An Arrow-Pratt Theory of Preference for Early Resolution of Uncertainty

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Abstract: This paper develops a theory of the elasticity of preference for early resolution of uncertainty (PER) that parallels the Arrow-Pratt measure of risk aversion in expected utility theory. We demonstrate that the local welfare gain of early resolution of uncertainty is equal to the product of the elasticity of PER and the conditional variance of continuation utility. We illustrate how asset market data can be used to estimate the elasticity of PER and how this measure can be used to compute the welfare gain for various experiments of early resolution of uncertainty.

Keywords: Preference for early resolution of uncertainty, risk aversion, generalized risk sensitivity, elasticity, long-run risk

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1 Introduction

In this paper, we develop a quantitative measure of preference for early resolution of uncertainty (PER) that parallels the Arrow-Pratt measure of risk aversion in expected utility theory. As emphasized by Kreps and Porteus (1978), choice under uncertainty in a dynamic environment requires decision makers to express a preference over the timing of the resolution of uncertainty. In fact, dynamic preferences used in economics often exhibit a non-trivial preference over the timing of resolution of uncertainty. Prominent examples are the recursive preference of Kreps and Porteus (1978) and Epstein and Zin (1989), and the robust control preference of Hansen and Sargent (2008). Despite its wide application, the literature lacks a quantitative measure of PER. This paper develops one.

A theory of an Arrow-Pratt measure of PER is needed to study the quantitative impact of PER, just like the Arrow-Pratt theory of risk aversion has been fundamental in the study of risk aversion. The Arrow-Pratt measure of risk aversion has an intuitive interpretation. It is the additional amount of consumption that is needed to compensate a risk-averse decision maker per unit of variance. Analogously, our notion of elasticity of PER, or the Arrow-Pratt measure of PER, is defined as the additional amount of utility needed to compensate a decision maker for delaying the resolution of uncertainty for one period. The main purpose of the paper is to demonstrate how such a measure can be used to quantify the impact of PER in preferences.

We focus on two applications, a welfare application, and an asset pricing application. We demonstrate how to use our elasticity measure to compute welfare gains of one-period early resolution of uncertainty and that of multiple-period early resolution of uncertainty through successive approximations. As in the case of risk aversion, the welfare gain computed using the Arrow-Pratt measure of risk aversion is a local approximation. It is fully accurate only under special cases such as the constant elasticity case. Our computation is also an approximation, but the advantage is that given the elasticity measure, one does not need the knowledge of the entire utility function to compute the welfare gain. In addition, such elasticity can be measured from the data without estimating the entire utility function. We also show that our local approximation is quite accurate by comparing it to global solutions.

In a second application, we show how to estimate our elasticity measure of PER from asset prices. Our procedure builds on the previous work of Ai, Bansal, Guo, and Yaron (2023). Ai, Bansal, Guo, and Yaron (2023) demonstrate that in a representative agent economy, PER is equivalent to the risk premium for resolution of information quality is positive. We show how to use this risk premium together with the volatility of the market portfolio to estimate the elasticity of PER. Taken together, our two examples provide a procedure to estimate the structural parameter of the elasticity of PER from the data and to use this parameter to answer welfare questions about the timing of resolution of uncertainty.

Literature review The concept of PER is developed by Kreps and Porteus (1978). Strzalecki (2013) studies PER in models with ambiguity aversion. As emphasized by Strzalecki (2013), many

non-expected utility models exhibit PER, such as the recursive preference developed by Epstein and Zin (1989), the robust control preference of Hansen and Sargent (2005, 2007), and the related multiplier preference of Strzalecki (2011), the variational ambiguity-averse preference of Maccheroni, Marinacci, and Rustichini (2006a,b), and the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005, 2009). The only model of ambiguity aversion that is indifferent towards the timing of resolution of uncertainty is the maxmin expected utility of Gilboa and Schmeidler (1989), the dynamic version of which is studied by Chen and Epstein (2002) and Epstein and Schneider (2003).

Our approach to identify and estimate the elasticity of PER builds on the methodological contributions of Ai and Bansal (2018) and Ai, Bansal, Guo, and Yaron (2023). Ai and Bansal (2018) develop a test for generalized risk sensitivity from asset market data, and Ai, Bansal, Guo, and Yaron (2023) provide an identification theorem for PER. Our estimation procedure for the elasticity of PER first requires estimating an elasticity of generalized risk sensitivity. Relatedly, several papers provide measures of PER in the context of recursive utility. Epstein, Farhi, and Strzalecki (2014) calculate the welfare gain of PER based on calibrated long-run risk models. Kadan and Manela (2019) estimate the value of information in a model with recursive utility. Schlag, Thimme, and Weber (2021) find suggestive evidence for PER using options market data. These papers assume the CES form of utility function and do not distinguish PER from GRS, or uncertainty aversion. In contrast, our procedure does not require an assumption on the functional form of utility, as the above papers do. However, our approach allows us to replicate the results of the above papers, in particular those in Epstein, Farhi, and Strzalecki (2014).

Many asset pricing models are based on intertemporal preferences that feature PER. We refer readers to Epstein and Schneider (2010) for a review of asset pricing studies with the maxmin expected utility model, Ju and Miao (2012) for an application of the smooth ambiguity-averse preference, Hansen and Sargent (2008) for the robust control preference, Routledge and Zin (2010) for an asset pricing model with disappointment aversion, and Bansal and Yaron (2004), Bansal (2007), and Hansen, Heaton, and Li (2008) for the long-run risk model that builds on recursive preferences. Borovicka and Stachurski (2020) provide necessary and sufficient conditions for the existence and uniqueness of recursive preferences with constant elasticities. Bhamara and Uppal (2006) study the role of risk aversion and intertemporal elasticity of substitution in portfolio choice problems. Bidder and Dew-Becker (2016) link ambiguity aversion to long-run risk models. Skiadas (2009) provides an excellent textbook treatment of recursive-preferences-based asset pricing theory.

Our asset pricing application makes use of risk premium realized around macroeconomic announcement days. The previous literature, surveyed in Ai, Bansal, and Guo (2024), documents that stock market returns and Sharpe ratios are significantly higher on days with macroeconomic news releases both in the United States (Savor and Wilson (2013)) and internationally (Brusa, Savor, and Wilson (2020)). Lucca and Moench (2015) find similar patterns and document a pre-FOMC announcement drift. Mueller, Tahbaz-Salehi, and Vedolin (2017) document an FOMC announcement premium on the foreign exchange market and attribute it to compensation to financially constrained intermediaries. Fisher, Martineau, and Sheng (2022), Liu, Tang, and Zhou (2022), and Johannes,

Kaeck, and Seeger (2023) develop ex-ante measures of macroeconomic announcement premium.

The rest of the paper is organized as follows. In Section 2, we develop the concept of Arrow-Pratt measure of PER. We demonstrate how to use our measure of the elasticity of PER to compute the welfare gain of early resolution of uncertainty in multiple periods and in infinite horizon settings in Section 3. Section 4 provides a procedure for estimating the elasticities of PER from asset prices. Section 5 concludes.

2 Elasticity of PER

2.1 An intuitive discussion

Our definition of the elasticity of PER parallels the development of the Arrow-Pratt measure of risk aversion. To illustrate the basic idea, consider an expected utility consumer who evaluates consumption by using a strictly increasing and weakly concave utility function u: $E\left[u\left(C\right)\right]$. We can ask, what is the utility gain for eliminating all uncertainty in C, that is, consuming $E\left[C\right]$ instead of C? The utility gain, measured in consumption terms, is denoted as ϵ and has to solve:

$$u\left(E\left[C\right]-\epsilon\right)=E\left[u\left(C\right)\right]. \tag{1}$$

Using a Taylor approximation for $u\left(C\right)$ around $\bar{C}=E\left[C\right]$, we can write the right hand of (1) as

$$E\left[u\left(C\right)\right]\approx E\left[u\left(\bar{C}\right)+u'\left(\bar{C}\right)\left(C-\bar{C}\right)+\frac{1}{2}u''\left(\bar{C}\right)\left(C-\bar{C}\right)^{2}\right]=u\left(\bar{C}\right)+\frac{1}{2}u''\left(\bar{C}\right)Var\left[C\right]. \tag{2}$$

Similarly, a Taylor approximation for the left hand side of (1) gives

$$u\left(E\left[C\right]-\epsilon\right)\approx u\left(\bar{C}\right)-u'\left(\bar{C}\right)\epsilon. \tag{3}$$

Comparing (2) with (3), we can write the risk compensation ϵ as

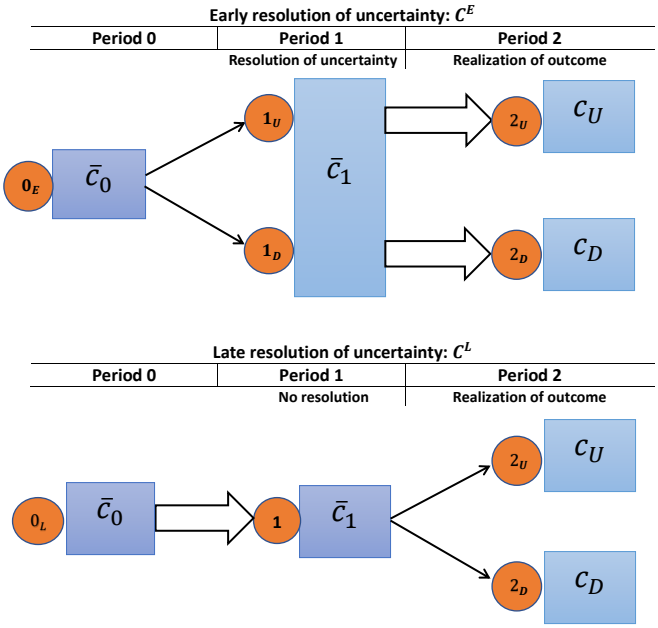
$$\epsilon\approx\frac{1}{2}\times\left[-\frac{u''\left(\bar{C}\right)}{u'\left(\bar{C}\right)}\right]Var\left[C\right] \tag{4}$$

The equation above motivates the definition of Arrow-Pratt measure of risk aversion as $\mathcal{A}u\left(\bar{C}\right)=-\frac{u''\left(\bar{C}\right)}{u'\left(\bar{C}\right)}$. It is the amount of consumption goods needed to compensate the agent per unit of variance.

To introduce an analogous notion of elasticity of PER, we consider the two-period example of Kreps and Porteus (1978), which we illustrate in Figure 1. In both the top panel and the bottom panel, consumption in period 2, denoted $C_2\left(Z\right)$, is the only source of uncertainty. We use $C_0=\bar{c}_0$ and $C_1=\bar{c}_1$ to illustrate that consumption in periods 0 and 1 is deterministic. We use circles to denote the agent's information node. In both panels, the state variable Z can take on two values, Z_U and Z_D, which determine consumption in period 2. The only difference between the two panels is that the top panel represents early resolution of uncertainty: the agent receives a signal s in

period 1 that fully reveals the value of Z. As a result, the consumer is able to distinguish node 1_U from 1_D, that is, she knows the value of C_2 at time 1. The bottom panel represents late resolution of uncertainty. Here, the agent does not know the value of Z, and hence, the value of C_2 until period 2. We can interpret late resolution of uncertainty as the case in which the signal in period 1 is trivial, $s = \varnothing$. In summary, the distribution of consumption in both panels is identical. They only differ in the timing of the resolution of uncertainty about Z.

Figure 1: Early versus late resolution of uncertainty



This future illustrates the notion of PER. The blue squares contain consumption and the orange circles represent information sets. Both panels have identical unconditional distributions of consumption. The top panel features early resolution, as the uncertainty about c_2 is resolved one period earlier in period 1. The bottom panel corresponds to the case of late resolution because the value of c_2 is not revealed to the consumer until period 2.

We assume that the consumer's preference can be represented recursively by

$$V_t = u\left(C_t\right) + \beta \mathcal{I}\left[V_{t+1}\right], \tag{5}$$

where u is a utility that computes the current-period utility and $\mathcal{I}$ computes the certainty equivalent of the continuation utility V_{t+1}. We further assume that the certainty equivalent functional $\mathcal{I}$ takes the following form:

$$\mathcal{I}\left[V_{t+1}\right] = h^{-1} E_t\left[h\left(V_{t+1}\right)\right], \tag{6}$$

where h is a strictly increasing function. The subscript t represents conditioning on information at time t. Under the above notation, the intertemporal preference is represented by $\{u, h, \beta\}$, where u and h are real valued functions and β is a discount rate.

Using the above setup, the period-0 utility associated with early resolution is

$$V_0^E = u(C_0) + \beta \mathcal{I}[u(C_1) + \beta u(C_2)] = u(C_0) + \beta \underbrace{h^{-1}\{E[h\{u(C_1) + \beta u(C_2)\}]\}}_{early\ resolution}, \tag{7}$$

where the utility at time 1 is computed by first aggregating across time and then computing its certainty equivalent using $\mathcal{I}$. The period-0 utility for late resolution is

$$V_0^L = u(C_0) + \beta\{u(C_1) + \beta I[u(C_2)]\} = u(C_0) + \beta \underbrace{\{u(C_1) + \beta h^{-1} E[h\{u(C_2)\}]\}}_{late\ resolution}, \tag{8}$$

where we first compute the certainty equivalent of $u(C_2)$ and then aggregate across time. As a result, to compare the utility for early resolution to that of late resolution, it is enough to compare the certainty equivalent of date-1 utility, which are bracketed in equations (7) and (8). We denote the certainty equivalent for early and late resolution to be $CE^E = h^{-1}\{E[h\{u(C_1) + \beta u(C_2)\}]\}$ and $CE^L = u(C_1) + \beta h^{-1} E[h\{u(C_2)\}]$, respectively. To simplify notation, we denote $u_1 = u(C_1)$ and $e = h(u(C_2))$. To compare early and late resolution, it is enough to compare their monotonic transformations:

$$h\left(CE^E\right) = E\left[h\left\{u_1 + \beta h^{-1} e\right\}\right] = E[f(e)], \tag{9}$$

and

$$h\left(CE^L\right) = h\left\{u_1 + \beta h^{-1} E[e]\right\} = f(E[e]), \tag{10}$$

where we define

$$f(e) = h\left\{u_1 + \beta h^{-1}(e)\right\}. \tag{11}$$

In what follows, we provide a heuristic discussion on how to use an appropriate notion of elasticity of PER to compute the difference between CE^E and CE^L via a local approximation.

Comparing Equation (9) with (10), by Jensen's inequality, preference for early resolution is equivalent to $f(e)$ being a convex function. Intuitively, the elasticity of PER should be related to the Arrow-Pratt measure of f. Using the risk compensation formula (4), we know that the solution to $E[f(e)] = f(E[e] + \epsilon)$, or

$$E\left[h\left\{u_1 + \beta h^{-1}(e)\right\}\right] = h\left\{u_1 + \beta h^{-1}(E[e] + \epsilon)\right\} \tag{12}$$

is approximately

$$\epsilon \approx \frac{1}{2}\frac{f''(E[e])}{f'(E[e])} Var[e] = -\frac{1}{2}\mathcal{A}f(E[e]|u_1)\, Var[e], \tag{13}$$

where we use the notation $\mathcal{A}f(E[e]) = -\frac{f''(E[e])}{f'(E[e])}$ for the Arrow Pratt measure of risk aversion of f.[1] The left-hand side of Equation (12) is the utility of early resolution and the right-hand side of Equation (12) is the compensated utility for late resolution. PER corresponds to a convex f,

[1] Here, ϵ has a negative sign, because Equation (1) and (12) differ in sign.

in which case $-\mathcal{A}f\left(E\left[e\right]\right) > 0$. For this reason, we use a different sign for ϵ in Equations (1) and (12) to keep the convention that the compensation is positive for the case of preference for early resolution.

Intuitively, in the case of late resolution, the consumer knows $E\left[e\right]$ but not the value of e at time 1. In the case of early resolution, the consumer knows the value of e. A preference for the early resolution is essentially a preference for the conditional variance of e. Equation (12) provide an approximate solution to the following question: suppose we compensate the consumer in units of $E\left[e\right]$, how much compensation is needed per unit of $Var\left[e\right]$?

$E\left[e\right]$ and $Var\left[e\right]$ are hard to interpret. It is more convenient to express this utility gain in terms of units of continuation utility. We first work on $Var\left[e\right]$. In period 1, after early resolution of uncertainty, the consumer's utility is $V_1^E = u\left(C_1\right) + \beta u\left(C_2\right)$, which, using the definition of e, can be written as:

$$V_1^E = u_1 + \beta h^{-1}\left(e\right). \tag{14}$$

Using a first-order approximation of (14), we can write $V_1^E = u_1 + \beta\left[h^{-1}\left(E\left[e\right]\right) + \frac{1}{h'\left\{h^{-1}\left(E[e]\right)\right\}}\left(e - E\left[e\right]\right)\right]$, which implies that, to a first-order approximation, $Var\left[V_1\right] = \left(\frac{\beta}{h'\left\{h^{-1}\left(E[e]\right)\right\}}\right)^2 Var\left[e\right]$. This allows us to write the solution ϵ in (13) as:

$$\epsilon \approx -\frac{1}{2}\left(\frac{h'\left\{h^{-1}\left(E\left[e\right]\right)\right\}}{\beta}\right)^2 \mathcal{A}f\left(E\left[e\right]|\, u_1\right) Var\left[V_1^E\right]. \tag{15}$$

To translate compensation in terms of $E\left[e\right]$ into certainty equivalent units, we apply a monotonic transformation h^{-1} to both sides of (12) to compute the difference between CE^E and CE^L using a first-order Taylor expansion:

$$\begin{aligned} h^{-1}\left\{E\left[h\left\{u_1 + \beta h^{-1}\left(e\right)\right\}\right]\right\} = & u_1 + \beta h^{-1}\left(E\left[e\right] + \epsilon\right) \\ \approx & u_1 + \beta h^{-1}\left(E\left[e\right]\right) + \beta \frac{1}{h'\left\{h^{-1}\left(E\left[e\right]\right)\right\}}\epsilon, \end{aligned}$$

where the second equality is a first-order approximation. Using the definition of CE^E and CE^L and replacing ϵ in the above equation using Equation (15), we have:

$$\begin{aligned} CE^E - CE^L &= \beta \frac{1}{h'\left\{h^{-1}\left(E\left[e\right]\right)\right\}}\epsilon \\ &= \frac{1}{2}\underbrace{\left[-\frac{h'\left\{h^{-1}\left(E\left[e\right]\right)\right\}}{\beta}\mathcal{A}f\left(E\left[e\right]\right)\right]}_{\eta_{PER}} Var\left[V_1^E\right]. \end{aligned}$$

This motivates defining the elasticity of PER as

$$\eta_{PER} = -\frac{h'\left\{h^{-1}\left(E\left[e\right]\right)\right\}}{\beta}\mathcal{A}f\left(E\left[e\right]\right), \tag{16}$$

which allows us to express the welfare gain of early resolution locally as a linear function of the variance of continuation utility.

In the next sections, we first provide an analytical framework to formalize the above definition of η_{PER}. We then demonstrate how η_{PER} can be used for various asset pricing and welfare calculations.

2.2 Elasticity of PER for dynamic preferences

We consider dynamic preferences defined by the recursive relation (5). Let $(\Omega, \mathcal{F}, P)$ be a probability space from which all uncertainty is generated. Let $\{Z_t\}_{t=1}^T$ be a Markov process where for each t, $Z_t : \Omega \to \boldsymbol{Z}$ where $\boldsymbol{Z}$ is the state space of the Markov chain. Let $C : \boldsymbol{Z} \to R$ be a bounded function that takes values in R. Let u be a real-valued, strictly increasing, and bounded utility function. With a slight abuse of notation, for any $z \in \boldsymbol{Z}$, we denote $u(C(z))$ as $u(z)$. That is, $u(z)$ is the per-period utility of consuming $C(z)$. Let h be an analytic function on its domain with strictly positive derivatives. We focus on recursive utility represented by $\{u, h, \beta\}$ constructed in the following way. Given a consumption process $\{C(Z_t)\}_{t=1}^T$, the continuation utility of the agent at time t is defined recursively using recursion (5). That is, $V_T(z) = u(z)$ for all $z \in \boldsymbol{Z}$, and for $t = T-1, T-2, \cdots 1$,

$$V_t(z) = u(z) + \beta h^{-1}\left\{E\left[h \circ V_{t+1}(Z_{t+1}) \middle| Z_t = z\right]\right\}, \tag{17}$$

where $h \circ V$ denotes the composition of h and V.

We think of the above construction of utility as a benchmark case where uncertainty about Z_t does not resolve until time t when consumption happens. In the language of Epstein, Farhi, and Strzalecki (2014), this is the utility for gradual resolution of uncertainty. To incorporate general patterns of early resolution of uncertainty, we define a utility process relative to a filtration. Let $\{\mathcal{F}_t\}_{t=1}^T$ be a filtration that is weakly larger than the filtration generated by $\{Z_t\}_{t=1}^T$, that is, for all t, $\mathcal{F}_t \supseteq \sigma\{Z_i : 1 \leq i \leq t\}$. We can define a utility process relative to filtration $\{\mathcal{F}_t\}_{t=1}^T$ recursively by

$$V_t = u(C(Z_t)) + \beta h^{-1}\left\{E\left[h \circ V_{t+1} \middle| \mathcal{F}_t\right]\right\}. \tag{18}$$

It is also useful to define the certainty equivalent process associated with the above construction of the utility process: $w_t = h^{-1}\left\{E\left[h \circ V_{t+1} \middle| \mathcal{F}_t\right]\right\}$. Clearly, $V_t = u(C_t) + \beta w_t$ by definition. Below, we provide a formal definition of preference for early resolution of uncertainty.

Definition. (Preference for early resolution of uncertainty)

A preference represented by $\{u, h, \beta\}$ is said to have preference for early resolution of uncertainty if for any two filtrations, $\{\mathcal{F}_t\}_{t=1}^T$ and $\left\{\bar{\mathcal{F}}_t\right\}_{t=1}^T$, such that i) both filtrations are weakly larger than the filtration generated by $\{Z_t\}_{t=1}^T$; and ii) $\mathcal{F}_t \subseteq \bar{\mathcal{F}}_t$ for all $t = 1, 2, 3, \cdots T$, $h^{-1}\left\{E\left[\left|h(V_1)\right| Z_1 = z\right]\right\} \leq h^{-1}\left\{E\left[\left|h\left(\bar{V}_1\right)\right| Z_1 = z\right]\right\}$ for all z, where $\{V_t\}_{t=1}^T$ is a utility process defined with respect to $\{\mathcal{F}_t\}_{t=1}^T$ and $\left\{\bar{V}_t\right\}_{t=1}^T$ is a utility process defined with respect to $\left\{\bar{\mathcal{F}}_t\right\}_{t=1}^T$.

The above definition generalizes the concept of the preference for early resolution of uncertainty

in Kreps and Porteus (1978) and defines it as a preference over filtrations. In the above definition, filtration $\bar{\mathcal{F}} = \left\{\bar{\mathcal{F}}_t\right\}_{t=1}^T$ has more information than $F = \{\mathcal{F}_t\}_{t=1}^T$, but to define the preference for early resolution of uncertainty, we need to compare the certainty equivalent of period-1 utility conditioning on the same information. If the preference can be represented by an additively separable utility function, that is, h is linear, then it is indifferent towards the timing of the resolution of uncertainty. In general, as we will see, PER depends on the properties of all three elements of $\{u, h, \beta\}$. In what follows, we first focus on one-period early resolution of uncertainty, but we also provide a recursive procedure to compute the welfare gain for an arbitrary number of periods.

To study one-period early resolution of uncertainty, we set $\mathcal{F}_t = \bar{\mathcal{F}}_t = \sigma\{Z_i : 1 \leq i \leq t\}$ for all $t = 2, 3, \cdots, T$. We set $\mathcal{F}_1 = \sigma\{Z_1\}$ as described above. To provide an approximation result, we consider a sequence of decreasing σ algebras, $\{\mathcal{G}_n\}_{n=0}^{\infty}$ with $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$ for all n and $\cap_{n=0}^{\infty}\mathcal{G}_n = \varnothing$. We set $\bar{\mathcal{F}}_1^n = \mathcal{G} \cup \mathcal{F}_1$ for all n. That is, for each n, $\bar{\mathcal{F}}_1^n$ contains more information than $\mathcal{F}_1$ and the information content in $\bar{\mathcal{F}}_1^n$ is a decreasing function of n. As a result, the filtration $\bar{\mathcal{F}}^n = \left\{\bar{\mathcal{F}}_t^n\right\}_{t=1}^T$ represents a sequence of early resolution of uncertainty experiments relative to $\mathcal{F}$, which, in the limit, has the same timing of resolution of uncertainty as $\mathcal{F}$. Our main result is to provide a linear approximation of welfare gain when n is large.

In the above setup, for each n, let $\left\{\bar{V}_t^n\right\}_{t=1}^{\infty}$ be the utility process associated with filtration $\bar{\mathcal{F}}^n$ and let $\{V_t\}_{t=1}^{\infty}$ be the utility process associated with filtration $\mathcal{F}$. Clearly, $\bar{V}_t^n(Z_t) = V_t(Z_t)$ for all $t \geq 2$. At time 1, the utility for early resolution can be computed as

$$\bar{V}_1^n = u(C(Z_1)) + \beta h^{-1}\left\{E\left[h\left(V_2(Z_2)\right)\middle|\bar{\mathcal{F}}_1^n\right]\right\}. \tag{19}$$

To compute welfare gains of PER, it is useful to define a certainty equivalent operator. Let $B(\boldsymbol{Z})$ be the space of bounded functions on $\boldsymbol{Z}$ (Because consumption is bounded, so are utility functions). For any $\bar{\mathcal{F}}_1^n$, we define the associated certainty equivalent operator $\bar{T}_n : B(\boldsymbol{Z}) \to B(\boldsymbol{Z})$ as:

$$\left[\bar{T}_n V\right](z) = h^{-1}\left\{E\left[h\left\{u(z) + \beta h^{-1}\left\{E\left[h \circ V(Z_2)\middle|\bar{\mathcal{F}}_1^n\right]\right\}\right\}\middle| Z_1 = z\right]\right\}. \tag{20}$$

That is, given any period-2 continuation utility V, we construct $\bar{V}_1^n$ as in Equation (19) with period-2 continuation utility V. $\left[\bar{T}_n V\right](z)$ is then defined as the certainty equivalent of $\bar{V}_1^n$, $h^{-1}\left\{E\left[h\left\{\bar{V}_1^n\right\}\middle| Z_1 = z\right]\right\}$ conditioning on period-1 information $Z_1 = z$. We also define a T operator as

$$[TV](z) = u(z) + \beta h^{-1}\left\{E\left[h \circ V(Z_2)\middle| Z_1 = z\right]\right\}, \tag{21}$$

for all $V \in B(\boldsymbol{Z})$. Here, $\bar{T}_n V(z)$ represents the certainty equivalent of a sequence of early resolution experiments, where as $n \to \infty$, the informativeness of the additional information, represented by $\bar{\mathcal{F}}_1^n$, converges to zero. As we will show in the appendix, in the limit, $\lim_{n\to\infty} \bar{T}_n V(z)$ converges to the certainty equivalent for the late resolution case, $[TV](z)$.

Both $\bar{V}_1^n$ and $\left[\bar{T}_n V\right](z)$ represent the consumer's life-time utility evaluated at time 1. The difference is that $\bar{V}_1^n$ is the utility after the uncertainty in $\bar{\mathcal{F}}_1^n$ is resolved and therefore measurable with respect to $\bar{\mathcal{F}}_1^n$. $\left[\bar{T}_n V\right](z)$ is the utility before the the uncertainty in $\bar{\mathcal{F}}_1^n$ is resolved. It is the

certainty equivalent of $\bar{V}_1^n$ computed conditioning on a coarser information set, $\mathcal{F}_1 = \sigma\{Z_1\}$. For this reason, we will refer to $\bar{V}_1^n$ as the ex post utility in period 1 and $\left[\bar{T}_n V\right](z)$ as the ex ante utility in period 1. Preference for the timing of resolution of uncertainty is a comparison of ex ante utilities for different flirtations. Below we demonstrate that as n becomes large enough, $\bar{T}_n V(z)$ can be obtained by a local linear approximation around $[TV](z)$. The coefficient for the linear approximation is the elasticity of PER. Using the definition of f in (11), it is not hard to show that the η_{PER} defined in Equation (16) can be written as a function of the Arrow-Pratt measure of risk aversion of h. Below is the definition.

Definition 1. Elasticity of PER

Give a dynamic preference represented as $\{u, h, \beta\}$, the Arrow-Pratt measure of absolute preference for early resolution of uncertainty, evaluated at $(u, w) \in R^2$ is defined as

$$\eta_{PER}(u, w|\beta) = \left\{\frac{1}{\beta}[\mathcal{A}h](w) - [\mathcal{A}h](u + \beta w)\right\}, \tag{22}$$

where $\forall w \in R, [\mathcal{A}h](w) = -\frac{h''(w)}{h'(w)}$ denotes the Arrow-Pratt measure of risk aversion of h evaluated at w. The Arrow-Pratt measure of relative PER, evaluated at (u, w), is defined as

$$\bar{\eta}_{PER}(u, w|\beta) = (u + \beta w)\,\eta_{PER}(u, w|\beta) = (u + \beta w)\left\{\frac{1}{\beta}[\mathcal{A}h](w) - [\mathcal{A}h](u + \beta w)\right\}. \tag{23}$$

It is clear that the elasticity of PER depends on the Arrow-Pratt measure of risk aversion of h and the discount rate β. To emphasize the dependence of η_{PER} on the discount rate β, we use the notation $\eta_{PER}(u, w|\beta)$, but we will suppress β below whenever the discount rate is clear from the context. As we will show below, $\eta_{PER}(u, w)$ provides a quantitative measure of the welfare gain of one-period resolution of uncertainty. The length of a period is encoded in the discount rate β.

Having defined η_{PER}, we first present a lemma that provides an approximation of the difference between the certainty equivalent for early resolution, $\bar{T}_n V$, and that for late resolution TV. As we show below, our approximation error is bounded by the third central moment of continuation utility around its conditional mean. To formally state our result, it is useful to introduce a notation for conditional central moments. For any random variable Z, we denote $M^{(n)}[Z|\mathcal{F}] = E[\{Z - E[Z|\mathcal{F}]\}^n|\mathcal{F}]$and the nth conditional central moment of Z given $\mathcal{F}$.

Proposition 1. *Let $\bar{T}_n V$ and TV be defined in (20) and (21), respectively. Then*

1. For any $z \in \boldsymbol{Z}$, $\bar{T}_n V(z) \to TV(z)$ a.s. and in L^2. In addition, let $w_1(n)$ be the certainty equivalent of $V(Z_2)$ given $\bar{\mathcal{F}}_1^n$,

$$w_1(n) = h^{-1}\left(E\left[h \circ V(Z_2)|\,\bar{\mathcal{F}}_1^n\right]\right), \tag{24}$$

then $\lim_{n\to\infty} Var[w_1(n)|\,Z_1 = z] = 0$.

2. $\bar{T}_n V(z)$ can be obtained from $TV(z)$ by

$$T_n V(z) = TV(z) + \frac{1}{2}\eta_{PER}(u(z), w_1(z)) Var[u(z) + \beta w_1(n) | Z_1 = z] + O\left(M^{(3)}[u(z) + \beta w_1(n) | Z_1 = z]\right), \quad (25)$$

where $w_1(z)$ is the certainty equivalent of $V(Z_2)$ given $Z_1 = z$:

$$w_1(z) = h^{-1}(E[h \circ V(Z_2) | Z_1 = z]). \quad (26)$$

and $\eta_{PER}(u, w)$ is defined in (1).

3. In addition,

$$\ln T_n V(z) = \ln TV(z) + \frac{1}{2}\bar{\eta}_{PER}(u(z), w_1(z)) Var[\ln\{u(z) + \beta w_1(n)\} | Z_1 = z] + O\left(M^{(3)}[\ln\{u(z) + \beta w_1(n)\} | Z_1 = z]\right). \quad (27)$$

Proof. See Section 6.1 in the appendix. □

In the above lemma, $u(z) + \beta w_n$ is the level of life-time utility upon early resolution of uncertainty and depends on the information in $\bar{\mathcal{F}}_1^n$. $T_n V(z)$ is the certainty equivalent of $u(z) + \beta w_n$ before the resolution of uncertainty, and therefore, is a function of z and does not depend on the information in $\bar{\mathcal{F}}_1^n$. $TV(z)$ is the certainty equivalent of later resolution, $u(z) + \beta w(z)$. PER corresponds to the condition $T_n V(z) \geq TV(z)$. The above lemma expresses the difference between $T_n V(z)$ and $TV(z)$ as a (locally) linear function of the conditional variance of $u(z) + \beta w_n$.

Proposition 1 allows for a very general notion of early resolution of uncertainty at time 1. That is, the additional information received by the agent is modeled by an abstract σ algebra, $\bar{\mathcal{F}}_1^n$. A special case of $\bar{\mathcal{F}}_1$ is of particular interest, that is, $\bar{\mathcal{F}}_1 = \sigma(Z_1, Z_2)$. That is, the consumer receives information about Z_2 one period earlier, at time 1. This corresponds to the classical example of early resolution of uncertainty of Kreps and Porteus (1978) as illustrated in Figure 1. Under the assumption of $\bar{\mathcal{F}}_1 = \sigma(Z_1, Z_2)$, the early resolution operator $\bar{T}V$ can be defined as in (20):

$$\left[\bar{T}V\right](z) = h^{-1}\{E[h\{u(z) + \beta V(Z_2)\} | Z_1 = z]\}. \quad (28)$$

In this case, Equation (25) can be written as

$$\left[\bar{T}V\right](z) - [TV](z) = \frac{1}{2}\eta_{PER}(u(z), w(z)) Var[u(z) + \beta V(Z_2) | Z_1 = z] + O\left(M^{(3)}[V(Z_2) | Z_1 = z]\right) \quad (29)$$

In addition, Equation (27) becomes

$$\ln\left[\bar{T}V\right](z)-\ln\left[TV\right](z)=\frac{1}{2}\bar{\eta}_{PER}\left(u\left(z\right),w\left(z\right)\right)Var\left[\ln\left\{u\left(z\right)+\beta V\left(Z_2\right)\right\}\middle|Z_1=z\right],\\ +O\left(M^{(3)}\left[\ln V\left(Z_2\right)\middle|Z_1=z\right]\right) \tag{30}$$

where $\bar{\eta}_{PER}\left(u,w\right)$ is the Arrow-Pratt measure of relative PER defined in (23).

Equations (29) and (30) express the welfare gain of early resolution of uncertainty as a linear function of the conditional variance of the ex post continuation utility in period 1, $u\left(z\right)+\beta V\left(Z_2\right)$. Here $u\left(z\right)$ is known in period 1. The only non-trivial random variable is $V\left(Z_2\right)$. In applications, it is sometimes more convenient to express welfare gain as a function of the variance of period-2 continuation utility, $Var\left[V\left(Z_2\right)\middle|Z_1=z\right]$. The following corollary provides such a formula.

Corollary 1. *Let the $\bar{T}V$ operator be as defined in (28) and the TV operator is as defined in (21),*

$$\left[\bar{T}V\right](z)-\left[TV\right](z)=\frac{1}{2}\beta^2\eta_{PER}\left(u\left(z\right),w\left(z\right)\middle|\beta\right)Var\left[V\left(Z_2\right)\middle|Z_1=z\right]+O\left(M^{(3)}\left[V\left(Z_2\right)\middle|Z_1=z\right]\right), \tag{31}$$

where $w\left(z\right)$ is as defined in (26). In addition, in log term

$$\ln\left[\bar{T}V\right](z)-\ln\left[TV\right](z)=\frac{1}{2}\left(\frac{\beta w\left(z\right)}{u\left(z\right)+\beta w\left(z\right)}\right)^2\bar{\eta}_{PER}\left(u\left(z\right),w\left(z\right)\middle|\beta\right)Var\left[\ln V\left(Z_2\right)\middle|Z_1=z\right]\\ +O\left(M^{(3)}\left[\ln V\left(Z_2\right)\middle|Z_1=z\right]\right) \tag{32}$$

2.3 Examples of the elasticity of PER

In this section, we provide several examples to illustrate how to use the concept of η_{PER} together with Lemma 1 to calculate the welfare gain for various experiments of early resolution of uncertainty.

An example with constant η_{PER} The multiplier robust control model of Hansen and Sargent (2008) features constant η_{PER}. As a result, the welfare gain of early resolution for this class of model is particularly simple to analyze. In the robust control model, the decision maker's subjective probability is represented by a probability distortion. Consider a discrete-time, infinite-horizon setup. Let the conditional probability distortion at time t be represented by m_{t+1}, in the sense that the time-t conditional expectation of any random variable Z, under the subjective probability, is computed as $E_t\left[m_{t+1}Z\right]$, where E_t represents expectation under the reference probability measure. The robust control preference can be constructed through the following recursive relationship:

$$V\left(z\right)=u\left(z\right)+\beta\min_{m_{t+1}}E\left[m_{t+1}V\left(Z_{t+1}\right)\middle|Z_t=z\right]-\theta E_t\left[m_{t+1}\ln m_{t+1}\middle|Z_t=z\right]\big|_{E_t[m_{t+1}]=1}.$$

The interpretation is that the nature chooses the worst-case probability, represented by m_{t+1}, to minimize the expected utility, $E\left[m_{t+1}V\left(Z_{t+1}\right)\middle|Z_t=z\right]$, but large probability distortions are subject to a penalty for the relative entropy $E_t\left[m_{t+1}\ln m_{t+1}\middle|Z_t=z\right]$. As shown in Hansen and Sargent

(2008), the above recursion can be conveniently represented in the form of (17) as:

$$V(z) = u(z) - \beta\theta \ln E\left[e^{-\frac{1}{\theta}V(Z_{t+1})}\middle| Z_t = z\right].$$

That is, $h(w) = -e^{-\frac{1}{\theta}w}$. Clearly, $\mathcal{A}h = \frac{1}{\theta}$, and $\eta_{PER} = \beta\{[\mathcal{A}h](w) - \beta[\mathcal{A}h](u+\beta w)\} = \beta(1-\beta)\frac{1}{\theta}$. As shown in Ai, Bansal, Guo, and Yaron (2023), this preference has a preference for early resolution of uncertainty if and only if $\beta < 1$. The Arrow-Pratt measure of PER is positive under the same condition.

Recursive preferences As shown in Kreps and Porteus (1978), the recursive preference with unit IES corresponds to the case where $h(w) = -e^{(1-\gamma)w}$ with $\gamma > 0$ and $\gamma \neq 1$ being the relative risk aversion of the recursive preference. From the definition above, because $\mathcal{A}h = \gamma - 1$ is constant, $\eta_{PER}(u,w) = \left(\frac{1}{\beta} - 1\right)(\gamma - 1)$ is also a constant. Clearly, $\bar{\eta}_{PER}(u,w) = (u+\beta w)\left(\frac{1}{\beta} - 1\right)(\gamma - 1)$.

The recursive preference with general IES is defined by the recursion

$$U(Z_t) = \left\{(1-\beta)C(Z_t)^{1-\frac{1}{\psi}} + \beta\left(E_t\left[U(Z_{t+1})^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right\}^{\frac{1}{1-1/\psi}},$$

where γ is the relative risk aversion parameter ($\gamma > 0$ and $\gamma \neq 1$), and ψ is the IES parameter ($\psi > 0$ and $\psi \neq 1$). Equivalently,

$$\frac{1}{1-\frac{1}{\psi}}U(Z_t)^{1-\frac{1}{\psi}} = \frac{1-\beta}{1-\frac{1}{\psi}}C(Z_t)^{1-\frac{1}{\psi}} + \beta\frac{1}{1-\frac{1}{\psi}}\left(E_t\left[U(Z_{t+1})^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}. \tag{33}$$

Using a monotonic transformation, $V = \frac{1}{1-\frac{1}{\psi}}U^{1-\frac{1}{\psi}}$, we can write the above recursion as:

$$V(Z_t) = u(C_t) + \beta\frac{1}{1-\frac{1}{\psi}}\left(E_t\left[\left(\left(1-\frac{1}{\psi}\right)V(Z_{t+1})\right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}},$$

that is, $V(Z_t)$ satisfies recursion (17) with

$$h(w) = \frac{1}{1-\gamma}\left(\left(1-\frac{1}{\psi}\right)w\right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}. \tag{34}$$

In this case, the elasticities are written as:

$$\eta_{PER}(u,w) = \frac{\gamma - \frac{1}{\psi}}{1-\frac{1}{\psi}}\frac{u}{\beta w(u+\beta w)}; \quad \bar{\eta}_{PER}(u,w) = \frac{\gamma-\frac{1}{\psi}}{1-\frac{1}{\psi}}\frac{u}{\beta w}. \tag{35}$$

In discounted utility models, the length of a period is encoded in the discount rate β. We can think of $\beta \to 1$ corresponds to the case in which the length of a period converges to zero.

In this case, the above elasticity measures all converge to zero as $\beta \to 1$. Clearly $\eta_{PER}(u,w) = \left(\frac{1}{\beta}-1\right)(\gamma-1) \to 0$ and $\bar{\eta}_{PER}(u,w) = (u+\beta w)\left(\frac{1}{\beta}-1\right)(\gamma-1) \to 0$ in the case of unit IES. In the case of general recursive utility, we also have $\bar{\eta}_{PER}(u,w) = \frac{\gamma-\frac{1}{\psi}}{1-\frac{1}{\psi}}\frac{u}{\beta w} \to 0$ as $\beta \to 1$ because the ratio of current period utility and the continuation utility $\frac{u}{\beta w} \to 0$ as the discount factor β converges to 1.

3 Long-horizon welfare gains

Multi-period welfare gain through successive approximations Corollary 1 provides a formula to calculate the one-period welfare gain under a preference for early resolution of uncertainty. In this section, we demonstrate how to apply this lemma recursively to compute the welfare gain of resolving uncertainty for multiple periods.

In an infinite-horizon Markov setup, we define the value function V as the fixed point of the following operator[2]

$$V(z) = u(z) + \beta h^{-1} E\left[h \circ V(Z_2) \middle| Z_1 = z\right]. \tag{36}$$

As in Epstein, Farhi, and Strzalecki (2014), the utility function $V(z)$ constructed this way will be called the utility associated with the gradual resolution of uncertainty. Our purpose is to compare the above utility with the utility of early resolution defined as:[3]

$$\bar{V}(z) = h^{-1} E\left[h\left\{\sum_{t=0}^{\infty}\beta^t u(Z_{t+1})\right\}\middle| Z_1 = z\right]. \tag{37}$$

The interpretation of $\bar{V}$ is that, at time 1, the decision maker receives an information that resolves all uncertainty in the future. As a result, the time-1 ex post utility is just an infinite sum of a deterministic stream of utilities, $\sum_{t=0}^{\infty}\beta^t u(Z_{t+1})$, where the sequence of $\{Z_{t+1}\}_{t=0}^{\infty}$ is known. $\bar{V}(Z_1)$ represents the certainty equivalent of this infinite sum before any uncertainty about the future, i.e., uncertainty about $\{Z_{t+1}\}_{t=1}^{\infty}$, resolves.

Our main idea for computing $\bar{V}(z) - V(z)$ is to construct a series of approximations and repeatedly apply Corollary 1. Consider the following one-period early resolution of uncertainty:

$$V^{(1)}(z) = h^{-1} E\left[h\left\{u(z) + \beta V(Z_2)\right\}\middle| Z_1 = z\right]. \tag{38}$$

Using Corollary 1, we can compare the utility for one-period early resolution of uncertainty in the above equation with the utility for gradual resolution of uncertainty in (36):

$$V^{(1)}(Z_1) \doteq V(Z_1) + \frac{1}{2}\beta^2 \eta_{PER}\left(u_1(Z_1), w(Z_1)\middle|\beta\right) Var\left[V(Z_2)\middle| Z_1\right],$$

[2] See Bloise, Van, and Vailakis (2024) and Werner (2024) for conditions for the existence and uniqueness of the fixed point.

[3] Our formulation differs from the EFS formulation by one period. EFS compares $u_{-1} + \beta h^{-1} E\left[h \circ V(z_0)\right]$ with $u_{-1} + \beta h^{-1} E\left[h \circ \bar{V}(z_0)\right]$. As a result, our welfare gain number will differ from that of EFS by a factor of β: $\left\{u_{-1} + \beta h^{-1} E\left[h \circ \bar{V}(z_0)\right]\right\} - \left\{u_{-1} + \beta h^{-1} E\left[h \circ V(z_0)\right]\right\} \approx \beta\left[\bar{V}(z_0) - V(z_0)\right]$.

where we use "$\doteq$" to indicate that this is an approximation with an error bounded by that implied by Proposition 1. In the above equation, $u_1(Z_1) = u(Z_1)$, and $w(Z_1) = h^{-1}E[h \circ V(Z_2)|\, Z_1]$ as in the Corollary.

Continue the above logic. Now compare one-period and two-period early resolution of uncertainty. To save notation, whenever the expectation is taken conditioning on date-1 information, that is, $Z_1 = z$, I ignore the conditional expectation notation and write it as unconditional expectation. Using (38) and writing $V(Z_2)$ as $V(Z_2) = u(Z_2) + \beta h^{-1}E[h \circ V(Z_3)|\, Z_2]$, the utility for one-period early resolution is given by:

$$V^{(1)}(z) = h^{-1}E\left[h\left\{u(z) + \beta u(Z_2) + \beta^2 h^{-1}E[h \circ V(Z_3)|\, Z_2]\right\}\right]. \tag{39}$$

Using the law of iterated expectations, the utility for two-period early resolution can be written as:

$$V^{(2)}(z) = h^{-1}E\left[h\left\{u(z) + \beta u(Z_2) + \beta^2 V(Z_3)\right\}\right] = h^{-1}E\left[h \circ h^{-1}E\left[h\left\{u(z) + \beta u(Z_2) + \beta^2 V(Z_3)\right\}\middle|\, Z_2\right]\right]. \tag{40}$$

Using Corollary 25, comparing the highlighted part in $V^{(1)}(z)$ and $V^{(2)}(z)$, we have:

$$h^{-1}E\left[h\left\{\underbrace{u(Z_1) + \beta u(Z_2)}_{u(z)\ in\ Corollary} + \underbrace{\beta^2}_{\beta\ in\ Corollary} V(Z_3)\right\}\middle|\, Z_2\right] \doteq \left\{u(Z_1) + \beta u(Z_2) + \beta^2 h^{-1}E[h \circ V(Z_3)|\, Z_2]\right\}$$
$$+\frac{1}{2}\left(\beta^2\right)^2 \eta_{PER}\left(u_2\left(Z^2\right), w(Z_2)\middle|\, \beta^2\right) Var[V(Z_3)|\, Z_2],$$

where we denote $u_2\left(Z^2\right) = u(Z_1) + \beta u(Z_2)$ and $w(Z_2) = h^{-1}E[h \circ V(Z_3)|\, Z_2]$. Due to the Markov property, $w(Z)$ does not need a subscript, but $u_2\left(Z^2\right)$ depends on the entire history of $Z^2 = (Z_1, Z_2)$. If the variance of the term $\eta_{PER}\left(u_2\left(Z^2\right), w(Z_2)\middle|\, \beta^2\right) Var[V(Z_3)|\, Z_2]$ is small, we can use Lemma 2 in the appendix to write[4]

$$V^{(2)}(z_0) - V^{(1)}(z_0) = \frac{1}{2}\left(\beta^2\right)^2 E\left[\eta_{PER}\left(u_2\left(Z^2\right), w(Z_2)\middle|\, \beta^2\right) Var[V(Z_3)|\, Z_2]\middle|\, Z_1 = z\right]. \tag{41}$$

Continuing the above logic and iterating forward, we can compute the welfare gain for n-period early resolution of uncertainty, which we summarize in the following proposition.

Proposition 2. *(PER through Successive Approximations) In an infinite-horizon setup, let the utility for gradual resolution of uncertainty be defined by (36).*

[4] In our application of unit IES case, $\eta_{PER}\left(u_2\left(Z^2\right), w(Z_2)\middle|\, \beta^2\right) Var[V(Z_3)|\, Z_2]$ is a constant.

1. The welfare gain for N-period early resolution of uncertainty is given by:

$$V^{(N)}(z) - V(z) = \frac{1}{2}\sum_{n=1}^{N}\beta^{2n}E\left[\eta_{PER}\left(u_n\left(Z^n\right), w\left(Z_n\right)|\,\beta^n\right)Var\left[V\left(Z_{n+1}\right)|\,Z_n\right]|\,Z_1 = z\right] \quad (42)$$
$$+ O\left(M^{(3)}\left[V\left(Z_{n+1}\right)|\,Z_n\right]\right),$$

and, in log terms,

$$\ln V^{(N)}(z) - \ln V(z) = \frac{1}{2}\sum_{n=1}^{N}\epsilon_n\left(Z^n\right) + O\left(M^{(3)}\left[\ln V\left(Z_{n+1}\right)|\,Z_n\right]\right) \quad (43)$$

where $\epsilon_n\left(Z^n\right)$ in the above expression is defined as:

$$\epsilon_n = \frac{1}{2}\left(\frac{\beta w\left(Z_n\right)}{u\left(Z^n\right) + \beta^{n+1}w\left(Z_n\right)}\right)^2 \bar{\eta}_{PER}\left(u\left(Z^n\right), w\left(Z_n\right)|\,\beta^n\right)Var\left[\ln V\left(Z_{n+1}\right)|\,Z_n\right]. \quad (44)$$

2. In the limit, $\lim_{N\to\infty} V^{(N)}(z) = \bar{V}(z)$, where $\bar{V}$ is the utility for one-time early resolution of uncertainty defined in (37).

Examples of welfare gain calculations In this section, we illustrate how to use Proposition 2 to compute the welfare gain of early resolution for several examples. We demonstrate that our approximation formulas are quite accurate by comparing local approximation results to global solutions for several well-studied economies in the literature, including those studied in Epstein, Farhi, and Strzalecki (2014).

The Epstein, Farhi, and Strzalecki (2014) example with unit IES We first consider the unit IES example studied in Epstein, Farhi, and Strzalecki (2014). The welfare gain of early resolution in this example has closed-form solutions. We show that our approximation formula produces a fully accurate solution in this example that is identical to the closed-form solution in Epstein, Farhi, and Strzalecki (2014). Unlike the Epstein, Farhi, and Strzalecki (2014) calculation, our method does not require knowledge of the shape of the utility function. It depends only on the elasticity parameter η_{PER}. It yields a fully accurate solution because the recursive preference with constant IES has a constant η_{PER}.

Epstein, Farhi, and Strzalecki (2014) consider a long-run risk model with recursive preference with unit IES. The consumption process is given by:

$$\ln C_{t+1} - \ln C_t = \mu + x_t + \sigma\epsilon_{C,t+1}, \quad (45)$$

where x_t is a AR (1) process with

$$x_{t+1} = \rho x_t + \phi\epsilon_{x,t+1}. \quad (46)$$

In this setup, the utility is defined by recursion (36) with $u(C) = \ln C$, and $h(w) = -e^{(1-\gamma)w}$.

Because the h function is CARA, $\mathcal{A}h = \gamma - 1$. As a result, $\eta_{PER}(u, w|\beta) = \left(\frac{1}{\beta} - 1\right)(\gamma - 1)$ is a constant. In addition, the value function for gradual resolution of uncertainty has a constant conditional variance. As a result, the expectation in Proposition 2 is not necessary. In this model the Markov state variable is $Z = (C, x)$. We can write (42) as

$$\begin{aligned} V^{(\infty)}(Z_1) - V(Z_1) &= \frac{1}{2}\sum_{n=1}^{\infty}\beta^n(1-\beta^n)(\gamma-1)\,Var\left[V(Z_{n+1})|\,Z_n\right] \\ &= \frac{1}{2}\frac{\beta(\gamma-1)}{(1-\beta)(1+\beta)}Var\left[V(Z_{n+1})|\,Z_n\right]. \end{aligned}$$

As we show in the appendix,

$$Var\left[V(Z_{n+1})|\,Z_n\right] = \sigma^2 + \left(\frac{\beta}{1-\beta\rho}\phi\sigma\right)^2 = \sigma^2\left[1 + \left(\frac{\beta}{1-\beta\rho}\phi\right)^2\right],$$

which implies

$$V^{(\infty)}(Z_1) - V(Z_1) = \frac{1}{2}\frac{\beta}{(1-\beta)(1+\beta)}\sigma^2\left[1 + \left(\frac{\beta}{1-\beta\rho}\phi\right)^2\right], \tag{47}$$

which is identical to the solution given in Epstein, Farhi, and Strzalecki (2014).[5]

General recursive utility We first consider the case of i.i.d. consumption growth. In the case of i.i.d. consumption growth, the value function for gradual resolution of uncertainty takes the form $V(C) = \frac{1}{1-1/\psi}HC^{1-\frac{1}{\psi}}$, where H is given in Equation (79) in the appendix. To apply the approximation formula (43), we use the expression of $\bar{\eta}_{PER}$ in Equation (35) to write ϵ_n in Equation (44) as

$$\epsilon_n = \frac{1}{2}\frac{\gamma - \frac{1}{\psi}}{1 - \frac{1}{\psi}}\lambda_n(1-\lambda_n)\,Var\left[\ln V(Z_{n+1})|\,Z_n\right] \tag{48}$$

where λ_n is defined as

$$\lambda_n = \frac{u(Z^n)}{u(Z^n) + \beta^n w(Z_n)}, \tag{49}$$

and $u(Z^n)$ and $w(Z_n)$ are given by:

$$u(Z^n) = \sum_{j=1}^{n}\beta^{j-1}\frac{1}{1-\frac{1}{\psi}}C_j^{1-\frac{1}{\psi}}, \quad w(Z_n) = \frac{1}{1-\frac{1}{\psi}}w(x_n)\,C_{n+1}^{1-\frac{1}{\psi}}.$$

[5] Our timing convention is different from Epstein, Farhi, and Strzalecki (2014). Epstein, Farhi, and Strzalecki (2014) computes the time-0 utility, whereas our Lemma 1 computes the certainty equivalent of time-1 utility. Using our notation, the welfare gain of early resolution in Epstein, Farhi, and Strzalecki (2014) is $\left\{u(C_0) + \beta V^{(\infty)}(z_0)\right\} - \{u(C_0) + \beta V(z_0)\}$. As a result, the welfare gain in our model, (47) differ from that in Epstein, Farhi, and Strzalecki (2014) by a factor of β.

Table 1: Welfare gain of early resolution

ψ	1.5					1				
γ	10	7.5	5	2	1	10	7.5	5	2	1
	Panel A: IID Growth									
Global solution	9.8%	7.4%	4.8%	1.5%	0.4%	7.6%	5.5%	3.4%	0.9%	0.0%
Approximation	10.1%	7.5%	4.9%	1.5%	0.4%	7.6%	5.5%	3.4%	0.9%	0.0%
	Panel B: Long-run Risk									
Global solution	31.0%	24.8%	17.5%	6.3%	1.7%	27.5%	20.7%	13.2%	3.5%	0.0%
Approximation	33.3%	26.3%	18.1%	6.2%	1.6%	27.3%	20.6%	13.2%	3.5%	0.0%
Approx w. adj	30.5%	24.7%	17.4%	6.2%	1.6%	27.3%	20.6%	13.2%	3.5%	0.0%

This table compares the welfare gain of early resolution of uncertainty obtained from global solutions and that using the linear approximation in Proposition 2. The welfare gain of early resolution of uncertainty is measured in consumption units as in Epstein, Farhi, and Strzalecki (2014). Panel A reports the results for the case in which consumption growth is i.i.d.. The parameter values for the consumption process are $\sigma = 0.0084$, $\beta = 0.998$, and $\mu = 0.0015$. Panel B reports the results for the case with persistent consumption growth. The parameters for the x_t process in (46) are $\phi = 0.044$, $\rho = 0.9790$. All other parameters are the same except we set $\sigma = 0.0078$ to be consistent with the example reported in Epstein, Farhi, and Strzalecki (2014).

Because consumption growth is i.i.d., $Var\left[\ln V\left(Z_{n+1}\right)\middle| Z_n\right] = \left(1-\frac{1}{\psi}\right)^2\sigma^2$ is a constant, and the infinite sum in Equation (43) can be computed in closed form. The details are provided in the appendix.

In the general case where x_t is time-varying, we provide a more accurate approximation that further improves upon Equation (43), and we compare our approximate result for all three case: unit IES, general recursive utility with i.i.d. consumption growth, and general recursive utility with persistent consumption growth, with those obtained from global solutions in Table 1.

As we demonstrate above, in the case of unit IES, the solution obtained from local approximation coincides with the global solution. As a result, in the columns where $\psi = 1$, global solutions and local solutions are identical. For the case $\psi = 1.5$, the global solution for the utility for gradual resolution of uncertainty is computed using value function iteration. The global solution for early resolution of uncertainty is computed by using Monte Carlo simulation to evaluate the expectation in (37). We use 1,000,000 simulated samples, each consisting of 5,000 (monthly) periods of consumption growth. In both panels, the approximations are computed according to equation 48 and summed across 5,000 monthly periods. In Panel A, $Var\left[\ln V\left(Z_{n+1}\right)\middle| Z_n\right]$ is a constant and is analytically computed, and λ_n is computed on the path of the steady state, assuming constant growth with no uncertainty. In Panel B, $Var\left[\ln V\left(Z_{n+1}\right)\middle| Z_n\right]$ is computed using a log-linearized value function. In the row "Approximation", λ_n is computed on the path of steady state which assumes constant growth with no uncertainty. The row Approximation with adjustment uses a more accurate approximation that takes into account of a covariance term that does not appear in the i.i.d. consumption growth case but does appear in the more general case. The details of this adjustment are in Section 6.2 of the appendix.

Propositions 1 and 2 express welfare in utility units. In Table 1, welfare gains are expressed

in percentage consumption units. We provide the details of the relationship between utility and consumption used in calculating Table 1 in Appendix 6.3.

The term structure of PER In Figure 2, we use Proposition 2 to plot the n-period welfare gain of early resolution of uncertainty as a function of n, where one period is one month. That is, we compute the fraction of life time consumption a representative consumer is willing to give up in exchange for information that resolves all uncertainty in the next n months. We consider four cases. The blue solid line ($\gamma = 10, \psi = 1.5$) and the blue dashed line ($\gamma = 10, \psi = 1$) represent long-run risk economies where consumption growth is assumed to follow (45) and (46). The red solid line ($\gamma = 10, \psi = 1.5$) and red dashed line ($\gamma = 10, \psi = 1$) represent economies with i.i.d. consumption growth. That is, the consumption dynamics are given by (45) and (46) with $\phi = 0$.

We make several observations. First, overall, the welfare gain are significantly larger in the long-run risk economy than those in the economy with i.i.d. consumption growth. This is because the quantity of risk is substantially higher in the long-run risk economy. Second, high IES parameter configurations typically feature a higher welfare gain of resolution of uncertainty. This is consistent with our calculation where $\bar{\eta}_{PER}$ is an increasing function of IES. In addition, as $n \to \infty$, the welfare gains converge to the infinite-horizon calculation listed in Table 1. However, the convergence takes a long time. In the long-run economy, for example, the welfare gain of resolving 50 years of uncertainty is roughly half of the infinite-horizon welfare gain. Even in the long-run risk economy, the welfare gain of resolving uncertainty at the 20-year horizon is moderate, about 5% of life-time consumption.

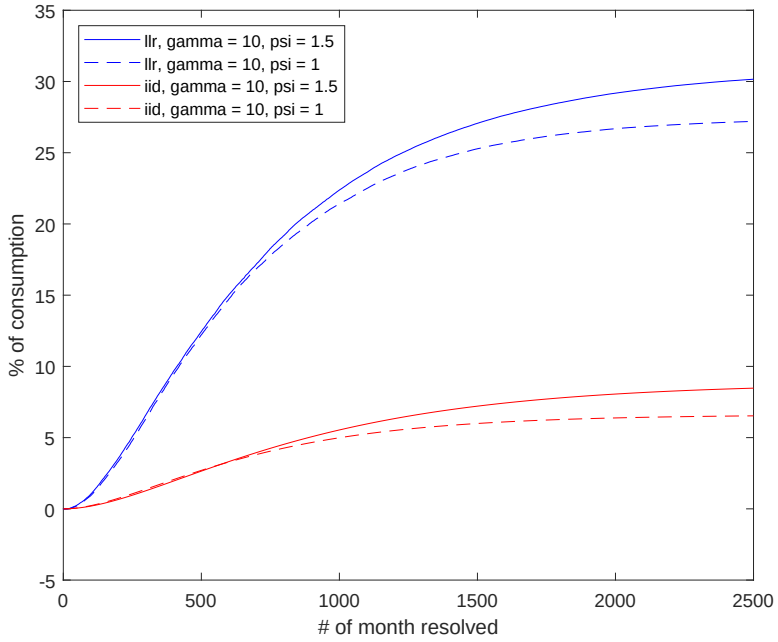
4 Infer η_{PER} from asset prices

In this section, we provide a log-linearization framework that can be used to estimate η_{PER} from asset prices. Our approach is based on the theoretical foundation for identifying PER developed in Ai, Bansal, Guo, and Yaron (2023). As shown in Ai, Bansal, Guo, and Yaron (2023), the key identification assumption for PER is GRS. Under the assumption of GRS, Ai, Bansal, Guo, and Yaron (2023) demonstrate that the risk premium for appropriately constructed option portfolios during the resolution of information quality (ROIQ) period identifies PER. Our estimation procedure parallels the above development to quantify GRS and PER. We first develop a quantitative measure of GRS, which we call the elasticity of GRS, $\bar{\eta}_{GRS}$. We demonstrate how to use the macroeconomic announcement premium to quantify $\bar{\eta}_{GRS}$. This step is a quantitative implementation of Ai and Bansal (2018). In the second step, we show that given $\bar{\eta}_{GRS}$, the risk premium for information quality during the ROIQ period identifies $\bar{\eta}_{PER}$. This step is a quantitative implementation of Ai, Bansal, Guo, and Yaron (2023).

4.1 Elasticity of GRS and the macroeconomic announcement premium

To illustrate the concept of GRS, we focus the top panel of Figure 1 and interpret the signal $s$that reveals uncertainty about Z as a macroeconomic announcement. The announcement in period 1

Figure 2: The term structure of PER for different economies



This figure plots, from top to bottom, the term structure of PER for four economies: i) $\gamma = 10, \psi = 1.5$ in a long-run risk economy; ii) $\gamma = 10, \psi = 1$ in a long-run risk economy; iii) $\gamma = 10, \psi = 1.5$ in an iid economy; $\gamma = 10, \psi = 1$ in an iid economy.

leads to early resolution of uncertainty. In the figure, because the announcement reveals the true value of Z at time 1, the utility at time 1 for the case of early resolution is a function of Z. In the two-period setting in the figure, $V_1(Z) = u(\bar{c}_1) + \beta u(C_2(Z))$ (See also Equation (14)). More generally, all the derivations below apply to the infinite-horizon case where $u(C_2(Z))$ is replaced by the continuation utility at time 2 as a function of the state variable Z. In both cases, the utility at time 0 can be defined as[6]

$$u(\bar{c}_0) + \beta h^{-1} E[h \circ V_1(Z)].$$

The stochastic discount factor that prices period-1 consumption units into period-0 consumption goods can be computed as ratios of marginal utilities:

$$ASDF_{0,1}(Z) = \beta \frac{h'(V_1(Z))}{h'(h^{-1}E[h \circ V_1(Z)])} \frac{u'(\bar{c}_1)}{u'(\bar{c}_0)}, \tag{50}$$

where we use $ASDF$ as the announcement stochastic discount factor. In the above expression, $ASDF_{0,1}(Z)$ depends on Z through the continuation utility $V_1(Z)$. The following proposition provides a log-linear approximation of the ASDF and the definition of $\bar{\eta}_{GRS}$.

Proposition 3. *The log ASDF can be written as*

$$\ln ASDF_{0,1}(Z) = \eta_0 - \bar{\eta}_{GRS}(w) \ln V_1(Z) + M^{(2)}[\ln V_1(Z)], \tag{51}$$

[6]In the two-period setting, this definition coincides with Equation (7).

where $\bar{\eta}_{GRS}(w) = w\mathcal{A}h(w)$ *is the Arrow-Pratt measure of relative risk aversion of* h *evaluated at* w*, and* $w = h^{-1}(E[h(V_1(Z))])$ *is the certainty equivalent of* $V_1(Z)$*.*

Proof. Consider the Taylor expansion of $h'(V_1)$ around w:

$$\ln h'(V_1) = \ln h'(w) + \frac{wh''(w)}{h'(w)}(\ln V_1 - \ln w) + O(\ln V_1 - \ln w)^2.$$

Taking log on both sides of Equation (50), we can write $\ln ASDF$ as in (51), where

$$\eta_0 = \ln\left[\beta\frac{u'(\bar{c}_1)}{u'(\bar{c}_0)}\right]$$

is a constant that does not depend on Z. □

The Theorem of Generalized Risk Sensitivity in Ai and Bansal (2018) implies that a positive announcement premium for all assets with pro-cyclical payoff is equivalent to the certainty equivalent functional, $\mathcal{I}$, being increasing in second-order stochastic dominance.[7] This condition on $\mathcal{I}$ is defined as GRS. In the special case where $\mathcal{I}$ has the representation (6), GRS is equivalent to the concavity of h. Whenever h is concave, $\bar{\eta}_{GRS} > 0$. Proposition 3 implies that the Arrow-Pratt measure of relative risk aversion of h is a quantitative measure of the magnitude of GRS. It represents the elasticity of $ASDF$ with respect to continuation utility.

Our main purpose is to use asset prices to infer properties of preferences, similar to Alvarez and Jermann (2004). As in Alvarez and Jermann (2004), we focus on the risk premium on the claim to aggregate consumption, or aggregate wealth, because aggregate wealth can be connected directly to the representative investors' utility and welfare. Let $R_{0,1}$ denote the return on the trading strategy of purchasing the aggregate wealth portfolio in period 0 and selling in period 1 upon the macroeconomic announcement. If we denote W_1 as the value of aggregate wealth in period 1, then $R_{0,1} = \frac{W_1}{E_0[ASDF_{0,1}W_1]}$. Assuming that $\ln ASDF_{0,1}$ and $\ln W_1$ are jointly normally distributed with variance and covariance matrix $\begin{bmatrix} \sigma_W^2 & \sigma_{W,ASDF} \\ \sigma_{W,ASDF} & \sigma_{ASDF}^2 \end{bmatrix}$. Using $E[ASDF_{0,1} \times R_{0,1}] = E[ASDF_{0,1} \times R_f] = 1$, we can write the (log) announcement premium as:

$$E[\ln R_{0,1}] - \ln R_f = -\sigma_{W,ASDF} - \frac{1}{2}\sigma_W^2. \tag{52}$$

To link the return on aggregate wealth to utility and ASDF, we provide a general result on the wealth-to-consumption ratio in representative-agent economies. We consider the general dynamic setup in Section 2.2, where the representative consumer's preference is defined through the recursive relation (17). We assume that, in a competitive equilibrium, the utility maximization problem of

[7] In the context of the above two-period model, a procyclical payoff is defined as a payoff that is an increasing function of continuation utility V_1^E.

the representative consumer is given by:

$$U_t(Z_t, W_t) = \max_{C_t, \{\xi_{j,t}\}_{j=1}^J} \{u(C_t) + \beta \mathcal{I}[U_{t+1}(Z_{t+1}, W_{t+1})]\} \tag{53}$$
$$C_t + \sum_{j=1}^J \xi_{j,t} = W_t$$
$$W_{t+1} = \sum_{j=1}^J \xi_{j,t} R_{j,t+1}(Z_t, Z_{t+1}),$$

with $U_T(Z_T, W_T) = u(W_T)$.[8] That is, the consumer simply consumes all wealth in the last period. The interpretation is that in each period t, the consumer optimally chooses consumption C_t and investment in J assets to maximize life-time utility.

Lemma 1. *Assume that the utility function defined through the recursive relationship (17) is homogenous. Then*

$$\frac{W_t}{C_t} = \frac{U_t(Z_t, W_t)}{u(C_t)}.$$

Proof. See Appendix 6.3. □

The above equation implies that in the two period model in Figure 1, the wealth-consumption ratio can be written as $\frac{W_1(Z)}{\bar{c}_1} = \frac{V_1(Z)}{u(\bar{c}_1)}$. As a result, $\ln V_1(Z) = \ln\left(\frac{u(\bar{c}_1)}{\bar{c}_1}\right) + \ln W_1(Z)$, where $\ln\left(\frac{u(\bar{c}_1)}{\bar{c}_1}\right)$ is a constant and does not depend on the state variable Z. Using Equation (51) and the log linear relationship between W and V, we can write

$$\ln ASDF_{0,1} = \eta_0 - \bar{\eta}_{GRS} \ln W_1(Z), \tag{54}$$

where $\eta_0 = \ln\left[\beta \frac{u'(\bar{c}_1)}{u'(\bar{c}_0)}\right] - \bar{\eta}_{GRS} \ln\left(\frac{u(\bar{c}_1)}{\bar{c}_1}\right)$ is a constant that does not depend on Z.[9] As a result, $Cov(\ln ASDF_{0,1}, \ln R_{0,1}) = -\bar{\eta}_{GRS} Var[\ln R_{0,1}]$. We can use Equations (52) and (54) to write

$$E[\ln R_{0,1}] - \ln R_f = \bar{\eta}_{GRS} Var[\ln R_{0,1}] - \frac{1}{2} Var[\ln R_{0,1}],$$

which implies that $\bar{\eta}_{GRS}$ can be written as a function of asset pricing moments:

$$\bar{\eta}_{GRS} = \frac{E[\ln R_{0,1}] - \ln R_f + \frac{1}{2} Var[\ln R_{0,1}]}{Var[\ln R_{0,1}]} = \frac{1}{2} + \frac{E[\ln R_{0,1}] - \ln R_f}{Var[\ln R_{0,1}]}. \tag{55}$$

[8] In this section, we use $U_t(Z_t, W_t)$ to emphasize that from the consumer's utility maximization perspective, the value function is a function of two state variables (Z_t, W_t). If, in equilibrium, $C_t = C(Z_t)$, then the equilibrium utility $\{V_t(Z_t)\}_{t=1}^T$ constructed from $\{C(Z_t)\}_{t=1}^T$ via the relation (17) must coincide with $U_t(Z_t, W_t)$, meaning $U_t(Z_t, W_t) = V_t(Z_t)$ for all t. We use $U_t(Z_t, W_t)$ here, but we will use $V_t(Z_t)$ instead in the rest of the paper without explicitly referencing $U_t(Z_t, W_t)$ to save notation.

[9] Because the value of the constant does not affect any of our calculations, we use η_0 as a generic notation for constants, even though the value of η_0 in Equations (51) and (54) differ.

This allows us to measure the structural parameter $\bar{\eta}_{GRS}$ from moments of the return on the aggregate wealth portfolio on macroeconomic announcement days.

4.2 PER premium

In this section, we demonstrate how to use PER premium in the data to provide a measure of the structural parameter $\bar{\eta}_{PER}$. We first briefly review the identification exercise in Ai, Bansal, Guo, and Yaron (2023). In Figure 3, we combine the top panel and the bottom panel in Figure 1 into one event tree and add a period -1. To link our theoretical development to empirical exercises, we interpret period 2 not as the terminal period of consumption but as a starting point of a continuation utility in an infinite-horizon setting. To accommodate this interpretation and to give this event tree empirical content, we make three changes to Figure 1. First, we denote the continuation utility in period 2 as $V_2(Z)$.

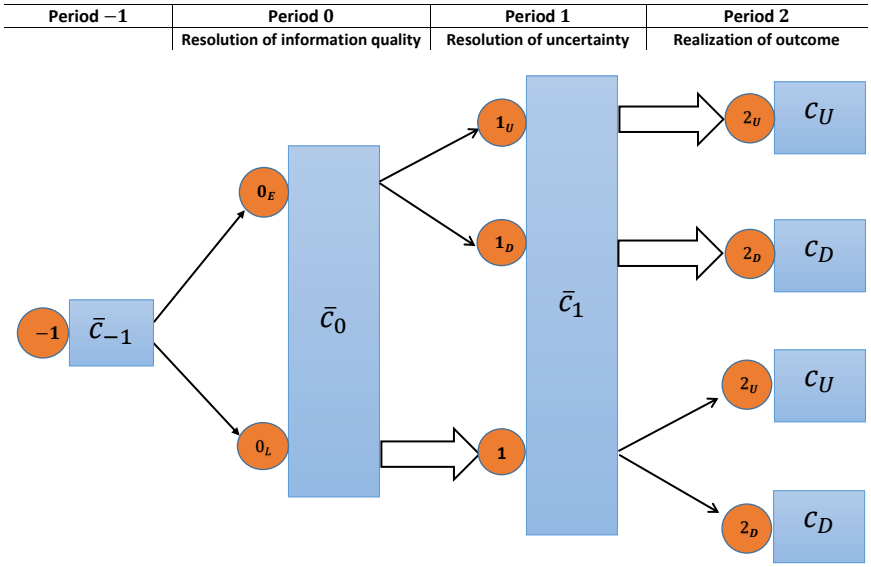
Second, we index the signal s in period 1 by an information quality parameter τ. A simple example of information quality is that Z follows a log-normal distribution and $s = \ln Z + \epsilon$, where $\epsilon \sim N\left(0, \tau^{-1}\right)$ is a noise with a Gaussian distribution and τ is the inverse of the variance of ϵ. The early resolution in the figure corresponds to the case in which $\tau \to \infty$ and s is fully revealing about Z. The case of late resolution is $\tau \to 0$ and the signal is infinitely noisy. In general, we think of the quality of the signal s as indexed by $\tau \in (0, \infty)$, which is revealed in period 0. As in Ai, Bansal, Guo, and Yaron (2023), period 1 is the announcement period where the signal s is revealed. Period 0 is the period of resolution of information quality, where the quality of s, τ is revealed, but the content of s is not known until the announcement period.

Third, in empirical applications, we think of period 1 as an FOMC announcement day, and we consider period 0 as the few days before FOMC announcements in which the information quality of announcements becomes known to the market. As a result, the empirically relevant case is that discounting across periods is small, and the flow utility in period -1, 0, and 1 is negligible compared to the continuation utility $V_2(Z)$. We nevertheless keep the notation of consumption and discounting just to be consistent with the general framework developed in Section 2.

As in Ai, Bansal, Guo, and Yaron (2023), the identification of PER requires an asset the payoff of which is a monotone function of τ. One such example is the variance of the log return on the aggregate wealth portfolio in period 1. Intuitively, more informative announcements have a larger impact on market prices and trigger higher realized volatility on announcement days. The variance of log return is a convenient test asset for the PER premium, because in the data, the variance of log return can be constructed from option prices (Bakshi, Kapadia, and Madan (2003)). We refer to this portfolio as the variance portfolio.

We denote the value of the aggregate wealth portfolio in period 1 upon announcement as $W_1(s, \tau)$. In period 1, both s and τ are revealed, and $W_1(s, \tau)$ is a function of both. We denote the period-0 market-expected variance of the return on $W_1(s, \tau)$ as $Var\left[\ln W_1(s, \tau) | \tau\right]$. The period-0 market-expected variance is a function of τ because information quality τ is revealed in

Figure 3: PER Premium



This figure represents our thought experiment of resolution of information quality. The blue squares contain consumption and the orange circles represent information sets. At node 0_E, the agent expects the uncertainty about $c_2(s)$ to be resolved in period 1 with an informative macroeconomic announcement that reveals s. Node 0_L represents the situation in which the upcoming announcement is expected to be uninformative about $c_2(s)$.

period 0. The present value of the variance portfolio evaluated as of period -1 can be computed as

$$PV_{-1} = E_{-1}\left[ASDF_{-1,0}(\tau)\, Var\left[\ln W_1(s,\tau)|\,\tau\right]\right], \tag{56}$$

where the announcement stochastic discount factor $ASDF_{-1,0}(\tau)$ prices the period 0 payoff into period -1 consumption units. The return on a strategy that purchases the variance portfolio in period -1 and sells it in period 0 upon the announcement of information quality τ is $\frac{Var[\ln W_1(s,\tau)|\tau]}{PV_{-1}}$. As in Ai, Bansal, Guo, and Yaron (2023), we call the risk premium of this strategy the PER premium, because the sign of this risk premium identifies PER.

In what follows, we use our approximation result in Proposition 3 to provide an expression that links the PER premium to the structural parameter $\bar{\eta}_{PER}$. We denote the agent's utility in period 1 as $V_1(s,\tau)$. The last subsection, Section 4.1, considers the special case where the signal fully reveals the value of Z and $s = Z$. In general, s may be a noisy signal of Z, and $V_1(s,\tau)$ can be constructed using the recursion (18) as:

$$V_1(s,\tau) = u(\bar{c}_1) + \beta h^{-1} E\left[h \circ u(Z)|\, s,\tau\right].$$

At time 0, information quality $\tau$is known but not s, and the time-0 utility of the agent is

$$V_0(\tau) = u(\bar{c}_0) + \beta h^{-1} E\left[h \circ V_1(s,\tau)|\,\tau\right].$$

First, using Proposition 3, we can write $\ln ASDF_{-1,0}$ as a linear function of log utility:

$$\ln ASDF_{-1,0}(\tau) = \eta_0 - \bar{\eta}_{GRS} \ln V_0(\tau). \tag{57}$$

Second, we use Equation 30 to relate period 0 utility $V_0(\tau)$ to the variance of continuation utility:[10]

$$\ln V_0(\tau) = \ln V_0(0) + \bar{\eta}_{PER} Var\left[\ln V_1(s,\tau) | \tau\right]. \tag{58}$$

Combing Equations (57) and (58), we can write

$$\ln ASDF_{-1,0}(\tau) = \eta_0 - \bar{\eta}_{GRS}\bar{\eta}_{PER} Var\left[\ln V_1(s,\tau) | \tau\right], \tag{59}$$

where η_0 is a constant that does not depend on the value of the random variables, τ, s or Z.

Using the wealth-utility relationship in Lemma 1, $\ln V_1(s,\tau) = \ln W_1(s,\tau) + constant$. This allows us to write an approximation of the present value calculation in (56) as

$$PV_{-1} = E_{-1}\left[e^{\eta_0 - \bar{\eta}_{GRS}\bar{\eta}_{PER} Var[\ln W_1(s,\tau)|\tau]} Var\left[\ln W_1(s,\tau) | \tau\right]\right].$$

The following lemma provides an expression for $\bar{\eta}_{PER}$ by assuming that the payoff of the variance portfolio follows a Gamma distribution. Consider the claim to the variance of the market return realized on the announcement day, $\ln W_1$ in Figure 3. We denote conditional variance of $\ln W_1$ in period 0 is $Var\left[\ln W_1(s,\tau) | \tau\right]$.

Proposition 4. *(Measurement of $\bar{\eta}_{PER}$) Assuming that $Var\left[\ln W_1(s,\tau) | \tau\right]$ follows a Gamma distribution, the Arrow-Pratt measure of relative PER is related to PER premium by:*

$$\bar{\eta}_{PER} = \frac{1}{\bar{\eta}_{GRS}} \frac{E\left[Var\left[\ln W_1(s,\tau) | \tau\right]\right]}{Var\left[Var\left[\ln W_1(s,\tau) | \tau\right]\right]} \left\{\frac{E\left[R_{-1,0}\right]}{E\left[R_f\right]} - 1\right\} \tag{60}$$

Proof. See appendix. □

In the above proposition, $E\left[R_{-1,0}\right]$ is the expected return of the variance portfolio, that is, $E\left[R_{-1,0}\right] = \frac{E[Var[\ln W_1(s,\tau)|\tau]]}{PV_{-1}}$. The term $\frac{E[R_{-1,0}]}{E[R_f]} - 1$ is the risk premium for the variance portfolio. The above lemma can therefore be interpreted as the quantitative version of the key result in Ai, Bansal, Guo, and Yaron (2023), that PER if and only if the risk premium on the variance portfolio, $\frac{E[R_{-1,0}]}{E[R_f]} - 1$ is positive. Equation (60) links this risk premium to the magnitude of the Arrow-Pratt measure of relative PER and is our key equation for the measurement of the structural parameter of the elasticity of PER.

[10] Computing the log deviation of $V_0(\tau)$ around $V_0(0)$, we have $\ln V_0(\tau) - \ln V_0(0) = \frac{\beta w}{u(\bar{c}_0)+\beta w}\left\{\ln h^{-1} E\left[h \circ V_1(s,\tau) | \tau\right] - \ln h^{-1} E\left[h \circ V_1(0)\right]\right\}$. Here $V_1(0) = u(\bar{c}_1) + \beta h^{-1} E\left[h \circ u(Z)\right]$ is the period-1 utility for the case where the signal s is completely uninformative. In our empirical exercise, the length of the macroeconomic announcement period is small, flow utility is negligible compared to continuation utility, $\frac{\beta w}{u(\bar{c}_0)+\beta w} = 1$. We can apply 30 to compute the term $\ln h^{-1} E\left[h \circ V_1(s,\tau) | \tau\right] - \ln h^{-1} E\left[h \circ V_1(0)\right]$ to obtain Equation (58).

5 Conclusion

In this paper, we develop a theory for the preference for early resolution of uncertainty that parallels the Arrow-Pratt theory of risk aversion. We demonstrate that the elasticity of PER is a key structural parameter that links the premium for resolution of information quality on asset markets and the welfare cost of late resolution of uncertainty. We show how this parameter can be used to compute the welfare gains of various thought experiments of early resolution of uncertainty and we demonstrate how to use asset prices to estimate this parameter. We argue that the empirical evidence for the macroeconomic announcement premium and that for the premium for resolution of information quality requires better estimates for the elasticity parameters such as the elasticity of GRS and elasticity of PER.

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6 Appendix

6.1 Proof of Lemma 1

Before we provide a proof for Lemma 1, we first state a result that computes the derivatives of f defined in (11).

Lemma. *(PER expansion)*
Let $f(e|u)$ be defined as in (11), then

$$f''(e|u) = \frac{\beta h'\left(u+\beta h^{-1}e\right)}{\left[h'\left(h^{-1}(e)\right)\right]^2}\left\{\left[\mathcal{A}h\right]\left(h^{-1}(e)\right) - \beta\left[\mathcal{A}h\right]\left(u+\beta h^{-1}(e)\right)\right\}, \tag{61}$$

where $[\mathcal{A}h](e) = -\frac{h''(e)}{h'(e)}$ is the Arrow-Pratt measure of absolute risk aversion evaluated at z.

Proof. By the definition of f, its first order derivative is:

$$f'(z;u,\beta) = h'\left(u+\beta h^{-1}(z)\right)\beta h^{-1\prime}(z) = \beta\frac{h'\left(u+\beta h^{-1}(z)\right)}{h'\left(h^{-1}(z)\right)}.$$

Taking second order derivative, we have:

$$\begin{aligned} f''(z;u,\beta) &= \beta\frac{h''\left(u+\beta h^{-1}(z)\right)\frac{\beta}{h'(h^{-1}(z))}h'\left(h^{-1}(z)\right) - h'\left(u+\beta h^{-1}(z)\right)h''\left(h^{-1}(z)\right)\frac{1}{h'(h^{-1}(z))}}{\left[h'\left(h^{-1}(z)\right)\right]^2} \\ &= \frac{\beta h'\left(u+\beta h^{-1}(z)\right)}{\left[h'\left(h^{-1}(z)\right)\right]^2}\left\{\beta\frac{h''\left(u+\beta h^{-1}(z)\right)}{h'\left(u+\beta h^{-1}(z)\right)} - \frac{h''\left(h^{-1}(z)\right)}{h'\left(h^{-1}(z)\right)}\right\}, \end{aligned}$$

as needed. □

Proof for Lemma 1

Proof. To simplify notation, we denote e_n as the conditional expectation of $V(Z_2)$ given $\bar{\mathcal{F}}_1^n$,

$$e_n = E\left[h\circ V(Z_2)|\,\bar{\mathcal{F}}_1^n\right], \tag{62}$$

and $e^*(z)$ as the conditional expectation of $V(Z_2)$ given $Z_1 = z$:

$$e^*(z) = E\left[h\circ V(Z_{t+2})|\,Z_1 = z\right]. \tag{63}$$

Using the above notation and the definition of f in (11), we can write $\bar{T}_n V(z)$ as

$$\begin{aligned} \bar{T}_n V(z) &= h^{-1}\left\{E\left[h\left\{u(z)+\beta h^{-1}\left\{E\left[h\circ V(Z_2)|\,\bar{\mathcal{F}}_1^n\right]\right\}\right\}\middle|\,Z_1 = z\right]\right\} \\ &= h^{-1}\left\{E\left[f(e_n|\,u(z))|\,Z_1 = z\right]\right\}, \end{aligned} \tag{64}$$

and $TV(z)$ as:

$$TV(z) = h^{-1}\{f(e^*|u(z))\}. \tag{65}$$

The law of iterated expectation implies that $E[e_n|Z_1 = z] = e^*$ for all n. Note that $\{e_n\}_{n=0}^{\infty}$ is a sequence of reverse martingales. As a result, $e_n \to e^*$ a.s. and in L^p (see Section 4.7 in Durrett (2019)). Convergence in L^1 implies that $\bar{T}_n V(z) \to TV(z)$ and converge in L^2 implies $\lim_{n\to\infty} Var[v_n|Z_1 = z] = 0$.

Because h is analytic with strictly positive derivatives, h^{-1} is analytic, and so is f, we can write (64) using Taylor expansion:

$$\begin{aligned}\bar{T}_n V(z) =& h^{-1}\left\{E\left[\sum_{j=0}^{\infty}\frac{1}{j!}f^{(j)}(e^*)(e_n - e^*)^j\middle| Z_1 = z\right]\right\} \\ =& h^{-1}\left\{E\left[f(e^*) + \frac{1}{2}f''(e^*)Var[e_n|Z_1 = z] + \sum_{j=3}^{\infty}\frac{1}{j!}f^{(j)}(e^*)E\left[(e_n - e^*)^j\middle| Z_1 = z\right]\right]\right\},\end{aligned}$$

where the second line uses the fact that e^* is Z_1 measurable. Because $e_n \to e^*$ a.s., $(e_n - e^*)^j = o(e_n - e^*)^2$ for $j \geq 3$, and $E\left[(e_n - e^*)^j\middle| Z_1\right] = o(Var[e_n|Z_1])$. As a result,

$$\begin{aligned}\bar{T}_n V(z) &= h^{-1}\{f(e^*)\} + \frac{1}{h'(h^{-1}\{f(e^*)\})} \times \frac{1}{2}f''(e^*)Var[e_n|Z_1 = z] + O\left(E\left[(e_n - e^*)^3\middle| Z_1 = z\right]\right) \\ &= TV(z) + \frac{1}{h'(u(z) + \beta h^{-1}(e^*))} \times \frac{1}{2}f''(e^*)Var[e_n|Z_1 = z] + O\left(M^{(3)}[e_n|Z_1 = z]\right).\end{aligned} \tag{66}$$

where the second equality uses the definition of $TV(z)$, (65), and uses (11) to simplify $h^{-1}\{f(e^*)\}$.

Because h^{-1} is analytical, we can write $h^{-1}\{e_n\} = h^{-1}(e^*) + \sum_{j=0}^{\infty}\frac{1}{j!}\left(h^{-1}\right)^{(j)}(e^*)(e_n - e^*)^j$. This implies that

$$h^{-1}\{e_n\} - E\left[h^{-1}\{e_n\}\middle| Z_1 = z\right] = \left(h^{-1}\right)'(e^*)(e_n - e^*) + O(e_n - e^*)^2.$$

Therefore,

$$Var\left[h^{-1}(e_n)\middle| Z_1 = z\right] = \left(\frac{1}{h'(h^{-1}(e^*))}\right)^2 Var[e_n|Z_1 = z] + O\left(M^{(3)}[e_n|Z_1 = z]\right),$$

and

$$Var\left[u(z) + \beta h^{-1}(e_n)\middle| Z_1 = z\right] = \left(\frac{\beta}{h'(h^{-1}(e^*))}\right)^2 Var[e_n|Z_1 = z] + O\left(M^{(3)}[e_n|Z_1 = z]\right)$$

Also, $O\left(M^{(3)}\left[e_n\right| Z_1=z\right])=O\left(M^{(3)}\left[h^{-1}\left(e_n\right)\right| Z_1=z\right])$. This allows us to write (66) as [11]

$$\begin{aligned}\bar{T}_nV(z)-TV(z)=&\frac{1}{2}\frac{f''(e^*)}{h'\left(u(z)+\beta h^{-1}(e^*)\right)}\times\left(\frac{h'\left(h^{-1}(e^*)\right)}{\beta}\right)^2 Var\left[u(z)+\beta h^{-1}(e_n)\right| Z_1=z]\\&+O\left(M^{(3)}\left[u(z)+\beta h^{-1}(e_n)\right| Z_1=z]\right)\Big].\end{aligned}\tag{67}$$

Using Lemma 6.1, we can write the coefficient in the above equation as

$$\begin{aligned}&\frac{f''(e^*)}{h'\left(u(z)+\beta h^{-1}(e^*)\right)}\times\left(\frac{h'\left(h^{-1}(e^*)\right)}{\beta}\right)^2\\=&\frac{\beta h'\left(u(z)+\beta h^{-1}e^*\right)}{\left[h'\left(h^{-1}(e^*)\right)\right]^2}\left\{[\mathcal{A}h]\left(h^{-1}(e^*)\right)-\beta[\mathcal{A}h]\left(u+\beta h^{-1}(e^*)\right)\right\}\times\frac{\left[h'\left(h^{-1}(e^*)\right)\right]^2}{h'\left(u(z)+\beta h^{-1}(e^*)\right)}\frac{1}{\beta^2}\\=&\frac{1}{\beta}[\mathcal{A}h]\left(h^{-1}(e^*)\right)-[\mathcal{A}h]\left(u+\beta h^{-1}(e^*)\right),\end{aligned}$$

and write the welfare gain in (67) as

$$\begin{aligned}T_nV(z)-TV(z)=&\frac{1}{2}\left\{\frac{1}{\beta}[\mathcal{A}h]\left(h^{-1}(e^*)\right)-[\mathcal{A}h]\left(u+\beta h^{-1}(e^*)\right)\right\}Var\left[u(z)+\beta h^{-1}(e_n)\right| Z_1=z]\\&+O\left(M^{(3)}\left[u(z)+\beta h^{-1}(e_n)\right| Z_1=z]\right)\Big].\end{aligned}$$

Because e_n and w_n are related by $w_n=h^{-1}(e_n)$, and e^* and w are related by $e^*=h^{-1}(w)$, this proves Equation (25).

To prove equation (27), we note that

$$\begin{aligned}\ln T_nV(z)-\ln TV(z)=&\frac{1}{TV(z)}\left[T_nV(z)-TV(z)\right]+o\left[T_nV(z)-TV(z)\right]\\=&\frac{1}{TV(z)}\frac{1}{2}\eta_{PER}\left(u(z),w_1(z)\right)Var\left[u(z)+\beta w_1(n)\right| Z_1=z]\\&+O\left(M^{(3)}\left[u(z)+\beta w_1(n)\right| Z_1=z]\right),\end{aligned}\tag{68}$$

by (25). Also, using a log approximation for $u(z)+\beta w_1(n)$ around $u(z)+\beta w(z)$, we have:

$$\begin{aligned}\ln\left[u(z)+\beta w_1(n)\right]=&\ln\left[u(z)+\beta w(z)\right]+\frac{1}{u(z)+\beta w(z)}\beta\left[w_1(n)-w(z)\right].\\&+o\left|w_1(n)-w(z)\right|\end{aligned}$$

[11] This may not be correct if $h^{-1}(e)=e^2$, for example.

We can write

$$Var\left[\ln\left[u\left(z\right)+\beta w_1\left(n\right)\right]\right|Z_1=z]=\frac{1}{\left(u\left(z\right)+\beta w\left(z\right)\right)^2}Var\left[u\left(z\right)+\beta w\left(z\right)\right|Z_1=z]$$
$$+O\left(M^{(3)}\left[\ln\left[u\left(z\right)+\beta w_1\left(n\right)\right]\right|Z_1=z]\right). \tag{69}$$

Because $TV\left(z\right)=u\left(z\right)+\beta w_1\left(z\right)$. Replacing the $Var\left[u\left(z\right)+\beta w_1\left(n\right)\right|Z_1=z]$ term in Equation (68) with (69), we have

$$\ln T_nV\left(z\right)-\ln TV\left(z\right)=\frac{1}{2}\eta_{PER}\left(u\left(z\right),w_1\left(z\right)\right)\left[u\left(z\right)+\beta w\left(z\right)\right]Var\left[\ln\left[u\left(z\right)+\beta w_1\left(n\right)\right]\right|Z_1=z]$$
$$O\left(M^{(3)}\left[\ln\left[u\left(z\right)+\beta w_1\left(n\right)\right]\right|Z_1=z]\right),$$

which is Equation (27). □

Proof for Corollary 1 Using Equation (25) Proposition 1,

$$\bar{T}V\left(z\right)-\left[TV\right]\left(z\right)=\frac{1}{2}\eta_{PER}\left(u\left(z\right),w\left(z\right)\right)Var\left[u\left(z\right)+\beta V\left(Z_2\right)\right|Z_1=z]+O\left(M^{(3)}\left[u\left(z\right)+\beta V\left(Z_2\right)\right|Z_1=z]\right)$$
$$=\frac{1}{2}\beta^2\eta_{PER}\left(u\left(z\right),w\left(z\right)\right)Var\left[V\left(Z_2\right)\right|Z_1=z]+O\left(M^{(3)}\left[V\left(Z_2\right)\right|Z_1=z]\right),$$

as needed.

To prove (32), note that the variance term in (68) can be written as:

$$Var\left[u\left(z\right)+\beta w_1\left(n\right)\right|Z_1=z]=\beta^2Var\left[w_1\left(n\right)\right|Z_1=z].$$

Using a first order approximation,

$$\ln w_1\left(n\right)=\ln w\left(z\right)+\frac{1}{w\left(z\right)}\left[w_1\left(n\right)-w\left(z\right)\right]+o\left|w_1\left(n\right)-w\left(z\right)\right|, \tag{70}$$

which implies

$$Var\left[\ln w_1\left(n\right)\right|Z_1=z]=\frac{1}{w^2\left(z\right)}Var\left[w_1\left(n\right)\right|Z_1=z]+O\left(M^{(3)}\left[\ln w_1\left(n\right)\right|Z_1=z]\right).$$

Combining this equation and (70), we have:

$$Var\left[u\left(z\right)+\beta w_1\left(n\right)\right|Z_1=z]=\beta^2w^2\left(z\right)Var\left[\ln w_1\left(n\right)\right|Z_1=z].$$

Using this Equation to replace the variance term in Equation (68), we can write

$$\begin{aligned}\ln T_n V(z) - \ln TV(z) = & \frac{1}{u(z)+\beta w(z)} \frac{1}{2} \eta_{PER}(u(z), w_1(z)) \beta^2 w^2(z) Var[\ln w_1(n) | Z_1 = z] \\ & + O\left(M^{(3)}[u(z) + \beta w_1(n) | Z_1 = z]\right) \\ = & \frac{1}{2} \bar{\eta}_{PER}(u(z), w_1(z)) \frac{\beta^2 w^2(z)}{(u(z)+\beta w(z))^2} Var[\ln w_1(n) | Z_1 = z] \\ & O\left(M^{(3)}[\ln w_1(n) | Z_1 = z]\right),\end{aligned}$$

as needed.

6.2 Successive approximations

Lemma 2. *(Successive Approximation) Suppose h is analytic, $u \in L^1$ and $\epsilon \to 0$ in L^2, then*

$$\begin{aligned}h^{-1}\{E[h(u+\epsilon)]\} = & h^{-1}\{E[h(u)]\} + E[\epsilon] \\ & + \frac{1}{2} \frac{h''(\bar{u})}{h'(\bar{u})} \{Var[\epsilon] + 2Cov[u, \epsilon]\} + O\left(M^{(3)}[\epsilon]\right),\end{aligned} \tag{71}$$

where $\bar{u} = E[u]$. In addition,

$$\begin{aligned}\ln h^{-1}\left\{E\left[h\left(e^{\ln u + \epsilon}\right)\right]\right\} = & \ln h^{-1}\{E[h(u)]\} + E[\epsilon] \\ & + \frac{1}{2}\left(1 + \frac{h''\left(e^{\ln \bar{u}} u\right) e^{\ln \bar{u}} u}{h'\left(e^{\ln \bar{u}} u\right)}\right) \{Var[\epsilon] + 2Cov[\ln u, \epsilon]\} + O\left(M^{(3)}[\epsilon]\right),\end{aligned} \tag{72}$$

Proof. Using Taylor expansion of $h(u+\epsilon)$ around $\bar{u} = E[u]$, and taking expectations on both sides, we can write $E[h(u+\epsilon)]$ as:

$$E[h(u+\epsilon)] = h(\bar{u}) + \sum_{j=1}^{\infty} \frac{1}{j!} h^{(j)}(\bar{u}) E[u - \bar{u} + \epsilon]^j.$$

This allows to compute the Taylor expansion of $h^{-1}\{E[h(u(Z) + \epsilon(Z))]\}$ around $h(\bar{u})$:

$$\begin{aligned}h^{-1}\{E[h(u+\epsilon)]\} = & h^{-1}\{h(\bar{u})\} + h^{-1\prime}(h(\bar{u})) \left\{\sum_{j=1}^{\infty} \frac{1}{j!} h^{(j)}(\bar{u}) E[u - \bar{u} + \epsilon]^j\right\} \\ & + \frac{1}{2} h^{-1\prime\prime}(h(\bar{u})) \left\{\sum_{j=1}^{\infty} \frac{1}{j!} h^{(j)}(\bar{u}) E[u - \bar{u} + \epsilon]^j\right\}^2 \\ & + \sum_{k=3}^{\infty} \frac{1}{k!} h^{-1(k)}(h(\bar{u})) \left\{\sum_{j=1}^{\infty} \frac{1}{j!} h^{(j)}(\bar{u}) E[u - \bar{u} + \epsilon]^j\right\}^k.\end{aligned} \tag{73}$$

Similarly, we can write the series representation for $h^{-1}\{E[h(u)]\}$ as:

$$\begin{aligned} h^{-1}\{E[h(u)]\} = & h^{-1}\{h(\bar{u})\} + h^{-1\prime}(h(\bar{u}))\left\{\sum_{j=1}^{\infty}\frac{1}{j!}h^{(j)}(\bar{u})E[u-\bar{u}]^j\right\} \\ & + \frac{1}{2}h^{-1\prime\prime}(h(\bar{u}))\left\{\sum_{j=1}^{\infty}\frac{1}{j!}h^{(j)}(\bar{u})E[u-\bar{u}]^j\right\}^2 \\ & + \sum_{k=3}^{\infty}\frac{1}{k!}h^{-1(k)}(h(\bar{u}))\left\{\sum_{j=1}^{\infty}\frac{1}{j!}h^{(j)}(\bar{u})E[u-\bar{u}]^j\right\}^k. \end{aligned} \tag{74}$$

Compute the difference between Equation (73) and (74), we have:

$$\begin{aligned} h^{-1}\{E[h(u+\epsilon)]\} = & h^{-1}\{E[h(u)]\} + h^{-1\prime}(h(\bar{u}))h'(\bar{u})E[\epsilon] \\ & + h^{-1\prime}(h(\bar{u}))\frac{1}{2}h''(\bar{u})E\left[(u-\bar{u}+\epsilon)^2-(u-\bar{u})^2\right] \\ & + \frac{1}{2}h^{-1\prime\prime}(h(\bar{u}))\left\{h'(\bar{u})E[u-\bar{u}+\epsilon]\right\}^2 + O\left(M^{(3)}[\epsilon]\right) \end{aligned}$$

Using the fact that $h^{-1\prime}(h(\bar{u})) = \frac{1}{h'(\bar{u})}$ and $h^{-1\prime\prime}(h(\bar{u})) = -\frac{h''(\bar{u})}{[h'(\bar{u})]^3}$, we can write the above as

$$\begin{aligned} h^{-1}\{E[h(u+\epsilon)]\} = & h^{-1}\{E[h(u)]\} + E[\epsilon] + \frac{1}{2}\frac{h''(\bar{u})}{h'(\bar{u})}E\left[\epsilon^2 + 2(u-\bar{u})\epsilon\right] \\ & - \frac{1}{2}\frac{h''(\bar{g})}{h'(\bar{g})}\{E[\epsilon]\}^2 + O\left(M^{(3)}[\epsilon(Z)]\right) \\ = & h^{-1}\{E[h(u)]\} + E[\epsilon] + \frac{1}{2}\frac{h''(\bar{u})}{h'(\bar{u})}\{Var[\epsilon] + 2Cov[\epsilon,u]\}, \end{aligned}$$

which is (71).

Finally, to prove (72), we can compare $\ln h^{-1}\left\{E\left[h\left(e^{u(Z)+\epsilon(Z)}\right)\right]\right\}$ with $\ln h^{-1}\left\{E\left[h\left(e^{u(Z)}\right)\right]\right\}$ by setting $g(x) = h(e^x)$ and applying (71). □

Proof for Proposition 2 We first consider Equation (42). We use the same operation as in (39) and (40). In general, the utility for $n+1$ period early resolution of uncertainty can be written as:

$$\begin{aligned} V^{(n+1)}(z) &= h^{-1}E_z\left[h\left\{u(Z_1) + \beta u(Z_2) + \cdots \beta^{n+1}V(Z_{n+2})\right\}\right]. \\ &= h^{-1}E\left[h \circ h^{-1}E\left[h\left\{u(Z_1) + \beta u(Z_2) + \cdots \beta^{n+1}V(Z_{n+2})\right\}\middle| Z_{n+1}\right]\right] \end{aligned} \tag{75}$$

and that for n period early resolution is:

$$V^{(n)}(z) = h^{-1}E_z\left[h\left\{u(Z_1) + \beta u(Z_2) + \cdots \beta^{n+1}h^{-1}E[hV(Z_{n+2})|Z_{n+1}]\right\}\right].$$

Using Corollary 25, where $u_0 + \beta u_1 + \cdots \beta^n u_n$ is interpreted as u and β^{n+1} is interpreted as β in the Corollary, we have:

$$\begin{aligned} & h^{-1}E\left[h\left\{\underbrace{u\left(Z_1\right)+\beta u\left(Z_2\right)+\cdots+\beta^n u\left(Z_{n+1}\right)}_{u_{n+1}}+\beta^{n+1}V\left(Z_{n+2}\right)\right\}\middle| Z_{n+1}\right] \\ & \doteq u\left(Z_1\right)+\beta u\left(Z_2\right)+\cdots+\beta^n u\left(Z_{n+1}\right)+\beta^{n+1}h^{-1}E\left[hV\left(Z_{n+2}\right)\middle| Z_{n+1}\right] \\ & +\frac{1}{2}\left(\beta^{n+1}\right)^2\eta_{PER}\left(u_{n+1}\left(Z^{n+1}\right),w\left(Z_{n+1}\right)\middle|\beta^{n+1}\right)Var\left[V\left(Z_{n+2}\right)\middle| Z_{n+1}\right] \end{aligned} \tag{76}$$

Denoting $u\left(Z^{n+1}\right) = u\left(Z_1\right)+\beta u\left(Z_2\right)+\cdots+\beta^n u\left(Z_{n+1}\right)$, and $w\left(Z_{n+1}\right) = h^{-1}E\left[hV\left(Z_{n+2}\right)\middle| Z_{n+1}\right]$. The above can be summarized as

$$V^{(n+1)}(z) = h^{-1}E\left[h\left(u\left(Z^{n+1}\right)+\beta^{n+1}w\left(Z_{n+1}\right)+\epsilon\right)\right],$$

with $\epsilon_{n+1} = \frac{1}{2}\left(\beta^{n+1}\right)^2\eta_{PER}\left(u_{n+1}\left(Z^{n+1}\right),w\left(Z_{n+1}\right)\middle|\beta^{n+1}\right)Var\left[V\left(Z_{n+2}\right)\middle| Z_{n+1}\right]$, and

$$V^{(n+1)}(z) = h^{-1}E\left[h\left(u\left(Z^{n+1}\right)+\beta^{n+1}w\left(Z_{n+1}\right)\right)\right].$$

We can apply Lemma 2 to get

$$V^{(n+1)}(z) - V^{(n)}(z) = E\left[\frac{1}{2}\left(\beta^{n+1}\right)^2\eta_{PER}\left(u_{n+1}\left(Z^{n+1}\right),w\left(Z_{n+1}\right)\middle|\beta^{n+1}\right)Var\left[V\left(Z_{n+2}\right)\middle| Z_{n+1}\right]\right].$$

We arrive at Equation (42) by summing up the first n terms of the above equation.

Equation (43) can be established similarly. Using (75), we can write the utility for $n+1$ period early resolution of uncertainty as:

$$\begin{aligned} V^{(n+1)}(z) &= h^{-1}E\left[h\circ e^{\ln h^{-1}E\left\{h\left\{u\left(Z^{n+1}\right)+\beta^{n+1}V\left(Z_{n+2}\right)\right\}\middle| Z_{n+1}\right\}}\right] \\ &= h^{-1}E\left[h\circ e^{\ln\left(u\left(Z^{n+1}\right)+\beta^{n+1}h^{-1}E\left[hV\left(Z_{n+2}\right)\middle| Z_{n+1}\right]\right)+\epsilon_{n+1}}\right], \end{aligned}$$

where the second line use the log approximation formula in (32), and ϵ_{n+1} is defined as in Equation (44). Using Lemma 2 and ignore the variance and covariance terms, we obtain

$$\ln V^{(n+1)}(z) - \ln V^{(n)}(z) = E\left[\epsilon_{n+1}\right], \tag{77}$$

We obtain Equation (43) by summing up the first n terms of the above equation.

Examples of successive approximations

The case of unit IES As shown in Epstein, Farhi, and Strzalecki (2014), in the case of unit IES, the value function takes the form of $V(x,C) = \ln C + \frac{\beta}{1-\beta\rho}x + cons$, where $cons$ is a constant.

To compute conditional variance, we write

$$\begin{aligned} V\left(x_{t+1}, C_{t+1}\right) &= \ln C_{t+1} + \frac{\beta}{1-\beta\rho} x_{t+1} + cons \\ &= \ln C_t + \ln\left(\frac{C_{t+1}}{C_t}\right) + \frac{\beta}{1-\beta\rho}\left(\rho x_t + \phi\sigma\epsilon_{x,t+1}\right) + cons \\ &= \ln C_t + \mu + x_t + \sigma\epsilon_{C,t+1} + \frac{\beta}{1-\beta\rho}\left(\rho x_t + \phi\sigma\epsilon_{x,t+1}\right) + cons. \end{aligned}$$

As a result, $Var\left[V\left(x_{t+1}, C_{t+1}\right)\middle| x_t, C_t\right] = \sigma^2 + \left(\frac{\beta}{1-\beta\rho}\right)^2 \phi^2\sigma^2$, as needed.

General IES In the general IES case, the value function for gradual resolution of uncertainty takes the form $V\left(x_t, C_t\right) = \frac{1}{1-\frac{1}{\psi}} H\left(x_t\right) C_t^{1-\frac{1}{\psi}}$, where the $H\left(x\right)$ satisfies the recursion (33):

$$\frac{1}{1-\frac{1}{\psi}} H\left(x_t\right) C_t^{1-\frac{1}{\psi}} = \frac{1}{1-\frac{1}{\psi}} C_t^{1-\frac{1}{\psi}} + \beta \frac{1}{1-\frac{1}{\psi}} \left(E_t\left[\left(H\left(x_{t+1}\right) C_{t+1}^{1-\frac{1}{\psi}}\right)^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}},$$

or equivalently, the H function satisfies the following functional equation:

$$H\left(x\right) = 1 + \beta\left(E\left[H^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\left(\rho x + \phi\epsilon_x\right) e^{(1-\gamma)(\mu+x+\sigma\epsilon_C)}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}, \tag{78}$$

where ϵ_C and ϵ_x are i.i.d. standard Normal distributions. In addition, the certainty equivalent of next period utility is $\frac{1}{1-\frac{1}{\psi}} w\left(x_t\right) C_t^{1-\frac{1}{\psi}}$, and $w\left(x\right)$ is given by:

$$w\left(x\right) = \left(E_t\left[H^{\frac{1-\gamma}{1-\frac{1}{\psi}}}\left(\rho x + \phi\epsilon_x\right) e^{(1-\gamma)(\mu+x+\sigma\epsilon_C)}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}.$$

In the case of i.i.d. consumption growth, that is, $\phi = 0$ in (46), the H function in 78 is a constant and does not depend on x. Equation (78) is simplified to

$$H = (1-\beta) + \beta H e^{\left(1-\frac{1}{\psi}\right)\left(\mu + \frac{1}{2}(1-\gamma)\sigma^2\right)}. \tag{79}$$

Because the conditional variance $Var\left[\ln V\left(Z_{n+1}\right)\middle| Z_n\right] = \left(1-\frac{1}{\psi}\right)^2 \sigma^2$ is a constant, we apply (43) directly. Here, we evaluate $u\left(Z^n\right)$ and $w\left(Z_n\right)$ at the steady state (which we denote as $\bar{u}\left(Z^n\right)$ and

$\bar{w}(Z_n)$):

$$\begin{aligned}\bar{u}(Z^n) =& \frac{1}{1-\frac{1}{\psi}}\left[C_1^{1-\frac{1}{\psi}} + \beta C_2^{1-\frac{1}{\psi}} + \cdots + \beta^{n-1} C_n^{1-\frac{1}{\psi}}\right] \\ =& \frac{1}{1-\frac{1}{\psi}}\left[C_1^{1-\frac{1}{\psi}} + \beta C_1^{1-\frac{1}{\psi}} e^{\left(1-\frac{1}{\psi}\right)(\mu+\sigma\epsilon_2)} + \cdots + \beta^{n-1} C_1^{1-\frac{1}{\psi}} e^{\left(1-\frac{1}{\psi}\right)\sum_{j=2}^{n}(\mu+\sigma\epsilon_j)}\right] \\ =& \frac{1}{1-\frac{1}{\psi}} C_1^{1-\frac{1}{\psi}}\left[1 + \beta e^{\left(1-\frac{1}{\psi}\right)\mu} + \cdots + \beta^{n-1} e^{\left(1-\frac{1}{\psi}\right)(n-1)\mu}\right],\end{aligned}$$

where the last line evaluate the utility sum at the deterministic steady state.[12] As a result, we have:

$$\bar{u}(Z^n) = \frac{1}{1-\frac{1}{\psi}} C_1^{1-\frac{1}{\psi}} \frac{1 - e^{n\left(\ln\beta + \left(1-\frac{1}{\psi}\right)\mu\right)}}{1 - e^{\ln\beta + \left(1-\frac{1}{\psi}\right)\mu}}.$$

Also, the certainty equivalent term can be computed as:

$$\begin{aligned}\beta^n \bar{w}(Z_n) =& \beta^n \frac{1}{1-1/\psi} H C_n^{1-\frac{1}{\psi}} e^{\left(1-\frac{1}{\psi}\right)\left(\mu + \frac{1}{2}(1-\gamma)\sigma^2\right)} \\ =& \beta^n \frac{1}{1-1/\psi} H C_1^{1-\frac{1}{\psi}} e^{(n-1)\left(1-\frac{1}{\psi}\right)\mu + \left(1-\frac{1}{\psi}\right)\left(\mu + \frac{1}{2}(1-\gamma)\sigma^2\right)}.\end{aligned}$$

This allows us to evaluate the weight $\bar{\lambda}_n$ in Equation (49) at steady state:[13]

$$\bar{\lambda}_n = \frac{\bar{u}(Z^n)}{\bar{u}(Z^n) + \beta^n \bar{w}(Z_n)} = \frac{\frac{1-e^{n\left(\ln\beta + \left(1-\frac{1}{\psi}\right)\mu\right)}}{1-e^{\ln\beta + \left(1-\frac{1}{\psi}\right)\mu}}}{\frac{1-e^{n\left(\ln\beta + \left(1-\frac{1}{\psi}\right)\mu\right)}}{1-e^{\ln\beta + \left(1-\frac{1}{\psi}\right)\mu}} + \beta^n H e^{\left(1-\frac{1}{\psi}\right)\left[n\mu + \frac{1}{2}(1-\gamma)\sigma^2\right]}}.$$

The total welfare gain of early resolution can therefore be computed as:

$$\ln \bar{V}(z) - \ln V(z) = \frac{1}{2}\left(\gamma - \frac{1}{\psi}\right)\left(1 - \frac{1}{\psi}\right)\sigma^2 \sum_{n=1}^{\infty} \bar{\lambda}_n \left(1 - \bar{\lambda}_n\right).$$

In the more general case with $\phi \neq 0$, we provide a more accurate approximation by including the variance and covariance terms in Lemma 2. We write Equation (77) as:

$$\begin{aligned}\ln V^{(n+1)}(z) =& \ln V^{(n)}(z) + E[\epsilon_{n+1}] \\ &+ \frac{1}{2}\frac{1-\gamma}{1-\frac{1}{\psi}}\left\{Var[\epsilon_{n+1}] + 2Cov\left[\ln\left(u\left(Z^{n+1}\right) + \beta^{n+1} w\left(Z^{n+1}\right)\right), \epsilon_{n+1}\right]\right\}\end{aligned}$$

[12] Hongye, we can instead write $\frac{1}{1-\frac{1}{\psi}} C_1^{1-\frac{1}{\psi}}\left[1 + \beta e^{\left(1-\frac{1}{\psi}\right)\mu + \frac{1}{2}\left(1-\frac{1}{\psi}\right)^2\sigma_C^2} + \cdots + \beta^{n-1} e^{\left(1-\frac{1}{\psi}\right)(n-1)\mu + \frac{1}{2}\left(1-\frac{1}{\psi}\right)^2(n-1)\sigma_C^2}\right]$. That is, taking expectations of $C_n^{1-\frac{1}{\psi}}$. Can you examine which approximation gives a better result? This is what I mean by stochastic steady state.

[13] Our previous formula missed a term? Again, please check if we want to include the vol term for consumption so that everything is evaluated at the stochastic steady state.

where we use the functional form of h in (34) to get $\frac{h''(e^{\bar{u}})e^{\bar{u}}}{h'(e^{\bar{u}})} = \frac{\frac{1}{\psi}-\gamma}{1-\frac{1}{\psi}}$. The term $Cov\left[\ln\left(u\left(Z^{n+1}\right)+\beta^{n+1}w\left(Z^{n+1}\right)\right),\right.$ is of the same order as ϵ_{n+1}. We focus on this term and use (48) to write a log linear approximation for ϵ_{n+1}: $\epsilon_{n+1} = \bar{\epsilon}_{n+1}\left\{1+\ln\epsilon_{n+1}-\ln\bar{\epsilon}_{n+1}\right\}$, where $\bar{\epsilon}_{n+1} = \frac{1}{2}\frac{\gamma-\frac{1}{\psi}}{1-\frac{1}{\psi}}\bar{\lambda}_n\left(1-\bar{\lambda}_n\right)Var\left[\ln V\left(Z_{n+2}\right)\middle|Z_{n+1}\right]$ is the steady-state level of ϵ_{n+1}, and $\ln\epsilon_{n+1}-\ln\bar{\epsilon}_{n+1}$ is the log deviation from steady state. The conditional variance term $Var\left[\ln V\left(Z_{n+2}\right)\middle|Z_{n+1}\right]$ is close to a constant. As a result, we focus on the log deviation of the term $\lambda_n\left(1-\lambda_n\right) = \frac{u(Z^n)\beta^n w(Z_n)}{\left[u(Z^n)+\beta^n w(Z_n)\right]^2}$ to write:

$$\ln\epsilon_{n+1}-\ln\bar{\epsilon}_{n+1} = \hat{u}\left(Z^{n+1}\right)+\hat{w}\left(Z_{n+1}\right)-2\left[\bar{\lambda}_n\hat{u}\left(Z^{n+1}\right)+\left(1-\bar{\lambda}_n\right)\hat{w}\left(Z_{n+1}\right)\right], \tag{80}$$

where for any random variable X, we use $\widehat{X} = \ln X - \ln\bar{X}$ to denote its log deviation.

Using the log linear approximation (80), the covariance term, $Cov\left[\ln\left(u\left(Z^{n+1}\right)+\beta^{n+1}w\left(Z^{n+1}\right)\right),\epsilon_{n+1}\right]$ can be approximated by:

$$\begin{aligned}
&Cov\left[\ln\left(u\left(Z^{n+1}\right)+\beta^{n+1}w\left(Z^{n+1}\right)\right),\epsilon_{n+1}\right]\\
=&\bar{\epsilon}_{n+1}Cov\left[\bar{\lambda}_n\hat{u}\left(Z^{n+1}\right)+\left(1-\bar{\lambda}_n\right)\hat{w}\left(Z_{n+1}\right),\hat{u}\left(Z^{n+1}\right)+\hat{w}\left(Z_{n+1}\right)-2\left[\bar{\lambda}_n\hat{u}\left(Z^{n+1}\right)+\left(1-\bar{\lambda}_n\right)\hat{w}\left(Z_{n+1}\right)\right]\right]\\
=&\bar{\epsilon}_{n+1}\left\{\bar{\lambda}_n\left(1-2\bar{\lambda}_n\right)Var\left[\hat{u}\left(Z^{n+1}\right)\right]+\left(1-\bar{\lambda}_n\right)\left(1-2\left(1-\bar{\lambda}_n\right)\right)Var\left[\hat{w}\left(Z_{n+1}\right)\right]\right.\\
&\left.+\left[1-4\bar{\lambda}_n\left(1-\bar{\lambda}_n\right)\right]Cov\left[\hat{u}\left(Z^{n+1}\right),\hat{w}\left(Z_{n+1}\right)\right]\right\}.
\end{aligned} \tag{81}$$

This leads to the following approximation formula:

$$\begin{aligned}
\ln V^{(n+1)}(z)-\ln V^{(n)}(z) \doteq& E\left[\epsilon_{n+1}\right]+\frac{1-\gamma}{1-\frac{1}{\psi}}Cov\left[\ln\left(u\left(Z^{n+1}\right)+\beta^{n+1}w\left(Z^{n+1}\right)\right),\epsilon_{n+1}\right]\\
=&\bar{\epsilon}_{n+1}\left\{1+\frac{1-\gamma}{1-\frac{1}{\psi}}\chi_{n+1}\right\},
\end{aligned}$$

where χ_{n+1} is defined using (81):

$$\begin{aligned}
\chi_{n+1} =& \bar{\lambda}_n\left(1-2\bar{\lambda}_n\right)Var\left[\hat{u}\left(Z^{n+1}\right)\right]-\left(1-\bar{\lambda}_n\right)\left(1-2\bar{\lambda}_n\right)Var\left[\hat{w}\left(Z_{n+1}\right)\right]\\
&+\left[1-4\bar{\lambda}_n\left(1-\bar{\lambda}_n\right)\right]Cov\left[\hat{u}\left(Z^{n+1}\right),\hat{w}\left(Z_{n+1}\right)\right].
\end{aligned}$$

6.3 Measurement of $\bar{\eta}_{PER}$

Proof for Lemma 1 First, under the assumption of the lemma, we can without loss generality assume that the utility function (viewed as a function of consumption sequences, $\{C_t\}_{t=1}^T$, is homogenous of degree ψ for some $\psi > 0$. This implies $u(C)$ must be homogenous of degree ψ and $\mathcal{I}$ must be homogenous of degree one. To see this, note that in the last period, $V_T(C_T) = u(C_T)$. Homogeneity of V implies the homogeneity of u. To see $\mathcal{I}$ must be homogenous of degree one, for any $\lambda > 0$,

$$V_{T-1}\left(\lambda C_{T-1},\lambda C_T\right) = \lambda^{\psi}V_{T-1}\left(C_{T-1},C_T\right) = \lambda^{\psi}\left\{u\left(C_{T-1}\right)+\beta\mathcal{I}\left[u\left(C_T\right)\right]\right\}$$

by homogeneity. But the homogeneity of u implies

$$V_{T-1}\left(\lambda C_{T-1}, \lambda C_T\right) = u\left(\lambda C_{T-1}\right) + \beta \mathcal{I}\left[u\left(\lambda C_T\right)\right] = \lambda^{\psi} u\left(C_{T-1}\right) + \beta \mathcal{I}\left[\lambda^{\psi} u\left(C_T\right)\right].$$

Combine the above two equations, we must have $\lambda^{\psi}\mathcal{I}\left[u\left(C_T\right)\right] = \mathcal{I}\left[\lambda^{\psi} u\left(C_T\right)\right]$ for all $\lambda > 1$, which establishes the homogeneity of $\mathcal{I}$.

The above properties imply that the value function in the dynamic program problem (53) must be homogenous of degree ψ in W: $U_t\left(z, \lambda W\right) = \lambda^{\psi} U_t\left(z, W\right)$ for all (z, W). Homogeneity of u and U implies that $u'\left(C\right)C = \psi u\left(C\right)$ and $\frac{\partial}{\partial W} U_t\left(z, W\right) W = \psi U_t\left(z, W\right)$ due to Euler's theorem.

We can now write the wealth-to-consumption ratio as:

$$\frac{W_t}{C_t} = \frac{W_t \frac{\partial}{\partial W} U_t\left(Z_t, W_t\right)}{C_t u'\left(C_t\right)} = \frac{\psi U_t\left(Z_t, W_t\right)}{\psi u\left(C_t\right)}.$$

The first equality is due to the envelope condition for the maximization problem in (53): $\frac{\partial}{\partial W} U_t\left(Z_t, W_t\right) = u'\left(C_t\right)$. The second equality applies Euler's theorem to both the value function $U_t\left(Z_t, W_t\right)$ and the utility function $u\left(C_t\right)$. This proves the lemma.

Proof for Proposition 4 To save notation, we denote $X = Var\left[\ln W_1\left(s, \tau\right) \middle| \tau\right]$ and assume that X follows a Gamma distribution with parameter (α, β). That is, the density of X is given by, for $x > 0$,

$$f\left(x \middle| \alpha, \beta\right) = \frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} x^{\alpha-1} e^{-\beta x},$$

where $\Gamma\left(\alpha\right)$ is the Gamma function. Using Equation (59), We can write $ASDF_{-1,0} = e^{\hat{\eta}_0 - \eta X}$, where $\eta = \bar{\eta}_{GRS}\bar{\eta}_{PER}$. This allows us to compute the expected payoff of the variance portfolio as:

$$E\left[X\right] = \frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} \int_0^{\infty} x \times x^{\alpha-1} e^{-\beta x} dx = \frac{\alpha}{\beta},$$

the present value of the variance portfolio as

$$E\left[e^{\hat{\eta}_0 - \eta X} X\right] = \frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} \int_0^{\infty} e^{\hat{\eta}_0 - \eta x} x \times x^{\alpha-1} e^{-\beta x} dx = e^{\hat{\eta}_0} \frac{\alpha \beta^{\alpha}}{\left(\eta + \beta\right)^{\alpha+1}},$$

and the risk-free rate as

$$R_f = \frac{1}{E\left[ASDF_{-1,0}\right]} = \left[\frac{\beta^{\alpha}}{\Gamma\left(\alpha\right)} \int_0^{\infty} e^{\hat{\eta}_0 - \eta x} \times x^{\alpha-1} e^{-\beta x} dx\right]^{-1} = \left[e^{\eta_0} \left(\frac{\beta}{\eta + \beta}\right)^{\alpha}\right]^{-1}.$$

The risk premium, $\frac{E[R_{-1,0}]}{R_f} - 1$ can therefore be computed as

$$\frac{E\left[R_{-1,0}\right]}{R_f} - 1 = \frac{E\left[X\right]}{E\left[e^{\hat{\eta}_0 - \eta X} X\right]} \frac{1}{R_f} - 1 = \frac{\eta}{\beta}.$$

As a result,

$$\eta = \bar{\eta}_{GRS}\bar{\eta}_{PER} = \beta \times \left\{ \frac{E\left[R_{-1,0}\right]}{R_f} - 1 \right\}.$$

To prove Equation (60), we note that $\beta = \frac{E[X]}{Var[X]}$ due to the property of the Gamma distribution.

Proof for Proposition, generalized 4 To save notation, we denote $X = Var\left[\ln W_1\left(s,\tau\right)|\,\tau\right] + c$ and assume that X follows a Gamma distribution with parameter (α, β). That is, the density of X is given by, for $x > 0$,

$$f\left(x|\,\alpha,\beta\right) = \frac{\beta^\alpha}{\Gamma\left(\alpha\right)} x^{\alpha-1} e^{-\beta x},$$

where $\Gamma\left(\alpha\right)$ is the Gamma function. Using Equation (59), We can write $ASDF_{-1,0} = e^{\hat{\eta}_0 - \eta(X-c)}$, where $\eta = \bar{\eta}_{GRS}\bar{\eta}_{PER}$. This allows us to compute the expected payoff of the variance portfolio as:

$$E\left[X\right] = \frac{\beta^\alpha}{\Gamma\left(\alpha\right)} \int_0^\infty x \times x^{\alpha-1} e^{-\beta x} dx = \frac{\alpha}{\beta},$$

the present value of the variance portfolio as

$$E\left[e^{\hat{\eta}_0 - \eta(X-c)} X\right] = \frac{\beta^\alpha}{\Gamma\left(\alpha\right)} \int_0^\infty e^{\hat{\eta}_0 - \eta(x-c)} x \times x^{\alpha-1} e^{-\beta x} dx = e^{\hat{\eta}_0 + \eta c} \frac{\alpha\beta^\alpha}{(\eta+\beta)^{\alpha+1}},$$

and the risk-free rate as

$$R_f = \frac{1}{E\left[ASDF_{-1,0}\right]} = \left[\frac{\beta^\alpha}{\Gamma\left(\alpha\right)} \int_0^\infty e^{\hat{\eta}_0 - \eta(x-c)} \times x^{\alpha-1} e^{-\beta x} dx\right]^{-1} = \left[e^{\hat{\eta}_0 + \eta c} \left(\frac{\beta}{\eta+\beta}\right)^\alpha\right]^{-1}.$$

The risk premium, $\frac{E[R_{-1,0}]}{R_f} - 1$ can therefore be computed as

$$\frac{E\left[R_{-1,0}\right]}{R_f} - 1 = \frac{E\left[X\right]}{E\left[e^{\hat{\eta}_0 - \eta(X-c)} X\right]} \frac{1}{R_f} - 1 = \frac{\frac{\alpha}{\beta}}{\frac{\alpha}{\eta+\beta}} - 1 = \frac{\eta}{\beta}.$$

As a result,

$$\eta = \bar{\eta}_{GRS}\bar{\eta}_{PER} = \beta \times \left\{ \frac{E\left[R_{-1,0}\right]}{R_f} - 1 \right\}.$$

To prove Equation (60), we note that $\beta = \frac{E[X]}{Var[X]}$ due to the property of the Gamma distribution.

Proof for Proposition 4 To save notation, we denote $X = Var\left[\ln W_1\left(s,\tau\right)|\,\tau\right]$ and assume that X follows a Gamma distribution with parameter (α, β). That is, the density of X is given by, for $x > 0$,

$$f\left(x|\,\alpha,\beta\right) = \frac{\beta^\alpha}{\Gamma\left(\alpha\right)} x^{\alpha-1} e^{-\beta x},$$

where $\Gamma(\alpha)$ is the Gamma function. Using Equation (59), We can write $ASDF_{-1,0} = e^{\hat{\eta}_0 - \eta X}$, where $\eta = \bar{\eta}_{GRS}\bar{\eta}_{PER}$. This allows us to compute the expected payoff of the variance portfolio as:

$$E[X+c] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty (x+c) \times x^{\alpha-1} e^{-\beta x} dx = \frac{\alpha}{\beta} + c,$$

the present value of the variance portfolio as

$$E\left[e^{\hat{\eta}_0 - \eta X}(X+c)\right] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{\hat{\eta}_0 - \eta x} x \times x^{\alpha-1} e^{-\beta x} dx = e^{\hat{\eta}_0} \frac{\alpha\beta^\alpha}{(\eta+\beta)^{\alpha+1}} + \frac{c}{R_f},$$

and the risk-free rate as

$$R_f = \frac{1}{E[ASDF_{-1,0}]} = \left[\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{\hat{\eta}_0 - \eta x} \times x^{\alpha-1} e^{-\beta x} dx\right]^{-1} = \left[e^{\hat{\eta}_0}\left(\frac{\beta}{\eta+\beta}\right)^\alpha\right]^{-1}.$$

The risk premium, $\frac{E[R_{-1,0}]}{R_f} - 1$ can therefore be computed as

$$\frac{E[R_{-1,0}]}{R_f} - 1 = \frac{E[X+c]}{E[e^{\hat{\eta}_0 - \eta X}(X+c)]} \frac{1}{R_f} - 1 = \frac{\frac{\alpha}{\beta} + c}{e^{\hat{\eta}_0} \frac{\alpha\beta^\alpha}{(\eta+\beta)^{\alpha+1}} R_f + c} - 1 = \frac{\frac{\alpha}{\beta} - \frac{\alpha}{\eta+\beta}}{\frac{\alpha}{\eta+\beta} + c} = \frac{\frac{\alpha}{\beta}\eta}{\alpha + c(\beta+\eta)}.$$

As a result,

$$\eta = \bar{\eta}_{GRS}\bar{\eta}_{PER} = rp \frac{\frac{\alpha}{\beta} + c}{\frac{\alpha}{\beta^2} - rp\frac{c}{\beta}}.$$

Here $rp = \left\{\frac{E[R_{-1,0}]}{R_f} - 1\right\}$. So does not look so nice.]

Proof for Proposition 4 To save notation, we denote $X = Var[\ln W_1(s,\tau)|\tau]$ and assume that X follows a Gamma distribution with parameter (α, β). That is, the density of X is given by, for $x > 0$,

$$f(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

where $\Gamma(\alpha)$ is the Gamma function. Using Equation (59), We can write $ASDF_{-1,0} = e^{\hat{\eta}_0 - \eta X}$, where $\eta = \bar{\eta}_{GRS}\bar{\eta}_{PER}$. This allows us to compute the expected payoff of the variance portfolio as:

$$E[X+c] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty (x+c) \times x^{\alpha-1} e^{-\beta x} dx = \frac{\alpha}{\beta} + c,$$

the present value of the variance portfolio as

$$E\left[e^{\hat{\eta}_0 - \eta X}(X+c)\right] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{\hat{\eta}_0 - \eta x} x \times x^{\alpha-1} e^{-\beta x} dx = e^{\hat{\eta}_0} \frac{\alpha\beta^\alpha}{(\eta+\beta)^{\alpha+1}} + \frac{c}{R_f},$$

and the risk-free rate as

$$R_f = \frac{1}{E[ASDF_{-1,0}]} = \left[\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{\hat{\eta}_0 - \eta x} \times x^{\alpha-1} e^{-\beta x} dx\right]^{-1} = \left[e^{\hat{\eta}_0}\left(\frac{\beta}{\eta+\beta}\right)^\alpha\right]^{-1}.$$

The risk premium, $\frac{E[R_{-1,0}]}{R_f} - 1$ can therefore be computed as

$$\frac{E\left[R_{-1,0}\right]}{R_f} - 1 = \frac{E\left[X + c\right]}{E\left[e^{\hat{\eta}_0 - \eta X}(X + c)\right]} \frac{1}{R_f} - 1 = \frac{\frac{\alpha}{\beta} + c}{e^{\hat{\eta}_0} \frac{\alpha\beta^\alpha}{(\eta+\beta)^{\alpha+1}} R_f + c} - 1 = \frac{\frac{\alpha}{\beta} - \frac{\alpha}{\eta+\beta}}{\frac{\alpha}{\eta+\beta} + c} = \frac{\frac{\alpha}{\beta}\eta}{\alpha + c(\beta + \eta)}.$$

As a result,

$$\eta = \bar{\eta}_{GRS}\bar{\eta}_{PER} = rp \frac{\frac{\alpha}{\beta} + c}{\frac{\alpha}{\beta^2} - rp\frac{c}{\beta}}.$$

Here $rp = \left\{\frac{E[R_{-1,0}]}{R_f} - 1\right\}$. So does not look so nice.

Utility and consumption ratio Welfare gains are calculated in utility units in Propositions 1 and 2. The translation between utility units and its consumption equivalent units is straightforward for homothetic preferences. We continue to use the Markov setup in Section 2.2. Let $\boldsymbol{C} = \{C(Z_t)\}_{t=1}^T$ be a consumption process. Denote $V(\boldsymbol{C}) = \{V_t(Z_t)\}$ be the associated utility process constructed from the recursion (17). Denote $\lambda\boldsymbol{C}$ be the consumption plan obtained by multiplying $C(Z_t)$ by λ in all periods, that is, $\lambda\boldsymbol{C} = \{\lambda C(Z_t)\}_{t=1}^T$. Clearly, if $V(\boldsymbol{C})$ is homogenous of degree $1 - \frac{1}{\psi}$, then $V(\lambda\boldsymbol{C}) = \lambda V(\boldsymbol{C})$. This allows us to translate utility into consumption units for general recursive utility with non-unit IES.

In the case of unit IES, utility function is homothetic but not homogenous. The following lemma provides a relationship between utility and its comsumption equivalent. The recursive utility with unit IES can be constructed from (17) with $u(C) = (1-\beta)\ln C$, and $h(V) = -e^{-(\gamma-1)V}$.

Lemma 3. *In the case of unit IES, under the above specification, $V(\lambda\boldsymbol{C}) = \ln\lambda + V(\boldsymbol{C})$. That is, increasing permanent consumption by λ times is equivalent to adding $\ln\lambda$ to life-time utility.*

Proof. The above relationship clearly holds for the last period T, where $V_T(\lambda\boldsymbol{C}) = \ln C_T$. Given the recursive structure, it is enough to show the following: suppose $V_{t+1}(\lambda\boldsymbol{C}) = \ln\lambda + V_{t+1}(\boldsymbol{C})$, then $V_t(\lambda\boldsymbol{C}) = \ln\lambda + V(\boldsymbol{C})$. We have

$$\begin{aligned}
V_t(\lambda\boldsymbol{C}) &= (1-\beta)\ln(\lambda C_t) - \beta(\gamma-1)\ln\left\{E\left[e^{-\frac{1}{\gamma-1}(V_{t+1}(\lambda\boldsymbol{C}))}\right]\right\} \\
&= (1-\beta)\ln\lambda + (1-\beta)\ln C_t - \beta(\gamma-1)\ln\left\{E\left[e^{-\frac{1}{\gamma-1}(\ln\lambda + V_{t+1}(\boldsymbol{C}))}\right]\right\} \\
&= (1-\beta)\ln\lambda + (1-\beta)\ln C_t + \beta\ln\lambda - \beta(\gamma-1)\ln\left\{E\left[e^{-\frac{1}{\gamma-1}V_{t+1}(\boldsymbol{C})}\right]\right\} \\
&= \ln\lambda + V_t(\boldsymbol{C}),
\end{aligned}$$

as needed. □