Climbing the Energy Ladder: How Energy Resources Hinder, Facilitate, and Fuel Economic Growth

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I show that the economy's potential for long-run growth depends on the characteristics of the energy resources it uses. As economies progressed from relying on harvested biomass resources to mined coal resources and tapped oil and gas resources, energy ceased being a drag on growth and became a facilitator of growth. A transition to a solar economy will not hinder long-run growth as long as the energy return on investment (EROI) of solar panels is sufficiently large. And if solar panels can eventually be produced from robots running on solar energy, then solar energy can drive economic growth even in the absence of technical change. Due to the upfront costs of initially developing solar panels, a solar economy may have been attainable only by transitioning through a fossil economy.

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1 Introduction

The ignition of sustained economic growth in the Industrial Revolution roughly coincided with the first sustained exploitation of fossil resources. Some attribute economic growth to that shift in the energetic basis of the economy (e.g., Wrigley, 1988; Pomeranz, 2000; Allen, 2009; Wrigley, 2010), whereas others argue that energy was not critical for growth (e.g., Mokyr, 1990; Clark and Jacks, 2007; Clark, 2014; Mokyr, 2016). Over the ensuing centuries, economic growth and fossil resource use have both continued, generating global wealth and global warmth that are both without precedent in human history. Now the global economy is in the midst of a transition from fossil resources to renewable resources, such as solar photovoltaics. Some fear that this transition will impede further economic growth by removing the fossil energy dividend that underpins it (e.g., King and van den Bergh, 2018; Jackson and Jackson, 2021).

I study the energetic basis of economic growth. In particular, I consider how fossil energy resources may have fostered growth and the implications of solar resources for growth. To this end, I extend a neoclassical Solow-Swan growth model to include energy as a factor of production. I show that the origin of that energy determines the types of growth outcomes that are possible.

Until the nineteenth century, all human societies were run on "harvested" resources in the form of biomass. These resources are extracted from an exogenous and renewable resource base via application of labor and capital. I show that the limited availability of land constituted a drag on growth in output per capita. Output per capita could tread water or grow only if technological change was sufficiently rapid. I call such an outcome "energy-hindered growth".

In the nineteenth century, several European economies began large-scale exploitation of "mined resources" in the form of coal. These resources require inputs of labor, capital, and energy, both to initially open a mine in a coal seam and to then extract coal from that seam over time. I show that mined resources cannot drive growth in output per capita but do enable output per capita to tread water even in the absence of technological change. As a result, positive growth arises with any degree of technological change. I call such an outcome "energy-enabled growth". The essential element of growth around the Industrial Revolution may well have been some factor other than energy, but the increasing use of coal energy may nonetheless have boosted the ability of that other factor to drive growth.

In the twentieth century, developed economies came to rely on "tapped resources" in the form of oil and gas. Initially tapping these resources requires an initial input of labor, capital, and energy to tap the resource, but once tapped, energy production does not scale with further inputs. In particular, oil and gas wells need inputs at the point of drilling, but once a well has been drilled, the pressure in the well drives production, without needing to send workers or machines into the well to coax the oil out. I show that such resources can sustain output per capita as mined resources do, but I also show that another outcome is

possible if energy is complementary to capital and labor in deposit tapping and each deposit does not produce enough energy. In that case, the availability of energy constrains the economy: energy cannot be reinvested at a high enough rate to ensure the continued supply of energy. I call such an outcome "energy-constrained degrowth".

In the twenty-first century, economies are increasingly reliant on solar energy. Solar panels have consistently outpaced expectations for installed capacity (Economist, 2024) and cost (Way et al., 2022), and solar power is on track to become the largest single source of electricity generation around 2030 (IEA, 2024). Solar panels are similar to oil and gas resources in that an initial investment produces energy for years on end. Solar panels' distinguishing feature is that they are manufactured, with machines and energy combining to convert sand into a standardized product.[1] I show that outcomes in a solar economy with machines produced from the capital stock are qualitatively similar to outcomes in an oil and gas economy. But many companies are working to automate both the production and installation of solar panels via use of robots and artificial intelligence (Liu, 2024; Plumer, 2024).[2] One day, these robots may themselves be produced in an automated fashion, from other robots and energy. In this case, I show that the economy can enter a regime of "energy-fueled growth", in which a self-replicating solar resource is able to fuel growth in output per capita even in the absence of technical change.

I tie each of these outcomes to energy return on energy investment, or EROI. This metric is much analyzed and discussed among scholars of energy systems.[3] EROI captures how much energy is produced for every unit of energy invested in making energy. Energy analysts have been especially interested in the possibility that the economy is approaching a "net energy cliff": because net energy production falls sharply as EROI approaches unity, society might eventually have to invest so much energy into producing energy that it becomes hard (or even impossible) to generate the energy required for the rest of civilization.[4] EROI looms especially large in current discussions for two reasons. First, the EROI of fossil fuels has been declining over time.[5] Second, many analysts worry that a shift to renewable energy

[1]Solar panels do require mining for minerals, including sand, but there is a salient difference with respect to a "mined resource" such as coal: the minerals embodied in solar panels subsequently produce a flow of energy that does not scale with ongoing inputs, whereas coal produces energy only at the moment of combustion.

[2]Nemet (2019) describes the key role that automation played in the declining cost of solar panels. The scope for automating production of solar panels potentially exceeds that of other sectors of the economy. For instance, the manager of a plant in Florida says, "City and state officials who have seen our plant contend this is likely the most automated factory in Florida" (Mendenhall, 2022).

[3]See Murphy and Hall (2010) and Brandt and Dale (2011) for histories of the concept. A small prior literature formally relates EROI and growth (Fagnart and Germain, 2016; Court et al., 2018; Fagnart et al., 2020).

[4]Analysts disagree about the minimum EROI required to sustain society: Hall et al. (2009) suggest that the minimum EROI is 5; Hall and Klitgaard (2012) say the minimum is probably around 10; Brandt (2017) argues for a minimum between 1.1 and 15; and Fizaine and Court (2016) suggest 11.

[5]See, among others, Cleveland (1992), Gagnon et al. (2009), Guilford et al. (2011), Murphy (2014), Court

would reduce EROI and risk approaching the net energy cliff.[6]

I show that EROI is endogenous to an economy, in the sense that it depends on input choices. EROI can even decrease in the physical productivity of an energy resource, depending on how energy is substituted for other inputs to energy production. I also show that EROI in an economy with a mined resource always exceeds unity plus the rate of population growth (i.e., $1 + g_L$). In an economy with a tapped resource having complementary inputs in its tapping or a solar resource having complementary inputs in its manufacture, I show that energy-constrained degrowth occurs when EROI cannot exceed $1 + g_L$ along a balanced growth path. This possibility substantiates fears of a net energy cliff. In an economy with a self-replicating solar resource fueling growth in output per capita, I show that EROI exceeds $1 + g_L$ by a margin that increases with the rate of economic growth. The EROI of the solar resource combines with the robotic returns to robot investment to drive energy-fueled growth, generating the promise of a "net energy ramp" to mirror the dread of the net energy cliff.

Even if a solar-fueled net energy ramp materializes, it will occur in a world full of carbon dioxide from prior fossil resource use. I show that a climate cost may have been an unavoidable byproduct of the path to a solar economy. Accessing a new type of resource and retooling the economy's capital stocks to work with it requires diverting a nontrivial portion of economic output from consumption. The upfront cost of learning to drill for oil and gas is not trivial, and the upfront cost of learning to harness and commercialize the photoelectric effect at the heart of solar panels is especially large. I show that when these fixed costs are high, an economy with a harvested resource may not generate enough output to be able to access oil/gas or solar resources, which would leave the economy trapped in a biomass world in the absence of an alternative. However, an alternative did exist, in the form of coal resources that were visible from the surface and readily usable for production. I show that when the coal resource's productivity is sufficiently large, the biomass economy optimally transitions to using the coal resource. Moreover, a productive coal resource can raise output per capita high enough to be able to pay the fixed cost of the oil and gas resource, so that the economy subsequently transitions to using the oil and gas resource. Finally, that oil and gas resource may be sufficiently productive to enable society to afford the fixed cost of the solar resource, and if that solar resource is sufficiently productive, the economy subsequently

and Fizaine (2017), and Brockway et al. (2019).

[6]There is much debate about the EROI of particular renewable technologies. Two analyses of the EROI for a renewable energy system suggest that it would be around 5, engendering concern about reaching the net energy cliff (Trainer, 2018; Capellán-Pérez et al., 2019). Others argue that the EROI for renewables will fall as the energy system changes (Trainer, 2018; Fabre, 2019) and that a transition to a renewable energy system would require a period in which so much energy is devoted to renewable energy development that energy becomes less abundant for the rest of society and energy sector emissions may even increase (King and van den Bergh, 2018; Sers and Victor, 2018; Slameršak et al., 2022). On the other hand, some argue that the EROI of renewable energy is now relatively large and will increase further over time as technology improves (e.g., Steffen et al., 2018; Diesendorf and Wiedmann, 2020).

transitions to the solar resource. The economy climbs an energy ladder towards the solar resource and the prospect of emission-free, energy-fueled growth. Emission-free solar panels thereby stand on the shoulders of dirty fossil resources.[7]

Economists have long analyzed the interaction between nonrenewable resources and economic growth. This literature is characterized by its focus on the scarcity of resources, with implications for the economy's ability to sustain long-run growth (e.g., Dasgupta and Heal, 1974; Krautkraemer, 1998) and for transitions between resources (e.g., Nordhaus, 1973; Chakravorty and Krulce, 1994; Tahvonen and Salo, 2001). Other literature endogenizes the role of innovation in determining the types of resources used (e.g., Acemoglu et al., 2012; Hart, 2019; Lemoine, 2024). The present paper is closer in spirit to a relatively small literature that studies the role of fixed costs in accessing deposits of natural resources (e.g., Hartwick et al., 1986; Holland, 2003; Venables, 2014), with the fixed costs here generalized and integrated into a growth model. Some recent work emphasizes that the flow of oil from a well is, for practical purposes, exogenous (Thompson, 2001; Mason and van't Veld, 2013; Anderson et al., 2018), based on a physical relationship known as Darcy's Law. My "tapped resource" is consistent with these papers' models.[8]

The "linearity critique" holds that sustained economic growth has fragile preconditions. To generate a trajectory with constant growth in output, economic models require a linear relationship somewhere under the hood: AK models place the linearity in physical capital accumulation and production, Lucas models place it in human capital accumulation, ideas-based endogenous growth models place it in productivity growth, and semi-endogenous growth models place it in population growth (Jones, 2005). These linear relationships are all knife-edge conditions, with small deviations ruling out sustained exponential growth (Growiec, 2010). The question is which type of linearity is well-founded, both in principle and empirically. Jones (2005) observes that there is no intuitive reason to prefer a model with linear growth in ideas to a model with either fishing-out or spillover effects, and Jones (1995) reports that linearity in idea accumulation is inconsistent with empirical evidence. Instead, Jones (2005) argues that population growth is sensibly linear since "people reproduce in proportion to their numbers", but many others think this linearity is contrary to both intuition and evidence (e.g., Solow, 2003; Growiec, 2010).

I show that energy can sustain growth when energy can be used to make more energy,

[7]Hansen and Prescott (2002) model a transition from a "Malthusian" land-based economy to a "Solow" economy that lacks a fixed factor. The decreasing returns to scale of the present paper's harvested resource are similar to the Malthusian economy of Hansen and Prescott (2002), and the constant returns to scale of the present paper's mined resource are similar to the Solow economy. In Hansen and Prescott (2002), the transition occurs because exogenous (and energy-free) technical progress raises productivity to the point where it is profitable to operate the Solow technology. I instead emphasize how the Solow technology can raise output per capita to the point where other resources become accessible.

[8]Moreno-Cruz and Taylor (2017) model how a resource's energetic density per unit of land affects the spatial distribution of economic activity. I abstract from space and study how a resource's energy produced per unit energy input affects the evolution of energy and output over time.

both directly as an input to solar panel production and by assisting in the production of robots that make solar panels. In that case, the stocks of robots and solar panels grow in tandem and the transition equation for solar panels becomes, in equilibrium, effectively linear. The mechanism has parallels in the recent literature on artificial intelligence and robots (see Trammell and Korinek, 2023). In particular, Mookherjee and Ray (2017) analyze when it is optimal for robots to crowd out labor in the production of robots and output. They show that this possibility depends on an additional robot producing sufficiently more than one extra robot even as robot production becomes entirely reliant on capital, which is an EROI-like condition (they refer to it as their "von Neumann singularity condition"). In that case, sustained growth is achievable, driven by capital accumulation. However, this literature generally ignores that operating robots requires energy. Incorporating a scarce energy input would tend to prevent sustained growth in economic output. I show that solar energy resources can restore the link between robots and growth if their production function permits self-sustaining feedback between energy and robots.[9]

The next section outlines the broader economic environment. Subsequent sections analyze harvested, mined, tapped, and solar resources, in turn. Section 7 analyzes the combination of solar panels and robots. Section 8 discusses actual energy transitions within the context of the model. Section 9 demonstrates the possibility of an energy ladder. The final section concludes. The appendix contains additional analysis and proofs.

2 Setting

I begin by introducing the elements of the model that will persist throughout the paper.

At each instant t, the economy is populated by $L(t)$ households who discount future utility at rate ρ. The population grows at rate $g_L \geq 0$, with $g_L < \rho$. Households supply a fixed unit of labor to the economy. They save a fixed share $s \in (0, 1)$ of the final good $Y(t)$ and consume the rest. They have utility $u(\cdot)$ over time t per-capita consumption, with $u(\cdot)$ increasing and concave. The aggregate capital stock is increased by savings:

$$\dot{K}(t) = sY(t) - \delta K(t),$$

with $\delta > 0$ the depreciation rate.

A representative firm produces the final good $Y(t)$ by combining labor $L_Y(t)$, capital $K_Y(t)$, and energy $E_Y(t)$ in a Cobb-Douglas production function:

$$Y(t) = A(t)L_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E},$$

where $\alpha_L + \alpha_K + \alpha_E = 1$. $A(t) > 0$ is productivity, which grows at rate $g_A \geq 0$.

[9]Nordhaus (2021) shows that artificial intelligence can increase the rate of economic growth if it makes capital substitutable for labor. This possibility is similar to my discussion of a roboticized solar economy in which solar panels are produced without labor inputs.

Available energy $E(t)$ depends on the time t resource base $R(t)$. Later sections will specify how energy is produced and how the resource base evolves. Depending on the section, the resource base will represent land, coal mines, oil and gas deposits, or solar panels.

I analyze the economy's optimal path, ranked according to the following welfare criterion:

$$\int_0^\infty e^{-\rho t} L(t)\, u\Big((1-s)Y(t)/L(t)\Big)\, \mathrm{d}t. \tag{1}$$

I am interested in the long-run evolution of output per capita. In particular, I am interested in the existence of a balanced growth path along which output is strictly positive and output, capital, and energy each grow at constant rates while maximizing (1) subject to resource constraints. Throughout, I use lower-case variables to indicate variables in per-capita form.

3 Harvested Resources: Biomass

Begin by considering a pre-industrial world in which energy derives from the land, whether via crops or trees. The land resource base R is fixed, so I drop the time argument. Energy produced from the land depends on labor $L_E(t)$ and capital $K_E(t)$ devoted to harvesting the resource:

$$E(t) = Q_H L_E(t)^{\phi_{HL}} K_E(t)^{\phi_{HK}} R^{\phi_{HR}},$$

where $\phi_{HL}+\phi_{HK}+\phi_{HR}=1$ and $Q_H>0$. There is only one use for energy, so $E(t)=E_Y(t)$. The labor market clears when $L(t)=L_Y(t)+L_E(t)$, and the capital market clears when $K(t)=K_Y(t)+K_E(t)$.

The following proposition establishes the growth rate of output per capita:

Proposition 1. *Along a balanced growth path, the growth rate of output per capita is* $[g_A - \alpha_E\phi_{HR}g_L]/[\alpha_L+\alpha_E(\phi_{HL}+\phi_{HR})]$.

Proof. See Appendix C. □

Technical change can increase output per capita, but it must overcome headwinds imposed by the fixed resource base. We can write the final good production function, in per-capita form, as

$$y(t) = A(t)\ell_Y(t)^{\alpha_L} k_Y(t)^{\alpha_K} \left[Q_H \ell_E(t)^{\phi_{HL}} k_E(t)^{\phi_{HK}}\right]^{\alpha_E} r^{\alpha_E\phi_{HR}}.$$

From the capital transition equation, capital and output must grow at the same rate on a balanced growth path. However, growth in capital is not sufficient for continual growth in output. First, the labor supply is constrained by population growth. Second, the land resource is fixed. The greater the value share of the land resource (i.e., the larger is $\alpha_E\phi_{HR}$), the greater the drag on growth in output per capita.

This economy exhibits two non-knife-edge types of long-run behavior, depending on the pace of technical progress. First, output per capita eventually collapses to zero if $g_A <$

$\alpha_E \phi_{HR} g_L$. If technical change is not sufficiently fast to overcome the diminishing returns in energy production, then output per capita collapses to zero. This is *energy-scarce degrowth* because the collapse is driven by lack of energy. Importantly, this case arises when technical change is nonexistent ($g_A = 0$). Second, output per capita grows forever if $g_A > \alpha_E \phi_{HR} g_L$. If technical change is sufficiently fast, then output per capita can grow forever. This is *energy-hindered growth* because growth occurs in spite of, not because of, the availability of the energy resource.

4 Mined Resources: Coal

Now consider an economy fueled by coal mines. There are two stages to obtaining coal from a mine. First, the mine must be opened: the hole must be bored and the shaft set up.[10] Second, coal must be extracted from the mine. The resource base $R(t)$ represents mines already opened and available for production. Opening a mine at time t uses labor $L_R(t)$, capital $K_R(t)$, and energy $E_R(t)$ inputs, and mining coal uses labor $L_E(t)$ and capital $K_E(t)$ inputs.[11] The energy market clears when $E(t) = E_Y(t) + E_R(t)$, the labor market clears when $L(t) = L_Y(t) + L_R(t) + L_E(t)$, and the capital market clears when $K(t) = K_Y(t) + K_R(t) + K_E(t)$.

Energy production from existing mines is

$$E(t) = Q_M L_E(t)^{\phi_{ML}} K_E(t)^{\phi_{MK}} R(t)^{\phi_{MR}}, \tag{2}$$

where $\phi_{ML} + \phi_{MK} + \phi_{MR} = 1$. The multiplier $Q_M > 0$ accounts for both the productivity of mining and the energy contained in each unit of coal.[12] Through appropriate rescaling, this production function is consistent with requiring energy to extract coal from a mine and transport it to market.[13]

[10] See Ashton and Sykes (1929, Chapter II) for a description of this process.

[11] Fouquet (2008, 222) observes, "At the pit face, the principal method of extraction was the miner's pick and shovel." And, of course, capital in the form of steam engines kept mines clear of water. See Ashton and Sykes (1929, Chapter III), among others. And see footnote 13 regarding the fuel for those steam engines.

[12] When describing the expansion of the coal industry, Wrigley (2010, 46) argues that labor productivity was static over 1700–1900, and Fouquet (2008, 57) writes, "...the main reason the industry expanded was simply because it used existing practices and multiplied the number of men and seams being exploited."

[13] First, imagine that time t energy production were instead $\tilde{Q}_M L_{Et}^{\tilde{\phi}_{ML}} K_{Et}^{\tilde{\phi}_{MK}} E_{Et}^{\tilde{\phi}_{ME}} R_t^{\tilde{\phi}_{MR}}$, with $\tilde{\phi}_{ML} + \tilde{\phi}_{MK} + \tilde{\phi}_{ME} + \tilde{\phi}_{MR} = 1$. The representative energy producer's first-order condition for E_{Et} would require that $E_{Et} = [\tilde{\phi}_{ME} \tilde{Q}_M L_{Et}^{\tilde{\phi}_{ML}} K_{Et}^{\tilde{\phi}_{MK}} R_t^{\tilde{\phi}_{MR}}]^{1/(1-\tilde{\phi}_{ME})}$. Substituting, time t energy production becomes $\left[\tilde{Q}_M (\tilde{\phi}_{ME})^{\tilde{\phi}_{ME}} L_{Et}^{\tilde{\phi}_{ML}} K_{Et}^{\tilde{\phi}_{MK}} R_t^{\tilde{\phi}_{MR}}\right]^{1/(\tilde{\phi}_{ML}+\tilde{\phi}_{MK}+\tilde{\phi}_{MR})}$. This is equivalent to our case with ϕ_{ML}, ϕ_{MK}, ϕ_{MR}, and Q_M appropriately defined. Second, imagine that production of energy does not require energy inputs but bringing energy to market imposes a fixed cost of c units of energy. If this production function has productivity $\tilde{Q}_M$, it is equivalent to our case if we define $Q_M \triangleq \tilde{Q}_M - c$. Moreno-Cruz and Taylor (2017) consider the implications of c varying with distance.

This energy production function is superficially similar to the one in Section 3, but there is an important difference: here the resource base is dynamic and endogenous, with the resource base at time $t+1$ composed of mines opened at time t and depreciated old mines. In turn, the quantity of new mines opened depends on labor, capital, and energy inputs. The transition equation for the resource base is:

$$\dot{R}(t) = Z(t)^{\omega} F(L_R(t), K_R(t), E_R(t))^{1-\omega} - \lambda R(t), \tag{3}$$

with $\lambda > 0$ and $\omega \in [0, 1)$. $Z(t)$ indicates the stock of available sites. When $\omega > 0$, mines are easier to open when more mine sites are available to choose from, in the spirit of models of resource extraction following Heal (1976), whereas when $\omega = 0$, mine sites are not depletable, as when coal resources are abundant. The stock of available sites evolves as

$$\dot{Z}(t) = -Z(t)^{\omega} F(L_R(t), K_R(t), E_R(t))^{1-\omega} + \Omega Z(t). \tag{4}$$

New deposits are discovered at rate $\Omega \in [0, \rho)$.[14] This discovery rate can also be interpreted as technical change that increases the economically recoverable resources from known deposits. Restricting $\Omega < \rho$ ensures that mine sites do not accumulate so rapidly that they are the ultimate source of economic growth.

The representative firm's production function for opening mines has the constant elasticity of substitution (CES) form:

$$F(L_R(t), K_R(t), E_R(t)) \triangleq \begin{cases} \left((1-\kappa_E)(A_{LK} L_R(t)^{\kappa_L} K_R(t)^{\kappa_K})^{\frac{\sigma-1}{\sigma}} + \kappa_E (A_E E_R(t))^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} & \text{for } \sigma \neq 1 \\ (A_{LK} L_R(t)^{\kappa_L} K_R(t)^{\kappa_K})^{1-\kappa_E} (A_E E_R(t))^{\kappa_E} & \text{for } \sigma = 1 \end{cases}, \tag{5}$$

where $\kappa_L + \kappa_K = 1$ and $\kappa_E \in (0, 1)$. $\sigma > 0$ is the elasticity of substitution between energy and labor inputs, and $A_{LK}, A_E > 0$ control the productivity of inputs in mine-opening.

Murphy and Hall (2010) define EROI as measuring energy gained over energy required to get it. But there are many possible metrics compatible with that definition—and the energy analysis literature has indeed used many types of metrics. My metric is in the spirit of the "net external energy ratio" (NEER), which is the ratio of the net energy produced in a system to the energy inputs from external sources (Brandt and Dale, 2011). I treat the system as the economy of an instant t. The external input is the energy devoted to mine opening (i.e., diverted from uses that assist in contemporaneous final good production) in the previous instant. Formally,

$$EROI(t) \triangleq \frac{E_R(t) + \dot{E}(t)}{E_R(t)}. \tag{6}$$

[14] For tractability, here deposits are easier to find when they are relatively abundant, but the opposite story is also plausible if economic incentives are stronger than geologic constraints on deposit discovery. Exhaustibility is not focus of my analysis, and I leave the study of endogenous discovery in a growth model to future work.

When $EROI(t) \geq 1$, the energy committed to opening mines is technically sustainable for some finite interval of time, without reducing energy available for final good production.

The following proposition establishes a lower bound on EROI:

Proposition 2 (EROI Lower Bound)**.**

$$EROI(t) > 1 + g_L + \frac{\dot{e}(t)}{e(t)}.$$

Proof. From definition (6) and the identity $\dot{E}(t)/L(t) = \dot{e}(t) + g_L e(t)$,

$$EROI(t) = 1 + \left(\frac{\dot{e}(t)}{e(t)} + g_L\right) \frac{e(t)}{e_R(t)}. \tag{7}$$

Energy market-clearing requires $e(t) \geq e_R(t) \geq 0$, with final good production strictly positive only if $e(t) > e_R(t)$. □

Economy-wide EROI must be large enough to generate surplus energy that can be dedicated to final good production, not just reinvested in mine opening. If energy grows only at the rate of population along a balanced growth path (i.e., if $\dot{e}(t) = 0$), then EROI must exceed $1 + g_L$ so that enough energy is produced from each energetic investment in resource extraction to keep up with population growth. If energy is growing faster than population along a balanced growth path (i.e., if $\dot{e}(t) > 0$), then EROI must be even larger so as to generate additional energy that can be invested in opening and extracting from yet more mines.

From here on, I eliminate exogenous productivity growth so as to highlight whether output growth is possible in the absence of technical change:

Assumption 1. $g_A = 0$.

The following propositions describe the potential for long-run growth that uses energy to make energy. Begin with the Cobb-Douglas case:

Proposition 3 (Cobb-Douglas Mine-Opening)**.** *Fix $\sigma = 1$ and let Assumption 1 hold. Any interior balanced growth path has all real variables growing at the same rate as population and the prices of unmined deposits and operating mines growing at rate $\rho - g_L$. Such a path exists if and only if either $\Omega > g_L$ or $\omega = 0$, and it has*

$$EROI(t) = 1 + \frac{1}{\phi_{MR}\kappa_E} \frac{\lambda + \rho}{\lambda + g_L} \frac{(1-\omega)(\rho - \Omega) + \omega(\rho - g_L)}{(1-\omega)(\rho - \Omega)} g_L.$$

The elasticity of output per capita on the balanced growth with respect to Q_M is strictly positive and constant in Q_M.

Proof. See Appendix E. □

Proposition 3 describes a case of *energy-enabled growth*. Whereas Proposition 1 showed that an economy with a harvested resource avoids a collapse in output per capita only if technical progress is fast enough, Proposition 3 shows that an economy with a mined resource can have output grow as fast as population even in the absence of any technical progress, as long as either resources are not depletable or deposit discoveries keep up with population growth. Because resource production has constant returns to scale, resources per capita are constant as long as inputs to mine opening increase at the same rate as population. And if the shares of labor and capital allocated to energy production are constant, then energy production will increase at the same rate as population. The foregoing implies constant per-capita inputs to final good production, so final good production grows at the rate of population growth. In this world, energy resources act like a second capital stock that must be produced from labor and capital inputs.

However, although energy enables output growth to match population growth, it cannot drive growth. Technical progress is still key for growth. Intuitively, energy per capita can grow no faster than capital per capita. In turn, through savings, capital per capita grows as fast as output per capita. However, labor constraints mean that output per capita cannot grow as fast as the growth rate of capital inputs. Formally, we have $g_e \leq g_k$, $g_k = g_y$, and $g_y = \alpha_K g_k + \alpha_E g_e$, which implies $g_k \leq (1 - \alpha_L) g_k$. The only solution has $g_k = 0$, which implies $g_e, g_y = 0$. Without an additional source of capital to drive growth in output per capita, sustained growth in capital per capita cannot happen, and thus neither can sustained growth in energy per capita.

In equilibrium, inputs adjust to ensure that EROI remains above $1 + g_L$, as required by Proposition 2. The precise EROI observed depends on economic forces. One might expect that increasing the productivity of energy in energy production (i.e., increasing A_E or Q_M) would increase EROI, but increasing either term also leads the economy to substitute towards energy inputs to mine opening: increasing A_E makes energy more productive relative to other inputs to mine opening, and increasing Q_M makes energy more abundant than other inputs to mine opening. This substitution works to reduce EROI through the declining returns to scale of increasing only one input in a CES production function.

In a Cobb-Douglas case ($\sigma = 1$), these two effects exactly offset. Based on engineering considerations, we may have intuitively expected EROI to depend on A_E and Q_M, but EROI is in fact independent of these technical parameters. EROI increases in ρ because less patient agents require a greater return on their investments in mine opening, and it decreases in ϕ_{MR} and κ_E because agents will invest more of their energy supplies in mine opening when the marginal energetic return of mines is larger (ϕ_{MR} is large) or energy inputs carry a larger weight in mine opening (κ_E is large).

The next proposition describes EROI under more general production functions for mine opening, using the assumption that capital and labor inputs have similar relative weights in mine opening, energy production, and final good production:

Assumption 2. $\kappa_K/\kappa_L = \phi_{MK}/\phi_{ML} = \alpha_K/\alpha_L$.

Proposition 4 (CES in Mine Opening)**.** *Let Assumptions 1 and 2 hold. Any interior balanced growth path has all real variables growing at the same rate as population and the prices of unmined deposits and operating mines growing at $\rho - g_L$. Such a path exists if and only if either $\Omega > g_L$ or $\omega = 0$. Along that path,*

i $EROI(t)$ is constant and is strictly greater than $1 + \frac{1}{\phi_{MR}} \frac{\lambda+\rho}{\lambda+g_L} \frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)} g_L$.

ii $EROI(t)$ increases in A_E and Q_M if $\sigma < 1$.

iii $EROI(t)$ decreases in A_E and Q_M if $\sigma > 1$.

Proof. See Appendix F. □

We again have energy-enabled growth. As before, EROI must be large enough to compensate for population growth (and indeed is so along a balanced growth path), but now EROI does depend on the technical parameters A_E and Q_M. When $\sigma < 1$, substitution towards energy inputs in mine opening is weak, leading the direct effects of A_E and Q_M on productivity to dominate the EROI calculation. EROI thus increases in A_E and Q_M, as many would expect to happen. But when $\sigma > 1$, substitution towards energy inputs in mine opening is strong, so that the effects of diminishing returns to energy investment dominate the direct effects of A_E and Q_M. Contrary to engineering intuition, EROI decreases in A_E and Q_M.

Among much else, the Industrial Revolution is known for a shift towards coal use. Many have debated whether this shift towards coal use spurred the takeoff in economic growth. We have obtain a more nuanced result. In the present setting, discovering the ability to use a coal resource saves the economy from the possibility that population growth erodes standards of living over time and does so irrespective of the productivity of that coal resource. Sustained growth in standards of living still requires technical change, but now any degree of technical change suffices, however small.

5 Tapped Resources: Oil/Gas

Now consider an economy with oil and gas resources. These require inputs when they are first tapped but do not require ongoing inputs to produce once they have been tapped: an oil or gas well must be drilled and connected, but pressure within the well then forces oil or gas to flow without requiring labor to haul it out of the well.[15]

The resource base $R(t)$ now represents deposits already tapped and producing, evolving as in (3). Tapping a resource deposit at time t uses labor inputs $L_R(t)$, capital inputs $K_R(t)$, and

[15] Anderson et al. (2018, 987) write, "...oil extraction is more akin to a "keg-tapping" problem than a cake-eating problem: extractors choose when to drill their wells (or tap their kegs), but the flow from these wells is (like the libation from a keg) constrained because of pressure and decays toward zero as more oil is extracted." Anderson et al. (2018) assume zero marginal costs of extracting from a tapped well, up to the physically determined constraint.

energy inputs $E_R(t)$, following (5). The energy market clears when $E(t) = E_Y(t) + E_R(t)$, the labor market clears when $L(t) = L_Y(t) + L_R(t)$, and the capital market clears when $K(t) = K_Y(t) + K_R(t)$. I again fix $g_A = 0$ in order to highlight the potential for growth in the absence of technical change.

Because oil and gas resources are depletable, I fix $\omega > 0$. And because oil and gas resources require energy, capital, and labor in drilling, I fix $\sigma \leq 1$.[16] Each tapped deposit produces $Q_D > 0$ units of energy:

$$E(t) = Q_D R(t). \tag{8}$$

Through appropriate rescaling, this production function is consistent with requiring energy to transport oil or gas to market (see footnote 13). The economy's EROI is as in (6), and Proposition 2 still holds.

This production function does not require that tapped deposits maintain constant productivity over time. Darcy's Law implies that the flow of oil from a tapped deposit decays over time, which is captured here by $\lambda > 0$.[17] Additional capital or labor inputs may be used to offset the decline in productivity by tapping new deposits, but energy production from already-tapped deposits does not scale with any such inputs.

I first delineate a set of familiar cases:

Proposition 5 (BGP with Tapped Resources)**.** *Let Assumption 1 hold and define*

$$\chi \triangleq (\lambda + g_L)\left(\frac{\lambda+\rho}{\lambda+g_L}\right)^{\frac{\sigma}{\sigma-1}}\left(\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right)^{\frac{\sigma}{\sigma-1}}(\Omega-g_L)^{\frac{\omega}{1-\omega}}.$$

Then any balanced growth path has all real variables growing at the rate of population and the prices of untapped deposits and operating wells growing at rate $\rho - g_L$.

i If $\sigma = 1$, such a balanced growth path exists if and only if $\Omega > g_L$. The elasticity of output per capita on the balanced growth with respect to Q_D is strictly positive and constant in Q_D.

ii If $\sigma < 1$, such a balanced growth path exists if and only if $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} > \chi$ and $\Omega > g_L$.

iii Along such a balanced growth path,

$$EROI(t) = 1 + \left(\frac{Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}}{\chi}\right)^{1-\sigma} g_L.$$

[16] Empirical work suggests that the elasticity of substitution between energy and value-added is strictly less than 1 in the energy supply sector: Koesler and Schymura (2015) estimate the elasticity of substitution between energy and value-added in resource mining as a (noisy) 0.42. Appendix A analyzes the case of $\sigma > 1$.

[17] The exponential form is consistent with prior work (Mason and van't Veld, 2013; Anderson et al., 2018).

Proof. See Appendix G. □

These are again cases of energy-enabled growth, in which the energy resource enables output to keep up with population but cannot drive output to grow faster than population. The results for $\sigma = 1$ in part i and the special case of $\sigma = 1$ in part iii are both familiar from the analysis of the mined resource. We obtain a similar result when $\sigma < 1$ as long as energy resources are sufficiently productive (part ii). The more that resource productivity exceeds the threshold χ, the more that EROI exceeds $1 + g_L$ (part iii).

The following corollary describes what happens when energy resources are not sufficiently productive:

Corollary 6 (Energy-Scarce Degrowth with Tapped Resources)**.** *Let $\sigma < 1$ and Assumption 1 hold. There is no balanced growth path if $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} \leq \chi$.*

Proof. See Appendix H. □

If energy resources are not sufficiently productive in making energy, then this economy cannot maintain constant output per capita. The requirement that $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} > \chi$ ensures that $EROI(t) > 1 + g_L$ along a balanced growth path. If a balanced growth path cannot have EROI greater than $1 + g_L$, then the inability to generate enough energy for deposit-tapping eventually binds the entire economy. This is a case of *energy-scarce degrowth*, as seen with the harvested resource. It is the type of outcome that analysts have in mind when they express concerns about a "net energy cliff" in EROI (Murphy and Hall, 2010). We here see that a net energy cliff requires that energy be an essential input in deposit tapping (because otherwise the economy could substitute capital and labor for energy) and that the resource be a tapped deposit and not a mined resource (because labor and capital can be used to produce more energy from already-opened mines).

6 Manufactured Resources: Solar

A solar resource is similar to oil and gas resources in that, once installed, solar photovoltaic cells produce energy for a period of time without ongoing inputs of labor or capital. However, a solar resource is different in that photovoltaic cells are manufactured, not found. The transition equation for the resource base becomes:

$$\dot{R}(t) = F(L_R(t), K_R(t), E_R(t)) - \lambda R(t).$$

We again maintain $\sigma \leq 1$, as manufacturing solar cells clearly requires both capital and energy inputs. Energy production is analogous to (8):

$$E(t) = Q_S R(t), \tag{9}$$

with $Q_S > 0$.

The following two corollaries describe outcomes in this economy:

Corollary 7 (BGP with Manufactured Resources)**.** *Let Assumption 1 hold and define*

$$\chi \triangleq (\lambda + g_L)\left(\frac{\lambda+\rho}{\lambda+g_L}\right)^{\frac{\sigma}{\sigma-1}}.$$

Then any interior balanced growth path has all real variables growing at the rate of population and the price of solar panels growing at rate $\rho - g_L$.

i If $\sigma = 1$, such a balanced growth path exists. The elasticity of output per capita on the balanced growth with respect to Q_S is strictly positive and constant in Q_S.

ii If $\sigma < 1$, such a balanced growth path exists if and only if $Q_S A_E \kappa_E^{\frac{\sigma}{\sigma-1}} > \chi$.

iii Along such a balanced growth path,

$$EROI(t) = 1 + \left(\frac{Q_S A_E \kappa_E^{\frac{\sigma}{\sigma-1}}}{\chi}\right)^{1-\sigma} g_L.$$

Proof. Follows proof of Proposition 5, with $\omega = 0$. □

Corollary 8 (Energy-Scarce Degrowth with Manufactured Resources)**.** *Let $\sigma < 1$ and Assumption 1 hold. Define χ as in Corollary 7. There is no balanced growth path if $Q_S A_E \kappa_E^{\frac{\sigma}{\sigma-1}} \leq \chi$.*

Proof. Follows proof of Corollary 6, with $\omega = 0$. □

These results are largely familiar from the case with a tapped resource, except that energy-enabled growth holds more generally (no longer requiring a sufficiently high rate of deposit discoveries). If inputs to solar panels are complementary and solar panels are not sufficiently productive, then we again risk energy-scarce degrowth.[18]

Many analysts have expressed concern about the lower EROI of solar photovoltaics compared to fossil energy resources (e.g., Trainer, 2018; Capellán-Pérez et al., 2019). Combining Corollary 8 with part iii of Corollary 7, we here see that while a decline in EROI may affect the level of consumption per capita, it does not affect the long-run growth rate of consumption as long as EROI remains above $1 + g_L$. However, if EROI falls any further, then the economy does not generate enough additional energy to have both energy and output keep pace with population growth.

[18] Appendix B shows that, if inputs are substitutable (so that energy can crowd out labor and capital in manufacturing), the economy can sustain growth in output per capita even in the absence of technical change.

7 Self-Replicating Resources: Roboticized Solar

We have seen that a solar economy has a qualitatively similar set of outcomes as an oil and gas economy, except with energy-enabled growth being more generic and with the risk that its smaller EROI may generate a qualitatively different particular outcome. I now assess the potential for interactions between solar panels and robotics to generate a qualitatively different set of outcomes.

Let energy production follow (9). The resource base of solar photovoltaics is now manufactured from energy $E_R(t)$ and from robots $B_R(t)$:

$$\dot{R}(t) = A_E B_R(t)^{\kappa_B} E_R(t)^{\kappa_E} - \lambda R(t),$$

with $\kappa_E, \kappa_B > 0$ and $\kappa_E + \kappa_B = 1$. Robots themselves are produced by combining energy $E_B(t)$ with robots $B_B(t)$. The total stock $B(t)$ of robots evolves as:

$$\dot{B}(t) = A_B B_B(t)^{\beta_B} E_B(t)^{\beta_E} - \Psi B(t),$$

with $A_B > 0$, $\beta_E, \beta_B > 0$, and $\beta_E + \beta_B = 1$. Robots depreciate at rate $\Psi > 0$. The robot market clears when $B_R(t) + B_B(t) = B(t)$, and the energy market clears when $E_Y(t) + E_R(t) + E_B(t) = E(t)$.

The following proposition describes outcomes in this economy:

Assumption 3 (Log Utility)**.**

$$u(Y_t/L_t) = \ln\left(Y_t/L_t\right).$$

Proposition 9 (Self-Replicating Resource)**.** *Let Assumptions 1 and 3 hold. Define g_e as the growth rate of energy per capita, g_y as the growth rate of output per capita, and g_k as the growth rate of capital per capita. There exists a balanced growth path with $R(t)/L(t)$, $E_R(t)/L(t)$, $E_Y(t)/L(t)$, and $B(t)/L(t)$ all growing at rate $g_e > 0$ if and only if*

$$\kappa_E^{1-\kappa_B} A_E Q_S > \chi_0, \text{ where } \chi_0 \triangleq (\rho + \lambda)\left(\frac{\beta_B}{\kappa_B \beta_E}\left(\frac{\rho + \Psi}{\beta_B A_B}\right)^{\frac{1}{\beta_E}}\right)^{\kappa_B}.$$

Along such a path:

i $g_y = g_k = \frac{\alpha_E}{\alpha_E + \alpha_L} g_e$.

ii g_e *increases in* Q_S, A_E, *and* A_B *and decreases in* ρ, λ, *and* Ψ.

iii There exists $\chi_1 > \chi_0$ *such that the prices of solar panels and robots decrease over time if and only if* $\kappa_E^{1-\kappa_B} A_E Q_S > \chi_1$.

iv $EROI(t)$ *increases in* A_E, Q_S, *and* A_B.

v $EROI(t) > 1 + \frac{1}{\kappa_E}\frac{\rho+\lambda}{g_L+\lambda}g_L$.

Proof. See Appendix I. □

We have *energy-fueled growth* when solar photovoltaic panels are sufficiently productive: the economy features strictly positive long-run growth in output per capita, despite the absence of technical progress.

To see why long-run growth is possible in the absence of technical progress, consider the dynamic equations governing the growth of the solar resource and of robots. The growth rate of the solar photovoltaic resource base is:

$$\frac{\dot{R}(t)}{R(t)} = Q_S A_E \left[\frac{B_R(t)}{E_R(t)}\right]^{\kappa_B} \frac{E_R(t)}{E(t)} - \lambda. \tag{10}$$

If robots could not keep up with the growth rate of energy, then the term in brackets would go to zero over time and the growth rate of energy must itself go to zero. If the shares s_R^B and s_R^E of robots and energy allocated to solar panel production are constant, then this becomes:

$$\frac{\dot{R}(t)}{R(t)} = Q_S A_E \left[\frac{s_R^B}{s_R^E}\frac{E(t)}{B(t)}\right]^{\kappa_B} s_R^E - \lambda. \tag{11}$$

The transition equation for $R(t)$ is linear in $R(t)$ when energy and robots grow at the same rate.

The growth rate of robots is:

$$\frac{\dot{B}(t)}{B(t)} = A_B \left[\frac{E_B(t)}{B_B(t)}\right]^{\beta_E} \frac{B_B(t)}{B(t)} - \Psi. \tag{12}$$

If energy could not keep up with the growth rate of robots, then the term in brackets would go to zero over time and the growth rate of robots must itself go to zero. Combining this result with the result from (10) that robots must grow at least as fast as energy, solar photovoltaics and robots must grow at the same rate on a balanced growth path. If the shares s_B^B and s_B^E of robots and energy allocated to robot production are constant, then this becomes:

$$\frac{\dot{B}(t)}{B(t)} = A_B \left[\frac{s_B^E}{s_B^B}\frac{B(t)}{E(t)}\right]^{\beta_E} s_B^B - \Psi.$$

The transition equation for $B(t)$ is linear in $B(t)$ when energy and robots grow at the same rate. So, combining with (11), when energy and robots grow at the same rate and maintain constant shares, we have an equilibrium with linear accumulation equations for solar panels and robots and therefore with the potential for strictly positive growth.

Combining equations (10) and (12) (by way of equation (A-43)), we find:

$$\frac{\dot{R}(t)}{R(t)} = Q_S A_E \left(\frac{A_B \frac{B_B(t)}{B(t)}}{\frac{\dot{B}(t)}{B(t)} + \Psi} \right)^{\kappa_B/\beta_E} \left(\frac{\beta_E \kappa_B}{\beta_B \kappa_E} \right)^{\kappa_B} \frac{E_R(t)}{E(t)} - \lambda.$$

The growth rate of the solar photovoltaic resource base declines in the growth rate of robots, because an increasing share of energy must be devoted toward robot production in order to sustain its high growth rate. The common growth rate g_e solves:

$$g_e = Q_S A_E \left(\frac{A_B \frac{B_B(t)}{B(t)}}{g_e + \Psi} \right)^{\kappa_B/\beta_E} \left(\frac{\beta_E \kappa_B}{\beta_B \kappa_E} \right)^{\kappa_B} \frac{E_R(t)}{E(t)} - \lambda.$$

If the solar resource is sufficiently productive, then the right-hand side is greater than zero when evaluated at the optimized factor allocation, in which case the unique g_e that solves the equation at that allocation is strictly positive. In line with this analysis, the proposition shows that the growth rate of energy per capita is strictly positive if and only if the solar resource is sufficiently productive, where the threshold is lower when robot production is more productive and robots depreciate more slowly.

The growth rate of output per capita is proportional to the growth rate of energy per capita (part i). The growth rates of energy per capita and output per capita increase in the productivity of the energy resource and the productivity of robot production and decrease in each depreciation rate and in the discount rate (part ii). The prices of solar panels and robots increase over time if solar panels are not too productive but decrease over time if solar panels are sufficiently productive (part iii). EROI increases in the productivity of the energy resource and of robot production (part iv), and strictly positive growth in energy per capita and output per capita requires that EROI exceed $1 + g_L$ by a sufficiently large margin (part v). Solar photovoltaics must generate enough surplus energy to outpace population growth.[19]

In this economy, the solar resource becomes self-replicating. Its energy is used to create the robots that combine with solar energy to produce more solar photovoltaics. Those additional solar panels generate additional energy that helps produce more robots, more photovoltaics, and more output. In contrast to the manufactured solar resource, solar photovoltaics here help produce all of their own inputs. If they are sufficiently productive, then their energy can generate economic growth. This story requires highly automated processes for fabricating solar photovoltaics, but once these exist, it does not require any ongoing technical change in any part of the economy. It also does not assume that robots directly affect final good production. Energy production here fuels its own growth and thereby fuels the growth of output per capita.

[19] Defining the robot return on robot investment (RROI) by analogy to EROI in (6), we have, from (A-50), that $g_e > 0$ only if $RROI(t) > 1 + \frac{1}{\beta_B} \frac{\rho + \Psi}{g_L + \Psi} g_L$.

8 Implications for Actual Energy Transitions

At the turn of the nineteenth century, the global economy was driven by biomass, a harvested resource. Per Section 3, technical change was necessary just to hold output per capita constant. Over the nineteenth century, the global economy came to be dominated by coal, a mined resource (Smil, 2010, Chapter 2). Within the model of Section 4, this transition generated a qualitative shift in behavior: energy-enabled growth became possible, with any degree of technical change now sufficient to induce growth in output per capita.

Over the course of the twentieth century, the global economy came to be driven by oil and gas (Smil, 2010, Chapter 2). It shifted from an economy that produced energy from coal with substantial ongoing labor and capital inputs to an economy that required inputs upfront, at the point of tapping an oil or gas well. Within the model of Section 5, the high EROI of oil and gas kept the economy within a regime of energy-enabled growth. However, the EROI of oil and gas deposits has declined over time—and may decline further—as oil and gas deposits have become increasingly scarce (see footnote 5). And a further reduction in EROI could occur as fossil plants are paired with technology to capture carbon emissions, whether onsite or through direct air capture. Within the model of Section 5, the oil and gas economy risks entering a regime of energy-scarce degrowth if the decline in EROI becomes substantial enough.

Within the next decades, solar energy may dominate the economy. Once solar panels are produced, they generate energy with little to no ongoing inputs, much as oil and gas resources do. But producing solar panels does require energy, and some literature suggests that solar panels have a smaller EROI than do contemporary oil and gas resources (see footnote 6). Two qualitative shifts are possible for growth behavior in a solar economy.

First, if energy inputs are complementary to labor and capital inputs in the production of solar panels, then the reduction in EROI could shift the economy from a regime in which any technical change generates growth to a regime in which substantial technical change would be necessary to overcome the drag from energy-scarce degrowth (Section 6). This pessimistic scenario manifests the "net energy cliff" that some analysts fear will be induced by a transition to a solar economy (e.g., Trainer, 2018; Capellán-Pérez et al., 2019).

Second, if energy and automated robotic inputs can substitute for other inputs in the production of solar panels, then solar panels may shift the economy from relying on resources with scarce deposits to relying on resources produced from ever-more abundant factors of production. In that case, a solar economy may experience energy-fueled growth, even in the absence of further technical change (Section 7). A fraction of the electricity generated from existing solar panels would be reinvested in producing more solar panels in automated factories and another fraction would be reinvested in producing more automated robots. This process would make energy increasingly abundant. This optimistic scenario manifests a "net energy ramp", which is the energetic counterpart of the growth takeoff scenarios pondered by scholars of artificial intelligence (e.g., Sandberg, 2013; Nordhaus, 2021).

9 The Energy Ladder

The switch from biomass resources to coal (and then to oil and gas) may have already transformed energy from a drag on growth to an enabler of growth, and the optimistic net energy ramp scenario for roboticized solar promises a further transformation in which energy could directly drive growth. However, even if the optimistic scenario comes to pass, that sequence of transitions had a real cost: the century-plus of intensive fossil fuel use loaded the atmosphere with carbon dioxide and thereby saddles the solar economy with climate change. I now consider the possibility that the world could not have achieved the solar economy without creating climate costs.

I demonstrate this possibility within a stylized model in which biomass (harvested), coal (mined), oil/gas (tapped), and solar (whether manufactured or self-replicating) resources all exist. The economy can use only one type of resource at a time, as if its capital stock is tooled to a certain type of resource, and the economy reaches a balanced growth path when one is available. At time 0, only the biomass resource is available for energy production. At the time that coal resources are first accessed, their available deposits are $\zeta_M > 0$, and at the time that oil/gas resources are first accessed, their available deposits are $\zeta_D > 0$. To highlight the main ideas, I assume that all production functions are Cobb-Douglas ($\sigma = 1$).

Developing a new type of resource requires paying a one-time fixed cost per capita.[20] This fixed cost reflects the need to find the new resources, figure out how to extract them, innovate in how to use them, and retool the capital stock to use them. The fixed cost ξ_M of first accessing a mined resource captures the cost of innovating in the techniques and steam engines necessary to avoid water seeping into mines. The fixed cost ξ_D of first accessing an oil/gas resource captures the cost of using steam engines and immature techniques in an unsuccessful sequence of initial drilling attempts. The fixed cost ξ_S of first accessing a solar resource captures the cost of the intensive R&D required to understand and commercialize the photoelectric effect. Reflecting the reliance of oil/gas and solar on progressively more advanced knowledge and specialized capital, assume $\xi_S > \xi_D > \xi_M > 0$. Up to and including the moment of paying the fixed cost, the resource base for an undeveloped resource is zero. The fixed cost must be paid out of production fueled by already-developed types of energy resources. The new resource base then begins growing as energy, labor, and capital are devoted to opening mines, tapping deposits, and/or producing solar panels.

The following proposition considers the interaction between various resources:[21]

[20] Defining fixed costs in per capita terms sharpens the story. If costs are fixed in aggregate rather than per capita, then they decline in per capita terms. The main results go through, once expressed in terms of timing rather than in terms of whether a resource is accessed at all.

[21] I study transitions in the absence of exogenous technical change, by imposing Assumption 1. If I instead assumed $g_A > 0$, then every resource could eventually be accessed, as in Hansen and Prescott (2002). However, to the extent that the new ideas embedded in $g_A > 0$ require energy, assuming $g_A > 0$ may beg the question of the feasibility of each transition.

Proposition 10 (Energy Ladder)**.** *Let Assumptions 1 and 3 hold. There exist ξ_D, ξ_S sufficiently large such that:*

i An economy that uses the biomass resource will not develop the oil/gas or solar resources.

ii If ξ_M is sufficiently small and Q_M is sufficiently large, then the economy eventually develops the coal resource.

iii If, in addition to the conditions in part ii, ξ_D is not too large and Q_D is sufficiently large, then the economy subsequently develops the oil/gas resource using energy from the coal resource.

iv If, in addition to the conditions in part iii, ξ_S is not too large and Q_S is sufficiently large, then the economy subsequently develops the solar resource using energy from the oil/gas resource.

Proof. See Appendix J. □

The economy climbs an *energy ladder*, advancing from rung to rung of resources. Developing a first type of fossil energy resource generates the extra energy necessary to enable the development of a more productive type of fossil energy resource. As Smil (2017, 230) writes, "Every transition to a new form of energy supply has to be powered by the intensive deployment of existing energies and prime movers: the transition from wood to coal had to be energized by human muscles, coal combustion powered the development of oil, and... today's solar photovoltaic cells and wind turbines are embodiments of fossil energies..."

On the first rung, if the fixed costs of developing the oil/gas and solar resources are large, then the economy with the biomass resource will not be able to produce the output required to begin developing those more advanced resources (part i). The biomass economy that constituted human history up to the early nineteenth century may not have been able to produce the capital stock and body of knowledge required to drill for oil and gas resources or to understand and commercialize the photoelectric effect at the heart of solar panels. In the absence of a coal resource, the economy would be stuck with the biomass resource and energy would always have constituted a drag on growth.

However, coal seams were in fact visible and readily available in places. As a result, the economy could transition from the biomass resource to the coal resource. Moreover, coal was found to offer a high energetic return relative to biomass. The economy began accessing coal in order to obtain its energetic return, and in so doing transitioned to a world of energy-enabled growth (part ii).

An economy with a sufficiently productive coal resource may raise output high enough to be able to afford the investments required to access subsurface oil and gas deposits. Indeed, a steam engine powered the drill for the first oil wells (Smil, 2017, 247). If the oil/gas resource has a high enough expected return, then the economy optimally pays that cost and transitions to using the oil/gas resource (part iii).

An economy with a sufficiently productive oil/gas resource may then raise output high enough to be able to afford the investments required to commercialize the photoelectric effect. A biomass economy would not have been able to generate the output required to figure out and commercialize the photoelectric effect.[22] In order to generate sufficient surplus energy to develop modern solar cells, the economy may have first needed to develop its fossil resources. In this way, the fossil age may have been necessary for the solar economy to emerge. If solar panels offer a sufficiently high energetic return, then the economy optimally pays that cost and transitions to using solar resources (part iv). The solutions to climate change may stand on the shoulders of the problem.

10 Conclusions

All economic activity requires energetic inputs. We have seen that the characteristics of energy resources used by an economy shape its growth possibilities. In particular, the characteristics of energy resources determine how fast technical change needs to be in order to achieve sustained growth. The important characteristics of energy resources are the nature of resource production and the productivity of energy in making additional energy, known in the literature as energy return on energy invested (EROI). We have seen that, for oil/gas and solar resources, EROI must exceed unity plus the growth rate of population for the economy to avoid energy-scarce degrowth.

We have also seen that the step-change from tapped oil/gas resources to manufactured solar resources may generate energy-fueled growth, even in the absence of further technical change, if paired with an additional step-change in using robots to produce and install solar panels. This mechanism for sustained growth is distinct from those proposed in prior literature and is consistent with robots requiring energy to operate. The production of solar panels is indeed being increasingly automated, and further automation is on the horizon. If these efforts succeed, the transition to low-carbon energy resources may end up stimulating an increase in the economy's long-run growth prospects.

This analysis does not consider how energy may affect the pace of technical change, whether by directly fueling research (see Schurr, 1984) or by absorbing scientists who may otherwise work on improving general productivity (see Arkolakis and Walsh, 2024). If roboticized solar makes energy more abundant and cheaper, then interactions with technical change could constitute a second growth dividend.

It may seem like a waste to have burned so much carbon when lower-carbon resources might actively drive growth, but we have also seen that accessing knowledge- and capital-intensive lower-carbon resources may have required burning fossil resources in order to raise

[22]Depending on Q_M and ζ_S, it is possible that even the coal economy would not have been able to investigate the photoelectric effect and commercialize it to the point where solar cells became productive enough to induce a transition to a solar economy.

output high enough. Future work should quantitatively assess the transition between resource types in order to understand how much earlier the world could have reached a solar economy. The excess emissions may be the salient ones when attributing historical responsibility for climate change.

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Appendix

The first two sections contain additional analysis. Subsequent sections contain proofs. In all cases, because markets are competitive and there are no market failures, equilibrium can be solved for via the social planner's problem. Throughout, I use ℓ, k, e, r, and y to denote per-capita variables, I use a ss subscript to denote a steady state, and I use g_x to indicate the instantaneous growth rate of variable x.

A Analysis of tapped resources with $\sigma > 1$

Modify Section 5 to assume $\sigma > 1$. First consider the counterpart of Proposition 5:

Proposition A-1 (BGP with Tapped Resources)**.** *Let Assumption 1 hold, fix $\sigma > 1$, and define χ as in Proposition 5. Then any interior balanced growth path has all variables grow at the rate of population. Such a balanced growth path exists if and only if $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} < \chi$ and $\Omega > g_L$. Along this path, $EROI(t)$ is as in Proposition 5.*

Proof. For $\sigma > 1$, inequality (A-36) holds if and only if

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D < (\lambda + g_L) \left(\frac{\lambda+\rho}{\lambda+g_L}\right)^{\frac{\sigma}{\sigma-1}} (\Omega - g_L)^{\frac{\omega}{1-\omega}} \left(\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right)^{\frac{\sigma}{\sigma-1}}.$$

The right-hand side is χ from Proposition 5. $EROI(t)$ follows from equations (A-20) and (A-35). □

When $\sigma > 1$, we again have the case of energy-enabled growth familiar from the analysis of the mined resource, but now only if energy resources are not too productive. The condition on χ again ensures that EROI on the interior balanced growth path exceeds $1+g_L$. However, here EROI decreases in the productivity of energy resources: as explained following Proposition 4, increasing the productivity of energy resources induces substitution towards energy inputs to deposit-tapping that works to reduce EROI. As a result, the necessary condition for the interior balanced growth path here caps the productivity of energy resources.

But the possibility of substitution suggests that the optimal path may not be interior. Instead, it may be a corner solution in which energy inputs to deposit-tapping completely crowd out other inputs to deposit-tapping (i.e., $L_R(t), K_R(t) = 0$). We analyze these "only-energy" balanced growth paths under log utility:

Proposition A-2 (Energy-Only Growth with Tapped Resources)**.** *Let Assumptions 1 and 3 hold and fix $\sigma > 1$. Define g_e as the growth rate of energy per capita, g_y as the growth rate of output per capita, g_k as the growth rate of capital per capita, and χ as in Proposition 5. If $\Omega > g_L$ and $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}$ is neither too large nor too small, then there exists an only-energy balanced growth path with $R(t)/L(t)$, $E_R(t)/L(t)$, $E_Y(t)/L(t)$, and $Z(t)/L(t)$ all growing at rate $g_e \neq 0$. Along this path:*

i $g_y = g_k = \frac{\alpha_E}{\alpha_E+\alpha_L} g_e$.

ii $g_e \in \big(-(\rho - \Omega), \Omega - g_L \big)$

iii g_e *and* $Z(t)/E_R(t)$ *each decrease in* A_E *and* Q_D

iv $g_e > 0$ *if, in addition,* $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} < \lim_{\sigma\to\infty} \chi$

v $EROI(t) = 1 + Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} \frac{g_e+g_L}{[g_e+g_L+\lambda](\Omega-g_L-g_e)^{\frac{\omega}{1-\omega}}}$

Proof. See Appendix K. □

We again have a case of energy-fueled growth, but now only if deposits are *not too* productive at making energy (e.g., we need Q_D small). When energy resources are extremely productive, reinvesting growing energy stocks in deposit tapping at a constant rate eats up the available deposits: $Z(t)/E_R(t)$ falls in the productivity of energy resources (part iii). The increasing scarcity of deposits restricts the ability to produce further energy, so the growth rate of energy must also fall until the rate of energy reinvestment comes into equilibrium with the rate of deposit finds (part iii).

If we continue to increase the productivity of energy resources, we eventually hit a point at which so much energy is produced from each unit of energy that a stock of deposits cannot be sustained if energy is reinvested in deposit-tapping at a constant rate. In particular, part iv of Proposition A-2 shows that growth turns negative once $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}$ exceeds $\lim_{\sigma\to\infty} \chi$.[23] This is a new scenario of *energy-abundant degrowth*. The economy accumulates a lot of spare energy very quickly. Deposits are scarce relative to energy, so their availability constrains how much of the spare energy can be devoted to tapping new deposits. As a result, a large share of the energy produced is devoted to final good production ($e_R(t)/e(t)$ is small) and thus to consumption. Nonetheless, the available deposits decline over time as they are exploited via energy inputs faster than they are found, and energy production falls with deposits. Consumption falls over time as available energy falls, albeit from a high level that was enabled by the high productivity of energy deposits.[24]

B Analysis of manufactured resources with $\sigma > 1$

Modify Section 6 to assume $\sigma > 1$. We now have:

[23] However, growth rates cannot be too negative along a balanced growth path because capital and resource stocks can only shrink so quickly. We therefore have a cap on how productive energy resources can be if the economy is to find a balanced growth path.

[24] Because χ decreases in σ and, from Proposition 5, χ is the cutoff for an interior balanced growth path to exist , we learn that productivity parameters support strictly positive growth per capita only if they also support an interior balanced growth path without any growth per capita.

Proposition A-3 (Energy-Only Growth with Manufactured Resources)**.** *Let Assumptions 1 and 3 hold and fix $\sigma > 1$. Define g_e as the growth rate of energy per capita, g_y as the growth rate of output per capita, g_k as the growth rate of capital per capita, and χ as in Proposition 5. If $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} > \max\left\{\rho - g_L,\ \rho + \lambda - \frac{\alpha_E+\alpha_L}{\alpha_E}(g_L+\delta)\right\}$, then there exists an only-energy balanced growth path with $R(t)/L(t)$, $E_R(t)/L(t)$, and $E_Y(t)/L(t)$ all growing at rate $g_e \neq 0$. Along such a path:*

i $g_y = g_k = \frac{\alpha_E}{\alpha_E+\alpha_L} g_e$.

ii $g_e = Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} - (\lambda + \rho) > 0$.

iii $g_e > 0$ *if, in addition,* $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} > \chi$

iv $EROEI(t) = 1 + \frac{Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}}{Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} - (\rho - g_L)}(g_e + g_L)$

Proof. See Appendix L. □

We have *energy-fueled growth* in those cases with energy resources too productive for an interior balanced growth path: the economy features strictly positive long-run growth in output per capita, despite the absence of technical progress. Along an only-energy path, the production function for solar panels becomes, from (5),

$$F(L_{Rt}, K_{Rt}, E_{Rt}) = A_E \kappa_E^{\frac{\sigma}{\sigma-1}} E_{Rt}.$$

Differentiating (8) with respect to time and using (3), the instantaneous change in energy production at time t is:

$$\dot{E}(t) = A_E Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} E_R(t) - Q_D \lambda R(t).$$

If $E_R(t)$ is a constant share s_E of $E(t)$, this becomes:

$$\dot{E}(t) = \left[A_E Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} s_E - \lambda\right] E(t).$$

We have a linear accumulation equation. The economy can grow by reinvesting energy in making energy as long as the return on that investment is sufficiently large.

The growth rate of output per capita is proportional to the growth rate of energy per capita (part i). That growth rate of energy per capita increases in the productivity Q_D of solar panels and the productivity A_E of energy in manufacturing solar panels (part ii) and is strictly positive only if these productivity terms are sufficiently large (part iii). Economy-wide EROI increases in both productivity parameters and, consistent with Proposition 2, is greater than $1 + g_L + g_e$ (part iv).

C Proof of Proposition 1

Equilibrium solves the following maximization problem:

$$\begin{aligned}
\max_{L_Y(\cdot),K_Y(\cdot),L_E(\cdot),K_E(\cdot)} & \int_0^\infty e^{-\rho t} L(t) u\left(\frac{(1-s)A(t)L_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t \\
\text{s.t. } \dot{A}(t) =& g_A A(t) \\
\dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& sAL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E(t)^{\alpha_E} - \delta K(t) \\
E(t) =& Q_H L_E(t)^{\phi_{HL}}K_E(t)^{\phi_{HK}}R^{\phi_{HR}} \\
L(t) =& L_Y(t) + L_E(t) \\
K(t) =& K_Y(t) + K_E(t).
\end{aligned}$$

Converting to per-capita and substituting, this is equivalent to:

$$\begin{aligned}
\max_{\ell_Y(\cdot),k_Y(\cdot)} & \int_0^\infty e^{-(\rho-g_L)t} u\left((1-s)A(t)\ell_Y(t)^{\alpha_L}k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}}r(t)^{\phi_{HR}}\right]^{\alpha_E}\right) \mathrm{d}t \\
\text{s.t. } \dot{A}(t) =& g_A A(t) \\
\dot{k}(t) =& sA\ell_Y(t)^{\alpha_L}k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}}r(t)^{\phi_{HR}}\right]^{\alpha_E} - (\delta+g_L)k(t) \\
\dot{r}(t) =& -g_L r(t).
\end{aligned}$$

The current-value Hamiltonian is:

$$\begin{aligned}
& u\left((1-s)A(t)\ell_Y(t)^{\alpha_L}k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}}r(t)^{\phi_{HR}}\right]^{\alpha_E}\right) \\
& + \nu(t)\left[sA\ell_Y(t)^{\alpha_L}k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}}r^{\phi_{HR}}\right]^{\alpha_E} - (\delta+g_L)k(t)\right] \\
& - \mu(t)g_L r(t) + \psi(t)g_A A(t).
\end{aligned}$$

The conditions to maximize the Hamiltonian are:

$$\begin{aligned}
0 =& \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\left[\frac{\alpha_L}{\ell_Y(t)} - \frac{\alpha_E\phi_{HL}}{1-\ell_Y(t)}\right]y(t), \\
0 =& \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\left[\frac{\alpha_K}{k_Y(t)} - \frac{\alpha_E\phi_{HK}}{k(t)-k_Y(t)}\right]y(t).
\end{aligned}$$

Together, these imply:

$$\begin{aligned}
\ell_Y(t) =& \frac{\alpha_L}{\alpha_L + \alpha_E\phi_{HL}}, \\
k_Y(t) =& \frac{\alpha_K}{\alpha_K + \alpha_E\phi_{HK}}k(t).
\end{aligned}$$

Substituting, we find

$$E(t) = Q_H \left(\frac{\alpha_E \phi_{HL}}{\alpha_L + \alpha_E \phi_{HL}} L(t)\right)^{\phi_{HL}} \left(\frac{\alpha_E \phi_{KL}}{\alpha_K + \alpha_E \phi_{HK}} K(t)\right)^{\phi_{HK}} R^{\phi_{HR}}$$

and

$$\begin{aligned} Y(t) =& A(t) \left(\frac{\alpha_L}{\alpha_L + \alpha_E \phi_{HL}}\right)^{\alpha_L} \left(\frac{\alpha_K}{\alpha_K + \alpha_E \phi_{HK}}\right)^{\alpha_K} \\ & \left[Q_H \left(\frac{\alpha_E \phi_{HL}}{\alpha_L + \alpha_E \phi_{HL}}\right)^{\phi_{HL}} \left(\frac{\alpha_E \phi_{KL}}{\alpha_K + \alpha_E \phi_{HK}}\right)^{\phi_{HK}} R^{\phi_{HR}}\right]^{\alpha_E} L(t)^{\alpha_L + \alpha_E \phi_{HL}} K(t)^{\alpha_K + \alpha_E \phi_{HK}}. \end{aligned}$$

Define

$$A_H(t) \triangleq A(t)^{\frac{1}{1-\alpha_K - \alpha_E(\phi_{HK} + \phi_{HR})}}.$$

Defining $y_H(t) \triangleq Y(t)/[A_H(t)L(t)]$ and analogously for the other variables, we find:

$$\begin{aligned} y_H(t) =& \left(\frac{\alpha_L}{\alpha_L + \alpha_E \phi_{HL}}\right)^{\alpha_L} \left(\frac{\alpha_K}{\alpha_K + \alpha_E \phi_{HK}}\right)^{\alpha_K} \\ & \left[Q_H \left(\frac{\alpha_E \phi_{HL}}{\alpha_L + \alpha_E \phi_{HL}}\right)^{\phi_{HL}} \left(\frac{\alpha_E \phi_{KL}}{\alpha_K + \alpha_E \phi_{HK}}\right)^{\phi_{HK}} r_H(t)^{\phi_{HR}}\right]^{\alpha_E} k_H(t)^{\alpha_K + \alpha_E \phi_{HK}}, \\ \dot{k}_H(t) =& s y_H(t) - \left(\delta + g_L + \frac{1}{1 - \alpha_K - \alpha_E(\phi_{HK} + \phi_{HR})} g_A\right) k_H(t), \\ \dot{r}_H(t) =& - \left(g_L + \frac{1}{1 - \alpha_K - \alpha_E(\phi_{HK} + \phi_{HR})} g_A\right) r_H(t). \end{aligned}$$

The growth rate of $k_H(t)$ is:

$$\frac{\dot{k}_H(t)}{k_H(t)} = s \chi_H Q_H^{\alpha_E} r_H(t)^{\alpha_E \phi_{HR}} k_H(t)^{\alpha_K + \alpha_E \phi_{HK} - 1} - \left(\delta + g_L + \frac{1}{1 - \alpha_K - \alpha_E(\phi_{HK} + \phi_{HR})} g_A\right),$$

where

$$\chi_H \triangleq \left(\frac{\alpha_L}{\alpha_L + \alpha_E \phi_{HL}}\right)^{\alpha_L} \left(\frac{\alpha_K}{\alpha_K + \alpha_E \phi_{HK}}\right)^{\alpha_K} \left[\left(\frac{\alpha_E \phi_{HL}}{\alpha_L + \alpha_E \phi_{HL}}\right)^{\phi_{HL}} \left(\frac{\alpha_E \phi_{KL}}{\alpha_K + \alpha_E \phi_{HK}}\right)^{\phi_{HK}}\right]^{\alpha_E}.$$

That growth rate is constant if and only if

$$r_H(t)^{\alpha_E \phi_{HR}} k_H(t)^{\alpha_K + \alpha_E \phi_{HK} - 1} = \frac{1}{s \chi_H Q_H^{\alpha_E}} \left[g_k + \delta + g_L + \frac{1}{1 - \alpha_K - \alpha_E(\phi_{HK} + \phi_{HR})} g_A\right] \tag{A-1}$$

is constant. Observe that:

$$r_H(t) = e^{-\left(g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)t} r_H(0).$$

Substitute into (A-1):

$$k_H(t) = \left[s\chi_H Q_H^{\alpha_E} \frac{e^{-\alpha_E\phi_{HR}\left(g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)t} \left(\frac{R}{L(0)A_H(0)}\right)^{\alpha_E\phi_{HR}}}{g_k + \delta + g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A} \right]^{\frac{1}{1-\alpha_K-\alpha_E\phi_{HK}}}.$$

And thus:

$$\frac{\dot{k}_H(t)}{k_H(t)} = -\frac{\alpha_E\phi_{HR}}{1-\alpha_K-\alpha_E\phi_{HK}}\left(g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right) < 0.$$

The growth rate of output per capita is:

$$\begin{aligned}\frac{\dot{y}(t)}{y(t)} =& \alpha_E\phi_{HR}\frac{\dot{r}_H(t)}{r_H(t)} + [\alpha_K + \alpha_E\phi_{HK}]\frac{\dot{k}_H(t)}{k_H(t)} + \frac{\dot{A}_H(t)}{A_H(t)} \\ =& \frac{g_A - \alpha_E\phi_{HR}g_L}{\alpha_L + \alpha_E(\phi_{HL}+\phi_{HR})}.\end{aligned}$$

The proposition follows.

D Preliminaries for Propositions 3 and 4

Equilibrium solves the following maximization problem:

$$\max_{L_Y(\cdot),K_Y(\cdot),E_Y(\cdot),L_E(\cdot),K_E(\cdot),L_R(\cdot),K_R(\cdot),E_R(\cdot)} \int_0^\infty e^{-\rho t}L(t)u\left(\frac{(1-s)AL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E}}{L(t)}\right)\mathrm{d}t$$

$$\begin{aligned}\text{s.t. } \dot{L}(t) =& g_L L(t) \\ \dot{K}(t) =& sAL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E} - \delta K(t) \\ \dot{Z}(t) =& \Omega Z(t) - Z(t)^{\omega}F(L_R(t),K_R(t),E_R(t))^{1-\omega} \\ \dot{R}(t) =& Z(t)^{\omega}F(L_R(t),K_R(t),E_R(t))^{1-\omega} - \lambda R(t) \\ E(t) =& Q_M L_E(t)^{\phi_{ML}}K_E(t)^{\phi_{MK}}R(t)^{\phi_{MR}} \\ L(t) =& L_Y(t) + L_E(t) + L_R(t) \\ K(t) =& K_Y(t) + K_E(t) + K_R(t) \\ E(t) =& E_Y(t) + E_R(t).\end{aligned}$$

Converting to per-capita and substituting, this is equivalent to:

$$\begin{aligned}
\max_{\ell_E(\cdot),k_E(\cdot),\ell_R(\cdot),k_R(\cdot),e_R(\cdot)} \int_0^\infty e^{-(\rho-g_L)t} u\Bigg(&(1-s)A\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K} \\
&\Big[Q_M \ell_E(t)^{\phi_{ML}} k_E(t)^{\phi_{MK}} r(t)^{\phi_{MR}} - e_R(t)\Big]^{\alpha_E}\Bigg)\,\mathrm{d}t \\
\text{s.t. } \dot{k}(t) =& sA\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K} \\
&\Big[Q_M \ell_E(t)^{\phi_{ML}} k_E(t)^{\phi_{MK}} r(t)^{\phi_{MR}} - e_R(t)\Big]^{\alpha_E} - (\delta+g_L)k(t) \\
\dot{z}(t) =& (\Omega-g_L)z(t) - z(t)^{\omega} F(\ell_R(t),k_R(t),e_R(t))^{1-\omega} \\
\dot{r}(t) =& z(t)^{\omega} F(\ell_R(t),k_R(t),e_R(t))^{1-\omega} - (\lambda+g_L)r(t).
\end{aligned}$$

The following lemma establishes that any balanced growth path with interior solutions must have variables growing at the rate of population:

Lemma A-4. *A balanced growth path with $L_R(t), K_R(t), E_R(t) > 0$ must have all variables growing at the rate of population.*

Proof. Use g_x to indicate the instantaneous growth rate of variable x. From the transition equation for $z(t)$, $\dot{z}(t)/z(t)$ is constant if and only if $F(\ell_R(t),k_R(t),e_R(t))/z(t)$ is constant. From the transition equation for $r(t)$, $\dot{r}(t)/r(t)$ can be constant with $F(\ell_R(t),k_R(t),e_R(t))/z(t)$ constant if and only if $F(\ell_R(t),k_R(t),e_R(t))/r(t)$ is constant. Therefore, if g_z and g_r are each constant, then $g_z = g_r = g_{F/L}$.

If $g_{F/L} \neq 0$ and e_R and k_R grow at constant rates, then ℓ_R must grow at a constant rate. The constant growth rate of ℓ_R cannot be strictly positive because ℓ_R is bounded above by 1. The constant growth rate of ℓ_R cannot be strictly negative because the (full employment) labor constraint implies that ℓ_Y and/or ℓ_E must grow at a strictly positive rate yet these variables are each bounded above by 1. Therefore a constant growth rate of ℓ_R must be zero. If $g_{F/L}$, g_{k_R}, and g_{e_R} are also to be constant, then $g_{F/L}, g_{k_R}, g_{e_R} = 0$. That implies $g_z, g_r = 0$. It follows straightforwardly that the remaining variables (in per capita form) also grow at rate zero. □

So we seek a steady state in per-capita variables. The current-value Hamiltonian is:

$$
\begin{aligned}
&u\Bigg((1-s)A\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K}\Big[Q_M\ell_E(t)^{\phi_{ML}}k_E(t)^{\phi_{MK}}r(t)^{\phi_{MR}}-e_R(t)\Big]^{\alpha_E}\Bigg)\\
&+\nu(t)\Bigg(sA\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K}\Big[Q_M\ell_E(t)^{\phi_{ML}}k_E(t)^{\phi_{MK}}r(t)^{\phi_{MR}}-e_R(t)\Big]^{\alpha_E}\\
&\qquad\qquad-(\delta+g_L)k(t)\Bigg)\\
&+\gamma(t)\Big[(\Omega-g_L)z(t)-z(t)^{\omega}F(\ell_R(t),k_R(t),e_R(t))^{1-\omega}\Big]\\
&+\mu(t)\Big[z(t)^{\omega}F(\ell_R(t),k_R(t),e_R(t))^{1-\omega}-(\lambda+g_L)r(t)\Big].
\end{aligned}
$$

The costate equations are:

$$
\begin{aligned}
(\rho-g_L)\nu(t)-\dot{\nu}(t)=&\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_K y(t)}{k_Y(t)}-(\delta+g_L)\nu(t)\\
\Leftrightarrow\dot{\nu}(t)=&-\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_K y(t)}{k_Y(t)}+(\rho+\delta)\nu(t), \qquad \text{(A-2)}\\
(\rho-g_L)\gamma(t)-\dot{\gamma}(t)=&\gamma(t)(\Omega-g_L)+\Big[\mu(t)-\gamma(t)\Big]\omega\left(\frac{F(\ell_R(t),k_R(t),e_R(t))}{z(t)}\right)^{1-\omega}\\
\Leftrightarrow\dot{\gamma}(t)=&\gamma(t)(\rho-\Omega)-\Big[\mu(t)-\gamma(t)\Big]\omega\left(\frac{F(\ell_R(t),k_R(t),e_R(t))}{z(t)}\right)^{1-\omega}, \qquad \text{(A-3)}\\
(\rho-g_L)\mu(t)-\dot{\mu}(t)=&-\mu(t)(\lambda+g_L)+\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_E\phi_{MR}y(t)}{e_Y(t)}\frac{e(t)}{r(t)}\\
\Leftrightarrow\dot{\mu}(t)=&\mu(t)(\rho+\lambda)-\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_E\phi_{MR}y(t)}{e_Y(t)}\frac{e(t)}{r(t)}. \qquad \text{(A-4)}
\end{aligned}
$$

The conditions to maximize the Hamiltonian are:

$$0 = \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\left[\frac{\alpha_E \phi_{ML}}{e_Y(t)}\frac{e(t)}{\ell_E(t)} - \frac{\alpha_L}{\ell_Y(t)}\right]y(t), \tag{A-5}$$

$$0 = \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\left[\frac{\alpha_E \phi_{MK}}{e_Y(t)}\frac{e(t)}{k_E(t)} - \frac{\alpha_K}{k_Y(t)}\right]y(t), \tag{A-6}$$

$$\begin{aligned} 0 = &- \frac{\alpha_L y(t)}{\ell_Y(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] \\ &+ \left[\mu(t) - \gamma(t)\right](1-\omega)\left(\frac{z(t)}{F(\ell_R(t), k_R(t), e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t), k_R(t), e_R(t))}{\partial \ell_R(t)}, \end{aligned} \tag{A-7}$$

$$\begin{aligned} 0 = &- \frac{\alpha_K y(t)}{k_Y(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] \\ &+ \left[\mu(t) - \gamma(t)\right](1-\omega)\left(\frac{z(t)}{F(\ell_R(t), k_R(t), e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t), k_R(t), e_R(t))}{\partial k_R(t)}, \end{aligned} \tag{A-8}$$

$$\begin{aligned} 0 = &- \frac{\alpha_E y(t)}{e_Y(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] \\ &+ \left[\mu(t) - \gamma(t)\right](1-\omega)\left(\frac{z(t)}{F(\ell_R(t), k_R(t), e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t), k_R(t), e_R(t))}{\partial e_R(t)}. \end{aligned} \tag{A-9}$$

Equations (A-5) and (A-6) imply:

$$\ell_Y(t) = \frac{\alpha_L}{\alpha_E \phi_{ML}}\frac{e_Y(t)}{e(t)}\ell_E(t), \tag{A-10}$$

$$k_Y(t) = \frac{\alpha_K}{\alpha_E \phi_{MK}}\frac{e_Y(t)}{e(t)}k_E(t). \tag{A-11}$$

Take ratios of equations (A-7), (A-8), and (A-9), substitute for $\ell_Y(t)$ and $k_Y(t)$, and cancel the $e_Y(t)$:

$$1 = \frac{\kappa_L}{\phi_{ML}}\frac{1-\kappa_E}{\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{k_R(t)}{\ell_R(t)}\right)^{\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_R(t)}{\ell_R(t)}\right)^{\frac{1}{\sigma}}\frac{\ell_E(t)}{e(t)}, \tag{A-12}$$

$$1 = \frac{\kappa_K}{\phi_{MK}}\frac{1-\kappa_E}{\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{\ell_R(t)}{k_R(t)}\right)^{\kappa_L\frac{\sigma-1}{\sigma}}\left(\frac{e_R(t)}{k_R(t)}\right)^{\frac{1}{\sigma}}\frac{k_E(t)}{e(t)}. \tag{A-13}$$

Solve (A-12) for $k_R(t)$:

$$k_R(t) = \left[\frac{\kappa_L}{\phi_{ML}}\frac{1-\kappa_E}{\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{\frac{\sigma-1}{\sigma}}\left(\frac{1}{\ell_R(t)}\right)^{\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_R(t)}{\ell_R(t)}\right)^{\frac{1}{\sigma}}\frac{\ell_E(t)}{e(t)}\right]^{-\frac{1}{\kappa_K\frac{\sigma-1}{\sigma}}}. \tag{A-14}$$

Substitute into (A-13) and solve for $\ell_R(t)$:

$$\ell_R(t) = \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} e_R(t) \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)+1}. \tag{A-15}$$

Substitute back into (A-14):

$$k_R(t) = \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)} \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} e_R(t) \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)+1}.$$

Substitute from (A-10) and (A-15) into the labor resource constraint and solve for $\ell_E(t)$:

$$\ell_E(t) = \Bigg[1 + \frac{\alpha_L}{\alpha_E \phi_{ML}} \frac{e_Y(t)}{e(t)} + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \frac{e_R(t)}{e(t)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \Bigg]^{-1}. \tag{A-16}$$

Solve for $e_R(t)/e(t)$:

$$\frac{e_R(t)}{e(t)} = \frac{[1 - \ell_E(t) - \frac{\alpha_L}{\alpha_E \phi_{ML}} \ell_E(t)]/\ell_E(t)}{-\frac{\alpha_L}{\alpha_E \phi_{ML}} + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}. \tag{A-17}$$

Substitute from (A-11) and (A-14) into the capital constraint:

$$\frac{k_E(t)}{k(t)} = \Bigg[1 + \frac{\alpha_K}{\alpha_E \phi_{MK}} \frac{e_Y(t)}{e(t)} + \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)} \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \frac{e_R(t)}{e(t)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \Bigg]^{-1}. \tag{A-18}$$

Substitute for $e_R(t)/e(t)$ from (A-17):

$$\begin{aligned}
&\frac{k_E(t)}{k(t)} \\
=&\Bigg[1 - \frac{\alpha_K}{\alpha_E \phi_{MK}} \frac{1}{\ell_E(t)} \\
&\quad \frac{1 - \ell_E(t) - \ell_E(t) \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}{-\frac{\alpha_L}{\alpha_E \phi_{ML}} + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}} \\
&\quad + \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)} \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \\
&\quad \frac{[1 - \ell_E(t) - \frac{\alpha_L}{\alpha_E \phi_{ML}} \ell_E(t)]/\ell_E(t)}{-\frac{\alpha_L}{\alpha_E \phi_{ML}} + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}\Bigg]^{-1}.
\end{aligned} \tag{A-19}$$

Finally, at a steady state, $\dot{e}(t) = 0$ in equation (7). Therefore,

$$EROI_{ss} = 1 + \frac{e_{ss}}{e_{Rss}} g_L. \tag{A-20}$$

E Proof of Proposition 3

Section D contains preliminaries. By Lemma A-4, any interior balanced growth path must have per-capita variables constant. If $\sigma = 1$, we have, from (A-16) and (A-18):

$$\begin{aligned}
\ell_E(t) =& \left[1 + \frac{\alpha_L}{\alpha_E \phi_{ML}} \left(1 - \frac{e_R(t)}{e(t)}\right) + \frac{\kappa_L}{\phi_{ML}} \frac{1-\kappa_E}{\kappa_E} \frac{e_R(t)}{e(t)}\right]^{-1}, \\
\frac{k_E(t)}{k(t)} =& \left[1 + \frac{\alpha_K}{\alpha_E \phi_{MK}} \left(1 - \frac{e_R(t)}{e(t)}\right) + \frac{\kappa_K}{\phi_{MK}} \frac{1-\kappa_E}{\kappa_E} \frac{e_R(t)}{e(t)}\right]^{-1}.
\end{aligned} \tag{A-21}$$

These are both interior as long as $e_R(t)/e(t) < 1$. With these:

$$\begin{aligned}
\ell_Y(t) =& \frac{\alpha_L}{\alpha_E \phi_{ML}} \frac{e_Y(t)}{e(t)} \ell_E(t), \\
\frac{k_Y(t)}{k(t)} =& \frac{\alpha_K}{\alpha_E \phi_{MK}} \frac{e_Y(t)}{e(t)} \frac{k_E(t)}{k(t)}.
\end{aligned} \tag{A-22}$$

And:

$$\begin{aligned}\ell_R(t) =& \frac{\kappa_L}{\phi_{ML}} \frac{1-\kappa_E}{\kappa_E} \frac{e_R(t)}{e(t)} \ell_E(t),\\ \frac{k_R(t)}{k(t)} =& \frac{\kappa_K}{\phi_{MK}} \frac{1-\kappa_E}{\kappa_E} \frac{e_R(t)}{e(t)} \frac{k_E(t)}{k(t)}.\end{aligned} \tag{A-23}$$

At a steady state, (A-4) becomes:

$$y_{ss}\Big[(1-s)u'((1-s)y_{ss}) + s\nu_{ss}\Big] = \mu_{ss}(\rho+\lambda)\frac{r_{ss}}{\alpha_E \phi_{MR}}\left(1 - \frac{e_{Rss}}{e_{ss}}\right).$$

Substitute into steady-state version of (A-9):

$$0 = -\frac{1}{\phi_{MR}}\mu_{ss}(\rho+\lambda)\frac{r_{ss}}{e_{ss}} + \Big[\mu_{ss} - \gamma_{ss}\Big](1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss}, k_{Rss}, e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E,$$

where

$$F_{ss} \triangleq F(\ell_{Rss}, k_{Rss}, e_{Rss}).$$

Solve for γ_{ss}:

$$\gamma_{ss} = \mu_{ss}\left(1 - \frac{\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}}{(1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss}, k_{Rss}, e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E}\right).$$

Substitute into (A-3):

$$\begin{aligned}0 =& (\rho-\Omega)\left((1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss}, k_{Rss}, e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E - \frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}\right)\\ &- \omega\left(\frac{F(\ell_{Rss}, k_{Rss}, e_{Rss})}{z_{ss}}\right)^{1-\omega}\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}.\end{aligned}$$

From the transition equation for $z(t)$,

$$\frac{z_{ss}}{F_{ss}} = (\Omega - g_L)^{-\frac{1}{1-\omega}}. \tag{A-24}$$

Using (A-24) in the expression prior to it yields:

$$0 = (\rho-\Omega)(1-\omega)\left(\Omega - g_L\right)^{-\frac{\omega}{1-\omega}}\kappa_E - \left[(\rho-\Omega) + \omega(\Omega - g_L)\right]\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{F_{ss}}\frac{e_{Rss}}{e_{ss}}.$$

The transition equation for $r(t)$ and (A-24) yield:

$$(\lambda + g_L)\frac{r_{ss}}{F_{ss}} = (\Omega - g_L)^{-\frac{\omega}{1-\omega}}. \tag{A-25}$$

Substitute into the prior expression:

$$\frac{e_{Rss}}{e_{ss}} = \phi_{MR}(1-\omega)\kappa_E \frac{\rho - \Omega}{(\rho - \Omega) + \omega(\Omega - g_L)} \frac{g_L + \lambda}{\rho + \lambda}. \tag{A-26}$$

The right-hand side is strictly positive because $\rho > \Omega, g_L$.

It remains to verify that the steady state is feasible. Equation (A-25) shows that a steady state with other variables strictly positive has $r_{ss} > 0$ if and only if either $\omega = 0$ or $\Omega > g_L$. From (A-24), $z_{ss} > 0$ with other variables strictly positive requires $\Omega > g_L$. Because $e_{Yss} = e_{ss} - e_{Rss}$, a steady state has strictly positive final good production only if $e_{Rss}/e_{ss} < 1$. Using $\rho > g_L$ and $\kappa_E, \phi_{MR} < 1$, it is clear from (A-26) that $e_{Rss}/e_{ss} < 1$ when either $\omega = 0$ or $\Omega > g_L$. Finally, from the capital transition equation, $y_{ss}/k_{ss} = (\delta + g_L)/s$, so $k_{ss} > 0$ if $y_{ss} > 0$, and from the final good production function and (A-22), $y_{ss} > 0$ when $\ell_{Yss}, e_{Yss} > 0$ if $k_{ss} > 0$. So it is internally consistent for all variables to be strictly positive if and only if either $\omega = 0$ or $\Omega > g_L$.

The claims about prices are implied by the existence of steady states in per-capita, present value terms for $\gamma(t)$ and $\mu(t)$. The claim about $EROI_{ss}$ follows from (A-20) and (A-26).

Finally, consider the elasticity of y_{ss} with respect to Q_M. From (A-25), (A-23), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}r_{ss}}{\mathrm{d}Q_M} = (\Omega - g_L)^{-\frac{\omega}{1-\omega}}(\lambda + g_L)^{-1}\left[\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta + g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M}\right]. \tag{A-27}$$

From (2), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M} = \frac{e_{ss}}{Q_M} + \phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta + g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \phi_{MR}\frac{e_{ss}}{r_{ss}}\frac{\mathrm{d}r_{ss}}{\mathrm{d}Q_M}.$$

Substitute from (A-27) and solve for $\mathrm{d}e_{ss}/\mathrm{d}Q_M$:

$$\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M} = \frac{\frac{e_{ss}}{Q_M} + \phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega - g_L)^{-\frac{\omega}{1-\omega}}(\lambda + g_L)^{-1}\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}}{1 - \phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega - g_L)^{-\frac{\omega}{1-\omega}}(\lambda + g_L)^{-1}\kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}. \tag{A-28}$$

From the final good production function, (A-22), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \alpha_K\frac{y_{ss}}{k_{Yss}}\frac{\mathrm{d}k_{Yss}}{\mathrm{d}k_{ss}}\frac{s}{\delta + g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \alpha_E\frac{y_{ss}}{e_{Yss}}\frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}}\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M}.$$

Substitute from (A-28) and rearrange to solve for $\mathrm{d}y_{ss}/\mathrm{d}Q_M$:

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \frac{\alpha_E \frac{y_{ss}}{e_{Yss}} \frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}} \frac{\frac{e_{ss}}{Q_M}}{1-\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}}{1-\alpha_K\frac{y_{ss}}{k_{Yss}}\frac{\mathrm{d}k_{Yss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}-\alpha_E\frac{y_{ss}}{e_{Yss}}\frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}}\frac{\phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}+\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}}{1-\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}}.$$

Substitute for y_{ss}/k_{ss} from the transition equation for $k(t)$ and simplify some other terms, using constancy of shares from (A-21), (A-22), (A-23), and (A-26):

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \frac{\alpha_E y_{ss}\frac{1}{Q_M}\frac{1}{1-\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{r_{ss}}}}{1-\alpha_K-\alpha_E\frac{\phi_{MK}+\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}(1-\kappa_E)\kappa_K\frac{F_{ss}}{r_{ss}}}{1-\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{r_{ss}}}}.$$

Substitute for F_{ss}/r_{ss} from (A-25):

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}\frac{Q_M}{y_{ss}} = \frac{\alpha_E\frac{1}{1-\phi_{MR}\kappa_E}}{1-\alpha_K-\alpha_E\frac{\phi_{MK}+\phi_{MR}(1-\kappa_E)\kappa_K}{\phi_{MK}+\phi_{ML}+\phi_{MR}(1-\kappa_E)}}.$$

The left-hand side is the elasticity of y_{ss} with respect to Q_M. The right-hand side is constant in Q_M and strictly positive. We have established the claim in the proposition about the elasticity.

F Proof of Proposition 4

Section D contains preliminaries. By Lemma A-4, any interior balanced growth path must have per-capita variables constant. If Assumption 2 holds, (A-19) simplifies to:

$$\frac{k_E(t)}{k(t)} = \ell_E(t).$$

Substitute this and (2) into (A-17):

$$\begin{aligned} e_R(t) = &\left[1-\ell_E(t)-\frac{\alpha_L}{\alpha_E\phi_{ML}}\ell_E(t)\right] \\ &\Bigg[-\frac{\alpha_L}{\alpha_E\phi_{ML}}\left(Q_M k(t)^{-\phi_{MK}} r(t)^{\phi_{MR}}\right)^{-1}\ell_E(t)^{\phi_{MR}} \\ &\quad+\left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)}\left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1}\left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma}\left(\frac{A_{LK}}{A_E}\right)^{\sigma-1}k(t)^{-(\kappa_K-\phi_{MK})\sigma-\kappa_K} \\ &\quad\ell_E(t)^{(1-\phi_{MR})\sigma}\left(Q_M r(t)^{\phi_{MR}}\right)^{-\sigma}\Bigg]. \end{aligned} \tag{A-29}$$

Use (A-4) in (A-9) and evaluate at a steady state:

$$0 = -\frac{\alpha_E}{e_{Yss}}\mu_{ss}(\rho+\lambda)\frac{e_{Yss}}{\alpha_E\phi_{MR}}\frac{r_{ss}}{e_{ss}} + \left[\mu_{ss}-\gamma_{ss}\right](1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}},$$

where

$$F_{ss} \triangleq F(\ell_{Rss}, k_{Rss}, e_{Rss}).$$

Solve for γ_{ss}:

$$\gamma_{ss} = \mu_{ss} - \mu_{ss}\frac{(\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}}{(1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}}}.$$

Substitute into the steady-state version of (A-3):

$$0 = (\rho-\Omega)\left[(1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}} - (\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}\right] - \omega\left(\frac{F_{ss}}{z_{ss}}\right)^{1-\omega}(\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}.$$

From the transition equation for $z(t)$:

$$\frac{z_{ss}}{F_{ss}} = (\Omega - g_L)^{-\frac{1}{1-\omega}}. \tag{A-30}$$

Use (A-30) in the foregoing expression:

$$0 = (\rho-\Omega)(1-\omega)(\Omega-g_L)^{-\frac{\omega}{1-\omega}}\kappa_E A_E^{\frac{\sigma-1}{\sigma}} - \left[(\rho-\Omega)+\omega(\Omega-g_L)\right](\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{F_{ss}}\frac{e_{Rss}}{e_{ss}}\left(\frac{F_{ss}}{e_{Rss}}\right)^{\frac{\sigma-1}{\sigma}}.$$

And use (A-30) in the transition equation for $r(t)$, evaluated at a steady state:

$$F_{ss} = (\Omega-g_L)^{\frac{\omega}{1-\omega}}(\lambda+g_L)r_{ss}. \tag{A-31}$$

Substitute that into the prior expression and rearrange:

$$r_{ss} = \left(\frac{\phi_{MR}\kappa_E}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}(\lambda+g_L)^{\frac{1}{\sigma}}\frac{e_{ss}}{e_{Rss}}\right)^{\frac{\sigma}{\sigma-1}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}A_E e_{Rss}. \tag{A-32}$$

When $\omega > 0$, equation (A-32) shows that a steady state requires $\Omega > g_L$. Using $\sigma \neq 1$, substitute for r_{ss} in (A-32) from (A-31), substitute for F_{ss} from (5), and rearrange:

$$e_{Rss} = \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\frac{\sigma}{\sigma-1}}\frac{A_{LK}}{A_E}\left(\frac{1}{\frac{\lambda+g_L}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}\phi_{MR}\frac{e_{ss}}{e_{Rss}}-1}\right)^{\frac{\sigma}{\sigma-1}}\ell_{Rss}^{\kappa_L}k_{Rss}^{\kappa_K}. \tag{A-33}$$

Rearrange (A-12) and (A-13) and evaluate at a steady state:

$$\ell_{Ess} = \frac{\phi_{ML}}{\kappa_L}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{k_{Rss}}{\ell_{Rss}}\right)^{-\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{\ell_{Rss}}\right)^{-\frac{1}{\sigma}} e_{ss},$$
$$k_{Ess} = \frac{\phi_{MK}}{\kappa_K}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{\ell_{Rss}}{k_{Rss}}\right)^{-\kappa_L\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{k_{Rss}}\right)^{-\frac{1}{\sigma}} e_{ss}.$$

Substitute into the per-capita version of (2) evaluated at the steady state and solve for e_{ss}:

$$e_{ss}^{\phi_{MR}} = Q_M r_{ss}^{\phi_{MR}}\left[\frac{\phi_{ML}}{\kappa_L}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{k_{Rss}}{\ell_{Rss}}\right)^{-\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{\ell_{Rss}}\right)^{-\frac{1}{\sigma}}\right]^{\phi_{ML}}$$
$$\left[\frac{\phi_{MK}}{\kappa_K}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{\ell_{Rss}}{k_{Rss}}\right)^{-\kappa_L\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{k_{Rss}}\right)^{-\frac{1}{\sigma}}\right]^{\phi_{MK}}.$$

Apply Assumption 2:

$$\frac{e_{ss}}{e_{Rss}} = r_{ss}\left[Q_M\left(\frac{\phi_{ML}}{\kappa_L}\right)^{\phi_{ML}}\left(\frac{\phi_{MK}}{\kappa_K}\right)^{\phi_{MK}}\right]^{\frac{1}{\phi_{MR}}}\left[\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\right]^{\frac{1-\phi_{MR}}{\phi_{MR}}}$$
$$\left[\ell_{Rss}^{\frac{1}{\sigma}\phi_{ML}} k_{Rss}^{\frac{1}{\sigma}\phi_{MK}} e_{Rss}^{-\phi_{MR}\frac{\sigma-1}{\sigma}-\frac{1}{\sigma}}\right]^{\frac{1}{\phi_{MR}}}.$$

Substitute for r_{ss} from (A-32):

$$\frac{e_{ss}}{e_{Rss}} = A_E(\Omega-g_L)^{-\frac{\omega}{1-\omega}}\left(\frac{1}{\lambda+\rho}\phi_{MR}\kappa_E\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}(\lambda+g_L)^{\frac{1}{\sigma}}\frac{e_{ss}}{e_{Rss}}\right)^{\frac{\sigma}{\sigma-1}}$$
$$\left[\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_E}{A_{LK}}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}}\left[Q_M\left(\frac{\phi_{ML}}{\kappa_L}\right)^{\phi_{ML}}\left(\frac{\phi_{MK}}{\kappa_K}\right)^{\phi_{MK}}\right]^{\frac{1}{\phi_{MR}}}$$
$$\left[\ell_{Rss}^{-(\phi_{ML}+\phi_{MK})\kappa_L\frac{\sigma-1}{\sigma}+\phi_{ML}} k_{Rss}^{-(\phi_{ML}+\phi_{MK})\kappa_K\frac{\sigma-1}{\sigma}+\phi_{MK}} e_{Rss}^{-\frac{1}{\sigma}(\phi_{ML}+\phi_{MK})}\right]^{\frac{1}{\phi_{MR}}}.$$

Substitute for e_{Rss} from (A-33) and rearrange to obtain:

$$\frac{e_{ss}}{e_{Rss}} = (\Omega-g_L)^{\frac{\omega}{1-\omega}(\sigma-1)}\left[\frac{1-\kappa_E}{\kappa_E}A_{LK}^{\frac{\sigma-1}{\sigma}}\right]^{\sigma\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}}\left[A_E Q_M\left(\frac{\phi_{ML}}{\kappa_L}\right)^{\phi_{ML}}\left(\frac{\phi_{MK}}{\kappa_K}\right)^{\phi_{MK}}\right]^{\frac{-(\sigma-1)}{\phi_{MR}}}$$
$$\left(\frac{1}{\lambda+\rho}\phi_{MR}\kappa_E\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}(\lambda+g_L)^{\frac{1}{\sigma}}\right)^{-\sigma}$$
$$\left[\frac{1}{\frac{\lambda+g_L}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}\phi_{MR}\frac{e_{ss}}{e_{Rss}}-1}\right]^{\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}}. \tag{A-34}$$

A strictly positive solution requires either $\omega = 0$ or $\Omega > g_L$. If $e_{ss}/e_{Rss} \in \left(0, \frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right)$, the left-hand side is strictly positive but the right-hand side is strictly negative, so the equation cannot hold. For $e_{ss}/e_{Rss} > \frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}$ and either $\omega = 0$ or $\rho > \Omega$, the right-hand side of (A-34) monotonically decreases in e_{ss}/e_{Rss}, going to 0 as $e_{ss}/e_{Rss} \to \infty$ and going to infinity as $e_{ss}/e_{Rss} \to \frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}$ from above. The left-hand side monotonically increases in e_{ss}/e_{Rss} and is strictly positive. So when either $\omega = 0$ or $\Omega > g_L$, there is a unique $e_{ss}/e_{Rss} > 0$ that solves the equation. That solution has:

$$\frac{e_{ss}}{e_{Rss}} > \frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}.$$

From (A-32), the steady state has $r_{ss} > 0$ if and only if $\Omega > g_L$. Because $e_{Yss} = e_{ss} - e_{Rss}$, the steady state has strictly positive final good production if and only if $e_{ss}/e_{Rss} > 1$, which holds because $\rho > \Omega, g_L$. From (A-20), $EROI(t)$ is constant along the balanced growth path and

$$EROI_{ss} > 1 + \frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}g_L.$$

We have proved part i of the proposition.

The right-hand side of equation (A-34) increases in A_E and Q_M if $\sigma < 1$ and decreases in A_E and Q_M if $\sigma > 1$. So the e_{ss}/e_{Rss} that solves (A-34)—and, by (A-20), $EROI_{ss}$—increases in A_E and Q_M if $\sigma < 1$ and decreases in A_E and Q_M if $\sigma > 1$. We have proved parts ii and iii of the proposition.

The claims about prices are implied by the existence of steady states in per-capita, present value terms for $\gamma(t)$ and $\mu(t)$.

G Proof of Proposition 5

The setting of Proposition 5 matches those of Propositions 3 and 4 as $\phi_{MR} \to 1$ (which implies $\phi_{ML}, \phi_{MK} \to 0$) and with Q_D in place of Q_M. Lemma A-4 still applies here. Therefore any interior balanced growth path must have per-capita variables constant and prices growing at $\rho - g_L$.

The case with $\sigma = 1$ (i.e., part i of the proposition) follows from taking the limit as $\phi_{MR} \to 1$ in the proof of Proposition 3.

Now consider $\sigma \neq 1$. As $\phi_{MR} \to 1$ (and $\phi_{ML}, \phi_{MK} \to 0$), equation (A-34) in the proof of Proposition 4 becomes:

$$\frac{e_{ss}}{e_{Rss}} = \left(\frac{(\lambda+\rho)\left[(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)\right]}{(\rho-\Omega)(1-\omega)\kappa_E}\right)^{\sigma} \frac{[A_E Q_D]^{1-\sigma}}{(\lambda+g_L)(\Omega-g_L)^{\frac{\omega}{1-\omega}(1-\sigma)}}. \tag{A-35}$$

From (A-31), an interior solution requires either $\omega = 0$ or $\Omega > g_L$. In either case, e_{ss}/e_{Rss} is interior.

The steady state has strictly positive final good production if and only if $e_{ss}/e_{Rss} > 1$. From (A-35), $e_{ss}/e_{Rss} > 1$ if and only if

$$\kappa_E^{-\sigma}[A_E Q_D]^{1-\sigma} > (\lambda + g_L)^{1-\sigma} \left(\frac{\lambda + g_L}{\lambda + \rho}\right)^{\sigma} (\Omega - g_L)^{\frac{\omega}{1-\omega}(1-\sigma)} \left(\frac{(1-\omega)(\rho - \Omega)}{(1-\omega)(\rho - \Omega) + \omega(\rho - g_L)}\right)^{\sigma}. \tag{A-36}$$

For $\sigma < 1$, inequality (A-36) holds if and only if

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D > (\lambda + g_L) \left(\frac{\lambda + \rho}{\lambda + g_L}\right)^{\frac{\sigma}{\sigma-1}} (\Omega - g_L)^{\frac{\omega}{1-\omega}} \left(\frac{(1-\omega)(\rho - \Omega) + \omega(\rho - g_L)}{(1-\omega)(\rho - \Omega)}\right)^{\frac{\sigma}{\sigma-1}},$$

where the right-hand side is χ from the proposition. We have proved part ii of the proposition.

$EROI(t)$ along a balanced growth path follows from equations (A-20) and (A-35). We have proved part iii of the proposition.

H Proof of Corollary 6

Fix $\sigma < 1$. Proposition 5 established that there is no balanced growth path with $L_R(t), K_R(t), E_R(t) > 0$ if $A_E Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} \leq \chi$.

Conjecture that there is a balanced growth path with $L_R(t)\, K_R(t)\, E_R(t) = 0$ from some time t_0 onward. Consider time $t > t_0$. We know that $F(L_R(t), K_R(t), E_R(t)) = 0$ if any of $L_R(t)$, $K_R(t)$, or $E_R(t)$ is zero. So $L_R(t) = K_R(t) = E_R(t) = 0$. In that case, $L_Y(t) = L(t)$, $K_Y(t) = K(t)$, and $E_Y(t) = E(t)$.

Because $\dot{r}(t) = -(\lambda + g_L) r(t)$ when $F(L_R(t), K_R(t), E_R(t)) = 0$, $r(t)$ declines monotonically towards zero, and because $e(t) = Q_D r(t)$, so too does $e(t)$. From $\dot{k}(t)/k(t) = sy(t)/k(t) - (\delta + g_L)$, $\dot{k}(t)/k(t)$ is constant if and only if $y(t)/k(t)$ is constant. The final good production function and $\dot{r}(t)/r(t) < 0$ imply that $\dot{y}(t)/y(t) < \alpha_K \dot{k}(t)/k(t)$ and thus that $y(t)/k(t)$ declines over time. Therefore $\dot{k}(t)/k(t)$ is not constant, which contradicts our assumption of a balanced growth path.

I Proof of Proposition 9

Equilibrium solves the following maximization problem:

$$\max_{L_Y(\cdot),K_Y(\cdot),E_Y(\cdot),E_R(\cdot),E_B(\cdot),B_R(\cdot),B_B(\cdot)} \int_0^\infty e^{-\rho t} L(t) u\left(\frac{(1-s)AL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t$$

$$\begin{aligned}
\text{s.t. } \dot{L}(t) =& g_L L(t)\\
\dot{K}(t) =& sAL(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E} - \delta K(t)\\
\dot{R}(t) =& A_E B_R(t)^{\kappa_B}E_R(t)^{\kappa_E} - \lambda R(t)\\
\dot{B}(t) =& A_B B_B(t)^{\beta_B}E_B(t)^{\beta_E} - \Psi B(t)\\
E(t) =& Q_S R(t)\\
L(t) =& L_Y(t)\\
K(t) =& K_Y(t)\\
E(t) =& E_Y(t) + E_R(t) + E_B(t)\\
B(t) =& B_B(t) + B_R(t).
\end{aligned}$$

Converting to per-capita and substituting, this is equivalent to:

$$\max_{e_Y(\cdot),e_R(\cdot),b_R(\cdot)} \int_0^\infty e^{-(\rho-g_L)t} u\left((1-s)A[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E}\right) \mathrm{d}t$$

$$\begin{aligned}
\text{s.t. } \dot{k}(t) =& sA[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E} - (\delta + g_L)k(t)\\
\dot{r}(t) =& A_E b_R(t)^{\kappa_B}e_R(t)^{\kappa_E} - (\lambda + g_L)r(t)\\
\dot{b}(t) =& A_B[b(t) - b_R(t)]^{\beta_B}[Q_S r(t) - e_Y(t) - e_R(t)]^{\beta_E} - (\Psi + g_L)b(t).
\end{aligned}$$

$\dot{k}(t)/k(t)$ is constant over time if and only if $y(t)/K(t)$ is constant over time. Therefore $g_k = g_y$, which, in the final good production function, implies that

$$\begin{aligned}
g_y =& \alpha_K g_k + \alpha_E g_{e_Y}\\
\Leftrightarrow g_y =& \frac{\alpha_E}{\alpha_E + \alpha_L} g_{e_Y}.
\end{aligned}$$

We have established part i of the proposition.

The current-value Hamiltonian is:

$$\begin{aligned}
&u\left((1-s)A[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E}\right) + \nu(t)\left(sA[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E} - (\delta + g_L)k(t)\right)\\
&+ \mu(t)\left[A_E b_R(t)^{\kappa_B}e_R(t)^{\kappa_E} - (\lambda + g_L)r(t)\right]\\
&+ \Upsilon(t)\left[A_B[b(t) - b_R(t)]^{\beta_B}[Q_S r(t) - e_Y(t) - e_R(t)]^{\beta_E} - (\Psi + g_L)b(t)\right].
\end{aligned}$$

The costate equations are:

$$\begin{aligned}(\rho - g_L)\nu(t) - \dot{\nu}(t) =&\alpha_K(1-s)u'((1-s)y(t))y(t)/k(t) + \alpha_K s\nu(t)y(t)/k(t) - (\delta + g_L)\nu(t)\\ \Leftrightarrow \dot{\nu}(t) =& -\frac{\alpha_K y(t)}{k(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] + (\rho+\delta)\nu(t), \qquad \text{(A-37)}\\ (\rho - g_L)\mu(t) - \dot{\mu}(t) =&Q_S\beta_E\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E} - \mu(t)(\lambda + g_L)\\ \Leftrightarrow \dot{\mu}(t) =& -Q_S\beta_E\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E} + \mu(t)[\rho+\lambda], \qquad \text{(A-38)}\\ (\rho - g_L)\Upsilon(t) - \dot{\Upsilon}(t) =&\beta_B\frac{\Upsilon(t)}{b_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E} - \Upsilon(t)(\Psi + g_L)\\ \Leftrightarrow \dot{\Upsilon}(t) =& -\beta_B\frac{\Upsilon(t)}{b_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E} + \Upsilon(t)[\rho+\Psi]. \qquad \text{(A-39)}\end{aligned}$$

The conditions to maximize the Hamiltonian are:

$$\begin{aligned}0 =&\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_E y(t)}{e_Y(t)} - \beta_E\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E}, \qquad \text{(A-40)}\\ 0 =&\kappa_E\frac{\mu(t)}{e_R(t)}A_E b_R(t)^{\kappa_B}e_R(t)^{\kappa_E} - \beta_E\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E}, \qquad \text{(A-41)}\\ 0 =&\kappa_B\frac{\mu(t)}{b_R(t)}A_E b_R(t)^{\kappa_B}e_R(t)^{\kappa_E} - \beta_B\frac{\Upsilon(t)}{b_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E}. \qquad \text{(A-42)}\end{aligned}$$

Equations (A-41) and (A-42) imply:

$$\frac{\kappa_E b_R(t)}{\kappa_B e_R(t)} = \frac{\beta_E b_B(t)}{\beta_B e_B(t)}. \qquad \text{(A-43)}$$

Rearrange equation (A-40):

$$\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E} = \left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_E y(t)}{\beta_E e_Y(t)}. \qquad \text{(A-44)}$$

Time-differentiating yields:

$$\begin{aligned}&\left[\frac{\dot{\Upsilon}(t)}{\Upsilon(t)} - \frac{\dot{e}_B(t)}{e_B(t)} + \beta_B\frac{\dot{b}_B(t)}{b_B(t)} + \beta_E\frac{\dot{e}_B(t)}{e_B(t)}\right]\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E}\\ =&\left[(1-s)^2\dot{y}(t)u''((1-s)y(t)) + s\dot{\nu}(t)\right]\frac{\alpha_E y(t)}{\beta_E e_Y(t)}\\ &+\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right]\frac{\alpha_E y(t)}{\beta_E e_Y(t)}\left[\frac{\dot{y}(t)}{y(t)} - \frac{\dot{e}_Y(t)}{e_Y(t)}\right].\end{aligned}$$

Substitute from (A-37), (A-39), and (A-44):

$$\begin{aligned}&-\beta_B A_B\left(\frac{e_B(t)}{b_B(t)}\right)^{\beta_E}+\rho+\Psi-\frac{\dot{e}_B(t)}{e_B(t)}+\beta_B\frac{\dot{b}_B(t)}{b_B(t)}+\beta_E\frac{\dot{e}_B(t)}{e_B(t)}\\=&\frac{(1-s)^2\dot{y}(t)u''((1-s)y(t))+s(\rho+\delta)\nu(t)}{(1-s)u'((1-s)y(t))+s\nu(t)}-s\frac{\alpha_K y(t)}{k(t)}+\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}.\end{aligned}$$

Substitute from the transition equation for $b(t)$, for $\nu(t)$ from (A-40), and for $\Upsilon(t)$ from (A-41):

$$\begin{aligned}&-\left[\frac{\dot{b}(t)}{b(t)}+\Psi+g_L\right]\beta_B\frac{b(t)}{b_B(t)}+\rho+\Psi-\frac{\dot{e}_B(t)}{e_B(t)}+\beta_B\frac{\dot{b}_B(t)}{b_B(t)}+\beta_E\frac{\dot{e}_B(t)}{e_B(t)}\\=&\frac{(1-s)^2\dot{y}(t)u''((1-s)y(t))+(\rho+\delta)\frac{e_Y(t)}{\alpha_E y(t)}\kappa_E\mu(t)A_E\left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B}-(\rho+\delta)(1-s)u'((1-s)y(t))}{\frac{e_Y(t)}{\alpha_E y(t)}\kappa_E\mu(t)A_E\left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B}}\\&-s\frac{\alpha_K y(t)}{k(t)}+\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}.\end{aligned}$$

Use Assumption 3 and rearrange:

$$\begin{aligned}&\frac{\kappa_E}{\alpha_E}\mu(t)e_Y(t)A_E\left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B}\\&\left[-\left(\frac{\dot{b}(t)}{b(t)}+\Psi+g_L\right)\beta_B\frac{b(t)}{b_B(t)}+\Psi-\delta-\frac{\dot{e}_B(t)}{e_B(t)}+\beta_B\frac{\dot{b}_B(t)}{b_B(t)}+\beta_E\frac{\dot{e}_B(t)}{e_B(t)}+s\frac{\alpha_K y(t)}{k(t)}-\frac{\dot{y}(t)}{y(t)}+\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\\=&-\left(\frac{\dot{y}(t)}{y(t)}+\rho+\delta\right).\end{aligned}$$

Substitute constant growth rates, recognizing that $e(t)$ and $b(t)$ must grow at the same rate g_e:

$$\begin{aligned}&\frac{\kappa_E}{\alpha_E}\mu(t)e_Y(t)A_E\left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B}\left[-\left(g_e+\Psi+g_L\right)\beta_B\frac{b(t)}{b_B(t)}+\Psi-\delta+s\frac{\alpha_K y(t)}{k(t)}-g_y+g_e\right]\\=&-\left[g_y+\rho+\delta\right].\end{aligned}$$

Substitute for $y(t)/k(t)$ from the transition equation for $k(t)$ and use $g_y=g_k=\frac{\alpha_E}{1-\alpha_K}g_e$:

$$\begin{aligned}&\frac{\kappa_E}{\alpha_E}\mu(t)e_Y(t)A_E\left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B}\left[-\left(g_e+\Psi+g_L\right)\beta_B\frac{b(t)}{b_B(t)}+\Psi-\delta+\alpha_K\left(\delta+g_L\right)+(1-\alpha_E)g_E\right]\\=&-\left[\frac{\alpha_E}{1-\alpha_K}g_e+\rho+\delta\right].\end{aligned}$$

Substitute from (A-43):

$$\begin{aligned}&\frac{\kappa_E}{\alpha_E}\mu(t)e_Y(t)A_E\left(\frac{\kappa_B\beta_E b_B(t)}{\kappa_E\beta_B e_B(t)}\right)^{\kappa_B}\\&\qquad\left[-\left(g_e+\Psi+g_L\right)\beta_B\frac{b(t)}{b_B(t)}+\Psi-\delta+\alpha_K\left(\delta+g_L\right)+(1-\alpha_E)g_E\right]\\=&-\left[\frac{\alpha_E}{1-\alpha_K}g_e+\rho+\delta\right].\end{aligned}\tag{A-45}$$

For this to hold, $\mu(t)e_Y(t)$ must be constant. Therefore $g_\mu=-g_e$. Use this in (A-38):

$$g_e+\rho+\lambda=Q_S\beta_E\frac{\Upsilon(t)}{\mu(t)}A_B\left(\frac{b_B(t)}{e_B(t)}\right)^{\beta_B}.\tag{A-46}$$

For this to hold, $\Upsilon(t)$ must grow at the same rate as $\mu(t)$. Use that in (A-39) and rearrange:

$$\frac{e_B(t)}{b_B(t)}=\left(\frac{g_e+\rho+\Psi}{\beta_B A_B}\right)^{\frac{1}{\beta_E}}.\tag{A-47}$$

Equation (A-41) implies:

$$\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E}=\frac{\kappa_E}{\beta_E}\frac{\mu(t)}{e_R(t)}A_E b_R(t)^{\kappa_B}e_R(t)^{\kappa_E}.$$

Substitute from (A-46) and (A-43), and then substitute from (A-47):

$$g_e+\rho+\lambda=\kappa_E A_E Q_S\left(\frac{\kappa_B\beta_E}{\kappa_E\beta_B}\left(\frac{g_e+\rho+\Psi}{\beta_B A_B}\right)^{-\frac{1}{\beta_E}}\right)^{\kappa_B}.\tag{A-48}$$

The left-hand side monotonically increases in g_e, and the right-hand side monotonically decreases in g_e. As $g_e\to\infty$, the left-hand side goes to ∞ and the right-hand side goes to zero. As $g_e\to-(\rho+\Psi)$ from above, the left-hand side goes to a finite value and the right-hand side goes to $-\infty$. So there is exactly one intersection at some $g_e\in\left(-[\rho+\Psi],\infty\right)$. That intersection has $g_e>0$ if and only if the right-hand side is greater than the left-hand side at $g_e=0$, so if and only if

$$\kappa_E A_E Q_S>(\rho+\lambda)\left(\frac{\kappa_E\beta_B}{\kappa_B\beta_E}\left(\frac{\rho+\Psi}{\beta_B A_B}\right)^{\frac{1}{\beta_E}}\right)^{\kappa_B}.$$

This is the condition given in the proposition. Because the right-hand side of (A-48) increases in Q_S while the left-hand side of (A-48) is independent of Q_S, g_e increases in Q_S. Similar

analysis holds that g_e increases in A_E and A_B and decreases in Ψ, ρ, and λ. We have established part ii of the proposition.

We established that $\mu(t)$ and $\Upsilon(t)$ grow at rate $-g_e$. Therefore the current-value prices of $R(t)$ and $B(t)$ each grow at rate $\rho-(g_e+g_L)$. These prices increase over time if and only if $g_e<\rho-g_L$. Part iii of the proposition follows from that observation and, from (A-48), that g_e increases in $\kappa_E^{1-\kappa_B}A_EQ_S$.

Now consider the feasibility of the solution. From the transition equation for $r(t)$,

$$g_e=Q_SA_E\left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B}\frac{e_R(t)}{e(t)}-(\lambda+g_L).$$

Substitute from (A-43):

$$g_e=Q_SA_E\left(\frac{\kappa_B\beta_Eb_B(t)}{\kappa_E\beta_Be_B(t)}\right)^{\kappa_B}\frac{e_R(t)}{e(t)}-(\lambda+g_L).$$

Rearrange:

$$\frac{e(t)}{e_R(t)}=Q_SA_E\left(\frac{\kappa_B\beta_Eb_B(t)}{\kappa_E\beta_Be_B(t)}\right)^{\kappa_B}\frac{1}{g_e+\lambda+g_L}.$$

And substitute from (A-47):

$$\frac{e(t)}{e_R(t)}=Q_SA_E\left(\frac{\kappa_B\beta_E}{\kappa_E\beta_B}\left(\frac{g_e+\rho+\Psi}{\beta_BA_B}\right)^{\frac{-1}{\beta_E}}\right)^{\kappa_B}\frac{1}{g_e+\lambda+g_L}. \tag{A-49}$$

Feasibility requires $g_e>-(\lambda+g_L)$ and

$$g_e+\lambda+g_L<Q_SA_E\left(\frac{\kappa_B\beta_E}{\kappa_E\beta_B}\left(\frac{g_e+\rho+\Psi}{\beta_BA_B}\right)^{\frac{-1}{\beta_E}}\right)^{\kappa_B}.$$

Substituting from (A-48), this is equivalent to:

$$g_e>-\frac{1}{1-\kappa_E}(\rho-\kappa_Eg_L)-\lambda.$$

This condition is satisfied when $g_e>0$.

From the transition equation for $b(t)$,

$$g_e=A_B\left(\frac{e_B(t)}{b_B(t)}\right)^{\beta_E}\frac{b_B(t)}{b(t)}-(\Psi+g_L).$$

Rearrange:

$$\frac{b(t)}{b_B(t)}=A_B\left(\frac{e_B(t)}{b_B(t)}\right)^{\beta_E}\frac{1}{g_e+\Psi+g_L}.$$

And substitute from (A-47):

$$\frac{b(t)}{b_B(t)} = \frac{g_e + \rho + \Psi}{\beta_B} \frac{1}{g_e + \Psi + g_L}. \tag{A-50}$$

Feasibility on the robot side requires

$$1 < \frac{1}{\beta_B} \frac{g_e + \Psi + \rho}{g_e + \Psi + g_L},$$

while holds for all $g_e > 0$.

Substitute (A-49) into (7):

$$EROI(t) = 1 + Q_S A_E \left(\frac{\kappa_B \beta_E}{\kappa_E \beta_B} \left(\frac{g_e + \rho + \Psi}{\beta_B A_B} \right)^{\frac{-1}{\beta_E}} \right)^{\kappa_B} \frac{g_e + g_L}{g_e + \lambda + g_L}.$$

Substitute from (A-48):

$$EROI(t) = 1 + \frac{1}{\kappa_E} \frac{g_e + g_L}{g_e + \lambda + g_L} (g_e + \rho + \lambda). \tag{A-51}$$

$EROI(t)$ depends on Q_S, A_E, and A_B only through g_e in (A-51). $EROI(t)$ increases in g_e in (A-51) because the fraction and the final term both increase in g_e. Part iv of the proposition follows from this observation and part ii. Part v also follows from using $g_e = 0$ in (A-51) and recalling that $EROI(t)$ increases in g_e in (A-51).

J Proof of Proposition 10

In a biomass economy in which Assumption 1 holds, Proposition 1 showed that there is no balanced growth path with output per capita growing forever at a strictly positive rate. Therefore output per capita in any economy using the biomass resource is bounded above. Define $\bar{Y}_H$ as the least upper bound on output in the biomass economy. Assume $\xi_D \geq \bar{Y}_H$. In that case, an economy that uses the biomass resource cannot begin accessing the oil/gas resource or, because $\xi_S > \xi_D$, the solar resource. We have established part i of the proposition.

If $\xi_M < \bar{Y}_H$, then it is feasible to develop the coal resource. From Proposition 3, there exists a balanced growth path in the economy with the coal resource in which all variables grow at the rate of population. Denote output per capita along this path as y^M_{ss}. Welfare on the balanced growth path is $u(s\, y^M_{ss})/(\rho - g_L)$. Using Assumption 3, welfare can be made arbitrarily large by making y^M_{ss} arbitrarily large. From Proposition 3, y^M_{ss} increases with Q_M in constant elasticity fashion, so y^M_{ss} and $u(s\, y^M_{ss})$ can be made arbitrarily large by making Q_M

arbitrarily large. Welfare at the time of first developing a coal resource may be lower than welfare along the balanced growth path but can also be made arbitrarily large by making Q_M arbitrarily large. In contrast, from Proposition 1, welfare in the biomass economy is bounded from above by $u(s\bar{Y})/(\rho - g_L)$, which is independent of Q_M. Therefore, the economy finds it optimal and feasible to eventually develop the coal resource if Q_M is sufficiently large and ξ_M is sufficiently small. We have established part ii of the proposition.

Now consider a case in which $y_{ss}^M \in (\xi_D, \xi_S)$, so that it is feasible to develop the oil/gas resource from an economy that uses the coal resource. From Proposition 5, there exists a balanced growth path in the economy with the oil/gas resource in which all variables grow at the rate of population. Denote output per capita along this path as y_{ss}^D. Following the argument of part ii, Q_D can be made large enough to make it optimal to develop the oil/gas resource from an economy that uses the coal resource. Therefore, an economy that uses the coal resource finds it optimal and feasible to eventually develop the oil/gas resource if Q_D is sufficiently large and ξ_D is not too large. We have established part iii.

Now consider a case in which $y_{ss}^D > \xi_S$, so that it is feasible to develop the solar resource from an economy that uses the oil/gas resource. Consider first a manufactured solar resource. From Corollary 7, there exists a balanced growth path in the economy with the solar resource in which all variables grow at the rate of population. Denote output per capita along this path as y_{ss}^S. Following the argument of part ii, Q_S can be made large enough to make it optimal to develop the solar resource from an economy that uses the oil/gas resource. Therefore, an economy that uses the oil/gas resource finds it optimal and feasible to eventually develop the solar resource if Q_S is sufficiently large and ξ_S is not too large. We have established part iv of the proposition for a manufactured solar resource.

For a self-replicating solar resource, observe from Proposition 9 that the growth rate increases in Q_S and, from (A-48), that it does so without bound. Welfare at the time of first developing a solar resource may be lower but can be made arbitrarily large by making Q_S arbitrarily large. Therefore, an economy that uses the oil/gas resource finds it optimal and feasible to eventually develop the solar resource if Q_S is sufficiently large and ξ_S is not too large. We have established part iv of the proposition for a self-replicating solar resource.

K Proof of Proposition A-2

Consider a case with $\ell_R(t), k_R(t) = 0$ and $e_R(t) > 0$. By labor and capital market-clearing, $\ell_Y(t) = 1$ and $k_Y(t) = k(t)$. Equilibrium solves the following maximization problem:

$$\begin{aligned}
&\max_{E_Y(\cdot)} \int_0^\infty e^{-\rho t} L(t) u\left((1-s)\frac{AL(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t \\
\text{s.t. } \dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& sAL(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E} - \delta K(t) \\
\dot{Z}(t) =& \Omega Z(t) - Z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t)\right)^{1-\omega} \\
\dot{R}(t) =& Z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t)\right)^{1-\omega} - \lambda R(t) \\
E(t) =& Q_D R(t) \\
E(t) =& E_Y(t) + E_R(t).
\end{aligned}$$

Converting to per-capita, substituting, and applying Assumption 3, this is equivalent to:

$$\begin{aligned}
&\max_{e_Y(\cdot)} \int_0^\infty e^{-(\rho - g_L)t} \ln\left((1-s)Ak(t)^{\alpha_K} e_Y(t)^{\alpha_E}\right) \mathrm{d}t \\
\text{s.t. } \dot{k}(t) =& sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} - (\delta + g_L)k(t) \\
\dot{z}(t) =& (\Omega - g_L)z(t) - z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E[Q_D r(t) - e_Y(t)]\right)^{1-\omega} \\
\dot{r}(t) =& z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E[Q_D r(t) - e_Y(t)]\right)^{1-\omega} - (\lambda + g_L)r(t).
\end{aligned}$$

The current-value Hamiltonian is:

$$\begin{aligned}
&\ln\left((1-s)Ak(t)^{\alpha_K} e_Y(t)^{\alpha_E}\right) + \nu(t)\left[sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} - (\delta + g_L)k(t)\right] \\
&+ \gamma(t)\left[(\Omega - g_L)z(t) - z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E[Q_D r(t) - e_Y(t)]\right)^{1-\omega}\right] \\
&+ \mu(t)\left[z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E[Q_D r(t) - e_Y(t)]\right)^{1-\omega} - (\lambda + g_L)r(t)\right].
\end{aligned}$$

The costate equations are:

$$\begin{aligned}(\rho - g_L)\nu(t) - \dot{\nu}(t) =&\alpha_K k(t)^{-1} + \alpha_K \nu(t) sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} k(t)^{-1} - \nu(t)[\delta + g_L] \\ \Leftrightarrow \dot{\nu}(t) =&(\rho + \delta)\nu(t) - \alpha_K k(t)^{-1} - \alpha_K \nu(t) sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} k(t)^{-1} \end{aligned} \tag{A-52}$$

$$\begin{aligned}(\rho - g_L)\gamma(t) - \dot{\gamma}(t) =&\gamma(t)(\Omega - g_L) + [\mu(t) - \gamma(t)]\omega z(t)^{\omega-1} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} \\ \Leftrightarrow \dot{\gamma}(t) =&\gamma(t)(\rho - \Omega) - [\mu(t) - \gamma(t)]\omega z(t)^{\omega-1} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} \end{aligned} \tag{A-53}$$

$$\begin{aligned}(\rho - g_L)\mu(t) - \dot{\mu}(t) =& - \mu(t)(\lambda + g_L) + [\mu(t) - \gamma(t)](1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D \\ \Leftrightarrow \dot{\mu}(t) =&\mu(t)(\lambda + \rho) - [\mu(t) - \gamma(t)](1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D. \end{aligned} \tag{A-54}$$

The condition to maximize the Hamiltonian is:

$$\begin{aligned}0 =&\alpha_E e_Y(t)^{-1} - [\mu(t) - \gamma(t)](1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \\ &+ \alpha_E \nu(t) sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} e_Y(t)^{-1}. \end{aligned} \tag{A-55}$$

Time-differentiate (A-55):

$$\begin{aligned}0 =& - \alpha_E e_Y(t)^{-1} \frac{\dot{e}_Y(t)}{e_Y(t)} - [\dot{\mu}(t) - \dot{\gamma}(t)](1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \\ &- [\mu(t) - \gamma(t)]\omega(1-\omega) z(t)^{\omega} \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{-\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \\ &\quad \left\{ \frac{\dot{z}(t)}{z(t)} - \frac{Q_D \dot{r}(t) - \dot{e}_Y(t)}{Q_D r(t) - e_Y(t)} \right\} \\ &+ \alpha_E \nu(t) sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} e_Y(t)^{-1} \left\{ \frac{\dot{\nu}(t)}{\nu(t)} + \alpha_K \frac{\dot{k}(t)}{k(t)} - (1 - \alpha_E) \frac{\dot{e}_Y(t)}{e_Y(t)} \right\} \end{aligned}$$

Substitute from the transition equation for $k(t)$ and from equations (A-52) and (A-55) and rearrange:

$$\begin{aligned}&\alpha_E e_Y(t)^{-1} \frac{1}{1-\omega} \frac{\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{\omega}}{z(t)^{\omega} \kappa_E^{\frac{\sigma}{\sigma-1}} A_E} \left\{ \alpha_E \frac{\dot{e}_Y(t)}{e_Y(t)} + (\rho - g_L) - \alpha_K k(t)^{-1} \nu(t)^{-1} \right\} \\ =& - [\dot{\mu}(t) - \dot{\gamma}(t)] \\ &- [\mu(t) - \gamma(t)] \left\{ \omega \frac{\dot{z}(t)}{z(t)} - \omega \frac{Q_D \dot{r}(t) - \dot{e}_Y(t)}{Q_D r(t) - e_Y(t)} \right. \\ &\qquad \left. - (\rho - g_L) - (1 - \alpha_K)(\delta + g_L) + \alpha_K k(t)^{-1} \nu(t)^{-1} + (1 - \alpha_E) \frac{\dot{e}_Y(t)}{e_Y(t)} \right\}. \end{aligned}$$

Substitute from the transition equations for $r(t)$ and $z(t)$ and solve for $\dot{\gamma}(t)$: for $\dot{\gamma}(t)$:

$$\begin{aligned}\dot{\gamma}(t) =&\dot{\mu}(t)\\ &+ \alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{e_R(t)}{z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}}\left\{\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}+(\rho-g_L)-\alpha_K k(t)^{-1}\nu(t)^{-1}\right\}\\ &+[\mu(t)-\gamma(t)]\Bigg\{\omega(\Omega-g_L)-\omega z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}\left(\frac{1}{z(t)}+\frac{Q_D}{e_R(t)}\right)\\ &\qquad+\omega(\lambda+g_L)\frac{e(t)}{e_R(t)}+\omega\frac{\dot{e}_Y(t)}{e_R(t)}-(\rho-g_L)-(1-\alpha_K)(\delta+g_L)+\alpha_K k(t)^{-1}\nu(t)^{-1}\\ &\qquad+(1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\Bigg\}.\end{aligned}$$

Substitute for $\gamma(t)$ from (A-55):

$$\begin{aligned}\dot{\gamma}(t) =&\dot{\mu}(t)\\ &+ \alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{e_R(t)}{z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}}\\ &\left\{\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}+(\rho-g_L)-\alpha_K k(t)^{-1}\nu(t)^{-1}\right.\\ &\quad+\left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]\\ &\quad\left[\omega(\Omega-g_L)-\omega z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}\left(\frac{1}{z(t)}+\frac{Q_D}{e_R(t)}\right)+\omega(\lambda+g_L)\frac{e(t)}{e_R(t)}\right.\\ &\quad\left.\left.+\omega\frac{\dot{e}_Y(t)}{e_R(t)}-(\rho-g_L)-(1-\alpha_K)(\delta+g_L)+\alpha_K k(t)^{-1}\nu(t)^{-1}+(1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\right\}.\end{aligned} \tag{A-56}$$

Substitute for $\gamma(t)$ from (A-55) into (A-54):

$$\dot{\mu}(t) =\mu(t)(\lambda+\rho)-\alpha_E e_Y(t)^{-1}Q_D\left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]. \tag{A-57}$$

Substitute $\dot{\mu}(t)$ into (A-56) and rearrange:

$$\begin{aligned}\dot{\gamma}(t) =&\mu(t)(\lambda+\rho) - \alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\left(\left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]Q_D + \omega\frac{e_R(t)}{z(t)}\right)\\ &+ \alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{e_R(t)}{z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t) - e_Y(t)]\right)^{1-\omega}}\\ &\left\{\omega(\Omega-g_L) + \omega(\lambda+g_L)\frac{e(t)}{e_R(t)} + \omega\frac{\dot{e}_Y(t)}{e_R(t)} + \frac{\dot{e}_Y(t)}{e_Y(t)}\right.\\ &\quad+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\\ &\quad\left[\omega(\Omega-g_L) - \omega z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t) - e_Y(t)]\right)^{1-\omega}\frac{1}{z(t)}\right.\\ &\quad+\omega(\lambda+g_L)\frac{e(t)}{e_R(t)} + \omega\frac{\dot{e}_Y(t)}{e_R(t)} - (\rho-g_L) - (1-\alpha_K)(\delta+g_L) + \alpha_K k(t)^{-1}\nu(t)^{-1}\\ &\quad\left.\left.+(1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\right\}. \end{aligned} \tag{A-58}$$

Solve (A-55) for $\gamma(t)$, substitute that and $\dot{\gamma}(t)$ from (A-58) into (A-53), and rearrange:

$$\begin{aligned}&\frac{\dot{e}_Y(t)}{e_Y(t)}\left\{1+\omega\frac{e_Y(t)}{e_R(t)} + \nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\left[(1-\alpha_E) + \omega\frac{e_Y(t)}{e_R(t)}\right]\right\}\\ =&-\frac{1}{\alpha_E}e_Y(t)\mu(t)\,(1-\omega)(\lambda+\Omega)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left(\frac{z(t)}{\kappa_E^{\frac{\sigma}{\sigma-1}}A_E e_R(t)}\right)^{\omega}\\ &+\left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]Q_D\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left(\frac{z(t)}{\kappa_E^{\frac{\sigma}{\sigma-1}}A_E e_R(t)}\right)^{\omega}\\ &-(\rho-\Omega) - \omega(\Omega-g_L) - \omega(\lambda+g_L)\frac{e(t)}{e_R(t)}\\ &-\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\\ &\quad\left[\rho-\Omega+\omega(\Omega-g_L) + \omega(\lambda+g_L)\frac{e(t)}{e_R(t)} - (\rho-g_L) - (1-\alpha_K)(\delta+g_L) + \alpha_K k(t)^{-1}\nu(t)^{-1}\right] \end{aligned} \tag{A-59}$$

Now consider a balanced growth path. From its transition equation, $k(t)$ grows at a constant rate if and only if $y(t)/k(t)$ is constant. So $g_k = g_y$. Substituting into the final good production function,

$$g_y = \frac{\alpha_E}{\alpha_E+\alpha_L}g_e. \tag{A-60}$$

From its transition equation, $z(t)$ grows at constant rate g_e if and only if:

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{e_R(t)}{z(t)} = (\Omega - g_L - g_e)^{\frac{1}{1-\omega}} . \tag{A-61}$$

Using this and the transition equation for $r(t)$, $r(t)$ grows at constant rate g_e if and only if:

$$\begin{aligned} g_e + \lambda + g_L =& \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right)^{1-\omega} \left(\frac{e_R(t)}{z(t)}\right)^{-\omega} \frac{e_R(t)}{r(t)} \\ =& \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left(\Omega - g_L - g_e\right)^{\frac{-\omega}{1-\omega}} \frac{e_R(t)}{r(t)} \\ \Leftrightarrow \frac{e_R(t)}{r(t)} =& \left[g_e + \lambda + g_L\right] \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right)^{-1} \left(\Omega - g_L - g_e\right)^{\frac{\omega}{1-\omega}} . \end{aligned} \tag{A-62}$$

Substituting for $y(t)/k(t)$ from the capital transition equation into (A-52), we find:

$$\frac{\dot{\nu}(t)}{\nu(t)} = (\rho + \delta) - \alpha_K k(t)^{-1} \nu(t)^{-1} - \alpha_K [g_k + \delta + g_L].$$

$\nu(t)k(t)$ is constant if and only if $\dot{\nu}(t)/\nu(t) = -g_k$. Using that condition, $g_k = g_y$, and (A-60) in the foregoing equation yields

$$\nu(t)k(t) = \frac{\alpha_K}{\alpha_E g_e + (\delta + \rho) - \alpha_K(\delta + g_L)}. \tag{A-63}$$

Substitute $y(t)/k(t)$ from the capital transition equation, $g_k = g_y$, g_y from (A-60), $\nu(t)k(t)$ from (A-63), and $\dot{e}_Y(t)/e_Y(t) = g_e$ into (A-59):

$$\begin{aligned} & g_e \left[1 + \omega \frac{e_Y(t)}{e_R(t)}\right] \frac{1}{1-\alpha_K} \frac{\alpha_E g_e + (1-\alpha_K)(\delta+\rho)}{\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta + g_L)} \\ =& -\frac{1}{\alpha_E} e_Y(t) \mu(t) (1-\omega)(\lambda + \Omega) \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right]^{1-\omega} \left(\frac{z(t)}{e_R(t)}\right)^{\omega} \\ & - \frac{1}{1-\alpha_K} \frac{\alpha_E g_e + (1-\alpha_K)(\delta+\rho)}{\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta + g_L)} \\ & \left[(\rho - \Omega) + \omega(\Omega - g_L) + \omega(\lambda + g_L) \frac{e(t)}{e_R(t)} - Q_D \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right]^{1-\omega} \left(\frac{z(t)}{e_R(t)}\right)^{\omega}\right]. \end{aligned} \tag{A-64}$$

Observe that

$$\frac{e_R(t)}{r(t)} = \frac{e_R(t)}{e_Y(t)} \frac{e_Y(t)}{r(t)} = \frac{e_R(t)}{e_Y(t)} Q_D \frac{e_Y(t)}{e(t)} = \frac{e_R(t)}{e_Y(t)} Q_D \left(1 - \frac{e_R(t)}{e(t)}\right) = \frac{e_R(t)}{e_Y(t)} \left(Q_D - \frac{e_R(t)}{r(t)}\right)$$

.

Therefore:

$$\frac{e_Y(t)}{e_R(t)} = \frac{Q_D - \frac{e_R(t)}{r(t)}}{\frac{e_R(t)}{r(t)}}.$$

Using that and (A-62),

$$\frac{e_Y(t)}{e_R(t)} + \frac{1}{\omega} = \frac{Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E + \frac{1-\omega}{\omega} \left[g_e + \lambda + g_L\right] \left(\Omega - g_L - g_e\right)^{\frac{\omega}{1-\omega}}}{\left[g_e + \lambda + g_L\right] \left(\Omega - g_L - g_e\right)^{\frac{\omega}{1-\omega}}}.$$

Substitute into (A-64), and also use $e_R(t)/r(t)$ from (A-62) and $e_R(t)/z(t)$ from (A-61):

$$\begin{aligned}
&\omega g_e \frac{Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E + \frac{1-\omega}{\omega} \left[g_e + \lambda + g_L\right] \left(\Omega - g_L - g_e\right)^{\frac{\omega}{1-\omega}}}{\left[g_e + \lambda + g_L\right] \left(\Omega - g_L - g_e\right)^{\frac{\omega}{1-\omega}}} \Big[\alpha_E g_e + (1-\alpha_K)(\delta+\rho)\Big] \\
=& -\frac{1-\alpha_K}{\alpha_E} e_Y(t) \mu(t) \left(1-\omega\right)(\lambda+\Omega) \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \Big[\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta+g_L)\Big] \left(\Omega - g_L - g_e\right)^{\frac{-\omega}{1-\omega}} \\
&- \Big[\alpha_E g_e + (1-\alpha_K)(\delta+\rho)\Big] \\
&\quad \Big[(\rho-\Omega) + \omega(\Omega - g_L) - Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left(\Omega - g_L - g_e\right)^{\frac{-\omega}{1-\omega}} \frac{g_e + (1-\omega)(\lambda+g_L)}{g_e + \lambda + g_L}\Big]. \quad \text{(A-65)}
\end{aligned}$$

This last equation requires $\mu(t)e_Y(t)$ be constant, which in turn requires that $g_\mu = -g_e$. Substitute that into the left-hand side of (A-57) and, in the right-hand side of (A-57), substitute $\dot{k}(t)$ from its transition equation, $g_k = g_y$ and g_y from (A-60), and $\nu(t)k(t)$ from (A-63), and then solve for $\mu(t)e_Y(t)$:

$$\mu(t)e_Y(t) = \alpha_E \frac{Q_D}{g_e + \lambda + \rho} \frac{1}{1-\alpha_K} \frac{\alpha_E g_e + (1-\alpha_K)(\delta+\rho)}{\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta+g_L)}. \quad \text{(A-66)}$$

Substitute into (A-65):

$$(g_e + \rho - \Omega) + \omega(\Omega - g_L - g_e) = (1-\omega) Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left(\Omega - g_L - g_e\right)^{\frac{-\omega}{1-\omega}} \frac{g_e + \rho - \Omega}{g_e + \rho + \lambda} \quad \text{(A-67)}$$

The left-hand side of (A-67) increases in g_e. The right-hand side of (A-67) is real-valued for $g_e < \Omega - g_L$ and, in those case, increases in g_e for $g_e > -(\rho+\lambda)$. As g_e approaches $-(\rho-\Omega)$ from above, the left-hand side of (A-67) approaches $\omega(\rho - g_L) > 0$ and the right-hand side of (A-67) approaches zero. As g_e approaches $\Omega - g_L$ from below, the left-hand side of (A-67) approaches $\rho - g_L > 0$ and the right-hand side of (A-67) approaches positive infinity. Therefore there exists $g_e \in \big(-(\rho-\Omega), \Omega - g_L\big)$ such that (A-67) holds. An intersection occurs

at strictly positive g_e if the left-hand side of (A-67) is strictly greater than the right-hand side of (A-67) when each is evaluated at $g_e = 0$. That sufficient condition is:

$$\begin{aligned} 0 >& (1-\omega) Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left(\Omega - g_L\right)^{\frac{-\omega}{1-\omega}} \frac{\rho - \Omega}{\rho + \lambda} - (\rho - \Omega) - \omega(\Omega - g_L) \\ \Leftrightarrow Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} <& \frac{1}{1-\omega} \left(\Omega - g_L\right)^{\frac{\omega}{1-\omega}} \frac{\rho + \lambda}{\rho - \Omega} \left\{ (\rho - \Omega) + \omega(\Omega - g_L) \right\} \end{aligned} \tag{A-68}$$

Increasing $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}$ increases the right-hand side of (A-67) when it is strictly positive, without affecting the left-hand side of (A-67). Because the right-hand side of (A-67) cuts the left-hand side of (A-67) from below around the intersection in g_e analyzed above, an increase in $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}$ moves that intersection to smaller g_e.

Now consider the feasibility of a balanced growth path. From the capital transition equation, $g_k = g_y$, and (A-60), $y(t)/k(t) > 0$ if and only if

$$0 < \frac{1}{s} \left[\frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \delta + g_L \right].$$

This is satisfied if and only if

$$g_e > -\frac{\alpha_E + \alpha_L}{\alpha_E} (\delta + g_L). \tag{A-69}$$

This holds for all $g_e \geq 0$. From (A-63), $\nu(t)k(t) > 0$ if and only if

$$g_e > -\frac{1}{\alpha_E} \left[(1 - \alpha_K)\delta + \rho - \alpha_K g_L \right].$$

Using that $\rho > g_L$, this inequality holds whenever inequality (A-69) holds. From (A-62), $e_R(t)/e(t) > 0$ if and only if

$$0 < \left[g_e + \lambda + g_L \right] \left(\Omega - g_L - g_e \right)^{\frac{\omega}{1-\omega}}, \tag{A-70}$$

which holds if and only if $g_e \in (-(\lambda + g_L), \Omega - g_L)$. From (A-66), $\mu(t) e_Y(t) > 0$ if and only if

$$0 < \frac{1}{g_e + \lambda + \rho} \frac{\alpha_E g_e + (1 - \alpha_K)(\delta + \rho)}{\alpha_E g_e + (\delta + \rho) - \alpha_K (\delta + g_L)}.$$

The first fraction is strictly positive for $g_e > -(\rho - \Omega)$, which is met by the analyzed solution to (A-67). Both the denominator and the numerator in the second fraction are strictly positive when inequality (A-69) holds. From (A-61), $e_R(t)/z(t) > 0$ if and only if $g_e < \Omega - g_L$, which is met by the analyzed solution to (A-67). Finally, from (A-62), $e_R(t)/e(t) < 1$ if and only if

$$Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} > \left[g_e + \lambda + g_L \right] \left(\Omega - g_L - g_e \right)^{\frac{\omega}{1-\omega}}. \tag{A-71}$$

From inequalities (A-69), (A-70), and (A-71), a balanced growth path with $g_e \in \big(-(\rho - \Omega), \Omega - g_L\big)$ is feasible if

$$g_e > \max\left\{-\frac{\alpha_E + \alpha_L}{\alpha_E}(\delta + g_L), -(\lambda + g_L)\right\} \tag{A-72}$$

and

$$Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} > \left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}.$$

The necessary conditions in the proposition recognize that the first of these two inequalities and g_e declining in $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}$ together imply a ceiling on $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}$. On a feasible balanced growth path with $g_e \in (0, \Omega - g_L)$, inequality (A-71) is satisfied if

$$Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} > \left[\Omega + \lambda\right](\Omega - g_L)^{\frac{\omega}{1-\omega}}. \tag{A-73}$$

That lower bound is below the upper bound in (A-68) because $\rho > \Omega$, so a path with $g_e > 0$ is feasible.

We have found a feasible path along which $\ell_R(t), k_R(t) = 0$ with $e_R(t) > 0$ and all variables grow at a constant rate. The conditions of the proposition ensure that $\Omega - g_L > 0$ and, using that the intersection that defines g_e declines in $Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}}$, that inequalities (A-72) and (A-71) hold. Part i of the proposition follows from (A-60). Part ii of the proposition follows from the analysis of (A-67). Part iiii of the proposition follows from that same analysis and from (A-61). Part iv of the proposition follows from (A-73).

Using (A-62), (7) becomes:

$$EROI(t) = 1 + Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} \frac{g_e + g_L}{\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}.$$

We have proved part v of the proposition.

L Proof of Proposition A-3

Consider a case with $\ell_R(t), k_R(t) = 0$ and $e_R(t) > 0$. By labor and capital market-clearing, $\ell_Y(t) = 1$ and $k_Y(t) = k(t)$. Equilibrium solves the following maximization problem:

$$\begin{aligned}
\max_{E_Y(\cdot), E_R(\cdot)} & \int_0^\infty e^{-\rho t} L(t) u\left(\frac{(1-s)AL(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t \\
\text{s.t. } \dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& sAL(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E} - \delta K(t) \\
\dot{R}(t) =& \kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t) - \lambda R(t) \\
E(t) =& Q_D R(t) \\
E(t) =& E_Y(t) + E_R(t).
\end{aligned}$$

Converting to per-capita and substituting, this is equivalent to:

$$\begin{aligned}\max_{e_Y(\cdot)} \int_0^\infty e^{-(\rho-g_L)t} u\Big((1-s)A[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E}\Big)\,\mathrm{d}t\\ \text{s.t. } \dot{k}(t) =& sA[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E} - (\delta+g_L)k(t)\\ \dot{r}(t) =& \kappa_E^{\frac{\sigma}{\sigma-1}} A_E[Q_D r(t) - e_Y(t)] - (\lambda+g_L)r(t).\end{aligned}$$

The current-value Hamiltonian is:

$$\begin{aligned}&u\Big((1-s)A[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E}\Big) + \nu(t)\Big(sA[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E} - (\delta+g_L)k(t)\Big)\\ &+ \mu(t)\Big[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E[Q_D r(t) - e_Y(t)] - (\lambda+g_L)r(t)\Big].\end{aligned}$$

The costate equations are:

$$\begin{aligned}(\rho-g_L)\nu(t) - \dot{\nu}(t) =& \Big[(1-s)u'((1-s)y(t)) + s\nu(t)\Big]\frac{\alpha_K y(t)}{k(t)} - (\delta+g_L)\nu(t)\\ \Leftrightarrow \dot{\nu}(t) =& -\Big[(1-s)u'((1-s)y(t)) + s\nu(t)\Big]\frac{\alpha_K y(t)}{k(t)} + (\rho+\delta)\nu(t), \qquad \text{(A-74)}\\ (\rho-g_L)\mu(t) - \dot{\mu}(t) =& \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - \mu(t)(\lambda+g_L)\\ \Leftrightarrow \dot{\mu}(t) =& \mu(t)\Big[\rho+\lambda-\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D\Big]. \qquad \text{(A-75)}\end{aligned}$$

The condition to maximize the Hamiltonian is:

$$0 = \Big[(1-s)u'((1-s)y(t)) + s\nu(t)\Big]\frac{\alpha_E y(t)}{e_Y(t)} - \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E. \qquad \text{(A-76)}$$

Substituting into (A-74), we obtain:

$$\dot{\nu}(t) = -\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{\alpha_K e_Y(t)}{\alpha_E k(t)} + (\rho+\delta)\nu(t). \qquad \text{(A-77)}$$

Time-differentiating (A-76), we obtain:

$$\begin{aligned}0 =& \Big[(1-s)u'((1-s)y(t)) + s\nu(t)\Big]\alpha_E\left[\frac{\dot{y}(t)}{e_Y(t)} - \frac{y(t)}{e_Y(t)}\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\\ &+ (1-s)^2 u''((1-s)y(t))\frac{\alpha_E y(t)}{e_Y(t)}\dot{y}(t) + s\dot{\nu}(t)\frac{\alpha_E y(t)}{e_Y(t)} - \dot{\mu}(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E.\end{aligned}$$

Substitute from (A-75), (A-76), and (A-77):

$$\begin{aligned}0 =&\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\\&+\alpha_E(1-s)^2u''((1-s)y(t))\frac{y(t)}{e_Y(t)}\dot{y}(t)\\&+sy(t)\left[-\alpha_K\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\frac{\mu(t)}{k(t)}+\alpha_E(\rho+\delta)\frac{\nu(t)}{e_Y(t)}\right]\\&-\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[(\rho+\lambda)-\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_D\right].\end{aligned}$$

Time-differentiate the final good production function and substitute for $\dot{y}(t)$ in the previous equation:

$$\begin{aligned}0 =&\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}\right]+\alpha_E(1-s)^2u''((1-s)y(t))\frac{y(t)}{e_Y(t)}y(t)\left[\alpha_K\frac{\dot{k}(t)}{k(t)}+\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\\&+sy(t)\left[-\alpha_K\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\frac{\mu(t)}{k(t)}+\alpha_E(\rho+\delta)\frac{\nu(t)}{e_Y(t)}\right]-\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[(\rho+\lambda)-\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_D\right].\end{aligned}$$

Use Assumption 3 and rearrange:

$$\begin{aligned}0 =&\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}-(\rho+\lambda)+\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_D-s\alpha_K\frac{y(t)}{k(t)}\right]\\&-\alpha_E\frac{1}{e_Y(t)}\left[\alpha_K\frac{\dot{k}(t)}{k(t)}+\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}\right]+\nu(t)s\alpha_E(\rho+\delta)\frac{y(t)}{e_Y(t)}.\end{aligned}\tag{A-78}$$

$\dot{k}(t)/k(t)$ is constant over time if and only if

$$\frac{y(t)}{k(t)}=\frac{g_k+\delta+g_L}{s}\tag{A-79}$$

Therefore $g_k=g_y$, which, in the final good production function, implies that

$$\begin{aligned}g_y =&\alpha_Kg_k+\alpha_Eg_{e_Y}\\\Leftrightarrow g_y =&\frac{\alpha_E}{\alpha_E+\alpha_L}g_{e_Y}.\end{aligned}$$

The resource transition equation implies:

$$\frac{\dot{r}(t)}{r(t)}=\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[Q_D-\frac{e_Y(t)}{r(t)}\right]-(\lambda+g_L).$$

The growth rate of r is constant if and only if $e_Y(t)$ grows at g_r. The growth rate of r is the same as the growth rate of e. Therefore $e_R(t)$ grows at rate g_e and

$$g_y = \frac{\alpha_E}{\alpha_E + \alpha_L} g_e. \tag{A-80}$$

Using this and $g_k = g_y$ in (A-79),

$$\frac{y(t)}{k(t)} = \frac{\frac{\alpha_E}{\alpha_E+\alpha_L} g_e + \delta + g_L}{s} \tag{A-81}$$

Use $g_k = g_y$ in (A-78) and substitute from (A-80) and (A-81):

$$\begin{aligned} 0 =& \mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[-\frac{\alpha_L}{\alpha_E + \alpha_L} g_e - (\rho + \lambda) + \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - \alpha_K \left(\frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \delta + g_L \right) \right] \\ & - \alpha_E \frac{1}{e_Y(t)} \frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \nu(t) s \alpha_E (\rho + \delta) \frac{y(t)}{e_Y(t)}. \end{aligned}$$

Multiply through by $e_Y(t)$ and multiply the final term by $k(t)/k(t)$, again using (A-81):

$$\begin{aligned} 0 =& \mu(t) e_Y(t) \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[-\frac{\alpha_L}{\alpha_E + \alpha_L} g_e - (\rho + \lambda) + \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - \alpha_K \left(\frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \delta + g_L \right) \right] \\ & - \alpha_E \frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \nu(t) k(t) \alpha_E (\rho + \delta) \left(\frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \delta + g_L \right). \end{aligned} \tag{A-82}$$

$\nu(t)k(t)$ is constant if and only if $\dot{\nu}(t)/\nu(t) = -g_k$. In that case, (A-77) implies

$$\nu(t)k(t) = \mu(t) e_Y(t) \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{\alpha_K}{\alpha_E} \frac{1}{g_k + \rho + \delta}. \tag{A-83}$$

Substitute into (A-82) and use (A-80):

$$\begin{aligned} 0 =& \mu(t) e_Y(t) \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[-\frac{\alpha_L}{\alpha_E + \alpha_L} g_e - (\rho + \lambda) + \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - \alpha_K \frac{\frac{\alpha_E}{\alpha_E+\alpha_L} g_e + g_L + \delta}{\frac{\alpha_E}{\alpha_E+\alpha_L} g_e + \rho + \delta} \frac{\alpha_E}{\alpha_E + \alpha_L} g_e \right] \\ & - \alpha_E \frac{\alpha_E}{\alpha_E + \alpha_L} g_e. \end{aligned} \tag{A-84}$$

If g_e is constant, so too is $\mu(t)e_Y(t)$. From (A-75), $\mu(t)$ grows at rate $\rho + \lambda - \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D$, so $\mu(t)e_Y(t)$ constant implies:

$$g_e = \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho + \lambda). \tag{A-85}$$

Substituting into (A-84), we find:

$$0 = \frac{\alpha_E}{\alpha_E + \alpha_L}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho+\lambda)\right) \left\{\mu(t) e_Y(t) \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[1 - \alpha_K \frac{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho+\lambda)\right] + g_L + \delta}{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho+\lambda)\right] + \rho + \delta}\right] - \alpha_E\right\}.$$

This holds if either $g_e = 0$ or

$$\mu(t) e_Y(t) = \frac{\alpha_E}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[1 - \alpha_K \frac{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho+\lambda)\right] + g_L + \delta}{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho+\lambda)\right] + \rho + \delta}\right]}.$$

Now consider the feasibility of a solution with $g_e \neq 0$. The right-hand side is strictly positive if and only if:

$$1 > \alpha_K \frac{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho+\lambda)\right] + g_L + \delta}{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho+\lambda)\right] + \rho + \delta}. \tag{A-86}$$

Observe that $y(t)/k(t) > 0$ requires, from (A-81), that the numerator on the right-hand side be strictly positive and that $\nu(t)k(t) > 0$ requires, from (A-83), that the denominator on the right-hand side be strictly positive. Using $\rho > g_L$, these two conditions are jointly satisfied if and only if the numerator is strictly positive, so if and only if

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D > \rho + \lambda - \frac{\alpha_E + \alpha_L}{\alpha_E}(g_L + \delta). \tag{A-87}$$

If this last inequality holds, then inequality (A-86) also holds, because $\alpha_K < 1$. Therefore we have an infeasible path in the case that $\mu(t)e_Y(t)$ is weakly negative. Finally, observe that $r(t)$ grows at a constant rate if and only if

$$\frac{e_R(t)}{e(t)} = \frac{g_e + \lambda + g_L}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D}. \tag{A-88}$$

Substituting from (A-85), we have $e_R(t)/e(t) \in (0,1)$ if and only if

$$\frac{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D - (\rho - g_L)}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D} \in (0,1).$$

Because $\rho > g_L$, this condition holds if and only if

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D > \rho - g_L. \tag{A-89}$$

We have found a feasible path along which $\ell_R(t), k_R(t) = 0$ with $e_R(t) > 0$ and all variables grow at a constant rate. The necessary condition in the proposition ensures that inequalities (A-87) and (A-89) hold. Part i follows from $g_y = g_k$ and (A-80). Part ii follows from (A-85). To derive part iii of the proposition, observe that when $\omega = 0$,

$$\chi \triangleq (\lambda + g_L)\left(\frac{\lambda+\rho}{\lambda+g_L}\right)^{\frac{\sigma}{\sigma-1}} > (\lambda + g_L)\frac{\lambda+\rho}{\lambda+g_L} = \lambda + \rho$$

and, from (A-85), $g_e > 0$ if $\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_D > (\rho + \lambda)$. Part iv of the proposition follows from (7), (A-85), and (A-88).