Liquidity Supply, Frequent Trading, and Stock Returns

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Abstract

This paper analyzes a dynamic equilibrium in a competitive economy where liquidity traders frequently trade stock shares with liquidity providers. Liquidity traders outnumber liquidity providers while facing small bid-ask spreads modeled as transaction costs. I show that differences in trading technologies between liquidity traders and liquidity providers can account for historically observed moments of stock returns. The presence of bid-ask spreads causes liquidity traders to significantly reduce their trading demand, making liquidity providers marginal investors. Consequently, the conditional risk premium of the stock market is defined by the current stock holdings of the liquidity providers. Furthermore, transaction costs make the liquidity traders hold suboptimal portfolios and control their trading volume. These features of the stock market lead to a strong asymmetric feedback effect between stock allocations of liquidity traders and the conditional risk premium causing the former to reduce their average stock allocations. This results in significantly longer average stock allocations of the liquidity providers inciting them to demand a major decline in stock prices.

The feedback effect between the liquidity traders' stock allocations and the conditional risk premium leads to a short-term reversal of stock returns, as well as a short-term positive comovement and a negative autocovariance between the liquidity providers' stock holdings and stock prices. Thus, the model relates the properties of the conditional and unconditional stock returns to the market microstructure. Finally, the model explains the conditional covariance between the aggregate consumption and the stock returns, the volatility of the stock returns, and the conditional Sharpe ratio.

Keywords: General Equilibrium; Liquidity Supply; Frequent Trading; Transaction Costs
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1 Introduction

Most of the trading volume in the stock market is driven by professional investors who trade on a daily basis. The majority of these investors prioritize liquidity and trade using market orders, while the remainder provide liquidity through limit orders. Liquidity providers possess the skills and equipment to place and cancel limit orders efficiently, minimizing their opportunity costs. These costs are typically assumed to be proportional to the trading demand from liquidity traders and are offset by the bid-ask spreads from market orders. Liquidity providers may include institutional investors, dealers, retail investors, and even market makers.[1] I assume that a liquidity provider is willing to hold inventory overnight and use it to hedge her future consumption stream. This study shows that the asymmetry in trading technologies between liquidity traders and liquidity providers arising from the presence of bid-ask spreads and opportunity costs significantly impacts stock returns, even when bid-ask spreads are small, provided that economic agents trade frequently, such as bi-weekly or more often.

The cost of immediacy for the liquidity traders is the bid-ask spread which is usually modeled by proportional transaction costs rendering the equilibrium analysis numerical. Furthermore, the stock market requires a lot of heterogeneous investors for the stock trading volume to remain positive.[2] To address the issue of the equilibrium tractability I reserve to use quadratic transaction costs. This choice allows the trading volume to be almost always positive over time even in the presence of two agents, and is also justified by the result of Isaenko (2024) who shows that changing the shape of transaction costs from linear to quadratic has a rather small effect on the statistical moments of stock returns.

I find an equilibrium in the stock market in which the price of one stock share trends at a constant rate δ. This stock market is consistent with the empirical findings of the seminal article by Campbell and Kyle (1993), henceforth CK, who determine that $\delta = 1.3\%$. I further assume that each economic agent has a utility function in which the coefficient of absolute risk aversion decreases over time at rate δ as $\gamma e^{-\delta t}$, where γ is constant. The agents also receive endowments and a dividend per share that have time-varying Gaussian distributions and whose expected values grow over time at a constant rate δ. Using numeraire $e^{\delta t}$ makes these processes stationary. Consequently, the price of one share becomes de-trended and is supported by agents with constant absolute risk aversions equal to γ's. This approach allows me to find an equilibrium where the stock price and the dividend per share grow, analyze it in a stationary framework, and consequently avoid an issue of non-survival of one of the agents which exists in the case when the relative risk aversions of investors are constant.[3] The agents in this equilibrium also have constant expected coefficients of the relative risk

[1] As long as they are willing to stay long in the stock market overnight.

[2] The presence of linear transaction costs makes an investor stop trading for extended type-periods when her state variables fall to the no-trade zone. See, for example, Constantinides (1986).

[3] See Wang (1996).

aversions.[4]

The rest of the introduction mostly discusses the returns of the de-trended stock. Its expected rate of return is lower than the expected rate of return of the trending stock by δ, but the volatilities of the rates of returns of the two stocks are the same. The lack of trending in the equilibrium let me relate the stock price and the dividends under the chosen numeraire with the de-trended stock price and the de-trended dividends studied in CK. The calibration based on the empirical analysis in CK is used with the assumption that liquidity traders and liquidity providers have the same coefficients of absolute risk aversions and that the weight of the liquidity provider in the economy is three times smaller than the weight of the liquidity trader. The last assumption is chosen to make the Sharpe ratio of the trending stock returns equal to 0.4. Under these settings, the model predicts that the risk premium and the standard deviation of the rate of return of the de-trended stock price in the market with transaction costs are 5.2% and 17.4%, respectively, and the agents consume as if their expected coefficients of relative risk aversion are equal to 5. In the market without transaction costs, these moments are 0.5% and 3.2%, respectively, and they can match their counterparts in the market with transaction costs only if the expected coefficients of relative risk aversion of agents are equal to 33.

In the model, the liquidity provider makes money by charging transaction costs and also from her inventory. The liquidity trader trades frequently with the liquidity provider and loses money on transaction costs. Since a transaction cost is charged per trade and can be recovered only over time, transaction losses make the liquidity trader dramatically reduce her demand per trade[5] resulting in the liquidity supplier becoming a marginal investor. Consequently, the conditional risk premium of the stock market is approximately proportional to the current stock allocations of the liquidity provider. Furthermore, the presence of transaction costs makes liquidity traders hold suboptimal portfolios and trade slowly to their optimal allocations causing the conditional risk premium depend on their current aggregate stock allocations. This leads to a strong positive feedback effect between these allocations and the conditional risk premium making the liquidity traders noteceably reduce their stock allocations. Consequently, the liquidity providers have to significantly increase their average stock market exposure while demanding a significantly higher Sharpe ratio of the de-trended stock returns. The increase in the Sharpe ratio is possible only if the risk premium on dollar returns rises multiple times. The latter leads to a lower stock price

[4] A different approach to model the equilibrium with the growing stock market in a stationary framework is undertaken in CK. Campbell and Kyle assume that investors have CARA utility functions, and their population and the number of outstanding shares are growing at rate δ. The dividend per share is required to be non-trending leading to the price of one stock share being non-trending as well.

[5] Appendix B shows in a single-period setting that the introduction of transaction costs at weekly trading reduces the trading demand of liquidity traders by almost fifty times, while the trading demand of liquidity providers does not change. The heterogeneity in trading demands in the intertemporal setting of continuous-time trading utilized in the paper is revealed in the infinite rate of trading demanded by the liquidity provider and trading at a finite speed demanded by the liquidity trader.

and, respectively, to a much higher risk in the rates of stock returns, which in turn feeds a further increase in the risk premium in the rates of stock returns.

Matching the expected losses on transaction costs with these losses should the transaction costs be linear suggests a fixed bid-ask spread of 0.06% which is in line with the current market practice. Furthermore, the present value of transaction costs discounted by the expected rate of stock returns is 0.078 dollars, while the effect of the costs on the stock price is about 76 dollars, indicating that the effects of transaction costs on stock returns are not defined by trading losses. Rather, they are explained by the weight of liquidity providers and the risk aversions of agents. Increasing absolute risk aversions and decreasing the weight of the liquidity providers quickly increase the moments of the rates of de-trended stock returns. Consequently, the target Sharpe ratio of the stock returns could be reached in the market with a smaller weight of the liquidity providers and a lower coefficient of risk aversion of the agents. For example, decreasing the weight of liquidity providers to 10.0% and decreasing the coefficient of absolute risk aversions such that the expected coefficients of relative risk aversions of the two agents become equal to 2, lead to the risk premium and the volatility on the rates of stock returns equal to 5.27% and 17.55%, while their counterparts in the market without transaction costs are equal to 0.16% and 2.45%, respectively. In this market, the liquidity providers are significantly more leveraged than in the market where agents are more risk averse and the weight of the liquidity providers is higher.

The equilibrium exists only if the aggregate endowment risk is absent in the economy. Consequently, the stock price is not affected by this uncertainty in the absence of transaction costs. However, transaction costs break the stock market dynamic completeness leading to long-term and short-term reversals in the stock prices. The long-term predictability arises from the reversal in the endowment volatility and is taken advantage of by liquidity traders. The short-term predictability arises from suboptimality for the liquidity trader to instantly trade to her utility-maximizing allocations and is beneficial to liquidity suppliers.[6] The benefits from long-term predictability exceed the losses on transaction costs making the liquidity traders better off in the presence of transaction costs. This presence makes the liquidity providers better off as well as long as the opportunity costs are not noticeably bigger than the transaction costs.

One of the key features of microstructure models is an aversion of market makers to taking on inventory. This aversion along with the necessity of investors to trade via market makers leads to a short-term price reversal which is beneficial to market makers in the sense that the longer their inventory today, the higher the stock return tomorrow.[7] The short-term reversal and a positive relation between market makers' inventories and the next-day

[6]The emergence of stock price reversals due to slow capital movement is predicted in a number of models. See, for example, Vayanos and Wooly (2013) and Buss and Dumas (2019).

[7]See Amihud and Mendelson (1980), Ho and Stoll (1981), Grossman and Miller (1988), and others for inventory models that lead to reversals.

returns are confirmed empirically in Lehmann (1990) and in Hendershott and Seasholes (2007), respectively, where Hendershott and Seasholes (2007) use the data on the specialists who are willing to keep the overnight inventory. This article makes similar predictions with the economic intuition related to the one in microstructure models. The necessity of investors to trade with market makers and the aversion to overnight inventory makes market makers marginal demanding a discount for buying and a premium for selling. Parts of the latter gains could be interpreted as transaction costs for liquidity traders, used in this article, which makes the liquidity provider marginal, leading to the short-term reversal and a positive relation between liquidity provider inventory and the next-day returns. These short-term traits in stock returns are linked to the feedback effect between the stock holdings of the liquidity trader and the conditional risk premium leading to a low stock allocation of the liquidity traders and low stock prices. Therefore, my model relates, in a stylized way, the market microstructure with the historical moments of the stock market returns.

In line with empirical results, the model of this article predicts a cyclical behavior of the stock price and counter-cyclical traits in the conditional risk premium on the rates of stock returns and the conditional Sharpe ratio. The countercyclical trends in the conditional Sharpe ratio are weakened by similar trends in the conditional standard deviation of the rates of stock returns. The latter prediction has mixed support in the empirical literature.[8] The model also predicts the persistence in the conditional risk premium on the rates of returns emerging from the persistence in the dividend flow and explains the predictability of the conditional variance on the rates of stock returns. These findings are aligned with the predictions of a few existing models.[9]

Literature. The effects of transaction costs on stock returns have been the subject of numerous theoretical studies. Many of these studies estimate the effect of these costs on the risk premium of stock returns by using a partial equilibrium approach in which the liquidity premium[10] is usually found from a change in the utility function of an investor arising from the presence of transaction costs.[11] The alternative approach relies on a general equilibrium analysis in which the liquidity premium is found from market clearing conditions.[12] With a couple of exceptions, both of these approaches predict that the impact of transaction costs on the risk premium should be in the order of a few percent of the risk premium in the absence of transaction costs or even smaller. The argument used to explain this conclusion is often as follows. Investors paying transaction costs demand a liquidity premium for their

[8]See Schwert (1989) and Chou, Engle, and Kane (1992), and Lettau and Ludvingston (2010) who report a countercyclicality in the conditional standard deviation, and Whitelaw (1994) and Brandt and Kang (2004), who find that the conditional risk premium and the conditional standard deviation have a negative correlation.

[9]See Cambell and Cochrane (1999), Guvenen (2009), and Chien, Cole, and Lusting (2011).

[10]The liquidity premium is the difference between the risk premium in the presence of transaction costs and the risk premium in the absence of these costs.

[11]See, for example, Constantinides (1986), Liu and Loewenstein (2002), and He and Mamaysky (2005).

[12]This literature will be reviewed below.

losses while reducing their trading volumes to be able to recover transaction costs over the time that they do not trade. Furthermore, they reduce their trading volume by so much that the required liquidity premium becomes much smaller than the risk premium. The reduction in the trading volume appears so significant due to the "flatness" of their utility functions. That is, investors prefer to stay far away from their optimal allocations to make significant portfolio adjustments. Consequently, an investor should be "inspired" to trade bigger volumes to suffer bigger losses on transaction costs and to demand a bigger liquidity premium. Amihud and Mendelson (1986) demonstrate that transaction costs can change the stock price by the order of ten percent by exogenously making investors undergo heavy losses on transaction costs.

Isaenko (2023), henceforth IS, shows in the equilibrium framework with a stationary stock market where all investors pay transaction costs, that these costs can have an impact on the first two unconditional moments of the rates of stock returns of the order of tens of percent from their values in the absence of transaction costs. The origin of the impact is shown to be new: in the presence of frequent trading and a significant heterogeneity in risk aversions among investors, the introduction of transaction costs substantially changes heterogeneity in trading demands across investors leading to noticeably different equilibrium prices. Even though the investors have to trade frequently, their losses on transaction costs make small fractions of their gains from the liquidity premium confirming that the metric guiding the effects of transaction costs on prices is different from the losses of investors on these costs which are used in the other studies to scale the liquidity premium. The intuition behind the present study is similar to the one in IS with the difference in the liquidity provision which leads to a much stronger heterogeneity in trading demands, as illustrated in Appendix B, much more substantial feedback effects on stock allocations, and the resulting much more significant impact on stock returns.

An equilibrium analysis of stock returns in the presence of transaction costs can also be found in the following articles. Heaton and Lucas (1996) solve a model in which agents trade at the annual frequency to share their labor income risk and conclude that transaction costs alone do not affect asset prices significantly. Vayanos (1998) considers an equilibrium in the presence of overlapping generations of investors who pay proportional transaction costs for trading stocks. Vayanos (1998) finds a small effect on the risk premium due to a lack of heterogeneity across investors. Huang (2003) assumes no heterogeneity in risk aversions or market shares across investors and predicts a very small effect of transaction costs on stock prices unless they are combined with borrowing constraints. The combination of borrowing constraints and transaction costs may lead to substantial losses on transaction costs resulting in a significant liquidity premium. Lo, Mamaysky, and Wang (2004) solve an equilibrium where investors with equal weights and the same risk aversions pay fixed transaction costs for trading a stock market. The authors establish the effect of transaction costs on stock prices constituting a few percent of the stock price in the absence of the costs unless investors have

very strong trading incentives and pay very high transaction costs. Buss and Dumas (2019) assume that investors cannot trade more often than once a year and, therefore, destine their predictions on the effects of transaction cost to be small.

The economic intuition for a high-risk premium used in this study could be related to the intuition used in the literature on limited market participation. According to the latter, investors have to take longer positions in the stock market than in the case of unlimited market participation and demand a higher Sharpe ratio.[13] The presence of an endogenous bond market is deemed to be important for stock returns since it allows to amplify consumption growth volatility of stockholders leading to a higher volatility of stock returns.[14] Chien, Cole, and Lustig (2011) analyze the model with investors who update their portfolios annually and investors who do not update their portfolios. They show that the fraction of stock shares held by active traders, not the fraction held by all participants, is critical for asset prices because only these traders respond to variations in state prices. The latter explanation also resonates with this article as the stock holdings of liquidity providers explain the stock returns. The origins of the absence of trading demand from households in the literature on the limited market participation and the reduced trading demand from passive investors in Chien, Cole, and Lustig (2011), and also a smaller trading demand of liquidity traders in this article are different: in the case of this article, the reduced demand emerges endogenously from the nature of transaction costs, while it is set exogenously in the first two cases.

Finally, the model in this article could be related to the representative intermediary-based models[15]. These models aim to explain the behavior of stock returns and the short-term interest rate during dramatic stock market episodes by applying financial constraints on intermediaries who cannot provide liquidity to the full extent when these constraints are binding. Consequently, this literature emphasizes conditional returns and strategies during financial turmoils. On the contrary, this article analyzes the stock market during the "regular times" when the liquidity providers are not severely bound by financial constraints and seeks to explain the unconditional moments of stock returns.

The rest of the article is organized as follows. Section 2 describes the model. Section 3 analyzes an equilibrium with transaction costs, and Section 4 concludes. Appendix A describes the solution of the equilibrium, Appendix B presents a single-period model for the basic intuition on the role of trading frequency for trading demand of the liquidity trader and the size of transaction costs, while the calculations of the covariances between the aggregate conditional consumption and the conditional moments of stock returns are found in Appendix

[13]See, for example, Mankiw and Zeldes (1991), Saito (1995) and Basak and Cuoco (1998), and Guvenen (2009).

[14]See Guvenen (2009) who provides a detailed intuition for the role of the bond market in generating high-risk premium and volatility of stock returns.

[15]See He and Krishnamurthy (2012), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014), and Kargar (2021).

C.

2 Economic Setting

I consider a Markovian economy with an infinite horizon and a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ where an uncertainty is generated by a standard three-dimensional Brownian motion $W_t = (W_{Dt}, W_{Yt}, W_{0t})$, which is also adapted.

Agents can borrow and lend at a constant interest rate r. Also, they can trade a stock that pays real dividend flow D^o and has a real post–dividend price S^o that follows the process:

$$dS^o_t + D^o_t dt = \mu_{S^o} dt + \sigma_{S^o_D}\, dW_{Dt} + \sigma_{S^o_Y}\, dW_{Yt}, \tag{1}$$

where μ_{S^o}, $\sigma_{S^o_D}$, and $\sigma_{S^o_Y}$ are the drift and diffusion coefficients to be determined endogenously in equilibrium, and Brownian motions W_D and W_Y are independent. The dividend rate follows the process

$$dD^o_t = [\delta D^o + \kappa_D(\bar{D}^o - D^o_t)]dt + \sigma_{D^o} dW_{Dt}, \tag{2}$$

where $\bar{D}^o = \bar{D}e^{\delta t}$, $\sigma^o_D = \sigma_D e^{\delta t}$ and δ, κ_D, $\bar{D}$, σ_D are positive constants.

Following the empirical findings in CK, I assume that the stock price and the dividend rate trend at rate δ, which is smaller than a risk-free rate, such that $S^o(t) = S(t)e^{\delta t}$, $D^o(t) = D(t)e^{\delta t}$. The processes $S(t)$ and $D(t)$ have no trends in time and are called de-trended stock and de-trended dividend, respectively. Equations (1) and (2) imply that $D(t)$ and $S(t)$ follow the processes:

$$dD_t = \kappa_D(\bar{D} - D_t)dt + \sigma_D dW_{Dt}, \tag{3}$$

$$dS_t + D_t dt = \mu_{St} dt + \sigma_{S_D t}\, dW_{Dt} + \sigma_{S_Y t}\, dW_{Yt}, \tag{4}$$

where $\mu_S = \mu_{S^o} e^{-\delta t} - \delta S$, $\sigma_{S_D} = \sigma_{S^o_D} e^{-\delta t}$, $\sigma_{S_Y} = \sigma_{S^o_Y} e^{-\delta t}$.

The processes in the trending market are measured in dollars and labeled with the symbol " ^{o} ". These processes can also be measured in terms of the price of a δ-note which at time t is equal to $e^{\delta t}$ dollars. Under this numeraire, the dividend rate and the stock price are equal to the de-trended dividend rate and the de-trended stock price measured in dollars, while a one-unit of debt changes over time as $e^{(r-\delta)t}$. For the most part, I study the de-trended stock market, and the term "de-trended" will often be omitted for bravity. As shown below, this approach allows me to analyze an equilibrium in a growing stock market in a stationary framework with viable heterogeneous infinitely-lived agents and relate this equilibrium to the empirical observations.

2.1 Economic Agents

The market includes competitive liquidity traders with constant weight λ and competitive liquidity providers with weight $1 - \lambda$. Liquidity traders trade with liquidity providers and

pay them quadratic transaction costs such that the loss of a liquidity trader from time t to $t+dt$ is $\alpha\, u^2\, dt$, where u is an average rate of share trading within this time interval and α is a constant measured in δ-notes.[16] This shape of transaction costs is chosen for the results' tractability and will be discussed in the next subsection. The transaction cost does not allow the rate of liquidity trader's trading to be infinite. Naturally, her shareholding N becomes absolutely continuous under continuous time and can be written as

$$dN_t = u_t dt. \tag{5}$$

Liquidity providers face opportunity costs which are short-term predictable and proportional to the costs faced by liquidity traders. They also receive transaction costs from liquidity traders which are used to cover opportunity costs. The resulting cost for the representative liquidity provider over time dt is $\psi u^2 dt$, where u is the rate of trading of the representative liquidity trader and ψ is a constant measured in δ-notes which is equal to $-\alpha\,\frac{\lambda}{1-\lambda}$ if the opportunity cost is zero, is equal to 0 if bid-ask spreads perfectly cover the opportunity costs, and is positive if opportunity costs exceed the cash flow from transaction costs.

A liquidity trader and a liquidity provider receive a non-tradable cumulative income $I_{it} = \int_0^t Y_{is} dW_{0s}$ denominated in δ-notes, where $Y_1 = Y$, $Y_2 = -\frac{\lambda}{1-\lambda}\, Y$, and Y follows the process measured in δ-notes:

$$dY_t = -\kappa_Y Y_t dt + \sigma_Y dW_{2t}, \quad Y(0) = Y_0, \tag{6}$$

with κ_Y and σ_Y being constant. The correlation coefficients of W_0 with W_D and W_Y are equal to one and zero for better tractability with little consequences to the model predictions. Furthermore, the aggregate endowment risk is set to zero: $\lambda Y_1 + (1-\lambda) Y_2 = 0$. This assumption allows the equilibrium to exist at least when σ_Y is small.[17]

2.2 About the Model Assumptions

This section addresses a few assumptions that are made to allow a linear-quadratic solution of the equilibrium to be presented below. First, a locally riskless bond market is set to be exogenous. This assumption is not important for the model predictions since they arise from the frequent trading of investors, while the time scale of changes in the locally risk-free rate is of the order of a few months.

[16] Coefficient in transaction costs is assumed independent from the current stock price for tractability. Assuming such dependence will have little bearing on the results.

[17] If the aggregate endowment is not zero then the conditional risk premium will depend on the endowment volatility inciting both types of agents to trade against changes in this volatility in the absence of transaction costs. Hence, the stock market will not clear. Making the endowment volatility two-dimensional will allow me to avoid this assumption but will not change the predictions of the model.

Second, the endowment volatility is introduced to make investors trade in the long run. The presence of frequent trading is pivotal for the model predictions on stock returns. The trading incentives, controlled by parameter σ_Y, strongly affect the trading volume but could be weak for the model predictions on stock returns to remain significant. The alternative way to make investors trade could be created by changing the distribution of dividend rate to non-Gaussian or/and by making the number of investors random. Both approaches will remove the tractability in the equilibrium.

Third, the transaction costs are set to be quadratic. As discussed in Garleanu and Pedersen (2016), these costs model the losses from a price impact where the first share in the market order is traded at the market price, that is the price in the middle between the bid and ask prices, while the other shares are traded at prices which are linearly increasing (decreasing) with the number of traded shares. The quadratic costs could be modified to the linear-quadratic $(\alpha_0|u| + \alpha u^2)dt$, where α_0 is a positive constant, by assuming that the first share in a market order is traded above in the case of an asking price and below in the case of a bid price, then the market price. Furthermore, assuming that coefficient α is very small makes transaction costs linear under the assumption that investors cannot trade arbitrarily large volumes instantly. The linear transaction costs are commonly used to model the losses from bid-ask spreads. Changing the costs from quadratic to linear-quadratic may have strong effects on trading volume, but, as shown in Isaenko (2024), will impose rather small consequences for stock returns. The origin of the last prediction lies in the observation that the liquidity provider wants to trade at a very high frequency, but the trading demand from the liquidity traders reduces by so much that the liquidity provider becomes marginal, and her stock holdings explain the stock returns.[18] The emphasis on stock returns and tractability of the equilibrium explains the use of quadratic costs.

Lastly, IS and Heston and Hu (2024) show that the risk premium in the presence of quadratic costs is not uniform with respect to taking the limits $\alpha \to 0$ and $\Delta t \to 0$, where Δt is the time between trades, and the limit $\alpha \to 0$ has always to be taken first. It follows that the predictions of continuous-time approximation hold for a chosen coefficient α only if investors trade frequently enough. Appendix B uses a single-period trading model and shows that the continuous-time model's predictions hold for the coefficients α values far below those utilized in this article as long as agents trade daily. In practice, most professional investors update their portfolios at least once per day.

2.3 Portfolio Choices

The liquidity trader (agent 1) and the liquidity provider (agent 2) maximize their expected utility functions that support intermediate consumption over an infinite time horizon. Each

[18]This logic also holds in the case of fixed transaction costs for the liquidity trader, suggesting a small difference between stock returns even when transaction costs change between the linear, linear-quadratic, and fixed. This conjecture remains to be verified.

agent has a utility function in which the coefficient of absolute risk aversion decreases with time and is equal to $\gamma_i e^{-\delta t}$, $i = 1, 2$, where γ_i is a positive constant.[19] The portfolio selection problem for agent i in the trending economy is

$$\max_{c_i^o, z_i \in R^2} E_0 \int_0^\infty \left(-\exp(-\zeta_i t - \gamma_i e^{-\delta t} c_{it}^o) \right) dt \tag{7}$$

$$\begin{aligned} dX_{it}^o &= [rX_{it}^o + N_{it}\, RP_t^o - c_{it}^o - \chi_i^o (u_{1t})^2]dt \\ &+ Y_{it}^o dW_0 + N_{it}(\sigma_{S_D^o t} dW_{Dt} + \sigma_{S_Y^o t} dW_{Yt}), \quad i = 1, 2, \end{aligned} \tag{8}$$

where ζ_i is a time discount rate, $RP^o = \mu_S^o - S^o r$ is the risk premium, $z_1 = u_1$, $z_2 = N_2$, $\chi_1^o = \alpha e^{\delta t}$, $\chi_2^o = \psi e^{\delta t}$, while u_1 and X_i^o stand for the liquidity trader's rate of trading and wealth of agent i. The initial wealth of each agent is X_{i0}.

Assuming δ-note as numeraire reduces the portfolio selection problem for agent i to

$$\max_{c_i, z_i \in R^2} E_0 \int_0^\infty \left(-\exp(-\zeta_i t - \gamma_i c_{it}) \right) dt \tag{9}$$

$$\begin{aligned} dX_{it} &= [(r - \delta)X_{it} + N_{it}\, RP_t - c_{it} - \chi_i (u_{1t})^2]dt \\ &+ Y_{it} dW_0 + N_{it}(\sigma_{S_D t} dW_{Dt} + \sigma_{S_Y t} dW_{Yt}), \quad i = 1, 2, \end{aligned} \tag{10}$$

where all processes denominated in dollars are changed to the processes in δ-notes by dropping label " ^{o} " and $RP = \mu_S - S(r - \delta)$ is the de-trended risk premium. It follows that the portfolio selection problem for each agent in the growing market becomes identical to this problem in the non-trending market where an agent has a constant absolute risk aversion γ_i. I will solve the latter problem and call γ_i erroneously as the coefficient of absolute risk aversion for the expositional convenience.

The problem faced by agent i is solved in a dynamic programming approach. For tractability, I assume that each individual liquidity trader has the same initial allocation to the stock market. This implies that each liquidity trader trades at an identical rate and has an identical allocation in the stock market. This assumption allows the removal of individual allocations from the list of state variables of a liquidity trader. Equations (3)-(6), and (10) suggest that the state variables for agent i's problem should include X_i, D, Y, and N_1. Furthermore, the de-trended dividend rate should be removed from this list due to the linearity of the stock price with respect to D, to be confirmed below. The dynamic programming problem is formulated in Appendix A. Solving this problem will result in the indirect utility function of agent i, $V_i(t, X_i, N_1, Y)$. Appendix A shows that

$$V_i(t, X_i, N_1, Y) = -\exp[-\zeta_i t - (r - \delta)\gamma_i X_i] \exp(g_i(N_1, Y)), \tag{11}$$

[19] The intuition behind this choice of preferences is as follows. It is known that investors with constant coefficients of absolute risk aversion support an equilibrium in the stock market where the standard deviation of stock returns is stationary. Supporting the equilibrium where the stock price returns have the standard deviation increasing over time at rate δ can only be possible if the investors become more risk tolerant at the same rate δ over time.

where function g_i is linear-quadratic in the state variables. The presence of transaction costs forces a liquidity trader to be in a state in which her expected utility is not maximal. Therefore, the liquidity trader would like to change her stock allocations even in the absence of the endowment volatility shocks in the economy. She trades in the direction of her target allocation $\hat{N}_1(Y)$ where the indirect utility function is maximal in a given state of the market.

Finally, to prevent agents from implementing a "Ponzi scheme", I impose the constraint on their policies:

$$\lim_{t\to\infty} E_0[-\exp[-\zeta_i t - (r-\delta)\gamma_i X_i]] = 0, \quad i = 1, 2. \tag{12}$$

This constraint is verified in equilibrium for each agent.

It is assumed that professional investors observe the dividend flow, the stock price, and the order book. Therefore, they can quickly learn the aggregate endowment volatility, under rational expectation, in the case if they do not observe it. Consequently, a complete information is assumed. A more general case where investors of a given type have different initial allocations and endowments, do not observe the aggregate stock allocations of liquidity traders and the aggregate endowment, with arrivals of new investors with different prior beliefs is left for future research.

3 Equilibrium

This section presents a rational expectations equilibrium.

Definition 1 *An equilibrium is a price system* (μ_S, σ_S) *and a set of trading strategies* (u_1, N_2) *of liquidity traders and liquidity suppliers, respectively, such that (i) individual agents choose their optimal portfolio strategies and (ii) stock market clears, that is* $\forall t \in [0, \infty)$

$$\lambda N_{1t} + (1-\lambda) N_{2t} = \frac{1}{2}.^{20} \tag{13}$$

Solving the equilibrium requires finding functions g_i, $i = 1, 2$ in (11) simultaneously with the stock price subject to the constraint (13). Appendix A shows that the stock price can be written as $S(D, N_1, Y) = \frac{D}{\nu+\kappa_D} + H(N_1, Y)$, where $\nu = r - \delta$. It follows that finding the equilibrium requires a simultaneous solution of the three partial differential equations (PDEs) for three functions of the two state variables with the market clearing constraint. These three functions are shown to be linear-quadratic in state variables allowing to reduce the three PDE's to the system of nonlinear algebraic equations. Appendix A proves the following proposition which provides the solution of the equilibrium with transaction costs.

Proposition 1 *Assume that the system of equations (A-9)-(A-14), (A-19)-(A-24), (A-27)-(A-29), (A-31)-(A-33) has a solution with respect to constants* R_0, R_1, R_2, H_0, H_1, H_2, A_0^i, A_1^i, A_2^i, B_0^i, B_1^i, C_0^i, $i = 1, 2$. *Then a linear–quadratic equilibrium exists. In*

this equilibrium the de-trended stock price, the conditional risk premium and volatilities of de-trended returns, functions $g_i\,(i=1,2)$*, the de-trended consumption rates, the trading rate of the liquidity trader, and her target stock allocation are given by*

$$\begin{aligned} S &= \frac{D}{\nu+\kappa_D} + H_0 + H_1 N_1 + H_2 Y, && (14)\\ RP_c &= R_0 + R_1 N_1 + R_2 Y, && (15)\\ \sigma_{S_D} &= \frac{\sigma_D}{\nu+\kappa_D},\ \ \sigma_{SY} = \sigma_Y H_2, && (16)\\ g_i &= A_0^i + A_1^i N_i + \frac{1}{2} A_2^i (N_i)^2 + (B_0^i + B_1^i N_i) Y + C_0^i Y^2, && (17)\\ c_i &= -\frac{1}{\gamma_i}[\ln(\nu) + g_i] + \nu X_i, && (18)\\ u_1 &= \xi(\hat{N}_1 - N_1), && (19)\\ \hat{N}_1 &= -\frac{A_1^1 + B_1^1 Y}{A_2^1}, && (20) \end{aligned}$$

where $N_2 = \frac{1/2-\lambda N_1}{(1-\lambda)}$, $\xi = \frac{A_2^1}{2\alpha\nu\gamma_1}$.

This equilibrium will be analyzed in the following sections.[21] The following corollary proves the existence and uniqueness of the linear-quadratic equilibrium in the limit of a very small value of parameter σ_Y. The existence and uniqueness of the equilibrium for other values of σ_Y is verified numerically.

Corollary 1 *Assume that volatility* σ_Y *becomes small enough for all contributions to the equilibrium coefficients proportional to* σ_Y^2 *to become negligibly small, then the unique linear-quadratic equilibrium exists. In this equilibrium, the conditional risk premium and the long-term expected stock allocation of the liquidity trader are equal to*

$$\begin{aligned} RP_c &= \nu\gamma_2(\sigma_S^2\, N_2 + \beta_2 \sigma_S\, Y), && (21)\\ \bar{N}_1 &= \frac{\gamma_2}{2[(1-\lambda)\gamma_1 + 2\lambda\,\gamma_2]}, && (22) \end{aligned}$$

where $\sigma_S = \sigma_{S_D}$ *and* $\beta_2 = -\frac{\lambda}{1-\lambda}$.

Appendix A shows that the system of equations (A-9)-(A-14), (A-19)-(A-24), (A-27)-(A-29), (A-31)-(A-33) could be solved in the closed form in the limit of a very small value of parameter σ_Y. The proof of Corollary 1 contains the other closed-form expressions for the equilibrium coefficients. Equation (21) shows that the liquidity provider is a marginal investor and her stock holdings and the exposure to the volatility risk define the conditional stock returns. Similarly, a single-period model in Appendix B predicts that the conditional

[21] It should be noticed that this equilibrium cannot be obtained from the equilibrium in IS by taking the limit of transaction costs paid by agent 2 to zero due to the non-uniformity of the equilibrium with respect to this limit and the limit of continuous-time trading.

dollar risk premium in the case of a short investment horizon should be $RP = \nu\gamma_2\sigma_S^2 N_{20}$, where N_{20} is the stock allocation of the liquidity supplier at the beginning of the period. This result confirms that the liquidity provider is a marginal investor independent of the nature of the trading incentives. This result also leads to an asymmetric feedback effect between the stock allocations of the liquidity traders and the conditional risk premium. Indeed, suppose that the allocation of an individual liquidity trader, N_i, decreases along with the allocations of the representative liquidity trader. This leads to a larger conditional risk premium. A smaller risk exposure and a higher risk premium make liquidity traders better off inciting further sales of stock shares. This positive feedback effect continues until the risk premium received by a liquidity trader, $N_i RP_c$, starts decreasing and the risk-return tradeoff starts deteriorating. On the other hand, taking a longer position in the stock market by a liquidity trader along with other liquidity traders leads to a smaller risk premium and a higher risk exposure. This negative feedback effect incites a liquidity trader to increase her position in the stock market only ever slightly. The asymmetry in the feedback effects results in excessive average stock allocations of the liquidity providers. That is, should the coefficients of risk aversions and the weights of the two agents be equal, the liquidity providers would still be more exposed to the market risk than the liquidity traders. Related, equation (22) establishes that the liquidity provider will not reduce her stock exposure to zero even if she becomes very risk-averse. In this limit, the stock holdings of the liquidity trader and the liquidity provider converge to $\frac{1}{4\lambda}$ and $\frac{1}{4(1-\lambda)}$, respectively. The stock holdings of the liquidity providers do not converge to zero due to an increasing risk premium.

The predictions of the model will be compared with the benchmark case in which transaction costs are absent. The benchmark case is solved in IS under more general settings and is presented here as a corollary.

Corollary 2 *In the market without transaction costs the conditional de-trended risk premium, the stock price, functions g_i, $i = 1, 2$, and the stock allocations of the agents, are given by*

$$
\begin{aligned}
RP_c &= \frac{\sigma_S^2}{2(\frac{\lambda}{\nu\gamma_1} + \frac{1-\lambda}{\nu\gamma_2})}, && (23)\\
S &= \frac{D}{\nu + \kappa_D} + \frac{1}{\nu}\left(-RP + \frac{\kappa_D \bar{D}}{\nu + \kappa_D}\right), && (24)\\
g_i &= B_i Y + A_i, && (25)\\
N_i &= \frac{RP}{\nu\gamma_i\sigma_S^2} - \frac{\beta_i}{\sigma_S} Y, \quad i = 1, 2, && (26)
\end{aligned}
$$

where $\sigma_S = \frac{\sigma_D}{\nu+\kappa_D}$, $B_i = RP\frac{\nu\gamma_i\beta_i}{(\nu+\kappa_Y)\sigma_S}$, $\beta_1 = 1$, $\beta_2 = \frac{\lambda}{1-\lambda}$, *and* $A_i = 1 - \ln\nu - \frac{\zeta_i}{\nu} - \frac{RP^2}{2\nu\sigma_S^2} + \frac{(B_i\sigma_Y)^2}{2\nu}$.

Corollary 33 implies that the economic agents can hedge the endowment risk perfectly, leading to the conditional risk premium of stock returns being constant and the stock price which depends on the dividends only.

The following lemma helps to establish the relations between the risk premiums and the stock prices across the markets with and without transaction costs.

Lemma 1 *Assume that the stock price is de-trended, that* $\frac{\bar{S}}{\bar{D}} = \omega$, $RP = m\,RP_0$, *and that* $\bar{S} = \chi \bar{S}_0$, *where* RP, $\bar{S}$ *and* RP_0, $\bar{S}_0$ *are the unconditional risk premiums and the expected stock prices in the presence and the absence of transaction costs, respectively. Then*

$$RP = \bar{D} - \nu \bar{S}, \tag{27}$$

$$\chi = \frac{\nu\omega\, m}{m + \nu\omega - 1}. \tag{28}$$

It follows from equation (27) that the risk premium is equal to the dividend received from the stock over the interest on the borrowed stock price. This equation also shows that an increase in the risk premium leads to a decrease in the average de-trended stock price. Equation (28) allows me to determine by how much the stock price decreases when the transaction costs are introduced for the empirically observed ratio ω. A decrease in the stock price has a leading impact on the rates of stock returns. It follows from the Corollary 1 that the risk premium of the de-trended stock returns in the limit of small σ_Y is

$$RP = \nu\gamma_2\sigma_S^2 \frac{(1-\lambda)\gamma_1 + \lambda\gamma_2}{2(1-\lambda)[(1-\lambda)\gamma_1 + 2\lambda\gamma_2])}. \tag{29}$$

Combining the last result with the prediction of Corollary 33 suggests that in the limit of small σ_Y

$$m = \frac{[(1-\lambda)\gamma_1 + \lambda\gamma_2]^2}{\gamma_1(1-\lambda)[(1-\lambda)\gamma_1 + 2\lambda\gamma_2])}. \tag{30}$$

It follows that the ratio of the coefficients of absolute risk aversions and the weight of the liquidity traders fully explain the increase in the risk premium of the detrended stock returns caused by transaction costs.

The presence of transaction costs leads to the short-term stock price reversal controlled by parameter ξ. The price reverts when the investors trade. Should the transaction costs be linear, they trade only outside the no-trade zone. Considering the trading volume, the quadratic transaction costs could be used as a proxy for the superposition of the linear transaction costs in a market populated with heterogeneous liquidity traders whose no-trade zones have a U-shaped distribution with the common utility-maximizing allocations. Consequently, the stock is always traded and the volume increases as the state variables move away from the optimal allocations. The smaller the coefficient α in quadratic costs, the smaller the average constant of proportionality in the linear transaction costs that are proxied by quadratic costs.

The parameter ξ could be found in terms of the standard deviation of stock returns.

Corollary 3 *Assume that coefficient* α *is small, then parameter* ξ *is approximated as*

$$\xi \approx \frac{\sigma_S}{\sqrt{2\alpha}} \sqrt{\nu\left(\gamma_1 + 2\gamma_2 \frac{\lambda}{1-\lambda}\right)}. \tag{31}$$

It follows that the stock price reversion slows down with higher transaction costs, less risky stock returns, lower risk aversions of investors, and lower weight of the liquidity traders. I verify that the difference between values of ξ predicted by formula (31) and found in the numerical analysis is negligible.

The following lemma establishes the autocorrelation coefficient for the de-trended stock returns between times t and t' as well as the comovement and the first-order autocorrelation between the stock allocations of the liquidity provider and the de-trended stock returns.

Lemma 2 *Consider the stock holdings and the de-trended stock returns over a small interval Δt for times t and t', such that $t' > t + \Delta t$. Then*

$$\rho(t,t') \equiv \frac{Cov_t(\Delta S_t + \int_t^{t+\Delta t} D_s ds, \Delta S_{t'} + \int_{t'}^{t'+\Delta t} D_s ds)}{\sqrt{Var_t(\Delta S_t + \int_t^{t+\Delta t} D_s ds) Var_{t'}(\Delta S_{t'} + \int_{t'}^{t'+\Delta t} D_s ds)}}$$
$$= \left(H_2 \sigma_Y^2 \left[-\kappa_Y \left(\frac{H_1 \xi b}{\xi - \kappa_Y} + H_2 \right) e^{-\kappa_Y (t'-t)} + \frac{H_1 \xi^2 b}{\xi - \kappa_Y} e^{-\xi(t'-t)} \right] + \frac{\nu \sigma_D^2}{(\nu + \kappa_D)^2} e^{-\kappa_D (t'-t)} \right) \frac{\Delta t}{\sigma_S^2}$$
$$+ o(\Delta t). \tag{32}$$

$$Cov_t(\Delta N_{2t}, \Delta S_t + \int_t^{t+\Delta t} D_s ds) = -\frac{\lambda}{2(1-\lambda)} b \xi \sigma_Y^2 H_2 \, \Delta t^2 + o(\Delta t^2), \tag{33}$$

$$Cov_t(\Delta N_{2t}, \Delta S_{t+\Delta t} + \int_{t+\Delta t}^{t+2\Delta t} D_s ds) = -\frac{\lambda}{2(1-\lambda)} (H_1 b \xi - H_2 \kappa_Y) b \xi \sigma_Y^2 \Delta t^3$$
$$+ o(\Delta t^3), \tag{34}$$

where $\sigma_S = \sqrt{\sigma_{SD}^2 + \sigma_{SY}^2}$, $b = -\frac{B_1^1}{A_2^1}$ and a change in a process M_t over time Δt is $\Delta M_t = M_{t+\Delta t} - M_t$.

The autocorrelation has a term proportional to $e^{-\xi(t'-t)}$ which arises due to delays in stock allocations due to transaction costs by the liquidity traders. The sign of this term is defined by the sign of the factor $\frac{H_1 H_2 b}{\xi - \kappa_Y}$ to be determined below. The contribution to the autoregression from the endowment volatility is defined by the factor $-H_2(\frac{H_1 \xi b}{\xi - \kappa_Y} + H_2)$. It emerges from the mean reversion of the endowment volatility and the slow trading of the liquidity traders. Finally, the contribution to the autocorrelation coefficient from the stock dividends appears positive even though the dividends are mean-reverting. The latter is due to the future dividends included in the de-trended stock returns. The contribution of the dividends is negative for the autocorrelation coefficient between changes in the de-trended stock prices. On the other hand, the right sides of equations (33) and (34) do not change if the de-trended stock returns on the left side are replaced with changes in the de-trended stock price.

The comovement of the stock holdings of the liquidity provider and the stock returns is defined by the uncertainty in the target allocations of the liquidity traders and the uncertainty in the contemporary stock price. The first uncertainty is controlled by parameter

b, while the second one depends on parameter H_2. On the other hand, the autocovariance depends on how uncertainty in the stock holdings today translates into the risk premium which then channels into uncertainty in the stock prices after time Δt. It follows that the sign of the comovement between the excess return and the stock allocation of the liquidity provider is defined by the sign of the product $-bH_2$, while the sign of their autocovariance is defined by the product $(H_1 b\xi - H_2\kappa_Y)b \approx R_2 b$. The letter is also smaller than the former by the order of magnitude of Δt.

3.1 Model Calibration

Table 1 presents the values of parameters used in the model. Most of these values are chosen to target certain market outputs. The Table shows the values for these outputs predicted by the model and also found empirically. Except for the bid-ask spread, the empirical values for the outputs that are not observed directly or have insufficient data are shown in the brackets.

The calibration of the economy is linked to the study by CK, who analyzes the behavior of the de-trended dividend and the de-trended real stock prices. The stock returns are influenced the most by the risk aversion of liquidity providers and their weight in the economy. The effects on the returns from each of these characteristics are very similar and this study emphasizes the role of the observable weight of the liquidity providers. The coefficients of absolute risk aversion are set to 9.73 and the weight of liquidity providers is set to 0.25 ($\lambda = 0.75$). These values are chosen to make the Sharpe ratio of the trending stock returns equal to 0.37[22] and, as shown below, result in the average coefficient of relative risk aversion of the investors equal to 5. Furthermore, the markets with the same Sharpe ratios but with other weights and the absolute risk aversions of agents are considered below. These markets have very similar statistical moments of the rate of stock returns. I further find that the ratio m of the risk premiums in the market with and without transaction costs is defined by the weight of liquidity traders and is only weakly influenced by the other parameters, suggesting a robust decrease in the average stock price once the transaction costs are introduced.

Matching the losses from the transaction costs with $\alpha = 0.01$ with those when the bid-ask spread is independent of the traded amount suggests a fixed bid-ask spread of no more than 0.059% of the traded dollar amount.[23] This bid-ask spread is consistent with

[22]See, for example, Linnainmaa and Roberts (2018), for the historical values of the S&P index Sharpe ratio.

[23]The value of coefficient α corresponds to the transitory price impact on trading 0.01 stock shares equal to 0.04% of the current stock price. Indeed, the expected annual trading rate of the liquidity trader in equilibrium is approximately 0.5 shares per year implying that she trades about 0.01 stock shares weekly. The quadratic transaction costs model the losses from the transitory impact on the stock price. The stock price faced by an investor who trades ΔN shares could be written as $S(1+\tilde{\alpha}\Delta N)$, implying that $\tilde{\alpha} = 2\frac{\alpha}{S\Delta t}$. Assuming that an investor trades every week, that $\Delta N = 0.01$, and using $\lim_{t\to\infty} E_0(S_t^{-1}) = 0.039$ from the analysis below suggest that $\tilde{\alpha}\Delta N = 0.00039$.

contemporary markets. Furthermore, a few other values for coefficient α are considered below. Changing α has a small effect on the stock prices but a significant impact on the trading volume. The value of ψ assumes that the opportunity costs faced by the liquidity provider make up two-thirds of the cash inflow received from bid-ask spreads. It will be shown that increasing ψ and, therefore, opportunity costs, will raise the liquidity premium and the risk of stock returns, but only insignificantly. The values of ζ_1 and ζ_2 are set to make the de-trended expected portfolio values drift-less implying that the weight of a stock position in the investment portfolio is steady over time for each agent.[24] The choices of ζ_1 and ζ_2 have no implications for the stock returns. The value of κ_Y makes the reversion in the endowment volatility last for a few years. Assuming that this reversion is aligned with the market cyclicality, this implies that the business cycle lasts at least 6.3 years.[25] The value of σ_Y leads to a portfolio turnover of investors which is consistent with historical observations.

Parameters κ_D, $\bar{D}$ and σ_D are set to match the empirical estimates of the processes for the de-trended dividend rate and the stock price determined by CK who analyze the real Standard and Poor Composite Stock Price Index in the period 1871-1986 by using annual observations. CK finds that $\sigma(\Delta D)/\bar{D} = 0.12$, $\bar{S}/\bar{D} = 20$, and $\sigma(\Delta D)/\sigma(\Delta S) = 0.032$, where ΔS and ΔD are the changes of the de-trended stock price and the de-trended dividend over one year and $\bar{S}$ is the average long-term de-trended stock price. The corresponding predictions of the model are $\sigma(\Delta D)/\bar{D} = \frac{\sigma_D}{\bar{D}}\sqrt{\frac{1-e^{-2\kappa_D}}{2\kappa_D}} = 0.12$, $\bar{S}/\bar{D} \approx 19.0$, and $\sigma(\Delta D)/\sigma(\Delta S) \approx \sigma_D/\sigma_S \approx 0.058$.[26] The biggest difference from the data appears in the ratio σ_D/σ_S. This ratio is bigger than its empirical counterpart partially because, in the model, the risk in the stock price emerges almost exclusively from the dividend risk, while in practice there could be significant contributions to the stock risk arising from other factors.[27]

The initial wealth of each agent affects neither the stock returns nor trading strategies and is set to 10.[28] This amount and the absence of a trend in the values of agents' portfolios

[24]The calculations of ζ_1 and ζ_2 are carried in closed forms in terms of equilibrium coefficients presented in Proposition 1. These calculations are not presented for brevity.

[25]The spectrum of the Ornstein-Uhlenbeck process puts more weight on low frequencies with a cut-off frequency of $\kappa_y/(2\pi) = 0.16$. See Wang and Uhlenbeck (1945). This frequency implies that the length of the business cycle is 6 years or longer.

[26]The approximation for $\sigma(\Delta D)/\sigma(\Delta S)$ takes into account that the dominant contribution to the stock risk comes from dividends.

[27]Campbell and Kyle (1993) reject the null hypothesis of a unit root in the Dickey-Fuller test at the 10% level for the de-trended dividend rate, suggesting that it is close to being stationary. The autocorrelation coefficients at different lags reported in Campbell and Kyle (1993) indicate that coefficient κ_D should be noticeably bigger than 0.05. Making it bigger, increases the ratio σ_D/σ_S, making me choose κ_D to be rather small. Nonetheless, I consider, without reporting, much bigger values for κ_D and find that the ratio between the risk premiums in the market with and without transaction costs, m, remains the same as long as $\lambda = 0.75$. This implies that the ratio of the average stock prices in the market with and without transaction costs, χ in equation (28), also remains the same.

[28]The amount of the initial wealth is explained by the total number of outstanding shares per investor, which is set to 0.5, and the average stock price, which is equal to 30.5. The calibration of the model could be renormalized by a different number of outstanding stock shares per investor, M, with the following

Parameter	Value	Targeted outputs	Model	Data
γ	9.73	Coefficients of RRA	5	$]0, 10]$
λ	0.75	Sharpe ratio	0.37	0.35 ± 0.1
r	0.023	Real interest rate	2.3%	$1.9 \pm 5.0\%$
α	0.01	Bid-ask spread	0.06%	$[0.01\%, 0.2\%]$
ψ	-0.01	NA	NA	$[-0.03, 0.1]$
κ_Y	1.0	Market cycle (years)	6.3	$[3.0, 10.0]$
σ_Y	1.4	portfolio turnover	82%	93%
ζ_1	0.016	NA	NA	[0.01,0.1]
ζ_2	0.023	NA	NA	[0.01,0.1]
$\bar{D}$	1.604	$\bar{S}/\bar{D}$	19	20
σ_D	0.2	$\sigma(\Delta D)/\bar{D}$	0.12	0.12
κ_D	0.05	$\sigma(\Delta D)/\sigma_S$	0.058	0.032
X_{10}, X_{20}	10	Weights of the stock allocations	0.91, 3.38	$[0.5, 20]$

Table 1: The model parameters and the values of the targeted moments. The term "NA" stands for not applicable.

allows me to estimate the long-term average coefficients of their relative risk aversions:

$$\begin{aligned} & \lim_{t\to\infty} E_0(CRRA_i(t)) = \lim_{t\to\infty} CARA_i(t) E_0(c_i^o(t)) = \gamma_i \lim_{t\to\infty} E_0(c_i(t)) \\ = \; & -\ln\nu - \lim_{t\to\infty} E_0(g_i(t)) + \nu\gamma_i \lim_{t\to\infty} E_0(X_i(t)) = -\ln\nu - \lim_{t\to\infty} E_0(g_i(t)) + \nu\gamma_i X_{i0}, \end{aligned}$$

where $CRRA_i$ and $CARA_i$ are the conditional coefficients of relative and absolute risk aversions of agent i. I find that the expected coefficients of relative risk aversions of the liquidity trader and the liquidity provider are 5.35 and 5.40, respectively. As shown in Section 3.3 below, the chosen initial value of the agent's portfolios also implies that the average stock holdings of the liquidity trader make 91% of her average portfolio and for the liquidity provider, this ratio is 338%.[29] It follows that increasing the initial wealth of each agent by 1 makes each agent more relatively risk-averse by 0.097, while the weight of the stock of the liquidity trader falls by 8.2% and it falls by 30.5% for the liquidity supplier.

The rates of return over time interval τ in the de-trending stock market is

$$R_S(\tau) = \lim_{t\to\infty} \frac{S_{t+\tau} + \int_t^{t+\tau} D_s ds - S_t}{S_t}, \tag{35}$$

where the stock price is assumed to be positive. If τ is small then $R_S^o(\tau) \approx R_S(\tau) + \delta\tau$, where $R_S^o(\tau)$ is the rate of return of the trending stock. This implies that the risk premium on the

replacements: $X_i \to 2X_iM$, $c_i \to 2c_iM$, $\gamma_i \to \gamma_i/(2M)$, $N_i \to 2N_iM$, $\alpha \to \alpha/(2M)$, $\beta_i \to 2\beta_iM$. The stock market clearing condition should be changed to $\lambda N_1 + (1-\lambda)N_2 = M$. These changes will not have any impact on stock returns.

[29] The liquidity provider controls her shareholdings rather than the portfolio fraction allocated to the stock. Should the investor have constant relative risk aversion and the stock returns have a normal distribution, her coefficient of the relative risk aversion would be 0.5 based on the moments of the rates of stock returns reported in Section 3.2.

short-term rate of return of the de-trended stock price is smaller than the short-term rate of return of the observed price by δ, while the volatilities on the rates of return of the two stock prices are the same. Assuming the Gaussian distribution for the dividend rate renders a small probability of negative stock prices which could be interpreted as the probability of company bankruptcy. In calculations of the rates of returns, I use simulations and follow the practice of excluding " penny " stocks, by replacing a stock path as soon as it falls below 1 with another path starting at 1.[30,31,32]

3.2 Conditional Moments

By using the calibration from section 3.1, I find the following equilibrium processes in the market with no transaction costs:

$$\begin{aligned} RP_c &= 0.54, \quad S = \frac{D}{0.06} + 79.65, && (36)\\ g_1 &= 2.68 + 0.016\,Y, \quad g_2 = 2.21 - 0.047\,Y, && (37)\\ N_1 &= 0.5 - 0.3\,Y, \quad N_2 = 0.5 + 0.9\,Y; && (38) \end{aligned}$$

and in the market with transaction costs:

$$\begin{aligned} RP_c &= 2.30 - 3.36\,N_1 - 1.00\,Y, && (39)\\ S &= \frac{D}{0.06} + 3.73 + 0.17\,N_1 + 0.45\,Y, && (40)\\ g_1 &= 0.23 - 0.011\,N_1 + 0.038\,(N_1)^2 + (0.036 + 0.0062\,N_1)Y - 0.0026\,Y^2, && (41)\\ u_1 &= 19.78(\hat{N}_1 - N_1), \quad \hat{N}_1 = 0.30 - 0.16\,Y, && (42)\\ g_2 &= 0.17 - 0.00068\,N_2 - 0.0024\,(N_2)^2 - (0.062 + 0.00031\,N_2)Y + 0.016\,Y^2. && (43) \end{aligned}$$

Predictability of stock returns.

The sign of the factor $\frac{H_1 H_2 b}{\xi - \kappa_Y}$, which defines the contribution of the slow, due to transaction costs, trading of the liquidity traders to the autocorrelation coefficient between the de-trended stock returns is negative implying the stock price reversal. This reversal is guided by the rate of the convergence of the liquidity trader to her current target allocation given by parameter $\xi \equiv \frac{A_2^1}{2\alpha\nu\gamma_1} = 19.78$ and is, therefore, short-term. The presence of short-term reversals in the stock price is confirmed empirically by Lehmann (1990) and Jegadeesh (1990).

[30] A few other small values of the cut-off price are considered with rather small differences in the moments on the rates of stock returns. Specifically, changing this price from 1 to 0.5 (or 1.5) makes the risk premium on the rates of returns increase (or decrease) by about 1% of its value and the standard deviation increase (or decrease) by about 5% of its value.

[31] The simulations show that the probability for the long-term stock price to be below 1 is 0.24% and the probability for the long-term positive stock price which is above 1 to fall below 1 over one year is 0.23%.

[32] Alternatively, the probability for the stock price to become negative could be removed by choosing an almost sure positive process, for example the Cox-Ingersoll-Ross process, for the de-trended dividend rate. This choice, however, requires the numerical solution of the equilibrium.

The contribution to the autocorrelation coefficient from the endowment volatility is defined by the sign of the factor $-H_2(\frac{H_1\xi b}{\xi-\kappa_Y}+H_2)$, which is also negative. This contribution is due to the mean-reversion of this volatility which is weakened by the liquidity trader's trading against changes in Y_t discussed next.

For ease of interpretation, suppose that endowment volatility shocks arrive at daily time intervals rather than continuously, as assumed in the model, while agents trade at a higher frequency than daily. Equations (40) and (42) suggest that a positive shock in the endowment volatility increases the stock price and decreases the target allocation, $\hat{N}_1$, for the liquidity trader causing her to sell stock shares. Consequently, the liquidity trader trades against instantaneous changes in the stock price arising from variations in process Y and takes advantage of a corresponding predictability in the stock prices. This predictability is defined by the reversal coefficient κ_Y and is, therefore, long-term. Given that the term proportional to allocation N_1 in equation (40) enters this equation with a positive sign, the gradual decrease in the stock allocations of the liquidity trader following a positive shock in Y, adjusts the stock price downwards introducing a reversal in the stock price. The stock price reverts slowly, while liquidity traders continue to sell their stock shares. Consequently, liquidity providers trade against slow market adjustments and their inventory should have a positive correlation with a contemporaneous stock price. This conclusion is confirmed by a positive sign of the covariance between changes in the stock holdings of the liquidity provider and the stock price found in equation (33). Furthermore, a positive price shock today leads to a longer inventory for liquidity providers at the end of the day and decreases the risk premium which in turn makes a smaller stock price and a bigger risk premium tomorrow more likely. Consequently, an increase in the inventory of liquidity providers over today predicts a decrease in the stock price over tomorrow. This conclusion is confirmed by a negative sign of the covariance between changes in the stock holdings of the liquidity provider and the stock price the next day, as found in equation (34). The last prediction as well as a positive correlation of the liquidity provider inventory with the contemporaneous stock price are both consistent with the existing empirical findings for liquidity provider inventories.[33]

Given a low sensitivity of the standard deviation of stock returns with respect to the weights of agents and coefficient α, to be confirmed below, equation (31) shows that the short-term mean reversion of the stock rises with a smaller weight of the liquidity providers and smaller transaction costs. The short-term reversal in the stock price benefits the liquidity provider who trades against it. The comparison of equations (37) with equations (41) and (43) shows that the long-term and the short-term stock return predictabilities, which are not observed in the market without transaction costs, make both agents better off in the economy with transaction costs.[34]

[33]See Hendershott and Seasholes (2007).

[34]The range of the values of state variable Y is found from the Orhstein-Ukhenbeck distribution, while

The positive feedback effect and small σ_Y.

Equation (40) suggests that $\sigma_{S_D} = 3.33$ and $\sigma_{SY} = 0.63$ implying that most of the risk in the stock emerges from the dividends. A small contribution of the endowment volatility to the standard deviation of stock returns suggests that the analytical solution of the equilibrium in the limit of small parameter σ_Y could be used as a reasonable approximation for the stock returns and allocations. In this approximation equation (21) can be written as

$$RP_c = \frac{\nu\gamma_2\sigma_S^2}{2(1-\lambda)} - \nu\gamma_2\,\sigma_S^2\,\frac{\lambda}{1-\lambda}\,N_1 + \beta_2\nu\gamma_2\sigma_S\,Y. \tag{44}$$

Under the calibration of section 3.1, the conditional risk premium (44) becomes

$$RP_c = 2.16 - 3.24\,N_1 - 0.97\,Y, \tag{45}$$

which is reasonably close to equation (39). The second term in equation (44) arises due to the inability of liquidity traders to change their stock allocations instantly in the presence of transaction costs and causes a feedback effect between their stock allocations and the risk premium. This feedback effect also propels the short-term price reversal explained above.[35] Furthermore, this effect is positive with respect to a decrease in the stock allocations of the liquidity traders and negative with respect to an increase. The positive feedback effect is strong even at a small trading volume. Indeed, decreasing σ_Y reduces the trading volume linearly, while the conditional risk premium barely changes. The conditional risk premium is explained by the feedback effect in the presence of heterogeneity in trading demands. The latter is comprised of infinitesimal amounts in comparison with the aggregate stock holdings. Furthermore, the feedback effect remains strong even if transaction costs are small conditioned that the trading frequency is high if investors trade at discrete times.

Stock return cyclicality.

I assume that the aggregate rate of consumption, $C = \lambda c_1 + (1-\lambda)c_2$ is perfectly aligned with the business cycle and, therefore, the correlation of its short-term changes with changes of the stock market moments of returns allows me to determine the short-term cyclicality of their movements. Appendix C shows that the aggregate consumption increases with the positive shocks in the dividend and the endowment volatility, where the letter plays a much less significant role than the former. Given that most of the risk in the stock emerges from the dividends, the short-term correlation between the aggregate consumption and the stock price is positive and is defined by the dividend risk. Assuming that τ is small in equation

the distribution of state variable N_1 also showing in functions g_i, $i = 1, 2$, is derived in IS .

[35]It should be noticed that the asymmetry in the feedback effects does not result in the asymmetry in the autocorrelation between the stock returns due to the linearity of the stock price and the trading rate of the liquidity traders.

(35), the annualized conditional risk premium on the rate of the de-trended stock return becomes

$$RP_{rt} = \frac{RP_{ct}}{S_t}, \tag{46}$$

where RP_c and S are given by equations (39) and (40), respectively. The last equation shows that the conditional risk premium on the rate of return increases with negative shocks in the dividend flow and the endowment volatility. Consequently, the rates of returns are counter-cyclical on average. The premium could be pro-cyclical in rare states where the stock price is significantly overpriced so that the dollar risk premium is negative. These states occur where the liquidity trader is too long in the stock for the current value of the endowment volatility. Equation (35) implies that, in the case of small τ, the annualized conditional standard deviation of the rate of stock return is approximately equal to

$$\sigma_{Rt} = \frac{\sigma_S}{S_t}. \tag{47}$$

The last equation shows that, like in the case of the conditional risk premium, the conditional standard deviation of the rates of stock returns increases with negative shocks in the dividends and the endowment volatility. Therefore, the risk in the conditional standard deviation is counter-cyclical. Finally, combining equations (46) and (47) suggests that the short-term annualized conditional Sharpe ratio on the rates of stock returns is equal to its counterpart for the dollar returns and is given by $\frac{RP_{ct}}{\sigma_S}$. In the short term, the dollar risk premium rises with a negative shock in the endowment volatility implying that the conditional Sharpe ratio is counter-cyclical.

The counter-cyclical traits of the conditional risk premium, the conditional standard deviation, and the conditional Sharpe ratio of the rates of stock returns are well-documented empirically and are predicted by a number of models.[36] The comovement of the conditional standard deviation and the risk premium of the rates of stock returns appears to be at odds with some empirical literature.[37] As argued by Lettau and Ludvingston (2010), "the disagreement in the empirical literature on the risk-return relation is likely to be attributable, in large part, to the relatively small amount of conditioning information that is typically used to model the conditional mean and conditional volatility of excess stock market returns."

The short-term reversal in the stock price arising from delayed trading of liquidity traders and also the long-term reversions arising from the reversals in the dividend and the endowment risk channel into corresponding reversals in the conditional risk premium and the standard deviation of the rates of stock returns. These reversals result in the superpositions of a few persistences in the risk premium and volatilities: over a few weeks, over a year, and over many years.[38]

[36] See, for example, Campbell and Cochrane (1999) and Guvenen (2009).

[37] See footnote 8.

[38] Eraker, Johannes, and Polson (2003) report that the mean-reversion coefficient in a univariate process

3.3 Unconditional Moments

In this section, I consider a few unconditional characteristics of the stock market and the optimal portfolios of agents. All prices are measured in δ-notes but will be labeled with a dollar sign for expositional convenience.

The unconditional moments of the stock rates of returns are found by using simulations in equation (35),[39] where τ is set equal to 1. Considering the size of the effects of transaction costs, the expected value of these costs might be an important characteristic of the stock market. The annualized trading cost paid by the liquidity trader is found in the formula

$$Cost = \lim_{t\to\infty} \alpha E_0 \int_t^{t+1} u_{1s}^2 ds = \alpha \lim_{t\to\infty} E_0(u_{1t}^2). \tag{48}$$

Apparently, the cost is based on the magnitude of the trading rate of the liquidity trader. The latter, in turn, increases with the rate of convergence of the liquidity trader to the optimal allocations, ξ, and with the uncertainty in the target allocations, $\hat{N}_1$. According to equation (20), the uncertainty in the target allocations increases with the risk in Y. The other two important characteristics of the economy, which are discussed below, are the portfolio turnover averaged across the two agents and parameter ξ. The turnover is defined as

$$\text{PT} = \frac{1}{2}\lim_{t\to\infty}\left(\lambda\frac{E_0|u_{1t}|}{E_0[N_{1t}]} + (1-\lambda)\frac{E_0|u_{2t}|}{E_0[N_{2t}]}\right) = \frac{\lambda}{2}\lim_{t\to\infty} E_0|u_{1t}|\left(\frac{1}{E_0[N_{1t}]} + \frac{1}{E_0[N_{2t}]}\right), \tag{49}$$

where $u_2 = -\frac{\lambda}{1-\lambda}u_1$.[40] This definition assumes that the portfolio turnover is equal to 100% if an agent liquidates all her stock positions and then acquires them to the original level. Parameter ξ defines the time scale of the short-term stock return reversal arising due to delayed trading of the liquidity trader. Specifically, the ratio $\ln(2)/\xi$ gives the time interval over which the magnitude of the autocorrelation coefficient between short-term returns decreases by half. The bigger parameter ξ, the faster the short-term reversal, and the greater the benefits of the liquidity provider.

The moments of stock returns.

Table 2 presents the unconditional expected dollar risk premium and the unconditional expected rate of de-trended stock returns, $E(RP)$ and $E(RP_r)$, the standard deviations of

of the stochastic variance of stock returns in the Heston model is equal to 5.8 corresponding to the stock volatility reversion over a few months.

[39]The standard deviations of the estimates of the moments of stock returns in the market with transaction costs are about 0.03% for the risk premiums and 0.25% for the standard deviations of the stock returns. In the market without costs, these errors are 0.002% and 0.005%, respectively. The errors of estimates are found based on a sample of simulations. The sample contains three simulations and each simulation has 400,000 paths.

[40]I find $\lim_{t\to\infty} E_0|u_{1t}|$ by using simulations, and use the results in IS that $\lim_{t\to\infty} E_0[N_{1t}] = -\frac{A_1^1}{A_2^1}$ and $Cost = \frac{\alpha(\xi B_1^1\sigma_Y)^2}{2(A_2^1)^2(\xi+\kappa_Y)}$.

	$RP, \$$	$\sigma_S, \$$	$RP_r, \%$	$\sigma_{Sr}, \%$	$E(N_1)$	Cost, $\$$	PT, %	ξ
No TC	0.540	3.333	0.52	3.20	0.500	0	$+\infty$	NA
With TC	1.299	3.393	5.22	17.42	0.297	0.00483	87.21	19.78

Table 2: The table reports the dollar risk premium, RP, the volatility of dollar returns, σ_S, the risk premium and the volatility of rates of returns, RP_r and σ_{Sr}, the average stock allocations of the liquidity trader, $E(N_1)$, and the trading cost for the economy without transaction costs and with trading costs. It is assumed that $\alpha = 0.01$, $\psi = -0.01$, $\kappa_Y = 1.0$, $\sigma_Y = 1.4$, $\kappa_D = 0.05$, $\sigma_D = 0.2$ and $\lambda = 0.75$, $\gamma_1 = \gamma_2 = 9.73$, $\bar{D} = 1.604$.

the dollar return and the rate of stock returns, σ_S and σ_{Sr}, the average stock allocation of the liquidity trader, $E(N_1)$, the annualized cost, the average portfolio turnover, and parameter ξ in the economies with and without transaction costs. Table 2 shows that the introduction of transaction costs increases the dollar risk premium by about 2.4 times while the risk premium on the stock rates of returns rises by ten times from 0.52% to 5.22%. Such a significant increase in the risk premium emerges from a few reasons. The presence of transaction costs makes the liquidity provider the marginal investor whose weight in the economy and/or risk aversion defines the unconditional moments of stock returns. Indeed, subject to trading frequency, an introduction of transaction costs in the economy dramatically reduces a trading demand from liquidity traders. For example, Appendix B shows in a single-period setting and under calibration of section 3.1, that the trading demand of liquidity traders drops by almost fifty times once transaction costs are introduced, assuming that agents trade every week. Costly portfolio updates lead to shorter positions for liquidity traders. The decrease in the stock holdings by the liquidity providers is strongly reinforced via feedback effects, introduced above, leading to a significantly longer stock allocation of the liquidity provider. Specifically, she increases her average stock allocations from 0.5 to 1.1. She is willing to do so only if the Sharpe ratio of the de-trended stock returns rises significantly. The Sharpe ratio increases only if the dollar risk premium does, which also causes the stock price to drop. According to Lemma 1, given that $\omega \approx 19$ and $m = 2.41$, the everage stock price decreases by factor 3.49 from \$106.38 to \$30.50. This drop feeds a multiple increase in the risk of the rates of stock returns, discussed below, which in turn makes the risk premium in the rates of stock returns increase more significantly than the dollar risk premium.

While most of the increase in dollar returns of the de-trended stock could be explained by a shift in the stock allocations of the liquidity provider, an additional contribution to the dollar risk premium aims to compensate for an increase in the standard deviation of dollar stock returns which rises from \$3.33 to \$3.39 as transaction costs are introduced to the liquidity traders. The liquidity provider is marginal in this increase since she trades in the direction of instantaneous changes in the stock price. The increase in the risk of dollar returns is rather small since the agents can share the dividend risk without updating their stock positions and trade only in response to shocks in endowment volatility. Nonetheless, the standard deviation of the rate of the stock returns undergoes a more than five-fold

increase from 3.20% to 17.42% mostly due to a decrease in the average stock price. This way, the liquidity provider makes a decisive contribution to the increase in the risk of the stock rates of returns.

The risk premiums and the standard deviations on the dollar returns and the rates of returns in the market with transaction costs could be approximately matched in the market without these costs by increasing the coefficient of absolute risk aversion of the agents by the factor equal to the ratio of the dollar risk premiums across the two markets. Specifically, increasing the coefficients of the absolute risk aversion of each agent by factor 2.41 leads to the dollar risk premium and the expected stock price in the market without transaction costs to \$1.299 and \$30.50, while the first two moments of the rates of stock returns become 5.03% and 14.39%. Furthermore, an increase in the absolute risk aversion by factor 2.41 translates into a rise of the expected values of the coefficients of the relative risk aversion of the agents from 6.2 to 32.8.[41]

The case of a small parameter σ_Y.

Under the approximation of a small value of σ_Y, the long-term expected stock holdings of the liquidity trader and the liquidity provider are $\bar{N}_1 = \frac{1}{2(1+\lambda)} = 0.286$ and $\bar{N}_1 = \frac{1}{2(1-\lambda^2)} = 1.143$, respectively. These allocations lead a risk premium of $RP = \frac{\nu\gamma_2\sigma_S^2}{2(1-\lambda^2)} = 1.233$. These equilibrium outcomes are close to their counterparts in Table 2. Furthermore, equation (30) implies that the ratio of the liquidity premiums in the market with and without transaction costs is $m = \frac{1}{1-\lambda^2} = 2.285$. This ratio is defined by the weight of the liquidity providers. Then it follows from lemma 1 that the expected de-trended stock prices should be approximately the same as under the main calibration, implying approximately the same moments of the rates of stock returns as in Table 2. I conclude that the average stock price and, therefore, the moments of the rates of stock returns, are almost fully explained by the weight of the liquidity providers.

The cost of trading and the portfolio turnover.

The conclusion that the impact of transaction costs follows from a change in heterogeneity in trading demands across agents, rather than from their losses on paying these costs per se, is consistent with the size of the expected annualized transaction cost paid by the liquidity trader. This cost is equal to \$0.0048 and is much smaller than the expected dollar liquidity premium received by the liquidity trader, which is equal to $E(N_1)RP - E(N_1^0)RP^0 = \$0.12$, where RP^0 and N_1^0 are the risk premium and stock holdings of the liquidity trader in the

[41] Calculations of these coefficients require making the expected drift in each agent's portfolio equal to zero which is achieved by adjusting the time discount rates of the agents 1 and 2 to 0.050 and 0.052 when $\gamma_1 = \gamma_2 = 9.73$, and to 0.25 and 0.31 when $\gamma_1 = \gamma_2 = 23.40$. The average portfolio growth rate due to a higher risk premium outpaces the consumption rate leading to an infinite wealth of each agent in the limit of an infinite time if the time discount rates are smaller than these values.

absence of transaction costs. Similarly, the present value of the annualized expected cost discounted by the expected rates of stock returns is equal to \$0.078, while the expected stock price changes by \$75.88 after transaction costs are introduced. It follows that the ratio of the annual cost to the average stock price in this model is about 0.016%. This ratio could be matched with proportional transaction costs, which are typically used to model the losses on a fixed bid-ask spread. If the liquidity trader pays proportional transaction costs with a portfolio turnover of 92% (see below), then her transaction losses over one year, as a fraction of the average stock holdings, are approximately $0.92 \times (bid - ask\ spread)$, where the bid-ask spread is measured in percent. Taking into account that the average stock holdings of the liquidity trader are 0.297 and matching the ratio of the annual cost to the average stock price to 0.016% suggests that the fixed bid-ask spread should be 0.059%.

Table 2 shows that the average portfolio turnover is 87.21% which is close to the average turnover in the U.S. stock market.[42] The portfolio turnover of agent i, $\lim_{t\to\infty} \frac{E_0|u_{it}|}{2E_0[N_{it}]}$, is equal to 92% for the liquidity trader and 74% for the liquidity supplier. A much faster rate of trading by the liquidity provider is offset by her significantly longer average position in the stock market. Overall, the portfolio turnover changes with parameter σ_Y. For example, decreasing σ_Y from 1.4 to 0.2 results in the average portfolio turnover of 12.75%, while the expected rates of return and the standard deviation of the stock's rates of returns become 3.75% and 11.67%, respectively, (versus 0.52% and 3.20% in the market with no transaction costs). The seven-fold decrease in the volatility of the endowment risk causes a multi-fold decrease in the portfolio turnover and the costs and also leads to a noticeable, but yet much less significant than in the portfolio turnover, reductions in the moments of the rates of stock returns. These reductions are due to a smaller heterogeneity in trading technologies across the two agents followed from smaller losses of the liquidity trader in her risk sharing. Finally, Table 2 shows that the value of parameter ξ corresponds to a decrease of the magnitude of the short-term component in the autocorrelation coefficient of stock returns by factor two within less than two weeks which is consistent with the observations that the first-order autocorrelation coefficients for weekly and monthly returns are significant.[43]

On the opportunity costs.

The transaction costs received, and opportunity costs faced by liquidity providers can only lead to a small effect on stock returns since they change the heterogeneity in trading demands across economic agents only ever slightly. Table 3 presents the characteristics of the de-trended stock market for the cases when transaction costs received by liquidity providers fully offset the costs of missed investment opportunities ($\psi = 0$), when these costs exceed

[42]For example, Lo and Wang (2000) report 92.56% for the New York and American Stock Exchange from 1962 to 1998. They utilize a different definition for the aggregate portfolio turnover than used in this paper: $PT = \lambda|u_1| + (1-\lambda)|u_2|$. According to this definition, the model predicts a portfolio turnover of 82.04%.

[43]See, for example, Jegadeesh (1990) and Lehmann (1990).

ψ	$RP_r, \%$	$\sigma_{Sr}, \%$	SR	$E(N_1)$	Cost, \$	PT, %	ξ	A_0^1	A_0^2
0	5.30	17.52	0.383	0.297	0.00487	87.57	19.79	0.22	0.89
0.01	5.37	17.71	0.384	0.297	0.00491	87.91	19.79	0.20	1.64
0.03	5.59	18.62	0.385	0.298	0.00502	88.70	19.81	0.16	3.23

Table 3: The table reports the risk premium and the volatility of rates of returns, RP_r and σ_{Sr}, the Sharpe ratio of de-trended stock rates of returns, SR, the average stock allocations of the liquidity trader, $E(N_1)$, the trading costs, parameter ξ, and the coefficients A_0^1, A_0^2 in the economies with different values of ψ. It is assumed that $\alpha = 0.01$, $\kappa_Y = 1.0$, $\sigma_Y = 1.4$, $\kappa_D = 0.05$, $\sigma_D = 0.2$ and $\lambda = 0.75$, $\gamma_1 = \gamma_2 = 9.73$, $\bar{D} = 1.604$.

the cash flow from transaction costs by one-third ($\psi = 0.01$), so that the loss of one liquidity provider is the same as the loss of one liquidity trader, and when the loss of one liquidity provider is three times bigger than that of one liquidity trader ($\psi = 0.03$). Tables 2 and 3 show that increasing the costs of missed investment opportunities makes liquidity providers reduce their stock market share and demand a higher Sharpe ratio of the de-trended stock returns for facing extra costs. They have to trade more actively for better risk sharing causing the stock market to become more volatile, as they trade along the market instantaneous changes, and require a higher risk premium for facing higher risk. Table 3 also reports coefficients A_0^1, A_0^2 which define the welfare of the agents, ceteris paribus. The table shows that the liquidity trader gets slightly better off with increasing ψ due to an increasing Sharpe ratio of the stock returns. On the other hand, the liquidity provider becomes noticeably worse off with higher ψ and her well-being deteriorates even in comparison to the market without costs when the costs faced by the liquidity provider noticeably exceed the costs faced by the liquidity trader.[44]

On the weight of the liquidity providers.

Taking into account that the weight of the liquidity providers may vary across markets, Table 4 presents the moments of de-trended stock returns for various weights of the liquidity trader starting from the market where the liquidity providers occupy 75% of the market and finishing with the case where they take only 5% of the market. To keep the moments on the dollar returns the same across the markets, I adjust the coefficients of absolute risk aversions, which are shown in the second column of the table, in each market. Inconvenient dimensional coefficients of absolute risk aversion are matched with the expected coefficients of relative risk aversions of agents presented in the third and the fourth columns of Table 4.[45] The table shows that the expected values of these coefficients change from about 14 when $\lambda = 0.25$ to about 1 when $\lambda = 0.95$. That is, a fifteen-fold decrease in the weight of

[44]This conclusion follows from the comparison of A_0^2 in the case when $\psi = 0.03$ with coefficient $A_2 = 2.21$ in equation (37) for agent 2 in the market with no costs.

[45]The calculations of these coefficients require small adjustments in the time-discount factors of agents to keep the expected drift in the value of their de-trended portfolios at zero.

λ	γ_i	$CRRA_1$	$CRRA_2$	$RP_r, \%$	$\sigma_{Sr}, \%$	$RP_r^0, \%$	$\sigma_{Sr}^0, \%$
0.250	21.88	13.73	13.89	5.27	17.42	3.45	10.54
0.500	17.34	9.95	10.05	5.23	17.22	1.54	5.46
0.750	9.73	5.35	5.40	5.24	17.42	0.52	3.20
0.833	6.63	3.61	3.64	5.24	17.46	0.29	2.74
0.9	4.02	2.18	2.20	5.27	17.55	0.16	2.45
0.95	2.02	1.09	1.10	5.31	17.67	0.08	2.26

Table 4: The table reports the coefficients of absolute risk aversions of the agents, γ_i, the expected coefficients of relative risk aversions, $CRRA_1$, $CRRA_2$, the risk premium on the rates of returns and volatility of rates of return in the market with transaction costs, RP_r, σ_{Sr}, and in the market without these costs the risk premium on the rates of returns, RP_r^0, σ_{Sr}^0, for different weights and different risk aversions of the liquidity trader in the economy. It is assumed that $\alpha = 0.01$, $\psi = -0.01$, $\kappa_D = 0.05$, $\kappa_Y = 1.0$, $\sigma_Y = 1.4$, $\sigma_D = 0.2$.

liquidity providers has the same effect on stock returns as an almost thirteen-fold increase in the coefficient of relative risk aversion of the agents. The equality of the moments of dollar returns warrants the close similarity in the moments of the rates on the stock returns across the markets. Indeed, Table 4 shows that the risk premiums and the standard deviations on the rates of stock returns stay within their standard errors.

The effect of the weight of the liquidity providers on stock returns is also seen from the comparison with the market without transaction costs. The statistical moments of the rates of stock returns in this market are presented in the last two columns of Table 4. The difference across the markets with and without transaction costs quickly increases with λ. If the risk premium in the market with costs is about fifty percent bigger than in the market without costs in the case where $\lambda = 0.25$, it becomes seventy-six times bigger in the market where $\lambda = 0.95$. A similar but much less pronounced relation is observed for the standard deviation which exceeds its counterpart in the market without costs by more than seven times when $\lambda = 0.95$, rather than by about sixty percent when $\lambda = 0.25$. As under the main calibration, the standard deviation on the rates of stock returns increases in the presence of transaction costs due to a decrease in the stock price, while the risk premium increases due to the decrease in the stock price and a significant increase in the dollar risk premium. It should be noticed that the Sharpe ratio of the dollar returns is always higher in the presence of transaction costs. The latter is not the case, however, for the Sharpe ratio on the rates of returns as these rates do not follow the normal distribution.

Table 5 presents trading strategies of the agents in the markets shown in Table 4. It confirms that in all markets the liquidity provider takes a longer position than in the absence of transaction costs, where $N_2^0 = 0.5$. The size of her position quickly increases with an increasing number of liquidity traders in the market and becomes almost ten times her position in the absence of transaction costs when $\lambda = 0.95$. Along the same line, the risk exposure of a liquidity supplier rises from 163% when $\lambda = 0.25$ and reaches $1,491\%$

λ	N_1	N_2	w_1	w_2	Cost, \$	PT,%	ξ
0.250	0.401	0.533	1.223	1.625	0.00662	34.83	14.23
0.500	0.340	0.660	1.037	2.013	0.00561	65.48	17.05
0.750	0.297	1.109	0.906	3.382	0.00483	87.21	19.78
0.833	0.284	1.580	0.866	4.819	0.00460	92.04	20.66
0.9	0.275	2.525	0.839	7.701	0.00447	95.01	21.31
0.95	0.269	4.889	0.820	14.911	0.00406	96.49	21.78

Table 5: The table reports the average allocations of the liquidity trader, N_1, and the liquidity provider, N_2, the weights of the stock in the portfolios of the two agents, w_1, w_2, the cost of trading, and parameter ξ for different weights of the liquidity trader in the economy. It is assumed that $\alpha = 0.01$, $\psi = -0.01$, $\kappa_D = 0.05$, $\kappa_Y = 1.0$, $\sigma_Y = 1.4$, $\sigma_D = 0.2$.

when $\lambda = 0.95$. Table 5 also reports the annualized trading costs paid by the liquidity trader. The liquidity traders trade a little slower as the number of liquidity providers and the risk aversion of agents decreases. The liquidity traders are willing to hedge less of their endowment risk by trading slower, and save on transaction costs, as they become more risk-tolerant. Despite slower trading by the liquidity trader, Table 5 shows that the average portfolio turnover increases with the decreasing weight of the liquidity providers. The average turnover rises due to the increasing weight of the liquidity traders whose portfolio turnovers slightly increase with λ (and smaller risk aversion) because of shorter stock allocations. The impact of liquidity providers on the average turnover decreases with bigger λ, even though the portfolio turnover of a liquidity provider quickly falls due to a significantly bigger size of her portfolio. Finally, Table 5 shows that parameter ξ slowly increases with the higher weight of the liquidity traders and lower risk aversions of investors. According to equation (31), this reversion speeds up with a decreasing number of liquidity suppliers and slows down with their lower risk aversions. It follows that the former trait slightly dominates the latter one. Given the allocations of agents and their expected coefficients of relative risk aversions presented in Tables 4 and 5, I conclude that the cases where λ is equal to 0.75, 0.83, and 0.9 are the most suitable to fit the historical data on the stock returns and strategies. I choose the case with $\lambda = 0.75$ for the main calibration as the most conservative one among the three cases in terms of the leverage of agent 2.

The role of the size of transaction costs.

The calibration of the model uses the value for the coefficient in transaction costs leading to a bid-ask spread consistent with today's practice. Historically, however, transaction costs were often higher imposing a need for a comparative static analysis with respect to α. Table 6 presents this analysis under the calibration of section 3.1 for a few values of α. The table shows that increasing coefficient α from 0.002 to 1.25 leads to about eight-fold rise in the transaction costs and more than nine times fall in the portfolio turnover. An increase

in α causes a much smaller cost increase as the liquidity traders significantly slow down their trading rate, revealing a smaller turnover. Table 6 also presents the bid-ask spread, should this spread be independent of the traded amount, implied by the transaction losses of the liquidity trader. Based on this spread, I conclude that the cases when $\alpha = 0.002$ and $\alpha = 0.01$ are consistent with the current markets, while the case when $\alpha = 0.05$ is more consistent with markets in the recent past where the bid-ask spreads were noticeably higher. The portfolio turnover of 54% in the latter case is also in the bulk of the observed turnovers during the recent historical past.[46] Furthermore, an increase in the trading rate at lower transaction costs, showing in a bigger turnover, is much less significant than the rise in the robustness of trading by the liquidity trader, measured by parameter ξ. An increase in ξ, leading to faster trading, appears to be partly offset by a smaller uncertainty in the target allocations. Table 6 shows values of ξ for different α's which all appear to be well approximated by the equation (31). The table reports that the short-term stock reversion should occur within less than one week when $\alpha = 0.002$ and within half a year when $\alpha = 1.25$. A faster stock reversion makes the liquidity provider better off, while the liquidity trader is better off by paying smaller bid-ask spreads when α is lower. This conclusion is confirmed from the comparison of coefficients A_0^i, $i = 1, 2$, in the utility functions. The latter are equal to $A_0^1 = 0.21$, $A_0^2 = 0.17$ when $\alpha = 0.002$ and $A_0^1 = 0.47$, $A_0^2 = 0.29$ when $\alpha = 1.25$. Finally, Table 6 reports the first two moments of the stock returns. Lower transaction costs imply smaller losses for the liquidity trader and a less significant heterogeneity in demand leading to a slightly lower Sharpe ratio of the dollar stock returns and smaller the first two moments on the dollar returns and the rates of stock returns. While the fall in the latter moments is noticeable, it is nowhere near the scale of the decrease in transaction costs. At discrete, rather than continuous, time trading lowering the transaction cost below its smallest value in Table 6 will continue to slightly decrease the moments of rates of stock returns until the trading frequency becomes restrictive in the way that the liquidity traders can recuperate the bid-ask spread over the trading time-interval. Then the moments of stock returns will start quickly converging to their values in the market without transaction costs.

On the trading frequency.

The impact of transaction costs on stock returns is significant only if agents trade frequently enough. Each trade of a liquidity trader arises from a risk-sharing demand and undergoes a substantial size reduction when transaction costs are introduced. The latter is conditioned on the assumption that trading is frequent enough since a liquidity trader is expected to recover trading losses over the time between trades. In the case of quadratic costs, the liquidity trader is better off by trading at a very high frequency which allows her to minimize the losses on these costs. However, in practice, this may not be the case and so it is of interest to estimate the trading frequency at which the predictions of this model

[46] See, for example, Lo and Wang (2000).

α	$RP, \$$	$\sigma_S, \$$	$RP_r, \%$	$\sigma_{Sr}, \%$	Cost, \$	PT, %	ξ	Spd,%
0.002	1.295	3.389	5.19	17.27	0.00234	133.99	44.21	0.018
0.05	1.306	3.402	5.47	18.20	0.00904	54.04	8.86	0.182
0.25	1.320	3.420	6.05	20.74	0.0142	30.33	3.97	0.528
1.25	1.345	3.453	7.12	23.82	0.0162	14.55	1.79	1.379

Table 6: The table reports the dollar risk premium, RP, the volatility of dollar returns, σ_S, the risk premium and the volatility of rates of returns, RP_r and σ_{Sr}, the trading cost, The portfolio turnover, parameter ξ, and the fixed bid-ask spread, *Spd*, for the economies with different values of coefficient α. The average stock allocations of the liquidity trader, $E(N_1)$ stays close to 0.297 for all values of α. It is assumed that $\psi = -0.01$, $\kappa_Y = 1.0$, $\sigma_Y = 1.4$, $\kappa_D = 0.05$, $\sigma_D = 0.2$, $\lambda = 0.75$, $\gamma_1 = \gamma_2 = 9.73$, $\bar{D} = 1.604$.

become practical. Appendix B utilizes a single-period model of the equilibrium which, based on the calibration of section 3.1, predicts that the effects of transaction costs on stock returns are close to the continuous-time approximations if agents trade with a bi-weekly frequency or higher. This prediction resonates with the conclusion that the short-term stock price reversal has a time scale of a few weeks. As shown in Appendix B, this minimal trading frequency should increase proportionally to the standard deviation of stock returns and the ratio $1/\sqrt{\alpha}$. This result implies that investors should trade at least every week for the impact of transaction costs on the stock price to be close to the model predictions, should the coefficient α be equal to the smallest value in Table 6. The last condition has been historically met by institutional investors and many retail investors.

The heterogeneity in the absolute risk aversions.

A bigger coefficient of absolute risk aversion of the liquidity provider implies a smaller dollar position in the stock and, therefore, a smaller weight in the stock market. Consequently, the stock market where agents have the same coefficients of absolute risk aversions equal to γ but different weights, equal to λ and $1-\lambda$, should be similar to the stock market where agents have the same weights but different coefficients of absolute risk aversions γ_1, γ_2 as long as $\frac{\lambda}{1-\lambda} = \frac{\gamma_2}{\gamma_1}$ and $\frac{1}{\gamma} \approx \frac{1}{2}(\frac{1}{\gamma_1} + \frac{1}{\gamma_2})$.[47] This expectation is confirmed in Table 7 which repeats Table 2 for the market where investors 1 and 2 have identical weights but different coefficients of risk aversions equal to 6.33 and 18.99, respectively. The first investor represents the liquidity trader and the second investor stands for the liquidity provider. Tables 2 and 7 show close matches between the moments of the rates of stock returns as well parameter ξ. The most noticeable difference between the two markets is found in the portfolio turnover which is smaller by about 38% in the case of the market with agents heterogeneous in their risk aversions. A bigger magnitude of Y_2 in Table 2 than in Table 7 creates incentives for the liquidity provider to trade faster for better hedging of the

[47]The second condition has approximate equality because of the difference in the agents' exposure to the endowment risk across the markets with the same risk aversions and with the same weights of agents.

	$RP, \$$	$\sigma_S, \$$	$RP_r, \%$	$\sigma_{Sr}, \%$	$E(N_1)$	Cost, \$	PT, %	ξ
No TC	0.555	3.333	0.53	3.24	0.75	0	$+\infty$	NA
With TC	1.299	3.361	5.30	17.92	0.436	0.00387	49.56	16.22

Table 7: The table reports the dollar risk premium, RP, the volatility of dollar returns, σ_S, the risk premium and the volatility of rates of returns, RP_r and σ_{Sr}, the average stock allocations of the liquidity trader, $E(N_1)$, and the trading cost for the economy without transaction costs and with trading costs. It is assumed that $\alpha = 0.01$, $\psi = -0.01$, $Y_1 = Y = -Y_2$, $\kappa_Y = 1.0$, $\sigma_Y = 1.4$, $\kappa_D = 0.05$, $\sigma_D = 0.2$ and $\lambda = 0.5$, $\gamma_1 = 6.33$, $\gamma_2 = 18.99$, $\bar{D} = 1.604$.

endowment risk. The liquidity trader follows the lead by increasing her portfolio turnover and faces bigger transaction losses.

The comparison with the market without liquidity providers.

It is worthwhile to compare the predictions of this model with the predictions in IS which assumes that all investors pay transaction costs. These predictions will be compared with Table 7 since IS relies on heterogeneity in the risk aversions of investors. Table 8 presents the outcomes of the model in IS where each agent pays the same quadratic costs with $\alpha = 0.01$ and utilizes the same calibration as in Table 7. Table 8 shows that the liquidity premium is very significant and makes 49 percent of the risk premium in the market without transaction costs. Yet, this premium is much smaller than suggested by this article. A smaller liquidity premium is due to a much smaller heterogeneity in demands across investors when both investors pay transaction costs. Appendix B shows in a single period economy that the introduction of transaction costs for both agents makes the more risk-averse agent demand three times more stock shares per trade than a less risk-averse agent does. This heterogeneity in demand is more than ten times smaller than in the case of this study if investors trade weekly. The difference in heterogeneities of demands is translated to a relatively shorter average allocation of the investor with stronger risk aversion in the model of IS. Indeed, according to Table 8, this investor increases her aggregate stock allocation from 0.125 to 0.185 as transaction costs are introduced, while in Table 7 she raises them from 0.125 to 0.277. Furthermore, because of a decrease in the average stock price, the volatility of the rates of stock returns in IS increases by a noticeable 17%, and yet this is a much smaller factor than in Table 7. Tables 7 and 8 also show reasonably close portfolio turnovers and that the costs paid by the investor with a smaller risk aversion are noticeably bigger in Table 8. The latter is because the weight of this agent is smaller than in the setting of the current model leading to faster trading and bigger costs as they increase quadratically with the trading rate. Moreover, the short-term mean reversion of the stock price, characterized by parameter ξ, is much slower in IS as the less risk-averse investor trades less robustly toward her optimal allocations.

$RP_r, \%$	$\sigma_{Sr}, \%$	$E(N_1)$	$RP_r^0, \%$	$\sigma_{Sr}^0, \%$	$E(N_1^0)$	Cost, \$	PT,%	ξ
0.79	3.79	0.630	0.53	3.24	0.75	0.00547	62.12	8.62

Table 8: The table reports the risk premium on the rates of returns, RP_r, volatility of rates of return, σ_{Sr}, and the average allocations of the liquidity trader, N_1, in the market with transaction costs; the risk premium on the rates of returns, RP_r^0, volatility of rates of return, σ_{Sr}^0, and the average allocations of the liquidity trader, N_1^0, in the market without transaction costs as well as the cost of trading, and parameter ξ in the model of IS, where both agents pay the same transaction costs. It is assumed that $\alpha = 0.01$, $\psi = -0.01$, $Y_1 = Y = -Y_2$, $\kappa_Y = 1.0$, $\sigma_Y = 1.4$, $\kappa_D = 0.05$, $\sigma_D = 0.2$, $\lambda = 0.5$, $\gamma_1 = 6.33$, $\gamma_2 = 18.99$, $\bar{D} = 1.604$.

4 Conclusion

This study finds a dynamic equilibrium in the trending stock market where liquidity providers trade frequently with liquidity traders who pay small transaction costs for the liquidity supply. Assuming that the absolute risk aversions of investors decrease at the market growth rate and using the notes growing at the same rate, as a numeraire, allows to reduce the economy to stationary. The combination of frequent trading and transaction costs makes liquidity providers marginal in the stock market. Furthermore, investors trade slowly allowing them to predict the future movement in the conditional risk premium. In turn, this leads to the asymmetric feedback effect between the stock allocations and the conditional risk premium resulting in significantly lower stock allocations of the liquidity traders, a much lower stock price, and a significantly higher de-trended risk premium, and the volatility of the rates of stock returns. Therefore, the presence of even small transaction costs can lead to a significant structural change in the stock market.

A few extensions of this study could be of interest. First, one could make the bond market endogenous to the economy, include the other major players in the bond market omitted in this model, and estimate the effect of transaction costs on the interest rate. It could also be of interest to see how diversification between liquid and less liquid stocks affects the stock returns. Finally, solving an equilibrium in the presence of multiple heterogeneous liquidity traders with incomplete information on the macro state variables, and possibly with a partial irrationality of portfolio selection decisions, will allow us to estimate the limits of the feedback effect between the stock returns and the aggregate stock allocations.

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Appendix A

In this Appendix, I prove Proposition 1, Corollaries 1, and 3, and Lemmas 1 and 2. The proof of Proposition 1 presents the Hamilton-Jacobi-Bellman (HJB) equation for each agent, the partial differential equation for the stock price, and their solutions in terms of equilibrium coefficients. The appendix also outlines the algorithm for finding these coefficients.

Proof of Proposition 1.

The indirect utility function of agent i, $V^i(t, X_i, N_1, Y)$, solves the following HJB equation:

$$\begin{aligned} 0 &= \max_{c_i,\, z_i \in R^2} \left\{ V_t^i + \frac{1}{2}\left[(Y_i)^2 + 2Y_i N_i \sigma_{S_D} + (N_i)^2 \sigma_S^2 \right] V^i_{X_i X_i} + \frac{1}{2}\sigma_Y^2 V^i_{YY} + \sigma_Y N_i \sigma_{S_Y} V^i_{X_i Y} \right. \\ &+ \left. u_1 V^i_{N_1} + \mu_Y V^i_Y + \left(RP\, N_i - \chi_i\, (u_1)^2 + \nu X_i - c_i \right) V^i_{X_i} - \exp(-\zeta_i t - \gamma_i c_i) \right\}, \end{aligned}$$

where $\mu_Y = -\kappa_Y Y$, $\sigma_S^2 = \sigma_{S_D}^2 + \sigma_{S_Y}^2$.

It follows that the optimal controls are

$$c_i^0 = -\frac{1}{\gamma_i}[1 + \ln(V^i_{X_i}/\gamma_i)], \quad u_1^0 = \frac{V^1_{N_1}}{2\alpha V^1_{X_1}}, \tag{A-1}$$

$$N_2^0 = -\frac{RPV^2_{X_2} + \sigma_{S_D} Y_2 V^2_{X_2 X_2} + \sigma_Y \sigma_{S_Y} V^2_{X_2 Y}}{\sigma_S^2 V^2_{X_2 X_2}}. \tag{A-2}$$

Assuming the separation of variables t and X_i as in

$$V^i(t, X_i, N_1, Y) = -\exp[-\zeta_i t - \gamma_i \nu X_i + g_i(N_1, Y)]$$

leads to:

$$c_i^0 = -\frac{1}{\gamma_i}[\ln(\nu) + g_i] + \nu X_i, \quad u_1^0 = -\frac{g^1_N}{2\alpha\nu\gamma_1}, \tag{A-3}$$

where $g_1(N_1, Y)$ solves the following PDE

$$\begin{aligned} 0 &= -\nu[\ln(\nu) + g_1] - \zeta_1 + \nu + \frac{(\nu\gamma_1)^2}{2}\left[(Y_1)^2 + 2Y_1 N_1 \sigma_{S_D} + (N_1)^2 \sigma_S^2 \right] \\ &- \nu\gamma_1 RP\, N_1 + \frac{1}{2}\sigma_Y^2 [g^1_{YY} + (g^1_Y)^2] - \frac{(g^1_N)^2}{4\alpha\nu\gamma_1} + \mu_Y g^1_Y - \sigma_Y \nu\gamma_1 N_1 \sigma_{S_Y} g^1_Y. \end{aligned} \tag{A-4}$$

Similar, function $g_2(N_1, Y)$ solves PDE:

$$\begin{aligned} 0 &= -\nu[\ln(\nu) + g_2] - \zeta_2 + \nu + \frac{(\nu\gamma_2)^2}{2}\left[(Y_2)^2 + 2Y_2 N_2^0 \sigma_{S_D} + (N_2^0)^2 \sigma_S^2 \right] \\ &- \nu\gamma_2 (RP\, N_2^0 - \psi u_1^2) + \frac{1}{2}\sigma_Y^2 [g^2_{YY} + (g^2_Y)^2] + u_1 g_{2N_1} + \mu_Y g^2_Y - \sigma_Y \nu\gamma_2 N_2^0 \sigma_{S_Y} g^2_Y, \end{aligned} \tag{A-5}$$

where N_2^0 is given by

$$N_2^0 = \frac{RP + \sigma_Y \sigma_{S_Y} g_{2Y} - Y_2 \nu\gamma_2 \sigma_{S_D}}{\nu\gamma_2 \sigma_S^2}. \tag{A-6}$$

Next, I will try the following quadratic function for $g_n(Y, N_n)$:

$$g_n(Y, N_n) = A_0^n + A_1^n N_n + \frac{1}{2}A_2^n N_n^2 + B_0^n Y + B_1^n N_n Y + C_0^n Y^2, \tag{A-7}$$

where $N_2 = \frac{1/2-\lambda N_1}{1-\lambda}$, A_0^n, A_1^n, A_2^n, B_0^n, B_1^n, C_0^n are constants to be determined numerically, and constant A_2^1 is positive. I assume that the stock price is a linear function of the state variables, to be found below, implying that the diffusion coefficients in the stock price are constant. Replacing function g_1 in equation (A-4) with quadratic form (A-7) leads to:

$$\begin{aligned}
- \quad & \zeta_1 - \nu\left[\ln(\nu) - 1 + A_0^1 + A_1^1 N_1 + \frac{1}{2}A_2^1 (N_1)^2 + (B_0^1 + B_1^1 N_1)Y + C_0^1 Y^2\right] \\
+ \quad & \frac{(\nu\gamma_1)^2}{2}\left[(Y_1)^2 + 2Y_1 N_1 \sigma_{S_D} + (N_1)^2\sigma_S^2\right] - \nu\gamma_1(R_0 + R_1 N_1 + R_2 Y)N_1 \\
+ \quad & \frac{1}{2}\sigma_Y^2[2C_0^1 + (B_0^1 + B_1^1 N_1 + 2C_0^1 Y)^2] + (B_0^1 + B_1^1 N_1 + 2C_0^1 Y)(\mu_Y - \sigma_Y \nu\gamma_1 N_1 \sigma_{S_Y}) \\
- \quad & \frac{(A_1^1 + A_2^1 N_1 + B_1^1 Y)^2}{4\alpha\nu\gamma_1} = 0.
\end{aligned} \tag{A-8}$$

Equation (A-8) holds only if the coefficients in the same powers of the product of variables Y and N_1 sum to zero, implying the following set of equations for the liquidity trader:

$$0 = -\nu A_0^1 + \nu[1 - \ln(\nu)] - \zeta_1 + \frac{1}{2}\sigma_Y^2[2C_0^1 + (B_0^1)^2] - \frac{(A_1^1)^2}{4\alpha\nu\gamma_1}, \tag{A-9}$$

$$0 = -\nu A_1^1 - \nu\gamma_1 R_0 + \sigma_Y^2 B_0^1 B_1^1 - \frac{A_1^1 A_2^1}{2\alpha\nu\gamma_1} - \sigma_Y^2 \nu\gamma_1 B_0^1 H_2, \tag{A-10}$$

$$0 = (2\sigma_Y^2 C_0^1 - \nu - \kappa_Y)B_0^1 - \frac{A_1^1 B_1^1}{2\alpha\nu\gamma_1}, \tag{A-11}$$

$$0 = -\nu A_2^1 + (\nu\gamma_1)^2\sigma_S^2 - 2\nu\gamma_1 R_1 + \sigma_Y^2 (B_1^1)^2 - \frac{(A_2^1)^2}{2\alpha\nu\gamma_1} - 2\sigma_Y^2 \nu\gamma_1 B_1^1 H_2, \tag{A-12}$$

$$0 = -(\nu + 2\kappa_Y)C_0^1 + \frac{(\nu\gamma_1\beta_1)^2}{2} + 2\sigma_Y^2 (C_0^1)^2 - \frac{(B_1^1)^2}{4\alpha\nu\gamma_1}, \tag{A-13}$$

$$\begin{aligned}
0 = \quad & -(\nu + \kappa_Y)B_1^1 + (\nu\gamma_1)^2\beta_1\sigma_{S_D} - \nu\gamma_1 R_2 + 2\sigma_Y^2 B_1^1 C_0^1 - 2\sigma_Y^2 \nu\gamma_1 C_0^1 H_2 \\
- \quad & \frac{A_2^1 B_1^1}{2\alpha\nu\gamma_1},
\end{aligned} \tag{A-14}$$

where $\beta_1 = 1$, $\beta_2 = -\frac{\lambda}{1-\lambda}$. In equilibrium $c_i = c_i^0$ and $u_1 = u_1^0$. It follows from equation (A-3) that

$$u_1 = A_2^1 \frac{\hat{N} - N}{2\alpha\nu\gamma_1}, \tag{A-15}$$

where $\hat{N} = -\frac{A_1^1 + B_1^1 Y}{A_2^1}$.

PDE (A-5) for the liquidity provider leads to the equation

$$\begin{aligned}
&- \quad \zeta_2 - \nu\Bigg[\ln(\nu) - 1 + A_0^2 + A_1^2 N_2 + \frac{1}{2}A_2^2(N_2)^2 + (B_0^2 + B_1^2 N_2)Y + C_0^2 Y^2\Bigg] \\
&+ \quad \frac{(\nu\gamma_2)^2}{2}[\beta_2^2 Y^2 + 2\beta_2 Y N_2^0 \sigma_{S_D} + (N_2^0)^2\sigma_S^2] - \nu\gamma_2\Bigg[(R_0 + R_1 N_1 + R_2 Y)N_2^0 \\
&- \quad \psi\Bigg(\frac{A_1^1 + A_2^1 N_1 + B_1^1 Y}{2\alpha\nu\gamma_1}\Bigg)^2\Bigg] + \lambda\frac{A_1^1 + A_2^1 N_1 + B_1^1 Y}{2(1-\lambda)\alpha\nu\gamma_1}(A_1^2 + A_2^2 N_2 + B_1^2 Y) \\
&+ \quad \frac{1}{2}\sigma_Y^2[2C_0^2 + (B_0^2 + B_1^2 N_2 + 2C_0^2 Y)^2] + [-\kappa_Y Y - \sigma_Y^2\nu\gamma_2 N_2^0 H_2] \qquad \text{(A-16)} \\
&\times \quad (B_0^2 + B_1^2 N_2 + 2C_0^2 Y) = 0,
\end{aligned}$$

where, in equilibrium,

$$N_2^0 = N_2 = (1/2 - \lambda N_1)/(1-\lambda). \qquad \text{(A-17)}$$

I replace N_2^0 with N_2 and N_1 with $[1/2 - (1-\lambda)N_2]/\lambda$ in the last equation results in

$$\begin{aligned}
&- \quad \zeta_2 - \nu\Bigg[\ln(\nu) - 1 + A_0^2 + A_1^2 N_2 + \frac{1}{2}A_2^2 N_2^2 + (B_0^2 + B_1^2 N_2)Y + C_0^2 Y^2\Bigg] \\
&+ \quad \frac{(\nu\gamma_2)^2}{2}[\beta_2^2 Y^2 + 2\beta_2 Y N_2 \sigma_{S_D} + N_2^2\sigma_S^2] - \nu\gamma_2\Bigg[(R_0 + R_1[1/2 - (1-\lambda)N_2]/\lambda + R_2 Y)N_2 \\
&- \quad \psi\Bigg(\frac{A_1^1 + A_2^1[1/2 - (1-\lambda)N_2]/\lambda + B_1^1 Y}{2\alpha\nu\gamma_1}\Bigg)^2\Bigg] + \lambda\frac{A_1^1 + A_2^1[1/2 - (1-\lambda)N_2]/\lambda + B_1^1 Y}{2(1-\lambda)\alpha\nu\gamma_1} \\
&\times \quad (A_1^2 + A_2^2 N_2 + B_1^2 Y) + \frac{1}{2}\sigma_Y^2[2C_0^2 + (B_0^2 + B_1^2 N_2 + 2C_0^2 Y)^2] \qquad \text{(A-18)} \\
&+ \quad [-\kappa_Y Y - \sigma_Y^2\nu\gamma_2 H_2 N_2](B_0^2 + B_1^2 N_2 + 2C_0^2 Y) = 0.
\end{aligned}$$

Considering N_2 as an independent variable in the last equation leads to the following six equations corresponding to different powers of variables Y and N_2 in their products:

$$\begin{aligned}
&- \quad \zeta_1 - \nu[\ln(\nu) - 1 + A_0^2] + \nu\gamma_2\psi\Bigg(\frac{A_1^1 + \frac{A_2^1}{2\lambda}}{2\alpha\nu\gamma_1}\Bigg)^2 + \frac{A_1^2(\lambda A_1^1 + \frac{A_2^1}{2})}{2(1-\lambda)\alpha\nu\gamma_1} \\
&+ \quad \frac{1}{2}\sigma_Y^2[2C_0^2 + (B_0^2)^2] = 0, \qquad \text{(A-19)} \\
&- \quad (\nu + \kappa_Y)B_0^2 + \gamma_2\psi\frac{(A_1^1 + \frac{A_2^1}{2\lambda})B_1^1}{2\nu(\alpha\gamma_1)^2} + \frac{(\lambda A_1^1 + \frac{A_2^1}{2})B_1^2 + \lambda B_1^1 A_1^2}{2(1-\lambda)\alpha\nu\gamma_1} \\
&+ \quad 2\sigma_Y^2 B_0^2 C_0^2 = 0, \qquad \text{(A-20)} \\
&- \quad \nu A_1^2 - \nu\gamma_2\Bigg[R_0 + \frac{R_1}{2\lambda} + \psi\frac{(1-\lambda)(A_1^1 + \frac{A_2^1}{2\lambda})A_2^1}{2\lambda(\alpha\nu\gamma_1)^2}\Bigg] + \frac{(\lambda A_1^1 + \frac{A_2^1}{2})\frac{A_2^2}{1-\lambda} - A_2^1 A_1^2}{2\alpha\nu\gamma_1} \\
&+ \quad \sigma_Y^2 B_0^2 B_1^2 - \sigma_Y^2\nu\gamma_2 H_2 B_0^2 = 0, \qquad \text{(A-21)}
\end{aligned}$$

$$
\begin{aligned}
- \quad & (\nu + 2\kappa_Y)C_0^2 + \frac{(\nu\gamma_2)^2}{2}\beta_2^2 + \nu\gamma_2\psi\left(\frac{B_1^1}{2\alpha\nu\gamma_1}\right)^2 + \frac{B_1^1 B_1^2}{2\alpha\nu\gamma_1} + 2\sigma_Y^2(C_0^2)^2 = 0, && \text{(A-22)}\\
- \quad & \frac{1}{2}\nu A_2^2 + \frac{(\nu\gamma_2)^2}{2}\sigma_S^2 + \nu\gamma_2\left[(1-\lambda)R_1/\lambda + \psi\left(\frac{(1-\lambda)A_2^1}{2\lambda\alpha\nu\gamma_1}\right)^2\right] - \frac{A_2^1 A_2^2}{2\alpha\nu\gamma_1} && \text{(A-23)}\\
+ \quad & \frac{1}{2}\sigma_Y^2(B_1^2)^2 - \sigma_Y^2\nu\gamma_2 H_2 B_1^2 = 0,\\
- \quad & (\nu + \kappa_Y)B_1^2 + (\nu\gamma_2)^2\beta_2\sigma_{S_D} - \nu\gamma_2\left(R_2 + \psi\frac{(1-\lambda)A_2^1 B_1^1}{2\lambda(\alpha\nu\gamma_1)^2}\right) - \frac{A_2^1 B_1^2 - B_1^1 A_2^2\frac{\lambda}{1-\lambda}}{2\alpha\nu\gamma_1}\\
+ \quad & 2\sigma_Y^2 B_1^2 C_0^2 - 2\sigma_Y^2\nu\gamma_2 H_2 C_0^2 = 0. && \text{(A-24)}
\end{aligned}
$$

Next, I consider the PDE for the stock price. Given that the dividend and the endowment volatility in the trending economy are equal to their de-trended counterparts multiplied by deterministic function, the state variables for the trending stock price could be chosen to be D, N_1, and Y. The PDE for the trending stock price is:

$$
rS^o + RP^o = D^o + S_t^o + S_D^o\mu_D + S_{N_1}^o u_1 + S_Y^o\mu_Y + \frac{1}{2}\sigma_D^2 S_{DD}^o + \frac{1}{2}\sigma_Y^2 S_{YY}^o. \quad \text{(A-25)}
$$

By using the substitutions $S^o(t, D, N_1, Y) = S(D, N_1, Y)e^{\delta t}$, $D^o = De^{\delta t}$, $Y^o = Ye^{\delta t}$, I arrive to the PDE for the de-trended stock price:

$$
\nu S + RP = D + S_D\mu_D + S_{N_1}u_1 + S_Y\mu_Y + \frac{1}{2}\sigma_D^2 S_{DD} + \frac{1}{2}\sigma_Y^2 S_{YY}. \quad \text{(A-26)}
$$

The Itô formula implies that $\sigma_{S_D} = S_D\sigma_D$ and $\sigma_{S_Y} = S_Y\sigma_Y$.

Let me assume that equation (14) holds, then I rewrite equation (A-26) as:

$$
R_0 + R_1N_1 + R_2Y + \nu(H_0 + H_1N_1 + H_2Y) = \frac{\kappa_D\bar{D}}{\nu + \kappa_D} - H_1\frac{A_1^1 + A_2^1N_1 + B_1^1Y}{2\alpha\nu\gamma_1} + H_2\mu_Y.
$$

The last identity leads to the following relations:

$$
\begin{aligned}
R_0 + \nu H_0 &= \frac{\kappa_D\bar{D}}{\nu + \kappa_D} - H_1\frac{A_1^1}{2\alpha\nu\gamma_1}, && \text{(A-27)}\\
R_1 + \nu H_1 &= -H_1\frac{A_2^1}{2\alpha\nu\gamma_1}, && \text{(A-28)}\\
R_2 + \nu H_2 &= -H_1\frac{B_1^1}{2\alpha\nu\gamma_1} - \kappa_Y H_2. && \text{(A-29)}
\end{aligned}
$$

Furthermore, equation (A-6) and the stock market clearing condition imply

$$
RP = \nu\gamma_2\sigma_S^2(1/2 - \lambda N_1)/(1-\lambda) + \beta_2\nu\gamma_2\sigma_{S_D}Y - \sigma_Y^2 H_2 g_{2Y}. \quad \text{(A-30)}
$$

Or,

$$
\begin{aligned}
R_0 + R_1N_1 + R_2Y &= \nu\gamma_2\sigma_S^2(1/2 - \lambda N_1)/(1-\lambda) + \beta_2\nu\gamma_2\sigma_{S_D}Y\\
&- \sigma_Y^2 H_2[B_0^2 + B_1^2(1/2 - \lambda N_1)/(1-\lambda) + 2C_0^2 Y].
\end{aligned}
$$

The last equation leads to the following three relations:

$$R_0 = \nu\gamma_2\sigma_S^2/[2(1-\lambda)] - \sigma_Y^2 H_2(B_0^2 + B_1^2/[2(1-\lambda)]), \quad \text{(A-31)}$$

$$R_1 = \lambda/(1-\lambda)(-\nu\gamma_2\sigma_S^2 + \sigma_Y^2 H_2\, B_1^2), \quad \text{(A-32)}$$

$$R_2 = \beta_2\nu\gamma_2\sigma_{S_D} - 2\sigma_Y^2 H_2 C_0^2. \quad \text{(A-33)}$$

I conclude that finding an equilibrium comes down to finding six coefficients for each function g_i, three coefficients for the risk premium, and three coefficients for the stock price. These coefficients should solve the system of equations (A-9)-(A-14), (A-19)-(A-24), (A-27)-(A-29), (A-31)-(A-33). Q.E.D.

Nest, I describe the steps that I follow to solve the equilibrium. This numerical approach allows me to control for the existence and the uniqueness of the linear-quadratic equilibrium. The system of equations for the unknown equilibrium coefficients could be isolated into sub-system of non-linear equations (A-12)-(A-14), (A-22)-(A-24), (A-28), (A-29), (A-32) and (A-33) and the subsystem of linear equations. The subsystem of non-linear equations does not depend on the solutions of the sub-system of linear equations and should be solved first.

I start by using equations (A-12)-(A-14) to find coefficients C_0^1, A_2^1 and B_1^1 in terms of coefficients H_1 and H_2. Specifically, I use equation (A-13) to determine C_0^1 in terms of B_1^1 taking into account that the sign in front of the radical should be negative. Then I use equation (A-12) to determine A_2^1 in terms of B_1^1, H_1 and H_2 keeping in mind that A_2^1 must be positive and using equation (A-28). Next, equation (A-14) is used to find B_1^1 in terms of H_1 and H_2, while keeping in mind equation (A-29). Second, I use equations (A-22)-(A-24) to find coefficients C_0^2, A_2^2 and B_1^2 in terms of coefficients H_1 and H_2. Specifically, I use equation (A-22) to determine C_0^2 in terms of B_1^2 keeping in mind that the sign in front of the radical should be negative. Then I use equation (A-23) to determine A_2^2 in terms of B_1^2, H_1 and H_2 and using equation (A-28). Next, equation (A-24) is used to find B_1^2 in terms of H_1 and H_2, while keeping in mind equation (A-29). Third, I utilize equations (A-28), (A-32) and (A-33) to find coefficients H_1 and H_2. The third step is finished by calculating coefficients R_1, R_2, C_0^1, A_2^1, B_1^1, C_0^2, A_2^2 and B_1^2. Fourth, two linear equations (A-10) and (A-11) are solved to determine A_1^1 and B_0^1 as linear functions of R_0. Fifth, I solve three linear equations (A-20), (A-21), and (A-31) with three unknowns A_1^2, B_0^2, and R_0. In these equations, I take into account that A_1^1 is a linear function of R_0. I finish this step by finding coefficients A_1^1 and B_0^1. Finally, equations (A-27), (A-9), and (A-19) are used to find coefficients H_0, A_0^1, and A_0^2, respectively.

The existence and the uniqueness of the linear-quadratic equilibrium are verified numerically for each set of parameters considered in Section 3.

Proof of Corollary 1.

One can remove all terms proportional to σ_Y^2 in equations (A-12)-(A-14), (A-22)-(A-24),

(A-28), (A-29), (A-32) and (A-33) if σ_Y is small enough. Then I find from equations (A-31)-(A-33):

$$R_0 = \frac{\nu\gamma_2\sigma_S^2}{2(1-\lambda)}, \tag{A-34}$$

$$R_1 = -\nu\gamma_2\,\sigma_S^2\,\frac{\lambda}{1-\lambda}, \tag{A-35}$$

$$R_2 = \beta_2\nu\gamma_2\sigma_S, \tag{A-36}$$

where $\sigma_S = \sigma_{S_D}$. Equation (21) follows after taking into account that $RP_c = R_0 + R_1\,N_1 + R_2\,Y$ and $N_2 = \frac{1/2-\lambda N_1}{1-\lambda}$. Equation (A-12) simplifies to

$$0 = \frac{(A_2^1)^2}{2\alpha\nu\gamma_1} + \nu A_2^1 - (\nu\gamma_1)^2\sigma_S^2 + 2\nu\gamma_1 R_1, \tag{A-37}$$

and has a unique positive solution given by

$$A_2^1 = \alpha\,\nu\gamma_1\left[-\nu + \sqrt{\nu^2 + 2\nu\frac{\sigma_S^2}{\alpha}\left(\gamma_1 + 2\gamma_2\frac{\lambda}{1-\lambda}\right)}\right]. \tag{A-38}$$

The remaining equations for the equilibrium coefficients are linear and could be solved in the following order. Use equation (A-14) to find B_1^1, then solve equation (A-28) for H_1 and then solve equation (A-29) for H_2. Coefficients C_0^1, B_0^1, A_1^1, and A_0^1 are found from equations (A-13), (A-11), (A-10), and (A-9), respectively. Next, use equation (A-23) to determine A_2^2 and equation (A-24) to find B_1^2. Finally, coefficients C_0^2, A_1^2, B_0^2, and A_0^2 are found from equations (A-22), (A-21), (A-20) and (A-19), respectively.

Equation (A-10) leads to

$$A_1^1 = -\frac{\nu^2\gamma_1\gamma_2\,\sigma_S^2}{2(1-\lambda)(\nu + \frac{A_2^1}{2\alpha\nu\gamma_1})}. \tag{A-39}$$

Taking into account equation (A-38) and that the long-term expected stock holding by the liquidity trader is $-\frac{A_1^1}{A_2^1}$, I arrive at equation (22). Q.E.D.

Proof of Lemma 1.

If the stock is de-trending, then $\lim_{t\to\infty} E_0(\Delta S_t) = 0$. Hence, equation (4) implies that $\bar{D} = RP + r\bar{S}$ and equation (27) follows.

Equation (27) implies that $\chi = \frac{\bar{D}-mRP_0}{\bar{D}-RP_0}$ and $\omega = \frac{\bar{S}}{\bar{D}} = \frac{1}{\nu}(1 - \frac{mRP_0}{\bar{D}})$. It follows that $RP_0 = \frac{1-\nu\omega}{m}\bar{D}$. Replacing RP_0 with the last expression in the formula for χ leads to equation (28). Q.E.D.

Proof of Corollary 3.

Similar to the corollary 4 in IS, the system of nonlinear equations (A-12)-(A-14), (A-22)-(A-24), (A-28), (A-29), (A-32) and (A-33) suggest that the equilibrium coefficients

A_1^n, A_2^n, B_1^n, $n = 1, 2$ and H_1 are proportional to $\sqrt{\alpha}$ when α is small. Solving equations (A-12), (A-23), (A-28) and (A-32) where all terms proportional to α and $\sqrt{\alpha}$ are neglected, leads to the following formulas for the coefficients A_2^1, H_1 and R_1 in terms of σ_S:

$$R_1 = -\frac{\lambda}{1-\lambda}\,\nu\gamma_2\sigma_S^2, \tag{A-40}$$

$$H_1 = \sqrt{\frac{2\alpha}{\nu\left(\gamma_1 + 2\gamma_2\frac{\lambda}{1-\lambda}\right)}}, \tag{A-41}$$

$$A_2^1 = \sqrt{\alpha}\,\nu\gamma_1\sigma_S\sqrt{2\nu\left(\gamma_1 + 2\gamma_2\frac{\lambda}{1-\lambda}\right)}. \tag{A-42}$$

The last result implies the formula (31) given that $\xi = \frac{A_2^1}{2\alpha\nu\gamma_1}$. Q.E.D.

Proof of Lemma 2. First, let me find the autocorrelation coefficient. The conditional covariance is

$$\begin{aligned}
&Cov_t(\Delta S_t + \int_t^{t+\Delta t} D_s ds, \Delta S_{t'} + \int_{t'}^{t'+\Delta t} D_s ds) \\
&= Cov_t\Bigg(H_1\Delta N_t + H_2\Delta Y_t + H_3\Delta D_t + \int_t^{t+\Delta t} D_s ds, H_1\Delta N_{t'} + H_2\Delta Y_{t'} + H_3\Delta D_{t'} \\
&+ \int_{t'}^{t'+\Delta t} D_s ds\Bigg) = Cov_t\Bigg(H_2\sigma_Y\Delta W_{Yt} + H_3\sigma_D\Delta W_{Dt}, H_1 u_{1t'} - H_2\kappa_Y Y_{t'} + D_{t'} \\
&+ H_3\kappa_D(\bar{D} - D_{t'})\Bigg)\Delta t + o(\Delta t) = Cov_t\Bigg(H_2\sigma_Y\Delta W_{Yt} + H_3\sigma_D\Delta W_{Dt}, H_1\xi(bY_{t'} - N_{1t'}) \\
&- H_2\kappa_Y Y_{t'} + D_{t'} - H_3\kappa_D D_{t'}\Bigg)\Delta t + o(\Delta t) = Cov_t\Bigg(H_2\sigma_Y\Delta W_{Yt} + H_3\sigma_D\Delta W_{Dt}, \\
&(H_1\xi b - H_2\kappa_Y)Y_{t'} - H_1\xi N_{1t'} + H_4 D_{t'}\Bigg)\Delta t + o(\Delta t),
\end{aligned} \tag{A-43}$$

where $H_3 = \frac{1}{\nu+\kappa_D}$ and $H_4 = \frac{\nu}{\nu+\kappa_D}$. To obtain the last covariance we have to consider the following differential equations

$$\begin{aligned}
dY_{t'} &= -\kappa_Y Y_{t'}dt', \qquad Y(t) = \sigma_Y\Delta W_{Yt}, \\
dN_{t'} &= -\xi(N_{t'} - bY_{t'})dt', \;\; N(t) = 0, \\
dD_{t'} &= -\kappa_D D_{t'}dt', \qquad D(t) = \sigma_D\Delta t W_{Dt},
\end{aligned}$$

The solutions of these equations are straightforward:

$$Y_{t'} = Y_t\exp[-\kappa_Y(t'-t)], \qquad Y_t = \sigma_Y\Delta W_{Yt}, \tag{A-44}$$

$$N(t') = \frac{\xi b Y_t}{\xi - \kappa_Y}\,[e^{-\kappa_Y(t'-t)} - e^{-\xi(t'-t)}], \tag{A-45}$$

$$D_{t'} = D_t\exp[-\kappa_D(t'-t)], \qquad D_t = \sigma_D\Delta W_{Dt}. \tag{A-46}$$

Substitution of equations (A-44)-(A-46) into the last line of expression (A-43) results in

$$
\begin{aligned}
& Cov_t(\Delta S_t + \int_t^{t+\Delta t} D_s ds, \Delta S_{t'} + \int_{t'}^{t'+\Delta t} D_s ds) \\
& = Cov_t\Big(H_2\sigma_Y \Delta W_{Yt} + H_3\sigma_D \Delta W_{Dt}, (H_1\xi b - H_2\kappa_Y) Y_t \exp[-\kappa_Y(t'-t)] \\
& - H_1\xi \frac{\xi b Y_t}{\xi - \kappa_Y} [e^{-\kappa_Y(t'-t)} - e^{-\xi(t'-t)}] + H_4 D_t \exp[-\kappa_D(t'-t)] \Big) \Delta t + o(\Delta t) \\
= & \Big(H_2\sigma_Y^2 \Big[(H_1\xi b - H_2\kappa_Y) e^{-\kappa_Y(t'-t)} - \frac{H_1\xi^2 b}{\xi - \kappa_Y} [e^{-\kappa_Y(t'-t)} - e^{-\xi(t'-t)}] \Big] \\
& + H_3 H_4 \sigma_D^2 e^{-\kappa_D(t'-t)} \Big) \Delta t^2 + o(\Delta t^2). \qquad \text{(A-47)}
\end{aligned}
$$

Rearranging the terms in the last equation and taking into account that $Var_t(\Delta S_t + \int_t^{t+\Delta t} D_s ds) Var_{t'}(\Delta S_{t'} + \int_{t'}^{t'+\Delta t} D_s ds) = \sigma_S^2 \Delta t + o(\Delta t)$ leads to equation (32).

The comovement between the de-trended stock return and the allocations of the liquidity provider is found as follows.

$$
\begin{aligned}
Cov_t(\Delta N_{2t}, \Delta S_t + \int_t^{t+\Delta t} D_s ds) &= -\frac{\lambda}{1-\lambda} Cov_t(\Delta N_{1t}, H_1 \Delta N_{1t} + H_2 \Delta Y_t), \\
&= -\frac{\lambda}{1-\lambda} Cov_t(\Delta N_{1t}, H_2 \Delta Y_t), \qquad \text{(A-48)}
\end{aligned}
$$

where it was taken in to account that W_D and W_Y are independent and Δt is small. The last covariance could be found by using the equations for process $Y_{t+\Delta t}$ and for the stock holding of the liquidity trader derived in IS:

$$
\begin{aligned}
Y_{t+\Delta t} &= Y_t e^{-\kappa_Y \Delta t} + \sigma_Y \int_t^{t+\Delta t} e^{-\kappa_Y(t+\Delta t-u)} dW_{Yu}, \qquad \text{(A-49)} \\
N_{1t+\Delta t} &= a + (N_{1t} - a) e^{-\xi \Delta t} + \frac{\xi b}{\xi - \kappa_Y} Y_t (e^{-\kappa_Y \Delta t} - e^{-\xi \Delta t}) \\
&+ \frac{\xi b \sigma_Y}{\xi - \kappa_Y} \int_t^{t+\Delta t} (e^{-\kappa_Y(t+\Delta t-s)} - e^{-\xi(t+\Delta t-s)}) dW_{Ys}, \qquad \text{(A-50)}
\end{aligned}
$$

where $a = -\frac{A_1^1}{A_2^1}$. It follows that

$$
\begin{aligned}
& Cov_t(\Delta N_{2t}, \Delta S_t + \int_t^{t+\Delta t} D_s ds) = -\frac{\lambda}{1-\lambda} \frac{b\xi\sigma_Y^2 H_2}{\xi - \kappa_Y} Cov_t\Big(\int_t^{t+\Delta t} (e^{-\kappa_Y(t+\Delta t-s)} \\
& - e^{-\xi(t+\Delta t-s)}) dW_{Ys}, \int_t^{t+\Delta t} e^{-\kappa_Y(t+\Delta t-u)} dW_{Yu} \Big) = -\frac{\lambda}{1-\lambda} \frac{b\xi\sigma_Y^2 H_2}{\xi - \kappa_Y} \int_t^{t+\Delta t} \Big(e^{-2\kappa_Y(t+\Delta t-u)} \\
& - e^{-(\xi+\kappa_Y)(t+\Delta t-u)} \Big) du = -\frac{\lambda}{1-\lambda} \frac{b\xi\sigma_Y^2 H_2}{\xi - \kappa_Y} \Big(\frac{1}{2\kappa_Y}(1 - e^{-2\kappa_Y \Delta t}) - \frac{1}{\xi + \kappa_Y}(1 - e^{-(\xi+\kappa_Y)\Delta t}) \Big) \\
& = -\frac{\lambda}{2(1-\lambda)} b\xi\sigma_Y^2 H_2 \, \Delta t^2 + o(\Delta t^2), \qquad \text{(A-51)}
\end{aligned}
$$

where the Itô isometry is applied.

The autocovariance is found as follows.

$$Cov_t(\Delta N_{2t}, \Delta S_{t+\Delta t} + \int_{t+\Delta t}^{t+2\Delta t} D_s ds) = -\frac{\lambda}{1-\lambda} Cov_t(\Delta N_{1t}, H_1 \Delta N_{1t+\Delta t} + H_2 \Delta Y_{t+\Delta t})$$
$$= -\frac{\lambda}{1-\lambda} Cov_t\Bigg(N_{1t+\Delta t}, H_1 \Bigg[N_{1t+\Delta t}(e^{-\xi \Delta t} - 1) + \frac{b\xi}{\xi - \kappa_Y} Y_{t+\Delta t}(e^{-\kappa_Y \Delta t} - e^{-\xi \Delta t}) \Bigg]$$
$$+ H_2 Y_{t+\Delta t}(e^{-\kappa_Y \Delta t} - 1) \Bigg) = -\frac{\lambda}{1-\lambda} \Bigg[-H_1(1 - e^{-\xi \Delta t}) Var_t(N_{1t+\Delta t})$$
$$+ \Bigg(H_1 \frac{b\xi}{\xi - \kappa_Y}(e^{-\kappa_Y \Delta t} - e^{-\xi \Delta t}) - H_2(1 - e^{-\kappa_Y \Delta t}) \Bigg) Cov_t(N_{1t+\Delta t}, Y_{t+\Delta t}) \Bigg], \quad \text{(A-52)}$$

where it is taken into account that ΔN_{1t} is independent from $W_{Dt+\Delta t}$ and $\Delta W_{Yt+\Delta t}$. With the help from calculations of $Cov_t(\Delta N_{2t}, \Delta S_t)$ and equation (A-50) I find:

$$Cov_t(N_{1t+\Delta t}, Y_{t+\Delta t}) = \frac{b\xi\sigma_Y^2}{\xi - \kappa_Y} \Bigg(\frac{1}{2\kappa_Y}(1 - e^{-2\kappa_Y \Delta t}) - \frac{1}{\xi + \kappa_Y}(1 - e^{-(\xi + \kappa_Y)\Delta t}) \Bigg), \quad \text{(A-53)}$$

$$Var_t(N_{1t+\Delta t}) = \left(\frac{\xi b \sigma_Y}{\xi - \kappa_Y} \right)^2 \int_t^{t+\Delta t} \left(e^{-\kappa_Y(t+\Delta t - u)} - e^{-\xi(t+\Delta t - u)} \right)^2 ds$$
$$= \left(\frac{b\xi\sigma_Y}{\xi - \kappa_Y} \right)^2 \Bigg(\frac{1}{2\kappa_Y}(1 - e^{-2\kappa_Y \Delta t}) + \frac{1}{2\xi}(1 - e^{-2\xi \Delta t}) - \frac{2}{\xi + \kappa_Y}(1 - e^{-(\xi + \kappa_Y)\Delta t}) \Bigg). \quad \text{(A-54)}$$

In the limit of a small Δt, the last covariance is proportional to $(\Delta t)^2$, while the last variance is proportional to $(\Delta t)^3$ and, therefore, could be neglected. Combining the last three equations and assuming that Δt is small (one day or one week) reveals:

$$Cov_t(\Delta N_{2t}, \Delta S_{t+\Delta t} + \int_{t+\Delta t}^{t+2\Delta t} D_s ds) = -\frac{\lambda}{2(1-\lambda)}(H_1 b\xi - H_2 \kappa_Y) b\xi \sigma_Y^2 \Delta t^3 + o(\Delta t^3). \quad \text{(A-55)}$$

Furthermore, equation (A-29) suggests that $R_2 = H_1 b\xi - (\kappa_Y + \nu) H_2$ implying that

$$Cov_t(\Delta N_{2t}, \Delta S_{t+\Delta t} + \int_{t+\Delta t}^{t+2\Delta t} D_s ds) \approx -\frac{\lambda}{2(1-\lambda)} R_2 b\xi \sigma_Y^2 \Delta t^3 + o(\Delta t^3). \quad \text{(A-56)}$$

Q.E.D.

Appendix B

This Appendix finds a conditional equilibrium in a single-period economy. By neglecting intertemporal hedging existing in the dynamic equilibrium, the investment horizon in a

single-period economy is naively interpreted as the time between trades. Consequently, this Appendix shows how the time between trades affects the trading demand of investors and the conditional risk premium in the presence of transaction costs. In the market, liquidity traders, who are represented by Agent 1, trade with liquidity providers, who are represented by Agent 2. Liquidity traders and liquidity providers maximize the expected exponential utility function from the consumption at the end of the period, which lasts Δt years. At the beginning of the period, agents 1 and 2 are endowed with N_{10}, N_{20} stock shares and wealth X_{10}, X_{20}, respectively. I assume that the stock price follow

$$S_{\Delta t} = S_0 + \mu_S \Delta t + \sigma_S \sqrt{\Delta t}\, \varepsilon_S, \tag{B-1}$$

where S_0 is an initial stock price, μ_S is the expected annualized dollar stock return, σ_S is a standard deviation of annualized dollar stock return, and ε is a standard normal variable. The agents can borrow and lend at a fixed annual interest rate r. Agent $i\,(i = 1, 2)$ trades ΔN_i stock shares at the beginning of the period, and the liquidity trader pays $\hat{\alpha}(\Delta N_1)^2$ dollars to the liquidity provider as transaction costs, where $\hat{\alpha}$ is a positive constant. On the other hand, the liquidity provider faces opportunity costs which together with transaction costs from the liquidity trader lead to the cash outflow of $\hat{\psi}(\Delta N_1)^2$, where $\hat{\psi}$ is a constant. Consequently, the portfolio selection problem for agent i in the de-trended market becomes:

$$\max_{\Delta N_i} E_0\bigg(-\exp(-\hat{\gamma}_i X_{i\Delta t})\bigg), \tag{B-2}$$

$$X_{i\Delta t} = X_{i0}(1+\nu\Delta t) + N_i[(\mu_S - \nu S_0)\Delta t + \sigma_S\sqrt{\Delta t}\,\varepsilon_S] - \hat{\chi}_i(\Delta N_1)^2(1+\nu\Delta t), \tag{B-3}$$

where $\hat{\gamma}_i = \nu\gamma_i > 0$ is a coefficient of absolute risk aversion, $N_i = N_{i0} + \Delta N_i$ and $\hat{\chi}_1 = \hat{\alpha}$, $\hat{\chi}_2 = \hat{\psi}$, where coefficient $\hat{\chi}_i$ under discrete time is related to its counterpart coefficient χ_i under continuous time as $\hat{\chi}_i = \chi_i/\Delta t$. Using budget constraint (B-3) for taking an expectation in (B-2) allows me to find an optimal number of traded shares by each agent:

$$\Delta N_1 = \left(\frac{RP}{\hat{\gamma}_1\sigma_S^2} - N_{10}\right)\frac{\hat{\gamma}_1\sigma_S^2}{2\hat{\alpha}(1+\nu\Delta t)/\Delta t + \hat{\gamma}_1\sigma_S^2}, \tag{B-4}$$

$$\Delta N_2 = \frac{RP}{\hat{\gamma}_2\sigma_S^2} - N_{20}, \tag{B-5}$$

where $RP = \mu_S - \nu S_0$ is the conditional dollar risk premium of the de-trended stock. Equation (B-4) shows that the trading demand from the liquidity trader decreases with smaller Δt as the trader tries to save on transaction costs. For example, let me consider the market without transaction costs in equilibrium where $\hat{\gamma}_1 = \hat{\gamma}_2 = 0.0973$, $\sigma_S = 3.333$, $\nu = 0.01$. These parameters match the predictions of the continuous-time model. Suppose now that the transaction costs for agent 1 have been turned on, but the market has not cleared yet. Then, equation (B-4) shows that, if agents trade once per week corresponding to $\Delta t = 0.02$

and $\hat{\alpha} = \alpha/\Delta t = 0.5$, the new aggregate trading demand from agent 1, $\lambda\Delta N_1$, decreases by about forty-seven times, while the aggregate trading demand from agent 2, $(1-\lambda)\Delta N_2$, does not change. For comparison with IS, should both agents pay the same transaction costs, then turning on these costs would dramatically decrease the trading demand from each agent, but the ratio between the aggregate trading demands of the agents, $|\frac{\lambda\Delta N_1}{(1-\lambda)\Delta N_2}|$, would be approximately equal to only $\frac{\hat{\gamma}_1}{\hat{\gamma}_2}$.

Taking into account that liquidity traders can differ in their initial stock allocations, weights, and the coefficients of risk aversion, one can use the stock market clearing condition $\lambda\Delta N_1 + (1-\lambda)\Delta N_2 = 0$ to find the dollar risk premium on the stock:

$$RP \;\; = \;\; \hat{\gamma}_2\sigma_S^2 \, \frac{2\hat{\alpha}(1-\lambda)(1+\nu\Delta t)N_{20}f + \frac{1}{2}\hat{\gamma}_1\sigma_S^2}{2(1-\lambda)\hat{\alpha}(1+\nu\Delta t)f + [(1-\lambda)\hat{\gamma}_1 + \lambda\hat{\gamma}_2]\sigma_S^2}, \tag{B-6}$$

where it is assumed that $f = (\Delta t)^{-1}$ and that the stock market is cleared before time zero: $\lambda N_{10} + (1-\lambda)N_{20} = \frac{1}{2}$. The last equation suggests that the conditional risk premium in the absence of transaction costs is

$$RP \;\; = \;\; \frac{\hat{\gamma}_1\hat{\gamma}_2}{2[(1-\lambda)\hat{\gamma}_1 + \lambda\hat{\gamma}_2]} \, \sigma_S^2. \tag{B-7}$$

The comparison of the conditional risk premiums in the presence and the absence of transaction costs suggests that the conditional liquidity premium could be quite significant.

Assuming that the investment horizon Δt is short enough so that $\nu\Delta t \ll 1$ and $2(1-\lambda)\hat{\alpha}(1+\nu\Delta t)f \gg [(1-\lambda)\hat{\gamma}_1 + \lambda\hat{\gamma}_2]\sigma_S^2$ results in

$$RP \;\; \approx \;\; \sigma_S^2\hat{\gamma}_2 \, N_{20}. \tag{B-8}$$

It follows that the conditional risk premium is defined by the current stock holdings of the liquidity provider and her risk aversion. On the other hand, should the investment horizon be far ahead such that $f \approx 0$, then the conditional risk premium would be approximately the same as in the market without transaction costs since the liquidity trader will get the premium over her investment horizon far more significant than the cost of the transaction.

The comparison of equations (B-7) and (B-8) shows that the risk premium is not uniform for taking the limits $\Delta t \to 0$ and $\alpha \to 0$ and the second limit has to be always taken first. Since the intertemporal model is solved under continuous-time approximation, the question remains: how frequently do investors have to trade for the predictions of the continuous-time approximation hold at a given choice of coefficient α? I answer this question under the approximation of a single-period model. Figure 1 shows the risk premium as a function of the logarithm of coefficient α for the investment horizons 1.0 0.1, 0.02, and 0.004, corresponding to the annually, monthly, weekly, and daily trading frequencies, for the parameters matching the predictions of the intertemporal model. Assuming that the allocation of the liquidity provider, N_{20}, is equal to its expected counterpart in the dynamic equilibrium, the figure

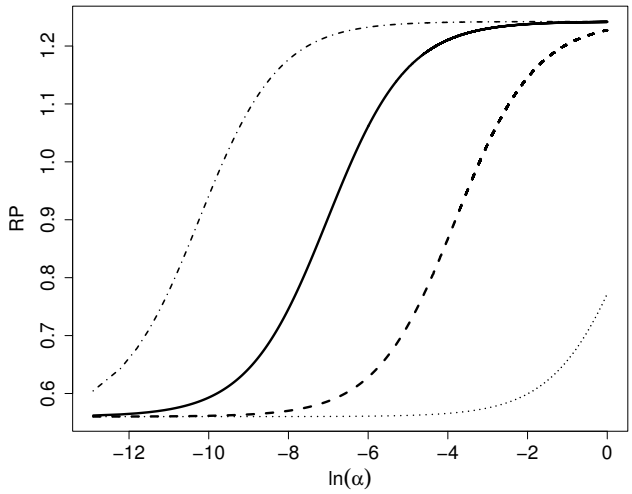


Figure 1: The figure depicts the risk premium versus $\ln(\alpha)$ when $\Delta t = 0.004$ (the dashed-dotted line), $\Delta t = 0.02$ (the solid line), $\Delta t = 0.1$ (the dashed line), and $\Delta t = 1.0$ (the dotted line). It is assumed that $\hat{\gamma}_1 = \hat{\gamma}_2 = 0.0973, \lambda = 0.75$, $N_{20} = 1.109$, and $\sigma_S = 3.393$.

shows that the continuous time limit $\sigma_S^2 \hat{\gamma}_2 \, N_{20} = 1.24$ is reached for the cases when Δt is equal to 0.1, 0.02 and 0.004. Furthermore, the limit is reached at a smaller value of α when Δt is smaller. Specifically, the coefficient α could be as low as 10^{-4} for the continuous time approximation to hold if investors trade daily.

The trading frequency at which the continuous-time approximation is close to its discrete-time counterpart could be estimated from equation (B-6). This equation shows that if $2(1-\lambda)\hat{\alpha}(1+\nu\Delta t)f$ is much bigger than $[(1-\lambda)\hat{\gamma}_1 + \lambda\hat{\gamma}_2]\sigma_S^2$ then the conditional liquidity premium is close to (B-8). It follows that the trading frequency should be much bigger than frequency f_c, where

$$f_c = \sqrt{\frac{\nu[(1-\lambda)\gamma_1 + \lambda\gamma_2]}{2(1-\lambda)\alpha}}\,\sigma_S, \tag{B-9}$$

where the term $\nu\Delta t$ is omitted due to its insignificance and $\hat{\alpha}$ is replaced with αf. By using the calibration from section 3.1 and the equilibrium predictions from Table 2, I find that $f_c \approx 15$. The numerical analysis suggests that trading at a frequency twice as high as f_c is enough to lead to the risk premium close to its continuous-time approximation.

Appendix C

This Appendix presents the short-term conditional covariances between the aggregate consumption rate and the annualized conditional moments of stock returns. Based on equations (10), (17), and (18), the aggregate de-trended consumption rate $C = \lambda c_1 + (1-\lambda)c_2$

follows the process

$$\begin{aligned} dC_t &= \mu_{Ct}dt + \frac{\nu}{2}\sigma_{S_D}\, dW_{Dt} + \left(\frac{1}{2}\nu H_2 - \frac{1}{\gamma}\left[\lambda B_0^1 + (1-\lambda)B_0^2 + \lambda N_{1t}(B_1^1 - B_1^2)\right.\right. \\ &+ \left.\left.\frac{1}{2}B_1^2 + 2Y_t[\lambda C_0^1 + (1-\lambda)C_0^2]\right]\right)\sigma_Y dW_{Yt}, \end{aligned} \tag{C-1}$$

where μ_C is a drift process and $\gamma = \gamma_i$, $i = 1, 2$. According to equations (40), (41), and (43) the last equation could be written as

$$dC_t = \mu_{Ct}dt + 0.0167\, dW_{Dt} + (0.0015 - 0.00070\, N_{1t} - 0.00059\, Y_t)\, dW_{Yt}. \tag{C-2}$$

Given the distributions of N_1 and Y and the values of the coefficients in the last equation, this equation implies that the aggregate consumption is very collinear with the dividend flow. Next, I find the short-term conditional covariances between the consumption rate and the processes describing the stock market. The conditional covariance between short-term changes in the aggregate consumption rate and the stock price is equal to $Cov_{1t} = \frac{1}{\tau}E_t(dC_t dS_t)$, where changes in the processes are taken place over small time-interval τ:

$$\begin{aligned} Cov_{1t} &\approx \frac{1}{\tau}\, E_t\Big([0.0170\, dW_{Dt} + (0.0015 - 0.00070\, N_{1t} - 0.00059\, Y_t)\, dW_{Yt}] \\ &\times (3.33\, dW_{Dt} + 0.63\, dW_{Yt})\Big) \approx 0.058 - 0.00046\, N_{1t} - 0.00037\, Y_t. \end{aligned}$$

The first term on the right side in the last formula is significantly bigger than the other terms. It arises from the dividend risk causing the stock price to be always procyclical.

The conditional covariance between the conditional aggregate consumption rate and the conditional risk premium on the rate of stock returns is equal to $Cov_{2t} = \frac{1}{\tau}E_t(dC_t\, d(\frac{RP_{ct}}{S_t}))$, where the risk premium follows the process

$$d\left(\frac{RP_{ct}}{S_t}\right) = \mu_{RPr}dt - \frac{RP_{ct}}{S_t^2}\sigma_{S_D}\, dW_{Dt} + \sigma_Y\, \frac{R_2 S_t - H_2 RP_{ct}}{S_t^2} dW_{Yt}$$

and μ_{RPr} is its drift. Therefore,

$$\begin{aligned} Cov_{2t} &\approx \frac{1}{\tau}\, E_t\left(\left[0.0170\, dW_{Dt} + (0.0015 - 0.00070\, N_{1t} - 0.00059\, Y_t)\, dW_{Yt}\right]\right. \\ &\times \left.\left[-3.33\, \frac{RP_t}{S_t^2}\, dW_{Dt} - \frac{1.40 S_t + 0.63 RP_{ct}}{S_t^2} dW_{Yt}\right]\right) \\ &\approx -(0.058 - 0.00044\, N_{1t} - 0.00037\, Y_t)\, \frac{RP_{ct}}{S_t^2} \\ &\quad -(0.0021 - 0.00098\, N_{1t} - 0.00083\, Y_t)\frac{1}{S_t} \end{aligned} \tag{C-3}$$

Both terms on the right side in the last equation are negative most of the time. This implies that the conditional risk premium is counter-cyclical and could be pro-cyclical in rare states

where the risk premium is negative which occurs when the liquidity trader is excessively long for a current level of the endowment volatility.

The conditional covariance between the consumption and the conditional Sharpe ratio is equal to

$$\begin{aligned} Cov_{3t} &= \frac{1}{\tau} E_t(dC_t \frac{1}{\sigma_S} d(RP_{ct})) \\ &\approx -\frac{1}{\tau} E_t\Big(\Big[0.0170\, dW_{Dt} + (0.0015 - 0.00070\, N_{1t} - 0.00059\, Y_t)\, dW_{Yt}\Big] 0.41\, dW_{Yt}\Big) \\ &\approx -0.00062 + 0.00029\, N_{1t} + 0.00024\, Y_t. \end{aligned} \tag{C-4}$$

The last covariance is negative almost all the time implying a counter-cyclical nature of the conditional Sharpe ratio. Finally, I find the covariance between the conditional standard deviation on the annualized rates of stock returns and the conditional risk premium on the rates of stock returns:

$$\begin{aligned} Cov_{4t} &= \frac{1}{\tau} E_t(d(\frac{\sigma_S}{S_t})\, d(\frac{RP_{ct}}{S_t})) \\ &\approx \frac{3.39}{\tau} E_t\Big(\Big[3.33 \frac{1}{S_t^2}\, dW_{Dt} + \frac{0.63}{S_t^2} dW_{Yt}\Big]\Big[3.33\, \frac{RP_{ct}}{S_t^2}\, dW_{Dt} + \frac{1.40 S_t + 0.63 RP_{ct}}{S_t^2} dW_{Yt}\Big]\Big) \\ &\approx \frac{39.04 RP_{ct} + 2.99 S_t}{S_t^4} \approx \frac{100.94 + 49.83 D_t - 130.67 N_{1t} - 37.69 Y_t}{S_t^4}. \end{aligned} \tag{C-5}$$

The last covariance is positive except for very rare events where the liquidity trader is very long in the stock market for the current level of the endowment volatility.

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