Financial Frictions Over the Early Life Cycle of Firms: A Cross-Country Perspective*

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Abstract

We study young firms' short- and long-term leverage dynamics across rich and poor countries to infer the severity of firm-level financial constraints. Using data from private European firms, we document two novel facts: short-term leverage increases while long-term leverage decreases over the early life cycle of firms, and this pattern persists longer in less developed economies. We build a model of endogenous borrowing constraints consistent with international corporate finance facts and leverage dynamics over the early life cycle of firms. The model implies that short-term funding gaps are biased measures of the severity of firm-level financial constraints, and cross-country data can be used to recover the bias. Guided by the model, we estimate that the bias accounts for 17.2% (10.6%) in low (high) contract-enforcement economies, suggesting the importance of cross-country data when measuring the severity of financial frictions.

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1 Introduction

Young firms fuel economic growth and dynamism as they account for the majority of employment creation in the economy (Haltiwanger et al., 2013; Ayyagari et al., 2014). Additionally, small and young firms are, on average, more likely to be financially constrained than mature firms (Hadlock and Pierce, 2010), which calls for government-assisted credit programs. Yet, the median young firm in a developed economy such as the U.S. does not appear to be financially constrained (Pugsley and Hurst, 2011).[1] To what extent then can we justify credit programs across the globe? How severe are the financial constraints over the early life cycle of firms? We provide a new theoretical and empirical framework to answer these questions.

The model features endogenous borrowing constraints emerging from the optimal financing problem under limited enforcement of a firm that faces an exogenous death shock, chooses scale (capital) at inception, and labor every period. The key mechanism of the theory results from the novel interaction between the contract-enforcement level of the economy, the capital/scale choice of firms at entry, and the corresponding sequence of endogenous borrowing constraints that firms face. Enforcement frictions directly and indirectly (through distortions in the initial scale choice) affect the borrowing constraint's severity. Moreover, the model uniquely pins down the firm's financial structure (i.e., the necessity of long- and short-term debt). This emerges from the interaction of transaction costs, the fact that short contracts are safe and long contracts are risky, and firms being unable to repay their total debt in the first period of operations. The main contribution of this paper is to show how, by looking at firms' capital structure over the early life cycle, one can infer the full effect of limited enforcement on the severity of financial constraints. The paper's central message is that while the direct effects of enforcement can be inferred using data from a single country,

[1] The authors argue that the median young firm in the U.S. does not grow and reports that it does not want to grow, which implies that it is not financially constrained.

the indirect effects that operate through the scale choice require the usage of a benchmark country. In this sense, measuring the severity of financial constraints is fundamentally a cross-country exercise.

An ideal setting to measure financial constraints would be the universe of firms divided into two groups (constrained and unconstrained firms) and study what firm characteristics are associated with belonging to the financially-constrained group. However, we don't typically have a classification criterion for assigning firms to each category; if we did, measuring financial constraints wouldn't be a problem. In this paper, we use the fact that enforcement institutions in less financially developed countries are relatively worse to solve the selection criterion problem. Inspired by the corporate finance literature, we study firms' capital structure to infer the presence of financing constraints. Precisely, we focus on a new moment, the joint dynamics of firms' short- and long-term leverage, to establish the duration of the early life cycle of firms and establish the short-term debt target that firms have. This allows us to construct firms' short-term debt funding gap over the early life cycle. However, this short-term funding gap does not capture the full effects of enforcement frictions. Firms' observable short-term debt target is also distorted since it is a function of the financially-constrained scale choice of firms decided at birth. Since a target distortion is a counterfactual object that cannot be calculated with data from a single country, we argue that cross-country comparisons are necessary to capture the full effects of financial constraints generated by limited enforcement. In short, this paper develops a theoretical and empirical framework to study financial constraints in the early life cycle of firms at different levels of development.

We define financial constraints as financing gaps—the difference between a firm's desired/target size of a loan and the effective loan obtained—that distort a firm's decision. Hence, one can study financial constraints by analyzing the firm's capital structure or by in-

specting the behavior of the firms' outcomes (investment, employment, profits, etc.). One of our contributions is to show how financing gaps map to outcome distortions, and by focusing on financing gaps, we are able to construct a monetary measure of the severity of financial constraints. Since our object of study are financial constraints, we cannot assume them, they need to be a result of the model. For this reason, we build on the endogenous borrowing constraints literature (Albuquerque and Hopenhayn (2004), Cooley et al. (2004)) to construct a model where financial constraints are a result of limited enforcement. This gives us a framework where we can get a notion of "more severe financial constraints" by comparing the model's behavior under different levels of enforcement. While an endogenous borrowing constraint can also result from agency problems (Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007), Verani (2018), etc.), we focus on enforcement frictions because a large literature in international corporate finance (discussed below) documents that cross country differences in capital structure are explained by differences in the quality of enforcement institutions.

In this paper, we solve an optimal contracting problem where the types of securities firms use emerge endogenously due to contracting costs and limited enforcement. One contribution of our framework is the elimination of the multiplicity of implementations that typically appear in the optimal contracting literature[2]. The solution to a dynamic contracting problem is a sequence of transfers between parties, and multiple implementations (codifications of those sequences of transfers into securities) are typically consistent with optimality. For example, instead of using long-term debt, one can implement a given sequence of transfers by rolling over short-term debt. This is usually the assumption of many models written

[2]For example in DeMarzo and Fishman (2007) they show how the optimal contract motivated by agency problems can be implemented with long-term debt, a line of credit and equity. Then, they proceed to clarify *"Of course, this implementation is not unique, I can aggregate the three securities into a single security and without changing the agent's incentives. However, the implementation I described is natural and corresponds well to securities observed in practice."*

in macro-finance. The issue is how to map the model to the data. Suppose one considers maturity considerations irrelevant and assumes them away by exogenously imposing short-term debt as the only security firms use. What would be the empirical counterpart of the model in that case? Should we map short-term debt in the model to total debt or short-term debt in the data? While for many research questions, maturity considerations might not be of first-order relevance, we argue that they are for studying the early life cycle of firms.

By focusing on the characterization of financing gaps, our approach is closer in spirit to the exercises done by the empirical capital structure literature; in this tradition (see Graham and Leary (2011) for a review), leverage is explained by outcomes (size, capital stock, profits, etc.). On the other hand, the empirical literature on financial frictions focuses on outcome distortions to infer financial constraints. In this second tradition, financial frictions are inferred from distortions in firms' outcomes, particularly forgone investment, employment, and profits.[3] The seminal paper of Fazzari et al. (1988) interprets cash flow sensitivity of investment as indicative of financial constraints; the critique of Kaplan and Zingales (1997) is also based on whether the firms identified as constrained are unable to fund their desired investment. However, the empirical implementation of Kaplan and Zingales (1997) done by Lamont et al. (2001) is based on whether there is a financial constraint factor that explains stock returns, therefore, while motivated by forgone investment approach, their implementation is closer to forgone profits. Whited and Wu (2006) index of financial constraints is constructed through a structural estimation of an Euler Equation governing firms' investment decisions. Dao and Liu (2017) study the effect of financial constraints on job creation. A recent paper by Cherchye et al. (2020) proposes a measure of financial constraint by study-

[3]A notable exception is Almeida et al. (2004), that studies the cash flow sensitivity of cash, in their model, financially constrained firms should increase their cash holdings in response to a cash flow innovation. However, the reason that motivates the savings decisions is an investment opportunity in the future; hence, while the test does not involve investment directly, increases in cash holdings reflect a reduction in future forgone investment.

ing forgone profits. Relative to these papers, our paper is the first to develop a framework to measure financial constraints that is applicable to private and young firms[4] and methodologically we are, to the best of our knowledge, the first to claim the need of cross country comparisons to measure financial constraints.

In our model, both long- and short-term debt emerge endogenously. Firms will optimally match the maturities of assets and liabilities: long-term debt will be used to fund the initial capital acquisition, and short-term debt will be used to fund working capital needs. This disciplines how to map the model to the data, since there is a direct correspondence between both. Crucially, we show that the joint dynamic behavior of short- and long-term leverage will give us information about financial constraints. In other words, long- and short-term leverage behave differently in young and constrained firms; therefore, focusing on total leverage as usually happens in models studies firms over the whole life cycle is with loss of generality when studying young firms. Specifically, according to our model, constrained firms during their early life cycle are characterized by decreasing long-term leverage and increasing short-term leverage as they age.

A relatively new literature that studies financing over the life cycle of firms documents for the US that private firms' short and long-term leverage is linearly decreasing in age, and when studying a quadratic specification for private firms, they find that short-leverage is always decreasing for US firms (Dinlersoz et al., 2019), moreover Derrien et al. (2021) document for French firms that in the first ten years of operations, total leverage decreases with age; and recently Kochen (2022) documents that leverage (defined as short plus long leverage[5]) is decreasing in age for European countries. Relative to this literature, we focus

[4] Cherchye et al. (2020) method is also applicable to private firms while the literature has typically study financial constraint on public firms

[5] See the data and variable definitions for definitions of these notions.

on the joint dynamics of short and long-term leverage in the early life cycle of firms and document a new fact: during the first years of firms' operations, short leverage increases while long leverage decreases.

Since financial constraints are endogenous in our model and emerge from enforcement frictions, we have a well-defined notion in the model of "more severe financial constraints". The natural counterpart of this comparative static of the model in the data is the capital structure of firms in different countries. This is precisely what the international corporate finance literature studies; they investigate the extent to which the traditional explanatory variable for firms' capital structure, developed using US data, translates to other countries. Two messages emerge from this literature: i) The same variables that explain capital structure choices in the US also explain capital structure choices in other countries. Rajan and Zingales (1995) document this for developed economies and Booth et al. (2001) for less developed countries. ii) Country's institutions affect financing patterns, and they do so more than industry affiliation (Demirgüç-Kunt and Maksimovic, 1999; Fan et al., 2012). Antzoulatos et al. (2016) study the convergence of capital structure, evaluating firm-specific, industry-specific, and economy-wide factors, and find evidence in favor of financial development (economy-wide factor) driving the firm's leverage ratio. Exploring firm size heterogeneity and financial development, Beck et al. (2008) find that small firms benefit relatively more than large firms from the institutional factors of a country. Demirgüç-Kunt and Maksimovic (1999) find that small firms obtain less long-term debt financing in countries with less-developed financial systems. Demirgüç-Kunt and Maksimovic (1998) find that a greater proportion of firms use long-term external financing in countries with higher legal system efficiency (better enforcement). Studying relatively large firms[6] Qian and Strahan (2007) find that stronger creditor rights are associated with longer-term lending by banks

[6]Those appearing in Worldscope and Dealscan.

and tangible assets correlation with debt maturity increases in countries with better creditor rights. Bae and Goyal (2009) argue that enforceability matter more than the existence of creditors protection law and find that better enforceability of contracts increases loan size and lengthens loan maturity. Haselmann et al. (2010) investigate the channel through which legal institutions get transmitted to the economy and find that collateral laws matter more than bankruptcy laws for banks lending.

A key moment to match the international corporate finance facts is that the model needs to produce higher long-term leverage in more financially developed countries. However, the standard model of endogenous financial constraints produces the contrary. The counterfactual prediction emerges from the assumption that all firms must pay the same unproductive set-up cost. This will generate that repaying the same face value of long-term debt in less developed countries (worse enforcement) takes longer. Furthermore, the value of the firm and the total surplus of the contract between the firm and financiers (the value of assets) is increasing in enforcement. A fixed face value of long-term debt for firms across development levels and the fact that assets value are increasing in enforcement imply that in less developed countries, leverage will be higher due to a lower value of assets of the firm. To fix this counterfactual prediction, we endogenize the unproductive set-up cost by making it a productive choice. In our preferred interpretation, young firms buy upfront the capital scale they want to operate during the early life cycle. This will generate that firms in less developed economies will face a higher cost of capital due to the endogenous financial frictions and choose a smaller scale. This will resolve the counterfactual prediction of the traditional model and introduce a new dimension to the measurement of financial frictions.

Financial frictions and firm dynamics literature typically distinguish between extensive (firm dynamism) margins and the intensive margin (factor allocation between existing firms).

Midrigan and Xu (2014) using detailed firm-level from Korea finds that the bulk of the effects of financial frictions goes through the extensive margin. The theoretical reason is that firms eventually grow out of their borrowing constraints by accumulating internal funds. In other words, the distortions along the intensive margin tend to disappear as time goes by while the extensive margin distortion persists. In our model the initial capital distortions will not exclude the firm from entering (hence we don't have distortions along the extensive marging[7]) but will generate a permanent distortion along the intensive margin. The scale distortion in our model is permanent by construction since we don't let firms adjust capital once they enter[8]. We will interpret the "persistency" of this scale distortion with care, our model is only appropriate for the early life cycle of firms (roughly from inception up to 7/8 years old), so a persistent distortion will mean, in the context of our model, a distortion that lasts the duration of the early life cycle of firms.

The remainder of the paper is organized as follows; section 2 presents the data and stylized facts, section 3 presents the theory, section 4 analyzes financial constraints through the lens of our model and present an illustrative example of our methodology. Section 5 discusses potential applications of our framework and concludes. All proofs, data description and alternative empirical specifications are presented in the appendix.

[7] A modification requiring a minimum investment in order to start operations could solve this.

[8] The idea that firms choose at inception a factor that after their future scale seems to be in line with the literature on persistent effects of entry conditions. See Moreira (2019).

2 Empirical Analysis

2.1 Data and Variables Definition

The data source is Orbis compiled by Moody's Bureau van Dijk (BvD). BvD compiles annual balance sheets and income statements of private and public firms and harmonizes them into an internationally comparable format. The data also report industry identifiers and the firm's year of incorporation, allowing us to construct the firm's age. While Orbis has extensive coverage of countries, the degree to which the data represents the population of firms varies across countries. Bajgar et al. (2020) shows that in an extended sample of countries, Orbis captures 60% of aggregate employment and output but 40% of value-added; they further identify a subset of countries (what they called "preferred sample") that is representative of the firm population. We use in our analysis those countries and restrict the analysis to companies with no consolidated accounts that have non zero debt; the summary statistics are presented in table 3 in the appendix.

Our objective is to document the patterns of young firms' capital structure. Guided by our model we focus on the life cycle patterns of short- and long-term leverage:

$$\text{Short Leverage} \equiv \frac{\text{Short Term Debt}}{\text{Short Term Debt} + \text{Long Term Debt} + \text{Net Worth}}$$

$$\text{Long Leverage} \equiv \frac{\text{Long Term Debt}}{\text{Short Term Debt} + \text{Long Term Debt} + \text{Net Worth}}$$

Where short-term debt are loans from financial institutions that mature in less than a year, and long-term debt are loans from financial institutions that mature in more than a year. One can define leverage in more than one way[9] (see Rajan and Zingales (1995) and Welch (2011)), the advantage of the definition adopted here is that it does not vary with omitted

[9]For example, defining leverage as $\frac{\text{Short Term Debt} + \text{Long Term Debt}}{\text{Net Worth}}$ captures the relative claims on the firms value held by equity and debt. If we were studying agency problems, this would be a more appropriate definition.

variables as it would be the case if we were to define short term leverage as $\frac{\text{Short Term Debt}}{\text{Total Assets}}$ where an increase in total assets funded by trade credit would result in lower leverage.

2.2 Fact 1: Financial Development, Long and Short-Leverage

Firms in more financially developed economies have higher levels of long leverage and lower levels of short term leverage. This patterns holds true for young and old firms, moreover in more developed economies the differences between young and old firms are attenuated. Figure 1 is in line with the previous literature that abstracting from life cycle considerations documents a similar patterns. Demirgüç-Kunt and Maksimovic (1999) finds that firms in developing countries have lower long-leverage ratios. Fan et al. (2012) finds that countries with more corruption and lower legal enforcement have lower debt maturities. Intuitively, in environments with lower law enforcement, short-term debt that allows insiders less discretion should dominate, as opposed to long-term debt that usually is accompanied by covenants and collateral clauses that rely more heavily on a country's legal institutions. Empirically, this intuition is corroborated by Fan et al. (2012) and Demirgüç-Kunt and Maksimovic (1999) that find higher low term leverage in less developed economies (weaker legal institutions, higher corruption, etc..). Arellano et al. (2012) study the relationship between financial development and firm dynamics along the size distribution of firms and finds that the difference between young and old firm outcomes is more pronounced in less developed economies. They report a negative relationship between leverage (measured as long leverage) and financial development. de Jong et al. (2008) study firm-specific and country-specific factors and document the indirect effect of institutions that results in the differences in levels of the typical factors explaining capital structure. They conclude that firms in countries with a better legal environment take more debt, and country-specific factors also reinforce the

effects of firm-level determinants of leverage.[10]

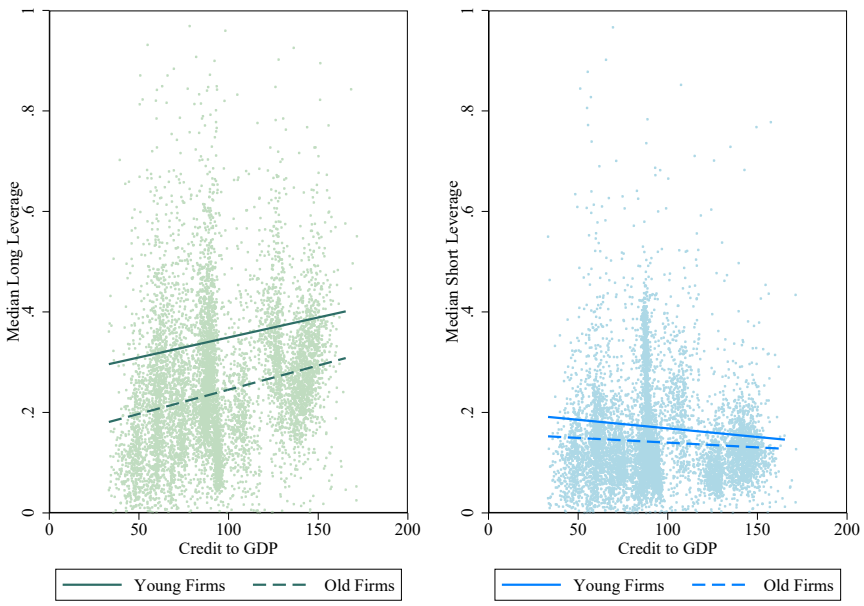


FIGURE 1: EACH OBSERVATION CORRESPONDS TO AN INDUSTRY COUNTRY AND AGE TRIPLET. YOUNG FIRMS ARE FIRMS THAT ARE LESS THAN 10 YEARS OLD, AND OLD FIRMS ARE FIRMS BETWEEN 10 AND 30 YEARS OLD. FITTED LINES CORRESPOND TO A LINEAR MODEL IN CREDIT TO GDP, AND INDUSTRY FIXED EFFECTS RUN OVER THE SUB-SAMPLE OF YOUNG AND OLD FIRMS.

2.3 Fact 2: The Early Life Cycle of Firms

To document the behavior of capital structure over the early life cycle of firms, we run the following model:

$$y_{i,t} = \eta_i + \sum_{a \in A} \beta_a \mathcal{D}^a_{i,t} + LEMP_{i,t} + COLLAT_{i,t} + PROFIT_{i,t} + LPROD_{i,t} + \epsilon_{i,t}$$

Where $y_{i,t}$ denotes the outcomes variable of interest (short- and long-term leverage), $\mathcal{A} = \{(0-1, 2-3, \ldots, 24-25, +25\}$ corresponds to 14 age categories, $\mathcal{D}^a_{i,t}$ is a dummy variable

[10]For example, collateral explains more long-term debt in more developed economies

that takes value one if firm i belongs to the age-bin a at time t. LEMP is log of employment, COLLAT is tangible assets over total assets, PROFIT is caculated as EBIT over total assets and LPROD is caculated as the log of the ratio of sales to employment. The main message of figure 2 is that firms up to 7 years old present a distinctive pattern, short-leverage increases while long-leverage decreases. We will refer to this pattern as the capital structure over the early life cycle (ELC) of firms, and we will refer to the last period where age has a positive effect on age as the *duration* of the ELC.

Note that beyond the duration of the ELC, short—and long-leverage decrease with firm age. Hence, the discussion of whether one should measure short-, long—or total leverage when studying life cycle effects is not of first-order importance. In this sense, models with only short-term debt (whose typical data counterpart is total leverage) are likely to be a good representation of firms' debt dynamics beyond, but not during, the ELC.

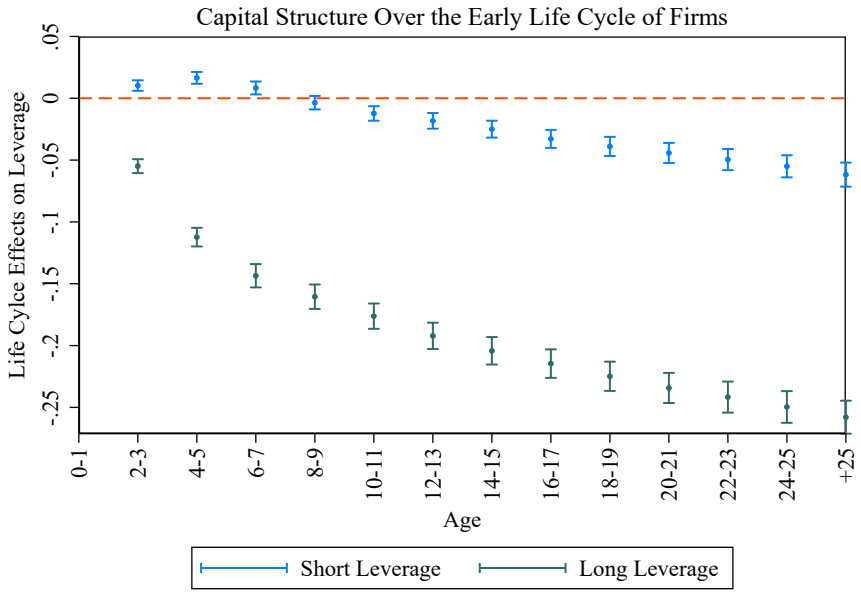


FIGURE 2: LIFE CYCLE PATTERNS OF CAPITAL STRUCTURE OF FIRMS FOR WHICH WE HAVE AT LEAST 3 OBSERVATIONS AND HAVE POSITIVE DEBT. CONFIDENCE INTERVALS ARE AT 90% AND STANDARD ERRORS ARE CLUSTERED AT COUNTRY-INDUSTRY-YEAR.

3 Model

Motivated by the previous stylized facts of international corporate finance and the empirical findings of persistence of capital and labor at the firm level, in this section, we present a model of firm dynamics, development and endogenous borrowing constraints. We show how the model replicates the stylized facts and use it to study the severity of financial constraints over the ELC of firms. We present the model in three steps. First we show that in the presence of transaction costs, the optimal financial structure requires the use of long and short term debt [11], and, firms engage in maturity matching[12] of liabilities and assets. Second, we present a benchmark economy without enforcement frictions, there the short term and long term debt design problems are separable. As a consequence, dividends and the maturity of long term debt is indeterminate. We use this section to highlight the trade-off involved with scale. Finally, we introduce enforcement constraints, this breaks down the separability of short and long term debt. As a result, the structure of long term debt is determinate as well as the dividends policy: firms optimally front-load payments of long term debt and back-load dividend payments.

3.1 Environment

We are going to study the optimal lending contract between competitive financiers and a firm that cannot commit to honor their obligations. Firms operate a DRS technology with a quasi-fixed factor of production (K) and a flexible factor (n). The firm can only choose capital[13] K when they are born. Every period t is divided into two sub-periods, morning and

[11] Note that typically in to the optimal contracting literature long term debt is one of many possible implementations of the optimal contract, here we show that other implementations that don't use long term are debt sub optimal.

[12] For empirical evidence of firms engaging in maturity matching see Graham and Harvey (2001) for U.S. and World Bank (2015) and references therein for developed and developing countries. For an alternative theoretical foundation see Hart and Moore (1995).

[13] We can interpret this as firms choosing capacity, or investing in technology given their productivity

night. During the morning, firms start the production plan, pay for the factors employed and make savings decisions. During the night, production and revenues are realized, then firms decide to abscond or continue in operations. Conditional on continuation, firms make dividends, savings and payments of liabilities decisions. At the end of t_N firms are hit with an exit shock with probability $1-\rho$.

Assumption 1 *(Property Rights) There is an enforcement technology ξ in the economy that can be applied to promises codified in contracts. Registering a transaction in a contract requires to pay a fix cost κ_C. The technology ξ imposes a penalty $(1-\xi)$ when the borrower engages in strategic default, but not to default induced by exit shocks.*

Assumption 1 says that verbal agreements cannot be enforced using a legal system of a country as opposed to agreements registered in a contract. Contracts then protect lenders against strategic default but not against exit/unavoidable defaults.[14] Formally registering a promise in a contract form it is not for free, it has an associated cost of κ_C.

Assumption 2 *(Collateral) A fraction $\phi \in [0,1)$ of the initial scale investment K can be used as collateral. Firms can give up ownership of ϕK to a lender in case of default*

Note that assumption 2 imposes that ϕK is indivisible, in other words the decision to pledge a durable assets as collateral is binary. In the appendix, we discuss a weaker version of the assumption where we allow the firm to pledge a fraction of the collateralize assets it possesses. We will refer to ϕ as the tangibility of capital of a firm.

Firms have access to competitive financiers with whom it can sign a loan contracts. We will denote by $\mathcal{C}^j_{t_g,m}$ the contract j signed at time t_g where $g \in \{M,N\}$ is the sub-period (Morning or Night) when the contract is signed and m is the maturity/duration of the contract. Contracts can be signed at morning or night, note that contracts signed at morning

[14] We will refer as exit default to the default on obligations induced by a an exit shock

that pay at night are not exposed to the exit shock but contracts signed at night that pay next day's morning are. Formally a contract specifies an initial transfer from financiers to firms R_{t_g} at the time the contract is signed t_g, and a sequence of transfer from firms to financiers $\{\tau_{t_{N+k}}\}_{k=0}^{m}, \{\tau_{t_{M+k}}\}_{k=0}^{m}$ where $m \leq \infty$ denotes the last period where $\tau > 0$. We will say that m is the maturity of the contract[15]. Finally if $\nu^j = 1$ we will say that the contract is collateralized by the durable assets of the firm.

$$\mathcal{C}^j_{t_M,m} = \{R_{t,M}; \{\tau_{t_N+k}\}_{k=0}^{m}, \{\tau_{t_M+k}\}_{k=0}^{m}; \nu^j\}$$
$$\mathcal{C}^j_{t_N,m} = \{R_{t,M}; \{\tau_{t_N+k}\}_{k=0}^{m}, \{\tau_{t_M+k}\}_{k=1}^{m}; \nu^j\}$$

We will denote by y_{t_M}, y_{t_N} the free cash-flow given production plans and liabilities. Note that the only relevant object once a loan contract is in place are the payments that the firm needs to make in the future that affects the free cash flow. Let $\bar{\tau}_t = \{\bar{\tau}_{t+k_M}, \bar{\tau}_{t+k_N}\}_{k=0}^{\infty}$ denote the sum of obligations that a firms faces from period t onward. For example, if there are 3 contracts in place indexed by $j \in \{a, b, c\}$ that requires payments: $\tau^a_{t,N}, \tau^b_{t,N}$ and $\tau^c_{t,N}$ at time t_N then $\bar{\tau}_{t_N} = \sum_{j\in\{a,b,c\}} \tau^j_{t_N}$

$$y_{t,N} = z\left(K^{\alpha} n_t^{1-\alpha}\right)^{\eta} - \bar{\tau}_{t_N} + (1 + r_f)b_{t,M} - b_{t,N} + R_{t,N}$$
$$y_{t,M} = -wn - \bar{\tau}_{t_M} + (1 + r_f)b_{t-1,N} - b_{t,M} + R_{t,M}$$

Given the sequence of production plans $\{(K, n_t)\}_{k=0}^{\infty}$, a savings policy $\{b_{t+k,M}, b_{t+k,N}\}_{k=0}^{\infty}$ and a plan for future contracts $\{C^j_{t+k_M,m}; C^j_{t+k_N,m}\}_{m\geq 0;k\geq 1}$ we will say that a contract signed at time t is feasible if it satisfies the participation constraint of financiers, limited liability of

[15] If $m = \infty$ we will say that the contract is a perpetuity. We will show later that this is never optimal

firms, non-negativity constraint of payments and a no dilution collateral constraint.

A contract $\mathcal{C}^j_{t_M,m} = \{R_{t,m}; \{\tau_{t+k,N}\}_{k=0}^m, \{\tau_{t+k,M}\}_{k=0}^m; \nu^j\}$ satisfies the PC of financiers if:

$$(1+r_f)R_{t,M} = \tau_{t,N} + \sum_{k=1}^{m} \beta^{2k-1}\rho^k \tau_{t+k,M} + \sum_{k=1}^{m} \beta^{2k}\rho^k \tau_{t+k,N} + (1-\rho)(1-\nu^j)\sum_{k=1}^{m} \beta^{2k}\rho^{k-1}\phi p_k K$$

A contract satisfies limited liability and non negativity constraint of payments if

$$0 \leq \tau_{t+k,N} \leq y_{t+k,N} \qquad \forall k \in [0,m]$$

$$0 \leq \tau_{t+k,M} \leq y_{t+k,M} \qquad \forall k \in [0,m]$$

Let $\mathcal{J}_t$ be the index of contracts in place at time t. Then a contract satisfies the no dilution of collateral constraint if:

$$v^j \geq 0 \qquad \text{if} \qquad v^k = 0 \ \ \forall\ k \neq j$$

$$v^j = 0 \qquad \text{if} \qquad \exists\ k \in \mathcal{J}_t \ : \ v^k = 1$$

Firms also have access to a bond/storage technology that pays intra-period $(1+\hat{r}_f)$. Hence if the firm saves $b_{t,M}$ in the morning of period t it will have $(1+\hat{r}_f)b_{t,M}$ in the night of period t , and similarly, if the firm saves $b_{t,N}$ in the night of period t it will have $(1+\hat{r}_f)b_{t,N}$ in the morning of period $t+1$. Importantly, firms cannot issue these bonds. The timing of the model is illustrated in figure 3:

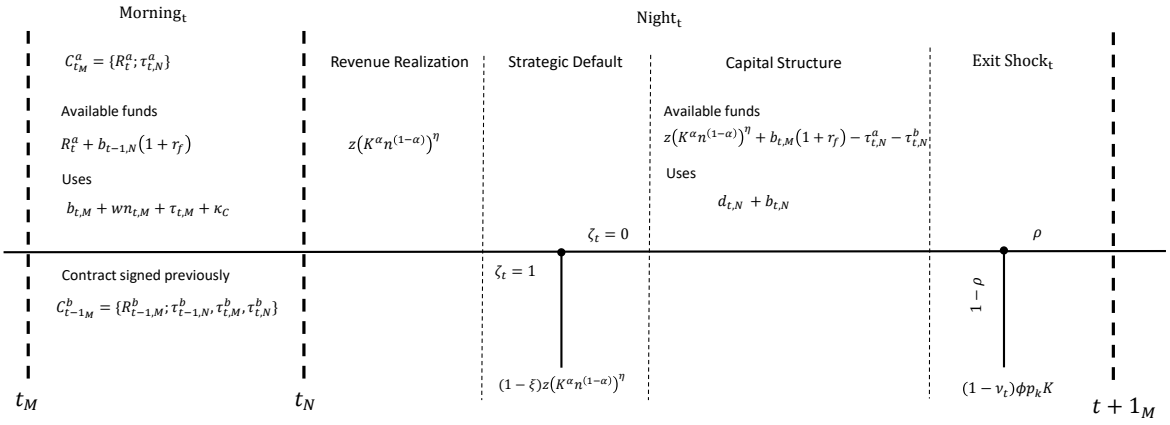


FIGURE 3: TIMING OF A MODEL OF A FIRM THAT INVESTED K AND HAS A CONTRACT IN PLACE C^b SIGNED IN PERIOD $t-1$ WITH MATURITY 1 THAT EXTENDS PAYMENTS INTO PERIOD t_M AND t_N. A CONTRACT C^a THAT GRANTS FUNDS R^a_t AND REQUIRES A PAYMENT $\tau^a_{t,N}$ AT t_N

3.2 Financial Structure: Long and Short Term Debt

3.2.1 No transaction costs

Proposition 1 *When transaction costs are zero ($\kappa_C = 0$):*

- *Firms cover intra-period funding needs with intra-period loans. (Short term maturity matching).*
- *Firms are indifferent between a multi-period loan contracts or a sequence of single-period loan contracts for funding needs that cannot be repaid by the cash flows generated in one period. (No long term maturity matching).*

Abstracting from feasibility, we can show that proposition 1 follows from minimizing the cost of funds. Consider that instead of taking a loan R_{t_M} at time t_M and R_{t+1_M} at time $t+1_M$, the firm decides to take a loan at time t, M of size $\hat{R}_{t_M} = R_{t_M} + \frac{R_{t+1_M}}{(1+\hat{r}_f)^2}$. Note that by investing $\frac{R_{t+1,M}}{(1+\hat{r}_f)^2}$ in the intra-period bonds, it can get R_{t+1_M} at time $t+1_M$.

$$R_{t,M} + \frac{R_{t+1,M}}{(1+\hat{r}_f)^2}$$

Financiers require:

- For a single period loan contract: $\{\hat{R}_{t,M}, \tau_{t,N}\}$

$$\tau_{t,N} = (1+\hat{r}_f)\left[R_{t,M} + \frac{R_{t+1,M}}{(1+\hat{r}_f)^2}\right]$$

- For an multi period loan contract: $\{\hat{R}_{t,M}, \tau_{t,N}, \tau_{t+1,N}\}$

$$\tau_{t,N} = (1+\hat{r}_f)R_{t,M}$$
$$\tau_{t+1,N} = (1+\hat{r}_f)\frac{R_{t+1,M}}{\rho}$$

- For a pair of single period loan contracts: $\{R_{t,M}, \tau_{t,N}\}, \{R_{t+1,M}, \tau_{t+1,N}\}$

$$\tau_{t,N} = (1+\hat{r}_f)R_{t,M}$$
$$\tau_{t+1,N} = (1+\hat{r}_f)R_{t+1,M}$$

Hence, firms will never choose $\{\hat{R}_{t,M}, \tau_{t,N}, \tau_{t+1,N}\}$ over a pair of single period loan contracts: $\{R_{t,M}, \tau_{t,N}\}$ and $\{R_{t+1,M}, \tau_{t+1,N}\}$. The problem of a multi period contract is that it exposes the financier to the default risk $(1-\rho)$, and consequently it needs to compensate the financier with a bigger payment in the state of no-default.

However, when the size of the loan is big enough such that the firm cannot repay it with the available cash flows, then it is forced to sign a multi period loan contract.

Assumption 3 *Parameters are such that:* $\left[\frac{\beta\rho(1+r_f)+\beta(1-\rho)\phi}{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}\right] > \frac{1-\eta}{\eta\alpha}$

Proposition 2 *Under assumption 3, funds generated by a firm in the first year of operations are not be enough to cover the acquisition of capacity K.*

Importantly, without transaction costs a multi-period contract can also be implemented by a sequence of short term contracts, which will amount to rolling over short term debt.

3.2.2 Positive transaction costs

Assumption 4 *The contracting costs $\kappa_C > 0$ are not too high*

$$\kappa_C < S_0(K(\xi_L), z_L)\frac{1-\rho}{\rho}$$

Where $S_0(K(\xi_L), z_L)$ denotes the short term debt that the least productive firm (z_L) firm would ask for in the first year of operations in the country with worst enforcement (ξ_L).

The existence of transaction costs breaks the indifference when financing a purchase that exceeds the cash flows generated in a single period. Now, a single multi-period loan contract dominates a sequence of single period loan contracts. While both financing strategies expose the financier to same unavoidable default risk, a single multiperiod contract saves on

transaction costs. However, notice that the introduction of transaction costs introduces a potential trade off for intra-period funding needs.

Let S_t denote the amount of short term debt borrowed, L_0 the amount of long term debt, $\{q_t\}_{t=0}^{\infty}$ the coupon structure, and $\nu \in \{0,1\}$ denote the collateralization policy where $\nu = 1$ denotes the decision to collateralize.

Proposition 3 *Under assumption 4 the optimal financial structure of firms in this economy is long term debt L in order to fund K and short term loans S to fund working capital.*

$$L = \left\{L_0 \, ; \{q_t\}_{t\geq}^{\infty} \, ; \, \nu\right\} \qquad and \qquad \mathcal{S} = \{\{S_t; (1+r_f)S_t\}\}_{t=0}^{\infty}$$

Compare the cost of $\{R_{t,M}, \tau_{t,N}\}, \{R_{t+1,M}, \tau_{t+1,N}\}$ and $\{\hat{R}_{t,M}, \tau_{t,N}, \tau_{t+1,N}\}$ with transaction costs:

$$\kappa_C + (1+\hat{r}_f)R_{t,M} + \beta\left[\kappa_C + (1+\hat{r}_f)R_{t+1,M}\right] < \kappa_C + (1+\hat{r}_f)R_{t,M} + \beta[(1+\hat{r}_f)\frac{R_{t+1,M}}{\rho}]$$

$$\kappa_C + (1+\hat{r}_f)R_{t+1,M} < (1+\hat{r}_f)\frac{R_{t+1,M}}{\rho}$$

$$\kappa_C < (1+\hat{r}_f)R_{t+1,M}\frac{1-\rho}{\rho}$$

Then if κ_C is not too high, it will still be optimal to use single period contracts that match the maturity of funding needs.

The takeaway from proposition 3 is that the optimal endogenous financial structure that emerges in an economy with enforcement frictions, positive underwriting fees and firms (borrowers) subject to exit shocks requires both long- and short-term debt. Firms (entrepreneurs)

that have to finance the initial investment ($p_k K$) that is partially pledgeable ($\phi < 1$), will optimally signed a long term contract and a sequence of short term contracts to finance working capital needs (wn).

An important difference with the literature of optimal financial contracts is that we get uniqueness in the implementation of the optimal contract. While an optimal contracting problem characterizes the sequence of optimal transfers between two parties, the implementation, i.e. how those transfers are translated into securities, is typically not unique. As we argued before, when $\kappa_C = 0$ financing K with long term debt or rolling over short term debt are equivalent ways of implementing the optimal sequence of transfers. The model then puts discipline on how to measure leverage, the fact that long and short term debt are used in equilibrium implies that we can define a notion of long and short leverage that can be transparently mapped to data. Since our assumptions the level of transaction costs don't affect qualitatively our results, we will assume $\kappa_C \to 0$ for the rest of the paper.

3.3 Perfect Enforcement Benchmark

Proposition 3 implies that the entry problem of a firm is, taking prices as given, choose a scale K, an employment sequence $\{n_t\}_{t=0}^{\infty}$, a sequence of short term debt $\{R_t\}_{t=0}^{\infty}$, and design long term debt with coupons structure $\{q_t\}_{t=0}^{\infty}$ and collateral policy $\{\nu_t\}_{t=0}^{\infty}$ in order to maximize the value of the firm given, the working capital constraint, the participation

constraint of financiers and the feasibility of coupons.

$$\Pi(z) = \max_{\{K,\{q_a,\nu_t,R_t,n_t\}_{a=0}^{\infty}\}} \sum_{t=0}^{\infty}(\beta\rho)^t \left[z\left(K^{\alpha}n_t^{1-\alpha}\right)^{\eta} - (1+r_f)R_t - q_t\right] + \beta(1-\rho)\sum_{t=0}^{\infty}(1-\nu_t)(\beta\rho)^t\phi p_k K$$

subject to

$$w_t n_t \leq R_t$$

$$p_k K(1+r_f) \leq \sum_{t=0}^{\infty}(\beta\rho)^a q_t + \beta(1-\rho)\sum_{t=0}^{\infty}(\beta\rho)^a\nu_t\beta(1-\rho)\phi p_k K$$

$$0 \leq q_t \leq z\left(K^{\alpha}n_t^{1-\alpha}\right)^{\eta} - (1+r_f)R_t$$

Without frictions, given K note that the design of long term debt $\{q_a, \nu_t\}_{a=0}^{\infty}$ does not affect the short term decisions n_t, R_t. The long-term decisions are "separable" from the short term decisions. Then, given a scale K, in a world of perfect enforcement ($\xi = 1$) firms are able to borrow the unconstrained level of short term debt $R^u(K,z)$ and to operate at the unconstrained level of employment $n^u(K,z)$ every period. Consequently, once the scale decision K has been made and the long term debt $\{q_a, \nu_t\}_{a=0}^{\infty}$ has been designed, the post-entry dynamic problem of a firm collapses to a a sequence of static problem given prices:

$$\max_{\{n_t,R_t\}} \quad z\left(K^{\alpha}n_t^{1-\alpha}\right)^{\eta} - (1+r_f)R_t$$

subject to

$$w_t n_t \leq R_t$$

Then, the unconstrained level of employment, short term debt and profits for a firm with scale K and productivity z are:

$$n^u(K,z) = K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}$$

$$R^u(K,z) = wn^u(K,z)$$

$$\pi^u(K,z) = [1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K,z)$$

The entry problem of a firm is then:

$$\Pi(z) = \max_{\{K,\{q_t,\nu_t\}_{t=0}^{\infty}\}} \sum_{a=0}^{\infty}(\beta\rho)^a\pi^u(K,z) + \beta(1-\rho)\sum_{a=0}^{\infty}(1-\nu_t)(\beta\rho)^a\phi p_k K - \sum_{t=0}^{\infty}(\beta\rho)^a q_t$$

subject to

$$p_k K(1+r_f) \leq \sum_{a=0}^{\infty}(\beta\rho)^a q_t + \beta(1-\rho)\sum_{a=0}^{\infty}(\beta\rho)^a\nu_t\beta(1-\rho)\phi p_k K$$

$$0 \leq q_t \leq [1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$$

Proposition 4 *With no enforcement frictions ($\xi = 1$):*

- *The scale choice (K) is independent of the maturity T and long term debt structure $\{q_t\}_{t=0}^{\infty}$.*
- *Maturity T and long term debt structure $\{q_t\}_{t=0}^{\infty}$ are indeterminate.*
- *The collateralization decision $\{\nu_t\}_{t=0}^{\infty}$ is indeterminate.*
- *The NPV of creating a firm with productivity z is given by:*

$$\Pi^u(z) = z^{\frac{1}{1-\eta}} \left[\frac{1-\eta}{1-\beta\rho}\right] \left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta}}$$

- *Employment of a firm with productivity z is:*

$$n^u(z) = z^{\frac{1}{[1-\eta]}} \left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{\eta\alpha}{1-\eta}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1-\eta+\eta\alpha\eta(1-\alpha)}{[1-\eta(1-\alpha)][1-\eta]}}$$

Without frictions, the firm's real choices (K, n) can be separated from the financial choices $\{q_t, \nu_t\}_{t=0}^{\infty}$. While the path of long term is indeterminate, the face value of long term debt during the firm's first year of operations is pinned down by K. Hence firm's value of long term debt at inception $p_k K(1+r_f)$ is closely related to firm size measured by employment. More productive firms start operations with higher long term debt.

3.4 Limited Enforcement and Capital Structure

When $\xi < 1$, firms have the possibility of engaging in strategic default and running away with $(1-\xi)$ of the realized revenues[16]. Financiers anticipating the possibility of firms engaging in strategic default, will only sign contracts that are enforceable, or equivalently, contracts that does not give incentives to the firm to engage in strategic default.

We will say that the productive and financial plan $\{q_t, R_t, \nu_t, K\}_{t=0}^{\infty}$ is enforceable if for all t:

$$(1-\xi)z\left[K^{\alpha}\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^{\eta} \leq \sum_{k=0}^{\infty}(\beta\rho)^k\left(z\left[K^{\alpha}\left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^{\eta} - (1+r_f)R_{t+k} - q_{t+k}\right) + \beta(1-\rho)\sum_{k=0}^{\infty}(1-\nu_{t+k})(\beta\rho)^t p_k \phi K$$

The presence of the enforceability breaks down the separability of the short $\{R_t\}_{t=0}^{\infty}$ and long term $\{q_t, \nu_t\}_{t=0}^{\infty}$ financial decisions. Now, given K two different designs of long term debt could potentially affect which short term debt contracts are feasible/enforceable. This interaction will rule out the indeterminacy of long term debt in the early stages of the firm, and when capital is high enough, the structure of long term is pinned down uniquely. we will discuss this in detail when we characterize the solution to the firm's problem.

Using proposition 3, the productive and financial plan, given prices, is the solution to following problem:

[16] Alternatively, we can interpret this as firm running away with all the revenues with probability $1-\xi$ and being caught with probability ξ

$$\max_{\{q_t,R_t,\nu_t,K\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}(\beta\rho)^t \left(z\left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)R_t - q_t \right) + \beta(1-\rho)\sum_{t=0}^{\infty}(1-\nu_t)(\beta\rho)^t p_k \phi K$$

subject to

$$(1+r_f)p_k K \leq \sum_{t=0}^{\infty}(\beta\rho)^t q_t + \beta(1-\rho)\sum_{t=0}^{\infty}(\beta\rho)^t \nu_t p_k \phi K$$

$$0 \leq q_t \leq z\left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)R_t \qquad \forall t$$

$$\nu_t \in \{0,1\}$$

$$(1-\xi)z\left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta \leq \sum_{k=0}^{\infty}(\beta\rho)^k \left(z\left[K^\alpha \left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)R_{t+k} - q_{t+k} \right) + \beta(1-\rho)\sum_{k=0}^{\infty}(1-\nu_{t+k})(\beta\rho)^k p_k \phi K \qquad \forall t$$

Where the first constraint is the participation constraint of financiers, the second constraint is the coupon payments constraint, the indivisibility of collateral and the enforceability constraint.

We solve this problem in two steps, first we show that for a given K we can characterize the financial contract. Note that once we fix K we are back in the canonical problem that the optimal dynamic contracting literature solves.[17] Then once we solve the optimal financial structure we can go back and study the optimal determination of K which will give us a theory of firm size and capital structure.

[17]See for example: Albuquerque and Hopenhayn (2004),Clementi and Hopenhayn (2006),DeMarzo and Fishman (2007),Verani (2018)

Proposition 5 *For any scale K, a solution to the problem features:*

- *Front-loading of the payment of long term debt*

$$q_t = \begin{cases} \pi_t(K) & \text{if} \quad t < T(K) \\ \in [0, \pi_t(K)] & \text{if} \quad t = T(K) \\ 0 & \text{if} \quad t > T(K) \end{cases}$$

Where $\pi_t(K) = z\left[K^{\alpha}\left(\frac{R_t(K)}{w}\right)^{1-\alpha}\right]^{\eta} - (1+r_f)R_t(K)$

- *Long term debt is collateralized.*

$$\nu_t = \begin{cases} 1 & \text{if} \quad t < T(K) \\ 0 & \text{if} \quad t \geq T(K) \end{cases}$$

- *Dividends are back-loaded*

$$d_t = \begin{cases} 0 & \text{if} \quad t < T(K) \\ \in [0, \pi^u(K)] & \text{if} \quad t = T(K) \\ \pi^u(K) & \text{if} \quad t > T(K) \end{cases}$$

Where $\pi^u(K)$ is the unconstrained level of profits given K.

$$R^u(K,z) = wK^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}$$

$$\pi^u(K,z) = [1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K,z)$$

- *Short term debt is strictly increasing for* $t \in [0, \hat{t}]$ *where* $\hat{t} \leq T(K)$

$$R_t(K) = \begin{cases} \left[\frac{(\beta\rho)^{T(K)+1-t}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \left\{\omega_0 + K^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\} R^u(K,z) & \text{if} \quad t \leq \hat{t} \\ R^u(K,z) & \text{if} \quad t > \hat{t} \end{cases}$$

Where $\omega_0 = [1-\eta(1-\alpha)]^{\frac{1}{\eta(1-\alpha)}}$ *and* $\omega_1(z) = \frac{[\beta(1-\rho)\phi p_k]^{\frac{1}{\eta(1-\alpha)}}}{z^{\frac{1}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}}$.

- *Employment* n_t *follows the dynamics of short term debt:*

$$n_t = \begin{cases} \frac{R_t(K)}{w} & \text{if} \quad t \leq \hat{t} \\ \frac{R^u(K)}{w} & \text{if} \quad t > \hat{t} \end{cases}$$

I relegate the full proof to the appendix, however most of the intuition of the proof can be grasped from the lagrange multiplier associated with the participation constraint of financiers λ. From the FOC we get:

$$\lambda = \left[1 + \sum_{j=0}^{\hat{t}} \psi_j \left[\frac{1}{\beta\rho}\right]^j\right]$$

Note that λ is the marginal cost of an extra unit of long term funding. By construction, optimal financial contract minimize λ, since the shadow price of the enforcement constraint ψ_j is weighted by $\left[\frac{1}{\beta\rho}\right]^j$, this imply that it is more costly to have $\psi_j > 0$ at later dates. Hence the optimal contract will try to relax the enforcement constraint as fast as possible. Intuitively, the main reason to front load debt payments is to increase the equity value of the firm in order to relax the short term borrowing constraint. Having a less tight short term borrowing constraint allows the firm to minimize the distortion on employment.

To match our empirical facts and study the financial constraints on young firms, having the capital structure fully pinned down till $\hat{t}$ is enough. A common feature of this class of models is that after the enforcement constraint cease to bind, then we are back to the perfect enforcement benchmark and we can have potentially many paths of coupons post $\hat{t}$ being consistent with the optimal contract.

At $\hat{t}+1$, the firm can borrow at the short term unconstrained level and the value of equity is:

$$V_{\hat{t}+1} = \sum_{k=0}^{\infty}(\beta\rho)^k \left([1-\eta(1-\alpha)] K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} - q_{\hat{t}+1+k} \right)$$
$$+ \beta(1-\rho)\sum_{k=0}^{\infty}(1-\nu_{\hat{t}+1+k})(\beta\rho)^k p_k \phi K$$

Therefore, if $\{q_t, R_t, \nu_t, K\}_{t=0}^{\infty}$ is a solution to the firms problem, then given the value of long term debt at $\hat{t}+1$ denoted $L_{\hat{t}+1}$, the sub-sequence $\{\{q_{\hat{t}+1+k}, \nu_{\hat{t}+1+k}\}_{k=0}^{\infty}\}$ must solve

$$V_{\hat{t}+1} = \max_{\{\{q_{\hat{t}+1+k}, \nu_{\hat{t}+1+k}\}_{k=0}^{\infty}\}} - \sum_{k=0}^{\infty}(\beta\rho)^k \left(q_{\hat{t}+1+k}\right) - \beta(1-\rho)p_k\phi K \sum_{k=0}^{\infty}(\beta\rho)^k \nu_{\hat{t}+1+k}$$

subject to

$$L_{\hat{t}+1} = \sum_{k=0}^{\infty}(\beta\rho)^k q_{\hat{t}+1+k} + \beta(1-\rho)p_k\phi K \sum_{k=0}^{\infty}(\beta\rho)^k \nu_{\hat{t}+1+k}$$

$$0 \leq q_t \leq [1-\eta(1-\alpha)] K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \qquad \forall t$$

$$\nu_t \in \{0,1\}$$

Where:

$$L_{\hat{t}+1} = \frac{(1+r_f)K}{(\beta\rho)^{\hat{t}+1}} - \sum_{t=0}^{\hat{t}} \frac{q_t}{(\beta\rho)^{\hat{t}+1-t}} - \beta(1-\rho)p_k\phi K \sum_{t=1}^{\hat{t}+1} \frac{1}{(\beta\rho)^t}$$

In a model without collateral $\phi = 0$, a perpetuity can be part of the optimal long term contract. However, the presence of collateral can help us to rule out potential cupons paths after $\hat{t}$. The next two propositions discusses the structure that collateral ($\phi > 0$) imposes to the model. A key assumption in what follows is that $\nu_t \in \{0, 1\}$. If we were in the world of $\nu \in [0, 1]$ we wouldn't get the results of proposition 6 and 7.

A perpetuity ($q_t > 0 \; \forall \; t$) will imply that the financier has a perpetual claim to the collateral, if the value of long term debt to be repaid after $\hat{t}$ is smaller than that, then a perpetuity cannot be optimal. The next proposition establishes a condition for that to happen:

Proposition 6 *Maturity is finite $T < \infty$ if the scale choice of the firm is "big enough":*

$$\left[\frac{[1-\eta(1-\alpha)]}{\beta(1-\rho)p_K\phi} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \left(\frac{1}{(\beta\rho)^{T(K)-\hat{t}(K)}} - 1\right)\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} < K$$

Proposition 6 establishes that no perpetuity can be part of the optimal contract but still leaves a multiplicity of paths for the coupons of long term debt. The next proposition gives conditions under which that multiplicity disappears:

Proposition 7 *For high enough pledgeability $\underline{\phi} < \phi$*

$$\underline{\phi} = \left[\frac{(1-\beta\rho)(1+r_f)[1-\eta(1-\alpha)] - \beta(1-\rho)\phi[1-\eta(1-\alpha)]}{\eta\alpha\beta\rho\beta(1-\rho)}\right]$$

Let $K(\xi, \phi)$ denote the solution to the problem when enforcement frictions are ξ and pledgeability is ϕ. Then, if ξ is such that $\underline{K}(\phi) < K(\xi, \phi)$, the contract defined in the propo-

sition 5 is unique.

Where:

$$\underline{K}(\phi) = \left[\frac{[1-\eta(1-\alpha)] z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{\beta\rho\beta(1-\rho)\phi p_k} \right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} < K^u$$

Proposition 7 says that when the collateral policy is $\{0, 1\}$, once the firm has pledge the ϕK increasing one period the maturity of long term debt implies also increasing by one period the rights of the financiers to size the collateral in case of unavoidable default. If the value of the insurance provided by the collateral is higher than the dividends generated by the firm in a given period, then it will be optimal to pay the long term debt as fast as possible and regain the rights to the collateral in case of unavoidable default.

Even when the results of proposition 7 does not apply, we will focus on the implementation that fully pays down long term debt[18].

[18]We will select this implementation on the basis that in an extended model where firms would have stochastic investment opportunities they would prefer to have more space in their balance sheet to take on debt in order to finance a new project.

3.5 Limited Enforcement and Optimal Scale

Applying the financial structure that emerges from proposition 5, the optimal scale K solves:

$$\max_{K} \ \frac{(\beta\rho)^{T(K)+1}}{1-\beta\rho}[1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}+\beta(1-\rho)\frac{(\beta\rho)^{T(K)+1}}{1-\beta\rho}p_k\phi K$$

subject to

$$\pi_t(K)=\begin{cases}\left(z\left[K^{\alpha}\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^{\eta}-(1+r_f)R_t\right) & \text{if} \quad t\leq\hat{t}(K)\\ [1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} & \text{if} \quad t>\hat{t}(K)\end{cases}$$

$$R_t(K)=\left[\frac{(\beta\rho)^{T(K)+1-t}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\left\{\omega_0+K^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}R^u(K,z)$$

$$\hat{t}(K)=\min\left\{x\ :\ \left[\frac{(\beta\rho)^{T(K)+1-x}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\left\{\omega_0+K^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}=1\right\}$$

$$T(K)=\min\left\{x\ :\ (1+r_f)p_kK\geq\sum_{t=0}^{x}(\beta\rho)^t\pi_t(K)+\beta(1-\rho)\sum_{t=0}^{x}(\beta\rho)^tp_k\phi K\right\}$$

The consequence of proposition 5 is that we can reduce the optimal capital structure decision and firm dynamics to the optimal choice of scale K. Optimality can be characterized as follows, consider two pairs of maturity and scale $(K_{(T)},T)$ and $(K_{(T+1)},T+1)$, then profits associated with these structures are:

$$\Pi(K_{(T)},T)=\frac{(\beta\rho)^{T+1}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)w}{\eta(1-\alpha)}K_{(T)}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}+\frac{(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)p_k\phi K_{(T)}$$

$$\Pi(K_{(T+1)},T+1)=\frac{(\beta\rho)^{T+2}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)w}{\eta(1-\alpha)}K_{(T+1)}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}+\frac{(\beta\rho)^{T+1}}{1-\beta\rho}\beta(1-\rho)p_k\phi K_{(T+1)}$$

Proposition 8 *$(K_{(T)}, T)$ is the optimal choice of scale-maturity if:*

$$\frac{\Pi(K_{(T+1)}, T+1)}{\Pi(K_{(T)}, T)} < 1 \qquad \frac{\Pi(K_{(T)}, T)}{\Pi(K_{(T-1)}, T-1)} \geq 1$$

$$\frac{(\beta\rho)[1-\eta(1-\alpha)]K_{(T+1)}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} + \beta(1-\rho)p_k\phi K_{(T+1)}}{(\beta\rho)[1-\eta(1-\alpha)]K_{(T)}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} + \beta(1-\rho)p_k\phi K_{(T)}} < \frac{1}{\beta\rho}$$

Proposition 8 captures the fundamental trade off involved in choosing long term debt maturity T (and consequently scale K_T). By increasing maturity of long term debt from T to $T+1$, the firm gives up the dividend that it would get in $T+1$, but increases the dividends perceived from $T+2$ onward due to the increase in scale $K_{T+1} > K_T$. In other words:

- Increasing maturity from T to $T+1$ costs loosing the dividend perceived at $T+1$:

$$(\beta\rho)^{T+1}[1-\eta(1-\alpha)]\frac{(1+r_f)w}{\eta(1-\alpha)}K_{(T)}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} + (\beta\rho)^T\beta(1-\rho)p_k\phi K_{(T)}$$

- Increasing maturity allows the firm to get to a higher scale and increase the value of dividends from $T+2$ onward:

$$\frac{(\beta\rho)^{T+2}}{1-\beta\rho}\left[[1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}\right]\left(K_{(T+1)}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K_{(T)}^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right)$$
$$+ \frac{(\beta\rho)^{T+1}}{1-\beta\rho}\beta(1-\rho)p_k\phi\left[K_{(T+1)} - K_{(T)}\right]$$

We calibrate the model (when $\xi = 1$) to the US economy and study how differences in enforcement affect the capital structure of firms over the ELC.

3.6 Scale-Maturity Choice and Financial Development

Firms in more developed economies have higher maturity of long term debt and higher scale compared to those in less developed economies.

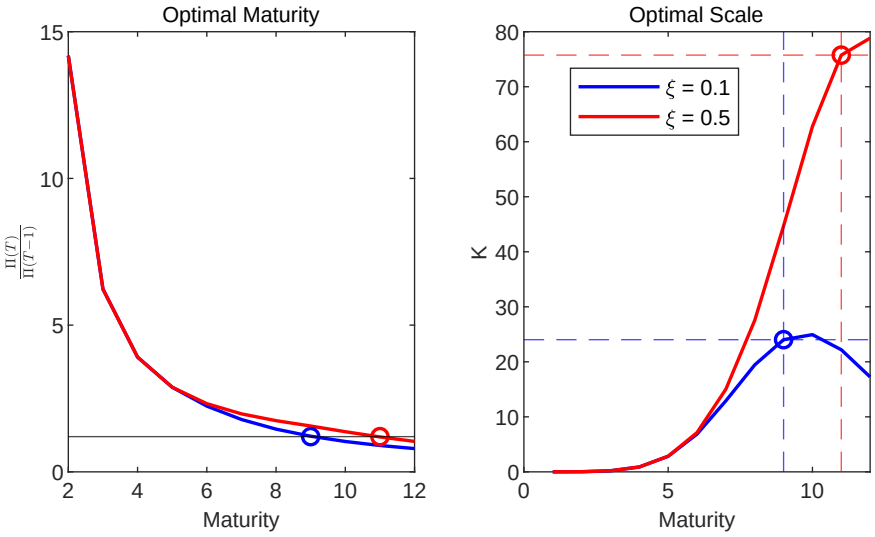


FIGURE 4: $\beta = 0.98, \rho = 0.85, \alpha = 0.35, \eta = 0.95, \phi = 0.4$

The optimal scale mapping is concave due to decreasing returns to scale ($\eta < 1$) and the enforcement of frictions. Consider the (K_T, T) maturity-scale choice, if the firm increases the maturity by one: i) it will be able to pay an extra coupon $q_{T+1} = \pi_{T+1}$ that increases the value of long-term debt. ii) A longer maturity will imply that the borrowing constraint during the early life cycle of firms will be tighter, which will be partially offset by the higher marginal productivity of labor. iii) For $t > \hat{t}$, the firm will be able to pay a higher coupon. Due to decreasing returns to scale and discounting, effects $i) - iii)$ will increase less than linearly while the cost of an extra unit of scale increases linearly. For firms in less developed countries, the tightening of the borrowing constraint due to a bigger scale-maturity choice will be stronger, and consequently, firms will find it too costly to implement the scale-maturity choice of more developed economies.

If we were to have an unproductive entry cost I_0 as is usual in the literature, we would have a counterfactual maturity prediction. Exogenous I_0 imply that the value of long-term debt that firms in economies with different degree of development ξ would have to pay is the same. However, firms in less developed economies will face tighter borrowing constraints during the early life cycle and will be able to pay a lower coupon, requiring them to increase maturity to compensate for the lower repayment they can generate early on.

3.7 Corporate Finance, Size and Financial Development

Proposition 9 *The capital structure of a firm of age a is characterized by (S_a, L_a, V_a) where S_a denotes short term liabilities, L_a long term liabilities and V_a net worth.*

$$L_a = \begin{cases} \sum_{t=a}^{T}(\beta\rho)^{t-a}\pi_a + \beta(1-\rho)\sum_{t=a}^{T-1}(\beta\rho)^{t-a}\phi p_k K & if\, a \leq T \\ 0 & if\, a > T \end{cases}$$

$$V_a = \begin{cases} \frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}[1-\eta(1-\alpha)]K_T^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} + \frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\beta(1-\rho)p_k\phi K_T & if\, a \leq T \\ \frac{1}{1-\beta\rho}[1-\eta(1-\alpha)]K_T^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} + \frac{1}{1-\beta\rho}\beta(1-\rho)p_k\phi K_T & if\, a > T \end{cases}$$

$$S_a = (1+r_f)\mathcal{M}(T-a,\xi,\phi,z)wK^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}$$

$$A_a = V_a + S_a + L_a$$

Where:

$$\mathcal{M}(T-a,\xi,\phi,z) = \min\left\{\left[\frac{(\beta\rho)^{T+1-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \left\{\omega_0 + K^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}\ ;\ 1\right\}$$

$$\pi_a = K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \left[[\mathcal{M}(T-a,\xi,\phi,z)]^{(1-\alpha)\eta} - (1-\alpha)\eta\mathcal{M}(T-a,\xi,\phi,z)\right]$$

Figure 5 captures the capital structure over the early life cycle of firms: long term debt decreases as short term debt increases. Moreover, long-term leverage is higher in more developed economies and target short-term leverage is lower. Note that total leverage mimics the behavior of long-term debt, in this sense the model is in line with the observation that empirical research make when studying total leverage as being equivalent to study long-leverage. The long-run level of short of total-leverage needs to be interpreted with care, our model only have unique predictions for the early life cycle (until the point short-leverage stops to grow while long leverage is decreasing). Another solution of the model would be to start paying dividends as soon as the early life cycle ends, this will imply that long-leverage becomes a straight line at the moment where the ELC ends, this would result in higher total leverage for more financially developed economies. In Albuquerque and Hopenhayn (2004), firms pay down the long-term debt they took up to finance an initial unproductive set-up cost and build up equity over time, which relaxes the constraint on short-term borrowing. The issue with models of endogenous borrowing constraints with unproductive set-up costs is that they are unsuitable to capture the international corporate finance facts. Financing a given set up cost in less developed economies is more expensive, since all firms are forced to pay this set up to operate, this result in higher long term leverage for firms in less developed economies. Our model show that by introducing endogenous scale choice, this problem can be solved.

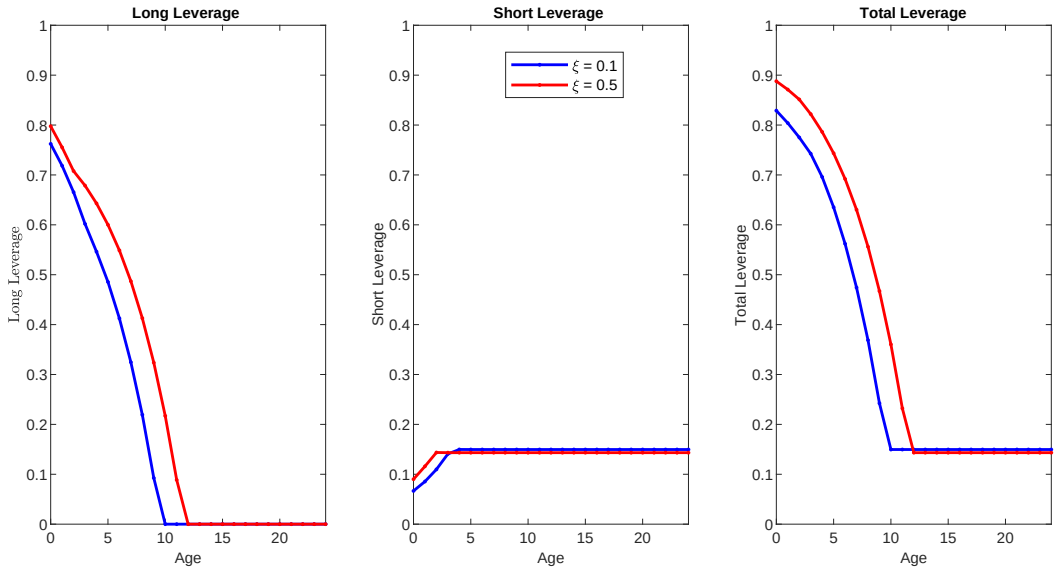


FIGURE 5: $\beta = 0.98, \rho = 0.85, \alpha = 0.35, \eta = 0.95, \phi = 0.4$

Figure 6 shows that short-term leverage is higher in less developed countries because while firms borrow less short term debt, the value of assets decreases even more than short-term debt with enforcement. This results in higher levels of short-term leverage at the end of ELC in less developed economies. Note that financial constraints are closely related to the duration of the ELC (period where short-term debt and leverage reach their maximum), and not to the maturity of long term debt. In fact, while the maturity of long-term debt is higher in more developed economies, the duration of the ELC is shorter. We corroborate this empirically in the next section.

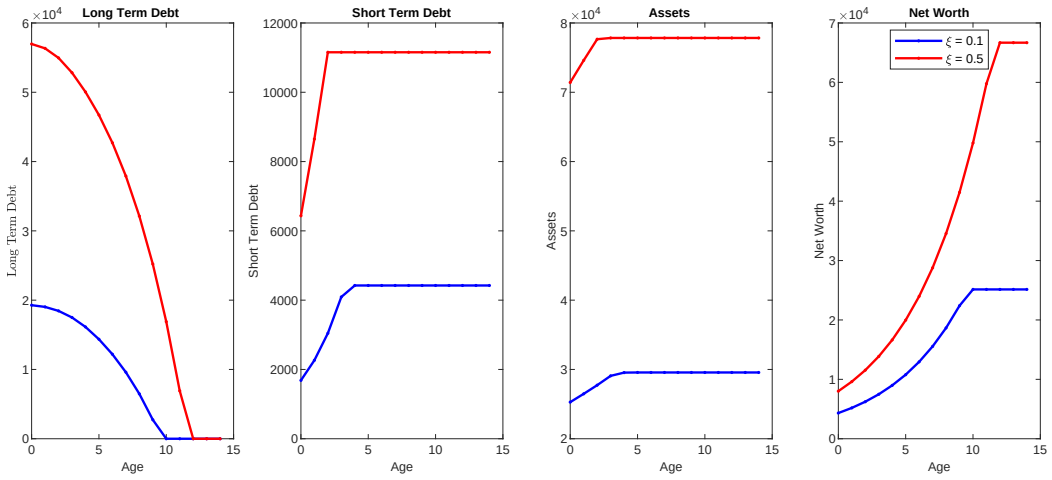


FIGURE 6: $\beta = 0.98, \rho = 0.85, \alpha = 0.35, \eta = 0.95, \phi = 0.4$

3.8 The Early Life Cycle of Firms and Financial Development

In order to test the effects of enforcement in the ELC of firms we divide our sample in two according to recovery rates from the World Banks Doing Business indicators. We will define as high (low) enforcement economies that have recovery rates above (below) the median. Figure 7 show the results, we can observe that the duration of ELC is higher in economies with lower enforcement. This is in line with our theory and will motivate how we construct our measure of financial constraints.

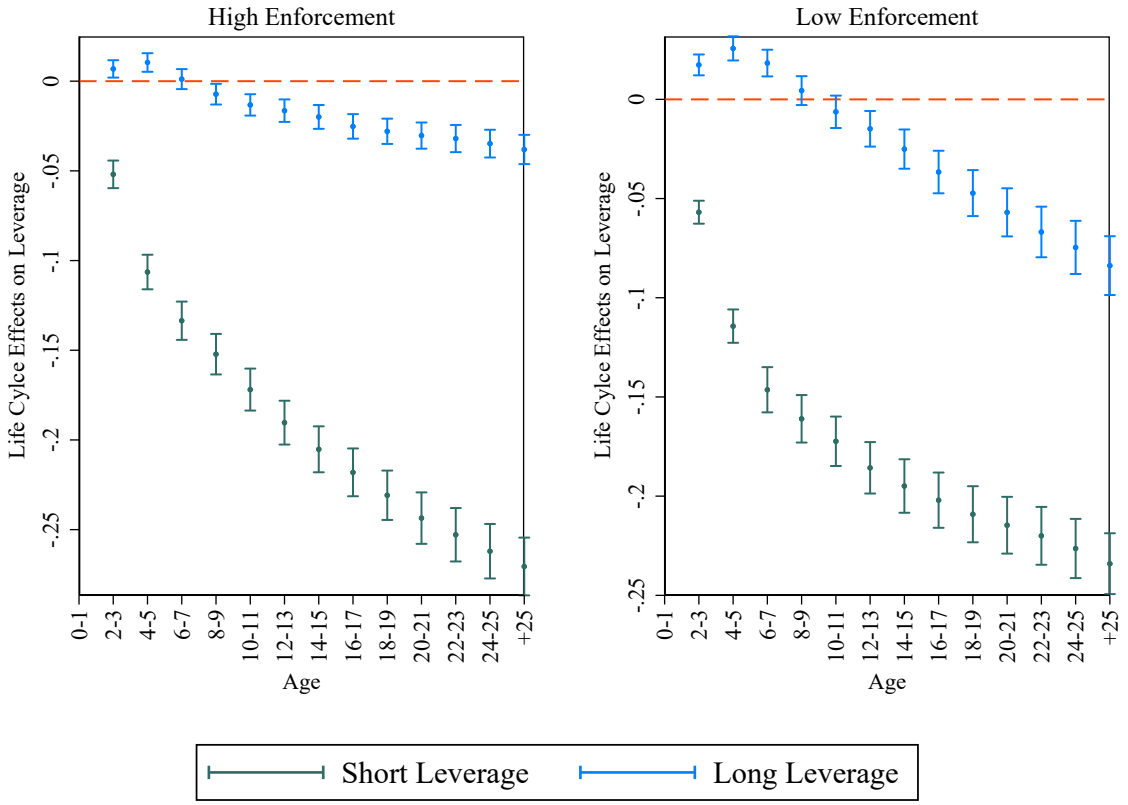


Figure 7: Life Cycle Patterns of Capital Structure of firms for which we have at least 3 observations and have positive debt. Confidence intervals are at 90% and standard errors are clustered at country-industry-year. Enforcement is defined as recovery rates (cents on the dollar) from World Banks Doing Business indicators. High enforcement means recovery rates above the median (62.3%)

4 Measurement of Financial Frictions: A Decomposition

Financial constraints are distortions in outcomes/firm decisions due to funding gaps. In our model, we can define a long and short term funding gap and the effects in initial investment (K), employment (n) and profits (π) induced by those funding gaps.

Let $R_a(K, z, \xi, p_k, \phi, \alpha)$ denote the level of short term debt obtained by a firm of age a, scale K, productivity z, in a country with enforcement ξ and capital prices p_k and in an industry characterized by capital intensity α and pledgeability ϕ. Let $R(z, 1, p_k, \phi, \alpha)$ denote the short term debt a firm would obtain when there are no enforcement frictions. We will define analogously $K(z, \xi, p_k, \phi, \alpha)$ and $K(z, 1, p_k, \phi, \alpha)$

The short and long funding gap (Γ_S, Γ_L) are defined as:

$$\Gamma_S = R(K(1, \cdot), z, 1, p_k, \phi, \alpha) - R_a(K(\xi, \cdot), z, \xi, p_k, \phi, \alpha)$$
$$\Gamma_L = p_k[K(1, z, p_k, \phi, \alpha) - K(\xi, z, p_k, \phi, \alpha)]$$

The employment, investment, profit and net-worth distortions (Λ_π, Λ_n) are defined as:

$$\Lambda_{n_a} = \rho^a \left[n^u(K, z, 1, p_k, \phi, \alpha) - n_a(K, z, \xi, p_k, \phi, \alpha)\right]$$
$$\Lambda_{\pi_a} = (\beta\rho)^a \left[\pi^u(K, z, 1, p_k, \phi, \alpha) - \pi_a(K, z, \xi, p_k, \phi, \alpha)\right]$$
$$\Lambda_{V_a} = (\beta\rho)^a \left[V^u(K, z, 1, p_k, \phi, \alpha) - V_a(K, z, \xi, p_k, \phi, \alpha)\right]$$

We will say that $\Lambda_x = \sum_{a=0}^{\infty} \Lambda_{x_a}$ is the total distortion of x

4.1 Distorted Outcomes

Our model allows us to study financial frictions by the distortion they generate in outcomes, we will show how the funding gaps relate to the distortions in employment, initial investment, profits and firm value. The literature mainly focus on the first three, we also calculate the effects on firm value since in our model the firm owns the capital.

Proposition 10 *The outcome gaps as a function of funding gaps are*

- *The total employment distortion* Γ_n *is a function of the short-term funding gap.*

$$\Lambda_n = \frac{1}{w}\frac{1}{1-\rho}\Gamma_{S,a}$$

- *The investment distortion* Γ_K *is a function of the long-term funding gap*

$$\Lambda_K = \Gamma_L$$

- *The total profits distortion*[19] Λ_π *is a function of the short-term funding gap*[20]

$$\begin{aligned}\Lambda_\pi &= \sum_{t=0}^{\infty}(\beta\rho)^t[\pi^u - \pi_t] \\ &= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t\left\{y^u(\xi,\cdot)\left[1-\left(1-\frac{\tilde{\Gamma}_{S,a}(K(\xi,\cdot))}{R(1,K(\xi,\cdot),\cdot)}\right)^{\eta(1-\alpha)}\right]-(1+r_f)\tilde{\Gamma}_{S,a}(K(\xi,\cdot))\right\} \\ &+ \frac{[1-\eta(1-\alpha)]}{1-\beta\rho}\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot))\end{aligned}$$

Where $y^u(\xi,\cdot) = zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R(1,K(\xi,\cdot),\cdot)}{w}\right)^{\eta(1-\alpha)}$ *are the revenues of a firm at end of the ELC.*

- *The total value of the firm distortion* Λ_V *is a function of the short- and long-term funding gap*

$$\Lambda_V = \Lambda_\pi + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma_L$$

In the model we made the stark assumption that all labor (variable cost) needs to be pay in-advance. This imply that we would be over-estimating the employment effect in industries where only a fraction of variable costs are financed up-front. The investment distortion is assuming that firms finance everything up-front with borrowed funds, if we were considering firms that star-up with some initial endowment, then we would be over-estimating the investment distortion with our model. However, since we know that capital is financed with long-term debt, if we observe the same scale and differences in long term debt we could infer the initial net-worth of the firm and adjust our measure accordingly. Moreover the distortion

[19] We can account for financial constraints as the forgone profits due to a distorted basket of inputs generated by financial constraints. See Cherchye et al. (2020) for a non-parametric implementation of this approach in a static context.

[20] Note that the unobservable target distortion (defined in the next section) is a part of the short term funding gap.

in total profits depends fully on the firm not having another investment opportunity in the future, while we can adjust our measure to net present values of distortion over early life cycle, it is likely that initial distortions will have persistent effects beyond the early life cycle that we should take into account.

4.2 Funding Gaps

The discussion in the previous paragraphs implies that the exact mapping from funding gaps to outcome gaps rely on details of the model. For this reason, we will focus on funding gaps that have magnitudes that are not model dependent. Consider the case of set-up costs (I_0) that are unproductive and exogenous, and let $(\tilde{\Gamma}_S(I_0), \tilde{\Gamma}_L(I_0))$ denote the funding gaps for that class of models [21], then:

$$\tilde{\Gamma}_{S,a}(I_0) = R(I_0, z, 1, p_k, \phi, \alpha) - R_a(I_0, z, \xi, p_k, \phi, \alpha)$$
$$\tilde{\Gamma}_L(I_0) = p_k \left[I_0 - \frac{L_0}{p_k(1+r_f)} \right]$$

By construction the long term funding gap can take two values $\tilde{\Gamma}_L \in \{0, p_k I_0\}$, it will be zero for firms that are active[22] and $p_k I_0$ for projects that are not undertaken. The funding gap $\tilde{\Gamma}_L$ maps to what the literature calls distortions along the "extensive margin", frictions in the environment increase the cost of paying a fixed entry cost upfront deterring frims to enter that would otherwise enter in an economy without frictions and lower financing costs. The short term funding gap can takes values in an interval $\tilde{\Gamma}_S \in [0, R(I_0, z, 1, p_k, \phi, \alpha)]$ and as firms mature it converges to zero. The funding gap $\tilde{\Gamma}_S$ maps to what the literature calls distortions along the "intensive margin"[23].

[21] Given that we have data for short term debt, the empirical implementation then would require us to establish/estimate/proxy the target short term debt in order to compute the funding gap.

[22] The value of long term debt obtained by active firms will be $L_0 = p_k(1+r_f)K$.

[23] If R is used to fund factors of production we will observe dispersion in the marginal productivity of those factors. In models with concave profits functions, old firms will be operating at their "efficient" level and

In our model, instead of fix entry costs, we have a flexible initial scale choice K so now the long term funding gap can take values in an interval $\Gamma_L \in [0, K(z, 1, p_k, \phi, \alpha)]$. Note that we could recover a notion of extensive margin if we assume that entering a sector requires a minimum scale $\underline{K}$ requirement. We will assume $\underline{K} = 0$ and focus on intensive margin distortions. Note that the long term funding gap $\Gamma_L = p_k[K(z, 1, p_k, \phi, \alpha) - K(z, \xi, p_k, \phi, \alpha)]$ is monotonic in ξ in a given industry with capital intensity α and collateralizability ϕ:

$$\frac{\partial \Gamma_L}{\partial \xi} = -p_k \frac{\partial K(z, \xi, p_k, \phi, \alpha)}{\partial \xi} < 0$$

Notice that to measure this funding gap, we need data from firms at inception; in general, having good quality data for firms at inception is hard. On the other hand, as shown above, if we are interested in employment creation, then we should focus on the short-term funding gap. We turn to this task now; first note that we can decompose the short-term funding gap in our model as follows:

$$\begin{aligned}
\Gamma_{S,a} &= R(K(1,\cdot), z, 1, p_k, \phi, \alpha) - R_a(K(\xi,\cdot), z, \xi, p_k, \phi, \alpha) \\
&= R(K(1,\cdot), z, 1, p_k, \phi, \alpha) - R(K(\xi,\cdot), z, 1, p_k, \phi, \alpha) \\
&+ R(K(\xi,\cdot), z, 1, p_k, \phi, \alpha) - R_a(K(\xi,\cdot), z, \xi, p_k, \phi, \alpha) \\
&= \left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] w z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \\
&+ R_a(K(\xi,\cdot), z, 1, p_k, \phi, \alpha) - R_a(K(\xi,\cdot), z, \xi, p_k, \phi, \alpha) \\
&= \underbrace{\Omega_S(K(\xi,\cdot))}_{\text{Short Term Debt Target Distortion}} + \underbrace{\tilde{\Gamma}_{S,a}(K(\xi,\cdot))}_{\text{Observable Short Funding Gap}}
\end{aligned}$$

The third lines follows from the property that firms converge to be unconstrained when they

young/constrained firms will be operating below the unconstrained level with a higher marginal productivity than their older counterparts.

mature $R(K(\xi,\cdot),z,1,p_k,\phi,\alpha) = R_a(K(\xi,\cdot),z,\xi,p_k,\phi,\alpha)$ for[24] $a \geq \hat{t}(\xi)$. And the last line follows from noting that $\tilde{\Gamma}_{S,a}(K(\xi,\cdot))$ would be the short term funding gap corresponding to a model with exogenous set up cost $I_0 = K(\xi,\cdot)$ and productivity $z[K(\xi,\cdot)]^{\eta\alpha}$. Since $R_a(1,K(\xi),\alpha,\phi,z) = R_{\hat{t}+j}(\xi,K(\xi),\alpha,\phi,z)$ for $a \geq \hat{t}(\xi)$, this implies that $\tilde{\Gamma}_S(K(\xi,\cdot))$ can, conditional on having enough data, be calculated from observables. However $\Omega_S(K(\xi,\cdot))$ is a counterfactual object, it captures the gap in the desired short term debt generated by the distortion in the scale choice due to being born in a country with enforcement frictions. Since $\Omega_S(K(\xi,\cdot))$ is fundamentally a cross country comparison we will need to establish a benchmark country where $\xi = 1$. We will follow the usual practice and assume that the US economy approximates the frictionless benchmark.

Proposition 11 *In a given industry (α,ϕ) the total funding gap $\Gamma_{S,a}$ and the short term debt target gap $\Omega_S(K(\xi,\cdot))$ are monotonic in financial development ξ but the observable short term funding gap is not $\tilde{\Gamma}_{S,a}(K(\xi,\cdot),\cdot)$*

An important takeaway from figure 8 is the behavior of the observable funding gap in different industries. In high capital intensity industries ($\alpha = 0.35$) we note that the observable funding gap is not monotonic in the level of enforcement frictions. This suggests that we cannot use how constrained firms are in less developed countries to bound how constrained are firms in more developed ones. More importantly, given that we typically don't have data for less developed economies, consider the following thought experiment: imagine we don't observe the blue lines (less developed economy), but we do have data for the light blue ones (more developed economy). Now, if we were to implement a program targeting young firms in high capital intensity industries ($\alpha = 0.35$) and we would use the size of the funding gap in a more developed economy ($\xi = 0.7$) and guess that the funding gap in a less developed economy ($\xi = 0.1$) should be higher, we would be over-estimating the size of the subsidized-loan that firms would require during the first years of operations. Now consider a second thought experiment, imagine that we have data for the high capital intensity industry for all levels of development, but we don't observe the less developed economies for the low capital intensity

[24] $\hat{t}(\xi)$ is the age at which the short term borrowing constraint cease to bind

industry ($\alpha = 0.25$). With this data, we would observe that the more developed economy over-estimates the (observable) short-term funding gap, can we infer that the funding gap in the more developed economy would also over-estimate the funding-gap in the low capital intensity ($\alpha = 0.25$) industry? The answer is no, for the low capital intensity industry the developed economy ($\xi = 0.7$) under-estimates the funding gap in less developed economies. Finally, focusing on a very high level of enforcement $\xi = 0.7$, we see that high capital intensity industries might be subject to observable funding gap distortion but no target distortion. Therefore we cannot infer from an observable distortion the existence of a target distortion. We can write the observable funding gap as follows:

$$\tilde{\Gamma}_S(K(\xi,\cdot)) = R^u(K(\xi,\alpha,\phi,z),\alpha,\phi,z)\left[1-\left[\frac{(\beta\rho)^{\hat{t}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}\right]$$

In order to give some intuition from the sources of non-monotinicty in the observable funding gap, consider the effects of increasing the level of development (ξ), this induces four effects:

1. A direct effect reducing the tightness of the borrowing constraint for a given scale $K(\xi,\cdot)$:

$$-\frac{1}{\eta(1-\alpha)}\left[\frac{(\beta\rho)^{\hat{t}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1-\eta(1-\alpha)}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}R^u(K(\xi,\cdot),\cdot)<0$$

2. An indirect effect that increases $K(\xi,\cdot)$ and the target funds:

$$\frac{\partial R^u(K(\xi,\cdot),\cdot)}{\partial K(\xi,\alpha,\phi,z)}\frac{\partial K}{\partial \xi}\left[1-\left[\frac{(\beta\rho)^{\hat{t}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}\right]>0$$

3. An increase in the debt maturity T and potentially on $\hat{t}(\xi)$ due to the increase in long term debt in order to finance $K(\xi,\cdot)$ that lowers the continuation value for a given scale.

$$\frac{\partial \hat{t}(\xi)}{\partial \xi}\frac{\ln(\frac{1}{\beta\rho})(\beta\rho)^{\hat{t}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\frac{1}{\eta(1-\alpha)}\left[\frac{(\beta\rho)^{\hat{t}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1-\eta(1-\alpha)}{\eta(1-\alpha)}}\left\{\omega_0+K(\xi,\cdot)^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}R^u(K(\xi,\cdot),\cdot)\geq 0$$

4. An indirect effect that increases $K(\xi,\cdot)$ and reduces the borrowing constraint due to a

higher collateral value:

$$-\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}R^u(K(\xi,\alpha,\phi,z),\alpha,\phi,z)\left[\frac{(\beta\rho)^{\hat{t}(\xi)-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}K(\xi,\cdot)^{\frac{1-\eta-\eta(1-\alpha)[1-\eta(1-\alpha)]}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z,\phi)<0$$

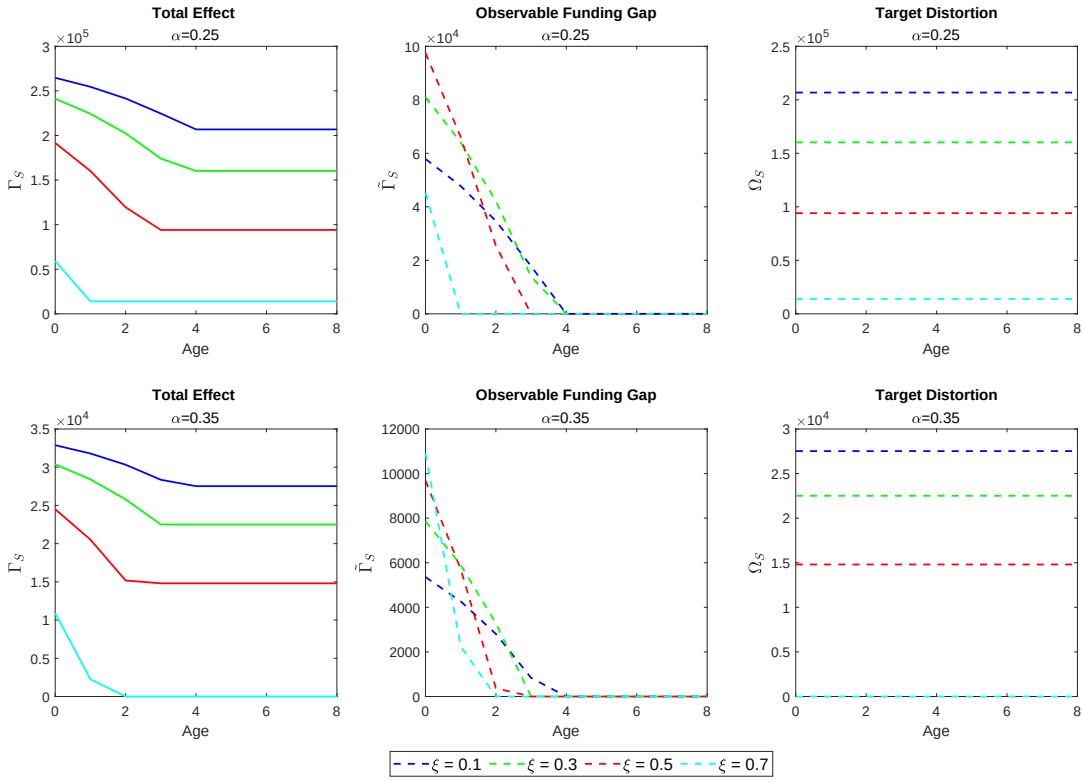


FIGURE 8: FUNDING GAPS ACROSS LEVELS OF DEVELOPMENT (ξ) FOR LOW AND HIGH CAPITAL INTENSITY ($\alpha \in \{0.25, 0.35\}$) INDUSTRIES.

Proposition 12 *Using a benchmark economy ($\xi = 1$), the unobservable target distortion can be inferred from the following three moments: initial scale K, initial long term debt (L_0) or mature size (n^u).*

$$\begin{aligned}\Omega_S(K(\xi,\cdot)) &= \left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \\ &= \left[\left[\frac{L_0(1)}{p_k(1+rf)}\right]^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - \left[\frac{L_0(\xi)}{p_k(1+rf)}\right]^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \\ &= w\left[n^u(1,K(1),\alpha,\phi,z) - n^u(1,K(\xi),\alpha,\phi,z)\right]\end{aligned}$$

While data at firm inception is difficult to come by, the level of employment at the end of a firm's early life cycle is not. We use this insight in the next section to suggest how to measure the severity of financial constraints.

4.3 Measuring Total Distortions

The conditionally unconstrained level of short term debt demanded by firms in a distorted economy reflects technological characteristics of the production process and institutional quality of the country. While the level is model free, how to identify it is not, we need to rely on our model to pin down $\hat{t}$. The strategy derived from our model is to identify the last period when age positively affected short-term leverage. We streamline below the empirical implementation of our approach:

1. Estimate the duration of the early life cycle $\hat{t}$ defined as the last period where age has a positive effect on leverage in a country and industry.

2. Use the estimate to define the *conditionally unconstrained*[25] average short-term debt in a given country-industry.

[25] Conditional on the distorted capital scale $K(\xi,\cdot)$.

3. Compute NPV of the sum of deviations (observed short term debt minus the target defined in the previous step). This is an estimate of the observable funding gap.

4. In order to compute the target distortion, compute the difference between the average employment of firms of age $a = \hat{t}$ in a given industry-country with the average employment of a firm in the US that is of age $a \leq \hat{t}$.

5. Scale the employment gap by the wage in the country. This is an estimate of the unobservable target distortion.

6. Since we want to consider the scale distortion only through the early life cycle of the firm, scale the previous estimate by $\frac{1-(\beta\rho)^{\hat{t}}}{1-\beta\rho}$

7. The estimate for the average funding gap in an industry-country is then the summation of the estimates we got in 3 and 6.

Our methodology can be applied to measure the severity of financial constraint for a given firm, the average firm in an industry, and the average firm in a country or group of countries. To illustrate the methodology we compute the severity of financial constraints for the average firm in high/low enforcement economies in our sample. Table 1 shows that in lower enforcement economies: i) The duration of the early life cycle is higher. ii) The target scale (proxied by $n_{\hat{t}}$) is lower despite looking at firms two years older than in high enforcement economies. iii) The target level of short-term debt is higher in less developed economies[26] iv) Note that in line with the simulations, the unobservable scale distortion during the early life cycle of firms[27] is relatively more important than the observable short-term funding gap at lower enforcement levels. This suggests that in developed economies inferring financial constraints using the observable funding gap will result in a less biased measure.

[26]This is not in line with our simulations. Our model predicts higher short-term leverage but lower short-term debt. Differences in industry composition can drive this discrepancy in the data.

[27]Scale distortion adjusted to the duration of the early life cycle.

The relative importance of the unobservable target distortion Ω_S at lower levels of develop-

Table 1: Severity of Financial Constraints

	High Enforcement	Low Enforcement
$\hat{t}$	5	7
$R_{\hat{t}}$	403947.5	525969.1
$n_{\hat{t}}$	16.01	14.59
n^{US}	16.9	16.9
w	34087	25883
R_1	279467.9	238098.5
R_2	256712.4	277746
R_3	310945.5	327072.1
R_4	343335.1	383022.1
R_5	403947.5	419104.8
R_6	-	481067.3
R_7	-	525969.1
$\tilde{\Gamma}_S$	917,979	1,239,722
Ω_S	108,801	258,381
Γ_S	1,026,780	1,498,104
$\frac{\tilde{\Gamma}_S}{\Gamma_S}$	89.4%	82.8%
$\frac{\Omega_S}{\Gamma_S}$	10.6%	17.2%

Notes: w denote annual wages in dollars (Source: International Labor Organization). n^{us} is the average employment over 1998-2017 of firms in US of age 6 to 10 with less than 500 employees. The calculations for the High and Low enforcement economies also consider firms with 500 employees or less (Source: US Census, Business Dynamics Statistics.). $n_{\hat{t}}$ denote the average employment of firms at age $\hat{t}$. The discount factor used comes from assuming $\beta = 0.98$ and a survival probability of $\rho = 0.85$.

ment points towards the relevance of distortions in firms' initial scale choice. In turn, scale distortions are closely linked to the long-term funding gap Γ_L. However, this does not imply that long-term funding is scarce in less developed economies; in our model, we assume that the finance supply is inelastic. In this sense, the size of the funding gap reveals the optimal behavior of firms and financiers, given the enforcement frictions in the environment. Firms don't get access to higher debt levels because they are not enforceable. While policymakers can affect the long-term funding gap by giving a lump-sump transfer upfront to firms at

inception, they can also affect it more subtly. For example, policymakers can announce a credit program that promises firms subsidized short-term credit[28]. If this policy is credible for firms and financiers (for the duration of the early life cycle), then this would affect the long-term debt that firms access in the market and their scale choice.

5 Conclusion

How severe are financial constraints in young firms? This paper proposes a methodology to estimate the full effect of financial constraints emerging from enforcement problems. The paper has four messages. First, studying long and short leverage, instead of total leverage, is important to study the severity of financial constraints over the early life cycle of firms. Moreover, the duration of the negative co-movement of short and long leverage defines the duration of the early life cycle. Second, a model of endogenous borrowing constraints that has any hope to match the international corporate finance stylized facts requires firms to make a productive initial investment choice. Third, a consequence of the initial productive choice of firms is that short-term borrowing constraints don't fully capture the severity of financial frictions in the economy. Fourth, to measure the full effect of financial frictions over the early life cycle of firms, we require international comparisons and the estimation of the duration of the early life cycle of firms.

We finish the paper by conjecturing future avenues of research. We illustrated the methodology by comparing the average firms in the relatively high and low enforcement economies in our sample. The model is flexible enough to implement our methodology across industries and countries; We leave for future work the estimation of the severity of financial constraints across industries and countries. In preliminary simulations, our model indicates that the ranking of the severity of financial constraints could not be stable across development levels.

[28] In our model, this would be that firms get access to short-term debt at an interest rate below r_f.

We take this as a theoretical possibility and leave the empirical plausibility of this for future work.

While our model can be used to calculate the funding gap that a credit program could target, note that it is only an ex-ante calculation to approximate the severity of financial constraints. Since we abstract from any asymmetric information problems, we caution against using the model to evaluate welfare gains from credit programs. However, we highlighted that credible announcements of subsidized short-term credit could reduce the long-term debt funding gap and increase the initial scale choice of firms. Exploring this implementation of a credit program might be interesting for international organizations that offer subsidized credit and require institutional reforms. Intuitively, our model suggests that one can get all the effects of a long-term debt program[29] with a credible subsidized short-term program sustained across time. However, the implementation using short-term debt gives a threat of discontinuing the program to international organizations. This threat could discipline the local government to implement institutional reforms.

The work of Lian and Ma (2021) and Cloyne et al. (2019) suggest that firms in their early life cycle are different: they rely more on asset-backed lending[30]. Building on the intuition that young firms are fundamentally different, we studied how the capital structure of a young firm buying capital with collateralized long-term debt reveals information about the severity of financial constraints. But what happens beyond the early life cycle? Here we considered only one investment opportunity; however, in a model where new projects arrive at later stages (for example, the opportunity of opening a new establishment), we could study whether the initial scale distortion disappears or persists. The literature on firm size and development (Hsieh and Klenow, 2014; Poschke, 2018; Bento and Restuccia, 2021) would suggest the ex-

[29]Conditional on financiers having deep pockets, or, in other words, conditional on the funding gap being purely driven by enforceability.

[30]While Lian and Ma (2021) documents this directly, Cloyne et al. (2019) find that this difference has important economic consequences, they show that no dividend paying and young firms investment account for more than %20 of the response of investment to monetary policy while accounting for less than %10 of aggregate investment;

istence of permanent effects of initial distortions. However, if collateral seems to be mainly relevant during the early life cycle, from which channel would the initial scale distortion be operating if it is going to be persistent beyond the early life cycle?

We leave the extension of this model in the previously highlighted directions for future work.

References

Albuquerque, Rui and Hugo A. Hopenhayn, "Optimal lending contracts and firm dynamics," *Review of Economic Studies*, 2004, *71* (2), 285–315.

Almeida, Heitor, Murillo Campello, and Michael S. Weisbach, "The Cash Flow Sensitivity of Cash," *The Journal of Finance*, aug 2004, *59* (4), 1777–1804.

Antzoulatos, Angelos A., Kostas Koufopoulos, Costas Lambrinoudakis, and Emmanuel Tsiritakis, "Supply of capital and capital structure: The role of financial development," *Journal of Corporate Finance*, 2016, *38*, 166–195.

Arellano, Cristina, Yan Bai, and Jing Zhang, "Firm dynamics and financial development," *Journal of Monetary Economics*, 2012, *59* (6), 533–549.

Ayyagari, Meghana, Asli Demirguc-Kunt, and Vojislav Maksimovic, "Who creates jobs in developing countries?," *Small Business Economics*, 2014, *43* (1), 75–99.

Bae, Kee Hong and Vidhan K. Goyal, "Creditor rights, enforcement, and bank loans," *Journal of Finance*, 2009, *64* (2), 823–860.

Bajgar, Matej, Giuseppe Berlingieri, Sara Calligaris, Chiara Criscuolo, and Jonathan Timmis, "Coverage and representativeness of Orbis data," *OECD Science, Technology and Industry Working Papers*, 2020, *2020* (6), 1–4,7–63.

Beck, Thorsten, Asli Demirgüç-Kunt, and Vojislav Maksimovic, "Financing patterns around the world: Are small firms different?," *Journal of Financial Economics*, 2008, *89* (3), 467–487.

Bento, Pedro and Diego Restuccia, "On average establishment size across sectors and countries," *Journal of Monetary Economics*, 2021, *117*, 220–242.

Booth, Laurence, Varouj Aivazian, Asli Demirguc-Kunt, and Vojislav Maksimovic, "Capital structures in developing countries," *Journal of Finance*, 2001, *56* (1), 87–130.

Cherchye, Laurens, Bram De Rock, Annalisa Ferrando, Klaas Mulier, and Marijn Verschelde, "Identifying financial constraints," *ECB Working Paper*, 2020.

Clementi, Gian Luca and Hugo A. Hopenhayn, "A Theory of Financing Constraints and Firm Dynamics," *The Quarterly Journal of Economics*, 2006, *121* (February), 229–265.

Cloyne, James, Clodomiro Ferreira, Maren Froemel, and Paolo Surico, "Monetary Policy, Corporate Finance and Investment," *SSRN Electronic Journal*, 2019.

Cooley, Thomas F., Ramon Marimon, and Vincenzo Quadrini, "Aggregate Consequences of Limited Contract Enforceability Enforceability Ramon Marimon Vincenzo Quadrini," *Journal of Political Economy*, 2004, *112* (4), 817–847.

Dao, Mai Chi and Lucy Qian Liu, "Finance and Employment in Developing Countries," *IMF Working Papers*, 2017, *17* (189).

de Jong, Abe, Rezaul Kabir, and Thuy Thu Nguyen, "Capital structure around the world: The roles of firm- and country-specific determinants," *Journal of Banking and Finance*, 2008, *32* (9), 1954–1969.

DeMarzo, Peter M. and Michael J. Fishman, "Agency and Optimal Investment Dynamics," *Review of Financial Studies*, jan 2007, *20* (1), 151–188.

Demirgüç-Kunt, Asli and Vojislav Maksimovic, "Law, Finance, and Firm Growth," *The Journal of Finance*, dec 1998, *53* (6), 2107–2137.

_ **and** _ , "Institutions, financial markets, and firm debt maturity," *Journal of Financial Economics*, dec 1999, *54* (3), 295–336.

Derrien, François, Jean-Stéphane Mésonnier, and Guillaume Vuillemey, "Set-Up Costs and the Financing of Young Firms," *SSRN Electronic Journal*, 2021, (December).

Dinlersoz, Emin, Sebnem Kalemli-Ozcan, Henry Hyatt, and Veronika Penciakova, "Leverage over the Firm Life Cycle, Firm Growth, and Aggregate Fluctuations," *Federal Reserve Bank of Atlanta, Working Papers*, 2019, (July).

Fan, Joseph P H, Sheridan Titman, and Garry Twite, "An International Comparison of Capital Structure and Debt Maturity Choices," *Journal of Financial and Quantitative Analysis*, feb 2012, *47* (1), 23–56.

Fazzari, Steven M, R Glenn Hubbard, and Bruce C Petersen, "Financing Constraints and Corporate Investment," *Brookings Papers on Economic Activity*, 1988, *19* (1), 141–206.

Graham, John R and Campbell R Harvey, "The theory and practice of corporate finance: evidence from the field," *Journal of Financial Economics*, 2001, *60* (2-3), 187–243.

Graham, John R. and Mark T. Leary, "A Review of Empirical Capital Structure Research and Directions for the Future," *Annual Review of Financial Economics*, 2011, *3* (1), 309–345.

Hadlock, Charles J. and Joshua R. Pierce, "New evidence on measuring financial constraints: Moving beyond the KZ index," *Review of Financial Studies*, 2010, *23* (5), 1909–1940.

Haltiwanger, John, Ron S. Jarmin, and Javier Miranda, "Who Creates Jobs? Small versus Large versus Young," *Review of Economics and Statistics*, may 2013, *95* (2), 347–361.

Hart, By Oliver and John Moore, "Debt and Seniority: An Analysis of the Role of Hard Claims in Constraining Management," *American Economic Review*, 1995, *85* (3), 567–585.

Haselmann, Rainer, Katharina Pistor, and Vikrant Vig, "How law affects lending," *Review of Financial Studies*, 2010, *23* (2), 549–580.

Hsieh, Chang-Tai and Peter J. Klenow, "The Life Cycle of Plants in India and Mexico *," *The Quarterly Journal of Economics*, aug 2014, *129* (3), 1035–1084.

Kaplan, S. N. and Luigi Zingales, "Do Investment-Cash Flow Sensitivities Provide Useful Measures of Financing Constraints?," *The Quarterly Journal of Economics*, feb 1997, *112* (1), 169–215.

Kermani, Amir and Yueran Ma, "Asset Specificity of Nonfinancial Firms," *The Quarterly Journal of Economics*, jul 2022, pp. 1–60.

Kochen, Federico, "Finance Over the Life Cycle of Firms," *Mimeo, New York University*, 2022, (February).

Lamont, Owen, Christopher Polk, and Jesús Saá-Requejo, "Financial constraints and stock returns," *Review of Financial Studies*, 2001, *14* (2), 529–554.

Lian, Chen and Yueran Ma, "Anatomy of Corporate Borrowing Constraints," *Quarterly Journal of Economics*, 2021, *136* (1), 229–291.

Midrigan, Virgiliu and Daniel Yi Xu, "Finance and misallocation: Evidence from plant-level data," *American Economic Review*, 2014, *104* (2), 422–458.

Moreira, Sara, "Firm Dynamics, Persistent Effects of Entry Conditions, and Business Cycles," 2019.

Poschke, Markus, "The firm size distribution across countries and skill-biased change in entrepreneurial technology," *American Economic Journal: Macroeconomics*, 2018, *10* (3), 1–41.

Pugsley, BW and E Hurst, "What Do Small Businesses Do?," *Brookings Papers on Economic Activity*, 2011, (Fall), 73–128.

Qian, Jun and Philip E. Strahan, "How laws and institutions shape financial contracts: The case of bank loans," *Journal of Finance*, 2007, *62* (6), 2803–2834.

Rajan, RAGHURAM G. and LUIGI Zingales, "What Do We Know about Capital Structure? Some Evidence from International Data," *The Journal of Finance*, 1995, *50* (5), 1421–1460.

Verani, Stéphane, "Aggregate consequences of dynamic credit relationships," *Review of Economic Dynamics*, jul 2018, *29*, 44–67.

Welch, Ivo, "Two common problems in capital structure research: The financial-debt-to-asset ratio and issuing activity versus leverage changes," *International Review of Finance*, 2011, *11* (1), 1–17.

Whited, Toni M. and Guojun Wu, "Financial Constraints Risk," *Review of Financial Studies*, 2006, *19* (2), 531–559.

World Bank, *Global Financial Development Report 2015/2016: Long-Term Finance*, The World Bank, aug 2015.

6 Appendix

6.1 Data Description and Alternative Empirical Specifications

Table 2:
Enforcement (Recovery Rates)

Low Enforcement	Recovery Rates
Estonia	38.56
Greece	39.88
Hungary	39.91
France	56.75
Slovenia	59.26
Italy	60.56
High Enforcement	
Portugal	69.87
Spain	73.52
Sweden	76.29
Germany	81.86
Belgium	87.50
Finland	88.87

Source: Doing Business Database, World Bank. Recovery rates are defined as cents on the dollar recovered by secured creditors through judicial reorganization, liquidation or debt enforcement (foreclosure or receivership) proceedings. The calculation takes into account the outcome: whether the business emerges from the proceedings as a going concern or the assets are sold piecemeal.

TABLE 3:
SUMMARY STATISTICS OF FIRM CHARACTERISTICS

	Young Firms					Old Firms				
	Long Leverage	Short Leverage	Age	Employment	N	Long Leverage	Short Leverage	Age	Employment	N
Belgium	0.317	0.230	4.282	59.437	1540	0.236	0.201	30.306	144.514	23821
	(0.253)	(0.210)	(1.452)	(100.686)		(0.209)	(0.191)	(18.568)	(668.328)	
Estonia	0.313	0.184	4.138	7.182	10367	0.250	0.151	14.418	16.901	25893
	(0.219)	(0.162)	(1.462)	(32.934)		(0.205)	(0.146)	(6.244)	(56.393)	
Finland	0.398	0.179	4.118	17.699	9478	0.327	0.152	21.778	32.309	44044
	(0.233)	(0.129)	(1.456)	(76.350)		(0.219)	(0.129)	(13.606)	(169.575)	
France	0.346	0.185	4.225	19.441	25982	0.169	0.136	24.511	73.752	156828
	(0.227)	(0.136)	(1.419)	(115.330)		(0.159)	(0.140)	(14.502)	(809.535)	
Germany	0.390	0.166	4.331	252.496	2272	0.283	0.153	41.419	554.288	31370
	(0.253)	(0.174)	(1.440)	(611.531)		(0.216)	(0.164)	(41.503)	(4866.912)	
Greece	0.262	0.272	4.563	35.892	1310	0.247	0.224	26.122	72.622	22501
	(0.206)	(0.212)	(1.334)	(71.588)		(0.204)	(0.190)	(15.748)	(267.882)	
Hungary	0.291	0.200	4.341	88.100	1849	0.186	0.170	18.961	152.434	15769
	(0.247)	(0.184)	(1.430)	(361.352)		(0.179)	(0.159)	(9.716)	(665.259)	
Italy	0.333	0.332	4.260	22.180	87141	0.257	0.312	23.019	49.384	482327
	(0.238)	(0.250)	(1.407)	(124.894)		(0.204)	(0.238)	(13.752)	(395.162)	
Portugal	0.398	0.215	4.083	14.370	22343	0.307	0.183	22.362	42.157	119223
	(0.247)	(0.196)	(1.504)	(33.348)		(0.216)	(0.167)	(14.338)	(181.730)	
Slovenia	0.378	0.207	3.981	6.832	29742	0.308	0.171	17.324	15.951	112192
	(0.247)	(0.189)	(1.518)	(31.608)		(0.228)	(0.166)	(6.602)	(103.615)	
Spain	0.399	0.182	4.248	18.616	84100	0.313	0.154	20.093	42.156	520001
	(0.256)	(0.180)	(1.432)	(177.656)		(0.241)	(0.161)	(11.194)	(373.392)	
Sweden	0.416	0.168	4.144	7.155	18633	0.359	0.120	21.850	15.560	68167
	(0.239)	(0.131)	(1.396)	(19.554)		(0.231)	(0.121)	(13.578)	(62.302)	

Notes: Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth.

6.2 Alternative Empirical Specification

6.2.1 Early Life Cycle

We estimate the following model where $\omega_{c,s,t}$ denote country-sector-time fixed effect.

$$\begin{aligned} y_{i,c,t} =& \omega_{c,s,t} + \beta_1 AGE_{i,t} + \beta_1^{ELC} AGE_{i,t} \times ELC_{i,c,t} + \beta_2 \ln EMP_{it} + \beta_2^{ELC} \ln EMP_{it} \times ELC_{i,c,t} \\ &+ \beta_3 COL_{i,c,t} + \beta_3^{ELC} COL_{i,c,t} \times ELC_{i,c,t} + \beta_4 PROFIT_{i,t} + \beta_4^{ELC} PROFIT_{i,t} \times ELC_{i,c,t} \\ &+ \beta_5 PROD_{i,t} + \beta_5^{ELC} PROD_{i,t} \times ELC_{i,c,t} + \epsilon_{i,c,t} \end{aligned}$$

This is equivalent to run the regression in two sub-samples of young and old firms, the above specification has the advantage that presents in a convenient way a hypothesis test of the coefficients for young and old firms being different.

TABLE 4:
LEVERAGE PATTERNS OF YOUNG FIRMS

	Short Leverage	Long Leverage
Young firms (Age 0-10)		
Age	-0.0009***	-0.0013***
	(0.0000)	(0.0000)
Age× ELC	0.0022***	-0.0152***
	(0.0003)	(0.0007)
Size	0.0013***	-0.0129***
	(0.0004)	(0.0003)
Size× ELC	0.0013***	-0.0023***
	(0.0005)	(0.0005)
Collateral	-0.1502***	0.2484***
	(0.0044)	(0.0036)
Collateral× ELC	0.0065***	0.0447***
	(0.0025)	(0.0036)
Profitability	-0.2455***	-0.2014***
	(0.0072)	(0.0050)
Profitability× ELC	0.0224***	-0.0867***
	(0.0062)	(0.0062)
Productivity	0.0078***	-0.0156***
	(0.0006)	(0.0004)
Productivity× ELC	0.0001	0.0096***
	(0.0002)	(0.0004)
Observations	1,647,120	1,647,120
Adjusted R^2	0.2340	0.2158

Notes: ELC (Early Life Cycle) is a dummy variable that takes the value 1 for firms that are 7 years old or younger. Size is defined as the log employment, Collateral as tangible assets over total assets. All regressions include Industry-Year-Country fixed effects. Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth. We consider firms for wich we have at least 3 observations and have positive debt.

$$
\begin{aligned}
y_{i,c,t} =& \omega_i + \beta_1 AGE_{i,t} + \beta_1^{ELC} AGE_{i,t} \times ELC_{i,c,t} + \beta_2 \ln EMP_{it} + \beta_2^{ELC} \ln EMP_{it} \times ELC_{i,c,t} \\
&+ \beta_3 COL_{i,c,t} + \beta_3^{ELC} COL_{i,c,t} \times ELC_{i,c,t} + \beta_4 PROFIT_{i,t} + \beta_4^{ELC} PROFIT_{i,t} \times ELC_{i,c,t} \\
&+ \beta_5 PROD_{i,t} + \beta_5^{ELC} PROD_{i,t} \times ELC_{i,c,t} + \epsilon_{i,c,t}
\end{aligned}
$$

Table 5:
Leverage Patterns of Young Firms

	Short Leverage	Long Leverage
Age	-0.0029***	-0.0044***
	(0.0002)	(0.0002)
Age× ELC	0.0041***	-0.0216***
	(0.0004)	(0.0008)
Size	0.0157***	-0.0112***
	(0.0008)	(0.0010)
Size× ELC	-0.0014***	-0.0023***
	(0.0004)	(0.0005)
Collateral	-0.0982***	0.2909***
	(0.0045)	(0.0051)
Collateral× ELC	-0.0128***	0.0546***
	(0.0021)	(0.0031)
Profitability	-0.2033***	-0.1948***
	(0.0062)	(0.0058)
Profitability× ELC	0.0612***	0.0032
	(0.0057)	(0.0064)
Productivity	0.0116***	-0.0115***
	(0.0007)	(0.0008)
Productivity× ELC	-0.0009***	0.0104***
	(0.0002)	(0.0005)
Observations	1,562,247	1,562,247
Adjusted R^2	0.7474	0.7325

Notes: ELC (Early Life Cycle) is a dummy variable that takes the value 1 for firms that are 7 years old or younger. Size is defined as the log employment, Collateral as tangible assets over total assets. All regressions include firm fixed effects. Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth. We consider firms for wich we have at least 3 observations and have positive debt.

6.2.2 ELC and Financial Development

Table 6:
Leverage Patterns of Young Firms and Financial Development

	Short Leverage	Long Leverage
Age	0.00835***	-0.02580***
	(0.00101)	(0.00119)
FD × Age	-0.00006***	0.00008***
	(0.00001)	(0.00001)
Size	0.00454***	-0.01665***
	(0.00039)	(0.00042)
Collateral	-0.15075***	0.28785***
	(0.00177)	(0.00232)
Profitability	-0.23712***	-0.29815***
	(0.00358)	(0.00457)
Productivity	0.00771***	-0.01513***
	(0.00059)	(0.00062)
Observations	233,921	233,921
Adjusted R^2	0.21105	0.22628
All firms		
Age	-0.00087***	0.00048***
	(0.00004)	(0.00005)
FD × Age	0.00000	-0.00002***
	(0.00000)	(0.00000)
Size	0.00117***	-0.01233***
	(0.00013)	(0.00014)
Collateral	-0.14960***	0.24804***
	(0.00067)	(0.00088)
Profitability	-0.24450***	-0.20091***
	(0.00413)	(0.00394)
Productivity	0.00782***	-0.01379***
	(0.00024)	(0.00025)
Observations	1,411,686	1,411,686
Adjusted R^2	0.23708	0.20030

Notes: Size is defined as the log employment, Collateral as tangible assets over total assets. All regressions include Industry-Year-Country fixed effects. Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth. FD is defined as Private Credit over GDP.

TABLE 7:
LEVERAGE PATTERNS OF YOUNG FIRMS AND FINANCIAL DEVELOPMENT

	Short Leverage	Long Leverage
Age	0.00382***	-0.02534***
	(0.00100)	(0.00116)
FD × Age	-0.00001	-0.00005***
	(0.00001)	(0.00001)
Size	0.01315***	-0.00893***
	(0.00128)	(0.00145)
Collateral	-0.09834***	0.33916***
	(0.00470)	(0.00620)
Profitability	-0.16750***	-0.23535***
	(0.00414)	(0.00551)
Productivity	0.01235***	-0.00934***
	(0.00121)	(0.00133)
Observations	191,970	191,970
Adjusted R^2	0.73422	0.73430
All firms		
Age	-0.00289***	-0.00443***
	(0.00005)	(0.00006)
FD × Age	-0.00000	0.00000***
	(0.00000)	(0.00000)
Size	0.01554***	-0.01294***
	(0.00049)	(0.00055)
Collateral	-0.10063***	0.29029***
	(0.00164)	(0.00212)
Profitability	-0.19556***	-0.18184***
	(0.00519)	(0.00554)
Productivity	0.01103***	-0.00967***
	(0.00052)	(0.00057)
Observations	1,333,407	1,333,407
Adjusted R^2	0.75993	0.74110

Notes: Size is defined as the log employment, Collateral as tangible assets over total assets. All regressions include Industry-Year-Country fixed effects. Short Leverage is defined as: Loans over Loans+Long Term Debt + Net Worth, and Long Leverage is defined as: Long Term Debt over Loans+Long Term Debt + Net Worth. FD is defined as Private Credit over GDP.

6.3 Alternative Specification: No Firm Controls

$$y_{i,t} = \eta_i + \sum_{a \in A} \beta_a \mathcal{D}^a_{i,t} + \epsilon_{i,t}$$

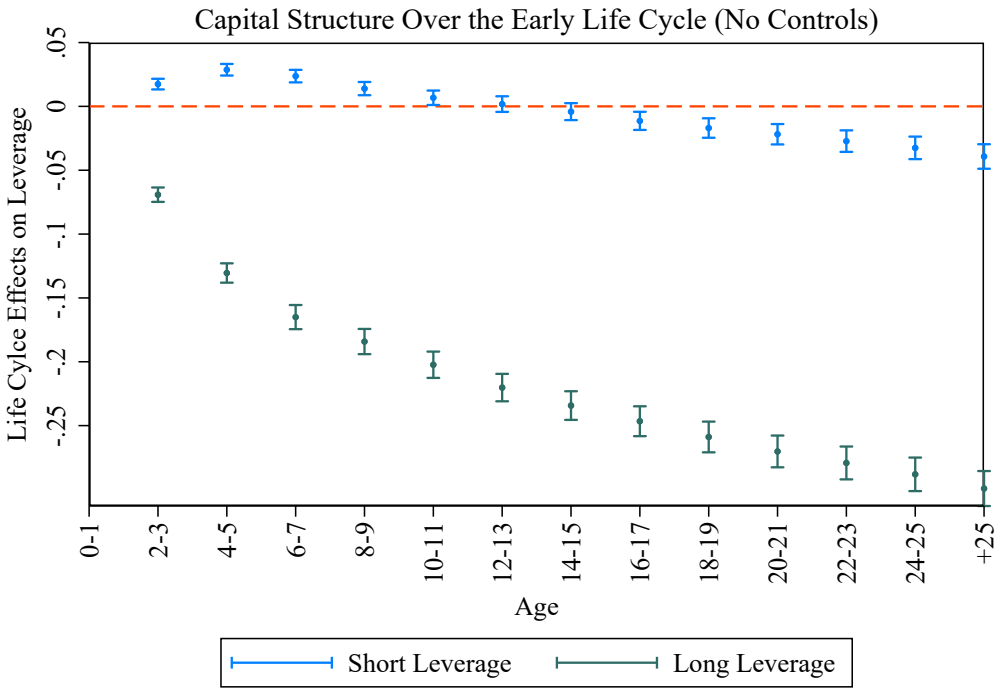


FIGURE 9: LIFE CYCLE PATTERNS OF CAPITAL STRUCTURE OF FIRMS FOR WHICH WE HAVE MORE THAN 4 OBSERVATIONS. CONFIDENCE INTERVALS ARE AT 90% AND STANDARD ERRORS ARE CLUSTERED AT COUNTRY-INDUSTRY-TIME LEVEL.

We run the regression over the sample of firms for which we have at least 3 observations.

6.3.1 Proof of proposition 2

We want parameter restrictions such that capital cannot be repaid back fully in one period.

As long as $p_k(1+r_f)K^u > \pi^u$ we need long term debt.

$$p_k(1+r_f)z^{\frac{1}{1-\eta}}\left[\frac{\eta\alpha\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$
$$> [1-\eta(1-\alpha)]z^{\frac{1}{1-\eta}}\left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]^{\frac{\eta\alpha}{1-\eta}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta}}$$

$$p_k(1+r_f)\left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$
$$> [1-\eta(1-\alpha)]\left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]^{\frac{\eta\alpha}{1-\eta}}$$

$$p_k(1+r_f)\left[\frac{1}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right] > 1+\frac{1-\eta}{\eta\alpha}$$

$$\left[\frac{(1+r_f)-(1-\beta\rho)(1+r_f)+\beta(1-\rho)\phi}{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}\right] > \frac{1-\eta}{\eta\alpha}$$
$$\left[\frac{\beta\rho(1+r_f)+\beta(1-\rho)\phi}{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}\right] > \frac{1-\eta}{\eta\alpha}$$

We need:

$$(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi>0$$

$$(1-\beta\rho)(1+r_f)>(\beta-\beta\rho)\phi$$

Note that $(1-\beta\rho)>(\beta-\beta\rho)$ and $(1+r_f)>1>\phi$

6.3.2 Proof of proposition 4

$$K^u=\left\{\frac{1}{1-\beta\rho}[1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}+\beta(1-\rho)\frac{1}{1-\beta\rho}\phi p_k K-p_k K(1+r_f)\right.$$

$$K^u=z^{\frac{1}{1-\eta}}\left[\frac{\eta\alpha\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

Given K^u note that multiply sequences $\{q_t,\nu_t\}_{t=0}^{\infty}$ satisfies the set of constraints:

$$p_k K^u(1+r_f)=\sum_{a=0}^{\infty}(\beta\rho)^a q_t+\beta(1-\rho)\sum_{a=0}^{\infty}(\beta\rho)^a\nu_t\beta(1-\rho)\phi p_k K^u$$

$$0\leq q_t\leq[1-\eta(1-\alpha)](K^u)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$$

In particular: a perpetuity, long term debt with increasing coupons $q_{t+1}>q_t$ is a solution

and a long term debt repayment contract that has $q_t = 0$. However, not anything is feasible, for example the promise to pledges the collateral ϕK^u to the financier for every period does not satisfies the PC of financiers, since:

$$\begin{aligned}
(1 + r_f)p_k K &\leq \frac{\beta(1-\rho)}{1-\beta\rho} p_K \phi K \\
(1 + r_f) &\leq \frac{\beta(1-\rho)}{1-\beta\rho} \phi \\
\beta^{-0.5} &\leq \frac{(1-\rho)}{1-\beta\rho} \phi \\
\frac{1-\beta\rho}{\phi(1-\rho)} &< \beta^{0.5} \\
\left[\frac{1-\beta\rho}{(1-\rho)}\right]^2 \frac{1}{\phi^2} &< \beta
\end{aligned}$$

Where we used that $(1 + r_f)(1 + r_f) = \beta$ and since $(1 - \rho) < 1 - \beta\rho$, $0 < (1 - \beta)\rho$ the LHS is greater than 1, a contradiction.

However, a perpetuity $q_t = q \; \forall \; t$ with no collateral $v_t = 0 \; \forall \; t.$ is feasible.

$$q = (1 - \beta\rho)p_k(1 + r_f)K^u$$

$$\Pi^u = \frac{1}{1-\beta\rho}[1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} + \beta(1-\rho)\frac{1}{1-\beta\rho}\phi p_k K - p_k K(1+r_f)$$

$$= \left[\frac{1}{1-\beta\rho}[1-\eta(1-\alpha)]K^{\frac{\eta-1}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} + \beta(1-\rho)\frac{1}{1-\beta\rho}\phi p_k - p_k(1+r_f)\right] K$$

$$K^u = z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

$$[K^u]^{\frac{1-\eta}{1-\eta(1-\alpha)}} = z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]$$

$$[K^u]^{\frac{\eta-1}{1-\eta(1-\alpha)}} = \frac{1}{z^{\frac{1}{1-\eta(1-\alpha)}}} \frac{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}{\eta\alpha\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}$$

$$
\begin{aligned}
\Pi^u &= \left[[1-\eta(1-\alpha)] K^{\frac{\eta-1}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} + \beta(1-\rho)\phi p_k - (1-\beta\rho)p_k(1+r_f) \right] \frac{1}{1-\beta\rho} K \\
&= \left[\frac{[1-\eta(1-\alpha)]}{\eta\alpha} \left[(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k \right] + \beta(1-\rho)\phi p_k - (1-\beta\rho)p_k(1+r_f) \right] \frac{1}{1-\beta\rho} K \\
&= \left[\frac{[1-\eta+\eta\alpha) - \eta\alpha}{\eta\alpha} \right] \frac{\left[(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k \right]}{1-\beta\rho} K \\
&= \left[\frac{1-\eta}{\eta\alpha} \right] \frac{\left[(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k \right]}{1-\beta\rho} z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k} \right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} \\
&= \left[\frac{1-\eta}{1-\beta\rho} \right] \frac{\left[(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k \right]}{\eta\alpha} z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k} \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} \\
&= \left[\frac{1-\eta}{1-\beta\rho} \right] z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k} \right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{\eta(1-\alpha)}{1-\eta}}
\end{aligned}
$$

Employment size of firms:

$$
n^u(K,z) = K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{1}{1-\eta(1-\alpha)}}
$$

$$
K^u = z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k} \right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}
$$

$$
\left[K^u\right]^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} = z^{\frac{\eta\alpha}{[1-\eta(1-\alpha)][1-\eta]}} \left[\frac{\eta\alpha \left[\frac{\eta(1-\alpha)}{(1+r_f)w} \right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k} \right]^{\frac{\eta\alpha}{1-\eta}}
$$

$$n^u(K^u,z)=n^u(z)=z^{\frac{1}{[1-\eta]}}\left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]^{\frac{\eta\alpha}{1-\eta}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1-\eta+\eta\alpha\eta(1-\alpha)}{[1-\eta(1-\alpha)][1-\eta]}}$$

Proof of proposition 5

$$\max_{\{q_t,R_t,\nu_t,K\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}(\beta\rho)^t\left(z\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta-(1+r_f)R_t-q_t\right)+\beta(1-\rho)\sum_{t=0}^{\infty}(1-\nu_t)(\beta\rho)^t p_k\phi$$

subject to

$$(1+r_f)p_kK\leq\sum_{t=0}^{\infty}(\beta\rho)^t q_t+\beta(1-\rho)\sum_{t=0}^{\infty}(\beta\rho)^t\nu_t p_k\phi K$$

$$0\leq q_t\leq z\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta-(1+r_f)R_t$$

$$\nu_t\in\{0,1\}$$

$$(1-\xi)z\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta\leq\sum_{k=0}^{\infty}(\beta\rho)^k\left(z\left[K^\alpha\left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^\eta-(1+r_f)R_{t+k}-q_{t+k}\right)+\beta(1-\rho)\sum_{k=0}^{\infty}(1-\nu_t$$

FOC:

$$\partial q_t\ :\ -(\beta\rho)^t+\lambda(\beta\rho)^t-\mu_t^H+\mu_t^L-\sum_{j=0}^{t}(\beta\rho)^j\psi_{t-j}=0$$

$$\partial R_t\ :\ \left[(\beta\rho)^t+\mu_t^H\right]\left[\frac{\eta(1-\alpha)}{R_t}z\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta-(1+r_f)\right]$$
$$-\psi_t(1-\xi)\frac{\eta(1-\alpha)}{R_t}z\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta+\left[\sum_{j=0}^{t}(\beta\rho)^j\psi_{t-j}\right]\left[\frac{\eta(1-\alpha)}{R_t}z\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta-(1+r_f)\right]$$

$$\begin{aligned}\partial K \ : \ & \sum_{t=0}^{\infty}(\beta\rho)^t z\frac{\eta\alpha}{K}\left[K^{\alpha}\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^{\eta}+\beta(1-\rho)\sum_{t=0}^{\infty}(1-\nu_t)(\beta\rho)^t p_k\phi-\lambda p_k(1+r_f)\\ & +\sum_{t=0}^{\infty}\mu_t^H z\frac{\eta\alpha}{K}\left[K^{\alpha}\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^{\eta}\\ & +\sum_{t=0}^{\infty}\psi_t\left[\sum_{k=0}^{\infty}(\beta\rho)^k\left(z\frac{\eta\alpha}{K}\left[K^{\alpha}\left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^{\eta}\right)+\beta(1-\rho)\sum_{k=0}^{\infty}(1-\nu_{t+k})(\beta\rho)^t p_k\phi\right]\\ & -(1-\xi)\sum_{t=0}^{\infty}\psi_t\left(z\frac{\eta\alpha}{K}\left[K^{\alpha}\left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^{\eta}\right)=0\end{aligned}$$

$$\partial\nu_t \ : \ -\beta(1-\rho)(\beta\rho)^t p_k\phi K+\lambda\beta(1-\rho)(\beta\rho)^t p_k\phi K-\mu_t^{H,\nu}+\mu_t^{L,\nu}-\beta(1-\rho)\sum_{j=0}^{t}(\beta\rho)^{t-j}\psi_j p_k\phi K=0$$

$$\begin{aligned}\partial\lambda \ : \ & \sum_{t=0}^{\infty}(\beta\rho)^t q_t+\beta(1-\rho)\sum_{t=0}^{\infty}(\beta\rho)^t\nu_t p_k\phi K-(1+r_f)p_k K\geq 0\\ & \left[\sum_{t=0}^{\infty}(\beta\rho)^t q_t+\beta(1-\rho)\sum_{t=0}^{\infty}(\beta\rho)^t\nu_t p_k\phi K-(1+r_f)p_k K\right]\lambda=0\end{aligned}$$

We know there exist a T such that $q_t=0$ for all $t>T$.

$$\begin{aligned}\partial\lambda \ : \ & \sum_{t=0}^{\infty}(\beta\rho)^t q_t+\beta(1-\rho)\sum_{t=0}^{\infty}(\beta\rho)^t\nu_t p_k\phi K-(1+r_f)p_k K\geq 0\\ & \left[\sum_{t=0}^{T}(\beta\rho)^t q_t+\beta(1-\rho)\sum_{t=0}^{\infty}(\beta\rho)^t\nu_t p_k\phi K-(1+r_f)p_k K\right]\lambda=0\end{aligned}$$

Proof.

Rewriting the FOC

$$\lambda = \left[1 + \mu_t^H \frac{1}{(\beta\rho)^t} - \mu_t^L \frac{1}{(\beta\rho)^t} + \frac{1}{(\beta\rho)^t} \sum_{j=0}^{t} (\beta\rho)^j \psi_{t-j}\right] \qquad \forall\ t$$

$$\psi_t = \frac{\left[(\beta\rho)^t + \mu_t^H + \left[\sum_{j=0}^{t} (\beta\rho)^j \psi_{t-j}\right]\right] \left[\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right]}{(1-\xi)\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta} \qquad \forall\ t$$

Note that $\psi_t = 0 \qquad \Longleftrightarrow \qquad R_t = R^u(K, z)$

Where

$$R^u(K,z) \equiv \left[\frac{\eta(1-\alpha)}{(1+r_f)} z \left[K^\alpha \left(\frac{1}{w}\right)^{1-\alpha}\right]^\eta\right]^{\frac{1}{1-\eta(1-\alpha)}}$$

$$\partial R_t \ :\ \left[(\beta\rho)^t + \mu_t^H\right] \left[\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right]$$
$$+ \left[\sum_{j=1}^{t} (\beta\rho)^j \psi_{t-j}\right] \left[\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right] - (1+r_f) = \psi_t(-\xi)\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R}{w}\right.\right.$$

$$\partial R_t \ :\ \left[(\beta\rho)^t + \mu_t^H + \sum_{j=1}^{t} (\beta\rho)^j \psi_{t-j}\right] \left[\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right] - (1+r_f) = \psi_t(-\xi)\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R_t}{w}\right)^{1-}\right.$$

$$\partial R_t \ :\ \left[(\beta\rho)^t + \mu_t^H + \sum_{j=0}^{t} (\beta\rho)^j \psi_{t-j}\right] \left[\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right] = \psi_t(1-\xi)\frac{\eta(1-\alpha)}{R_t} z \left[K^\alpha \left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta$$

$$\lambda = \left[1 + \mu_0^H - \mu_0^L + \psi_0\right]$$
$$\lambda = \left[1 + \mu_1^H \frac{1}{(\beta\rho)^1} - \mu_1^L \frac{1}{(\beta\rho)^1} + \psi_0 + \psi_1 \frac{1}{\beta\rho}\right]$$
$$\lambda = \left[1 + \mu_2^H \frac{1}{(\beta\rho)^2} - \mu_2^L \frac{1}{(\beta\rho)^2} + \psi_0 + \psi_1 \frac{1}{\beta\rho} + \psi_2 \frac{1}{(\beta\rho)^2}\right]$$
$$\lambda = \left[1 + \mu_t^H \frac{1}{(\beta\rho)^t} - \mu_t^L \frac{1}{(\beta\rho)^t} + \sum_{j=0}^{t} \psi_j \left[\frac{1}{\beta\rho}\right]^j\right]$$

Combining the expressions for λ we get:

$$\begin{aligned}
\beta\rho(\mu_0^H - \mu_0^L) - \psi_1 &= (\mu_1^H - \mu_1^L) \\
\beta\rho(\mu_1^H - \mu_1^L) - \psi_2 &= (\mu_2^H - \mu_2^L) \\
(\beta\rho)^2(\mu_0^H - \mu_0^L) - \beta\rho\psi_1 - \psi_2 &= (\mu_2^H - \mu_2^L) \\
(\beta\rho)^3(\mu_0^H - \mu_0^L) - (\beta\rho)^2\psi_1 - (\beta\rho)\psi_2 - \psi_3 &= (\mu_3^H - \mu_3^L) \\
&\vdots \\
(\beta\rho)^t(\mu_0^H - \mu_0^L) - \sum_{j=1}^{t} (\beta\rho)^{t-j}\psi_j &= (\mu_t^H - \mu_t^L)
\end{aligned}$$

Note that we need $(\mu_0^H - \mu_0^L) > 0$ which implies $\mu_0^H > 0$ and $\mu_0^L = 0$ because if $(\mu_0^H - \mu_0^L) < 0$ since $\psi_t \geq 0 \ \forall \ t$ we would have $(\mu_t^H - \mu_t^L) < 0 \ \forall \ t$ and this would imply $\mu_t^L > 0$ for all t which does not satisfy the participation constraint of the financier.

Then, given $\mu_0^H > 0$, now we have:

$$(\beta\rho)^t\mu_0^H - \sum_{j=1}^{t}(\beta\rho)^{t-j}\psi_j = (\mu_t^H - \mu_t^L)$$

Note the following

- Furthermore notice that once $\mu_{\hat{t}}^H = 0$ (this needs to happen, otherwise the firm value is 0) then it will be zero for all $t \geq \hat{t}$ and $\mu_{\hat{t}}^L > 0$ and $\mu_t^L > 0$ for all $t \geq \hat{t}$.
- Note that if $\mu_{\hat{t}}^H > 0$ then $\mu_t^H > 0$ for all $t < \hat{t}$ this is immediate from the recursion $(\beta\rho)^t(\mu_0^H - \mu_0^L) - \sum_{j=1}^{t}(\beta\rho)^{t-j}\psi_j = (\mu_t^H - \mu_t^L)$.
- From the previous observations, there is a $\hat{t}$ such that $\mu_t^H > 0$ for all $t < \hat{t}$ and $\mu_t^H = 0$ for all $t \geq \hat{t}$

Let $\hat{t}$ be the first time that $\mu_{\hat{t}}^H = 0$ then:

$$\mu_0^H - \left[\frac{1}{(\beta\rho)}\psi_1 + \cdots + \frac{1}{(\beta\rho)^{\hat{t}-1}}\psi_{\hat{t}-1}\right] = \frac{1}{(\beta\rho)^{\hat{t}-1}}\mu_{\hat{t}-1}^H > 0$$

$$\mu_0^H - \left[\frac{1}{(\beta\rho)}\psi_1 + \cdots + \frac{1}{(\beta\rho)^{\hat{t}-1}}\psi_{\hat{t}-1} + \frac{1}{(\beta\rho)^{\hat{t}}}\psi_{\hat{t}}\right] = -\frac{1}{(\beta\rho)^{\hat{t}}}\mu_{\hat{t}}^L$$
$$\mu_0^H - \left[\frac{1}{(\beta\rho)}\psi_1 + \cdots + \frac{1}{(\beta\rho)^{\hat{t}-1}}\psi_{\hat{t}-1}\right] - \frac{1}{(\beta\rho)^{\hat{t}}}\psi_{\hat{t}} = -\frac{1}{(\beta\rho)^{\hat{t}}}\mu_{\hat{t}}^L$$

Since $\mu_0^H - \left[\frac{1}{(\beta\rho)}\psi_1 + \cdots + \frac{1}{(\beta\rho)^{\hat{t}-1}}\psi_{\hat{t}-1}\right] \geq 0$ and $\psi_{\hat{t}} > 0$ then: We can have either $\mu_{\hat{t}}^L = 0$ and $\psi_{\hat{t}} > 0$ or $\mu_{\hat{t}}^L > 0$ and $\psi_{\hat{t}} > 0$

- Case 1 $\mu_{\hat{t}}^L = 0$ and $\psi_{\hat{t}} > 0$

$$\mu_0^H - \left[\frac{1}{(\beta\rho)}\psi_1 + \cdots + \frac{1}{(\beta\rho)^{\hat{t}-1}}\psi_{\hat{t}-1}\right] - \frac{1}{(\beta\rho)^{\hat{t}}}\psi_{\hat{t}} - \frac{1}{(\beta\rho)^{\hat{t}+1}}\psi_{\hat{t}+1} = -\frac{1}{(\beta\rho)^{\hat{t}+1}}\mu_{\hat{t}+1}^L$$

  Then $\mu_{\hat{t}+1}^L = 0 = \psi_{\hat{t}+1}$

- Case 2 $\mu_{\hat{t}}^L > 0$ and $\psi_{\hat{t}} > 0$

  Note that this case doesn't make sense since the borrowing constraint would be binding and we wouldn't be using funds to reduce the liabilities that has a bigger marginal return than paying back dividends.

In a solution we will have a $\hat{t}$ such that $\mu_t^H > 0$ for all $t < \hat{t}$ and $\mu_t^H = 0$ for all $t \geq \hat{t}$ and $\mu_t^L = 0$ for all t. Then:

$$q_t = \begin{cases} \pi_t & \forall\ t \leq \hat{t} \\ [0, \pi_t] & \forall\ t > \hat{t} \end{cases}$$

$$\psi_t = \begin{cases} > 0 & \forall\ t \leq \hat{t} \\ = 0 & \forall\ t > \hat{t} \end{cases}$$

$$R_t = \begin{cases} R_a(K,z) & \forall\ t \leq \hat{t} \\ R^u(K,z) & \forall\ t > \hat{t} \end{cases}$$

$$\lambda = \left[1 + \sum_{j=0}^{\hat{t}} \psi_j \left[\frac{1}{\beta\rho}\right]^j\right]$$

$$\partial\nu_t \ :\ [\lambda-1]\beta(1-\rho)(\beta\rho)^t p_k\phi K - \mu_t^{H,\nu} + \mu_t^{L,\nu} - \beta(1-\rho)\sum_{j=0}^{t}(\beta\rho)^{t-j}\psi_j p_k\phi K = 0$$

$$[\lambda-1]\beta(1-\rho)(\beta\rho)^t p_k\phi K - \beta(1-\rho)\sum_{j=0}^{t}(\beta\rho)^{t-j}\psi_j p_k\phi K = \mu_t^{H,\nu} - \mu_t^{L,\nu}$$

$$[\lambda-1] - \sum_{j=0}^{t}(\beta\rho)^{-j}\psi_j = \frac{1}{(\beta\rho)^t\beta(1-\rho)p_k\phi K}\left[\mu_t^{H,\nu} -$$

Using :

$$\lambda = \left[1 + \sum_{j=0}^{\hat{t}} \psi_j \left[\frac{1}{\beta\rho}\right]^j\right]$$

$$\sum_{j=0}^{\hat{t}} \psi_j \left[\frac{1}{\beta\rho}\right]^j - \sum_{j=0}^{t}(\beta\rho)^{-j}\psi_j = \frac{1}{(\beta\rho)^t\beta(1-\rho)p_k\phi K}\left[\mu_t^{H,\nu} - \mu_t^{L,\nu}\right]$$

For any $t < \hat{t}$ then: $\left[\mu_t^{H,\nu} - \mu_t^{L,\nu}\right] > 0$ hence $\mu_t^{H,\nu} = 1$ and for any $t \geq \hat{t}$ we get $\mu_t^{H,\nu} = \mu_t^{L,\nu} = 0$

Now consider:

$$(1+r_f)p_kK = \sum_{t=0}^{\infty}(\beta\rho)^t q_t + \beta(1-\rho)\sum_{t=0}^{\infty}(\beta\rho)^t \nu_t p_k \phi K$$

$$(1+r_f)p_kK = \sum_{t=0}^{\hat{t}}(\beta\rho)^t q_t + \sum_{t=\hat{t}+1}^{\infty}(\beta\rho)^t q_t + \beta(1-\rho)\sum_{t=0}^{\hat{t}}(\beta\rho)^t \nu_t p_k \phi K + \beta(1-\rho)\sum_{t=\hat{t}+1}^{\infty}(\beta\rho)^t \nu_t p_k \phi K$$

$$(1+r_f)p_kK - \sum_{t=0}^{\hat{t}}(\beta\rho)^t q_t - \beta(1-\rho)\sum_{t=0}^{\hat{t}}(\beta\rho)^t \nu_t p_k \phi K = \sum_{t=\hat{t}+1}^{\infty}(\beta\rho)^t q_t + \beta(1-\rho)\sum_{t=\hat{t}+1}^{\infty}(\beta\rho)^t \nu_t p_k \phi K$$

■

- χ consider the case where the liquidation value of capital is less in the hands of the firm since it has to sell the capital "quickly" compared to the lender that can claim possession of the capital and sell it whenever is a good time.

- Consider the decision of pledging capital being $0, 1$. $v_t > 0$ till $q_t = 0$
- Consider the case of the firm being more impatient than the lenders $\beta_F < \beta_L$

For the capital decision:

$$\begin{aligned}\partial K \ : \ & \sum_{t=0}^{\infty}(\beta\rho)^t z\frac{\eta\alpha}{K}\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta + \beta(1-\rho)\sum_{t=0}^{\infty}(1-\nu_t)(\beta\rho)^t p_k\phi - \lambda p_k(1+r_f) \\ & + \sum_{t=0}^{\infty}\mu_t^H z\frac{\eta\alpha}{K}\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta \\ & + \sum_{t=0}^{\infty}\left[\sum_{k=0}^{\infty}\psi_{t+k}\left[\sum_{k=0}^{\infty}(\beta\rho)^k\left(z\frac{\eta\alpha}{K}\left[K^\alpha\left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^\eta\right) + \beta(1-\rho)\sum_{k=0}^{\infty}(1-\nu_{t+k})(\beta\rho)^t p_k\phi\right]\right] \\ & - \sum_{t=0}^{\infty}\psi_t(1-\xi)\left(z\frac{\eta\alpha}{K}\left[K^\alpha\left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^\eta\right)\end{aligned}$$

Using that for $t > \hat{t}$ we have $\psi_t = 0$ and $\mu_t = 0$ and for $t \leq \hat{t}$ $v_t = 1$:

$$\begin{aligned}\partial K \ : \ & \sum_{t=0}^{\infty}(\beta\rho)^t z\frac{\eta\alpha}{K}\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta + \beta(1-\rho)\sum_{t=\hat{t}}^{\infty}(1-\nu_t)(\beta\rho)^t p_k\phi - \lambda p_k(1+r_f) \\ & + \sum_{t=0}^{\hat{t}}\mu_t^H z\frac{\eta\alpha}{K}\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta \\ & + \sum_{t=0}^{\hat{t}}\left[\sum_{k=0}^{\hat{t}-t}\psi_{t+k}\left[\sum_{k=0}^{\infty}(\beta\rho)^k\left(z\frac{\eta\alpha}{K}\left[K^\alpha\left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^\eta\right)\right]\right] \\ & - \sum_{t=0}^{\hat{t}}\psi_t(1-\xi)\left(z\frac{\eta\alpha}{K}\left[K^\alpha\left(\frac{R_{t+k}}{w}\right)^{1-\alpha}\right]^\eta\right)\end{aligned}$$

Using

$$\lambda = \left[1 + \sum_{j=0}^{\hat{t}}\psi_j\left[\frac{1}{\beta\rho}\right]^j\right]$$

Solving for ψ_t:

$$\psi_t = \frac{\left[(\beta\rho)^t + \mu_t^H + \left[\sum_{j=0}^{t}(\beta\rho)^j \psi_{t-j}\right]\right]\left[\frac{\eta(1-\alpha)}{R_t} z\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right]}{(1-\xi)\frac{\eta(1-\alpha)}{R_t} z\left[K^\alpha\left(\frac{R_t}{w}\right)^{1-\alpha}\right]^\eta} \qquad \forall\, t$$

Note ψ_0 is:

$$\begin{aligned}
\psi_0 &= \frac{\left[1+\mu_0^H+\psi_0\right]\left[\frac{\eta(1-\alpha)}{R_0} z\left[K^\alpha\left(\frac{R_0}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right]}{(1-\xi)\frac{\eta(1-\alpha)}{R_0} z\left[K^\alpha\left(\frac{R_0}{w}\right)^{1-\alpha}\right]^\eta} \\
&= \frac{\lambda\left[\frac{\eta(1-\alpha)}{R_0} z\left[K^\alpha\left(\frac{R_0}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right]}{(1-\xi)\frac{\eta(1-\alpha)}{R_0} z\left[K^\alpha\left(\frac{R_0}{w}\right)^{1-\alpha}\right]^\eta} \\
&= \frac{\lambda}{1-\xi}\left[1 - \frac{(1+r_f)R_0}{\eta(1-\alpha)z\left[K^\alpha\left(\frac{R_0}{w}\right)^{1-\alpha}\right]^\eta}\right] \\
&= \frac{\lambda}{1-\xi}\left[1 - \frac{w^{\eta(1-\alpha)}(1+r_f)R_0^{1-\eta(1-\alpha)}}{\eta(1-\alpha)zK^{\eta\alpha}}\right]
\end{aligned}$$

Where we used that $\mu_0^L = 0$ and $1 + \mu_0^H + \psi_0 = \lambda$

Note ψ_1 is:

$$\begin{aligned}
\psi_1 &= \frac{\left[(\beta\rho) + \mu_1^H + [\psi_1 + (\beta\rho)\psi_0]\right]\left[\frac{\eta(1-\alpha)}{R_1} z\left[K^\alpha\left(\frac{R_1}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right]}{(1-\xi)\frac{\eta(1-\alpha)}{R_1} z\left[K^\alpha\left(\frac{R_1}{w}\right)^{1-\alpha}\right]^\eta} \\
&= \frac{\beta\rho\left[1 + \frac{1}{\beta\rho}\mu_1^H + \left[\frac{1}{\beta\rho}\psi_1 + \psi_0\right]\right]\left[\frac{\eta(1-\alpha)}{R_1} z\left[K^\alpha\left(\frac{R_1}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right]}{(1-\xi)\frac{\eta(1-\alpha)}{R_1} z\left[K^\alpha\left(\frac{R_1}{w}\right)^{1-\alpha}\right]^\eta} \\
&= \frac{\beta\rho\lambda\left[\frac{\eta(1-\alpha)}{R_1} z\left[K^\alpha\left(\frac{R_1}{w}\right)^{1-\alpha}\right]^\eta - (1+r_f)\right]}{(1-\xi)\frac{\eta(1-\alpha)}{R_1} z\left[K^\alpha\left(\frac{R_1}{w}\right)^{1-\alpha}\right]^\eta} \\
&= \beta\rho\frac{\lambda}{1-\xi}\left[1 - \frac{w^{\eta(1-\alpha)}(1+r_f)R_1^{1-\eta(1-\alpha)}}{\eta(1-\alpha)zK^{\eta\alpha}}\right]
\end{aligned}$$

$$\begin{aligned}
\lambda &= \left[1 + \sum_{j=0}^{\hat{t}} \frac{\lambda}{1-\xi}\left[1 - \frac{w^{\eta(1-\alpha)}(1+r_f)R_j^{1-\eta(1-\alpha)}}{\eta(1-\alpha)zK^{\eta\alpha}}\right]\right] \\
\lambda &= \frac{1}{1 - \sum_{j=0}^{\hat{t}} \frac{1}{1-\xi}\left[1 - \frac{w^{\eta(1-\alpha)}(1+r_f)R_j^{1-\eta(1-\alpha)}}{\eta(1-\alpha)zK^{\eta\alpha}}\right]} \\
\lambda &= \frac{1}{1 - \sum_{j=0}^{\hat{t}} \frac{1}{1-\xi}\left[1 - \frac{w^{\eta(1-\alpha)}(1+r_f)R_j^{1-\eta(1-\alpha)}}{\eta(1-\alpha)zK^{\eta\alpha}}\right]} \\
\lambda &= \frac{1}{1 - \sum_{j=0}^{\hat{t}} \frac{1}{1-\xi}\left[1 - \left(\frac{R_j}{R^u(K,z)}\right)^{1-\eta(1-\alpha)}\right]}
\end{aligned}$$

Then we can rewrite λ as:

$$R^u(K,z)^{1-\eta(1-\alpha)} \equiv \frac{\eta(1-\alpha)}{(1+r_f)} z\left[K^\alpha\left(\frac{1}{w}\right)^{1-\alpha}\right]^\eta$$

For $\lambda > 0$ we need:

$$\sum_{j=0}^{\hat{t}} \left[1 - \left(\frac{R_j}{R^u(K,z)} \right)^{1-\eta(1-\alpha)} \right] < 1 - \xi$$

Proof of proposition 6

$$\begin{aligned}
\frac{(\beta\rho)^{\hat{t}} - (\beta\rho)^T}{1-\beta\rho}\pi^u + \beta(1-\rho)\frac{(\beta\rho)^{\hat{t}} - (\beta\rho)^T}{1-\beta\rho}p_K\phi K &< \beta(1-\rho)\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}p_K\phi K \\
\frac{(\beta\rho)^{\hat{t}} - (\beta\rho)^T}{1-\beta\rho}\pi^u &< \beta(1-\rho)\frac{(\beta\rho)^T}{1-\beta\rho}p_K\phi K \\
[1-\eta(1-\alpha)]\frac{(\beta\rho)^{\hat{t}} - (\beta\rho)^T}{1-\beta\rho}K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} &< \beta(1-\rho)\frac{(\beta\rho)^T}{1-\beta\rho}p_K\phi K \\
\frac{[1-\eta(1-\alpha)]}{\beta(1-\rho)p_K\phi}\left(\frac{1}{(\beta\rho)^{T-\hat{t}}} - 1\right)z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} &< K^{\frac{1-\eta}{1-\eta(1-\alpha)}} \\
\left[\frac{[1-\eta(1-\alpha)]}{\beta(1-\rho)p_K\phi}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}\left(\frac{1}{(\beta\rho)^{T(K)-\hat{t}(K)}} - 1\right)\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} &< K
\end{aligned}$$

Indeterminacy of coupons after $\hat{t}$

The value of long term debt used in the main body of the paper comes from:

$$\begin{aligned}
L_0 &= \sum_{t=0}^{T}(\beta\rho)^t q_t + \beta(1-\rho)\sum_{t=0}^{T-1}(\beta\rho)^t\phi p_k K \\
L_1 &= \sum_{t=1}^{T}(\beta\rho)^{t-1} q_t + \beta(1-\rho)\sum_{t=1}^{T-1}(\beta\rho)^{t-1}\phi p_k K \\
&\vdots \\
L_{\hat{t}} &= \sum_{t=\hat{t}}^{T}(\beta\rho)^{t-\hat{t}} q_t + \beta(1-\rho)\sum_{t=\hat{t}}^{T-1}(\beta\rho)^{t-\hat{t}}\phi p_k K \\
&\vdots
\end{aligned}$$

$$L_{\hat{t}} = \frac{(1+r_f)K}{(\beta\rho)^{\hat{t}}} - \sum_{t=0}^{\hat{t}-1} \frac{q_t}{(\beta\rho)^{\hat{t}-t}} - \beta(1-\rho)p_k\phi K \sum_{t=1}^{\hat{t}} \frac{1}{(\beta\rho)^t}$$

6.4 Proof of outcome gaps as functions of funding gaps

The investment and employment gap are straightforward:

$$\begin{aligned}\Gamma_n &= \frac{1}{w}\left[R(K(1,\cdot),z,1,p_k,\phi,\alpha) - R_a(K(\xi,\cdot),z,\xi,p_k,\phi,\alpha)\right] \\ &= \frac{1}{w}\Gamma_S\end{aligned}$$

$$\begin{aligned}\Gamma_K &= p_k[K^u(1,\cdot) - K(\xi,\cdot)] \\ &= \Gamma_L\end{aligned}$$

The expression from profits and firm value gap are obtained as follows:

$$
\begin{aligned}
\Gamma_\pi &= \sum_{t=0}^{\infty}(\beta\rho)^t[\pi^u - \pi_t] \\
&= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(1)) - \pi_t(\xi,K(\xi))] + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[\pi^u(1,K(1)) - \pi_t(1,K(\xi))] \\
&= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(1)) - \pi_t(\xi,K(\xi))] + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}\left[K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - \right. \\
&= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(1)) - \pi_t(\xi,K(\xi))] + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot)) \\
&= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(1)) - \pi^u(1,K(\xi)) + \pi^u(1,K(\xi)) - \pi_t(\xi,K(\xi))] + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(\\
&= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(\xi)) - \pi_t(\xi,K(\xi))] + \frac{1}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot))
\end{aligned}
$$

Where the last term comes from:

$$
\Omega_S(K(\xi,\cdot)) = \left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}
$$

$$
\Omega_S(K(\xi,\cdot)) = \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]\left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right] wz^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}
$$

$$
\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot)) = z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}\left[K(1)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} - K(\xi)^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}\right]
$$

In order to get an expression ofr $\sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(\xi))-\pi_t(\xi,K(\xi))]$, note that:

$$
\begin{aligned}
&[\pi^u(1,K(\xi))-\pi_t(\xi,K(\xi))]= \\
&zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R(1,K(\xi,\cdot),\cdot)}{w}\right)^{\eta(1-\alpha)}-(1+r_f)R(1,K(\xi,\cdot),\cdot)-zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R_a(\xi,K(\xi,\cdot),\cdot)}{w}\right)^{\eta(1-\alpha)}+(1+r_f)R_a(\xi,K(\xi,\cdot),\cdot) \\
&zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R(1,K(\xi,\cdot),\cdot)}{w}\right)^{\eta(1-\alpha)}-zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R_a(\xi,K(\xi,\cdot),\cdot)}{w}\right)^{\eta(1-\alpha)}-(1+r_f)\tilde{\Gamma}_S(K(\xi,\cdot)) \\
&zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R(1,K(\xi,\cdot),\cdot)}{w}\right)^{\eta(1-\alpha)}\left[1-\left(\frac{R_a(\xi,K(\xi,\cdot),\cdot)}{R(1,K(\xi,\cdot),\cdot)}\right)^{\eta(1-\alpha)}\right]-(1+r_f)\tilde{\Gamma}_S(K(\xi,\cdot)) \\
&zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R(1,K(\xi,\cdot),\cdot)}{w}\right)^{\eta(1-\alpha)}\left[1-\left(1-\frac{\tilde{\Gamma}_{S,a}(K(\xi,\cdot))}{R(1,K(\xi,\cdot),\cdot)}\right)^{\eta(1-\alpha)}\right]-(1+r_f)\tilde{\Gamma}_{S,a}(K(\xi,\cdot))
\end{aligned}
$$

Then:

$$
\begin{aligned}
&\sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(\xi))-\pi_t(\xi,K(\xi))]= \\
&zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R(1,K(\xi,\cdot),\cdot)}{w}\right)^{\eta(1-\alpha)}\sum_{t=0}^{\hat{t}-1}(\beta\rho)^t\left[1-\left(1-\frac{\tilde{\Gamma}_{S,a}(K(\xi,\cdot))}{R(1,K(\xi,\cdot),\cdot)}\right)^{\eta(1-\alpha)}\right]-\sum_{t=0}^{\hat{t}-1}(\beta\rho)^t(1+r_f)\tilde{\Gamma}_{S,t}(K(\xi
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_V &= \sum_{t=0}^{\infty}(\beta\rho)^t[\pi^u-\pi_t]+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)p_k\phi\left[K(1,\cdot)-K(\xi,\cdot)\right] \\
&= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(1))-\pi_t(\xi,K(\xi))]+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[\pi^u(1,K(1))-\pi_t(1,K(\xi))] \\
&= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(1))-\pi_t(\xi,K(\xi))]+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}\left[K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}-\right. \\
&= \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(1))-\pi_t(\xi,K(\xi))]+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot))+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma
\end{aligned}
$$

$$\sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\pi^u(1,K(1))-\pi_t(\xi,K(\xi))]=$$

$$zK(1,\cdot)^{\eta\alpha}\left(\frac{R(1,\cdot)}{w}\right)^{\eta(1-\alpha)}-(1+r_f)R(1,\cdot)-zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R_a(\xi,\cdot)}{w}\right)^{\eta(1-\alpha)}+(1+r_f)R_a(\xi,K(\xi,\cdot),\cdot)$$

$$zK(1,\cdot)^{\eta\alpha}\left(\frac{R(1,\cdot)}{w}\right)^{\eta(1-\alpha)}-zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R_a(\xi,\cdot)}{w}\right)^{\eta(1-\alpha)}+(1+r_f)[R_a(\xi,K(\xi,\cdot),\cdot)-R(1,\cdot)]$$

$$zK(1,\cdot)^{\eta\alpha}\left(\frac{R(1,\cdot)}{w}\right)^{\eta(1-\alpha)}-zK(\xi,\cdot)^{\eta\alpha}\left(\frac{R_a(\xi,\cdot)}{w}\right)^{\eta(1-\alpha)}-(1+r_f)[\Gamma_{S,a}]$$

$$zK(1,\cdot)^{\eta\alpha}\left(\frac{R(1,\cdot)}{w}\right)^{\eta(1-\alpha)}\left[1-\left[\frac{K(\xi,\cdot)}{K(1,\cdot)}\right]^{\eta\alpha}\left(\frac{R_a(\xi,\cdot)}{R(1,\cdot)}\right)^{\eta(1-\alpha)}\right]-(1+r_f)[\Gamma_{S,a}]$$

We can derive an alternative expression for Λ_V as follows:

$$\Lambda_V=\sum_{t=0}^{\hat{t}-1}(\beta\rho)^t\left[zK(1,\cdot)^{\eta\alpha}\left(\frac{R(1,\cdot)}{w}\right)^{\eta(1-\alpha)}\left[1-\left[\frac{K(\xi,\cdot)}{K(1,\cdot)}\right]^{\eta\alpha}\left(\frac{R_a(\xi,\cdot)}{R(1,\cdot)}\right)^{\eta(1-\alpha)}\right]-(1+r_f)[\Gamma_{S,a}]\right]$$
$$+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot))+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma_L$$

$$\Gamma_V=\sum_{t=0}^{\hat{t}-1}(\beta\rho)^t\left[zK(1,\cdot)^{\eta\alpha}\left(\frac{R(1,\cdot)}{w}\right)^{\eta(1-\alpha)}\left[1-\left[\frac{K(\xi,\cdot)}{K(1,\cdot)}\right]^{\eta\alpha}\left(\frac{R_a(\xi,\cdot)}{R(1,\cdot)}\right)^{\eta(1-\alpha)}\right]\right]$$
$$-(1+r_f)\sum_{t=0}^{\hat{t}-1}(\beta\rho)^t[\Gamma_{S,a}]$$
$$+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot))+\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma_L$$

$$\Gamma_V = \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \left[zK(1,\cdot)^{\eta\alpha} \left(\frac{R(1,\cdot)}{w} \right)^{\eta(1-\alpha)} \left[1 - \left[\frac{K(\xi,\cdot)}{K(1,\cdot)} \right]^{\eta\alpha} \left(\frac{R_a(\xi,\cdot)}{R(1,\cdot)} \right)^{\eta(1-\alpha)} \right] \right]$$
$$- (1+r_f) \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t [\Omega_S(K(\xi,\cdot)) + \tilde{\Gamma}_{S,a}(K(\xi,\cdot))]$$
$$+ \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot)) + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma_L$$

$$\Gamma_{S,a} = \Omega_S(K(\xi,\cdot)) + \tilde{\Gamma}_{S,a}(K(\xi,\cdot))$$

$$\Gamma_V = \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \left[zK(1,\cdot)^{\eta\alpha} \left(\frac{R(1,\cdot)}{w} \right)^{\eta(1-\alpha)} \left[1 - \left[\frac{K(\xi,\cdot)}{K(1,\cdot)} \right]^{\eta\alpha} \left(\frac{R_a(\xi,\cdot)}{R(1,\cdot)} \right)^{\eta(1-\alpha)} \right] \right]$$
$$- (1+r_f) \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \tilde{\Gamma}_{S,a}(K(\xi,\cdot))] - (1+r_f)\frac{1-(\beta\rho)^{\hat{t}}}{1-\beta\rho}\Omega_S(K(\xi,\cdot))$$
$$+ \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot)) + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma_L$$

$$\Gamma_V = \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \left[zK(1,\cdot)^{\eta\alpha} \left(\frac{R(1,\cdot)}{w} \right)^{\eta(1-\alpha)} \left[1 - \left[\frac{K(\xi,\cdot)}{K(1,\cdot)} \right]^{\eta\alpha} \left(\frac{R_a(\xi,\cdot)}{R(1,\cdot)} \right)^{\eta(1-\alpha)} \right] \right]$$
$$- (1+r_f) \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \tilde{\Gamma}_{S,a}(K(\xi,\cdot))] - (1+r_f)\frac{1}{1-\beta\rho}\Omega_S(K(\xi,\cdot)) + (1+r_f)\frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\eta(1-\alpha)\frac{\Omega_S(K(\xi,\cdot))}{\eta(1-\alpha)}$$
$$+ \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot)) + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma_L$$

$$\Gamma_V = \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \left[zK(1,\cdot)^{\eta\alpha} \left(\frac{R(1,\cdot)}{w} \right)^{\eta(1-\alpha)} \left[1 - \left[\frac{K(\xi,\cdot)}{K(1,\cdot)} \right]^{\eta\alpha} \left(\frac{R_a(\xi,\cdot)}{R(1,\cdot)} \right)^{\eta(1-\alpha)} \right] \right]$$
$$- (1+r_f) \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \tilde{\Gamma}_{S,a}(K(\xi,\cdot))] - \eta(1-\alpha)\frac{(1+r_f)}{\eta(1-\alpha)}\frac{1}{1-\beta\rho}\Omega_S(K(\xi,\cdot))$$
$$+ \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot)) + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma_L$$

$$\Gamma_V = zK(1,\cdot)^{\eta\alpha} \left(\frac{R(1,\cdot)}{w} \right)^{\eta(1-\alpha)} \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \left[1 - \left[\frac{K(\xi,\cdot)}{K(1,\cdot)} \right]^{\eta\alpha} \left(\frac{R_a(\xi,\cdot)}{R(1,\cdot)} \right)^{\eta(1-\alpha)} \right]$$
$$- (1+r_f) \sum_{t=0}^{\hat{t}-1}(\beta\rho)^t \tilde{\Gamma}_{S,a}(K(\xi,\cdot))]$$
$$+ \frac{[(\beta\rho)^{\hat{t}} - \eta(1-\alpha)]}{1-\beta\rho}\frac{(1+r_f)}{\eta(1-\alpha)}\Omega_S(K(\xi,\cdot)) + \frac{(\beta\rho)^{\hat{t}}}{1-\beta\rho}\beta(1-\rho)\phi\Gamma_L$$

6.5 Binary Collateral Decision $\nu \in \{0, 1\}$

From the divisible collateral case, we know that when $\psi_0 > 0$ then it will be optimal to pledge collateral to the financial contract.

Now, conditional on $\nu = 1$ we get:

Consider two contracts $L = \{p_k K, \{q_t\}_{k=0}^{T}, \nu = 1\}$ and $\tilde{L} = \{p_k K, \{\hat{q}_t\}_{k=0}^{T+\tilde{t}}, \nu = 1\}$ where L is a contract that finances $p_K K$ and is of minimum maturity T, and, $\tilde{L}$ is a contract that finances $p_k K$ with longer maturity than T.

- We know from $t \in [0, \hat{t}]$ the coupons should be $q_t = \pi_t$
- Now compare what happens from $\hat{t} + 1$ onward.

$$\sum_{t=0}^{T} (\beta\rho)^t q_t + \beta(1-\rho)\frac{1-(\beta\rho)^T}{1-\beta\rho}\phi p_k K = \sum_{t=0}^{T+\tilde{t}} (\beta\rho)^t \tilde{q}_t + \beta(1-\rho)\frac{1-(\beta\rho)^{T+\tilde{t}}}{1-\beta\rho}\phi p_k K$$

Let $\Delta_{q_t} = q_t - \tilde{q}_t$

$$\sum_{t=\hat{t}+1}^{T} (\beta\rho)^t \Delta_{q_t} = \sum_{t=T+1}^{T+\tilde{t}} (\beta\rho)^t \tilde{q}_t + \beta(1-\rho)\frac{(\beta\rho)^T - (\beta\rho)^{T+\tilde{t}}}{1-\beta\rho}\phi p_k K$$

- Consider a maturity T , if we increase maturity to $T + 1$ we gain:
- We gain $[1 - \eta(1-\alpha)] K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$ today
- The financier requires compensation next period of $\frac{1}{\beta\rho}[1-\eta(1-\alpha)] K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$

- By increasing maturity one period we loose having the rights to collateral $\beta(1-\rho)\phi p_k K$ next period.
- If $\beta(1-\rho)\phi p_k K > \frac{1}{\beta\rho}[1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$ then the value of of pledging collateral an extra period is too big compared to the required dividend that we would need to pay.

Alternative proof, consider a LTD of maturity T. And the last coupon is $q_T = \pi^u$, then consider the deviation of increasing maturity of long term debt to $T+1$ and perceiving dividends in T:

$$
\begin{aligned}
n^u(K,z) &= K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}} \\
R^u(K,z) &= wn^u(K,z) \\
\pi^u(K,z) &= [1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K,z) \\
\pi^u(K,z) &= [1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}
\end{aligned}
$$

There is overcompensation when increasing maturity if:

$$\beta(1-\rho)\phi p_k K > \frac{1}{\beta\rho}[1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$$

$$\beta\rho\beta(1-\rho)\phi p_k K^{\frac{1-\eta}{1-\eta(1-\alpha)}} > [1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$$

$$K > \left[\frac{[1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{\beta\rho\beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

Let

$$\underline{K} = \left[\frac{[1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{\beta\rho\beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

Unconstrained level of capital:

$$K^u = z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

$$= \underline{K}\left[\frac{\beta\rho\beta(1-\rho)\phi p_k}{[1-\eta(1-\alpha)]}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} \left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

And then we can claim that $\underline{K} < K^u$ if:

$$1 < \left[\frac{\beta\rho\beta(1-\rho)\phi p_k}{[1-\eta(1-\alpha)]}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}} \left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

$$1 < \left[\frac{\beta\rho\beta(1-\rho)\phi}{[1-\eta(1-\alpha)]}\right]\left[\frac{\eta\alpha}{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}\right]$$

$$\left[\frac{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}{\eta\alpha}\right] < \left[\frac{\beta\rho\beta(1-\rho)\phi}{[1-\eta(1-\alpha)]}\right]$$

$$(1-\beta\rho)(1+r_f) < \eta\alpha\frac{\beta\rho\beta(1-\rho)\phi}{[1-\eta(1-\alpha)]}+\beta(1-\rho)$$

$$1+r_f-\beta < \eta\alpha\frac{\beta\rho\beta(1-\rho)\phi}{[1-\eta(1-\alpha)]}+\beta\rho r_f$$

We can get a condition on pledgeability:

$$\left[\frac{(1-\beta\rho)(1+r_f)[1-\eta(1-\alpha)]-\beta(1-\rho)\phi[1-\eta(1-\alpha)]}{\eta\alpha\beta\rho\beta(1-\rho)}\right] < \phi$$

6.6 R^u implementable

In principle it could be that ξ is so low that the conditional unconstrained level of short-term funding $R^u(K,\xi,\cdot)$ is no implementable, we show below that for reasonable parameterizations this is not the case.

$$\frac{1}{1-\beta\rho}[1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u \geq (1-\xi)zK^{\eta\alpha}\left[\frac{R^u}{w}\right]^{\eta(1-\alpha)}$$

$$\frac{1}{1-\beta\rho}[1-\eta(1-\alpha)]K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}} \geq (1-\xi)zK^{\eta\alpha}\left[K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}}z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}\right]^{\eta(1-\alpha}$$

$$\frac{1}{1-\beta\rho}[1-\eta(1-\alpha)] \geq (1-\xi)$$

Let ξ^u be the minimum enforcement level for which the unconstrained level of financing is enforceable.

$$\xi^u = 1 - \frac{[1-\eta(1-\alpha)]}{1-\beta\rho} = \frac{(1-\beta\rho)-1+\eta(1-\alpha)}{1-\beta\rho} = \frac{-\beta\rho+\eta(1-\alpha)}{1-\beta\rho}$$

Note that for $\beta\alpha \approx 0.7$ and $\alpha = 0.30$ since $\eta < 1$ $\xi^u \to 0$. Hence for reasonable parametrizations $\beta \to 1$ and $\rho = [0.75, 0.85]$ the unconstrained level of financing would always be implementable. Hence we will focus on this case.

Using $\frac{1+r_f}{\eta(1-\alpha)}R^u = K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}$

6.7 Price of long term debt First Best

The cum dividend price of long term debt with time to maturity $T-a$, loan size $p_k K$ and collateralizability ϕ is:

$$\begin{aligned} p(T-a,K,\phi) &= \sum_{a=1}^{T}(\beta\rho)^a[1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K) + \frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi p_k K \\ &= \beta\left[\rho\sum_{a=0}^{T-1}(\beta\rho)^a[1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K) + (1-\rho)\frac{1-(\beta\rho)^T}{1-\beta\rho}\phi p_k K\right] \end{aligned}$$

So we can equivalently write:

$$\begin{aligned} p_k K(1+r_f) &= [1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K) + \sum_{a=1}^{T}(\beta\rho)^a[1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K) + \frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho \\ &= [1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K) + p(T-a,K,\phi) \end{aligned}$$

$$p_k K(1+r_f) = \sum_{a=0}^{T} (\beta\rho)^a [1-\eta(1-\alpha)] \frac{1+r_f}{\eta(1-\alpha)} R^u(K) + \frac{1-(\beta\rho)^T}{1-\beta\rho} \beta(1-\rho)\phi p_k K$$
$$= \frac{1-(\beta\rho)^{T+1}}{1-\beta\rho} [1-\eta(1-\alpha)] \frac{1+r_f}{\eta(1-\alpha)} R^u(K) + \frac{1-(\beta\rho)^T}{1-\beta\rho} \beta(1-\rho)\phi p_k K$$

6.8 Algorithm: Matlab Implementation

Let

$$\mathcal{M}(T,\xi,\phi,z) = \min\left\{ \left[\frac{(\beta\rho)^{T-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \left\{\omega_0 + K^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\} \;;\; 1 \right\}$$

If frictions are not relevant then:

$$\left[\frac{(\beta\rho)^{T-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \left\{\omega_0 + K^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\} \geq 1$$

Given T we have the $K_{T,u}$, so we can ask:

$$\left[\frac{(\beta\rho)^{T-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \left\{\omega_0 + K_{T,u}^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\} \geq 1 \qquad \forall\, a \;\in\; \{0,\ldots,T-a\}$$

Note that if this is true for $a=0$, then it's true for all the rest.

We will go on till we find a T for which $a=0$

$$\left[\frac{(\beta\rho)^{T-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\left\{\omega_0+K_{T,u}^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\}<1 \qquad \forall\ a\ \in\ \{0,\ldots,T-a\}$$

Note:

$$\left[\frac{\frac{1-(\beta\rho)^{T+1}}{1-\beta\rho}[1-\eta(1-\alpha)]z^{\frac{1}{1-\eta(1-\alpha)}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{p_k(1+r_f)-\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}=K_{T,u}$$

And $\omega_0=[1-\eta(1-\alpha)]^{\frac{1}{\eta(1-\alpha)}}$ and $\omega_1(z)=\frac{[\beta(1-\rho)\phi p_k]^{\frac{1}{\eta(1-\alpha)}}}{z^{\frac{1}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}}$

$$K_{T,u}^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}=\left[\frac{\frac{1-(\beta\rho)^{T+1}}{1-\beta\rho}[1-\eta(1-\alpha)]}{p_k(1+r_f)-\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi p_k}\right]^{\frac{1}{\eta(1-\alpha)}}z^{\frac{1}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}$$

Then:

$$\begin{aligned}\omega_1(z)K_{T,u}^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}&=[\beta(1-\rho)\phi p_k]^{\frac{1}{\eta(1-\alpha)}}\left[\frac{\frac{1-(\beta\rho)^{T+1}}{1-\beta\rho}[1-\eta(1-\alpha)]}{p_k(1+r_f)-\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi p_k}\right]^{\frac{1}{\eta(1-\alpha)}}\\&=[\beta(1-\rho)\phi]^{\frac{1}{\eta(1-\alpha)}}\left[\frac{\frac{1-(\beta\rho)^{T+1}}{1-\beta\rho}[1-\eta(1-\alpha)]}{(1+r_f)-\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi}\right]^{\frac{1}{\eta(1-\alpha)}}\end{aligned}$$

$$\left[\frac{(\beta\rho)^{T-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\left\{[1-\eta(1-\alpha)]^{\frac{1}{\eta(1-\alpha)}}+[\beta(1-\rho)\phi]^{\frac{1}{\eta(1-\alpha)}}\left[\frac{\frac{1-(\beta\rho)^{T+1}}{1-\beta\rho}[1-\eta(1-\alpha)]}{(1+r_f)-\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi}\right]^{\frac{1}{\eta(1-\alpha)}}\right\}<$$

$$\left[\frac{(\beta\rho)^{T-a}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}[1-\eta(1-\alpha)]^{\frac{1}{\eta(1-\alpha)}}\left\{1+[\beta(1-\rho)\phi]^{\frac{1}{\eta(1-\alpha)}}\left[\frac{\frac{1-(\beta\rho)^{T+1}}{1-\beta\rho}}{(1+r_f)-\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi}\right]^{\frac{1}{\eta(1-\alpha)}}\right\}<$$

Then, solve the equation for $\pi_0^c+\pi_1^u+\cdots+\pi_T^u$, this will give us $K_{T,c0}$

$$p_kK(1+r_f)=\pi_0+\sum_{a=1}^{T}(\beta\rho)^a\pi^u+\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi p_kK$$

$$p_kK(1+r_f)=\pi_0+\beta\rho\frac{1-(\beta\rho)^T}{1-\beta\rho}\pi^u+\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi p_kK$$

$$p_kK(1+r_f)=\frac{1}{[1-\eta(1-\alpha)]}\left[\frac{(\beta\rho)^T}{1-\beta\rho}\frac{[1-\eta(1-\alpha)]}{1-\xi}-[\eta(1-\alpha)]\left[\frac{(\beta\rho)^T}{1-\beta\rho}\frac{[1-\eta(1-\alpha)]}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\right]\pi^u+\beta\rho\frac{1-(\beta\rho)^T}{1-\beta\rho}\pi^u+\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi p_kK$$

$$p_kK(1+r_f)=\frac{(\beta\rho)^T}{1-\beta\rho}\frac{1}{1-\xi}\left[1-[\eta(1-\alpha)]\left[\frac{(\beta\rho)^T}{1-\beta\rho}\frac{[1-\eta(1-\alpha)]}{1-\xi}\right]^{\frac{1-\eta(1-\alpha)}{\eta(1-\alpha)}}\right]\pi^u+\beta\rho\frac{1-(\beta\rho)^T}{1-\beta\rho}\pi^u+\frac{1-(\beta\rho)^T}{1-\beta\rho}\beta(1-\rho)\phi p_kK$$

Using $\pi_a=\min\left\{\frac{1}{[1-\eta(1-\alpha)]}\left[\frac{(\beta\rho)^{T-a}}{1-\beta\rho}\frac{[1-\eta(1-\alpha)]}{1-\xi}-[\eta(1-\alpha)]\left[\frac{(\beta\rho)^{T-a}}{1-\beta\rho}\frac{[1-\eta(1-\alpha)]}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\right],1\right\}\pi^u$
and $\pi^u(K,z)=[1-\eta(1-\alpha)]\frac{1+r_f}{\eta(1-\alpha)}R^u(K,z)$

6.8.1 Coupon adjustment

- Case of no collateral in the last period:

  If the code maturity is T and we need to adjust it to $T-1$ full coupons

$$p_k(1+r_f)K_*=\frac{1-(\beta\rho)^T}{1-\beta\rho}\pi(K_*)+\frac{1-(\beta\rho)^{T-1}}{1-\beta\rho}\phi p_kK_*+(\beta\rho)^T\theta$$

If the code maturity is T and we need to adjust it to T full coupons plus a payment in $T+1$

$$p_k(1+r_f)K_* = \frac{1-(\beta\rho)^{T+1}}{1-\beta\rho}\pi(K_*) + \frac{1-(\beta\rho)^{T}}{1-\beta\rho}\phi p_k K_* + (\beta\rho)^{T+1}\theta$$

- Case of collateral in the last period and the last 2 coupons are lower than the max coupon:

6.8.2 Scale adjustment in case of frictions

First we want to get $\hat{t}$ and T from the code.

If the code maturity is T and we need to adjust it to $T-1$ full coupons

$$p_k(1+r_f)K_* = \frac{1-(\beta\rho)^{T}}{1-\beta\rho}\pi(K_{T-1}+\Delta) + \frac{1-(\beta\rho)^{T-1}}{1-\beta\rho}\phi p_k(K_{T-1}+\Delta) + (\beta\rho)^{T}\theta$$

$$\Pi^F(K_{T-1})+$$

6.8.3 Duration of the ELC

$$R_t(K) = \begin{cases} \left[\frac{(\beta\rho)^{T(K)+1-t}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}} \left\{\omega_0 + K^{\frac{1-\eta}{\eta(1-\alpha)[1-\eta(1-\alpha)]}}\omega_1(z)\right\} R^u(K,z) & \text{if} \quad t \leq \hat{t} \\ R^u(K,z) & \text{if} \quad t > \hat{t} \end{cases}$$

Assume $\phi = 0$ then:

$$\left[\frac{(\beta\rho)^{T(K)+1-\hat{t}}}{1-\beta\rho}\frac{1}{1-\xi}\right]^{\frac{1}{\eta(1-\alpha)}}\omega_0 = 1$$

$$(\beta\rho)^{T(K)+1-t} = (1-\beta\rho)(1-\xi)\left[\frac{1}{\omega_0}\right]^{\eta(1-\alpha)}$$

$$(T(K)+1-\hat{t})\ln((\beta\rho)) = \ln\left((1-\beta\rho)(1-\xi)\left[\frac{1}{\omega_0}\right]^{\eta(1-\alpha)}\right)$$

$$(T(K)+1-\hat{t})\ln((\beta\rho)) = \ln\left((1-\beta\rho)\left[\frac{1}{\omega_0}\right]^{\eta(1-\alpha)}\right) + \ln\left((1-\xi)\right)$$

$$T(K)-\hat{t} = \frac{\ln\left((1-\beta\rho)\left[\frac{1}{\omega_0}\right]^{\eta(1-\alpha)}\right)}{\ln((\beta\rho))} + \frac{\ln\left((1-\xi)\right)}{\ln((\beta\rho))} - 1$$

$$\hat{t} = T(K)+1-\frac{\ln\left((1-\beta\rho)\left[\frac{1}{\omega_0}\right]^{\eta(1-\alpha)}\right)}{\ln((\beta\rho))} - \frac{\ln\left((1-\xi)\right)}{\ln((\beta\rho))}$$

We want to know:

$$\frac{\partial\hat{t}(\xi)}{\partial\xi} = \frac{\partial T(K)}{\partial\xi} + \frac{1}{1-\xi}\frac{1}{\ln((\beta\rho))}$$

The direct effect $\frac{1}{1-\xi}\frac{1}{\ln((\beta\rho))}$ is negative since $\beta\rho < 1$.

$$\frac{\partial T(K)}{\partial\xi} = \frac{\partial T(K(\xi))}{\partial K(\xi)}\frac{\partial K(\xi)}{\partial\xi} \geq 0$$

6.9 Calibration

We calibrate the model to the US economy assuming that $\xi = 1$.

Employment size of firms:

$$n^u(K,z) = K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}$$

$$K^u = z^{\frac{1}{1-\eta}} \left[\frac{\eta\alpha \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{1-\eta(1-\alpha)}{1-\eta}}$$

$$\left[K^u\right]^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} = z^{\frac{\eta\alpha}{[1-\eta(1-\alpha)][1-\eta]}} \left[\frac{\eta\alpha \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{\eta(1-\alpha)}{1-\eta(1-\alpha)}}}{(1-\beta\rho)p_k(1+r_f) - \beta(1-\rho)\phi p_k}\right]^{\frac{\eta\alpha}{1-\eta}}$$

$$
\begin{aligned}
\frac{K}{n^u(K,z)} &= \frac{K}{K^{\frac{\eta\alpha}{1-\eta(1-\alpha)}} z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}} \\
&= \frac{K^{\frac{1-\eta}{1-\eta(1-\alpha)}}}{z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}} \\
\frac{K^u}{n^u(K^u,z)} &= \frac{z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}\right]}{z^{\frac{1}{1-\eta(1-\alpha)}} \left[\frac{\eta(1-\alpha)}{(1+r_f)w}\right]^{\frac{1}{1-\eta(1-\alpha)}}} \\
&= \frac{(1+r_f)w}{\eta(1-\alpha)} \frac{\eta\alpha}{(1-\beta\rho)p_k(1+r_f)-\beta(1-\rho)\phi p_k}
\end{aligned}
$$

If we target firms of 10 yrs old:

- w is $\frac{\text{Wage bill}}{\text{Employment}}$
- $\rho = 0.85$ (Average exit rate of US economy is roughly 10% according to Business Dynamics Statistics.)
- β to match a risk free rate of less than a year $r_f = \frac{1-\sqrt{\beta}}{\sqrt{\beta}}$
- $\eta = 0.95$.
- $\frac{K}{n^u(K,z)}$ is the average capital to employment ratio of firms
- $\phi = 0.4$ is in the range of reasonable values according to Kermani and Ma (2022).
- We can get $p_k = \frac{n^u(K,z)}{K} \frac{(1+r_f)w}{\eta(1-\alpha)} \frac{\eta\alpha}{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}$

We will calibrate p_k to be the price of capital in US, we have in the data:

$$\frac{p_k K}{n}$$

6.10 Capital Labor Ratios US

Code	Sector	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
01-99	Total	0.38	0.38	0.38	0.37	0.37	0.37	0.37	0.37	0.37	0.37
01-03	Agriculture, hunting, forestry and fishing [A]	0.43	0.43	0.43	0.44	0.46	0.44	0.45	0.45	0.45	0.45
05-09	Mining and quarrying [B]	3.01	2.77	2.62	2.65	2.64	2.97	3.54	3.43	3.20	3.21
10-33	Manufacturing [C]	0.29	0.29	0.29	0.29	0.29	0.29	0.30	0.30	0.30	0.30
35-39	Electricity, gas (...) and remediation activities [D-E]	4.06	4.10	4.15	4.19	4.21	4.22	4.29	4.29	4.27	4.30
41-43	Construction [F]	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
45-47	Wholesale and retail trade, repair of motor vehicles [G]	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
49-53	Transportation and storage [H]	0.24	0.24	0.24	0.24	0.24	0.24	0.24	0.24	0.23	0.23
55-56	Accommodation and food service activities [I]	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
58-63	Information and communication [J]	0.45	0.45	0.45	0.46	0.47	0.47	0.48	0.50	0.51	0.52
69-75	Professional, scientific and technical activities [M]	0.11	0.11	0.11	0.11	0.11	0.10	0.10	0.11	0.11	0.11
77-82	Administrative and support service activities [N]	0.07	0.07	0.07	0.07	0.07	0.08	0.08	0.08	0.08	0.09
87-88	Residential care and social work activities [QB]	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
90-93	Arts, entertainment and recreation [R]	0.30	0.30	0.29	0.29	0.28	0.29	0.28	0.29	0.29	0.29
45-99	Total services	0.36	0.36	0.36	0.35	0.35	0.35	0.35	0.35	0.35	0.35

Table 8: United States Capital-Labor ratio based on OECD STAN number of employees and the volume of net capital stock.

	Local Currency	2017 PPP $	U.S. dollars	2017 PPP $	U.S. dollars
Belgium	48201	53938	63901	1.33	1.57
Estonia	9828	16762	13029	0.41	0.32
Finland	36528	37341	48426	0.92	1.19
France	30804	34347	40837	0.84	1.00
Germany	34584	40598	45849	1.00	1.13
Greece	21588	27558	28620	0.68	0.70
Hungary	2692080	18835	12946	0.46	0.32
Italy	27432	33529	36367	0.82	0.89
Portugal	9372	13397	12425	0.33	0.31
Slovenia	17724	25390	23497	0.62	0.58
Spain	22066	27572	29253	0.68	0.72
Sweden	33650	3681	4669	0.09	0.11
United States	40668	40668	40668	1.00	1.00

Table 9: Annualized wages and relative wages compared to US for the year 2010. *Source: International Labor Organization.*

We get a price of capital for US in dollar using

$$p_k = \frac{n^u(K,z)}{K}\frac{(1+r_f)w}{\eta(1-\alpha)}\frac{\eta\alpha}{(1-\beta\rho)(1+r_f)-\beta(1-\rho)\phi}$$

Denote this p_k^{US} in dollars, we take the wages in dollars w^{US} and for each country convert

them to local currency and divide them by the price of consumption (private consumption deflator) of that country.

Table 10: Alternative calibrations of prices for the countries in the sample

Country	Local Currency	2017 PPP $	U.S. dollars	(2017 PPP $)	(U.S. dollars)	Deflator	pk	Relative p_k	p_k/w	Relative p_k/w
Belgium	48200.88	53937.60	63900.72	1.33	1.57	0.89	491372.38	0.89	9.11	0.67
Estonia	9828.00	16762.08	13029.12	0.41	0.32	0.59	322392.28	0.59	19.23	1.42
Finland	36528.00	37340.88	48425.76	0.92	1.19	0.98	537884.14	0.98	14.40	1.07
France	30804.00	34347.24	40837.32	0.84	1.00	0.90	493131.41	0.90	14.36	1.06
Germany	34584.00	40597.80	45848.64	1.00	1.13	0.85	468403.48	0.85	11.54	0.85
Greece	21588.00	27558.00	28619.52	0.68	0.70	0.78	430736.93	0.78	15.63	1.16
Hungary	2692080.00	18834.96	12946.20	0.46	0.32	142.93	78590608.63	142.93	4172.59	308.61
Italy	27432.00	33528.84	36367.08	0.82	0.89	0.82	449869.28	0.82	13.42	0.99
Portugal	9372.00	13396.56	12424.56	0.33	0.31	0.70	384668.28	0.70	28.71	2.12
Slovenia	17724.00	25390.20	23496.96	0.62	0.58	0.70	383833.62	0.70	15.12	1.12
Spain	22065.84	27572.40	29253.00	0.68	0.72	0.80	440041.15	0.80	15.96	1.18
Sweden	33649.80	3681.00	4668.72	0.09	0.11	9.14	5026481.20	9.14	1365.52	101.00
United States	40668.24	40668.24	40668.24	1.00	1.00	1.00	549854.01	1.00	13.52	1.00

Let K, N denote the aggregate factors of production used in the economy, N_f the number of firms, then once we consider firms as a reproducible factor we can recover an homogenized aggregate production function. We can write GDP in our model economy as:

$$Y = \left(K^{\alpha} N^{1-\alpha}\right)^{\eta} \left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{N_f} \right)^{1-\eta} N_f^{1-\eta}$$

We get then a transformation of the average firm productivity $\left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{N_f} \right)$ that can be calculated:

$$\left[\frac{Y}{(K^{\alpha} N^{1-\alpha})^{\eta}} \right]^{\frac{1}{1-\eta}} \frac{1}{N_f} = \left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{N_f} \right)$$

Using US GDP in 2010, the number of establishments in 2010 from BDS, capital stock and number of employees in USA we get a value for $\left(\frac{\sum_i z_i^{\frac{1}{1-\eta}}}{N_f} \right)$. Assuming that all firms have the same productivity then: $\bar{z} N_f = \sum_i z_i^{\frac{1}{1-\eta}} = z_i^{\frac{1}{1-\eta}} N_f$ and $\bar{z}^{1-\eta} = z_i$. In our calibration: $z_i^{US} = 170089$