC1–3	0.931287	0.075334						
	0.090478	0.031365						
PC1–5	0.950424	0.115397						
	0.070940	0.039986						
PC1–6	0.956448	0.130616						
	0.062670	0.041685						

Table 7: **Time-series Fit ($\mathbf{1 - Ave(\sigma^2_{\epsilon n})/Ave(\sigma^2_n)}$): non-traded factors and principal components of returns**. The test assets are the 25 Portfolios Formed on Size and Book-to-Market and 30 Industry Portfolios from Kenneth French's site. In **Panel A**, the factors used are the excess return of the Market (Mkt.Rf) downloaded from French's site, and the mimicking portfolios of the following non-traded factors: the growth rate in real per capita nondurable consumption (CG), Lettau and Ludvigson (2001)'s consumption-wealth ratio (CAY), the interaction between CG and CAY (CG×CAY), the growth rate in per capita labor income (LAB), and the lagged yield spread between Baa- and Aaa-rated corporate bonds from FRED (Baa–Aaa). The Optimal Factor Portfolios analyzed are the tangency portfolio of CG, CAY, and CG×CAY (CC–CAY), and the tangency portfolio of Mkt.Rf, LAB and Baa–Aaa (C–LAB). For non-traded factors, the data cover the period from July 1963 to September 2019. In **Panel B**, the factors used are the first 6 Principal components of the test assets returns. We also display the results for the Optimal Factor Portfolios formed with the first three, five and six PCs (PC1–3, PC1–5 and PC1–6). The data cover the period from July 1963 to December 2022. The Root Mean Square Error of the alternative hypothesis and the percentiles of the null hypotheses are determined using a block-bootstrap approach with a block dimension of 3 and 10,000 resamples. The null hypothesis tested is that the factor is priced but does not explain excess returns ($H_0 : \beta = 0$). The cells are colored when the null is rejected. The level of significance is 5%. In the GLS case, when the factors are included in the test assets, the Time-series fit is equal to 1 divided by the number of test assets (*).

	Estimate (RMSE)		$H_0 : \beta = 0$	
	OLS	GLS	OLS	GLS
Panel A: Non-traded factors				
Mkt.RF	0.6337	0.0181		
	(0.0228)	(0.0000)		
CG	0.1518	0.0182		
	(0.0273)	(0.0000)		
CAY	0.0056	0.0182		
	(0.0044)	(0.0000)		
CG×CAY	0.0297	0.0182		
	(0.0108)	(0.0000)		
LAB	0.0126	0.0182		
	(0.0050)	(0.0000)		
Baa–Aaa	0.0247	0.0182		
	(0.0137)	(0.0000)		
CC–CAY	0.4500	0.0181		
	(0.1761)	(0.0000)		
C–LAB	0.0988	0.0182		
	(0.1726)	(0.0000)		
Panel B: Principal Components of test assets returns				
PC1	0.6849	0.0182		
	(0.0197)	(0.0000)		
PC2	0.0512	0.0182		
	(0.0055)	(0.0000)		
PC3	0.0441	0.0182		
	(0.0091)	(0.0000)		
PC4	0.0270	0.0182		
	(0.0059)	(0.0000)		
PC5	0.0216	0.0182		
	(0.0042)	(0.0000)		
PC6	0.0180	0.0182		
	(0.0038)	(0.0000)		
PC1–3	0.5609	0.0182		
	(0.1514)	(0.0000)		
PC1–5	0.3737	0.0182		
	(0.1273)	(0.0000)		
PC1–6	0.3322	0.0182		
	(0.1187)	(0.0000)		

Table 8: **Ratio of squared Sharpe Ratios $\frac{\hat{\lambda}^2/\sigma_f^2}{\mathbf{Ave}(\mu_n^2)/\mathbf{Ave}(\sigma_n^2)}$: non-traded factors and principal components of returns**.

The test assets are the 25 Portfolios Formed on Size and Book-to-Market and 30 Industry Portfolios from Kenneth French's site. In **Panel A**, the factors used are the excess return of the Market (Mkt.Rf) downloaded from French's site, and the mimicking portfolios of the following non-traded factors: the growth rate in real per capita nondurable consumption (CG), Lettau and Ludvigson (2001)'s consumption-wealth ratio (CAY), the interaction between CG and CAY (CG×CAY), the growth rate in per capita labor income (LAB), and the lagged yield spread between Baa- and Aaa-rated corporate bonds from FRED (Baa–Aaa). The Optimal Factor Portfolios analyzed are the tangency portfolio of CG, CAY, and CG×CAY (CC–CAY), and the tangency portfolio of Mkt.Rf, LAB and Baa–Aaa (C–LAB). For non-traded factors, the data cover the period from July 1963 to September 2019. In **Panel B**, the factors used are the first 6 Principal components of the test assets returns. We also display the results for the Optimal Factor Portfolios formed with the first three, five and six PCs (PC1–3, PC1–5 and PC1–6). The data cover the period from July 1963 to December 2022. The Root Mean Square Error of the alternative hypothesis and the percentiles of the null hypotheses are determined using a block-bootstrap approach with a block dimension of 3 and 10,000 resamples. The null hypotheses tested are as follows: the factor prices expected excess returns exactly ($H_0 : \hat{\alpha} = 0$); the factor is not priced ($H_0 : \hat{\lambda} = 0$); and the factor is priced but does not explain excess returns ($H_0 : \beta = 0$). The cells are colored when the null that the factor prices expected excess returns exactly is not rejected, and when the nulls that the factor is not priced and the factor is priced but does not explain excess returns are rejected. The level of significance is 5%.

	Estimate (RMSE)		$H_0 : \alpha = 0$		$H_0 : \beta = 0$		$H_0 : \lambda = 0$	
	OLS	GLS	OLS	GLS	OLS	GLS	OLS	GLS
Panel A: Non-traded factors								
Mkt.RF	1.428401	2.839747						
	(0.2284)	(1.4854)						
CG	5.808942	0.156591						
	(1.3452)	(0.5442)						
CAY	21.344114	8.565156						
	(15.6607)	(3.0422)						
CG×CAY	22.516086	0.798932						
	(6.4649)	(0.8044)						
LAB	11.948606	5.774808						
	(9.0568)	(2.2303)						
Baa–Aaa	29.068621	0.11427						
	(9.2711)	(0.4838)						
CC–CAY	2.028615	9.416475						
	(2.3267)	(3.0305)						
C–LAB	9.168324	10.403195						
	(6.9210)	(3.1664)						
Panel B: Principal Components of test assets returns								
PC1	1.339033	3.341233						
	(0.1884)	(1.6720)						
PC2	0.001941	0.004843						
	(0.8138)	(0.4265)						
PC3	0.319515	0.797271						
	(1.7356)	(0.7916)						
PC4	0.013332	0.033268						
	(3.0696)	(0.3970)						
PC5	0.869731	2.170203						
	(4.1545)	(1.1969)						
PC6	0.33545	0.837035						
	(4.6057)	(0.7511)						
PC1–3	1.660489	4.143347						
	(0.8992)	(1.7251)						
PC1–5	2.543552	6.346818						
	(1.7623)	(2.1993)						
PC1–6	2.879002	7.183853						
	(2.2170)	(2.2927)						

Table 9: **Factorization of the cross-sectional R-square: traded factors – differential results across test assets**. The test assets, downloaded from Kenneth French's site, include the 25 Portfolios Formed on Size and Book-to-Market and 30 Industry Portfolios, which serve as the baseline model. Additional portfolios are incrementally added, including 25 Portfolios Formed Monthly on Size and Momentum, 25 Portfolios Formed on Size and Operating Profitability, and 25 Portfolios Formed on Size and Investment. The factors used are the five factors from Fama and French (2015)—the excess return of the Market (Mkt.Rf), Size (SMB: small-minus-big), Value (HML: high-minus-low), Profitability (RMW: robust-minus-weak), and Investment (CMA: conservative-minus-aggressive)—along with Momentum (UMD: up-minus-down). The rows Size-Mom, Size-OP, and Size-Inv indicate that the original 25 size-B/M and 30 industry portfolios are augmented with the respective additional 25 portfolios based on Momentum, Operating Profitability, and Investment. The log differentials are calculated relative to FF6 for Size-Mom (i.e., results for 25 Size-B/M + 30 Industry + 25 Size-Mom compared to those for only 25 Size-B/M + 30 Industry) and relative to FF5 for Size-OP and Size-Inv. The Optimal Factor Portfolios analyzed are the tangency portfolio of all six factors (FF6) for Size-Mom and the tangency portfolio of five factors (FF5) for Size-OP and Size-Inv. The data cover the period from July 1963 to December 2022. Estimates of the measures used to compute the log differentials presented in the table can be found in the Appendix.

	$\Delta \ln R^2_{CS}$	$\Delta \ln \left(\mathbf{1} - \frac{\mathbf{Ave}(\sigma^{\mathbf{2}}_{\epsilon \mathbf{n}})}{\mathbf{Ave}(\sigma^{\mathbf{2}}_{\mathbf{n}})}\right)$	$\Delta \ln \left(\frac{\hat{\lambda}^{\mathbf{2}}/\sigma^{\mathbf{2}}_{\mathbf{f}}}{\mathbf{Ave}(\mu^{\mathbf{2}}_{\mathbf{n}})/\mathbf{Ave}(\sigma^{\mathbf{2}}_{\mathbf{n}})}\right)$
OLS			
Size-Mom	0.0027	0.3778	−0.3751
Size-OP	0.0138	0.0867	−0.0729
Size-Inv	0.0091	−0.0476	0.0567
GLS			
Size-Mom	−0.5514	0.3580	−0.9094
Size-OP	−0.1792	−0.1783	−0.0009
Size-Inv	−0.0793	−0.1746	0.0953

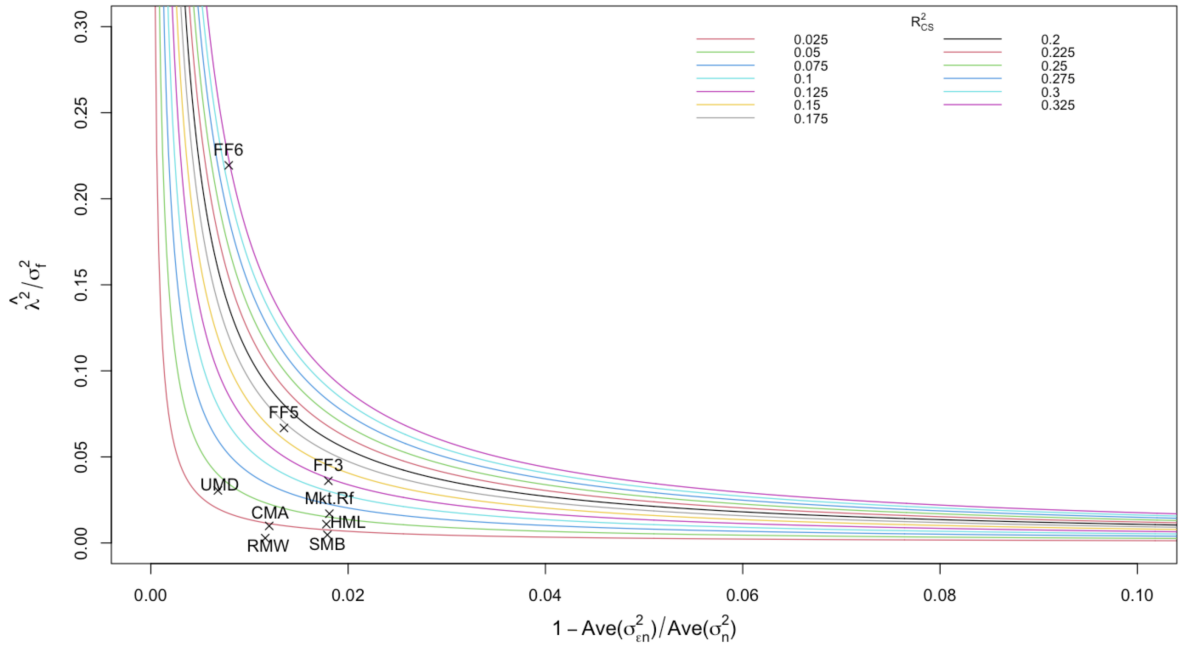


Figure 1: Iso-R2 curves – GLS. Given $\text{Ave}(\mu_n^2)/\text{Ave}(\sigma_n^2)$, in Equation (1), we plot $\hat{\lambda}^2/\sigma_f^2$ as a function of the Time-series Fit $\left[1 - \frac{\text{Ave}(\sigma^2_{\epsilon n})}{\text{Ave}(\sigma^2_n)}\right]$ for different values of the Cross-sectional Fit $\bar{R}^2_{CS}$. In black, we plot the position of the single factors, which are Mkt.Rf (Market), SMB (Size), HML (Value), RMW (Profitability), CMA (Investment), and UMD (Momentum), as well as the Optimal Factor Portfolios of some combinations of the factors (*e.g.*, FF3 is the tangency portfolio of the Mkt-Rf, SMB, and HML factors). The test assets are the 25 Fama-French (FF) size-B/M portfolios and the 30 FF industry portfolios.

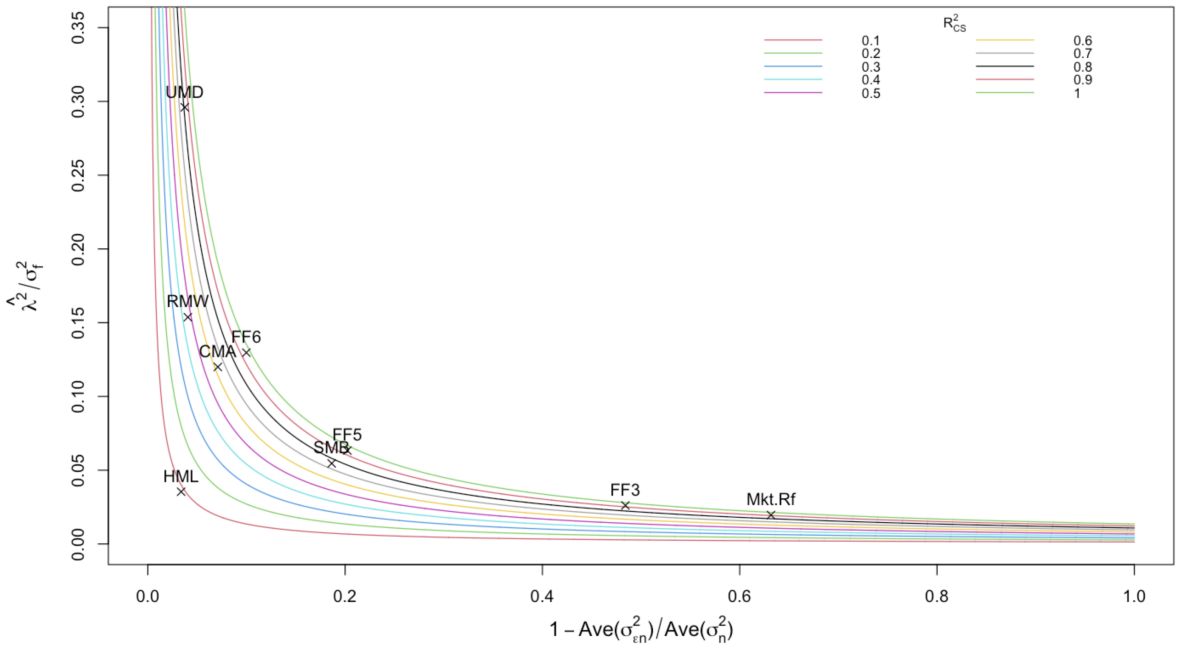


Figure 2: Iso-R2 curves – OLS. Given $\text{Ave}(\mu_n^2)/\text{Ave}(\sigma_n^2)$, in Equation (1), we plot $\hat{\lambda}^2/\sigma_f^2$ as a function of the Time-series Fit $\left[1 - \frac{\text{Ave}(\sigma^2_{\epsilon n})}{\text{Ave}(\sigma^2_n)}\right]$ for different values of the Cross-sectional Fit $\bar{R}^2_{CS}$. In black, we plot the position of the single factors, which are Mkt.Rf (Market), SMB (Size), HML (Value), RMW (Profitability), CMA (Investment), and UMD (Momentum), as well as the Optimal Factor Portfolios of some combinations of the factors (*e.g.*, FF3 is the tangency portfolio of the Mkt-Rf, SMB, and HML factors). The test assets are the 25 Fama-French (FF) size-B/M portfolios and the 30 FF industry portfolios.

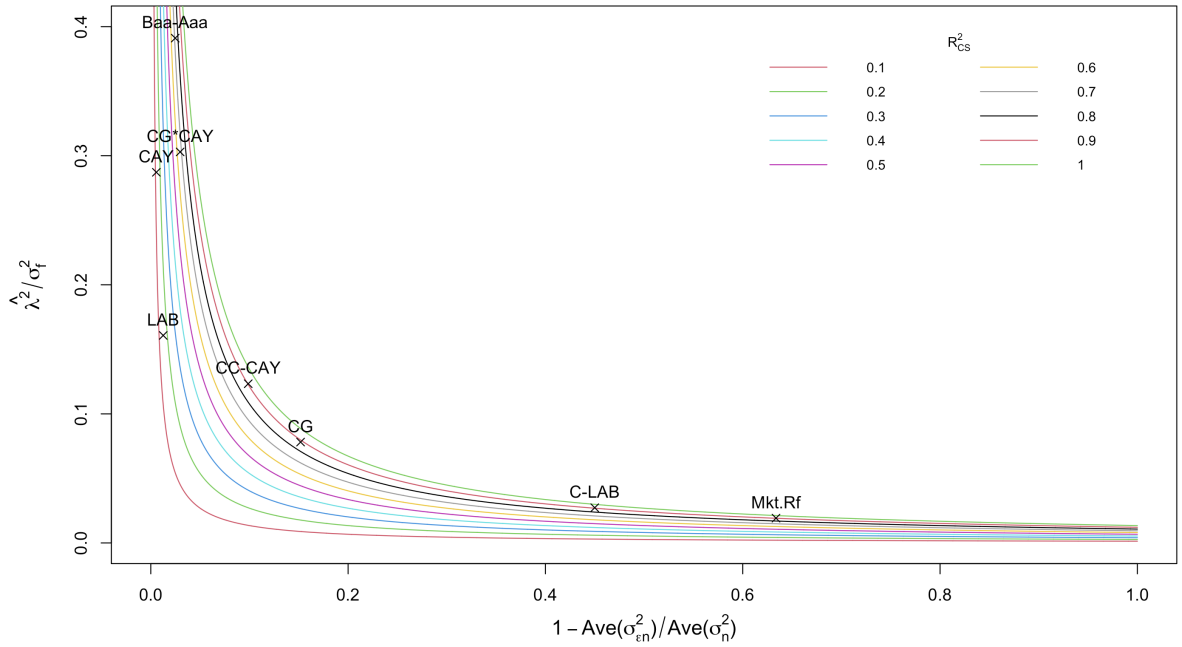


Figure 3: Iso-R2 curves – OLS. Non-traded factors. Given $\text{Ave}(\mu_n^2)/\text{Ave}(\sigma_n^2)$, in Equation (1), we plot $\hat{\lambda}^2/\sigma_f^2$ as a function of $\left[1 - \frac{\text{Ave}(\sigma^2_{\varepsilon n})}{\text{Ave}(\sigma^2_n)}\right]$ for different values of $\bar{R}^2_{CS}$. In black, we plot the position of the mimicking portfolios of 5 non-traded factors from Jagannathan and Wang (1996) and Lettau and Ludvigson (2001), and the market risk premium (Mkt.Rf). The non-traded factors are CG (consumption growth), CAY (consumption-wealth ratio), CG×CAY (interaction term of CG and CAY), LAB (growth rate in per capita labor income), Baa–Aaa (lagged yield spread between Baa- and Aaa-rated corporate bonds). We also plot two Optimal Factor Portfolios: CC–CAY is the tangency portfolio of the CG, CAY, and CG×CAY factors, while C–LAB is the tangency portfolio of the Mkt.Rf, LAB and Baa–Aaa factors. The test assets are the 25 Fama-French (FF) size-B/M portfolios and the 30 FF industry portfolios.

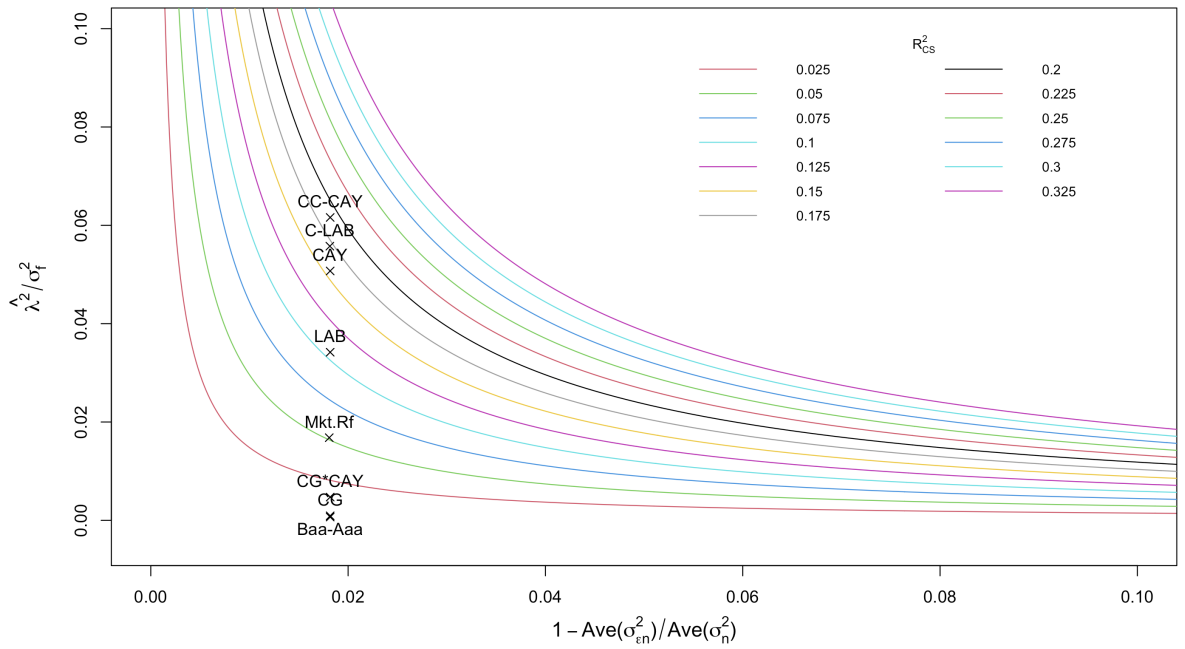


Figure 4: Iso-R2 curves – GLS. Non-traded factors. Given $\text{Ave}(\mu_n^2)/\text{Ave}(\sigma_n^2)$, in Equation (1), we plot $\hat{\lambda}^2/\sigma_f^2$ as a function of $\left[1 - \frac{\text{Ave}(\sigma^2_{\varepsilon n})}{\text{Ave}(\sigma^2_n)}\right]$ for different values of $\bar{R}^2_{CS}$. In black, we plot the position of the mimicking portfolios of 5 non-traded factors from Jagannathan and Wang (1996) and Lettau and Ludvigson (2001), and the market risk premium (Mkt.Rf). The non-traded factors are CG (consumption growth), CAY (consumption-wealth ratio), CG×CAY (interaction term of CG and CAY), LAB (growth rate in per capita labor income), Baa–Aaa (lagged yield spread between Baa- and Aaa-rated corporate bonds). We also plot two Optimal Factor Portfolios: CC–CAY is the tangency portfolio of the CG, CAY, and CG×CAY factors, while C–LAB is the tangency portfolio of the Mkt.Rf, LAB and Baa–Aaa factors. The test assets are the 25 Fama-French (FF) size-B/M portfolios and the 30 FF industry portfolios.

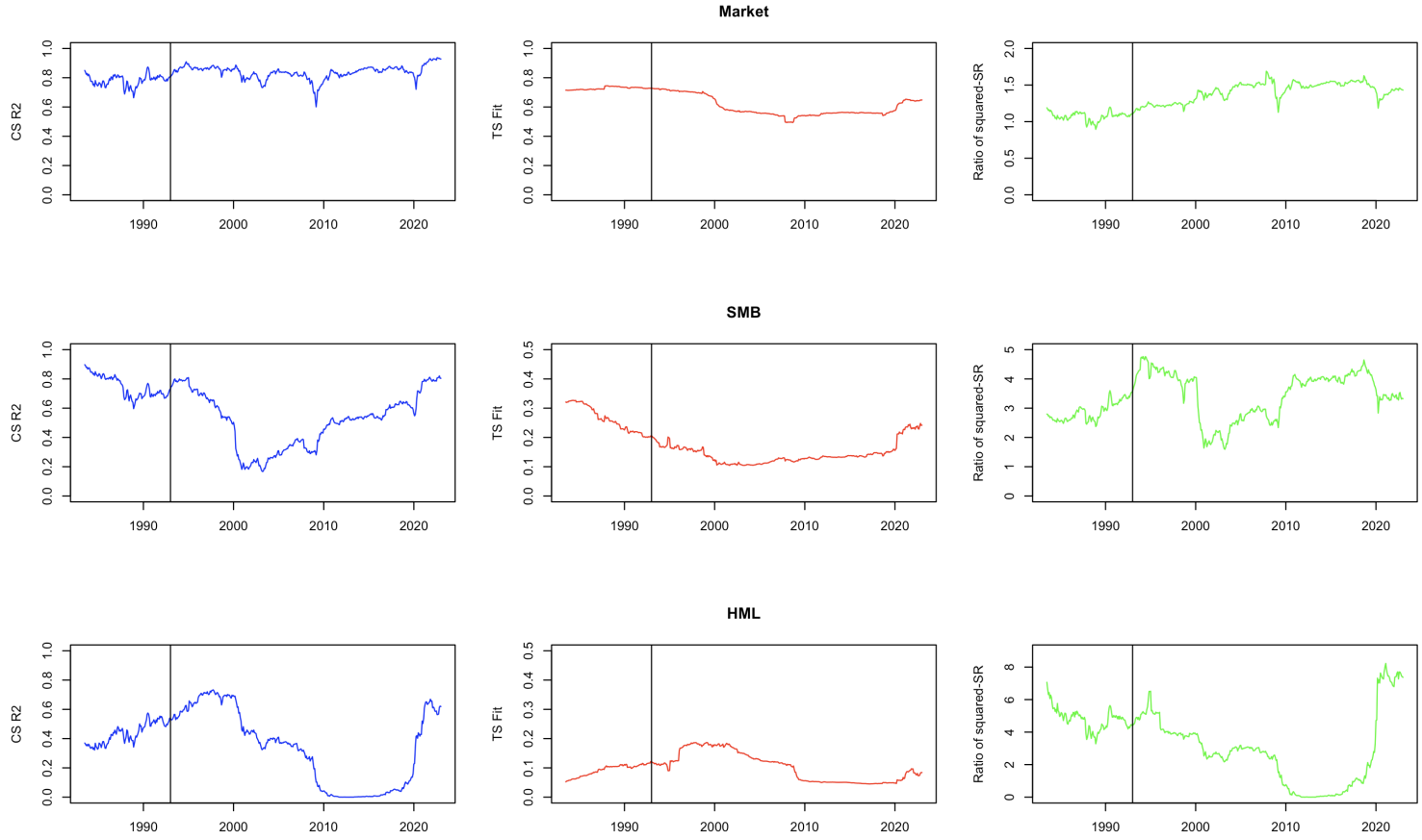


Figure 5: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the Market (Mkt.Rf), Small (SMB), and Value (HML) factors in the OLS case. Calculations are based on a 240-month moving window.

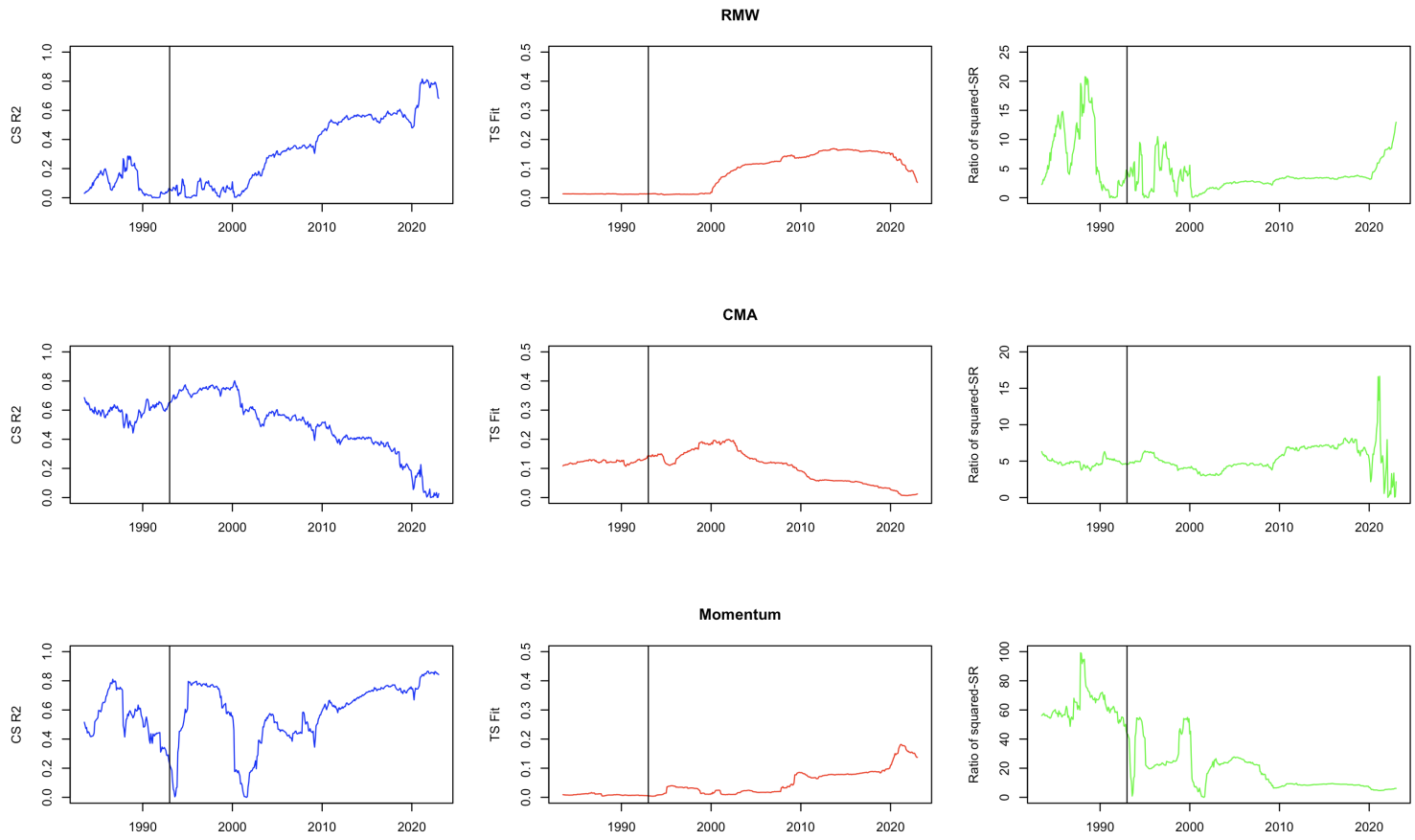


Figure 6: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the Profitability (RMW), Investment (CMA), and Momentum (UMD) factors in the OLS case. Calculations are based on a 240-month moving window.

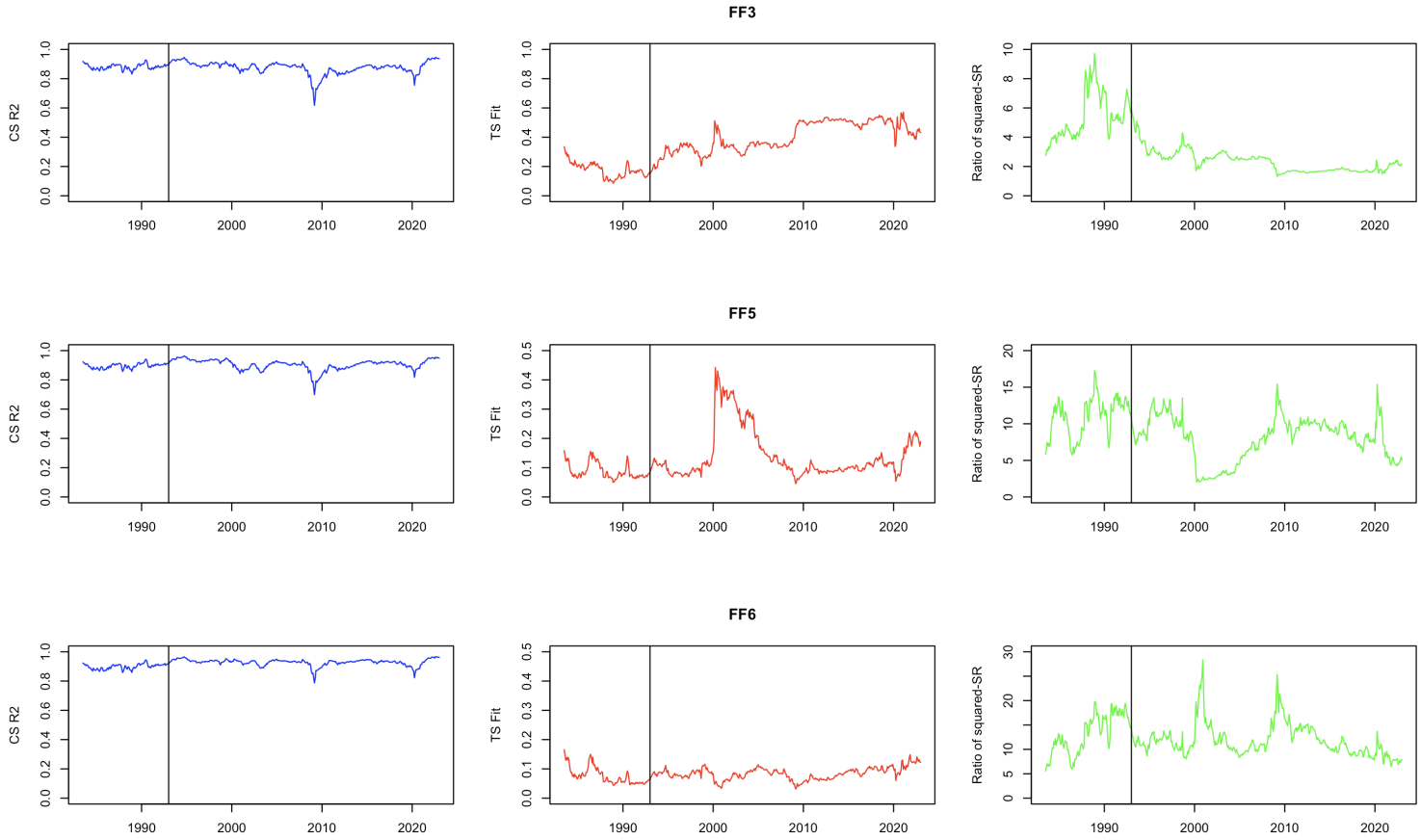


Figure 7: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the Optimal Factor Portfolios related to the Fama-French 3-, 5- and 6-factors models in the OLS case. Calculations are based on a 240-month moving window.

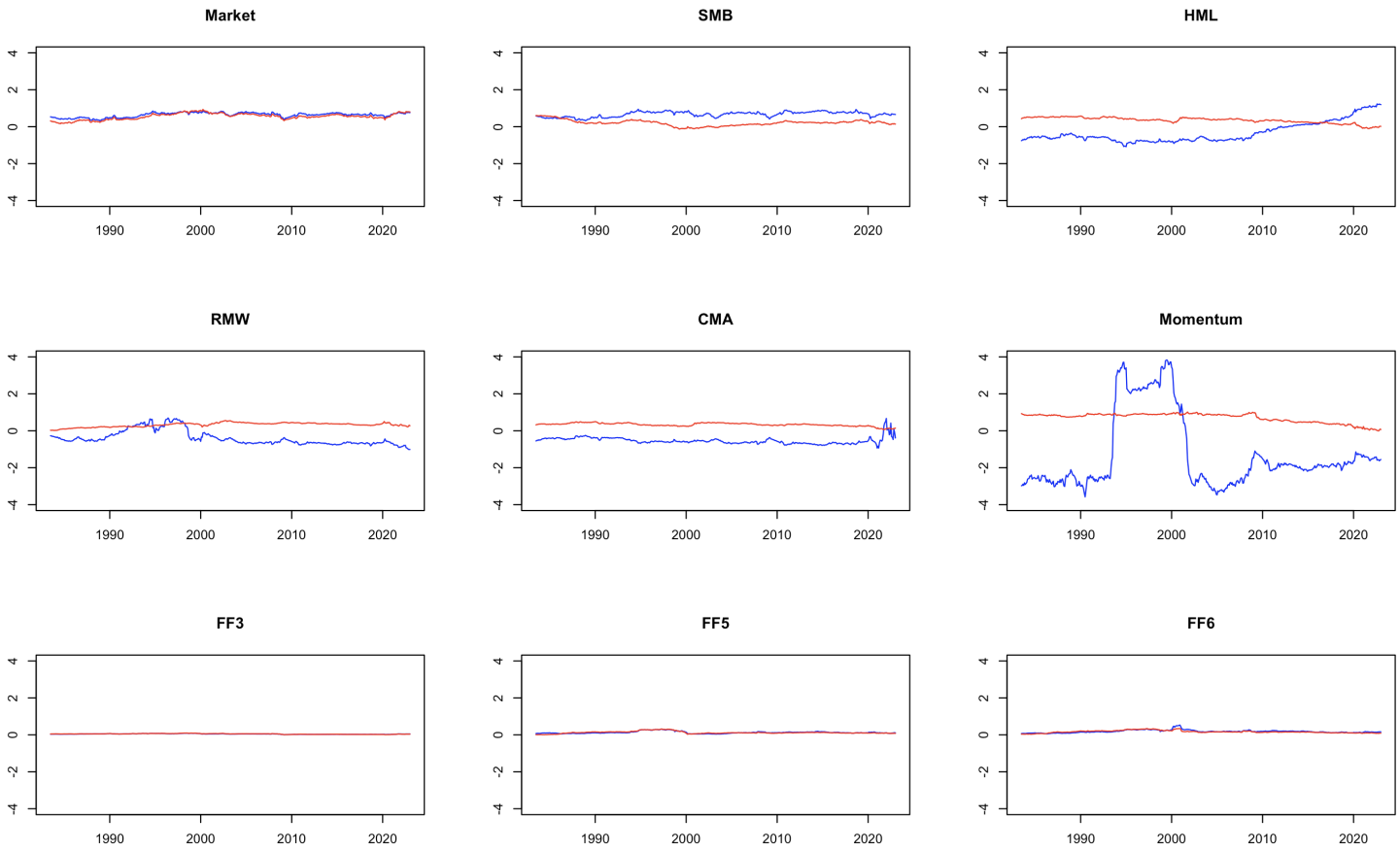


Figure 8: Behavior of the risk premium (blue) and the factor's expected value (red) over time for the Fama and French (2015) 5 factors, the Momentum factor, and the OFPs related to the FF 3-, 5-, and 6-factor models in the OLS case, using a 240-month moving window.

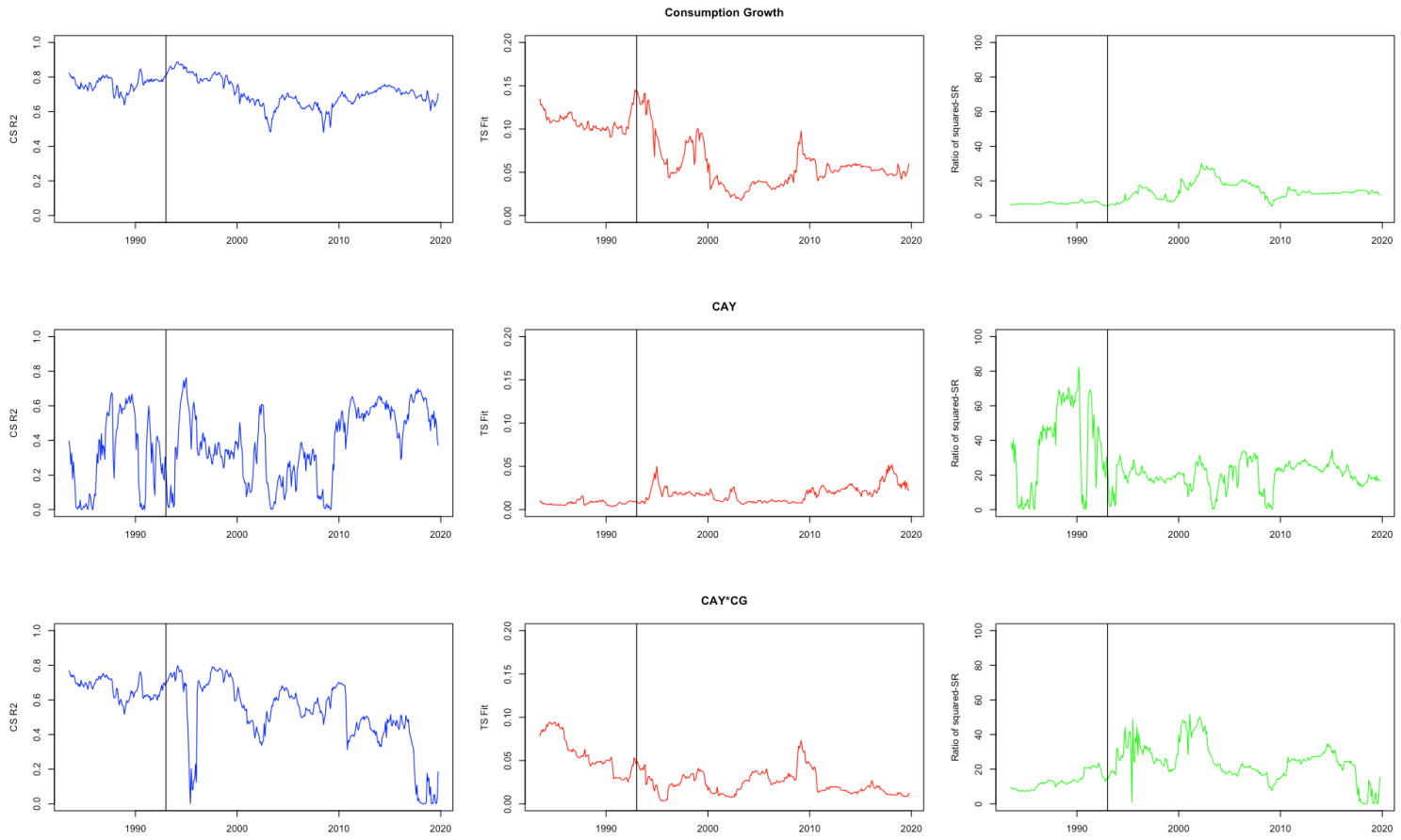


Figure 9: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the mimicking portfolios of Consumption Growth (CG), consumption-wealth ratio (CAY), and their interaction (CG×CAY) in the OLS case. Calculations are based on a 240-month moving window.

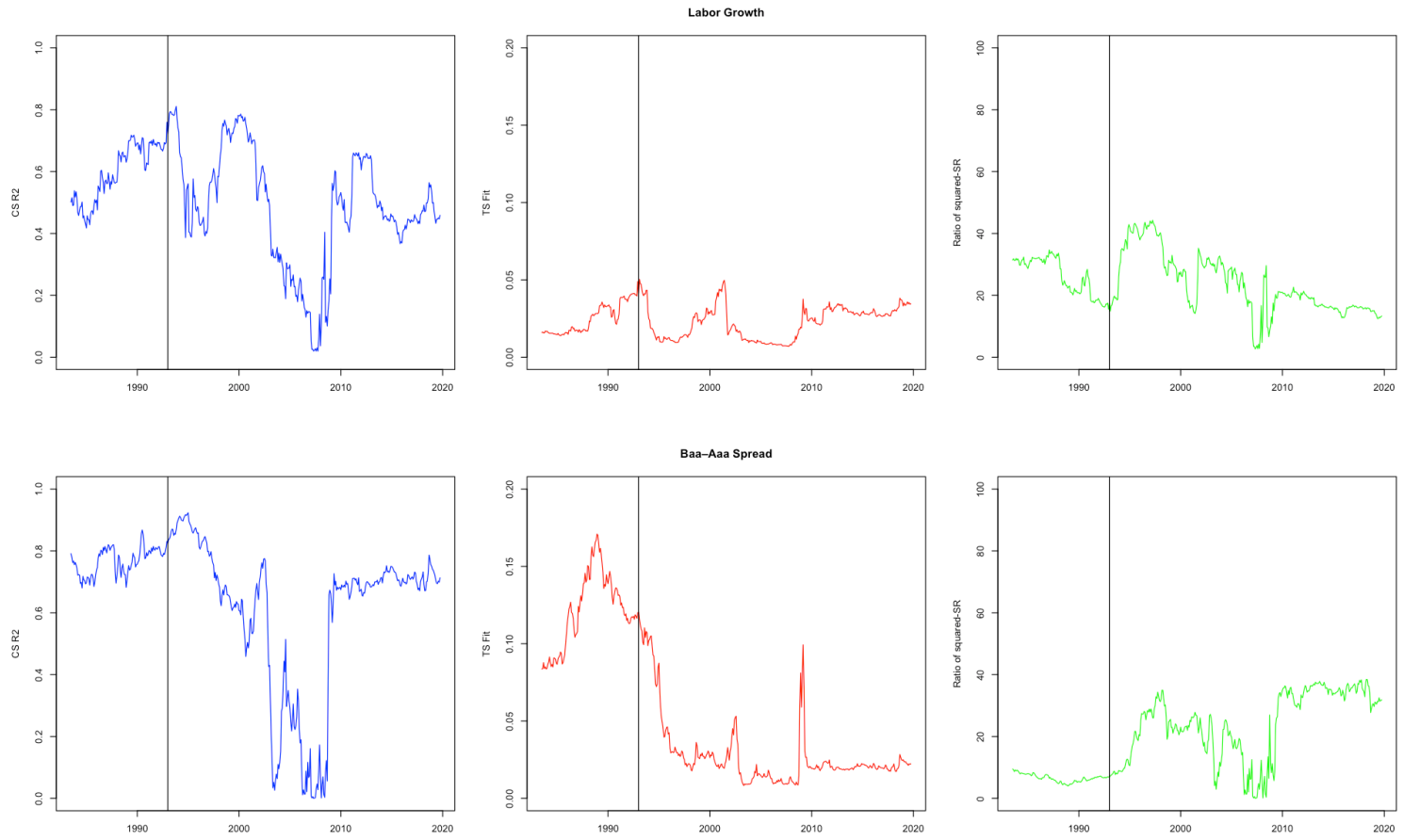


Figure 10: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the mimicking portfolios of the growth rate in per capita labor income (LAB) and the Baa-Aaa Spread in the OLS case. Calculations are based on a 240-month moving window.

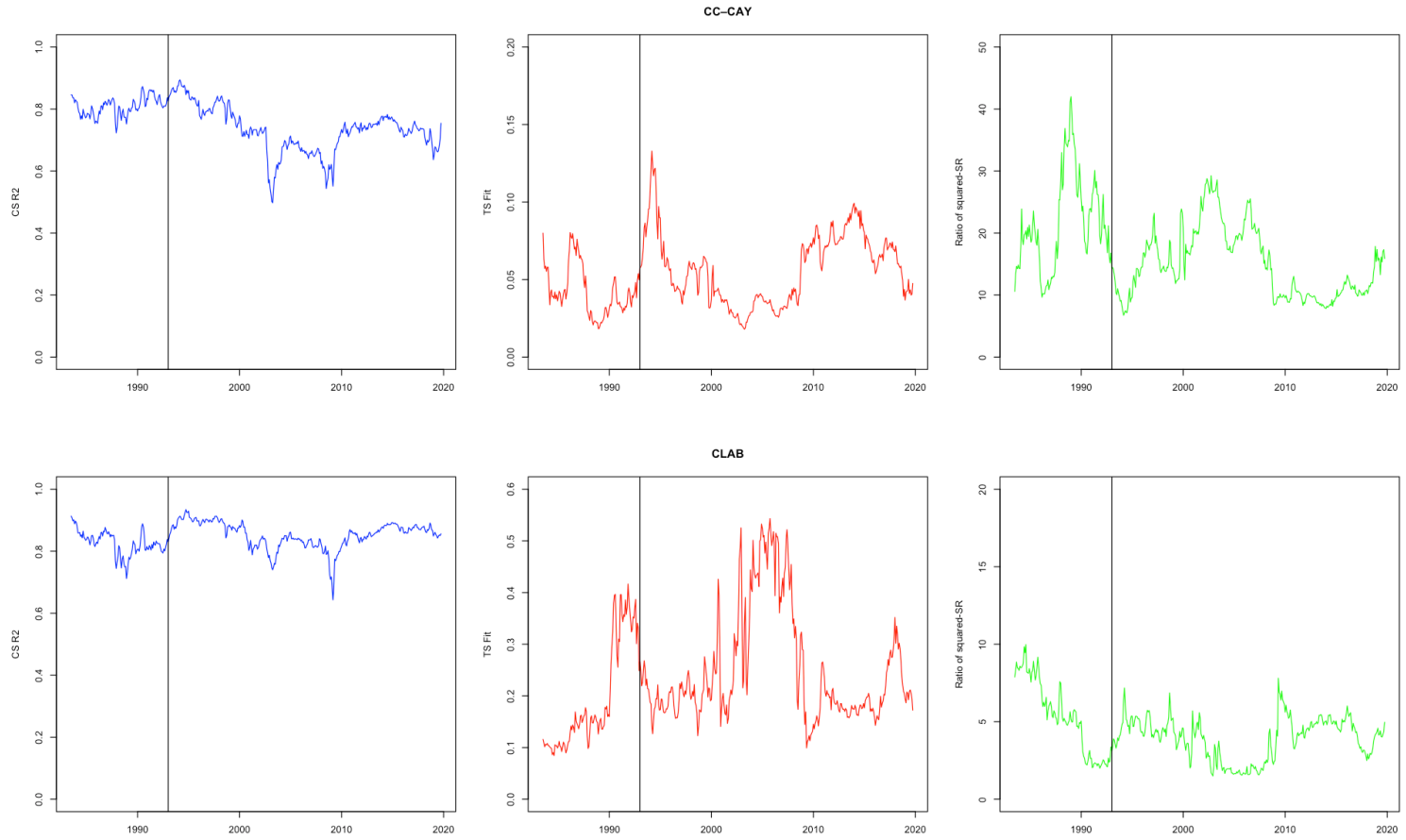


Figure 11: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the Optimal Factor Portfolios related to CC–CAY and C–LAB in the OLS case. Calculations are based on a 240-month moving window.

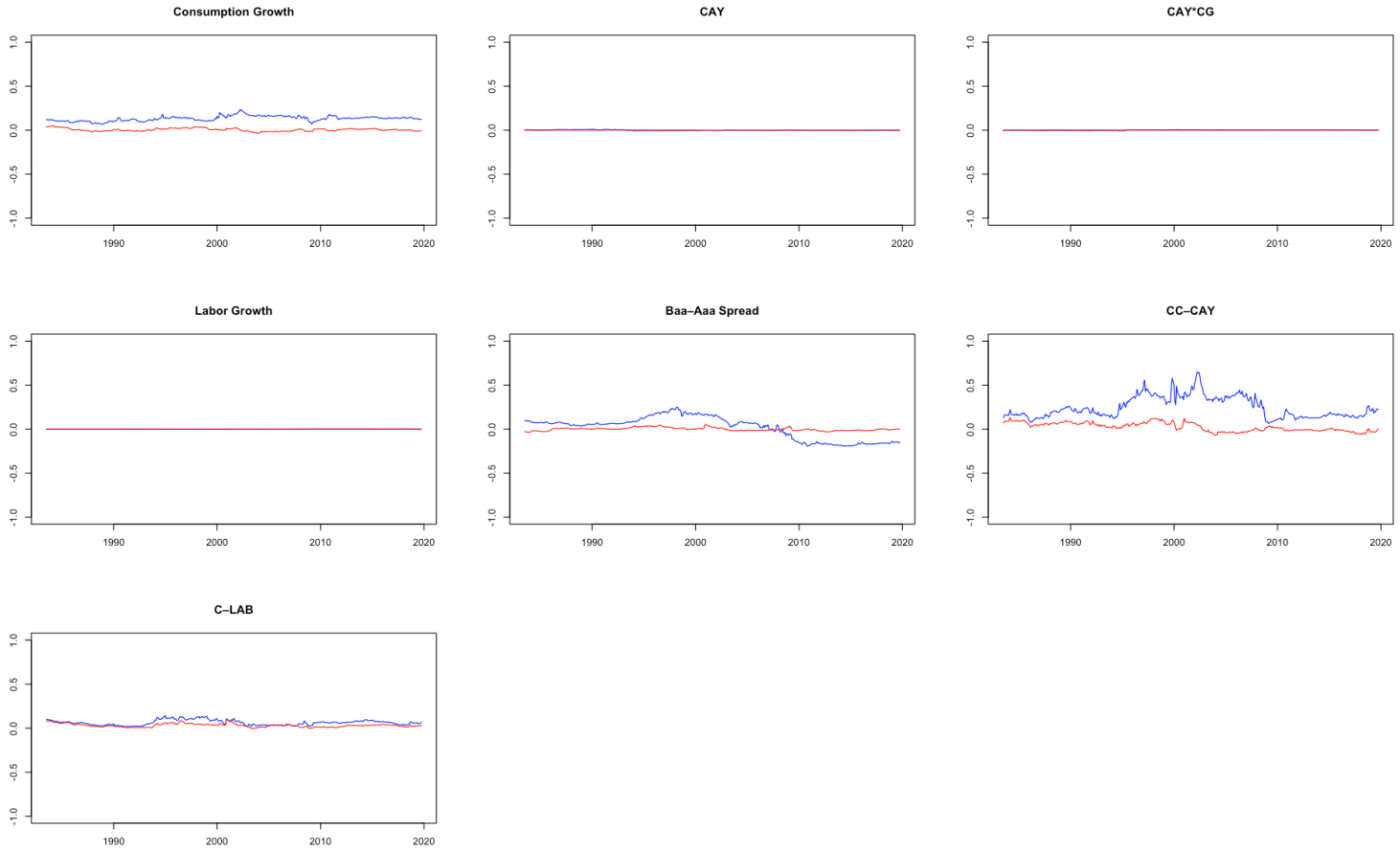


Figure 12: Behavior of the risk premium (blue) and the factor's expected value (red) over time for the mimicking portfolios of the non-traded factors analyzed, and the OFPs related to the CC–CAY and CLAB models in the OLS case, using a 240-month moving window.

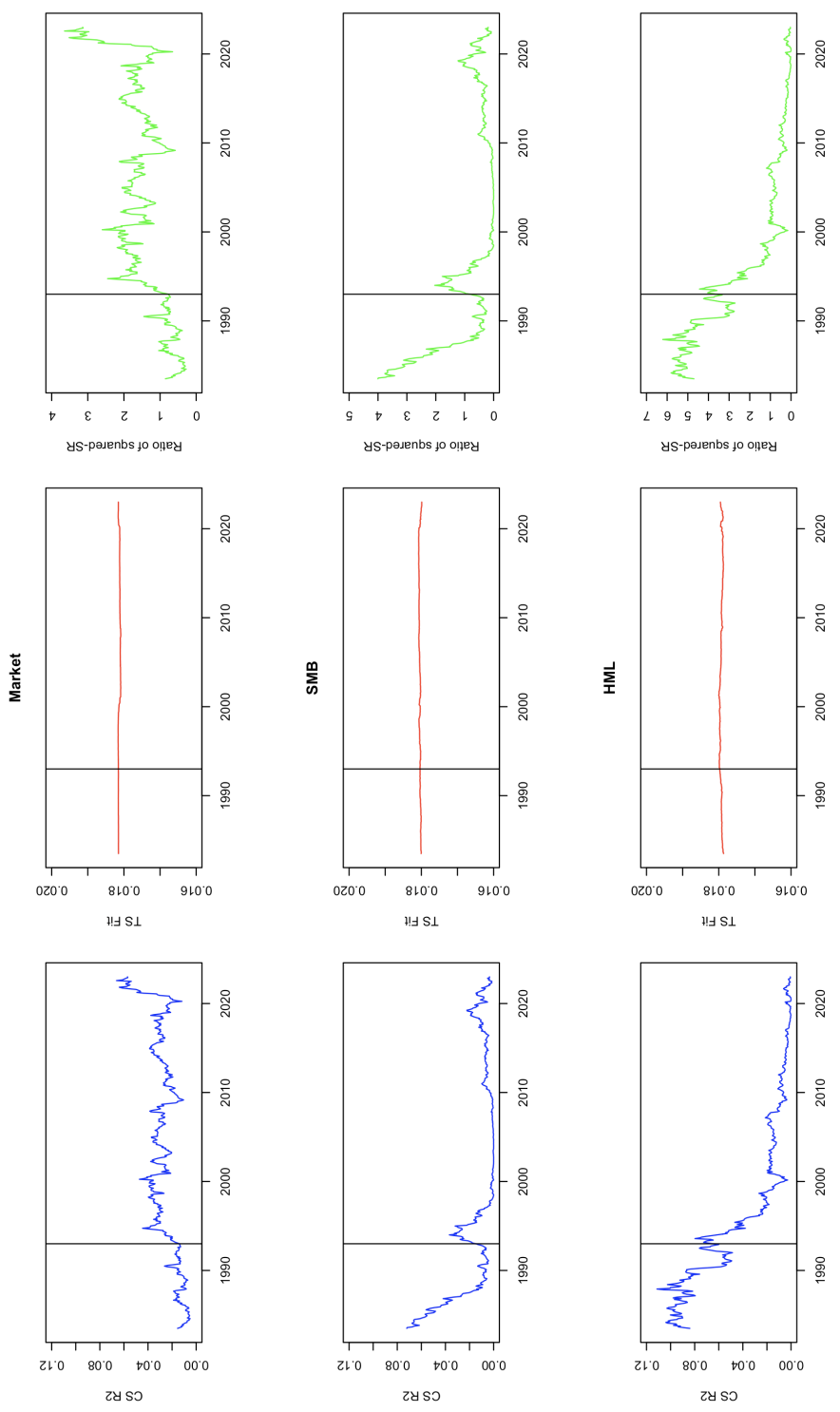


Figure 13: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the Market (Mkt.Rf), Small (SMB), and Value (HML) factors in the GLS case. Calculations are based on a 240-month moving window.

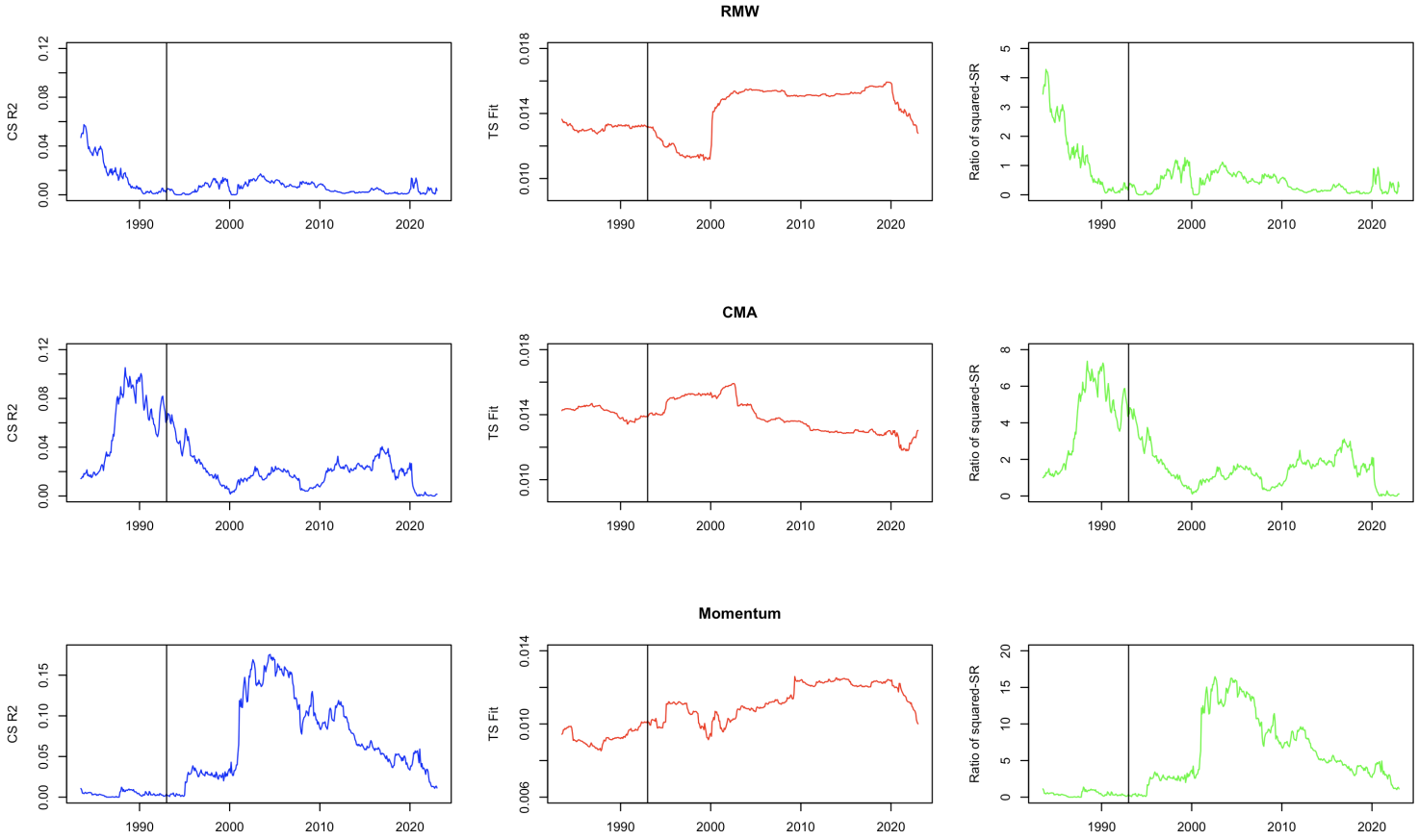


Figure 14: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the Profitability (RMW), Investment (CMA), and Momentum (UMD) factors in the GLS case. Calculations are based on a 240-month moving window.

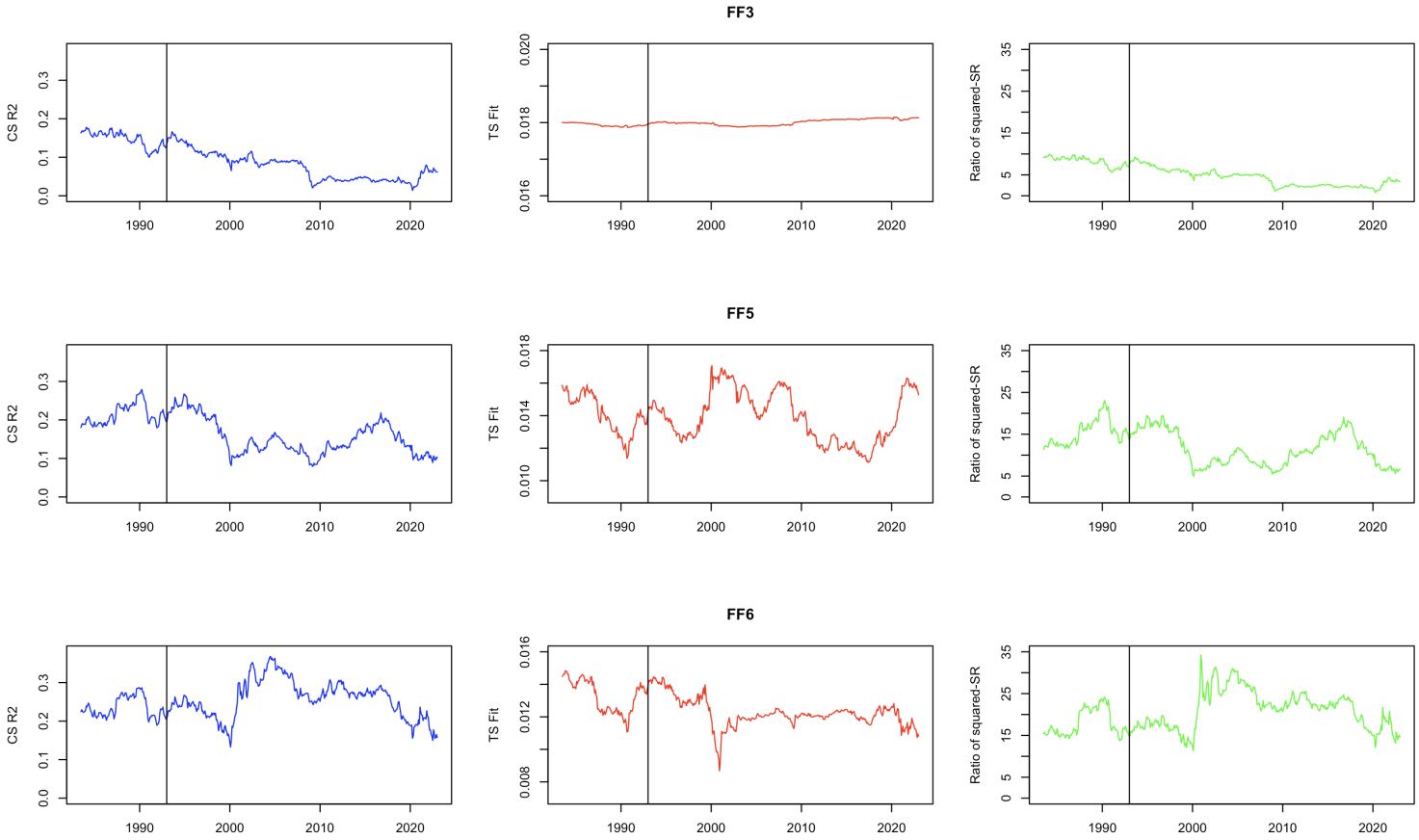


Figure 15: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the Optimal Factor Portfolios related to the Fama-French 3-, 5- and 6-factors models in the GLS case. Calculations are based on a 240-month moving window.

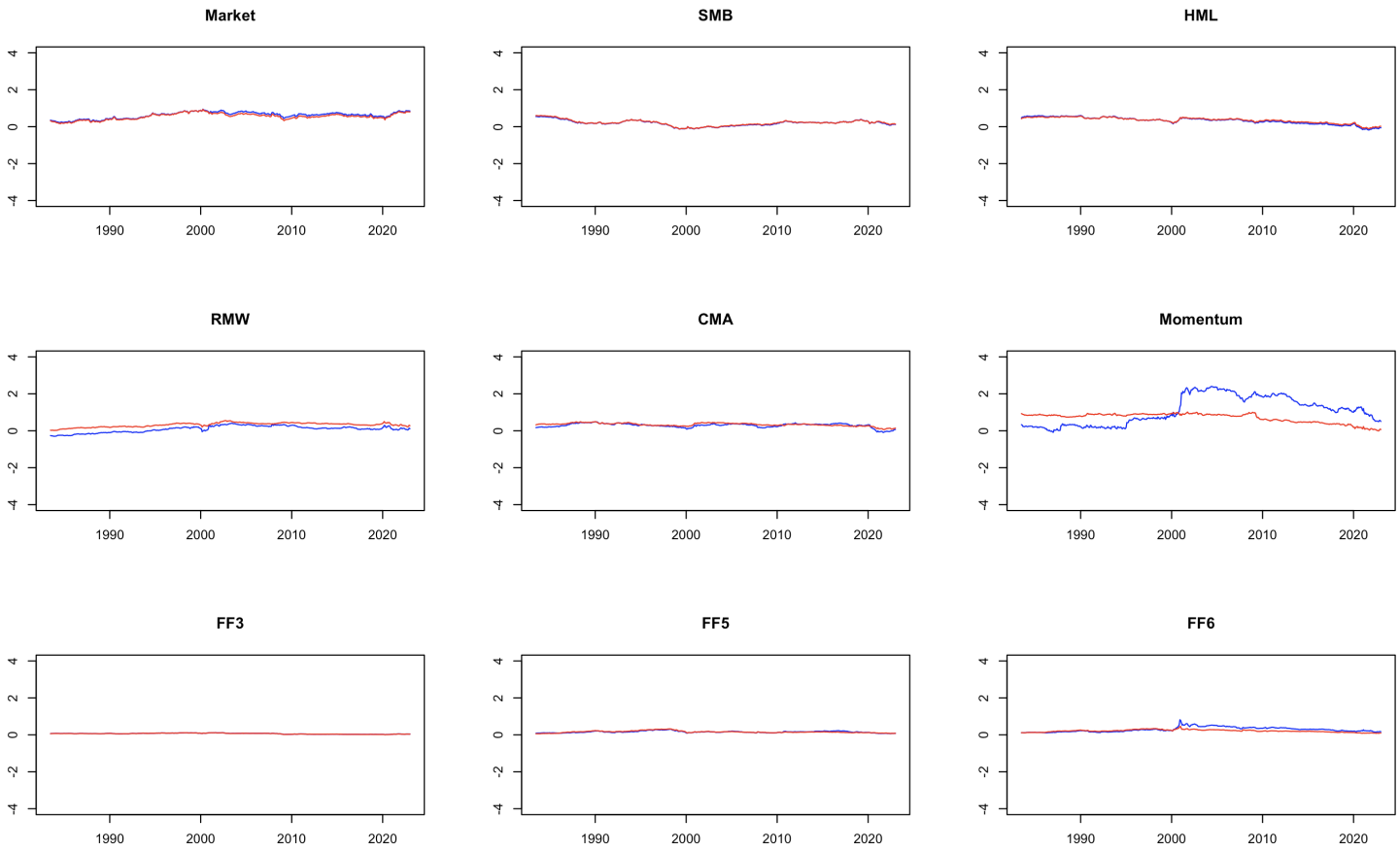


Figure 16: Behavior of the risk premium (blue) and the factor's expected value (red) over time for the Fama and French (2015) 5 factors, the Momentum factor, and the OFPs related to the FF 3-, 5-, and 6-factor models in the GLS case, using a 240-month moving window.

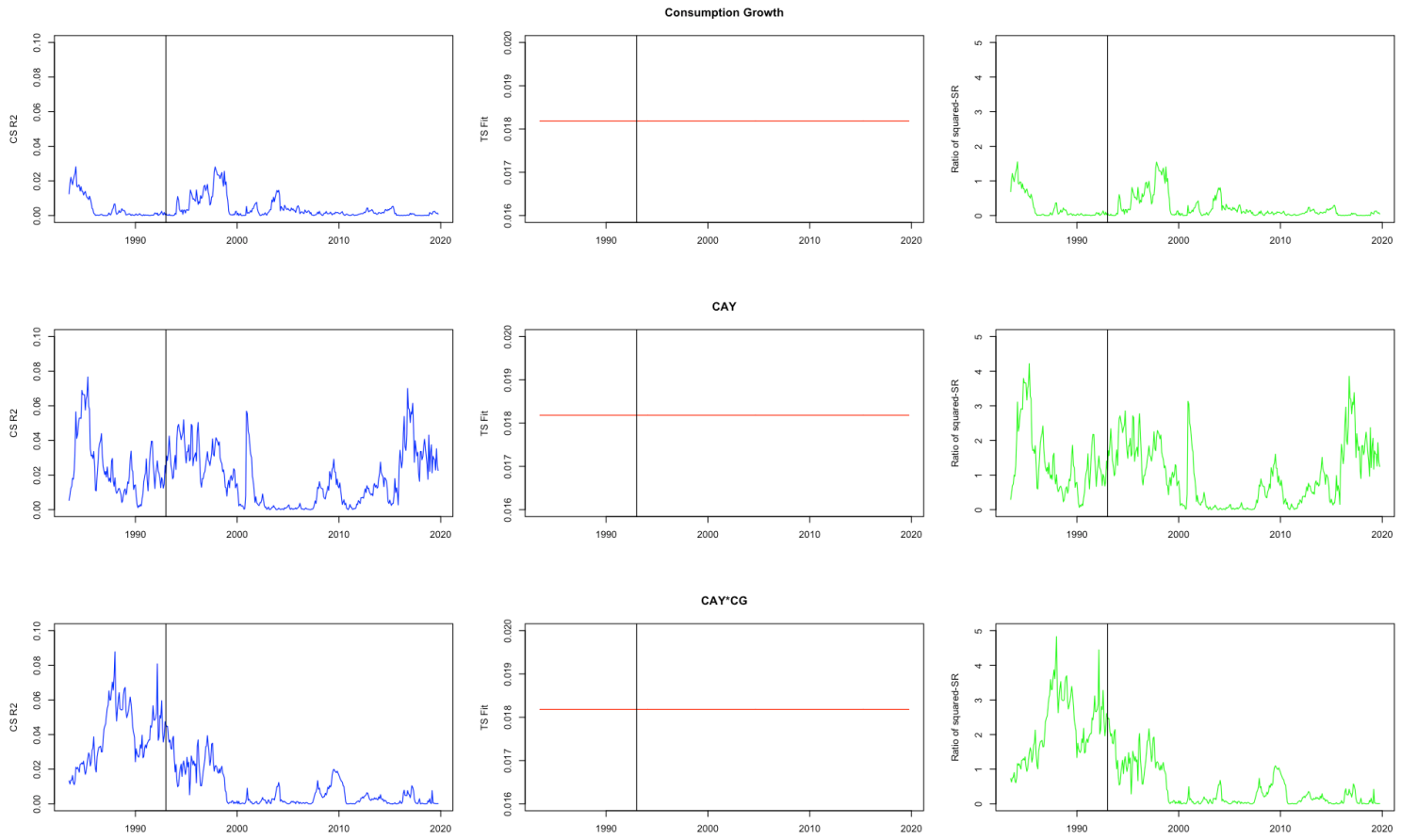


Figure 17: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the mimicking portfolios of Consumption Growth (CG), consumption-wealth ratio (CAY), and their interaction (CG×CAY) in the GLS case. Calculations are based on a 240-month moving window.

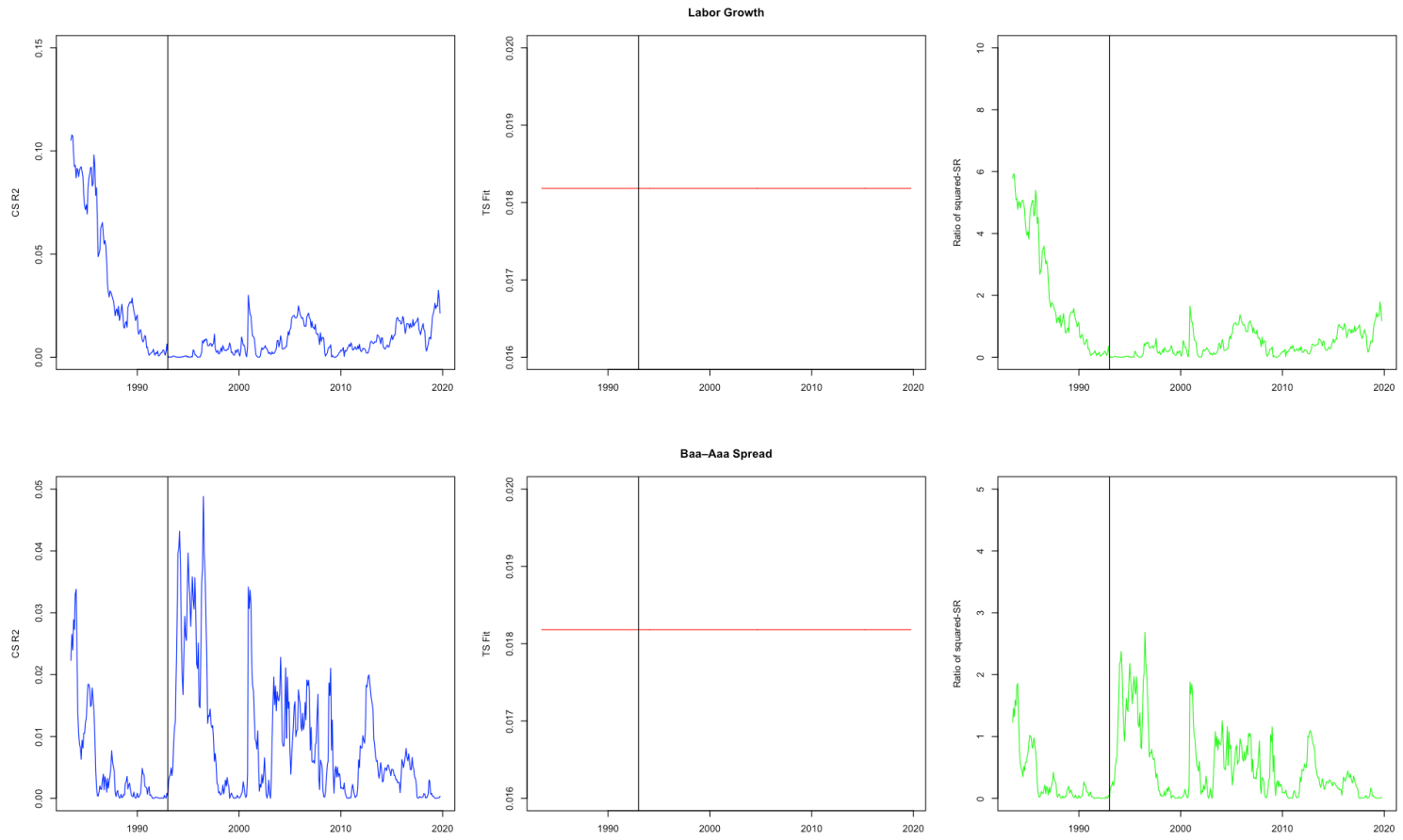


Figure 18: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the mimicking portfolios of the growth rate in per capita labor income (LAB) and the Baa-Aaa Spread in the GLS case. Calculations are based on a 240-month moving window.

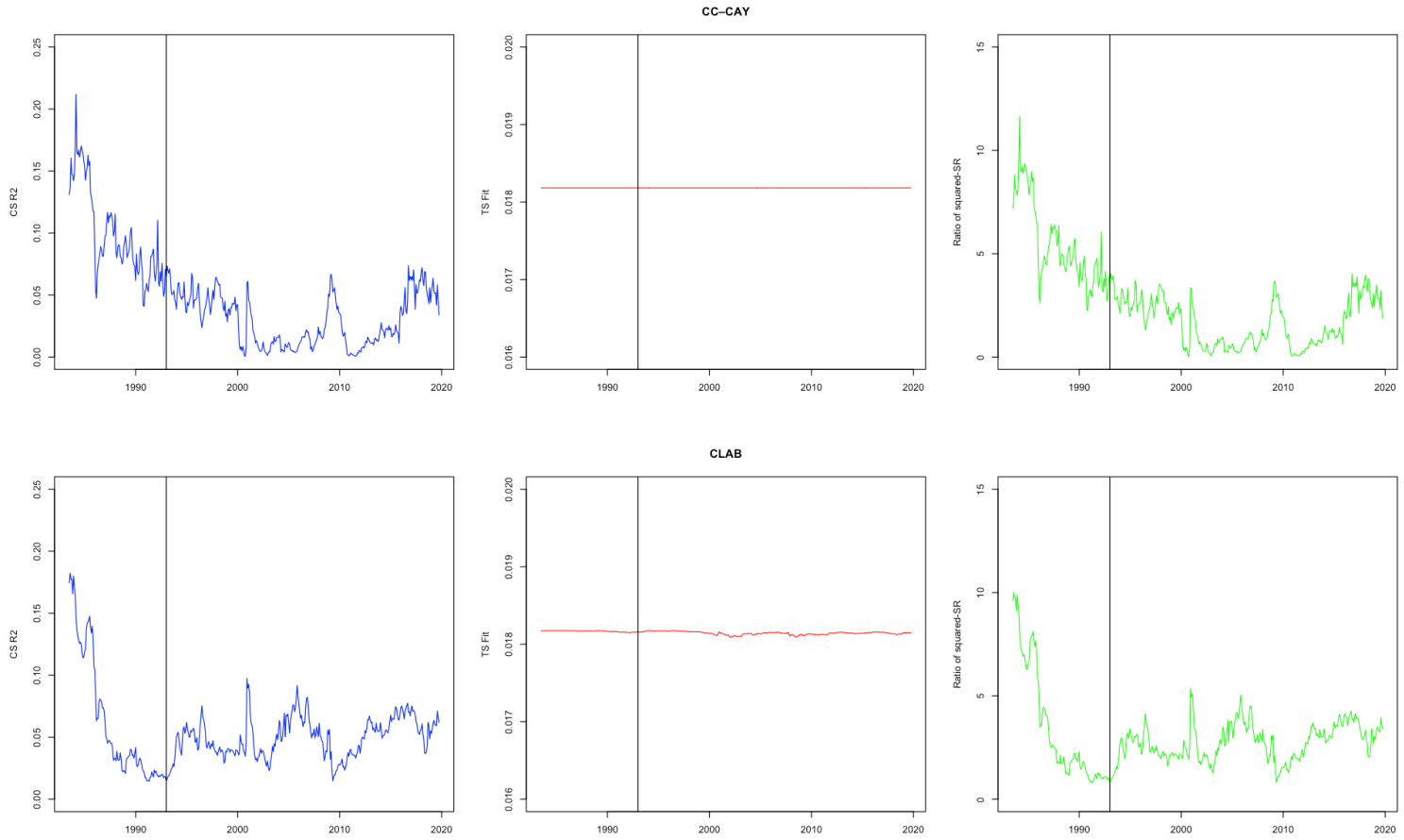


Figure 19: Cross-Sectional R^2 (blue, left column) and its components—the Time-Series Fit (red, middle column) and the squared-SR ratio (green, right column)—over time for the Optimal Factor Portfolios related to CC–CAY and C–LAB in the GLS case. Calculations are based on a 240-month moving window.

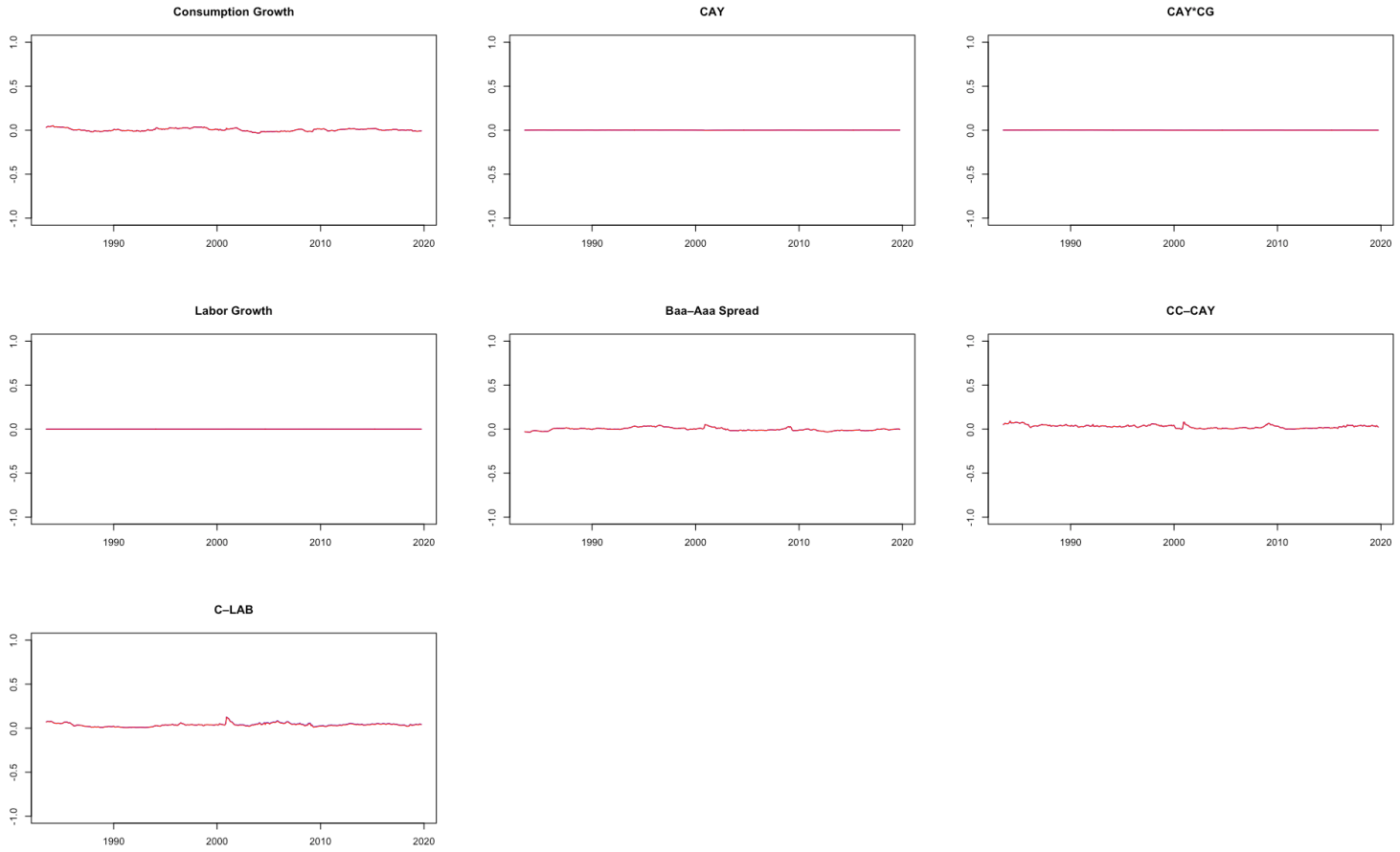


Figure 20: Behavior of the risk premium (blue) and the factor's expected value (red) over time for the mimicking portfolios of the non-traded factors analyzed, and the OFPs related to the CC–CAY and CLAB models in the GLS case, using a 240-month moving window.