FROM PORTER'S VALUE CHAINS TO HOTELLING'S LAW: A NETWORK-STRUCTURED EQUILIBRIUM FOR COMPLEX ORGANIZATIONAL DYNAMICS AND MARKET COMPETITION

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Abstract

Value chains, understood as the process of sequential activities towards the realization of outputs by organizations, are a key idea in Michael Porter's theory of management due to their ability to informally describe arbitrarily complex operational structures. We propose a formalization of this concept by defining value chains as network structures (graphs) composed by bipartite subnetworks (subgraphs) which represent the operations of each activity as a game played by agents performing their corresponding roles in an organization. Agents' and organizations' pay-offs increase as a function of the number of successfully completed value chain activities, so we obtain a single equilibrium solution for all subgraph games which consists in minimizing uncertainty on the successful performance of each agent. We take this framework to the specific case of market competition dynamics, such that our equilibrium solution implies First-Order Stochastic Dominance across firms based on the joint prior probability distributions for firms' successfully completed value chain activities, which we call 'organizational technologies'. This result leads to a reinterpretation of Hotelling's Law, since the optimal strategy for all firms implies structuring their operations in a way that is the closest they can afford to the practices of the least uncertain value chain within the industry (at least while random market outcomes redefine firms' priors on successful performance). In our contribution, the tendency to similarities across firms in terms of spatial distances stated by Hotelling is reinterpreted in terms of distances between probability distributions describing arbitrarily complex industry practices.

Keywords: Bayesian Non-Parametrics, Dirichlet Process, First-order Stochastic Dominance, Hotelling's Law, Network Analysis, Value Chains.

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1. Introduction

Value chains are understood as the process of sequential activities for the progressive transformation of inputs towards the realization of final outputs to be delivered to consumers/costumers. This is a key concept in Michael Porter's theory of management (1985) due to its ability to describe arbitrarily complex operational structures under a systematic view of organizations and economic interactions, which allows his analytic framework to yield informal contributions for virtually any economic sector or organizational setting, even in present day.

On the other hand, models of market competition are a cornerstone of formal microeconomic analysis, since they are abstract conceptualizations capable to explain the relations between the basic economic agents (firms and consumers/costumers) and their motivations towards the formation of economic activities that turn more complex in reality, but remain with the essence that these classical models present regarding our relation with scarce resources.

As the last basic market competition model, the Hotelling model (1929) emerged with a simple mathematical structure, but with a powerful conclusion: Hotelling's Law. This postulate, based on the conceptualization of the Hotelling Line, dictates that firms in competition tend to locate the closest possible to each other, since in this way they can optimally exploit the technical properties revealed by the first located firm in the market place, in this way, industrial organization is understood as a process of spatial clustering. Hotelling's insights predicted that this result was just a simplified expression of a fundamental property of industrial organization that he considered quite complex to summarize given the mathematical tools of his time, that is, the tendency of firms to present similarities in all aspects of their production processes.

In this article, we build a bridge between Porter's value chains and Hotelling's Law, by providing a reinterpretation of this same fundamental property under a network-structured and game-theoretical framework, which formalizes Porter's conceptualization yielding a model that allows for an embedding of all market competition features of firms in prior beliefs structures in correspondence with their organizational structures, associated to the value chains they build to produce outputs.

We call these prior beliefs structures 'organizational technologies', they describe the relation between the average successful value chain assets and practices within the industry and the firm's capabilities seeking to match them. By defining a measure for the differences in efficiency between firms' organizational technologies, Hotelling's Law arises in the form of a partial order of First-order Stochastic Dominance (FSD), implying, in equilibrium, a tendency to develop organizational and technological similarities, represented by the closeness between firms' organizational technologies.

Our approach gathers instruments from network-structured Bayesian games and information distances analysis in microeconomics (Van Zandt, 2004; Burbano-Gómez & Sinisterra-Rodríguez, 2023), as well as Bayesian non-parametric modeling (Teh, 2017), yielding a market competition model driven by firms' strategies to manage uncertainty on their

organizational settings based on incomplete industry-related information, and suggesting a flexible framework for estimation and optimization of complex economic scenarios through Bayesian non-parametric techniques.

The upcoming section shows a summary of literature review on some related contributions, some applications of the Hotelling model, and the methodological sources for our approach. In the third section we present our model, showing its network structuring, the essence of its game strategies and equilibrium, and the implications that lead to our reinterpretation of Hotelling's Law. Concluding remarks and interesting research gaps are presented in the final section.

2. Literature review

In Economics there is a set of theoretical models that explain how firms relate to competitors and consumers in specific markets. The Cournot, Bertrand and Stackelberg models are classical game theory approaches for analyzing how firms compete given their optimal contrivance of prices or output quantities to maximize profits (Colombo & Labrecciosa, 2019). Although they show different characteristics and assumptions, they are similar by assuming complete information in demand functions, cost functions, and prices, allowing to reach simplified optimal results for firms under Nash equilibrium.

Nonetheless, much of these assumptions fail in real markets because it is impossible for firms to know at all times and places the full behavior of actions, strategies, or even payoffs in market competition. As a response to the latter, Bayesian games arise to deal with scenarios where players possess incomplete information on game variables, both for themselves and the rest of players (Zamir, 2009). In this sense, it is more reasonable to assume a generalized information incompleteness within games, and, therefore, that players choose optimal strategies based on probabilistic beliefs (Harsanyi, 1995). Probability is subjective and represents agents' individual beliefs about the occurrence of all possible states in game-theoretical settings, these beliefs are updated using Bayes' theorem as new information is obtained (Zaks, 2021).

Bayesian games' applications have been prolific around the last classical market competition model, the Hotelling Model. This model analyses competition between two firms selling the same product in a linear market, the widely used Hotelling Line, in which firms choose their position and compete for customers located within. By assuming that firms have equal costs and that customers' preferences are represented by their location, firms tend to cluster in order to optimize their market share (Hotelling, 1929). Since in real-world cases information about the characteristics of other players may also be incomplete, firms' interdependence is also modelled by Bayesian games theory (Harsanyi, 1968). The incorporation of uncertainty can arise for various reasons, which are explored in the intersection between the Hotelling model and Bayesian games, such as changes in consumer preferences (Harter, 1997); uncertainty in competitors' costs (Pinto, & Parreira, 2015), fluctuations in the supply of finite raw materials (Gaugler, 2015), and other economic topics as demand (Gorelkina, 2020; Balvers & Szerbn, 1996) or rates of return (Gaudet & Howitt, 1989), among others.

Nonetheless, most of these models follow a Bayesian parametric modelling. That is, a specific shape is assumed for the data distribution, and the number of parameters is fixed regardless of the size of information (Rossi, 2014). In contrast, Bayesian non-parametric models allow model complexity to increase with the size of the data, making them suitable for problems where the amount of information is variable or where the data has more complicated structures (Papageorgiou, et al., 2015). In this regard, a commonly used method in Bayesian non-parametric modelling is the Dirichlet Process, where the number of data groups/clusters is automatically determined as more data is observed, allowing the adaptation of models to flexible clustering structures (Elliot, et al., 2019; Teh, 2010) such as network arrangements, from which we take an analytical basis for this paper.

In our design, the Hotelling model industry can be seen as an aggregate network structure where the pattern of connections/relationships between firms and consumers is represented as firm-level and industry-level information flows along firms' network structured value chains. These information flows track the behavior of firms' performance through their underlying Networks of Targeted Communication (NTC's), which model the joint flow of acceptance or rejection of finished value chain activities through targeted messages across value chain components, in a framework of incomplete information à la Van Zandt (2004). We complete this approach in order to obtain a sequential game with a generalization of Burbano-Gómez & Sinisterra-Rodríguez (2023), and derive a Bayesian non-parametric stochastic framework using the properties of the Dirichlet distribution (Lin, 2016) and the Dirichlet Process (Teh, 2017).

The applications of Bayesian non-parametric techniques are also abundant in different economic topics. For instance, in public goods games (Beggs, 2009); in data annotation games (Servajean; et al., 2018); in neural games (Zhou, 2020) or, in more recent contributions, reformulations of the Bayesian-Nash equilibrium (Chen, et al., 2021).

3. Model

Organizations are understood as structures composed by one or more agents performing arbitrarily complex activities towards the achievement of particular objectives. This transversal concept encompasses almost all features of modern (and probably ancient) society, involving companies, corporations, institutions, associations, or any kind of outfit ranging between profit and non-profit activities, or even ranging between legal and illegal activities.

In his renowned theory of business management, Michael Porter (1985) presents value chains as the process of sequential activities for the progressive transformation of inputs towards the realization of outputs to be delivered to final clients, defining an informal framework to understand the operational mechanisms of organizations while preserving transversality. So, in order to take advantage of the transversality of both the concepts of organizations and value chains in terms of generality for our conclusions, our model proposes a formalization of Porter's value chains as its cornerstone.

Since the purposes of this research are within the study of Microeconomics and Industrial Organization, the language of this article explains our framework and implications regarding the specific case of organizations as firms engaging in market competition, however, it is important to stress that our formalization of Porter's value chains can be applied to conceptualize activities and contexts within the wide spectrum of organizations as defined above.

We conceive industries as the integration of the organizational processes that compose firms' value chains towards their interaction within the context of market competition. In this sense, we define our representation of competing firms as network structures by providing a framework to track the probabilistic behavior of their value chains as a stochastic process.

Firms' value chains are modeled as Networks of Targeted Communication (NTC's), an analytical structure where each completed value chain activity is communicated as a "message" (as in Van Zandt's framework). In this way, it is possible to provide a model that shows the interaction of the agents that compose firms' organizational structures along the sequences of activities that are necessary to yield a commodity or service. Hence, this interaction is represented from the initial actions of the firm within its organizational structure to its interaction with demand, where all firms participate in market competition.

Our framework is designed to describe arbitrarily complex organizational structures, so we kindly ask the reader to bear with an initially saturated notation, which will get simpler as we move towards the model's results and implications.

3.1 Value chains and network structures.

First, we define the representation of a single firm's value chain as a NTC. Let $G_j(V_j, E_j)$ be the firm's *value chain graph*, which shows its organizational (network) structure within an industry in which participate $j = 1, \dots, J$ firms. $G_j(V_j, E_j)$ is a *directed graph*, in which $V_j \subseteq \mathbb{Z}_+$ is the set of agents[1] that carry out the firm's value chain activities, such that their *sequential activities*[2] lead to the production and sale of goods or services. $E_j \subseteq V_j \times V_j$ is the set of pairs of these agents, which indicate the directed edges (connections) that ultimately form the firm's value chain. We can note the firm's value chain graph simply as G_j when its original notation is not required.

As stated before, sequential activities are represented by the connections between the firm's agents, in this sense, the described graph structure requires an additional specification in order to capture the essence of the model. Sequentiality along a value chain implies that the

[1] In graph theory, the elements of V_j are also called vertices or nodes. The labeled agents may represent individuals as well as groups of individuals conducting activities within a network-structured value chain.

[2] Since our model admits production structures for any kind of goods or services, these activities involve any kind of interaction of labor with other required production factors.

completion of an activity leads to the beginning of the one that follows, and so on up to the final activity, which is the sale of the obtained commodity.

The latter is modeled in this article as the communication of the culmination of an activity by its agent in charge, or *sender*, as a *message*. This *message* is directed towards the agent in charge of the next activity, or *receiver*, who processes this *message* and decides whether to accept it as an input to carry on with its own activity, or communicating its rejection to the *sender*[3]. In this sense, we find an analogy of interactions of offer and demand within the firm's value chain, as well as the common offer and demand interaction in market competition at the final activity of the value chain, where *receivers* are consumers, and the criteria for the acceptance or rejection of the offer contained in the *message* are subject to the matching of the firm's output to the consumer type's preferences.

Let $k = 1, \dots, r_j$ be the index for the sequential activities of firm j's value chain. Notice that r_j is the *terminal activity* of the value chain[4], in which the firm interacts with consumers within the frame of market competition. Each k-th activity of firm j's value chain requires to be modeled as a *bipartite subgraph* of G_j, i.e., a subgraph in which the participating agents are divided into two disjoint subsets, a subset $O_{jk} \subseteq V_j$ for *senders* who offer their culminated activity through a message, and a subset $D_{jk} \subseteq V_j$ for *receivers* who expect to receive a culminated activity as an input to their own activity, or as a commodity to purchase in the case of the terminal activity r_j. Sequentiality along the value chain implies the following[5]:

$$D_{jk} \equiv O_{j(k+1)} \ for\ k = 1, \dots, r_j - 1. \quad (1)$$

Since our upcoming analysis focuses in a single k-th component in order to draw conclusions for all $k = 1, \dots, r_j$ and ultimately for all firms in the industry $j = 1, \dots, J$, we can adopt the following notation for the elements of each k-th subgraph:

$$O_{jk} \equiv i_O = 1, \dots, z_O \qquad (2)$$

$$D_{jk} \equiv i_D = 1, \dots, z_D.$$

This notation provides the indices that allow us to construct the game design showing the stochastic behavior across the edges representing the sequential activities that compose firms' value chains towards market competition. We retake the sender-receiver terminology from Van Zandt (2004), we also take the O-D notation from Graham & de Paula (2020), who

[3] The latter leads to loops in firms' organizational processes. The occurrence of loops is compatible with this model; however, we neglect modeling loops since they are not crucial for the purposes of this paper and would make more difficult the understanding of its implications.

[4] We index the terminal activity as r_j since it is not mandatory that all firms' value chains have the same number of components.

[5] We place this scheme in order to give shape to a generalized representation of industries as the integration of value chains, however, the use of (1) is not a necessary condition for the model. Clearly, neglecting sequentiality for this game theoretical setting would imply representing other kinds of non-sequential economic interactions.

develop orientations for network data models under their alter-ego (a-e) notation in order to describe agents within directed graphs. We stress the O-D notation in order to emphasize the fact that an interaction between offer and demand is given at all scales of economic activities. Figure 1 below shows a graphic representation of value chain activities as bipartite subgraphs.

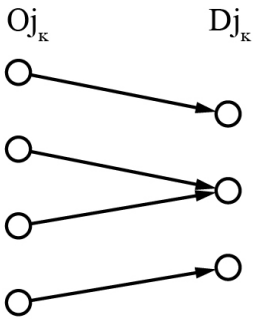


Fig. 1. Example of a value chain's arbitrary k-th activity represented as a bipartite subgraph, with a set of senders $O_{jk} \equiv i_O = 1,2,3,4 = z_O$, a set of receivers $D_{jk} \equiv i_D = 1,2,3 = z_D$, and a set of edges $E_{jk} \equiv \{(1,1),(2,2),(3,2),(4,3)\}$.

Notice that the decomposition of G_j into bipartite subgraphs implies an alternative notation for the firm's value chain graph, which provides a better description of its sequential essence:

$$G_j \equiv G_j\left(O_j \equiv \bigcup_{k=1}^{r_j} O_{jk}\,, D_j \equiv \bigcup_{k=1}^{r_j} D_{jk}\,, E_j \equiv \bigcup_{k=1}^{r_j} E_{jk}\right). \quad (3)$$

Naturally, the degree of complexity of a firm's value chain is given by the complexity of its value chain graph, described by (3). An example of a very simple value chain is the process of growing, harvesting and selling agricultural products in local markets of rural Colombia, such that the resulting value chain graph would be a one-to-one in line sequence of agents towards the terminal activity of sales. Conversely, complex examples of value chains such as the overseas sale of products via platforms like Amazon or Alibaba result in value chain graphs shaped as complex spanning trees with several branches. Figure 2 below provides an arbitrary example of a value chain graph.

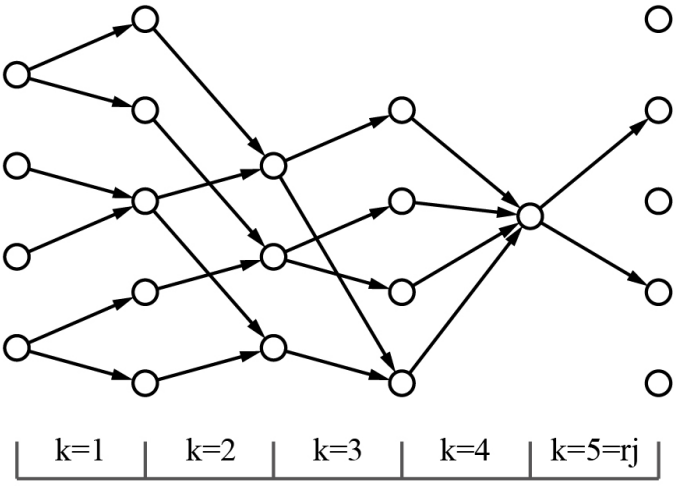


Fig. 2. Example of a firm's value chain with 5 activities.

Given (3), we can obtain a graph representation for the whole industry as an aggregate network:

$$\mathfrak{G} \equiv \bigcup_{j=1}^{J} G_j . \quad (4)$$

We interpret the above definition by saying that the aggregate *industry graph* $\mathfrak{G}$ is the union of all agents within each firm's value chain graph G_j, shaped by their corresponding activities represented by the edges E_j.

3.2 Game design

Since we have already provided proper notation for firms' value chain graphs composed by corresponding activities, now it is possible to design a game-theoretical framework that represents the strategic interaction within firms as value chains. Following the rationale of the edges of G_j being a representation of the connections of sequential activities in firm j's value chain, we provide a game design in which the completion of each k-th activity is communicated through messages between agents.

In this sense, game strategies consist in senders selecting the orientation of their messages towards specific receivers based on the maximization of pay-offs, which is given as a function of the matching of their offer to the receivers' preferences.

For $k \neq r_j$, these preferences are related to the acceptance of the finished activity as a proper input to carry on with the next value chain activity. Hence, we treat the efficiency in the use of production factors as the performance of the underlying set of probability distributions that describe the ratio of successful completion of k-th activities along the firm's organizational setting of its value chain. For $k = r_j$, the maximization of pay-offs regarding sales (revenue) implies a matching of the firm's finished product with costumers'

preferences, this is well-explored in Burbano-Gómez & Sinisterra-Rodríguez (2023). This article presents a generalization of the latter contribution to model industrial organization as a sequential game-theoretical framework, also presenting an adaptation of Van Zandt's logical restrictions (2004), which we explain below.

The following conditions will allow us to construct the set of games that characterize this model as a strategic NTC representation of industries as integrated value chains: (a) It is a network-structured frame composed by r_j sequential *subgraph games* for each firm j (each value chain activity is seen as a game corresponding to a bipartite subgraph k), communication is unidirectional (only from senders to receivers), and the realization of the outcome for each subgraph game k is given after all sent messages are processed by all receivers $i_D \in D_{jk}$[6]. (b) Each sender from O_{jk} can communicate with any subset of receivers from D_{jk}. (c) Receivers are passive agents, i.e., their role is relegated exclusively to the acceptance or rejection of the received offers given their preferences as parameters for the considered k-th activity, hence, strategic interaction is carried out only by senders. (d) Production factors and all observable and non-observable features of firms in market competition are embedded in the probabilistic behavior of each activity k, we could then say that this model presents a sort of "organizational competition", where the key to success is the articulation of information on industry practices to its structured application by firms' agents. (e) Market competition is closed with respect to $\mathfrak{G}$, i.e., r_j is the common terminal activity for each firm $j = 1, \dots, J$.

Consider some activity k of G_j and its corresponding subgraph game. Within activity k, there is a finite number of senders $i_O = 1, \dots, z_O \in O_{jk}$. Each sender communicates its offer as a single *unstructured message*. Recalling Van Zandt (2004), we say that the message containing i_O's offer is unstructured in the sense that it is received entirely or not received at all, the terms of the offer are fixed and display full information of the offered finished activity, and receivers cannot choose which message to process from among all those received.

Let $m_{i_D} \in \mathbb{Z}_+$ be the attention span of receiver i_D, i.e., the number of messages i_D can process until this action starts being avoided due to exhaustion. Since z_O is the maximum number of messages i_D may receive, the attention span is limited by the following inequality:

$$1 \leq m_{i_D} \leq z_O. \qquad (5)$$

Let $\ell_{i_D} \in \mathbb{Z}_+$ be the average amount of messages received by i_D in a fixed time point for activity k, such that $0 \leq \ell_{i_D} \leq z_O$. Considering the properties of unstructured messages, for simplicity, the probability of i_D processing a message is measured as a uniform distribution under the following expression:

[6] In general, the required time lapse for all sent messages to be processed is associated with the maximum estimated time for the completion of each k-th activity, this condition leads to compatibility of the game design with a characterization of industrial activities and market competition as a stochastic process. A direct interpretation of the latter is that the outcomes of each k-th activity are stored and read as a specific data structure for determined time points.

$$min\left\{1, \frac{m_{i_D}}{\ell_{i_D}}\right\}. \qquad (6)$$

(6) is subject to the relation between m_{i_D} and ℓ_{i_D}, which shows the level of information overload in the communication dynamics given limited attention among the agents in activity k. The importance of introducing limited attention to NTC games is well explored in Van Zandt (2004) and Burbano-Gómez & Sinisterra-Rodríguez (2023). Expression (6) can also be seen as a probability subject to the average fraction of available time for i_D to receive and process messages within a round of subgraph game k, this useful interpretation defines attention as a subset of time, being analogous to the concepts of attention schedules or business/office hours.

As senders possess incomplete information on receivers' preferences, it is necessary to model this information as *receiver types* in the sense of Bayesian games. Let $T_{i_D} \subseteq [0,1] \equiv \Theta_{i_D}^k$ be the *receiver type* for agent i_D, such that the behavior of all receivers in activity k is treated as a $(z_D + 1)$-tuple of Borel subsets $\Theta^k \equiv [0,1]^{z_D+1}$, each one of these measurable sets defining a probability space $\Theta_{i_D}^k$, which acts as the support for T_{i_D}, *which is a random variable that yields the number of acceptance outcomes generated by receiver i_D given all received messages, as a proportion of all acceptance outcomes in a round of subgraph game k*. For $k \neq r_j$, this is interpreted as acceptance decisions on the communicated finished activities; for $k = r_j$, i.e., market competition, this is interpreted as sales.

We define the dimension of each Θ^k as $z_D + 1$ since we need an additional "residual type" $T_{\overleftarrow{i_D}} \subseteq [0,1] \equiv \Theta_{\overleftarrow{i_D}}^k$, which yields the *proportion of rejected outputs by types T_{i_D} for all $i_D = 1, \dots, z_D \in D_{jk}$*. Rejections lead to losses for the corresponding senders i_O, inducing loops represented as edges with orientation $i_D i_O$[7], which we neglect in this work without loss of generality. As a consequence of the latter, let $T_k \subseteq \Theta^k \equiv [0,1]^{z_D+1}$ be a random vector representing the behavior of all receiver types in activity k.

In order to obtain a well-defined mapping for the probability of successful transactions for each sender, we define a measurable partition of each $\Theta_{i_D}^k$ into subsets corresponding to each i_O:

$$\Theta_{1i_D}^k, \dots, \Theta_{z_O i_D}^k \subseteq \Theta_{i_D}^k \quad for \ i_D = 1, \dots, z_D. \qquad (7)$$

This partition construction is ruled by the interpretation that the values for each $\Theta_{i_O i_D}^k$ imply that the outcome on T_{i_D} goes in favor of i_O for a round of subgraph game k. Given (7), probability measures over each $\Theta_{i_O i_D}^k$ are a representation of receivers' preferences for senders' outputs through *density clustering* behavior within each $\Theta_{i_O i_D}^k$, in the following way: when a significant density cluster is formed around values in some $\Theta_{i_O i_D}^k$, the high likelihood

[7] Which is why we note the residual type for rejections with an arrow pointing with opposite direction to the orientation of the edges that compose firms' value chain graphs.

for the acceptance of i_O's message indicates high preference for i_O's finished activity/product; conversely, when no significant density cluster is formed around values in some $\Theta^k_{i_O i_D}$, the low likelihood for the acceptance of i_O's message indicates low preference for i_O's finished activity/product.

It is worth noting that by seeing receivers' preferences as density clustering on $\Theta^k_{i_D}$, we treat all observable and unobservable parameters determining the industry's dynamics, including consumers' preferences, as embedded in the underlying probabilistic behavior of each value chain's k-th activity, providing a perspective for both consumers and producers choice theory from Bayesian non-parametric modeling[8]. Under this conceptualization we construct proper *success functions* for each sender i_O.[9]

Let $\gamma_{jk}(T_{i_D})$ be a function determining the density/mass for firm j's prior probability measure for a receiver type T_{i_D} in subgraph game k. Hence, $\gamma_{jk}(T_k)$ is defined as firm j's prior beliefs regarding the average behavior of industry standards on each k-th activity of its value chain. In this sense, each $\gamma_{jk}(T_k)$ represents firm j's prior beliefs with respect to the parameters (observable and unobservable) that rule each activity towards the completion and sale of its finished product, considering the standard behavior of all known activities within the industry. Microeconomically, we stress that each $\gamma_{jk}(T_k)$ contains all available prior information on the average efficiency in the use of production factors within the industry ($k \neq r_j$), as well as prior information on the average behavior of consumers' preferences ($k = r_j$).

For each subgraph game $k = 1, \dots, r_j$, let the following expression be firm j's density/mass function for its prior probability measure for successful transactions of sender i_O on receiver i_D, for all $i_O = 1, \dots, z_O$ and $i_D = 1, \dots, z_D$:

$$\gamma_{jk}\left(T_{i_O i_D}\right) = \psi^k_{i_O i_D} = \Pr[T_{i_O i_D} = t_{i_D} \in \Theta^k_{i_O i_D}]. \quad (8)$$

With $T_{i_O i_D} = t_{i_D} \in \Theta^k_{i_O i_D}$ being its associated random variable, and t_{i_D} a realization of T_{i_D} [10]. By (7), there is:

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[8] The seed for this perspective can be tracked in the rationale of Bayesian mixture non-parametric models. Ickstadt et. al. (2010) provide a view of these models in systems biology that is aligned with our research.

[9] Notice that this construction is possible for an unaccountably large number of measurable partitions, an example of a symmetric partition construction representing an initial state of perfect competition is presented in Burbano-Gómez & Sinisterra-Rodríguez (2023). This theoretical partition rationale is close to clustering techniques and algorithms within machine learning and Data Science, especially regarding clustering analysis based on the Polya Urn scheme and Blackwell-McQueen urn scheme, as it will be seen further in this article. For a survey on approximations towards cluster analysis see Duran & Odell (2013).

[10] Since our construction applies for integrable functions in general, (8) may refer both to p.m.f's or p.d.f's. Also, (8) is easily computable for a chosen probability measure, since (7) guarantees that measuring $\psi^k_{i_O i_D}$ consists in integrating $\gamma_{jk}$with $\Theta^k_{i_O i_D}$ as the domain.

$$supp\left(T_{i_O i_D}\right) = \Theta^{k}_{i_O i_D} \;\; for\; i_O = 1, \dots, z_O \;\; within\; each \;\; i_D = 1, \dots, z_D. \quad (9)$$

Such that the previously explained relation between *density clustering* on each T_k and receiver preferences is understood.

Since each $\psi^{k}_{i_O i_D}$ represents firm j's prior beliefs on the behavior of successful transactions between its network-structured senders and receivers, we call them *success functions*. They describe firm j's agents' capabilities to successfully carry out all of its value chain's activities within the industry, and provide a view of the firm's average efficiency in the use of production factors towards market competition. We can comprise all firm j's success functions as a set of $z_O \times z_D$ matrices for each subgame k:

$$\boldsymbol{\psi^{k}} = \boldsymbol{\psi^{k}}_{(z_O \times z_D)} = \left[\psi^{k}_{i_O i_D}\right] \; for\; k = 1, \dots, r_j. \qquad (10)$$

In conjunction with each $\gamma_{jk}(T_k)$, (10) describes the relation between the organizational (network) structure of firm j and the industry in the sense of the matching between firm j's potential finished activities/products and the technological/consumption standards for each asset and practice involved in the industry's dynamics. Thus, we can define the following expression:

$$\tau\left(G_j\right) \equiv \left\{\left(\gamma_{j1}, \boldsymbol{\psi^{1}}\right), \dots, \left(\gamma_{jr_j}, \boldsymbol{\psi^{r_j}}\right)\right\}. \quad (11)$$

We call $\tau\left(G_j\right)$ firm j's *organizational technology*, since it describes the (network) structure of information that the firm possesses regarding the parameters that determine the average organizational, technological and consumption behavior of the industry (γ_{jk}), as well as the (network) structure of information regarding the firm's capabilities to perform successfully within the industry ($\boldsymbol{\psi^{k}}$). $\tau\left(G_j\right)$ can be interpreted as firm j's approximation to the technological state of all assets and practices within the industry, involving the efficiency in the use of production factors as a whole, but specified through the concept of informational and organizational network structures. This conceptualization is clearly related to firms' production functions. (11) is the instrument through which firm j develops the set of strategies for all its participating agents, which we describe below.

Given our game design, the NTC structure of firms leads agents' game strategies to be represented as edges of G_j, these strategies compose the sequential functioning of the firm's value chain within the industry, and are determined in each subgraph game k by $\tau\left(G_j\right)$. Let $X_{i_O} \subseteq E_j$ be the set of *sender strategies* for a sender i_O, that is, the connections agent i_O forms by orienting its messages (communicating its finished activities) towards receivers given the matching structure in (10). For each subgraph game k, we define the set of *sender strategies* in the following way:

$$\boldsymbol{X_k} \equiv \left\{X_1, \dots, X_{z_O}\right\}. \quad (12)$$

Retaking Van Zandt's terminology, we call (12) firm *j*'s *strategy profile* for subgraph game k, and, considering all of the value chain's sequential activities, we construct the following collection, which we call firm *j*'s *organizational strategy*:

$$\xi(G_j) \equiv \left\{\boldsymbol{X_1}, \ldots, \boldsymbol{X_{r_j}}\right\}. \quad (13)$$

$\xi(G_j)$ represents strategical settings for joint activities of firm *j* within the industry given the network-structured actions of its participating agents. (13) can be punctually defined employing practical matrix representations of graphs, such as the graph Laplacian or the adjacency matrix.

Notice that in our construction firms' success is determined by the individual actions of each participating agent, if well these actions are subject to the industry information contained in the firm's organizational technology, optimality must be reached individually by each agent within each activity of the value chain.

Given a strategy profile $\boldsymbol{X_k}$ belonging to an organizational strategy $\xi(G_j)$, we build the expression for the expected number of successfully completed activities for each agent i_O within firm *j*'s value chain, as a result of our previous construction of the probability of processing (6) and acceptance (8) of messages:

$$\sigma_{i_O}(\boldsymbol{X_k}) = \int_{X_{i_O}} T_{i_O i_D}\left(min\left\{1, \frac{m_{i_D}}{\ell_{i_D}}\right\}\right) dF\left(\gamma_{jk}(T_{i_D})\right). \quad (14)$$

Where $F\left(\gamma_{jk}(T_{i_D})\right)$ is the CDF for $\gamma_{jk}(T_{i_D})$. The set of edges belonging to the sender strategy X_{i_O} indicates the selected subsets $\Theta^k_{i_O i_D}$ for each $\psi^k_{i_O i_D}$ to be evaluated given each $\gamma_{jk}(T_{i_D})$. (14) is a generalization of the expression for the expected value of sales presented in Burbano-Gómez & Sinisterra-Rodríguez (2023), and an adaptation of Van Zandt (2004) for our network-structured framework.

Let $S_{i_O} > 0$ be the *surplus coefficient* per successfully completed activity for a sender i_O, S_{i_O} is taken as a fixed exogenous coefficient per time period which gathers all technical properties regarding i_O's revenue or income from its achieved transactions. For $k \neq r_j$, S_{i_O} involves the behavior of concepts related to the return to production factors such as wages, salaries, or return to capital assets; for $k = r_j$, S_{i_O} is related to subjects such as market prices and firm revenue strategies.

Let $c_{i_O} = c_{i_O}(\gamma_{jk}(T_k)) > 0$ be firm *j*'s *transaction cost coefficient* for each sender i_O within an activity k. Each c_{i_O} composes the cost (network) structure of firm j based on its technological approximation of the assets and practices necessary to perform within the industry, provided by $\gamma_{jk}(T_k)$. Like S_{i_O}, transaction costs shown by each c_{i_O} are related to the efficiency in the use of production factors of firm j in each k-th activity of its value chain.

Now we can obtain the objective function for our game design, which is the expected net pay-off for each one of firm j's participating agents:

$$\pi_{i_O}(\boldsymbol{X_k}; c_{i_O}) = S_{i_O}\sigma_{i_O}(\boldsymbol{X_k}) - c_{i_O}\gamma_{jk}(X_{i_O}). \quad (15)$$

Notice that, as a consequence of (14), expected pay-offs increase strictly as a function of the expected number of successful transactions for all participating agents. The latter leads to the fact that degrees of certainty on successfully completed activities across value chains, measured by $\tau(G_j)$, is the driver of firm efficiency in this model. In this way, our condition of all aspects of industrial organization being embedded in firms' organizational technologies makes sense: sequential activities involving the network-structured use of production factors across firms' value chains determine the dynamics of market competition, and the level of control over uncertainty on these dynamics, reflected in the expected value of successful transactions for each firms' participating agent, is the key to efficiency and market dominance.

3.3 Equilibrium solution for all network-structured value chain activities

As sender strategies are driven by the matching mechanism between each γ_{jk} and $\boldsymbol{\psi^k}$, given the information provided by $\tau(G_j)$, our view of receiver types' preferences as density clustering leads to an intuitive equilibrium solution by pure dominant strategies for all activities composing the value chains of competing firms.

Consider the Tchebishev interval for receiver types under $\gamma_{jk}(T_{i_D})$ within a subgraph game k:

$$I^k_{i_D} \equiv I(T_{i_D}) \equiv \left(E[T_{i_D}] - \rho_{i_D}SD[T_{i_D}], E[T_{i_D}] + \rho_{i_D}SD[T_{i_D}]\right) for\ all\ i_D = 1, \dots, z_D. \quad (16)$$

With $SD[T_{i_D}]$ the standard deviation for T_{i_D} under $\gamma_{jk}(T_{i_D})$, and $\rho_{i_D} > 1$ a fixed parameter that graduates how many standard deviations the interval uses for its limit values. This general property of probability measures provides a unique criterion under which we can define the significance of density clusters for outcomes of receiver types T_{i_D} in favor of senders i_O, since it yields the values with highest probabilities for each $\Theta^k_{i_D} \equiv supp(T_{i_D})$.

Since, by (9), the success function $\psi^k_{i_O i_D}$ allows to compute the probability measure for $\gamma_{jk}(T_{i_D})$ supported at the subset of $\Theta^k_{i_D}$ that corresponds to each sender i_O, i. e., $\Theta^k_{i_O i_D} \equiv supp(T_{i_O i_D})$, we can use the Radon-Nikodym theorem (Lang, 2012) to obtain Tchebishev's interval for $T_{i_O i_D}$ in a subgraph game k, with its associated fixed parameter $\rho_{i_O} > 1$[11]:

$$I^k_{i_O i_D} \equiv I(T_{i_O i_D}) \equiv \left(E[T_{i_O i_D}] - \rho_{i_O}SD[T_{i_O i_D}], E[T_{i_O i_D}] + \rho_{i_O}SD[T_{i_O i_D}]\right) \quad (17)$$

[11] We specify the construction of (17) and suggest some of its microeconomic interpretations in the Appendix.

$$for\ all\ \ i_O = 1, \dots, z_O\ \ within\ each\ \ \ i_D = 1, \dots, z_D.$$

Such that this interval yields the highest certainty range for the priors regarding successfully completed activities by each sender i_O towards receiver i_D, represented by $\psi^k_{i_O i_D}$. For each subgraph game k, we fix $\rho_{i_D} \geq \rho_{i_O}$ since senders' uncertainty parameterization ρ_{i_O} should not exceed its analogous regarding firm j's knowledge of average industry behavior ρ_{i_D}. A gap between ρ_{i_D} and ρ_{i_O} can be fixed in order to provide a stable "tolerance to uncertainty" level, however, this is not mandatory for our model.

Given (7) and (9), we can view each $\Theta^k_{i_D} \equiv supp(T_{i_D})$ as a topological space such that all of the senders' Tchebishev intervals $I^k_{i_O i_D}$ are neighborhoods belonging to its chosen topology:

$$I^k_{1 i_D}, \dots, I^k_{z_O i_D} \in \mathcal{N}^k_{i_D}\ for\ all\ i_D = 1, \dots, z_D. \quad (18)$$

Where $\mathcal{N}^k_{i_D}$ is the chosen topology for each topological space $\Theta^k_{i_D}$. (18) guarantees the topological stability of our construction and game equilibrium.

The following assumption not only encloses the significance of density clusters within each $\Theta^k_{i_D}$ under a simple binary criterion, but also provides a quantitative definition for the network-structured opportunity costs for firms with respect to the uncertainty on their capabilities to successfully complete transactions across their value chains.

Assumption 1. Existence of non-negligible transaction costs for organizational technologies.

Given an organizational technology $\tau(G_j)$, for each subgame $k = 1, \dots, r_j$, each $c_{i_O}(\gamma_{jk}(T_k))$ is sufficiently large, in the following way:

$$c_{i_O}\left(\gamma_{jk}(T_k)\right) > S_{i_O}\vartheta_k\ \ with\ \ \vartheta_k = \max_{A_{i_O} \in A_k} \frac{\sigma_{i_O}(\boldsymbol{A_k})}{\gamma_{jk}(A_{i_O})}$$

$$for\ all\ i_O = 1, \dots, z_O\ and\ i_D = 1, \dots, z_D.$$

Where A_{i_O} are sender strategies within a strategy profile $\boldsymbol{A_k}$ composed only by types such that $I^k_{i_O i_D} \not\subseteq I^k_{i_D}$.

By stating the existence of a lower bound for agents' transaction costs in relation to the transactions' surplus based on the limits of $I^k_{i_D}$ (which summarizes the average successful behavior of assets and practices within the industry), non-negligible opportunity costs regarding uncertainty on the completion each k-th value chain activity are shaped, since the pay-offs agents obtain for successfully completing transactions are not large enough to compensate the costs of carrying out activities for receiver types for which certainty of acceptance is not sufficiently guaranteed ($I^k_{i_O i_D} \not\subseteq I^k_{i_D}$).

A direct interpretation of the latter can be found in any set of structured activities within firms or organizations in general, personnel is hired for determined labors based on their signaled skills conveying sufficiently high likelihood of performing successfully, otherwise, if they do not guarantee the completion of their tasks they should be replaced since they cause inefficiency via opportunity cost on the value chain and, therefore, losses to the firm or organization. Machinery, technological platforms or any kind of investment in capital assets are carried out by firms and organizations based on their features as a requirement for the completion of activities within value chains, their underlying opportunity cost is based on a sufficiently high level of certainty regarding the completion of these activities in a timely fashion.[12]

The study of NTC frameworks with heavily small costs, where Assumption 1 is removed, is well explored by Van Zandt (2004) and Hefti & Liu (2020), leading to scenarios with information overload in network settings. The usual analogous of information overload concerning industrial organization in general, given this article's framework, is organizational inefficiency in incipient or large scaled firms. The first ones must tolerate some level of inefficiency while they standardize their organizational processes and get to a stable level of market placement, the second ones, due to their large structure and market share, can tolerate persistent resource wasting in some activities of their value chains, not all of them, since opportunity costs are inherent to firms' structures, so Assumption 1 cannot be removed for the whole value chain. However, following the sequential rationale of our game design, the analysis of these firm structures should be focused on their strategies based on the capability to sustain resource wasting caused by organizational inefficiency through extended business cycles. The latter subject exceeds the purposes of this paper. Assumption 1 paves the way to the equilibrium solution of our game design.

Proposition 1. Network-structured game equilibrium.

Given Assumption 1, the equilibrium organizational strategy for each firm $j = 1, \dots, J$, $\xi(G_j) \equiv \{\boldsymbol{X_1}, \dots, \boldsymbol{X_{r_j}}\}$, is given by the following optimization rule for all senders $i_O = 1, \dots, z_O$, within each subgraph game $k = 1, \dots, r_j$:

$$arg \max_{T_{i_D} \in \mathcal{B}} \pi_{i_O}(\boldsymbol{X_k}; c_{i_O}) \equiv X_{i_O} \equiv \{i_O i_D \in E_j \mid I^k_{i_O i_D} \cap I^k_{i_D} \equiv I^k_{i_O i_D}\} \ for\ fixed\ \rho_{i_D} \geq \rho_{i_O}.$$

According to Proposition 1, the equilibrium organizational strategy for each firm $j = 1, \dots, J$, $\xi(G_j) \equiv \{\boldsymbol{X_1}, \dots, \boldsymbol{X_{r_j}}\}$, consists in all senders orienting their messages for completed activities only towards receiver types with the highest values for their success functions $\psi^k_{i_O i_D}$, which, given the corresponding organizational technology $\tau(G_j)$, are defined by the containment of $I^k_{i_O i_D}$ inside $I^k_{i_D}$.

[12] For cases where there are no senders with Tchebishev intervals outside of $I^k_{i_D}$ within the analyzed subgame, ϑ_k can simply be obtained based on a technological reference interval belonging to $\mathcal{N}^k_{i_D}$, such that the lower bound provided by Assumption 1 always exists.

As shown by the proof of Proposition 1, presented in the Appendix, and seen graphically in Figure 3, the pay-off maximizing sender strategies that compose firm j's equilibrium organizational strategy $\xi(G_j)$ consist in all agents using the containment of $I^k_{i_O i_D}$ in $I^k_{i_D}$ as the criterion to minimize uncertainty on the successful performance of their corresponding value chain activities. Any other collection of sender strategies yields lower expected pay-offs for agents and a waste of resources for the firm. In this way, containment of $I^k_{i_O i_D}$ in the interval $I^k_{i_D}$ defines the significance of density clusters as a fixed criterion to evaluate the preference of receivers for the finished activities/products communicated in the received messages. The equilibrium organizational strategy $\xi(G_j)$ given the available organizational technology $\tau(G_j)$, induces the optimal configuration for the sequential activities that compose firm j's value chain, being represented as a well-defined optimal network structure, or, in merely graph-theoretical terms, an optimal spanning tree.

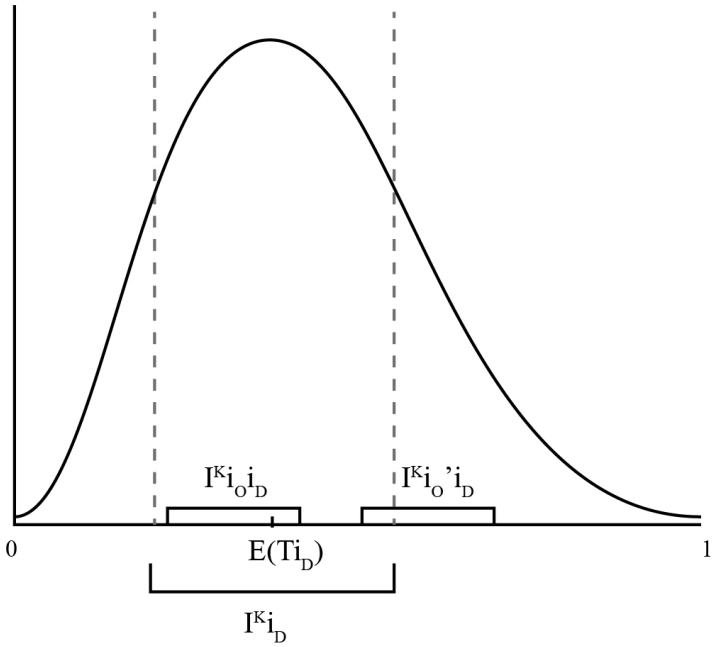


Fig. 3. Graphic example of Proposition 1 for an arbitrary receiver type T_{i_D}. Since $I^k_{i_O i_D}$ is contained inside T_{i_D}'s Tchebishev interval $I^k_{i_D}$, sending an offer to receiver i_D is optimal for agent i_O given its significantly high likelihood for success. On the other hand, $I^k_{i_O' i_D}$ is not contained in $I^k_{i_D}$, hence, it is not optimal for agent i_O' to send an offer to receiver i_D, since there is no significantly high likelihood for success, implying an average waste of resources.

Simple examples of value chains, such as one-to-one sequences of agents, should make the reader believe that our equilibrium solution is trivial most of the time, however, as firms' organizational structures and value chains grow larger, more complex, and decentralized, this equilibrium solution is anything but trivial, since it yields an optimal spanning tree $\xi(G_j)$ from among countably many possible spanning trees lying in the set of possible combinations of the vertices of G_j.

The latter is analogous to well-known matching mechanisms for the efficient use of production factors in economics. Acquisition of capital assets for specific production processes and labor market signaling towards job posts across value chains are a clear example of our rationale: capital assets with known features for specific production processes and job market candidates with proper profiles for job posts are more likely to succeed in firms' selection processes, since they lead to minimizing uncertainty on optimal performance by firms and organizations. Our equilibrium solution in Proposition 1 translates these conceptualizations as prior beliefs on the capabilities of production factors represented by $\boldsymbol{\psi^k}$ matching with the firm's prior information on the average behavior of efficiency within the industry, represented by γ_{jk}.

A remaining comment on the nature of our equilibrium solution is necessary. Notice that, since equilibrium in this game design is obtained with respect to minimization of uncertainty on the completion of value chain activities, our solution is independent of the distribution of sales profits among the firm's participating agents, the allocation of pay-offs per transaction across agents is exogenously defined in the relation between each S_{i_O} and its corresponding c_{i_O}, this issue, involving discussions on bargaining power between agents and organizations in specific areas of labor economics and industrial organization, and even in inequality analysis, is not of interest to this research.

3.4 Parameterizing $\boldsymbol{\tau(G_j)}$ through Beta/Dirichlet distributions

Having provided a well-defined game design and its mechanism towards our network-structured equilibrium solution, the purpose of this research leads us to construct a general parameterization that allows for a consistent probability measure of the behavior of industry dynamics according to the structure and essence of our model, specifically regarding firm success based on the minimization of uncertainty on the completion of value chain activities. The latter unequivocally implies that we must somehow quantify the difference in the level of uncertainty control by each firm j, so we arrive at a need for measuring distances between firms' organizational technologies $\tau(G_j)$. The Beta distribution, or its generalized form, the Dirichlet distribution, responds perfectly to our needs.

The Dirichlet distribution, also called Continuous Categorical distribution or Multivariate Beta distribution, is the most general probability measure for proportions or compositional random variables, i.e., normalized under the [0,1] interval. It provides a probability distribution for sum 1 random vectors, given a parameterization based on the positive real numbers. Below we present the definition of the Dirichlet according to our notation, based on Lin (2016), and outline its fitting to our model based on its structural properties to measure densities on probability simplices, which provide a simple parameterization for *density clustering* on each $\Theta^k \equiv [0,1]^{z_D+1}$.

For each activity $k = 1, \dots, r_j$ of firm j's value chain, let receiver types T_k be defined as a random vector $T_k = \langle T_1, \dots, T_{z_D}, T_{\overleftarrow{\iota_D}} \rangle$ of $z_D + 1$ components, with $T_{i_D} \geq 0$ for $i_D = 1, \dots, z_D, \overleftarrow{\iota_D}$ and $\sum_{i_D=1}^{\overleftarrow{\iota_D}} T_{i_D} = 1$. Then, let $\alpha^k = \langle \alpha_1^k, \dots, \alpha_{z_D}^k, \alpha_{\overleftarrow{\iota_D}}^k \rangle$, where $\alpha_{i_D}^k > 0$ for each

$i_D = 1, \ldots, z_D, \overleftarrow{i_D}$, being $\overleftarrow{i_D}$ our label for the additional "type" representing rejections. The Dirichlet probability density function is:

$$f^k(T_k) = \frac{\Gamma(\alpha_0^k)}{\prod_{i_D=1}^{\overleftarrow{i_D}} \Gamma(\alpha_{i_D}^k)} \prod_{i_D=1}^{\overleftarrow{i_D}} {T_{i_D}}^{(\alpha_{i_D}^k - 1)}. \quad (19)$$

Where $\alpha_0^k = \sum_{i_D=1}^{\overleftarrow{i_D}} \alpha_{i_D}^k$, $T_1 + \cdots + T_{z_D} < 1$ and $T_{\overleftarrow{i_D}} = 1 - T_1 - \cdots - T_{z_D}$. Thus, it is denoted that the receiver types' random vector T_k is Dirichlet distributed. According to our previous construction, we have:

$$T_k \sim \gamma_{jk}(T_k) = Dir(\alpha^k) \; for \; all \; k = 1, \ldots, r_j. \quad (20)$$

We can immediately check that the Dirichlet complies with the previously defined structure for receiver types in our game design, i.e., $supp(T_k) \equiv \Theta^k \equiv [0,1]^{z_D+1}$ for all $k = 1, \ldots, r_j$. Given (19), we obtain the expressions for the expected value and variance of each T_{i_D} in a subgraph game k, which are very simple expressions and will be the cornerstone of our analysis:

$$E\left[T_{i_D}\right] = \frac{\alpha_{i_D}^k}{\alpha_0^k} \quad (21)$$

$$Var\left[T_{i_D}\right] = \frac{\alpha_{i_D}^k \left(\alpha_0^k - \alpha_{i_D}^k\right)}{\left(\alpha_0^k\right)^2 \left(\alpha_0^k + 1\right)}. \quad (22)$$

Let us quote the following summary on the behavior of the Dirichlet distribution adjusted to our notation:

> "The Dirichlet distribution operates mapping the probability density for every possible combination of the $z_D + 1$ components for the sum 1 vector, i.e., density is distributed over a z_D-dimensional simplex. This distribution depends entirely on the configuration of the parametric vector α^k, also called concentration parametric vector. Reference intervals for α^k are explained below.
>
> When each $\alpha_{i_D}^k$ parameter has the same value, it is called symmetrical Dirichlet distribution. In this case, density is symmetrically distributed on the z_D-dimensional simplex, inside a space of z_D dimensions. When $0 < \alpha_{i_D}^k < 1$, density congregates at the edges of the simplex. As the values of $\alpha_{i_D}^k$ increase towards 1, density turns uniformly distributed all over the simplex, behaving as a "uniform Dirichlet". When $\alpha_{i_D}^k > 1$, density tends to concentrate at the central values of the simplex. Density is asymmetrical when the values of $\alpha_{i_D}^k$ are not identical (Lin, 2016), however, for asymmetrical configurations, density tends to concentrate around the expected value

correspondent to the components with the highest $\alpha_{i_D}^k$." (Burbano-Gómez & Sinisterra-Rodríguez, 2023, p. 13).

The Beta distribution is the univariate form of the Dirichlet ($z_D = 1$), as well as its marginal distribution $Beta\left(\alpha_{i_D}^k, \alpha_0^k - \alpha_{i_D}^k\right)$ for $i_D = 1, \ldots, z_D, \overleftarrow{i_D}$, for each $k = 1, \ldots, r_j$, such that it is determined by a concentration parametric vector of dimension 2. Its expected value and variance are the same as in (21) and (22) respectively, this will be a key concept for our upcoming construction.

The expression for variance in (22) provides a key feature for our model: as the parameters $\alpha_{i_D}^k$ increase, they reduce variance such that $Var\left[T_{i_D}\right] \to 0$ as $\alpha_{i_D}^k \to \infty$.

The above property yields the needed parameterization for density clustering in our model, which justifies the use of the Beta/Dirichlet distribution, together with its wide generality. We explain this by paraphrasing Burbano-Gómez & Sinisterra-Rodríguez summary on the "concentration property" of the Dirichlet (2023).

Given the expression in (22), increasing values of $\alpha_{i_D}^k$ yield progressively smaller standard deviations for T_{i_D}, and, therefore, smaller Tchebishev intervals $I_{i_D}^k$, which means less variability on the predictions by firms' prior beliefs that compose their organizational technologies $\tau\left(G_j\right)$, since the behavior of each T_{i_D} tends to concentrate around its expected value. In this sense, as the values of each $\alpha_{i_D}^k$ increase, so does certainty on the successful completion of value chain activities structured from the firm's organizational technology $\tau\left(G_j\right)$, and, considering our game design, there is increasing efficiency in the use of production factors given the equilibrium organizational strategy $\xi\left(G_j\right)$.

The "concentration property" of the Dirichlet provides a quantifiable relation between firms' levels of efficiency in their standards for the use of production factors (parameterized by each $\alpha^k \subseteq \mathbb{R}_+^{z_D+1}$) and their consequent structured beliefs on the average behavior of the industry contained in the organizational technologies $\tau\left(G_j\right)$. In this way, firms in competition present "informational" distances between their organizational technologies. Specifically, we are talking about distances between probability distributions. In Burbano-Gómez & Sinisterra-Rodriguez (2023) we present an application of this concept employing the Kullback-Liebler divergence as a distance. The KL divergence and similar instruments belong originally to the field of information geometry, properly summarized by Nielsen (2020).

The latter conceptualization is our generalized analogous to spatial distances in Hotelling's model, and is the key to our reinterpretation of Hotelling's Law. Moving forward, let us specify organizational technologies with Dirichlet distributions as their functional forms:

$$\tau\left(G_j\right) \equiv \left\{\left(\gamma_{j1} = Dir(\alpha^1), \boldsymbol{\psi^1}\right), \ldots, \left(\gamma_{jr_j} = Dir(\alpha^{r_j}), \boldsymbol{\psi^{r_j}}\right)\right\}. \quad (23)$$

3.5 The Dirichlet Process as the joint stochastic setting of the game design

We endowed our game design with a proper parameterization provided by the Dirichlet distribution, such that we can quantify the differences in the efficiency of firms' organizational technologies based on the size of progressively smaller Tchebishev intervals. However, our sequential game design still lacks a fundamental component in order to yield the main contribution of this research, that is, a mechanism that allows us to analyze the behavior of the joint activities of firms' value chains within industry dynamics. So, now we need to characterize the model as a stochastic process, for this purpose we appeal to the Dirichlet Process (DP).

"The Dirichlet Process (DP) is a stochastic process used in Bayesian non-parametric models of data, particularly in Dirichlet Process Mixture Models. It is a distribution over distributions, that is, each draw from a Dirichlet Process is itself a distribution. It is called a Dirichlet Process because it has Dirichlet distributed finite dimensional marginal distributions, just as the Gaussian process, another popular stochastic process used for Bayesian non-parametric regression, has Gaussian distributed finite dimensional marginal distributions. Distributions drawn from a Dirichlet Process are discrete, but cannot be described using a finite number of parameters, thus the classification as a non-parametric model." (Teh, 2017)

As placed in the above summary, the Dirichlet Process (DP), as a stochastic generalization of the Dirichlet distribution, possesses wide generality in statistical theory, particularly in Bayesian non-parametric modeling. It allows to model random distributions as a weighted composition of probability distributions, in the same way as the previously explained Dirichlet distribution models random vectors as a composition of random proportions.

Apart from its generality, we appeal to the DP to model our game design due to two fundamental properties. First, as it will be shown below, its structure allows us to present in the simplest way the behavior through time of firms' value chains and industry dynamics as a whole in the light of our game design, perfectly fitting to our notation and allowing us to use its conjugate prior properties to yield the implications of our work under simple proofs.

Second, being this reason very important for economic theory, it allows us to link the modeling of industrial organization dynamics to Bayesian non-parametric analysis. This means, as explained in the last statement of the quoted summary, that draws from the DP represent distributional outcomes that cannot be described using a finite number of parameters, hence, the DP provides a framework to study industrial organization as a stochastic process without appealing to factorial production functions or other parametric conventional schemes, so, the implications of this approach are based on the behavior of the participating agents and, specially, on the behavior of uncertainty regarding the successful completion of activities across value chains.

The following construction is based on Teh (2017), adjusted to our notation. Let $\widetilde{G}_j$ be the random distribution that characterizes the functioning of firm j's value chain within the industry, we note it by $\widetilde{G}_j$ since its structuring is in direct correspondence with firm j's value

chain graph G_j, composed by its bipartite subgraphs, which contain the participating agents for each subgraph game k, as seen in (3). The random distribution $\widetilde{G_j}$ is supported by a stem probability space Θ.

Let H be a probability measure over Θ and β be a positive real number. Now, we partition Θ into the probability spaces $\Theta^k \equiv [0,1]^{z_D+1}$ corresponding to the supports of receiver types T_k for each k-th subgraph game of firm j's value chain:

$$\Theta^1, \ldots, \Theta^{r_j} \subseteq \Theta. \quad (24)$$

We say that $\widetilde{G_j}$ is Dirichlet Process distributed with base distribution H and a strength parameter of the base distribution β, written $\widetilde{G_j} \sim DP(\beta, H)$, if $\widetilde{G_j}$ is defined such that its marginal distributions are Dirichlet distributed:

$$\left(\widetilde{G_j}(\Theta^1), \ldots, \widetilde{G_j}(\Theta^{r_j})\right) \sim Dir\left(\beta H(\Theta^1), \ldots, \beta H(\Theta^{r_j})\right). \quad (25)$$

Let us adjust (25) to our previous construction given (23):

$$\left(\widetilde{G_j}(\Theta^1), \ldots, \widetilde{G_j}(\Theta^{r_j})\right) \sim Dir\left(\beta Dir(\alpha^1), \ldots, \beta Dir(\alpha^{r_j})\right). \quad (26)$$

Since firm j operates with its organizational technology $\tau(G_j)$ describing its prior beliefs on the average behavior of the industry, the base distributions $H = Dir(\alpha^k)$ are the "mean" distributions of $\widetilde{G_j}$ for each k-th activity of firm j's value chain, such that for all $\Theta^1, \ldots, \Theta^{r_j}$, we have:

$$E\left[\widetilde{G_j}(\Theta^k)\right] = Dir(\alpha^k). \quad (27)$$

So, it is understood that given the priors on $\tau(G_j)$, firm j carries out its value chain activities under its equilibrium strategy $\xi(G_j)$ as a Dirichlet Process, such that the outcomes of each subgraph game k, i.e., the realizations of $\widetilde{G_j}(\Theta^k)$ that yield pay-offs across the value chain, show their behavior around $T_k \sim Dir(\alpha^k)$ for all $k = 1, \ldots, r_j$.

The role of β is similar to the concentration parameters of the Dirichlet distribution, considering the expression for the variance of $\widetilde{G_j}(\Theta^k)$ (being a random variable) for each subgraph game k under the DP:

$$Var\left[\widetilde{G_j}(\Theta^k)\right] = \frac{Dir(\alpha^k)\left(1 - Dir(\alpha^k)\right)}{\beta + 1}. \quad (28)$$

As β increases, variability on the outcomes of each $\widetilde{G_j}(\Theta^k)$ around their corresponding $Dir(\alpha^k)$ is progressively smaller, such that as $\beta \longrightarrow \infty$ there is $\widetilde{G_j}(\Theta^k) \longrightarrow Dir(\alpha^k)$, which is why β is called the strength parameter of the base distribution. Given the rationale of our

model, β is interpreted as the strength of the prior beliefs on the average successful assets and practices within the industry, since it measures how industry dynamics are determined by the consistency of the average behavior of value chains' activities, implying the previously related embedding of the technical aspects of the use of production factors and consumers' behavior in firms' organizational technologies $\tau(G_j)$. Since market competition implies the simultaneous operation of all firms based on their organizational technologies, we fix β for all firms $j = 1, \dots, J$.

The reader should be a little confused with the expressions in (27) and (28), since $Dir(\alpha^k)$ is a distribution for sum 1 random vectors and these expressions appear as univariate. The strict definition of the DP for our model at each k-th component's receiver type requires a univariate distribution, which in this case is the Beta distribution as the marginal distribution of the Dirichlet, applied to each vector component of each base distribution $Dir(\alpha^k)$; however, we maintain the notation with $Dir(\alpha^k)$ to emphasize that the realizations of $\widetilde{G_j}$ are value chain activities' outcomes given our game design (draws from $k = 1, \dots, r_j$ probability z_D-simplices) without this change being harmful to our construction. So, when analyzing these expressions, the reader should remember they apply to each univariate marginal distribution of each $Dir(\alpha^k)$.[13]

The notation in this article can lead the reader to consider our model as a Dirichlet Process Finite Mixture Model, since the random behavior of each $\widetilde{G_j}$ is represented as a mixture of r_j fixed network-structured distributions modeled by each corresponding $Dir(\alpha^k)$ base distribution. Nevertheless, we can construct each $\widetilde{G_j}$ and their interaction as a Dirichlet Process Infinite Mixture Model, which allows for the incorporation of new firms and value chain activities through time into the market design, the latter being an additional advantage of our framework towards a stochastic analysis of business cycles considering complex network structures for organizations, industries and economic clusters. We provide this construction in the Appendix.

Manouchehri et. al. (2021) present the estimation of a DP mixture model for Dirichlet-distributed data through Variational Learning (VL) towards medical applications, which can work as an initial reference for further empirical developments under our framework.

[13] The DP also requires that each component in (26) behaves as outcomes of random vectors of equal dimension. Given the Dirichlet as the prior base distribution, the latter can be achieved by considering a basis vector for all components, such that the dimension of this vector is the maximum $z_D + 1$ from among all $k = 1, \dots, r_j$. Then, each concentration parametric vector α^k is adjusted by reducing the parameter values of the i_D types that must be neglected in each component (if required) due to the difference with respect to the component's original dimension, such that their realization is near 0 a. c. In this sense, our previous construction is not harmed and the almost certainly null proportion of these types can be neglected in order to present a clearer notation for this article.

3.6 Terminal activities r_j, market competition and a reinterpretation of Hotelling's Law.

It is quite obvious that firms' success within the industry is measured at the terminal activity r_j, where all previous activities across firms' value chains are led towards market competition, so, all efforts by the agents participating in each firm $j = 1, \dots, J$, executing equilibrium organizational strategies $\xi(G_j)$ given the respective organizational technologies $\tau(G_j)$, are evaluated at the r_j component of each firm's DP. This fact simplifies our analysis without loss of generality.

Given the construction in the previous section, let us provide a description of the DP for the market competition subgame r_j as a component of the DP in (26), for all firms $j = 1, \dots, J$:

$$\widetilde{G}_j(\Theta^{r_j}) \sim DP\left(\beta, H_{i_D}^{r_j} = Beta\left(\alpha_{i_D}^{r_j}, \alpha_0^{r_j} - \alpha_{i_D}^{r_j}\right)\right) for \quad i_D = 1, \dots, z_D, \overleftarrow{i_D}. \quad (29)$$

Such that the base marginal distributions $H_{i_D}^{r_j} = \gamma_{jr_j}(T_{i_D})$, for all $i_D = 1, \dots, z_D, \overleftarrow{i_D}$, compose the prior base distribution for receiver types in subgraph game r_j: $T_{r_j} \sim Dir(\alpha^{r_j}) \in \tau(G_j)$.

For the market competition activity of firms' value chains, we have that the base distribution $T_{r_j} \sim Dir(\alpha^{r_j})$ measures firm j's prior beliefs on the behavior of the population of consumers of the goods/services provided by firms within the industry, so, each $H_{i_D}^{r_j} = \gamma_{jr_j}(T_{i_D})$ measures the prior beliefs on the behavior of consumer segments, such that the offered outputs produced as a result of each network-structured activity of firms' value chains seek to match with consumer preferences based on these priors, with the matching mechanism being determined by our previous density clustering construction described by $\tau(G_j)$ for all $j = 1, \dots, J$.

In this way, when we use the concentration property of the Dirichlet distribution to show the measurable efficiency differences between each firm's priors for the market competition subgraph game, i.e., $T_{r_j} \sim Dir(\alpha^{r_j})$[14], we are actually presenting a measure of the "informational" distance between firms' organizational technologies $\tau(G_j)$.

Following the previous rationale, and considering that the structure and asymmetries within the partitions $\Theta_{1i_D}^{r_j}, \dots, \Theta_{z_O i_D}^{r_j} \subseteq \Theta_{i_D}^{r_j}$, for $i_D = 1, \dots, z_D$, determine the intrinsic behavior of market shares within the industry based on costumers' preferences, and, therefore, how monopolistic/competitive it is; we stress that our game equilibrium and the upcoming

[14] The model in Burbano-Gómez & Sinisterra-Rodríguez (2023) presents a simplified form of our construction to provide an analysis on informative advertising and its consequences on market structure. It can be seen as an analysis of a terminal activity r_j of firms' value chains under an initial state of perfect competition, which is why it is quite similar to the analysis of prior beliefs on consumer preferences for this section. This model shows that significant efficiency asymmetries regarding consumer preferences priors among firms lead to alterations in average market shares.

implications are independent of the market structure of the industry, as well as of each firm's size and average market share. Since our framework is a stochastic process, the latter applies even in cases where the related partition is not fully known, for example, when the model is approached as a Dirichlet Process Infinite Mixture Model.

Notice that each firm may have more than one sender i_O in subgame r_j as "sales representatives", however, it is easier to view senders in the market competition subgraph game as $i_O = j = 1, \ldots, J$. Let us note the previous construction for $T_{i_D} \sim H_{i_D}^{r_j} = \gamma_{jr_j}(T_{i_D})$ as $T_{i_D}^{j} \sim H_{i_D}^{r_j} = \gamma_{jr_j}(T_{i_D})$ so that the indexation for the priors of each competing firm as a sender in subgraph game r_j is understood.
Following Burbano-Gómez & Sinisterra-Rodríguez (2023), we place Assumption 2 in order to facilitate measurable comparisons between organizational technologies $\tau(G_j)$ for the analysis of subgraph game r_j.

Assumption 2. Unbiased organizational technologies at subgraph game r_j.

The average behavior of consumer segments in subgraph game r_j, centralized at $E[T_{i_D}^{j}]$, is unbiased for all firms $j = 1, \ldots, J$:

$$E[T_{i_D}^{1}] = \ldots = E[T_{i_D}^{J}] = \mu_{i_D} \quad for\ all\ i_D = 1, \ldots, z_D.$$

According to Assumption 2, the consumers' segments expected proportions within the market μ_{i_D} are known and fixed. Economically, Assumption 2 is interpreted as firms knowing at least the average proportion of each segment i_D within the population of consumers as receivers in subgraph game r_j, so the differences in the efficiency of their organizational technologies $\tau(G_j)$ are given only by the variability of the priors $H_{i_D}^{r_j} = \gamma_{jr_j}(T_{i_D})$ they use to contrive their equilibrium organizational strategies $\xi(G_j)$ by evaluating the matching of γ_{jr_j} and $\boldsymbol{\psi}^{r_j}$ (recall (11)). The latter leads us to the specification of the role of the concentration property of the Dirichlet distribution in our model.

Consider the following statement for firms' priors $T_{i_D}^{1}, \ldots, T_{i_D}^{J}$ at subgraph game r_j, given $H_{i_D}^{r_j} = \gamma_{jr_j}(T_{i_D}^{j}) = Beta(\alpha_{i_D}^{r_j}, \alpha_0^{r_j} - \alpha_{i_D}^{r_j})$ according to (29):

$$\alpha_{i_D}^{r_1} > \ldots > \alpha_{i_D}^{r_J} \implies Var[T_{i_D}^{1}] < \ldots < Var[T_{i_D}^{J}]\ for\ all\ i_D = 1, \ldots, z_D, \overleftarrow{i_D}. \quad (30)$$

Given Assumption 2, larger parameters $\alpha_{i_D}^{r_j}$ imply smaller variance for the priors $T_{i_D}^{j} \sim H_{i_D}^{r_j} = \gamma_{jr_j}(T_{i_D}^{j})$[15], meaning that $\tau(G_j)$ is more capable to reduce uncertainty regarding the achievement of sales. Given its value chain network structure, this is equivalent to firm j's

[15] A proof for (30), involving its own version of Assumption 2, can be found in Appendix A of Burbano-Gómez & Sinisterra-Rodríguez (2023).

priors being more efficient towards the use of production factors in market competition. This relation is formally stated through the concept of First-order Stochastic Dominance (FSD).

Proposition 2. A partial order of First-order Stochastic Dominance (FSD) on the priors of $\tau(G_j)$ *at the market competition activity* r_j.

In equilibrium, a finite set of well-defined Dirichlet-distributed priors belonging to each firm's organizational technology $\tau(G_j)$ at subgraph game r_j, subject to Assumption 2 and ordered by their concentration parameters as

$$\alpha_{i_D}^{r_1} > \ldots > \alpha_{i_D}^{r_J} \quad for\ all\ \ i_D = 1, \ldots, z_D, \overleftarrow{\iota_D},$$

induces a partial order in the sense of First-order Stochastic Dominance (FSD):

$$\boldsymbol{H}_{i_D}^{r_1} \leq \ldots \leq \boldsymbol{H}_{i_D}^{r_J} \ for\ all\ \ i_D = 1, \ldots, z_D, \overleftarrow{\iota_D}.$$

With strict inequality for each pair $\{\boldsymbol{H}_{i_D}^{r_j}(T_{i_D}^{j}), \boldsymbol{H}_{i_D}^{r_{j+1}}(T_{i_D}^{j+1})\}$ for some $T_{i_D}^{j}$. We use $\boldsymbol{H}_{i_D}^{r_j}$ to note the CDF of $H_{i_D}^{r_j}$ for all $i_D = 1, \ldots, z_D, \overleftarrow{\iota_D}$.

FSD for $\boldsymbol{H}_{i_D}^{r_j}$, for all $i_D = 1, \ldots, z_D$, $\overleftarrow{\iota_D}$, over other competing firms' prior distributions implies, given (16) and the equilibrium solution $\xi(G_j)$, that firm j contrives its equilibrium network-structured value chain based on smaller Tchebishev intervals $I_{i_D}^{r_j}$, hence, planning its activities more accurately based on beliefs with less uncertainty, which provide higher expected profits than if it used any of the other dominated organizational technologies.

The latter can be seen directly with respect to the variance of $\widetilde{G}_j(\Theta^{r_j})$ seen as a random variable. Based on the expressions in (27) and (28), FSD yields less uncertainty for $\tau(G_j)$ and its consequent equilibrium organizational strategy $\xi(G_j)$ at r_j than for any other of the dominated organizational technologies.

Corollary 1. A partial order of First-order Stochastic Dominance (FSD) for variances based on the priors of $\tau(G_j)$ *at the market competition subgraph game* r_j.

Given Proposition 2 and the expression for the variance of $\widetilde{G}_j(\Theta^{r_j})$ as a random variable $V_{i_D}^{j}$, there is a partial order of FSD:

$$\boldsymbol{Var}\left[\Theta_{i_D}^{r_1}\right] \leq \ldots \leq \boldsymbol{Var}\left[\Theta_{i_D}^{r_J}\right] for\ all\ i_D = 1, \ldots, z_D, \overleftarrow{\iota_D}.$$

With strict inequality for each pair $\{\boldsymbol{Var}[\Theta_{i_D}^{r_j}], \boldsymbol{Var}[\Theta_{i_D}^{r_{j+1}}]\}$ for some $V_{i_D}^{j}$. We use $\boldsymbol{Var}[\Theta_{i_D}^{r_j}]$ to note the CDF of $V_{i_D}^{r_j}$ for all $i_D = 1, \ldots, z_D, \overleftarrow{\iota_D}$.

Proposition 2 yields a first view of our reinterpretation of Hotelling's Law concerning the prior distributions for each $\tau(G_j)$ at subgraph game r_j. We can check this by defining an arbitrary neighborhood $\boldsymbol{B}(\alpha^{r_j})$ centered at the concentration parametric vector α^{r_j} for the priors of each firm $j = 1, \ldots, J$, which works as an affordable efficiency range for $\tau(G_j)$ at subgraph game r_j. Given the lattice in Proposition 2, the optimal choice for each firm is to contrive its equilibrium organizational strategy $\xi(G_j)$ based on the largest element in $\boldsymbol{B}(\alpha^{r_j})$, which is the same as firms' organizational technologies $\tau(G_j)$ being in closest possible resemblance to the least uncertain of them, developing a pattern of technological and organizational similarities in their r_j-th value chain activity. Notice that this exercise is easy due to Assumption 2, generalizations removing this assumption imply operations considering diffeomorphisms on $\tau(G_j)$ seen as a set of statistical manifolds, which are more complex objects found outside of Economics, in the field of information geometry[16]. Parameterizations in the light of information geometry yielding $\boldsymbol{B}(\alpha^{r_j})$ can be seen as a sort of analogous to production functions under our framework. This is an interesting subject for further research.

Proposition 2 provides a good summary of a reinterpretation of Hotelling's Law in the light of our model; however, it cannot work as a proper reinterpretation for two reasons: it applies only to priors, and it does not directly consider the activities previous to r_j in firms' value chains.

We overcome these obstacles by appealing to Bayesian non-parametric modeling of the DP. Taken from Teh (2017) and adapting to our notation, consider the form for both the posterior base distribution and predictive distribution for the market competition subgraph game r_j, for all firms $j = 1, \ldots, J$, provided by the property of conjugacy between the Dirichlet and Multinomial distributions:

$$\widetilde{G}_j(\Theta^{r_j})|\theta_1, \ldots, \theta_{n_j} \sim DP\left(\beta + n_j, \frac{\beta H_{i_D}^{r_j}}{\beta + n_j} + \frac{\sum_{i=1}^{n_j} \delta_{\theta_i}(\Theta^{r_j})}{\beta + n_j}\right) \quad for\ all\ i_D \in \boldsymbol{X}_{\boldsymbol{r_j}}|\xi(G_j). \quad (31)$$

With $H_{i_D}^{r_j} = Beta\left(\alpha_{i_D}^{r_j}, \alpha_0^{r_j} - \alpha_{i_D}^{r_j}\right)$ the prior base distribution for $i_D = 1, \ldots, z_D, \overleftarrow{i_D}$ given $\tau(G_j)$ under the DP construction. θ_i for $i = 1, \ldots, n_j$ is a sequence of independent draws from firm j's DP, $\widetilde{G}_j$, which yields previous outcomes across firm j's value chain, involving the behavior of all k-th activities previous to r_j and previous results of market competition[17]; and $\delta_{\theta_i}(\Theta^{r_j})$ a point mass located at $\theta_i \in \Theta^{r_j}$. (31) describes the stochastic behavior of the market competition subgraph game r_j for firm j, it is quite straightforward that this distribution is a weighted sum of the prior base distribution $H_{i_D}^{r_j}$ given $\tau(G_j)$ and the empirical

[16] For a detailed reference, see Nielsen (2020).

[17] The outcomes $\theta_1, \ldots, \theta_{n_j}$ may be drawn following the sequentiality condition imposed by (1), however, this is not mandatory, since the construction of the posterior base distribution and predictive distribution for $\widetilde{G}_j$ verifies that $\theta_1, \ldots, \theta_n$ is an infinitely exchangeable sequence, subject to De Finetti's theorem for exchangeable sequences (Teh 2017, p. 6).

distribution obtained from n_j realizations of $\widetilde{G_J}$, in this way, both the posterior base distribution and the predictive distribution of $\widetilde{G_J}(\Theta^{r_j})$, (31), are determined by the predominance between β and n_j. (31) is also known as the Blackwell-McQueen urn scheme.

The following assumption leads to a simple proof of our reinterpretation of Hotelling's Law, also being a key statement regarding the essence of Industrial Organization.

Assumption 3. Industry dynamics with limited time horizons for analysis.

Given industry dynamics representations as $\widetilde{G_J} \sim DP$ and the consequent behavior of market competition described by (31), value chain and market outcomes are analyzed for a finite set of time points $\boldsymbol{\theta_I}$, such that:

$$\beta >> n_j \;\; for\ all\ j = 1, \ldots, J.$$

With $\beta >> n_j$ denoting β is much greater than n_j.

Microeconomically, Assumption 3 states that industry dynamics rely on the consistency of the prior beliefs around the use of production factors across the value chains that integrate in market competition, summarized by each $\tau(G_j)$. Although the future is unpredictable and may change the structure and practices of industries at any of their characteristic value chain standard activities, the very essence of firms and industries lies in the organizational processes they structure based on prior beliefs, in order to produce goods and services within market competition (a similar rationale is provided by Coase, 1937). So, this assumption is both convenient and microeconomically necessary.

The failure of Assumption 3 leads to the implication that, at some point, priors contained in $\tau(G_j)$ will become outdated and non-informative to this analytical structure ($n_j > \beta$), paving the way for the restructuring of industries based on new regularities represented in (31) by the empirical distribution, for which the implications of this model will apply again under a new structure of priors contained in updated organizational technologies for a new set of time points $\boldsymbol{\theta_{I'}}$. The latter is analogous to business cycles and the emergence of phenomena such as new consumption patterns, macroeconomic shocks or technological disruptions in the sense of Schumpeter's so-called creative destruction.

Let us note the posterior base distribution and predictive distribution for $\widetilde{G_J}(\Theta^{r_j})$ at (31) as $H_{i_D}^{r_j P}$ for all firms $j = 1, \ldots, J$.

Proposition 3. Reinterpreted Hotelling's Law.

In equilibrium, a finite set of well-defined Dirichlet distributed priors belonging to each firm's organizational technology $\tau(G_j)$, implying Proposition 2 and subject to Assumption 3, induces a partial order in the sense of First-order Stochastic Dominance (FSD) for $H_{i_D}^{r_j P}$,

which is both the posterior base distribution and predictive distribution of the outcomes of market competition r_j for all firms $j = 1,\ldots,J$, in the following way:

$$\boldsymbol{H}_{i_D}^{r_1 P} \leq \ldots \leq \boldsymbol{H}_{i_D}^{r_J P} \;\; for\ all\ i_D = 1,\ldots,z_D.$$

With strict inequality for each pair $\{\boldsymbol{H}_{i_D}^{r_j P}(T_{i_D}^{j}), \boldsymbol{H}_{i_D}^{r_{j+1} P}(T_{i_D}^{j+1})\}$ for some $T_{i_D}^{j}$. As a consequence of the latter, and indexing the least uncertain organizational technology as $\tau(G_1)$, all attainable expected profits for any firm $j = 1,\ldots,J$ given the extant organizational technologies in the industry are ordered as:

$$\pi_{jP}\left(\boldsymbol{X}_{\boldsymbol{r_j}}|\tau(G_1);\ c_j\right) \geq \ldots \geq \pi_{jP}\left(\boldsymbol{X}_{\boldsymbol{r_j}}|\tau\left(G_J\right);\ c_j\right).$$

We note $\boldsymbol{H}_{i_D}^{r_j P}$ as the CDF of $H_{i_D}^{r_j P}$ and π_{jP} as the posterior expected pay-offs of the market competition subgraph game r_j, for $j = 1\ldots,J$.

Hotelling's Law is now reinterpreted in the sense that the dominant strategy for all firms is to optimally operate under an organizational technology that is affordably closest to the least uncertain organizational technology, with distance being measured in an "informational sense" that embeds all aspects of the industry and market competition in our network structure approach, instead of the previous interpretation of spatial closeness.

Proposition 3 provides the pursued reinterpretation of Hotelling's Law as an extension of our previous arguments in Proposition 2, obtaining its implications considering the whole structure of firms' value chain activities through time and beyond the concept of FSD based only on priors. In this sense, the main conclusion of the constructed market competition model is the tendency of firms' value chains to develop organizational and technological similarities within the industry in order to minimize the uncertainty regarding the successful completion of their value chain activities, since the optimal management of uncertainty by firms, as conceptualized in this research, is capable to embed all key aspects of market competition treated in previous models.

Recalling the traditional interpretation of FSD describing a partial order for progressively safer gambles, Proposition 3 expresses market competition dynamics transversally, seeing it as the result of a set of gambles regarding each value chain activity, such that these gambles are contrived with different levels of optimality for each firm $j = 1\ldots,J$ based on uncertainty. This result makes sense as all economic activities performed by agents motived by individual interest in an uncertain world can be seen as gambles with diverse risk frames, such that each risk frame corresponds to the characteristic value chains that compose each economic sector or organizational ecosystem.

4. Concluding remarks

We have developed a framework in which progressive efficiency concerning the management of uncertainty with respect to the behavior of industry dynamics is seen as a

conceptualization of productivity, capable of embedding all observable and non-observable aspects of market competition. The latter led us to a reinterpretation of Hotelling's Law, i.e., firms' optimal behavior implies a tendency to organizational and technological similarities within industries. Thus, our analogous of Hotelling's spatial clustering is organizational and technological clustering, in the sense that closeness between firms is given by adopting similar value chain assets and practices, involving issues such as prices, outputs, locations, supply chains, contract figures, labor and capital asset structures, or any other set of aspects that determine performance within an analyzed industry.

We can borrow some of Hotelling's examples to advocate for our rationale regarding transversal similarity, such as shoe factories, political parties and religious organizations; or think of diverse scenarios from contemporary industry developments to check that, as unusual as they appear, similarity is an inevitable pattern in market competition. Competing apps tend to structure similar features in order to gain and retain users; social media marketing dynamics is mostly about repeating trends, K-pop entertainment agencies tend to promote similar acts, to sell similar merchandise, to organize similar creative processes and to train idols under similar standards[18]; even drug cartels tend to stablish similar trafficking routes and organized crime structures.

Organizational and technological similarities, which define industrial and maybe social dynamics as a whole, can be more accurately approached through Bayesian non-parametric or semi-parametric analysis, since it allows us to capture the relation between the set of observable and non-observable key aspects of any considered market, and the stochastic behavior of the agents that perform on their basis.

We believe that this framework suggests some gaps for further research which could result in important contributions for Economics. Our model implicitly considers firm interdependence given the structure of organizational technologies, however, models making it explicit could yield instruments that, given the stochastic nature of our proposal and our novel representation of Hotelling's similarity, lead to general tools for industrial policy, or for the understanding of the development of industries from the consolidation of pioneer firms.

Another important gap is a further consideration of price, elasticity and discrimination analysis in our model. We believe this matter can be engaged through an extension based on the geometrical characterization of spaces with structure-preserving operations over the variables composing π_{i_o}. Similarly, a characterization of these spaces, based on the S_{i_o} and c_{i_o} coefficients, and considering the average time frames for value chain activities, can result in an instrument for the identification of cluster scales, types of commodities and industries under this framework. Regarding these endeavors, we see huge potential in contributions belonging to the fields of Statistical Learning and information geometry, which are boosted by computational resources in the Information Age, and methodologies in the light of Data Science research.

[18] The ongoing plagiarism scandal between K-pop groups NewJeans and Illit can work as a very interesting example of Hotelling's similarity in market competition.

The key relation in our model between priors in organizational technologies and empirical results showed a hint for the study of dynamic clusters, business cycles, and changes in preference patterns through time, which leads us to, in our opinion, the most important gap of this article. That is, the development empirical and inferential proposals around network-structured Dirichlet Process Mixture Models, since, as the reader should notice, our framework can be applied outside industrial organization, in any frame of economic/organizational clusters with traceable NTC's. We provide an incipient suggestion for this matter in the Appendix.

Finally, given the conjecture of an underlying NTC for every economic/organizational cluster, how far can we take this type of model and its implications in Economics, game theory and social sciences? Maybe Harold Hotelling was right, and his Law can explain the majority of aspects of social organization. After all, we just discovered that Hotelling's Law has a much deeper meaning when analyzed from the scope of Porter's value chains.

Appendix.

Construction of Tchebishev intervals for senders' density clusters.

For each $T_{i_O i_D}$ in each k-th subgraph game, let $\varphi^k_{i_O i_D}: \psi^k_{i_O i_D} \longrightarrow \mathbb{R}$ be an integrable function inducing a probability measure given by the CDF $F(\varphi^k_{i_O i_D})$, such that $F(\varphi^k_{i_O i_D}; \Theta^k_{i_O i_D}) = 1$. $F(\varphi^k_{i_O i_D})$ is absolutely continuous with respect to $F(\psi^k_{i_O i_D})$, then, by the Radon-Nikodym theorem, $\varphi^k_{i_O i_D}$ is unique, and $F(\varphi^k_{i_O i_D}; \mathbb{s})$ can be computed as:

$$F\left(\varphi^k_{i_O i_D}; \mathbb{s}\right) = \int_{\mathbb{s}} \varphi^k_{i_O i_D}\, dF\left(\psi^k_{i_O i_D}\right). \quad (32)$$

For any measurable subset $\mathbb{s} \subseteq \Theta^k_{i_O i_D}$. $\varphi^k_{i_O i_D}$ turns $\psi^k_{i_O i_D}$ into the density/mass function of an alternative probability measure, allowing to treat $\Theta^k_{i_O i_D}$ as a probability space. Given (32), we can obtain the following adjusted expressions:

$$E\left[T_{i_O i_D}\right]\left(\varphi^k_{i_O i_D}\right) = E\left[T_{i_O i_D}\right] \quad (33)$$

$$SD\left[T_{i_O i_D}\right]\left(\varphi^k_{i_O i_D}\right) = SD\left[T_{i_O i_D}\right].$$

Which, applying Tchebishev's theorem, allow for the construction of the pursued interval in (17):

$$I^k_{i_O i_D} \equiv I\left(T_{i_O i_D}\right) \equiv \left(E\left[T_{i_O i_D}\right] - \rho_{i_O} SD\left[T_{i_O i_D}\right], E\left[T_{i_O i_D}\right] + \rho_{i_O} SD\left[T_{i_O i_D}\right]\right)$$

$$for\ all\ \ i_O = 1, \dots, z_O\ \ within\ each\ \ \ i_D = 1, \dots, z_D.$$

Additionally, in order to ensure that the analyzed density clusters have stable distributional behavior in relation to our model, we require that, for a subset $\Theta^k_{i_O i_D}$ to be considered as the

support of a density cluster, its associated random variable $T_{i_O i_D}$ has bounded measure with respect to $\gamma_{jk}(T_{i_D})$, in the following way:

$$\int_{I^k_{i_D}} \gamma_{jk}(T_{i_D})dt_{i_D} \geq \int_{I^k_{i_O i_D}} \psi^k_{i_O i_D}\, dt_{i_D} \geq \mathcal{L}_k \;\; for\ all\ i_O = 1, \dots, z_O \;\; within\ i_D = 1, \dots, z_D. \quad (34)$$

On the right-hand side, the arbitrary lower bound $\mathcal{L}_k \in (0,1)$ guarantees that the measure for $\Theta^k_{i_O i_D} = supp(T_{i_O i_D})$ is large enough to define an eligible density cluster within T_{i_D}. The choice of $\mathcal{L}_k$ should be closely related to the real scale of the receiver type T_{i_D} and the k-th value chain activity in which it is placed, being important for further empirical procedures under our framework. On the left-hand side, the upper bound defined by the measure of T_{i_D}'s Tchebishev interval $I^k_{i_D}$ has two roles, first, it allows to discard highly atypical cases of clusters with very high variances. Second, given the uniqueness of $I^k_{i_D}$ due to Tchebishev's inequality, it allows to describe the interaction between senders and receiver types as a matching scheme, by implying that for $\int_{I^k_{i_D}} \gamma_{jk}(T_{i_D})dt_{i_D} = \int_{I^k_{i_O i_D}} \psi^k_{i_O i_D}\, dt_{i_D}$ there is $I^k_{i_D} \equiv I^k_{i_O i_D}$, and, therefore, a limit case of perfect matching such that $T_{i_D} \equiv T_{i_O i_D}$. The latter translates into a frame where $T_{i_D} \equiv T_{i_O i_D}$ describes a sort of monopoly for sender i_O eith respect to receiver type i_D. Following the latter rationale, (34) provides a frame for the stochastic analysis of the level of concentration in economic activities, analogous to the analysis of market structures under our framework.

Proof of Proposition 1.

Consider any sender $i_O = 1, \dots, z_O$ in some subgame $k = 1, \dots, r_j$, then, fix $\rho_{i_D} \geq \rho_{i_O}$. Consider Tchebishev's inequality for any receiver type T_{i_D} under firm j's prior probability measure described by $\gamma_{jk}(T_{i_D})$:

$$Pr\left[t_{i_D} \in I^k_{i_D}\right] > 1 - \frac{1}{{\rho_{i_D}}^2}.$$

Since γ_{jk} integrated at any subset of T_{i_D}'s support is non-decreasing, the latter inequality and (18) imply:

$$F\left(I^k_{i_O i_D} \subseteq I^k_{i_D}\right) = Pr\left[t_{i_D} \in I^k_{i_O i_D} \subseteq I^k_{i_D}\right] > F\left(I^k_{i_O i_D} \nsubseteq I^k_{i_D}\right) = Pr\left[t_{i_D} \in I^k_{i_O i_D} \nsubseteq I^k_{i_D}\right].$$

Hence,

$$\int_{\Theta^k_{i_O i_D}} \left(\psi^k_{i_O i_D} \middle| I^k_{i_O i_D} \subseteq I^k_{i_D}\right) dt_{i_D} > \int_{\Theta^k_{i_O i_D}} \left(\psi^k_{i_O i_D} \middle| I^k_{i_O i_D} \nsubseteq I^k_{i_D}\right) dt_{i_D}. \quad (35)$$

So, the receiver types T_{i_D} with highest probability for any sender i_O are those for which $I^k_{i_O i_D}$ is completely contained in $I^k_{i_D}$. Since pay-offs for each sender i_O in any subgraph game k increase strictly on the expected value of successful transactions, and the latter increases strictly on $\psi^k_{i_O i_D}$, define an organizational strategy $\xi(G_j)$ for firm j, such that the strategy profiles for each subgraph game k are given in the following way:

$$\boldsymbol{X_k} \equiv \{X_1, \dots, X_{z_O} | X_{i_O} \equiv \{i_O i_D \in E_j |\, I^k_{i_O i_D} \subseteq I^k_{i_D}\}\} \quad for\ all\ k = 1, \dots, r_j.$$

And define an organizational strategy $\xi'(G_j)$, such that there is at least one k for which:

$$\boldsymbol{Y_k} \equiv \left\{\mathrm{Y}_1, \dots, Y_{z_O} \middle| \exists Y_{i_O}\ s.t\ \exists i_O i_D \in Y_{i_O} \quad s.t \quad I^k_{i_O i_D} \nsubseteq I^k_{i_D}\right\}.$$

An organizational strategy $\xi'(G_j)$ can occur under two possible cases. In the first one, at least one sender i_O in at least one subgame k neglects at least one $i_O i_D \in E_j$ for which $I^k_{i_O i_D} \subseteq I^k_{i_D}$, and, instead, selects an $i_O i_D \in E_j$ for which $I^k_{i_O i_D} \nsubseteq I^k_{i_D}$. Given (35) and considering that pay-offs increase strictly as a function of σ_{i_O}, it follows:

$$\pi_{i_O}(\boldsymbol{X_k}; c_{i_O}) > \pi_{i_O}(\boldsymbol{Y_k}; c_{i_O}) \quad for\ all\ \ k = 1, \dots, r_j.$$

In the second case, senders i_O may select all $i_O i_D \in E_j$ for which $I^k_{i_O i_D} \subseteq I^k_{i_D}$, and, additionally, select at least one $i_O i_D \in E_j$ where $I^k_{i_O i_D} \nsubseteq I^k_{i_D}$. Then, the expected transactions for these senders can be expressed as:

$$\sigma_{i_O}(\boldsymbol{Y_k}) = \sigma_{i_O}(\boldsymbol{X_k} \cup \boldsymbol{A_k})\ for\ at\ least\ one \quad k = 1, \dots, r_j.$$

Where $\boldsymbol{A_k}$ is a strategy profile such that $A_{i_O} \equiv \{i_O i_D \in E_j | I^k_{i_O i_D} \nsubseteq I^k_{i_D}\}$, then:

$$\sigma_{i_O}(\boldsymbol{Y_k}) = \sigma_{i_O}(\boldsymbol{X_k}) + \sigma_{i_O}(\boldsymbol{A_k}).$$

Following the above expression, expected pay-offs for any sender i_O under strategy profile $\boldsymbol{Y_k}$ are stated as:

$$\pi_{i_O}(\boldsymbol{Y_k}; c_{i_O}) = \pi_{i_O}(\boldsymbol{X_k}; c_{i_O}) + \pi_{i_O}(\boldsymbol{A_k}; c_{i_O}) = \pi_{i_O}(\boldsymbol{X_k}; c_{i_O}) + \left(S_{i_O}\sigma_{i_O}(\boldsymbol{A_k}) - c_{i_O}\gamma_{jk}(A_{i_O})\right).$$

Let us consider $\vartheta_k = \max_{A_{i_O} \in A_k} \frac{\sigma_{i_O}(\boldsymbol{A_k})}{\gamma_{jk}(A_{i_O})}$, the highest ratio between the expected value of transactions among sender strategies in $\boldsymbol{A_k}$, and the measure of its corresponding receiver types. Notice that, by (35), there is $\frac{\sigma_{i_O}(\boldsymbol{X_k})}{\gamma_{jk}(X_{i_O})} > \vartheta_k$ for all $i_O = 1, \dots, z_O$. Then, under Assumption 1, the following holds:

$$\pi_{i_O}(\boldsymbol{A_k}; c_{i_O}) = S_{i_O}\sigma_{i_O}(\boldsymbol{A_k}) - c_{i_O}\gamma_{jk}(A_{i_O}) < 0.$$

Implying:

$$\pi_{i_O}(\boldsymbol{Y_k}; c_{i_O}) = \pi_{i_O}(\boldsymbol{X_k}; c_{i_O}) + \pi_{i_O}(\boldsymbol{A_k}; c_{i_O}) < \pi_{i_O}(\boldsymbol{X_k}; c_{i_O}).$$

Hence,

$$arg\max_{T_{i_D}\in\mathcal{B}} \pi_{i_O}(\boldsymbol{X_k}; c_{i_O}) \equiv X_{i_O} \equiv \{i_O i_D \in E_j |\, I^k_{i_O i_D} \cap I^k_{i_D} \equiv I^k_{i_O i_D}\}$$
$$for\ all\ i_O = 1, \ldots, z_O\ \ in\ any\ \ k = 1, \ldots, r_j.$$

QED.

Proof of Proposition 2.

Initially, fix $\rho_{i_D} \geq \rho_{i_O}$ and let us consider only two organizational technologies $\tau(G_j)$ and $\tau(G_{j+1})$. Given (30), we obtain the following inequality for standard deviations:

$$SD(T^j_{i_D}) < SD(T^{j+1}_{i_D})\ \ for\ all\ i_D = 1, \ldots, z_D, \overleftarrow{i_D}\ \ in\ subgraph\ game\ r_j.$$

Now, regarding the Tchebishev interval, let the difference between $\rho_{i_O} SD(T^j_{i_D})$ and $\rho_{i_O} SD(T^{j+1}_{i_D})$ be defined as:

$$\rho_{i_O} SD(T^{j+1}_{i_D}) = \rho_{i_O} SD(T^j_{i_D}) + b.$$

With $b > 0$ a constant. Then, consider Tchebishev's inequality for $T^j_{i_D}$:

$$Pr\left[t_{i_D} \in I^{r_j}_{i_D}\right] > 1 - \frac{1}{{\rho_{i_D}}^2}.$$

Since all values on the support of T_{i_D} inside the intervals of size b do not belong to $I^{r_j}_{i_D}$, the following inequality is deducted:

$$Pr\left[t_{i_D} \in I^{r_{j+1}}_{i_D}\right] \leq 1 - \frac{1}{{\rho_{i_D}}^2}.$$

For all,

$$T^j_{i_D} \in \left(\mu_{i_D} - \rho_{i_O} SD\left(T^{j+1}_{i_D}\right), \mu_{i_D} - \rho_{i_O} SD\left(T^{j+1}_{i_D}\right) + b\right)$$
$$\cup \left(\mu_{i_D} + \rho_{i_O} SD\left(T^{j+1}_{i_D}\right) - b, \mu_{i_D} + \rho_{i_O} SD\left(T^{j+1}_{i_D}\right)\right).$$

In terms of cumulative distribution functions, the latter is equivalent to:

$$\boldsymbol{H}_{i_D}^{r_j} \leq \boldsymbol{H}_{i_D}^{r_{j+1}} \ for\ all\ \ i_D = 1,\ldots,z_D,\overleftarrow{\iota_D}.$$

With strict inequality at:

$$\left(\mu_{i_D} - \rho_{i_O} SD\left(T_{i_D}^{j+1}\right), \mu_{i_D} - \rho_{i_O} SD\left(T_{i_D}^{j+1}\right) + b\right) \cup \left(\mu_{i_D} + \rho_{i_O} SD\left(T_{i_D}^{j+1}\right) - b, \mu_{i_D} + \rho_{i_O} SD\left(T_{i_D}^{j+1}\right)\right).$$

This is equivalent to the definition of First-order Stochastic Dominance (FSD). By applying induction on $\alpha_{i_D}^{r_1} > \ldots > \alpha_{i_D}^{r_J}$ for all $i_D = 1,\ldots,z_D,\overleftarrow{\iota_D}$ using what we just proved, we obtain:

$$\boldsymbol{H}_{i_D}^{r_1} \leq \ldots \leq \boldsymbol{H}_{i_D}^{r_J} \ for\ all\ \ i_D = 1,\ldots,z_D,\overleftarrow{\iota_D}.$$

With strict inequality for each pair $\{\boldsymbol{H}_{i_D}^{r_j}(T_{i_D}^{j}), \boldsymbol{H}_{i_D}^{r_{j+1}}(T_{i_D}^{j+1})\}$ for some $T_{i_D}^{j}$.

QED.

Proof of Corollary 1.

Given (28), let us provide the expression for the variance of the random distributions at the market competition subgraph game r_j:

$$Var\left[\widetilde{G}_J\left(\Theta_{i_D}^{r_j}\right)\right] = \frac{Beta\left(\alpha_{i_D}^{r_j}, \alpha_0^{r_j} - \alpha_{i_D}^{r_j}\right)\left(1 - Beta\left(\alpha_{i_D}^{r_j}, \alpha_0^{r_j} - \alpha_{i_D}^{r_j}\right)\right)}{\beta + 1}$$

$$for\ all\ i_D = 1,\ldots,z_D,\overleftarrow{\iota_D}.$$

Since $Var\left[\widetilde{G}_J\left(\Theta_{i_D}^{r_j}\right)\right]$ depends on its prior base distributions, given Proposition 2 and the fact that $supp\left(\Theta_{i_D}^{k}\right) \equiv [0,1]$ for all $i_D = 1,\ldots,z_D,\overleftarrow{\iota_D}$, we immediately have a partial order of FSD:

$$\boldsymbol{Var}\left[\Theta_{i_D}^{r_1}\right] \leq \ldots \leq \boldsymbol{Var}\left[\Theta_{i_D}^{r_J}\right] for\ all\ i_D = 1,\ldots,z_D,\overleftarrow{\iota_D}.$$

With strict inequality for each pair $\{\boldsymbol{Var}[\Theta_{i_D}^{r_j}], \boldsymbol{Var}[\Theta_{i_D}^{r_{j+1}}]\}$ for some $V_{i_D}^{j}$.

QED.

Proof of Proposition 3.

Given Assumption 3, $\beta >> n_j$ for all $j = 1,\ldots,J$ allows us to neglect the weight of the empirical distributions in the posterior base distributions and predictive distributions at the

market competition subgraph game, $H_{i_D}^{r_j P}$ for $j = 1, \ldots, J$, such that Proposition 2 can be applied:

$$\boldsymbol{H}_{i_D}^{r_1 P} \leq \ldots \leq \boldsymbol{H}_{i_D}^{r_J P} \ for\ all\ i_D = 1, \ldots, z_D.$$

With strict inequality for each pair $\{\boldsymbol{H}_{i_D}^{r_j P}(T_{i_D}^{j}), \boldsymbol{H}_{i_D}^{r_{j+1} P}(T_{i_D}^{j+1})\}$ for some $T_{i_D}^{j}$. Given this partial order of FSD, consider an arbitrary neighborhood $\boldsymbol{B}(\alpha^{r_j})$ and (15), then, there is an underlying partial order for the equilibrium expected profits of any of the $j = 1, \ldots, J$ firms based on the efficiency of the extant organizational technologies:

$$\pi_{jP}\left(\boldsymbol{X}_{\boldsymbol{r_j}} | \tau(G_1);\ c_j\right) \geq \ldots \geq \pi_{jP}\left(\boldsymbol{X}_{\boldsymbol{r_j}} | \tau(G_J);\ c_j\right).$$

QED.

Notice that as n_j grows towards $n_j > \beta$, the weight of the prior base distributions progressively declines turning them uninformative, so Proposition 3 can no longer be guaranteed. However, obtaining updated priors based on a smoothing of the posterior base distribution and predictive distribution in (31) returns the implications of Proposition 3 in a new business cycle described by a set of time points $\boldsymbol{\theta_{I'}}$.

Our construction as a Dirichlet Process Infinite Mixture Model.

We provide a construction of our framework as a Dirichlet Process Infinite Mixture Model, which maintains the previously obtained implications for "infinite" firms' value chain activities and "infinite" firms participating in the industry[19]. This version provides an initial view towards a further empirical proposal from our approach, and allows for the incorporation of new firms and value chain components through time into the market design. Particularly, we build the model as a Hierarchical Dirichlet Process Infinite Mixture Model.

The first level of this hierarchical construction corresponds to industry-level activities, i.e., the common market competition frame associated to each firm's subgraph game r_j, which operate under their equilibrium organizational strategies $\xi(G_j)$. This level consists in a set of outcomes $\mathcal{Y}$ whose observations are the results for each firm given rounds of subgraph game r_j, so observations tend to group in J clusters corresponding to each firm $j = 1, \ldots, J$. For $\mathcal{Y}$, we specify a distribution corresponding to each cluster, $\widetilde{G*_j}$, it tracks the posterior results for each firm's market share performance through time as a product of their previous value chain activities. Each $\widetilde{G*_j}$ is a DP, however, they draw results within a common market competition frame, which is why they share a base distribution:

[19] Although the Infinite DP considers infinite components, suggesting infinite firms or value chain activities, its own properties lead outcomes to progressively group into J (industry-level analysis) or r_j (firm-level analysis) clusters depending on the size of the analyzed "dataset" (Teh, 2017).

$$\widetilde{G*_j}\,|G_0 \sim DP(\beta_j, G_0). \quad (36)$$

Where β_j is the associated strength parameter for the shared base distribution G_0, which is itself a Dirichlet Process:

$$G_0|H \sim DP(\beta, H). \quad (37)$$

(37) characterizes the second level of our hierarchical construction, which corresponds to the network-structured firm-level activities reflected in the terminal subgraph game r_j of each firm, which is why the base distribution for G_0 is composed by the posterior base distributions of each $\widetilde{G_j}(\Theta^{r_j})|\theta_1, \dots, \theta_{n_j}$, according to (31):

$$H = \begin{cases} H_{i_D}^{r_j P} \; if \; i_D \in \boldsymbol{X_{r_j}}|\xi(G_j) \\ 0 \; if \; otherwise. \end{cases} \; for \; all \; i_D = 1, \dots, z_D \;\; at \; each \;\; j = 1, \dots, J. \quad (38)$$

The elements for the dataset $\mathcal{Y} = \{\mathcal{y}_1, \dots, \mathcal{y}_N\}$ will consist in z_D-dimensional discrete observations of firms' market performance, each dimension corresponding to the consumer segments in market competition. Considering their grouping for $j = 1, \dots, J$ we note them as $\mathcal{y}_{ji}$, and define their structure:

$$\mathcal{y}_{ji} = (\mathcal{y}_{ji,1}, \dots, \mathcal{y}_{ji,z_D}) \; such \; that \; \mathcal{y}_{ji,i_D} = \begin{cases} 0 \; if \; i_D \notin \boldsymbol{X_{r_j}}|\xi(G_j) \vee \; t_{ji,i_D} \notin \; \Theta^{r_j}_{(i_O=j)i_D} \\ t_{ji,i_D} \;\; if \;\; t_{ji,i_D} \in \; \Theta^{r_j}_{(i_O=j)i_D} \end{cases}. \quad (39)$$

Each $\mathcal{y}_{ji}$ is associated with a latent parameter θ_{ji}, which assigns observations to their corresponding firm clusters following the hierarchical DP construction, in the following way:

$$\begin{aligned} &\theta_{ji}|\widetilde{G*_j} \sim \widetilde{G*_j} \\ &\mathcal{y}_{ji}|\theta_{ji} \sim F(\theta_{ji}). \end{aligned} \quad (40)$$

Where $F(\theta_{ji})$ is the distribution for $\mathcal{y}_{ji}$ given the assignation parameter θ_{ji}.[20]

We say this construction is a Dirichlet Process Infinite Mixture Model due to its clustering mechanism:

$$G_0 = \sum_{\mathcal{k}=1}^{\infty} \pi_{0\mathcal{k}} \delta_{\theta^*_{\mathcal{k}}}. \quad (41)$$

[20] Notice that (39) can be modified in order to allow designs where firms can obtain information regarding successful results by their competitors, being this a resource towards research on firm interdependence within our framework.

Where $\pi_{0\hbar} = \pi_{0\hbar}|\beta \sim GEM(\beta)$[21] is the weighting proportion for the point masses $\delta_{\theta^*_\hbar}$ located at $\theta^*_\hbar$, a latent parameter for $\hbar = 1, \ldots, \infty$, such that it is assumed that the support of the base distribution H is infinite, implying that this DP has infinite $\hbar$ components. This property is transferred to the first level of the hierarchy, such that the weights of each $\widetilde{G*_J}$ are given by those of their shared base distribution:

$$\widetilde{G*_J} = \sum_{\hbar=1}^{\infty} \pi_{j\hbar}\delta_{\theta^*_\hbar}. \qquad (42)$$

The point masses for $\theta^*_\hbar$ are shared across all of the clusters $j = 1, \ldots, J$, weighted by the mixing proportions of the model $\pi_{j\hbar}$. The latter fully describes the behavior of the dataset $\mathcal{Y}$ as a Dirichlet Process Infinite Mixture Model:

$$\mathcal{y}_{ji}|\widetilde{G*_J} \sim \sum_{\hbar=1}^{\infty} \pi_{j\hbar}F(\theta^*_\hbar). \qquad (43)$$

By the structure of H in (38), we have that the infinite $\hbar$-th components' latent parameters $\theta^*_\hbar$ will tend to present repeated values, such that there will be $\theta^*_\hbar = \theta_{ji}$ for $j = 1, \ldots, J$, as expressed in (40). [22]

Notice that an additional required entry for this model are the specific value chain activities' results for each firm, since they update the posterior base distributions that compose H. For further empirical contributions, these data can be obtained and structured through primary firm-level information, approximated in correspondence with $\mathcal{Y}$ by optimization/characterization techniques, or an intersection between both previous options. This property enriches our approach by giving place to broad modeling and optimization methods and techniques applied to the network structures of the considered firms/clusters, in the light of Data Science, Machine Learning or Information Geometry, among other disciplines that can help in improving this proposal.

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[21] $GEM(\beta)$, named after Griffiths, Engen & McCloskey (Teh, 2017), is the distribution for weights on the DP known as the stick-breaking construction. Do not confuse its π notation with the expression for pay-offs in the article.

[22] This Infinite DP construction can also be applied to each $\widetilde{G}_J$.

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