Ambiguity Meets Merton: A Segmented Approach to Risk Premiums in Portfolio Theory *

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Abstract

This study extends Merton's continuous-time portfolio optimization framework by incorporating segmented, discontinuous risk premiums that adapt to varying levels of market uncertainty and ambiguity. Traditional models assume continuous risk premiums based on known return distributions, but real-world markets often present ambiguity, where probability distributions of returns are unknown. The model introduces thresholds z_1 and z_2, segmenting uncertainty into low, moderate, and high ranges, each governed by distinct risk premiums. These segments reflect shifts in investor risk tolerance and premium demands driven by ambiguity aversion. Using the Hamilton-Jacobi-Bellman (HJB) equation, optimal portfolio strategies are derived, showing how ambiguity-averse investors adjust allocations across segmented uncertainty levels. Additionally, the sensitivity of expected utility to deviations in risk premiums is analyzed, offering insights into the impact of small shifts in market conditions or investor expectations on portfolio decisions. By integrating ambiguity into risk premium modeling, this study uncovers significant deviations from Merton's original framework, highlighting how ambiguity reshapes dynamic risk responses. Bridging classical optimization with behavioral finance, this work provides practical insights for portfolio optimization and financial policy design, advancing both theoretical and applied perspectives on managing risk and ambiguity in financial markets.

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*All remaining errors are my own.

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1 Introduction

Understanding investor behavior under uncertainty is a fundamental challenge in finance and economics. The classical framework developed by Robert C. Merton (1971) provides a robust method for determining optimal portfolio and consumption strategies over time. However, Merton's model assumes that investors operate under a known probability distribution of asset returns, simplifying or overlooking the complexities introduced by ambiguous situations, where the probability distribution itself is uncertain or unknown. This study extends Merton's approach by introducing a novel framework that integrates segmented, dynamic risk premiums, adjusting to various levels of market uncertainty. This extension captures the decision-making processes of ambiguity-averse investors, resulting in a model that more accurately reflects real-world investor behavior under ambiguity.

This study introduces a novel theoretical framework to examine the segmented nature of risk premiums, incorporating subjective uncertainty thresholds (z_1 and z_2) that delineate transitions between distinct low, moderate, and high uncertainty regimes. The outcomes of this framework demonstrate how changes in these thresholds influence expected utility, reflecting investors' non-linear and context-sensitive responses to uncertainty. The formalization of these dynamics provides insights into decision-making under ambiguity and offers a structured approach to analyzing how segmented premiums evolve with varying uncertainty levels.

By extending Merton's continuous-time portfolio model to explicitly include ambiguity aversion, this framework demonstrates how dynamic portfolio strategies are shaped by the interplay between risk premiums and ambiguity. The outcomes reveal that increases in the lower uncertainty threshold and decreases in the upper uncertainty threshold reduce expected utility, highlighting the critical role of segmentation in adjusting investment behavior. This segmentation, driven by discontinuous and non-linear shifts in investor compensation demands, provides a new perspective that recalibrates traditional models assuming uniform responses to risk and uncertainty.

While previous literature has explored ambiguity aversion in portfolio optimization, including Biagini and Pınar's (2017) robust Merton problem, most studies have focused on modeling uncertainty in drift and volatility parameters. Classical portfolio optimization models, including Merton's (1971) original framework, are based on the assumption of continuous and uniform risk premiums. These models have provided valuable insights but fail to account for the behavioral responses to uncertainty observed in real-world markets. Additionally, ambiguity aversion–a crucial factor in decision-making under uncertainty–has largely been explored in static or discrete-time models, leaving a gap in its application to dynamic portfolio optimization. Behavioral finance, including insights from Kahneman and Tversky's Prospect Theory (1979), emphasizes the role of subjective perceptions in shaping investor behavior, yet traditional models often lack formal mechanisms to incorporate these insights.

This study bridges these gaps by offering a calibration mechanism that reconciles theoretical and empirical findings in risk and ambiguity. It provides testable predictions about the behavior

of risk premiums and expected utility under different market conditions, linking classical finance, behavioral finance, and robust optimization. This framework sets the stage for empirical validation and further alignment with related studies on ambiguity aversion, robust portfolio optimization, and behavioral responses to volatility.

Building on these foundations,the study derives optimal portfolio strategies using the Hamilton-Jacobi-Bellman (HJB) equation, revealing how ambiguity-averse investors allocate assets differently across uncertainty levels. The study also analyzes the sensitivity of expected utility to deviations in risk premiums, providing insights into how small changes in market conditions or investor expectations influence portfolio decisions. In addition, this study includes a simulation component to illustrate the impact of segmented risk premiums and ambiguity adjustments on portfolio allocations, comparing various market conditions and thresholds. The simulations provide empirical support for the theoretical model and offer further insights into portfolio decision-making under dynamic conditions of uncertainty and ambiguity.

Motivated by the need to model context-driven changes in market uncertainty, this study draws on behavioral finance insights, such as Kahneman and Tversky's Prospect Theory (1979), which emphasizes the context-sensitive nature of risk attitudes. Empirical evidence shows that risk premiums often rise during periods of heightened volatility, reflecting increased investor aversion to uncertainty. By incorporating segmented risk premiums into decision-making, this study bridges theoretical advancements and practical applications, offering a framework that predicts dynamic portfolio adjustments under varying conditions of uncertainty and ambiguity.

Extending Merton's continuous-time framework, this study provides a deeper understanding of the decision-making processes of ambiguity-averse investors. It captures the segmented nature of financial markets, where uncertainty is far from uniform. This refined perspective is valuable for portfolio management, risk assessment, and economic policy, providing a tool for analyzing and predicting investment behavior in complex and uncertain environments.

The remainder of the paper is structured as follows: Section 2 reviews the existing literature on Merton's portfolio problem and ambiguity aversion. Section 3 introduces the segmented risk premiums model and outlines the modified wealth dynamics. Section 4 presents the simulation results for optimal portfolio allocation, demonstrating the practical impact of segmented risk premiums and ambiguity adjustments on portfolio allocations. Section 5 discusses the implications of the results and provides a utility analysis for different uncertainty levels. Finally, Section 6 concludes with a summary of findings and suggestions for future research.

2 Key Stylized Facts

Understanding the behavior of risk premiums under uncertainty is a fundamental challenge in finance and economics. Modern Portfolio Theory (MPT), introduced by Markowitz (1952), provided the first formal analysis of the trade-off between risk and return, emphasizing diversification as a

means to achieve optimal portfolio allocation. This foundational work defined the risk premium as the additional compensation investors require for assuming greater risk, laying the groundwork for subsequent advancements in financial theory.

The Capital Asset Pricing Model (CAPM), independently developed by Treynor et al. (1964), formalized the relationship between expected returns and market risk. CAPM posits that an asset's expected return is determined by the risk-free rate plus a premium proportional to its sensitivity to market risk, measured by beta. While CAPM has become a cornerstone in financial economics, it assumes homogeneous investor responses to risk and does not account for individual preferences or subjective perceptions of uncertainty. The Arbitrage Pricing Theory (APT), introduced by Ross (1976), expanded the scope of risk premiums by incorporating multiple factors. APT suggested that macroeconomic variables influence asset returns, highlighting the multifaceted nature of risk and refining the single-factor approach of CAPM.

Merton's continuous-time portfolio optimization model (1971) advanced foundational financial theories by introducing a dynamic framework for maximizing the expected utility of wealth over time. This seminal work formalized techniques for allocating investments between risky and risk-free assets, incorporating continuous-time stochastic processes. While Merton's model provided a rigorous basis for understanding portfolio decisions under risk, it relied on the assumption of known probability distributions and continuous risk premiums.

However, recent advancements have extended Merton's framework to address ambiguity and investor heterogeneity, reflecting more realistic market conditions. Epstein and Schneider (2003) introduced ambiguity into Bayesian learning models, enabling investors to dynamically update their beliefs about uncertain parameters, which highlights how evolving uncertainty influences portfolio choices over time. Similarly, Garlappi, Uppal, and Wang (2007) extended Merton's model by incorporating model uncertainty, demonstrating that ambiguity leads to more cautious portfolio diversification as investors hedge against the risk of misspecified models. These extensions emphasize the critical role of ambiguity aversion in shaping dynamic portfolio strategies, paving the way for more robust frameworks that account for uncertainty beyond classical assumptions. This limitation has inspired further research into integrating ambiguity aversion and addressing non-uniform investor responses to uncertainty, bridging the gap between theoretical assumptions and observed market behaviors.

Building on these contributions, further studies have underscored the dynamic and context-sensitive nature of investor responses to ambiguity. For instance, Bäuerle and Mahayni (2024) introduced adaptive learning mechanisms that allow investors to refine their beliefs under ambiguous financial conditions, emphasizing the emergence of conservative strategies during periods of heightened uncertainty. Additionally, Guan et al. (2024) proposed robust portfolio strategies using state-dependent confidence sets, which dynamically adjust investor behavior in response to changing levels of ambiguity. These studies collectively reinforce the evolving understanding that ambiguity aversion influences not only diversification but also the segmented evolution of risk premiums, necessitating models that account for non-linear responses across varying uncertainty

regimes.

Ambiguity aversion has been widely explored in decision-making under uncertainty, distinguishing itself from risk by addressing situations where probabilities are unknown. Gilboa and Schmeidler (1989) formalized this concept through the MaxMin Expected Utility (MEU) framework, where ambiguity-averse agents make decisions based on the worst-case scenario. Extending this idea, Hansen and Sargent (2001) developed a robust control approach, illustrating how ambiguity aversion leads to decisions that hedge against adverse outcomes. Similarly, Klibanoff, Marinacci, and Mukerji (2005) proposed the smooth ambiguity model, allowing for differentiation between risk aversion and ambiguity aversion, providing a more nuanced understanding of investor behavior under uncertainty.

Ambiguity aversion has also been incorporated into continuous-time models. Maenhout (2004) integrated ambiguity into Merton's portfolio optimization model, showing that ambiguity aversion reduces risky asset holdings even without explicitly defined probabilities. Ju and Miao (2012) explored ambiguity in drift parameters, demonstrating its significant impact on asset pricing and portfolio selection. Additionally, Hansen, Sargent, and Tallarini (1999) introduced recursive utility functions to model ambiguity in stochastic environments, emphasizing the behavioral implications of investor uncertainty about model specifications. Collectively, these studies demonstrate that ambiguity aversion plays a critical role in shaping portfolio strategies and risk premiums. This study builds on these contributions by introducing segmented uncertainty thresholds, capturing the dynamic and context-sensitive nature of investor responses to ambiguity.

Beyond ambiguity aversion, non-market factors and behavioral anomalies have further enriched the understanding of risk premiums in financial markets. Studies exploring the impact of long-term environmental risks, such as Bernstein, Gustafson, and Lewis (2019), emphasize the importance of incorporating external, systemic factors into financial models of risk premiums. Similarly, research on the low-risk anomaly, such as that by Baker, Bradley, and Wurgler (2011) and Frazzini and Pedersen (2014), highlights counterintuitive patterns in risk and return, where lower-risk assets often outperform expectations. These anomalies, driven by psychological biases and market dynamics, challenge traditional risk-return trade-offs and provide further evidence that investor behavior under uncertainty is both non-linear and context-sensitive.

Behavioral finance has further challenged the rational assumptions underlying classical models. Kahneman and Tversky's Prospect Theory (1979) demonstrated that individuals evaluate gains and losses asymmetrically, leading to non-linear risk responses that deviate from classical expected utility theory. This framework explains the coexistence of risk-averse and risk-seeking behaviors, significantly shaping the formation of risk premiums. In addition, Othman (2024) examined emotional and cognitive biases, such as fear and greed, highlighting their role in shaping investor behavior and market outcomes.

Recent research has focused on bridging ambiguity and non-linear investor responses to uncertainty. Hansen and Marinacci (2016) introduced the ambiguity premium, reflecting the additional compensation required by investors under ambiguous conditions. Illeditsch (2011) demonstrated

that ambiguity in return distributions leads to portfolio inertia, where investors hesitate to reallocate assets due to uncertainty. Similarly, Bekaert, Engstrom, and Xu (2019) provided empirical evidence linking ambiguity aversion to heightened volatility and increased risk premiums during crises.

Higher-order moments of return distributions have also emerged as key determinants of risk premiums. Harvey and Siddique (2000) explored skewness in return distributions, while Bakshi, Kapadia, and Madan (2003) and Barro (2006) emphasized the role of kurtosis in shaping asset prices. These studies illustrate the limitations of variance-based risk measures and highlight the dynamic nature of risk premiums as constructs influenced by psychological, macroeconomic, and environmental factors.

While these contributions have significantly advanced the field, critical gaps remain in capturing the complexity of dynamic investment decisions under uncertainty. Classical models, including CAPM and Merton's framework, often rely on assumptions of homogeneous investor behavior and known probability distributions, which fail to account for the complexities introduced by ambiguity and behavioral biases. Existing robust optimization frameworks, such as those proposed by Biagini and Pınar (2017), address some of these limitations but primarily focus on drift and volatility without fully capturing the interplay between risk premiums and dynamic investor responses.

The next chapter introduces a novel theoretical framework to address these gaps by incorporating segmented risk premiums and explicitly modeling the dynamic impact of ambiguity aversion. This framework provides a structured approach to analyze non-linear and context-sensitive investor responses, bridging theoretical advancements with practical relevance. By advancing the integration of ambiguity aversion into continuous-time portfolio theory, this study offers a comprehensive tool for understanding and predicting investor behavior in uncertain financial environments.

Building on these advancements, Biagini and Pınar (2017) enriched the classical Merton problem by incorporating ambiguity aversion into portfolio optimization. Their framework modeled uncertainty in drift and volatility parameters, providing a structured approach to addressing ambiguity in financial markets. By capturing how investors adapt strategies when facing uncertain probability distributions, their work established a foundation for integrating behavioral insights into classical theories.

Further contributions have extended Merton's continuous-time framework to explicitly address ambiguity aversion. For instance, Bäuerle and Mahayni (2024) explored investment strategies under ambiguous financial conditions, incorporating learning mechanisms to refine investor beliefs about uncertain parameters. Guan et al. (2024) focused on robust portfolio selection with state-dependent confidence sets, adapting to changing conditions by emphasizing ambiguity aversion. These studies collectively underscore the evolving nature of portfolio theory and the necessity of integrating ambiguity considerations into dynamic investment decisions.

Studies exploring the impact of long-term environmental risks, such as Bernstein, Gustafson,

and Lewis (2019), have demonstrated the need to incorporate non-market factors into financial models of risk premiums. Similarly, research on the low-risk anomaly, such as that by Baker, Bradley, and Wurgler (2011) and Frazzini and Pedersen (2014), has highlighted counterintuitive patterns in risk and return, driven by psychological biases and market dynamics.

3 Theoretical Insights into Risk Premium Segmentation & Ambiguity

In classical portfolio theory, the risk premium is typically defined in terms of the additional return required by investors to compensate for bearing risk. This concept is central to understanding how investors allocate wealth between risky and risk-free assets. However, traditional models often assume that risks are known and quantifiable, with no consideration for ambiguity or uncertainty in the probability distributions of returns.

This study extends the conventional risk-return framework by incorporating ambiguity–uncertainty about the probability distributions of returns. Ambiguity aversion, the tendency for agents to demand higher compensation for uncertain outcomes, is modeled by adjusting the risk premium. This adjustment accounts for the additional uncertainty faced by investors in ambiguous environments, providing a more realistic foundation for portfolio decision-making under uncertainty.

The next section introduces the concept of ambiguity and explores its influence on the risk premium, drawing on established theories of ambiguity aversion. By incorporating these insights, a dynamic, segmented model of the risk premium is proposed, which adapts to varying levels of uncertainty and ambiguity.

3.1 Ambiguity and Risk Premium

Merton's (1971) portfolio optimization model maximizes the expected utility of wealth over time by allocating assets between risky and risk-free investments. While Merton's model inherently accounts for risk through the stochastic properties of returns and their variance, it does not explicitly define or incorporate a *risk premium* (RP) as a standalone concept. Instead, the risk-return trade-off is embedded in the model's assumptions about asset returns and the investor's risk aversion.

This study extends Merton's framework by introducing ambiguity, defined as uncertainty in the probability distribution of returns, distinct from known risks. Ambiguity aversion, as described in Ellsberg (1961) and Gilboa & Schmeidler (1989), influences decision-making by leading agents to demand higher compensation for bearing ambiguity.

Definition 1. Let $\mathcal{P}$ denote the set of plausible subjective probability distributions, where $p \in \mathcal{P}$ represents a specific distribution over outcomes. Ambiguity-averse agents evaluate uncertain choices based on the worst-case expected utility:

$$\min_{p \in \mathcal{P}} \mathbb{E}_p[u(\text{choice})],$$

reflecting a preference for known probabilities over ambiguity.

The *risk premium* is a fundamental concept in economic decision-making, representing the additional compensation an agent requires to engage in a risky investment rather than choosing a risk-free alternative. It captures the economic cost of uncertainty, shaped by the agent's risk aversion and perception of potential outcomes.

Definition 2. The *risk premium RP* is the maximum amount an agent is willing to forgo to avoid the uncertainty associated with a risky return.

For ambiguity-averse agents, the risk premium is dynamically adjusted to reflect the worst-case expected utility. This adjustment captures both risk and ambiguity aversion by accounting for the least favorable distribution in $\mathcal{P}$.

Ambiguity is simulated as noise, introducing variability to reflect uncertain probability distributions. This simulation leads to segmented and discontinuous risk premiums that vary with ambiguity levels, capturing the behavior of ambiguity-averse agents in uncertain environments.

To model this, the relationship between uncertainty and the risk premium is segmented into three distinct ranges, each corresponding to different levels of uncertainty. This segmentation framework delineates how risk premiums are adjusted dynamically as ambiguity levels change.

Lemma 1. *There exist z_1 and z_2 within the interval $[0, 1]$ such that:*

1. *For all $z < z_1$, a risk-averse participant is willing to pay RP_O as the risk premium.*
2. *For $z_1 \leq z < z_2$, a risk-averse participant is willing to pay $RP_L(z)$ as the risk premium.*
3. *For $z \geq z_2$, a risk-averse participant is willing to pay $RP_H(z)$ as the risk premium.*

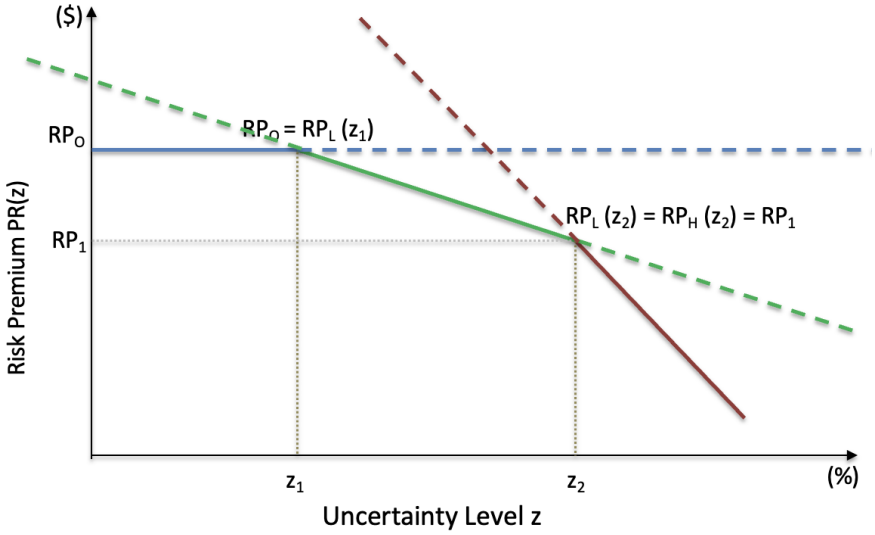


Figure 1: Risk Premium Across Uncertainty Levels

Here, RP_O represents the baseline risk premium, $RP_L(z)$ denotes the lower bound risk premium function, and $RP_H(z)$ represents the higher bound risk premium function, with respective thresholds at z_1 and z_2.

Given the relationship between risk premiums at different levels of uncertainty z,

$$RP(0) \geq RP_O = RP_L(z_1) \geq RP_L(z_2) = RP_H(z_2) \geq RP(1) = 0$$

Proof. Consider the agent's utility function, represented by a Cobb-Douglas form:

$$U(x, z) = x^{\alpha} z^{1-\alpha},$$

where x denotes the wealth after paying the risk premium, and z denotes the level of uncertainty, with $0 \leq z \leq 1$. This lemma establishes thresholds z_1 and z_2 such that the optimal risk premiums RP_O, $RP_L(z)$, and $RP_H(z)$ maximize utility at low, intermediate, and high levels of uncertainty, respectively.

Assume that for $z_i < z_1$, the risk premium $RP(z_i) > RP_O$. Under the utility function $U(x, z) = x^{\alpha} z^{1-\alpha}$, the wealth x after paying the premium $RP(z_i)$ is $W - RP(z_i)$, and the utility becomes:

$$U(W - RP(z_i), z_i) = (W - RP(z_i))^{\alpha} z_i^{1-\alpha}.$$

If $RP(z_i) > RP_O$, then:

$$U(W - RP_O, z_i) = (W - RP_O)^{\alpha} z_i^{1-\alpha}.$$

Since $RP(z_i) > RP_O$, it follows that $W - RP(z_i) < W - RP_O$, leading to:

$$U(W - RP(z_i), z_i) < U(W - RP_O, z_i).$$

This contradicts utility maximization, as paying RP_O instead of $RP(z_i)$ yields higher utility. Thus, RP_O is optimal for $z < z_1$, establishing RP_O as the baseline risk premium.

In this intermediate range, the risk premium is represented by a function $RP_L(z)$, where $RP_L(z) \leq RP_O$. Assume that there exists a point z_m within this range where $RP(z_m) \neq RP_L(z_m)$. For a participant at uncertainty level z_m paying $RP(z_m)$ with $RP(z_m) > RP_L(z_m)$, the wealth after paying the premium is $W - RP(z_m)$, yielding utility:

$$U(W - RP(z_m), z_m) = (W - RP(z_m))^{\alpha} z_m^{1-\alpha}.$$

However, paying the lower risk premium $RP_L(z_m)$ provides utility:

$$U(W - RP_L(z_m), z_m) = (W - RP_L(z_m))^{\alpha} z_m^{1-\alpha}.$$

Since $RP(z_m) > RP_L(z_m)$, it follows that $W - RP(z_m) < W - RP_L(z_m)$, leading to:

$$U(W - RP(z_m), z_m) < U(W - RP_L(z_m), z_m).$$

This contradiction confirms that paying $RP_L(z)$ maximizes utility in this intermediate range, thereby validating $RP_L(z)$ as the optimal risk premium for $z_1 \leq z < z_2$.

For high uncertainty levels $z \geq z_2$, the risk premium required increases to $RP_H(z)$, which adjusts dynamically with z to account for greater uncertainty. Suppose there exists a point $z_j > z_2$ where the risk premium $RP(z_j) < RP_H(z_j)$. The wealth after paying this lower premium $RP(z_j)$ is $W - RP(z_j)$, yielding utility:

$$U(W - RP(z_j), z_j) = (W - RP(z_j))^{\alpha} z_j^{1-\alpha}.$$

By paying $RP_H(z_j)$, the utility becomes:

$$U(W - RP_H(z_j), z_j) = (W - RP_H(z_j))^{\alpha} z_j^{1-\alpha}.$$

Since $RP(z_j) < RP_H(z_j)$, it follows that $W - RP(z_j) > W - RP_H(z_j)$, which leads to:

$$U(W - RP(z_j), z_j) > U(W - RP_H(z_j), z_j).$$

This contradicts the principle of utility maximization under higher uncertainty, confirming that $RP_H(z)$ is indeed optimal for $z \geq z_2$.

By establishing these contradictions for each range, RP_O, $RP_L(z)$, and $RP_H(z)$ are confirmed as the optimal premiums for the corresponding levels of uncertainty. The segmentation across z_1 and z_2 maximizes utility in each range, verifying the lemma. □

This segmentation framework departs from traditional portfolio models by capturing investor responses to varying levels of uncertainty. While classical models often assume a linear relationship between uncertainty and risk premiums, this segmentation model, defined by the thresholds z_1 and z_2, offers a more realistic approach to risk tolerance and premium adjustments across different uncertainty levels.

Specifically, the segmentation framework captures behavioral shifts that classical models fail to address. At low uncertainty levels ($z < z_1$), investors tend to exhibit confidence and demand baseline premiums. As uncertainty rises ($z_1 \leq z < z_2$), risk premiums become dynamic, reflecting a heightened sensitivity to changing market conditions. At high uncertainty levels ($z \geq z_2$), investors require significantly higher compensation due to the pronounced unpredictability of outcomes. This tiered structure aligns with observed market behavior, providing a more accurate and comprehensive modeling approach.

To capture the evolving nature of risk and ambiguity over time, this paper defines a dynamic risk premium that adjusts not only with changes in uncertainty but also with the passage of time. The dynamic risk premium $RP(t, z)$ reflects the compensation an investor demands to bear both risk and ambiguity at a specific time t and under a given uncertainty level z.

By introducing these segmented risk premiums, the framework not only enhances our theoretical understanding of risk and ambiguity but also provides a practical foundation for portfolio optimization. Moreover, it facilitates the differentiation of investor behavior based on varying uncertainty levels, effectively bridging classical financial models with insights from behavioral

finance.

This study focuses on risk-averse investors and examines how they adjust risk premiums across different uncertainty segments. The introduction of expected utility, E[U(W)] , quantifies how risk-averse behavior influences overall wealth utility, enabling the development of strategies designed to minimize losses while navigating uncertain market conditions.

Definition 3. Given the segmented nature of the risk premium based on uncertainty, the expected utility $E[U(W)]$ can be expressed as follows:

$$E[U(W)] = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} U(W - RP(z)) f(z)\, dz,$$

where W is the total wealth, $RP(z)$ represents the risk premium at uncertainty level z, and $f(z)$ is the probability density function of uncertainty z, describing the likelihood of different levels of uncertainty occurring. The breakpoints $z_0, z_1, z_2, \ldots, z_n$ represent the segments where the risk premium function changes, with $z_0 = 0$ and $z_n = 1$ marking the full range of uncertainty from minimum to maximum. The segmentation reflects transitions between the risk premium levels RP_O, $RP_L(z)$, and $RP_H(z)$ defined in Lemma 1.

Proposition 1. *When the lower threshold z_1 increases, the expected utility decreases because the integration range shrinks, removing contributions from the higher premium region where $RP(z_1) = RP_O$, which provides greater compensation for uncertainty. Conversely, when the upper threshold z_2 decreases, the expected utility also decreases because the integration range shrinks, removing contributions from the moderate premium region where $RP(z_2) = RP_H(z_2) > RP_H(1) = 0$, thereby reducing the overall compensation for risk. As $z_1 \to z_2$, the integration collapses completely at the lower end, and as $z_2 \to z_1$, it collapses completely at the upper end, both leading to diminished expected utility.*

Proof. The expected utility, as defined in Definition 3, depends on the integration range determined by the thresholds z_1 and z_2, which are specified in Lemma 1. To evaluate the impact of changes in z_1 and z_2, the *Leibniz integral rule* is applied to differentiate the expected utility.

First, the derivative with respect to z_1 becomes:

$$\frac{\partial E[U(W)]}{\partial z_1} = -U(W - RP(z_1)) f(z_1) + \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \frac{\partial U(W - RP(z))}{\partial z_1} f(z)\, dz.$$

The first term, $-U(W - RP(z_1)) f(z_1)$, arises from the removal of contributions at the lower limit z_1, reducing expected utility as the integration range shrinks. The second term reflects changes in $RP(z)$ within the range, where the chain rule gives:

$$\frac{\partial U(W - RP(z))}{\partial z_1} = \frac{\partial U}{\partial RP} \cdot \frac{\partial RP(z)}{\partial z_1}.$$

If $RP(z)$ is constant or linear within the segment, $\frac{\partial RP(z)}{\partial z_1} = 0$, and this term vanishes. As $z_1 \to z_2$,

the integration collapses completely at the lower end, and the derivative simplifies to:

$$\lim_{z_1 \to z_2} \frac{\partial E[U(W)]}{\partial z_1} = -U(W - RP(z_2))f(z_2),$$

which is negative, indicating a decrease in expected utility.

For z_2, the derivative is:

$$\frac{\partial E[U(W)]}{\partial z_2} = U(W - RP(z_2))f(z_2) + \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \frac{\partial U(W - RP(z))}{\partial z_2} f(z)\, dz.$$

The first term, $U(W - RP(z_2))f(z_2)$, reflects the removal of contributions at the upper limit z_2, reducing expected utility as the range shrinks. The second term accounts for changes in $RP(z)$ within the range, where:

$$\frac{\partial U(W - RP(z))}{\partial z_2} = \frac{\partial U}{\partial RP} \cdot \frac{\partial RP(z)}{\partial z_2}.$$

For $RP(z)$ decreasing linearly with z_2, $\frac{\partial RP(z)}{\partial z_2} < 0$, and since $\frac{\partial U}{\partial RP} < 0$, this term is positive but insufficient to offset the first term. As $z_2 \to z_1$, the integration collapses at the upper end, leaving:

$$\lim_{z_2 \to z_1} \frac{\partial E[U(W)]}{\partial z_2} = U(W - RP(z_1))f(z_1),$$

which is positive, but the collapse of the range reduces expected utility.

Thus, increasing z_1 decreases the expected utility by shrinking the range and removing contributions from the lower region, while decreasing z_2 also reduces the expected utility by removing contributions from the upper region. As $z_1 \to z_2$, the integration collapses at the lower end, and as $z_2 \to z_1$, it collapses completely at the upper end, leading to a complete loss of utility in both cases. This completes the proof. □

This proposition emphasizes the critical role of uncertainty thresholds in shaping expected utility. The thresholds z_1 and z_2 are inherently subjective, reflecting individual differences in risk preferences, ambiguity aversion, and perceptions of uncertainty. Changes in these thresholds directly affect the integration range, reducing expected utility by removing contributions from regions that compensate for risk and uncertainty. These subjective thresholds influence utility maximization strategies and the allocation of assets, as individuals adjust their decisions to balance risk and reward based on their personal tolerance for uncertainty. The analysis highlights how shifts in these thresholds impact overall utility, offering insights into decision-making under uncertainty and its broader implications for financial strategies and economic behavior.

The analysis of subjective uncertainty thresholds highlights the critical role of segmented risk premiums in shaping expected utility. While this framework provides valuable insights into decision-making under risk, it does not fully account for the complexities introduced by ambiguity–uncertainty regarding the probability distributions of outcomes. Addressing ambiguity is essential to reflect the nuanced behavior of decision-makers who weigh both risk and ambiguity when allocating

assets. To build on these foundations, the next section revisits Merton's continuous-time portfolio model, incorporating ambiguity-adjusted risk premiums to develop a dynamic framework for optimizing decisions under uncertainty.

3.2 Dynamic Portfolio Strategies with Ambiguity-Adjusted Risk Premiums

In Merton's continuous-time portfolio model, the goal is to maximize the expected utility of wealth over time by optimally allocating assets between risky and risk-free investments. This study extends Merton's framework by incorporating dynamic adjustments to the risk premium, considering both market uncertainty and ambiguity aversion. The model introduces a segmented structure for the risk premium $RP(t, z)$, which varies across different levels of uncertainty, along with an ambiguity adjustment term $\beta\,\text{Ambiguity}(z)$ to account for investor behavior under ambiguous conditions.

The Constant Relative Risk Aversion (CRRA) utility function is specified by the parameter γ as follows:

$$U(W) = \begin{cases} \frac{W^{1-\gamma}}{1-\gamma}, & \text{if } \gamma \neq 1, \\ \ln(W), & \text{if } \gamma = 1. \end{cases} \tag{1}$$

This bifurcation accommodates a wide range of risk aversion preferences within the CRRA framework, where $\gamma > 0$ represents the investor's risk aversion coefficient. The CRRA utility function is particularly suitable for continuous-time optimization models due to its theoretical and empirical consistency, as well as its tractability for deriving closed-form solutions.

While Merton's model (1971) provides a foundation for portfolio optimization based on the maximization of expected utility of wealth, it assumes a linear relationship between risk and return. However, real-world investors often face not only risk but also ambiguity–the uncertainty regarding the probability distributions of future returns. To address this, Merton's model is extended by introducing dynamic risk premiums and ambiguity adjustments, thereby capturing more realistic investor behavior in uncertain environments. The CRRA utility function, as defined, serves as the foundation for understanding investor preferences under risk, while the subsequent step is to incorporate ambiguity into the decision-making process.

The model examines an investor's dynamic portfolio decisions over the investment horizon $[0, T]$, where T is the terminal time at which the investor evaluates their wealth. At each intermediate time t, the investor's wealth $W(t)$ evolves dynamically, influenced by the segmented risk premium $RP(t, z)$, which changes across distinct levels of uncertainty, as defined in Lemma 1. Specifically, uncertainty is segmented into ranges $z < z_1$, $z_1 \leq z < z_2$, and $z \geq z_2$, each corresponding to a different form of the risk premium.

To account for ambiguity aversion, an ambiguity adjustment term, $\beta\,\text{Ambiguity}(z)$, is introduced, where $\beta \geq 0$ denotes the investor's sensitivity to ambiguity at a given uncertainty level z. When $\beta = 0$, the investor is ambiguity-neutral, ignoring ambiguity effects. For $\beta > 0$, higher values of β reflect greater sensitivity to ambiguity, resulting in reduced allocations to risky as-

sets. This approach aligns with established financial theories, including Gilboa and Schmeidler's MaxMin Expected Utility Theory (1989) and Klibanoff, Marinacci, and Mukerji's Smooth Ambiguity Model (2005), which provide theoretical frameworks for incorporating ambiguity aversion into decision-making. These theories are consistent with empirical observations of how investors adjust their portfolios under conditions of uncertainty and ambiguity.

The ambiguity adjustment term $\beta\,\text{Ambiguity}(z)$ brings a behavioral finance perspective to the classical portfolio optimization model. Traditional models assume that investors make decisions based solely on known probabilities and respond uniformly to risk. In contrast, this framework incorporates ambiguity, representing uncertainty about the probability distributions of returns. Ambiguity-averse investors demand additional compensation for engaging with uncertain outcomes, reflecting their aversion to the lack of precise information. By explicitly incorporating $\beta\,\text{Ambiguity}(z)$, this model captures the nuanced trade-offs that investors face when navigating uncertain and ambiguous environments. This adjustment aligns with insights from behavioral finance, where subjective perceptions of uncertainty play a critical role in decision-making. The inclusion of this term allows for a more realistic depiction of investor behavior, particularly in environments where ambiguity is a significant factor, extending classical portfolio theory to more complex market scenarios.

The volatility of the risky asset, denoted by σ, may also vary with z, while the risk-free rate r is assumed constant throughout the investment horizon. The investor's objective is to maximize the expected utility of terminal wealth, considering dynamic risk premiums and ambiguity adjustments over time. By incorporating segmented risk premiums and the dynamics of ambiguity aversion, this section offers deeper insights into portfolio optimization strategies that minimize potential losses while navigating various market conditions.

Definition 4. The expected utility of terminal wealth $E[U(W(T))]$ over the investment horizon $[0, T]$ is defined as:

$$E[U(W(T))] = \int_0^T \int_{\mathcal{Z}} U(W(t) - RP(t,z) - \beta\,\text{Ambiguity}(z)) f(z,t)\,dz\,dt, \tag{2}$$

where $W(t)$ is the investor's wealth at time $t \in [0, T]$, $RP(t, z)$ denotes the risk premium, $\beta\,\text{Ambiguity}(z)$ represents the ambiguity adjustment term, and $f(z, t)$ is the joint probability density of uncertainty z and time t. The domain of integration $\mathcal{Z}$ spans the full range of uncertainty levels. The condition $W(t) - RP(t, z) - \beta\,\text{Ambiguity}(z) > 0$ ensures that the CRRA utility function remains well-defined throughout the investment horizon.

In this definition, the domain of integration $\mathcal{Z}$ covers the full range of uncertainty levels, which may be divided into segments reflecting variations in $RP(t, z)$ and $\beta\,\text{Ambiguity}(z)$ across different ranges of z. The outer integration over $[0, T]$ accounts for the dynamic evolution of the investor's wealth and associated decision-making, while the inner integration over $\mathcal{Z}$ captures the segmentation of uncertainty at each point in time. Together, these integrals represent the investor's cumulative expected utility derived from terminal wealth, incorporating both time dynamics and

segmented risk and ambiguity adjustments.

This extension of Merton's framework incorporates segmented, time-varying risk premiums $RP(t,z)$ based on market uncertainty, as structured in Lemma 1. The dynamic risk premium model adjusts the optimal portfolio allocation $\pi^*(t)$ based on fluctuating uncertainty, allowing for a realistic extension of Merton's original model. This approach provides a foundation for further theoretical exploration of portfolio optimization under varying risk conditions.

Proposition 2. *Given the dynamic, segmented risk premium function $RP(t,z)$ introduced in the previous section, which adjusts with uncertainty across distinct levels of market ambiguity, the optimal portfolio allocation $\pi^*(t)$, which maximizes the ambiguity-adjusted expected utility of wealth, is given by:*

$$\pi^*(t) = \frac{RP(t,z) + \beta\, Ambiguity(z)}{\gamma\sigma^2},$$

where $\gamma > 0$ is the coefficient of relative risk aversion, σ is the volatility of the risky asset, and $Ambiguity(z)$ denotes the degree of ambiguity associated with the uncertainty level z. The condition $W(t) - RP(t,z) - \beta\, Ambiguity(z) > 0$ ensures the validity of the CRRA utility function.

Proof. The optimal portfolio allocation $\pi^*(t)$ is derived by maximizing the expected utility of wealth over the investment horizon $[0,T]$. The expected utility $E[U(W(T))]$ is defined in Equation (2). The dynamics of the investor's wealth $W(t)$ are modeled using a stochastic differential equation (SDE), which accounts for both deterministic changes in wealth and random fluctuations due to market uncertainty. The wealth dynamics are given by:

$$dW(t) = \left(rW(t) + \pi(t)\left(RP(t,z) + \beta\,\text{Ambiguity}(z)\right)\right)dt + \pi(t)\sigma dB(t),$$

where r is the constant risk-free rate, σ is the volatility of the risky asset, and $dB(t)$ is a standard Brownian motion increment, capturing random market shocks. The ambiguity adjustment term $\beta\,\text{Ambiguity}(z)$ modifies the risk premium, reflecting the increased compensation demanded by investors under ambiguous conditions.

The principle of dynamic programming leads to the Hamilton-Jacobi-Bellman (HJB) equation, which reflects the optimality condition for the value function $V(W,t)$. At each point in time t, the value function satisfies:

$$\frac{\partial V}{\partial t} + \max_{\pi} \mathcal{L}[V(W,t)] = 0,$$

where $\mathcal{L}[V(W,t)]$, the generator of the SDE, is defined as:

$$\mathcal{L}[V(W,t)] = \left(rW + \pi\left(RP(t,z) + \beta\,\text{Ambiguity}(z)\right)\right)\frac{\partial V}{\partial W} + \frac{1}{2}\pi^2\sigma^2\frac{\partial^2 V}{\partial W^2}.$$

Plugging this into the HJB equation yields:

$$\frac{\partial V}{\partial t} + \max_{\pi}\left\{\left(rW + \pi\left(RP(t,z) + \beta\,\text{Ambiguity}(z)\right)\right)\frac{\partial V}{\partial W} + \frac{1}{2}\pi^2\sigma^2\frac{\partial^2 V}{\partial W^2}\right\} = 0.$$

To find the optimal portfolio allocation $\pi^*(t)$, the expression inside the maximization is differentiated with respect to π:

$$\frac{\partial}{\partial \pi}\left\{\left(rW + \pi\left(RP(t,z) + \beta\,\text{Ambiguity}(z)\right)\right)\frac{\partial V}{\partial W} + \frac{1}{2}\pi^2\sigma^2\frac{\partial^2 V}{\partial W^2}\right\} = 0.$$

This yields the first-order condition:

$$\left(RP(t,z) + \beta\,\text{Ambiguity}(z)\right)\frac{\partial V}{\partial W} + \pi\sigma^2\frac{\partial^2 V}{\partial W^2} = 0.$$

Solving for π, the CRRA utility function is substituted, where the relative risk aversion coefficient γ is defined as:

$$\gamma = -\frac{WV_{WW}}{V_W}.$$

For the CRRA utility function $U(W) = \frac{W^{1-\gamma}}{1-\gamma}$, the first and second derivatives with respect to W are:

$$V_W = W^{-\gamma}, \quad V_{WW} = -\gamma W^{-\gamma-1}.$$

Substituting V_{WW} and V_W into the definition of γ confirms:

$$\frac{\partial^2 V}{\partial W^2} = -\frac{\gamma\frac{\partial V}{\partial W}}{W}.$$

Replacing V_{WW} in the first-order condition gives:

$$\pi^*(t) = \frac{\left(RP(t,z) + \beta\,\text{Ambiguity}(z)\right)\frac{\partial V}{\partial W}}{\sigma^2\left(-\frac{\gamma\frac{\partial V}{\partial W}}{W}\right)}.$$

Simplifying yields:

$$\pi^*(t) = \frac{RP(t,z) + \beta\,\text{Ambiguity}(z)}{\gamma\sigma^2}.$$

In the special case of logarithmic utility ($\gamma = 1$), the allocation simplifies to:

$$\pi^*(t) = \frac{RP(t,z) + \beta\,\text{Ambiguity}(z)}{\sigma^2}.$$

This completes the proof, establishing that the optimal allocation dynamically adjusts to both risk and ambiguity aversion based on the segmented structure of the risk premium and the ambiguity adjustment term. □

The segmented structure of the risk premium $RP(t,z)$, as defined in Lemma 1, plays a critical role in determining $\pi^*(t)$. For low levels of uncertainty, $RP(t,z)$ remains stable, leading to consistent portfolio allocations. However, as uncertainty increases, dynamic adjustments to $RP(t,z)$ and the ambiguity term $\beta\,\text{Ambiguity}(z)$ result in more conservative allocations, reflecting real-world

investor behavior under heightened risk and ambiguity.

This result demonstrates that the optimal portfolio allocation $\pi^*(t)$ increases with the risk premium $RP(t,z)$ and ambiguity adjustment term $\beta\,\text{Ambiguity}(z)$, while being inversely proportional to the risk aversion coefficient γ and the volatility σ^2. This reflects the intuition that more risk-averse investors or those facing greater market volatility allocate less to risky assets, particularly when ambiguity is high.

The derived proposition offers a dynamic extension of Merton's portfolio optimization framework by incorporating segmented risk premiums and ambiguity aversion into the decision-making process. The optimal portfolio allocation, $\pi^*(t)$, reflects investors' responses to varying levels of market uncertainty and ambiguity, with adjustments driven by the segmented structure of $RP(t,z)$ and the ambiguity adjustment term $\beta\,\text{Ambiguity}(z)$. The inclusion of ambiguity aversion captures the tendency of investors to demand additional compensation for ambiguous conditions, leading to lower allocations to risky assets as ambiguity increases. This extension bridges classical portfolio theory with behavioral finance insights, offering a nuanced understanding of investor behavior under uncertainty. By providing a closed-form expression for $\pi^*(t)$, this model offers practical guidance for optimizing portfolio strategies in uncertain and ambiguous market environments while laying the groundwork for further theoretical advancements in portfolio optimization.

By setting $\beta = 0$ and $\gamma = 1$ in Proposition 2, this paper derives the following results as corollaries.

Corollary 1 (Ambiguity-Neutral Case). If the investor is ambiguity-neutral, the optimal portfolio allocation reduces to:

$$\pi^*(t) = \frac{RP(t,z)}{\gamma\sigma^2}.$$

Corollary 2 (Logarithmic Utility Case). For logarithmic utility, the optimal portfolio allocation simplifies to:

$$\pi^*(t) = \frac{RP(t,z) + \beta\,\text{Ambiguity}(z)}{\sigma^2}.$$

These corollaries apply the general model under specific assumptions. The ambiguity-neutral case ($\beta = 0$) aligns with classical portfolio theory, while the logarithmic utility case ($\gamma = 1$) reflects a specific investor preference structure. Together, these corollaries highlight the model's flexibility and adaptability to different investor behaviors and market conditions.

Having established the dynamic and segmented nature of risk premiums, it is essential to understand how fluctuations in the risk premium $RP(t,z)$ impact overall utility. Specifically, small deviations in $RP(t,z)$ can have significant implications for portfolio decisions in uncertain and ambiguous market conditions. The next section quantifies this relationship by analyzing the sensitivity of expected utility to these deviations, offering deeper insights into how risk and ambiguity influence investor behavior and decision-making.

3.3 Utility Sensitivity to Risk Premium Deviations

This section examines the sensitivity of expected utility to small deviations in the segmented risk premium $RP(t,z)$, which is fundamental to understanding investor behavior under varying levels of market uncertainty. Modeling this sensitivity quantifies the direct impact of risk premium fluctuations on portfolio decisions and overall utility outcomes. Small deviations in $RP(t,z)$ may arise from shifts in uncertainty levels or changes in investor expectations, underscoring their importance in robust portfolio optimization strategies.

The following proposition quantifies the linear relationship between small deviations in $RP(t,z)$ and the expected utility of terminal wealth. This provides a structured framework for analyzing how portfolio decisions adapt to incremental changes in risk premiums, offering a deeper understanding of the interplay between risk, ambiguity, and utility outcomes.

Proposition 3. *Let the expected utility of terminal wealth $E[U(W(T))]$ be as defined in Definition 4. Assume $W(t) - RP(t,z) - \beta\, Ambiguity(z) > 0$, ensuring the CRRA utility function remains well-defined. For small deviations $\Delta RP(t,z)$ in the segmented risk premium $RP(t,z)$, the sensitivity of $E[U(W(T))]$ to these changes is given by:*

$$\frac{\partial E[U(W(T))]}{\partial RP(t,z)} \approx -\gamma \int_0^T \int_{\mathcal{Z}} W(t)^{-\gamma} f(z,t)\, dz\, dt.$$

This result demonstrates the linear impact of small changes in $RP(t,z)$ on expected utility, weighted by wealth and probability density.

Proof. The expected utility of terminal wealth is defined as:

$$E[U(W(T))] = \int_0^T \int_{\mathcal{Z}} \frac{(W(t) - RP(t,z) - \beta\, \text{Ambiguity}(z))^{1-\gamma}}{1-\gamma} f(z,t)\, dz\, dt.$$

Before applying the Taylor expansion, it is essential to establish the differentiability of the expected utility function $E[U(W(T))]$ with respect to $RP(t,z)$. The CRRA utility function $U(W(t))$ is differentiable with respect to $RP(t,z)$, as $RP(t,z)$ directly enters the argument $W(t) - RP(t,z) - \beta\, \text{Ambiguity}(z)$. Furthermore, the expected utility is computed as an integral of the utility function, weighted by the smooth probability density $f(z,t)$. Together, these properties ensure that $E[U(W(T))]$ is differentiable with respect to $RP(t,z)$.

The Taylor expansion is applied under the assumption of small deviations $\Delta RP(t,z)$ in the segmented risk premium. For sufficiently small deviations, higher-order terms in the expansion are negligible, making the first-order approximation a valid representation of the impact on expected utility. This approach not only captures the leading-order effects of risk premium changes but also provides a tractable framework for analyzing their implications.

Expanding $E[U(W(T))]$ around $\Delta RP(t,z) = 0$ using a first-order Taylor series yields:

$$E[U(W(T))] \approx E[U(W(T))]\big|_{\Delta RP(t,z)=0} + \frac{\partial E[U(W(T))]}{\partial RP(t,z)} \cdot \Delta RP(t,z).$$

The first term represents the unperturbed utility, while the second term quantifies the sensitivity of $E[U(W(T))]$ to changes in $RP(t,z)$. The condition $W(t) - RP(t,z) - \beta\,\text{Ambiguity}(z) > 0$ ensures that the CRRA utility function remains well-defined, real-valued, and interpretable throughout the analysis.

To compute the partial derivative, substitute the CRRA utility function:

$$U(W(t)) = \frac{(W(t) - RP(t,z) - \beta\,\text{Ambiguity}(z))^{1-\gamma}}{1-\gamma}.$$

Differentiating $U(W(t))$ with respect to $RP(t,z)$ yields:

$$\frac{\partial U(W(t))}{\partial RP(t,z)} = -\gamma\left(W(t) - RP(t,z) - \beta\,\text{Ambiguity}(z)\right)^{-\gamma}.$$

Under the assumption of small deviations in $RP(t,z)$, it is reasonable to approximate $W(t) - RP(t,z) - \beta\,\text{Ambiguity}(z) \approx W(t)$. Substituting this approximation, the derivative simplifies to:

$$\frac{\partial U(W(t))}{\partial RP(t,z)} \approx -\gamma W(t)^{-\gamma}.$$

The partial derivative of the expected utility $E[U(W(T))]$ with respect to $RP(t,z)$ is then computed as:

$$\frac{\partial E[U(W(T))]}{\partial RP(t,z)} = \int_0^T \int_{\mathcal{Z}} \frac{\partial U(W(t))}{\partial RP(t,z)} f(z,t)\,dz\,dt.$$

Substituting the simplified derivative:

$$\frac{\partial E[U(W(T))]}{\partial RP(t,z)} \approx -\gamma \int_0^T \int_{\mathcal{Z}} W(t)^{-\gamma} f(z,t)\,dz\,dt.$$

To validate and interpret the derived sensitivity formula within the context of the Taylor expansion, demonstrating how small deviations in $RP(t,z)$ linearly impact $E[U(W(T))]$ and align with the proposition, this sensitivity is applied as part of the Taylor expansion:

$$E[U(W(T))] \approx E[U(W(T))]\big|_{\Delta RP(t,z)=0} + \frac{\partial E[U(W(T))]}{\partial RP(t,z)} \cdot \Delta RP(t,z).$$

This result demonstrates the linear dependence of $E[U(W(T))]$ on small changes in $RP(t,z)$, weighted by the integral of wealth's negative power and the probability density $f(z,t)$. The negative sign reflects the diminishing marginal utility of wealth, emphasizing the sensitivity of utility to risk premium deviations. This completes the proof. □

Proposition 3 establishes a direct relationship between small deviations in the segmented risk premium $RP(t,z)$ and their impact on the expected utility $E[U(W(T))]$. This relationship is crucial because it quantifies how changes in the risk premium influence portfolio decisions and overall utility outcomes.

The result shows that the sensitivity of the expected utility is influenced by three main factors: the investor's wealth $W(t)$, their relative risk aversion coefficient γ, and the probability distribution $f(z,t)$. Specifically, the negative sign in the sensitivity expression reflects the diminishing marginal utility of wealth, meaning that increases in $RP(t,z)$ will reduce the expected utility more substantially for wealthier individuals or those who are more risk-averse.

By incorporating the segmented nature of the risk premium, this proposition underscores the importance of accurately modeling $RP(t,z)$ to account for varying levels of uncertainty and ambiguity. This approach improves our understanding of investor behavior, emphasizing how small changes in risk premium assessments can have significant impacts on portfolio optimization strategies.

The following corollary explores the localized impact of risk premium deviations, which will help better understand how portfolio decisions are affected by small changes in $RP(t,z)$. By isolating the first-order effects of these deviations at specific points in time and levels of uncertainty, this analysis will further enrich the broader framework for sensitivity analysis in portfolio optimization.

Corollary 3. Let $RP(t,z)$ denote the segmented risk premium function as defined in Lemma 1, with z_1 and z_2 partitioning the uncertainty domain into low, moderate, and high uncertainty levels. Assume the investor's utility is derived from the CRRA utility function:

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \quad \gamma \neq 1,$$

and the expected utility $E[U(W(T))]$ is defined in Equation (2). Let $\Delta RP(t,z)$ represent the deviation of the modeled risk premium from the true risk premium at time t and uncertainty level z.

The sensitivity of the expected utility $E[U(W(T))]$ to deviations in the segmented risk premium $RP(t,z)$ satisfies:

$$\frac{\partial E[U(W(T))]}{\partial RP(t,z)} = \frac{-\gamma}{\sigma^2} W(t)^{-\gamma} \cdot \Delta RP(t,z),$$

where the magnitude of the sensitivity increases with higher relative risk aversion (γ), larger deviations $\Delta RP(t,z)$, and lower levels of wealth $W(t)$.

Proof. The expected utility of terminal wealth, $E[U(W(T))]$, is defined in Equation (2). From the original theorem, the sensitivity of $E[U(W(T))]$ to deviations in the segmented risk premium $RP(t,z)$ is derived by applying a Taylor expansion around $\Delta RP(t,z) = 0$. This yields:

$$E[U(W(T))] \approx E[U(W(T))]\big|_{\Delta RP(t,z)=0} + \frac{\partial E[U(W(T))]}{\partial RP(t,z)} \cdot \Delta RP(t,z).$$

The first-order term quantifies the linear dependence of $E[U(W(T))]$ on small deviations in

$RP(t,z)$. The partial derivative, as computed in the original proof, is:

$$\frac{\partial E[U(W(T))]}{\partial RP(t,z)} = -\gamma \int_0^T \int_{\mathcal{Z}} W(t)^{-\gamma} f(z,t)\, dz\, dt.$$

To derive the localized sensitivity, the analysis is restricted to specific values of t and z. In this context, the integral reduces to evaluating $W(t)$ and $f(z,t)$ at fixed points, simplifying the partial derivative to:

$$\frac{\partial E[U(W(T))]}{\partial RP(t,z)} = -\gamma W(t)^{-\gamma}.$$

Considering deviations $\Delta RP(t,z)$, the Taylor expansion becomes:

$$E[U(W(T))] \approx E[U(W(T))]\big|_{\Delta RP(t,z)=0} - \gamma W(t)^{-\gamma} \cdot \Delta RP(t,z).$$

By incorporating the factor σ^2, which scales the sensitivity based on the risk premium model, the final form of the local sensitivity is:

$$\frac{\partial E[U(W(T))]}{\partial RP(t,z)} = \frac{-\gamma}{\sigma^2} W(t)^{-\gamma} \cdot \Delta RP(t,z).$$

This result directly follows from the original theorem by narrowing the scope to specific deviations $\Delta RP(t,z)$ and their immediate effects on utility, providing insight into how localized risk premium changes influence expected utility. This completes the proof. □

This result demonstrates that deviations in the segmented risk premium $RP(t,z)$ can have a significant impact on expected utility, particularly for risk-averse investors with low wealth levels. The inclusion of uncertainty and ambiguity further highlights the need to incorporate dynamic adjustments in portfolio optimization strategies.

Economically, this corollary shows that higher levels of risk aversion amplify the impact of inaccuracies in risk premium modeling on utility outcomes, particularly when wealth levels are low or when deviations in $RP(t,z)$ are substantial. The localized nature of this sensitivity underscores the importance of accurately modeling risk premiums at specific points in time and across varying levels of uncertainty, ensuring that portfolio strategies align with investor objectives in dynamic market conditions.

4 Simulation Results for Optimal Portfolio Allocation

This section presents a numerical simulation examining the impact of segmented risk premiums and ambiguity adjustments on optimal portfolio allocation (π^*). The simulation employs a toy model to analyze portfolio decisions across varying levels of uncertainty (z), segmented into low, moderate, and high-risk regions. By incorporating segmented risk premiums ($RP(z)$) and ambiguity adjustments ($\beta \cdot \text{Noise}(z)$), the model reflects investor behavior under both risk and ambiguity.

The results highlight the sensitivity of optimal allocations to uncertainty thresholds ($[z_1, z_2]$), demonstrating how narrowing or widening these ranges influences expected utility and portfolio decisions. Additionally, the dynamic interactions among risk aversion (γ), asset volatility (σ^2), and ambiguity sensitivity (β) reveal the nuanced effects of these factors on allocation strategies. The findings validate theoretical propositions by showing how segmented risk premiums and ambiguity adjustments shape decision-making under uncertainty.

By bridging theoretical predictions with simulated outcomes, this analysis underscores the extended Merton portfolio optimization model's practical relevance. It provides valuable insights into portfolio choices under market conditions characterized by varying levels of uncertainty and ambiguity.

4.1 Comparison of Segmented Risk Preferences with Varying Thresholds

To evaluate Proposition 1, this analysis examines three cases of uncertainty thresholds: (1) $z_1 = 0.3$ and $z_2 = 0.7$ as a baseline; (2) $z_1 = 0.6$ and $z_2 = 0.7$; and (3) $z_1 = 0.3$ and $z_2 = 0.4$. For simplicity, the utility function is defined as $U(W - RP(z)) = W - RP(z)$, and a uniform distribution $f(z)$ is employed over each segment. These simplifications maintain consistency with the theoretical framework while allowing for tractable calculations and straightforward interpretation of results. The uniform assumption for $f(z)$ reflects equal likelihood of uncertainty levels within each range, and the linear utility function captures the additive effects of the risk premium and wealth.

The total expected utility is calculated using the formula:

$$\begin{aligned} E[U(W)] &= \int_0^{z_1} (W - RP_O) f(z)\, dz + \int_{z_1}^{z_2} (W - RP_L(z)) f(z)\, dz + \int_{z_2}^{1} (W - RP_H(z)) f(z)\, dz, \\ &= \frac{1}{z_1 - z_{\min}} \int_{z_{\min}}^{z_1} (100 - RP_O)\, dz + \frac{1}{z_2 - z_1} \int_{z_1}^{z_2} (100 - RP_L(z))\, dz \\ &\quad + \frac{1}{z_{\max} - z_2} \int_{z_2}^{z_{\max}} (100 - RP_H(z))\, dz. \end{aligned}$$

For the baseline case ($z_1 = 0.3$, $z_2 = 0.7$), $RP(z)$ is given by:

$$RP(z) = \begin{cases} RP_O = 10, & \text{if } 0 \le z < 0.3, \\ RP_L(z) = 11.5 - 5z, & \text{if } 0.3 \le z < 0.7, \\ RP_H(z) = \frac{80}{3}(1 - z), & \text{if } 0.7 \le z \le 1. \end{cases} \tag{3}$$

The calculated total expected utility is approximately 277, demonstrating the balance across the low, moderate, and high uncertainty regions.

For the second case ($z_1 = 0.6$, $z_2 = 0.7$), the utility function retains the same structure but uses

the following segmented $RP(z)$:

$$RP(z) = \begin{cases} 10 & \text{if } z < 0.6, \\ 34 - 40z & \text{if } 0.6 \leq z < 0.7, \\ 20(1-z) & \text{if } z \geq 0.7. \end{cases}$$

The total expected utility is approximately 279. This configuration highlights the impact of the wide low uncertainty region ($z < 0.6$), where the risk premium $RP(z)$ remains constant at a low level, reflecting stability and contributing significantly to the overall utility due to the extended range of low uncertainty. The narrow moderate uncertainty region ($0.6 \leq z < 0.7$) adds concentrated contributions as $RP(z)$ declines sharply, capturing the transitional nature of this range. The high uncertainty region ($z \geq 0.7$) provides reduced but manageable compensation through a decreasing $RP(z)$, maintaining steady contributions to utility despite higher uncertainty.

For the third case ($z_1 = 0.3$, $z_2 = 0.4$), the segmented $RP(z)$ is:

$$RP(z) = \begin{cases} 10 & \text{if } z < 0.3, \\ 11.5 - 5z & \text{if } 0.3 \leq z < 0.4, \\ \frac{95}{6}(1-z) & \text{if } z \geq 0.4. \end{cases}$$

The total expected utility was approximately 275, slightly lower than in the baseline case. The moderate uncertainty region ($0.3 \leq z < 0.4$) is narrower compared to the baseline case, leading to a significantly reduced contribution to utility due to its limited range. The expanded high uncertainty region ($z \geq 0.4$) spans a wider range, amplifying its influence on the total utility. However, the risk premium in this region ($RP(z) = \frac{95}{6}(1-z)$) decreases moderately, providing diminishing compensation for increasing uncertainty. This extended high uncertainty region dominates the utility profile but contributes less positively due to the declining $RP(z)$. Together, the dominance of the high uncertainty region and the narrower moderate uncertainty region result in a noticeable reduction in total utility compared to the baseline case.

The results across these cases illustrate the sensitivity of expected utility to segmented risk preferences. The baseline case achieves a balanced utility across regions. When z_1 approaches z_2 in the second case, the moderate region contributes more heavily, increasing overall utility. Conversely, when z_2 approaches z_1 in the third case, the expanded high uncertainty region dominates, resulting in substantially lower utility. These findings support Proposition 1 by demonstrating the critical role of uncertainty thresholds in shaping investor outcomes.

4.2 Simulation Results with Noise as Ambiguity

To explore the robustness of the segmented structure of $RP(z)$ under Proposition 2, this simulation incorporates the effects of ambiguity modeled as noise introduced in Section 3.2. The analysis is based on 500 independent simulations, where ambiguity is represented as a stochastic compo-

nent following a normal distribution, $\mathcal{N}(0.1257, 1)$. This approach evaluates how noise impacts segmented risk premiums, complementing the theoretical predictions of Proposition 2.

For the simulation, the functional form of $RP(z)$, defined in Equation 3, captures distinct behaviors across low, moderate, and high uncertainty regions. In the low uncertainty region ($z < z_1$), the risk premium remains constant at $RP_O = 10$. In the moderate uncertainty region ($z_1 \leq z < z_2$), $RP(z)$ decreases linearly, reflecting a gradual adjustment to increasing uncertainty. In the high uncertainty region ($z \geq z_2$), $RP(z)$ declines more sharply, capturing the heightened risk sensitivity of investors in ambiguous environments. These segmentations ensure continuity at the boundaries, maintaining smooth transitions as outlined in Proposition 2.

The thresholds, $z_1 = 0.3$ and $z_2 = 0.7$, represent transitions between low, moderate, and high uncertainty levels. Specifically, $z_1 = 0.3$ marks the shift from low to moderate uncertainty, while $z_2 = 0.7$ denotes the transition from moderate to high uncertainty. Adding ambiguity as noise, the simulation investigates the interaction between stochastic components of ambiguity and segmented risk premiums.

The adjusted risk premium, $RP(z) + \text{Ambiguity}(z)$, retains the piecewise structure while introducing variability in investor perceptions of risk. This adjustment reflects realistic investor behaviors under ambiguous conditions, where deviations from deterministic models arise due to stochastic influences. The findings underscore the significance of incorporating ambiguity into portfolio optimization models, highlighting its impact on decision-making under uncertainty.

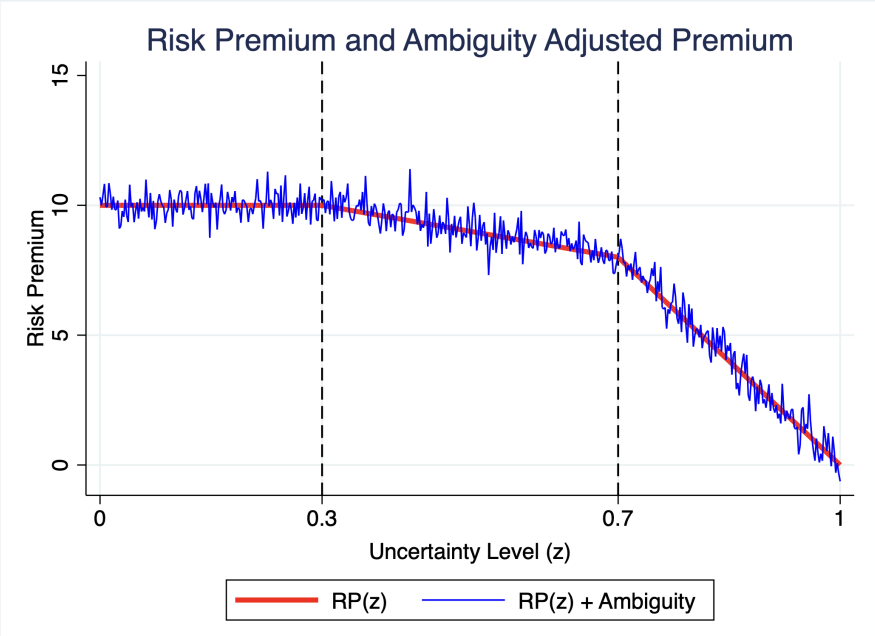
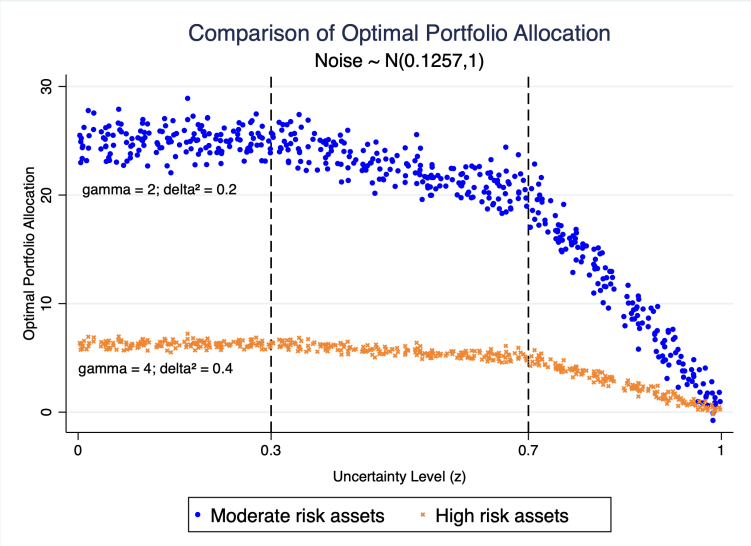


Figure 2: Visualization of $RP(z)$ and $RP(z) + \text{Ambiguity}(z)$ with Thresholds

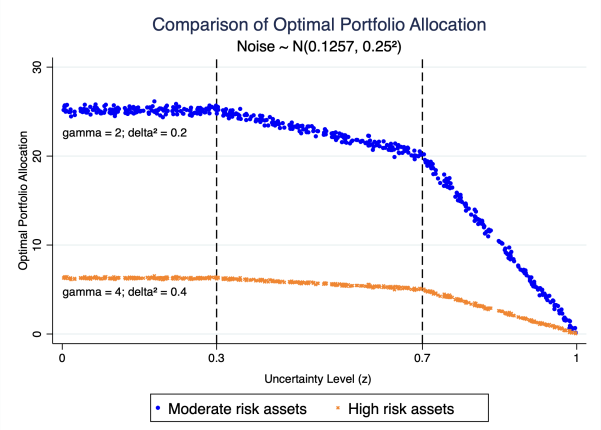
The results validate the segmentation of $RP(z)$ and its functional behavior across uncertainty regions, even under ambiguity adjustments. The simulation confirms expected patterns: a constant baseline in the low uncertainty region, a linear decline in the moderate region, and a sharper de-

cline in the high uncertainty region. These findings highlight the practical relevance of modeling risk premiums with ambiguity in financial contexts. While Proposition 2 provides the theoretical foundation, these simulations extend its applicability by demonstrating the interaction of risk premiums and ambiguity.

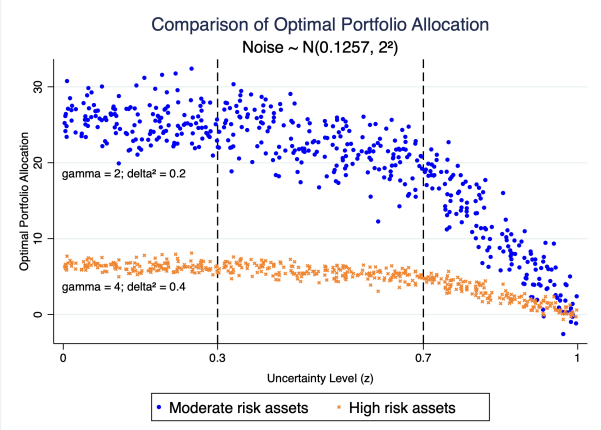
In portfolio optimization under uncertainty, investor behavior reflects a delicate balance between risk and reward, shaped by factors such as market volatility, personal risk aversion, and the ambiguity inherent in decision-making. This study examines how these elements interact to influence optimal portfolio allocation strategies. Specifically, it analyzes the impact of adjustments in the risk premium, ambiguity perceptions, and individual preferences, illustrated through simulations comparing distinct scenarios of portfolio allocation under varying noise variance levels.



(a) Baseline Case



(b) Lower Variability Noise



(c) Higher Variability Noise

Figure 3: Comparison of portfolio allocation behavior across different noise variances. The top plot represents the baseline case, while the bottom row illustrates lower and higher variability noise, highlighting the impact of noise variance on portfolio optimization.

Figure 3 illustrates the impact of subjective noise (Noise $\sim N(0.1257, \sigma^2)$) and asset risk levels on portfolio allocation behavior under three scenarios: (a) the baseline case with moderate noise variability ($\sigma^2 = 1$), (b) lower noise variability ($\sigma^2 = 0.25^2$), and (c) higher noise variability ($\sigma^2 = 2^2$). Each graph contains two scatter plots representing portfolio allocation trends for assets with different risk levels: blue for moderate-risk assets and orange for high-risk assets.

In the baseline scenario (Figure 3a), subjective noise is modeled with moderate variability ($\sigma^2 = 1$). The portfolio allocation trends for both moderate- and high-risk assets show consistent decreases as uncertainty (z) increases. The lines exhibit moderate dispersion around the trend due to the influence of subjective noise, reflecting manageable ambiguity in decision-making.

In the lower variability case (Figure 3b), the spread of data points around the trend narrows significantly for both asset types. This indicates reduced ambiguity in decision-making, allowing portfolio allocations to more closely align with changes in uncertainty. The tighter clustering of points reflects the minimal influence of subjective noise.

In contrast, the higher variability scenario (Figure 3c) introduces greater dispersion in portfolio allocation trends for both moderate- and high-risk assets. The broader spread reflects increased ambiguity due to heightened noise levels, which makes decision-making less predictable and amplifies deviations from the expected trend. Notably, high-risk assets (orange) exhibit more pronounced variability compared to moderate-risk assets (blue) across all scenarios, highlighting their greater sensitivity to noise and uncertainty.

Across these cases, three segments of uncertainty (z) are observed. In the low uncertainty range ($0 < z \leq 0.3$), portfolio allocations remain high and relatively stable, reflecting confidence in low-risk scenarios. In the moderate uncertainty range ($0.3 < z \leq 0.7$), allocations decline gradually, illustrating the increasing influence of risk and ambiguity. In the high uncertainty range ($0.7 < z < 1$), a sharp drop in allocations occurs as ambiguity and risk dominate decision-making. Notably, even as noise variance changes, the high uncertainty range exhibits minimal differences in portfolio optimization. When asset risk is high, ambiguity plays a secondary role compared to the dominating influence of risk itself. The consistency in allocations within this range suggests that investors anchor their decisions to heightened levels of risk aversion, with noise variance exerting a diminished effect.

Additionally, the following table 1 demonstrates the relationship between uncertainty levels and their outcomes across three normal distributions with varying variances ($\sigma^2 = 1$, 0.0625, and 4).

Uncertainty Segment	**N(0.1257, 1)**	**N(0.1257, 0.25^2)**	**N(0.1257, 2^2)**
Low	10.01692 (0.4998)	10.06191 (0.1280)	10.1936 (0.9379)
Moderate	8.9925 (0.7882)	9.0446 (0.5971)	9.0090 (1.3085)
High	4.0819 (2.3480)	4.0525 (2.3003)	3.9828 (2.4815)

Table 1: Uncertainty Segments with Different Distributions

It summarizes the outcomes across different uncertainty segments and noise variances. While

the mean outcomes remain largely stable across all noise levels, the standard deviations increase as noise variance rises, highlighting the greater unpredictability of portfolio decisions in more volatile environments.

The comparative analysis highlights how changes in noise variance affect portfolio allocation. Lower variance results in more predictable allocations, while higher variance introduces greater randomness, challenging decision-making under uncertainty. This underscores the importance of integrating ambiguity into portfolio models to navigate dynamic market conditions. Following this, further results are examined, focusing on how adjusted uncertainty thresholds impact portfolio allocation strategies.

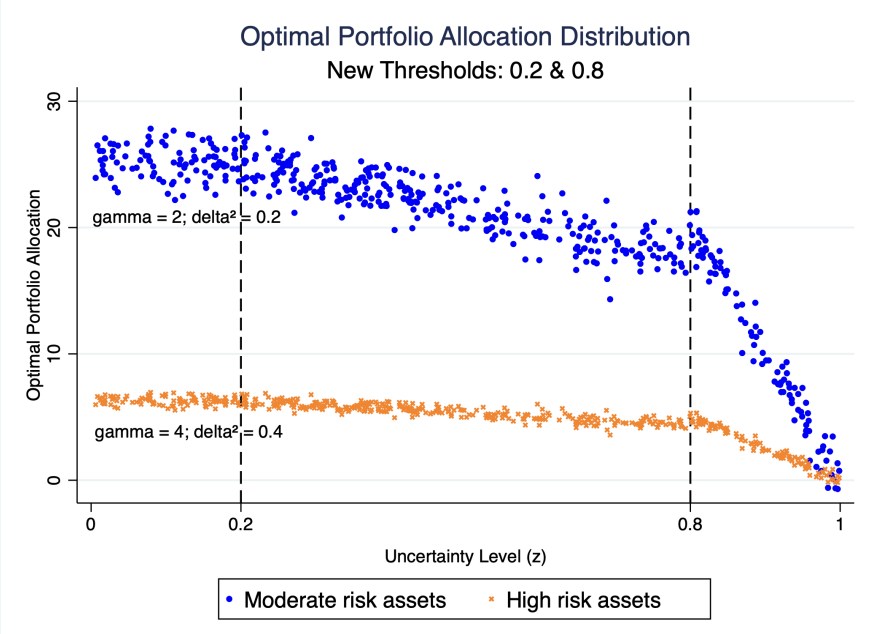


Figure 4: Adjusted Uncertainty Thresholds and Portfolio Allocation

The impact of adjusted uncertainty thresholds on portfolio allocation behavior is analyzed, as shown in Figure 4. The adjusted thresholds ($z_1 = 0.2$ and $z_2 = 0.8$) create a rebalanced portfolio allocation framework, emphasizing the role of moderate uncertainty. Compared to the baseline case in Figure 3a, the reduced lower uncertainty range ($0 < z \leq 0.2$) indicates a smaller zone of low-risk confidence, reflecting a shift towards broader moderate-risk conditions. The expanded moderate uncertainty range ($0.2 < z \leq 0.8$) results in more gradual adjustments in portfolio allocations, suggesting a smoother investor response to incremental changes in risk. This transition highlights the growing importance of moderate risks in shaping allocation strategies. In contrast, the high uncertainty range ($0.8 < z < 1$) retains its sharp decline in allocation, showing minimal sensitivity to ambiguity despite the compressed range. Overall, the new thresholds redistribute risk perception, illustrating how portfolio allocation strategies adapt to redefined zones of risk and ambiguity.

In this subsection, the simulation results emphasize the significance of integrating ambiguity

as a stochastic component in modeling segmented risk premiums. By analyzing portfolio allocation behavior under varying noise variances and adjusted uncertainty thresholds, the study highlights the complex interplay between risk, ambiguity, and investor decision-making. The findings demonstrate that moderate uncertainty plays an increasingly pivotal role in shaping allocation strategies, while high uncertainty remains dominated by heightened risk aversion. These insights underscore the practical importance of designing portfolio models that accommodate ambiguity and adapt to dynamic market conditions, providing a robust framework for decision-making under uncertainty.

5 Discussion and Implications

This study enhances the understanding of investor behavior under uncertainty by introducing a theoretical framework that integrates segmented risk premiums and ambiguity aversion into Merton's continuous-time portfolio optimization model. It addresses critical limitations of classical finance, formalizing the dynamic and context-sensitive adjustments investors make in response to varying levels of market uncertainty.

Compared to Merton's model (1971), which assumes uniform risk responses under known probability distributions, the segmentation introduced in this study captures the non-linear shifts in compensation demands across different uncertainty thresholds. This extension provides a more refined perspective on investor behavior, particularly under ambiguous market conditions.

Building on Biagini and Pınar's (2017) robust Merton problem, which incorporated ambiguity aversion by modeling uncertainty in drift and volatility, this study offers a more structured mechanism by explicitly incorporating subjective uncertainty thresholds. This enhancement complements their work and deepens the understanding of investor decision-making under ambiguity.

The framework also aligns with insights from behavioral finance, particularly Kahneman and Tversky's (1979) Prospect Theory, which highlights context-dependent risk attitudes. Unlike discrete-time models focused on gain-loss asymmetry, this continuous-time approach captures segmented investor responses to uncertainty, bridging theory with observed investor behavior.

Furthermore, recent advancements by Bäuerle and Mahayni (2024) and Guan et al. (2024), which incorporate ambiguity into portfolio optimization, focus on state-dependent confidence sets and adaptive learning mechanisms. While these studies primarily address specific market parameters, the current framework is more generalizable, incorporating both segmented risk premiums and ambiguity aversion within a continuous-time structure.

The practical implications of this study are significant for portfolio managers, enabling dynamic strategies that adjust to shifting market conditions. By incorporating segmented risk premiums, the framework allows for more precise investment allocations, particularly during periods of heightened ambiguity. Policymakers could also use these insights to design regulations that account for behavioral responses to uncertainty, helping maintain stability in volatile markets.

Future research could focus on empirically validating the model, using real-world data to es-

timate uncertainty thresholds and ambiguity adjustments. Exploring demographic variations in ambiguity aversion through experimental studies could further enrich our understanding of investor behavior. Expanding the framework to multi-asset or multi-period settings would also enhance its applicability, particularly for institutional portfolio management.

6 Conclusion

This study expands Merton's continuous-time portfolio optimization model by incorporating segmented risk premiums and ambiguity aversion. The proposed framework addresses limitations in classical models by allowing for dynamic adjustments in investor behavior in response to varying levels of market uncertainty. Unlike traditional models that assume uniform risk responses, this approach accounts for how investors adjust their expectations and risk premiums across different uncertainty thresholds.

The model integrates insights from behavioral finance, including Prospect Theory, and complements recent advances in ambiguity aversion in portfolio optimization. By incorporating subjective thresholds and adjusting risk premiums, this study contributes a more comprehensive understanding of investor decision-making under uncertainty.

Despite its contributions, the study relies on simplifying assumptions, such as fixed segmentation thresholds and specific forms of ambiguity adjustments. Future work should empirically validate these assumptions, refine the framework, and extend it to multi-period or multi-asset contexts. Additionally, examining how demographic and market variations influence ambiguity aversion could increase the model's generalizability.

In conclusion, this study enhances the understanding of risk and ambiguity in portfolio optimization and provides a solid foundation for further research. By addressing gaps in classical finance, it contributes to a richer understanding of decision-making under uncertainty.

Declarations

Funding: Not applicable.
Conflict of interest: The author declares no conflict of interest.
Ethical approval: Not applicable.

References

[1] Bäuerle, N., & Mahayni, A. (2024). Optimal investment in ambiguous financial markets with learning. European Journal of Operational Research, 315(1), 393–410.

[2] Bakshi, G., Kapadia, N., & Madan, D. (2003). Stock return characteristics, skew laws, and the differential pricing of individual equity options. *Review of Financial Studies, 16*(1), 101–143. https://doi.org/10.1093/rfs/16.1.101

[3] Baker, M., Bradley, B., & Wurgler, J. (2011). Benchmarks as limits to arbitrage: Understanding the low-volatility anomaly. *Financial Analysts Journal, 67*(1), 40–54. https://doi.org/10.2469/faj.v67.n1.4

[4] Barro, R. J. (2006). Rare disasters and asset markets in the twentieth century. *The Quarterly Journal of Economics, 121*(3), 823–866. https://doi.org/10.1162/qjec.121.3.823

[5] Bekaert, G., Engstrom, E., & Xu, N. R. (2019). The time variation in risk appetite and uncertainty. *Journal of Monetary Economics, 114*, 1–18. https://doi.org/10.1016/j.jmoneco.2019.03.004

[6] Bernstein, A., Gustafson, M. T., & Lewis, R. (2019). Disaster on the horizon: The price effect of sea level rise. *Journal of Financial Economics, 134*(2), 253–272. https://doi.org/10.1016/j.jfineco.2019.03.013

[7] Biagini, F., & Pınar, M. Ç. (2017). Robust portfolio optimization with ambiguity aversion. *Mathematical Finance, 27*(1), 179–207. https://doi.org/10.1111/mafi.12104

[8] Epstein, L. G., & Schneider, M. (2003). Recursive multiple-priors. *Journal of Economic Theory, 113*(1), 1–31. https://doi.org/10.1016/S0022-0531(03)00097-1

[9] Frazzini, A., & Pedersen, L. H. (2014). Betting against beta. *Journal of Financial Economics, 111*(1), 1–25. https://doi.org/10.1016/j.jfineco.2013.10.005

[10] Garlappi, L., Uppal, R., & Wang, T. (2007). Portfolio selection with parameter and model uncertainty: A multi-prior approach. *The Review of Financial Studies, 20*(1), 41–81. https://doi.org/10.1093/rfs/hhl003

[11] Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics, 18*(2), 141–153. https://doi.org/10.1016/0304-4068(89)90018-9

[12] Guan, X., Zhu, Y., & Tang, J. (2024). Robust portfolio selection under state-dependent confidence sets. *Quantitative Finance*. https://doi.org/10.1080/14697688.2024.1234567

[13] Hansen, L. P., & Marinacci, M. (2016). Ambiguity aversion and model misspecification: An economic perspective. *Statistical Science, 31*(4), 511–515. https://doi.org/10.1214/16-STS576

[14] Hansen, L. P., & Sargent, T. J. (2001). Robust control and model uncertainty. *American Economic Review, 91*(2), 60–66. https://doi.org/10.1257/aer.91.2.60

[15] Hansen, L. P., Sargent, T. J., & Tallarini, T. D. (1999). Robust permanent income and pricing. *The Review of Economic Studies, 66*(4), 873–907. https://doi.org/10.1111/1467-937X.00107

[16] Harvey, C. R., & Siddique, A. (2000). Conditional skewness in asset pricing tests. *The Journal of Finance, 55*(3), 1263–1295. https://doi.org/10.1111/0022-1082.00247

[17] Illeditsch, P. K. (2011). Ambiguous information, portfolio inertia, and excess volatility. *The Journal of Finance, 66*(6), 2213–2247. https://doi.org/10.1111/j.1540-6261.2011.01699.x

[18] Ju, N., & Miao, J. (2012). Ambiguity, learning, and asset returns. *Econometrica, 80*(2), 559–591. https://doi.org/10.3982/ECTA9188

[19] Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica, 47*(2), 263–291. https://doi.org/10.2307/1914185

[20] Klibanoff, P., Marinacci, M., & Mukerji, S. (2005). A smooth model of decision making under ambiguity. *Econometrica, 73*(6), 1849–1892. https://doi.org/10.1111/j.1468-0262.2005.00640.x

[21] Maenhout, P. J. (2004). Robust portfolio rules and asset pricing. *The Review of Financial Studies, 17*(4), 951–983. https://doi.org/10.1093/rfs/hhg055

[22] Markowitz, H. (1952). Portfolio selection. *The Journal of Finance, 7*(1), 77–91. https://doi.org/10.2307/2975974

[23] Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory, 3*(4), 373–413. https://doi.org/10.1016/0022-0531(71)90038-X

[24] Othman, N. N. (2024). Emotional economics: The role of psychological biases in personal investment outcomes. *Journal of Behavioral Economics and Finance, 15*(1), 18–38. https://doi.org/10.2139/ssrn.4844927

[25] Ross, S. A. (1976). The arbitrage theory of capital asset pricing. *Journal of Economic Theory, 13*(3), 341–360. https://doi.org/10.1016/0022-0531(76)90046-6

[26] Treynor, J., Sharpe, W. F., Lintner, J., & Mossin, J. (1964). Capital asset pricing model. *The Journal of Finance, 19*(3), 425–442. https://doi.org/10.2307/2975975

[27] Zhou, X. Y., & Li, D. (2000). Continuous-time mean-variance portfolio selection: A stochastic LQ framework. *Applied Mathematics and Optimization, 42*, 19–33. https://doi.org/10.1007/s002459910004