Do Equity and Options Markets Agree about Volatility?*

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Abstract

We derive tight pricing kernel restrictions from zero-date options, which are options that expire on the same day they are traded. These restrictions concern the volatility of small and frequent asset price moves that the equity and options markets must agree on in a frictionless economy where the two markets are integrated. Violations of such restrictions lead to pseudo-arbitrage opportunities that can be exploited using a static portfolio of zero-date options and a dynamic position in the asset. These pseudo-arbitrage opportunities are characterized by arbitrarily high reward-to-risk ratios and cause local explosion of conditional moments of the aggregate pricing kernel.

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1 Introduction

The absence of arbitrage is a fundamental concept in financial economics. Applied to option pricing, it is equivalent to the existence of a positive stochastic discount factor that determines the current price of an option as the conditional expectation of its discounted payoff given present information. The classical option pricing formula of Black and Scholes (1973) as well as its many extensions falls into this asset pricing framework.

Yet, earlier work has documented various puzzling features in observed option data relative to standard asset pricing models. These include a non-monotonic U-shape pattern of the pricing kernel when projected onto the asset return space as well as the large returns, compared to investing in the stock market, from selling put options. To rationalize the observed option data, the literature has primarily adopted two different approaches. The conventional approach, based on the premise of market efficiency, involves adding more complicated features to the underlying asset dynamics or generalizing investors' risk preferences.

An alternative approach is to view the equity and options markets as partially segmented (Bates (2022)). In the partially segmented story, the law of one price need not hold, and mispricing of options can occur due to various factors limiting investors' ability, or willingness, to fully eliminate arbitrage opportunities.

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Investors might not be *able* to eliminate arbitrage violations because of margin constraints (Gârleanu and Pedersen (2011)), transaction costs (Pontiff (1996)) or estimation risk (Da et al. (2024)). For example, if there is 5% disagreement between equity and options markets about diffusive variance, option bid–ask spreads of the same magnitude will make exploiting this discrepancy not profitable. Similarly, estimation risk is non-negligible, as an arbitrageur must first device an econometric tool to detect such a gap, using backward-looking asset price data and forward-looking option price data. Due to the strong intraday variation of volatility, there is a considerable statistical limit to arbitrage.

Investors might not be *willing* to eliminate arbitrage violations because doing so typically involves risk in practice. As risk-adverse agents, arbitrageurs might thus forego arbitrage opportunities exceeding their acceptable risk level (Shleifer and Vishny (1997)). Consider for example noise traders willing to buy put options at an irrationally high price. Informed arbitrageurs, seeking to profit from selling these options, face downside risk: first, in the form of fundamental un-hedgeable risk (Figlewski (1979), Gârleanu et al. (2009)), due to potentially large losses resulting from a drop of the underlying stock price; and second, in the form of noise trader risk (De Long et al. (1990)), mediated by the arbitrageurs' own activities. Namely, if the stock drops significantly before the option expires, to limit their risk exposure, arbitrageurs might start to delta hedge their positions by selling the stock, which further pushes down the stock price. This in turn confirms noise traders in their initial buying decision and further increases their demand for put options, which arbitrageurs are more likely to meet only with delta hedged positions and therefore depresses the stock even more.[1] Similarly, strategies that exploit disagreements between equity and options markets about diffusive volatility involve risk and when the disagreements are small, the Sharpe ratios of these strategies can be even below that of the standard benchmark of holding the market portfolio.

Set against this background, the goal of this paper is a model-free assessment of the integration between equity and options markets by focusing on so-called *zero-date options*. These options, called *0DTEs* for short, expire on the same day they are traded and have gained immense popularity recently.[2] We show that integration of the two markets implies strong no-arbitrage restrictions on the prices of zero-date options and we evaluate in economic terms the consequences of their violations.[3] As mentioned in the previous two paragraphs, such violations might exist if the costs and risks involved in implementing strategies that exploit them are too high. Moreover, arguments for mispricing due to limits to arbitrage are arguably most compelling for options with such short time to maturity, as the underlying asset has minimal opportunity to recover from significant price movements before expiration. Recent evidence reported by Bryzgalova et al. (2023) that a significant proportion of trading in short-dated options is due to retail investors, who act as noise traders, prefer cheap speculative investments and lose money on average, further reinforces the plausibility of a market segmentation hypothesis for 0DTEs.

Our economic analysis consists of two steps. In a first step, we use the classical method of Hansen and Jagannathan (1991) to form pricing kernel restrictions from Sharpe ratios of

[1] Such feedback mechanisms are considered capable of generating a "Volmageddon 2.0" in the options market, see the *Reuters* article "Rise of near-term options trading carries risk of Volmageddon 2.0 -JPM's Kolanovic" from February 16, 2023.

[2] Currently, nearly half of the trading volume of all options written on the S&P 500 index that are traded on the CBOE options exchange is in the zero-date maturity domain, see the *Financial Times* article "Zero-day options now account for 50% of S&P options volumes" from September 25, 2023.

[3] One can also develop a formal statistical test for the integration of the equity and option markets. We leave this for future work and we focus instead in this paper on quantifying in economic terms the consequences of market separation.

portfolios involving options and the underlying asset. We show, in our Theorem 1, that the existence of an aggregate pricing kernel rationalizing both equity and options prices implies the absence of so-called *pseudo-arbitrage* opportunities. A portfolio with investment horizon T is called a pseudo-arbitrage if its Sharpe ratio remains above a fixed value as T shrinks to zero. Pseudo-arbitrage can be viewed as an asymptotic version of Cochrane and Saa-Requejo (2000)'s notion of good deals. While good deals simply refer to investments with a large Sharpe ratio, pseudo-arbitrage opportunities require a fixed Sharpe ratio level be maintained even as the investment horizon shrinks to zero.

Theorem 1 is a consequential result. For options with nontrivial time to expiration, e.g., monthly options, there is a large gap between classical no-arbitrage bounds and the good-deal bounds of Cochrane and Saa-Requejo (2000), mainly due to potentially complicated underlying asset dynamics. This gap, however, is much smaller for 0DTEs: as Theorem 1 shows, the no-arbitrage and good-deal bounds converge and become equivalent as the time to maturity shrinks to zero. This yields a powerful economic tool to test theoretical (risk-free) arbitrage conditions concerning market integration by considering practical (risky) pseudo-arbitrage strategies and their reward-to-risk ratios.

In a second step, we design such an economic test by constructing an explicit pseudo-arbitrage in case the equity and options markets are segmented. To this end, we recast the existence of a joint pricing kernel for equity and option prices as a requirement that the statistical probability measure, $\mathbb{P}$, and the risk-neutral probability measure, $\mathbb{Q}$ (under which option prices are determined), agree about the small moves in asset prices. These can come from diffusive moves but also from small and frequent jumps in asset prices. As a result, market integration imposes restrictions on the diffusion coefficient in the asset price (diffusive volatility) and the small asset price jumps under $\mathbb{P}$ and $\mathbb{Q}$. The law of the big asset price jumps, on the other hand, remains essentially unrestricted and may differ under $\mathbb{P}$ and $\mathbb{Q}$.

To remove the contribution to price volatility due to big and infrequent asset moves, we consider portfolios that replicate short-horizon *truncated volatility*. Risk-neutral truncated volatility can be spanned from a portfolio of zero-date options with different strikes in a manner similar to the construction of the popular VIX index but with significantly less weight assigned to deep out-of-the-money options.[4] Truncated volatility under $\mathbb{P}$, on the other hand, can be recovered from high-frequency records of the underlying price in a short time window prior to observing the options. Market integration then boils down to an agreement between option-implied and return-based truncated volatilities in the limit as the tenor of the considered options shrinks to zero. If the equity and options markets are segmented, we show in Theorem 2 that a static (long or short) position in the option portfolio spanning $\mathbb{Q}$-truncated volatility combined with a dynamic position in the underlying asset until the options expire constitutes pseudo-arbitrage and, therefore, an implementable test for market integration.

Our empirical analysis shows that the answer to the question posed in the title is a qualified "Yes": While we detect occasional mild disagreements between equity and options markets about volatility due to small and frequent moves in the S&P 500 market index, transaction costs, estimation risk and short-term volatility risk make implementing volatility arbitrage strategies to exploit these differences not profitable. Indeed, when implemented on S&P 500 index options from 2020–2024 with a time-to-expiration of 30 minutes, our truncated volatility portfolio has an annualized Sharpe ratio of 0.90 if transaction costs are neglected. But once transaction costs are accounted for, the annualized Sharpe ratio drops to -0.75, signifying the absence of economically significant pseudo-arbitrage opportunities in practice.

[4]Investigating potential arbitrage violations by focusing on options at or near the money is also reasonable from a practical point of view: as Bryzgalova et al. (2023) document, retail investors mostly trade at the money.

The same portfolio without truncation achieves high Sharpe ratios, even after accounting for transaction costs, consistent with earlier literature documenting high risk premia embedded in 0DTEs. This short volatility portfolio is therefore a good deal in the classical sense, albeit a very risky one as it derives its profits from returns with a large negative skew. It does not, however, indicate any arbitrage violations as its Sharpe ratio declines with the time to maturity of the options.

Our empirical result is a positive one from a market efficiency and asset pricing perspective, as the rich information contained in 0DTE options can be readily used for studying properties of the aggregate pricing kernel. It is also a positive one from an econometric perspective, as it justifies using estimators of volatility based on short-dated options which are significantly more efficient than their return-based counterparts, see e.g., Andersen et al. (2017), Todorov (2019) and Bandi et al. (2023b), among others.

Related Literature

Many previous works have studied occurrences of arbitrage opportunities, including Pontiff (1997) for mutual funds, Du et al. (2018), Augustin et al. (2024) and Dávila et al. (2024) for currencies and Lamont and Thaler (2003) for stocks. Similarly, retail investors' tendency to engage in risky and losing investments has been analyzed by Barber and Odean (2001) and Barberis and Huang (2008) for the stock market and by Xiong and Yu (2011) and Boyer and Vorkink (2014) for derivatives markets. The focus of this paper is on whether arbitrage violations exist in the 0DTE market, where retail investors are particularly active (Bryzgalova et al. (2023)).

More generally, this paper establishes an important relationship between arbitrage pricing theory (see e.g., Cochrane (2005), Duffie (2010) and Back (2010) for textbook treatments) and the good-deal pricing kernel bounds of Cochrane and Saa-Requejo (2000) (see also Bernardo and Ledoit (2000) and Almeida and Freire (2022) for variations of this method). While in economies with fixed time horizon, good-deal pricing kernel bounds are much tighter than no-arbitrage bounds (in particular, a violation of the former does not imply arbitrage), we show that the two bounds collapse in an economy with shrinking time horizon: the existence of good deals with Sharpe ratios bounded away from zero as the time horizon shrinks implies the existence of arbitrage. This enables the design of economic tests for integration between the equity and options markets by considering Sharpe ratios of portfolios involving 0DTEs and the underlying asset.

Our concept of pseudo-arbitrage is not a form of *risk-free arbitrage* in the classical sense but a form of *risky arbitrage*. It is related to Ledoit (1995)'s notion of δ-arbitrage, which refers to an investment with Sharpe ratio exceeding δ, and to Huberman and Stanzl (2004)'s notion of quasi-arbitrage, which refers to investments with Sharpe ratios tending to infinity. Like quasi-arbitrage, pseudo-arbitrage is an asymptotic concept and can be viewed as a sequence of δ-arbitrage opportunities where $\delta > 0$ is uniform over shrinking horizons.

Our paper is also related to previous literature discussing option pricing puzzles. Coval and Shumway (2001), Bollen and Whaley (2004) and Bondarenko (2014) document a so-called *over-priced puts puzzle*, which describes the high returns from selling out-of-the-money put options and similar strategies that derive profits from a discrepancy between historical $\mathbb{P}$-volatility and option-implied $\mathbb{Q}$-volatility. For options with a nontrivial time to expiration (e.g., monthly options), this gap is explicable through the time-variation of volatility in asset prices for which investors demand a variance risk premium (see e.g., Bakshi and Kapadia (2003), Bollerslev et al. (2009), Carr and Wu (2009) and Drechsler and Yaron (2011)). Without jumps in asset prices, this gap will disappear as the time-to-maturity of the options shrinks to zero. This, however, is

not true empirically, as there is still a considerable gap between $\mathbb{P}$- and $\mathbb{Q}$-measures of total return variance in 0DTEs (Bandi et al. (2023a), Vilkov (2023), Almeida et al. (2024)), which shows that asset price jumps and the associated jump risk premium is another important explanatory factor (see e.g., Broadie et al. (2009) Bollerslev and Todorov (2011), Du (2011), Bollerslev et al. (2015) and Seo and Wachter (2019)).

Our empirical findings are consistent with this strand of literature as we also find high Sharpe ratios in selling volatility strategies using 0DTEs. Our focus, however, is on how these findings relate to market segmentation. We show that high returns from selling volatility do not imply market segmentation. Instead, one has to consider option portfolios that focus on the small moves in asset prices—such as truncated volatility—in order to draw conclusions about market segmentation. Our finding that selling truncated volatility is *not* profitable, although selling untruncated volatility is, shows that large disaster risk (Barro (2006)) and intermediate disaster risk (Beason and Schreindorfer (2022)) can rationalize the overpriced puts puzzle within the framework of integrated equity and options markets.

Previous works have also found a *pricing kernel puzzle*, which refers to the non-monotonicity of the empirical pricing kernel when projected onto the return space and which is inconsistent with representative agent models (see e.g., Aït-Sahalia and Lo (2000), Jackwerth (2000) and Rosenberg and Engle (2002)). In fact, Constantinides et al. (2008) report[5] substantial violations of stochastic dominance bounds even for at-the-money options, which Almeida et al. (2024) confirm for 0DTEs. These violations imply that there is no marginal investor in the market who has a concave and state-independent utility of wealth, with wealth at the end of each period being an increasing function of the stock return over that period.

Our focus, by contrast, is not on stochastic dominance bounds but on the existence of an aggregate pricing kernel for the equity and options markets. Combining Constantinides et al. (2008) and Almeida et al. (2024)'s empirical results with ours, we conclude that the weaker assumption of integrated equity and options markets cannot be rejected, while the stronger assumptions about investors' preferences in these markets that underlie stochastic dominance bounds are rejected empirically.

Outline

The rest of the paper is organized as follows. We establish the no-arbitrage conditions relating the $\mathbb{P}$- and $\mathbb{Q}$-dynamics for the asset price in Section 2. We introduce the notion of pseudo-arbitrage in Section 3 and of truncated volatility in Section 4. Section 5 shows how to construct pseudo-arbitrage portfolios of options and the underlying asset that exploit differences in perceptions about volatility in the equity and options markets due to small and frequent asset moves. Section 6 makes the pseudo-arbitrage portfolios of Section 5 feasible in practice. We evaluate the various volatility-related trading strategies in Section 7 on simulated data. Section 8 contains our empirical application. Section 9 concludes. The Appendix contains the derivation of the theoretical results in the paper as well as additional technical details about the implementation of the volatility strategies.

2 Asset Price Dynamics and Market Integration

We consider an asset on a finite time interval $[0, \mathcal{T}]$, where $\mathcal{T} > 0$. Its price at time t is denoted by X_t and is defined on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a right-continuous

[5] See also Wallmeier (2021) and Constantinides et al. (2021).

filtration $(\mathcal{F}_t)_{t\in[0,\mathcal{T}]}$. Under the physical probability measure $\mathbb{P}$, we assume that the asset price evolves according to a general semimartingale process given by

$$\frac{dX_t}{X_{t-}} = \alpha_t dt + \sigma_t^{\mathbb{P}} dW_t + \int_{\mathbb{R}} (e^z - 1)\widetilde{\mu}^{\mathbb{P}}(dt, dz), \tag{2.1}$$

where W_t is a standard $\mathbb{P}$-Brownian motion, μ is an integer-valued measure on $[0,\mathcal{T}]\times\mathbb{R}$ that counts the jumps in the log-asset price, $\nu^{\mathbb{P}}(dt,dz) = \nu_t^{\mathbb{P}}(dz)dt$ is the jump compensator measuring the intensity of jumps of different sizes, and $\widetilde{\mu}^{\mathbb{P}} = \mu - \nu^{\mathbb{P}}$ is the associated martingale jump measure.

We further consider standard European-style options written on this asset. As the goal of this paper is to examine whether the equity and options markets are integrated, we are interested in the family $\mathcal{Q}$ of all risk-neutral probability measures $\mathbb{Q}$ that can jointly price the underlying asset and the associated options, which means that $\mathcal{Q}$ is the class of probability measures $\mathbb{Q}$ such that both asset and option prices are martingales under $\mathbb{Q}$ if discounted at the risk-free rate. For simplicity of exposition, and because we only work with short time horizons, we set the risk-free rate and the dividend yield of the underlying asset to zero throughout the rest of the paper. In this case, a risk-neutral measure $\mathbb{Q}\in\mathcal{Q}$ satisfies

$$X_t = \mathbb{E}_t^{\mathbb{Q}}(X_{t+T}), \quad P_{t,T}(K) = \mathbb{E}_t^{\mathbb{Q}}(K - X_{t+T})^+ \text{ and } C_{t,T}(K) = \mathbb{E}_t^{\mathbb{Q}}(X_{t+T} - K)^+, \tag{2.2}$$

for $K>0$ and $t,T\geq 0$ with $t+T\leq\mathcal{T}$, where $P_{t,T}(K)$ and $C_{t,T}(K)$ denote the prices at time t of a put and a call option, respectively, with strike K and time to maturity T.[6]

It is well-known, see e.g., Cochrane (2005), Duffie (2010) and Back (2010), that the absence of arbitrage implies the existence a positive *aggregate pricing kernel* M_t such that (2.2) can be rewritten as

$$X_t = \mathbb{E}_t^{\mathbb{P}}(D_{t,T}X_{t+T}),\ P_{t,T}(K) = \mathbb{E}_t^{\mathbb{P}}(D_{t,T}(K-X_{t+T})^+),\ C_{t,T}(K) = \mathbb{E}_t^{\mathbb{P}}(D_{t,T}(X_{t+T}-K)^+), \tag{2.3}$$

where $D_{t,T} = M_{t+T}/M_t$ denotes the *stochastic discount factor*. We denote by $\mathcal{Q}_e$ the subset of risk-neutral measures in $\mathcal{Q}$ for which (2.3) holds with a stochastic discount factor $D_{t,T}$.[7]

The risk-neutral measures in $\mathcal{Q}_e$ may differ but they must all generate the set of observable stock prices and options, that is, satisfy (2.2) and (2.3) above. If we assume that we observe the options on a continuum of strikes covering $(0,\infty)$, then due to a well-known result by Breeden and Litzenberger (1978), all risk-neutral measures $\mathbb{Q}$ generate the same $\mathcal{F}_t$-conditional distribution of X_{t+T}. This motivates

Definition 1. *If $\mathcal{Q}_e\neq\emptyset$, we say that the equity and options markets are* integrated. *If $\mathcal{Q}_e=\emptyset$, we say that they are* segmented.

Therefore, market integration means that there is an aggregate pricing kernel that can be used to price the considered asset and options simultaneously. The two markets are segmented in the absence of such a joint pricing kernel.[8]

[6] A put/call option with these parameters is the right, without the obligation, to sell/buy the asset at time $t+T$ at price K.

[7] In probability theory terms, $\mathcal{Q}_e$ is precisely the set of risk-neutral measures $\mathbb{Q}\in\mathcal{Q}$ for which $\mathbb{P}$ and $\mathbb{Q}$ are equivalent on $\mathcal{F}_{\mathcal{T}}$. In this case, M_t is the Radon–Nikodym derivative of $\mathbb{Q}|_{\mathcal{F}_t}$ with respect to $\mathbb{P}|_{\mathcal{F}_t}$. Note that both M_t and $D_{t,T}$ depend on $\mathbb{Q}$ but we suppress the dependence in the notation.

[8] Even if equity and option markets are not segmented, there can be nevertheless partial segmentation where a segment of the option market, e.g., the one for deep out-of-the-money puts, is separated from the equity market and the rest of the option market, cf. Almeida et al. (2024). Such partial segmentation of deep out-of-the-money puts, however, cannot be separately identified using equity and option data alone as there exists a valid aggregate pricing kernel that can rationalize all observed equity and option prices.

Whether the equity and options markets are integrated or segmented can be characterized in terms of the asset price dynamics under $\mathbb{P}$ and $\mathbb{Q}$. This is the content of Girsanov's theorem, which we now recall. Suppose that the asset price X_t has the following dynamics under a candidate measure $\mathbb{Q} \in \mathcal{Q}$:

$$\frac{dX_t}{X_{t-}} = \sigma_t^{\mathbb{Q}} dW_t^{\mathbb{Q}} + \int_{\mathbb{R}} (e^z - 1)\widetilde{\mu}^{\mathbb{Q}}(dt, dz), \tag{2.4}$$

where $W_t^{\mathbb{Q}}$ is $\mathbb{Q}$-Brownian motion, the jump compensator of μ under $\mathbb{Q}$ is denoted by $\nu^{\mathbb{Q}}$ and $\widetilde{\mu}^{\mathbb{Q}} = \mu - \nu^{\mathbb{Q}}$ is the martingale jump measure under $\mathbb{Q}$. Note that the drift in the dynamics of X under $\mathbb{Q}$ is absent because X is a martingale under $\mathbb{Q}$. Then, whether the candidate measure $\mathbb{Q}$ belongs to $\mathcal{Q}_e$ and hence admits a stochastic discount factor satisfying (2.3) is essentially equivalent to the conditions

$$\sigma_t^{\mathbb{Q}} = \sigma_t^{\mathbb{P}} \quad \text{and} \quad \nu^{\mathbb{Q}}(dt, dz) = Y_t(z)\nu_t^{\mathbb{P}}(dz)dt \ \text{ a.s. for } t \in [0, \mathcal{T}], \tag{2.5}$$

where $Y_t(z)$ is a positive predictable function such that

$$\int_0^{\mathcal{T}} \int_{\mathbb{R}} (|Y_t(z) - 1|^2 \wedge |Y_t(z) - 1|)\nu_t^{\mathbb{P}}(dz)dt < \infty \ \text{ a.s. for } t \in [0, \mathcal{T}]. \tag{2.6}$$

The precise statement is given in Theorem 3 in the appendix. The first of the two conditions in (2.5) is the well-known requirement that the diffusion coefficients in the dynamics of X must be the same under $\mathbb{P}$ and $\mathbb{Q}$. The second condition, together with (2.6), imposes restrictions on how the distribution of asset price jumps can change from $\mathbb{P}$ to $\mathbb{Q}$. This is captured by the nonnegative predictable function $Y_t(z)$ that reweighs the intensity of jumps of various sizes under the two probability measures. The requirement for Y mainly concerns the frequent and small jumps of X_t (of absolute value less than 1): If $\int_{|z|<1} \nu_t^{\mathbb{P}}(dz)$ is finite, then the integral condition in (2.6) is automatically fulfilled as long as Y satisfies minimal boundedness conditions. If $\int_{|z|<1} \nu_t^{\mathbb{P}}(dz)$ is infinite, however, then Y must not deviate too much from 1 for small values of $|z|$.

Both conditions in (2.5) are intuitive as payoffs that depend on the small and frequent moves in X should earn no risk premium as the time horizon of these payoffs shrinks. This is because most of the risk in these payoffs can be hedged by a dynamic position in the underlying asset at no cost, as we show in the subsequent sections.

Definition 2. *If for some $\mathbb{Q} \in \mathcal{Q}$, we have (2.5) together with (2.6), we say that* the equity and options markets agree about volatility.

This definition explains the question posed in the title of this paper: we are interested in whether there is a risk-neutral measure $\mathbb{Q}$ that prices both asset and options and that agrees with $\mathbb{P}$ about both diffusive volatility and the small jumps in the asset prices. By Girsanov's theorem, market integration is equivalent to the agreement of equity and options markets about volatility. This enables us to draw conclusions about market integration based on testing the conditions in (2.5). We also remark that both notions introduced above are local in the sense that they are about what happens in a local neighborhood of time t.

3 Generalized Sharpe Ratios and Pseudo-Arbitrage

In this section, we devise a practical criterion that not only can be used to verify from observed asset and option prices whether (2.5) is satisfied or not, but also allows us to conclude whether

a potential segmentation of the equity and options markets is *economically* significant. To this end, we start from a variant of the classical method of Hansen and Jagannathan (1991) to derive lower bounds on the stochastic discount factor. Instead of only estimating the standard deviation of the stochastic discount factor using the Sharpe ratio of portfolio returns, we consider an extension due to Snow (1991) that uses reward-to-risk ratios of risky asset positions based on pth moments.

Definition 3. *Given a portfolio with simple return $R_{t,T}$ from time t to time $t+T$ and given some exponent $p \in [1,\infty)$, we define the* (conditional) generalized Sharpe ratio (GSR) *of order p of this portfolio as*

$$GSR_{t,T}(p) = \frac{\mathbb{E}_t^{\mathbb{P}}(R_{t,T})}{\left(\mathbb{E}_t^{\mathbb{P}}\left(\left|R_{t,T} - \mathbb{E}_t^{\mathbb{P}}(R_{t,T})\right|^p\right)\right)^{1/p}}. \tag{3.1}$$

The case $p = 2$ corresponds to the Sharpe ratio, $SR_{t,T}$, that is used in classical mean–variance portfolio analysis. The case $p = 1$ corresponds to the mean absolute deviation ratio of Konno and Yamazaki (1991). By Minkowski's inequality, we trivially have

$$GLR_{t,T} \geq GSR_{t,T}(1) \geq GSR_{t,T}(p_1) \geq SR_{t,T} \geq GSR_{t,T}(p_2), \tag{3.2}$$

for any $1 \leq p_1 \leq 2 \leq p_2 < \infty$, where $GLR_{t,T}$ denotes the conditional gain-loss ratio of Bernardo and Ledoit (2000) associated with the return $R_{t,T}$. The following result, obtained by Hansen and Jagannathan (1991) for $p = 2$ and by Snow (1991) for general p, gives a lower bound on the variability of the pricing kernel in terms of generalized Sharpe ratios. We remind the reader that for simplicity, the risk-free rate in our analysis is set to zero.

Proposition 1. *Consider a portfolio with simple return $R_{t,T}$ from time t to time $t+T$ and assume that this portfolio can be priced by some risk-neutral measure $\mathbb{Q} \in \mathcal{Q}_e$, that is,*

$$\mathbb{E}_t^{\mathbb{Q}}(R_{t,T}) = 0. \tag{3.3}$$

For any $p \in (1,\infty)$, we have

$$\left(\mathbb{E}_t^{\mathbb{P}}\left(|D_{t,T} - 1|^q\right)\right)^{1/q} \geq |GSR_{t,T}(p)|, \tag{3.4}$$

where $D_{t,T}$ is the stochastic discount factor associated with $\mathbb{Q}$ and $q = \frac{p}{p-1}$ is the Hölder conjugate of p.

We reproduce a short proof in Appendix A.2. The pricing equation (3.3) automatically holds as a consequence of (2.2) if the payoff of the portfolio is linear in the considered asset and option prices.

If $p = 2$, inequality (3.4) reduces to the classical bound of Hansen and Jagannathan (1991): the conditional standard deviation of the stochastic discount factor $D_{t,T}$ exceeds the Sharpe ratio attained by any portfolio that can be priced by $\mathbb{Q}$. For general $p \in (1,\infty)$, inequality (3.4) asserts that the qth-order central moment of the stochastic discount factor is bounded from below by the GSR of order p of any such portfolio, where p and q are related to each other through the equation $\frac{1}{p} + \frac{1}{q} = 1$. In particular, generalized Sharpe ratios with $p \in (1,2)$ provide bounds on the moments of the stochastic discount factor of higher order than variance.

The generalized Hansen–Jagannathan bound of Proposition 1 means that portfolios with high GSR imply highly varying pricing kernels. Such portfolios are good deals in the sense of Cochrane and Saa-Requejo (2000) but, if T is fixed, do not imply arbitrage violations. As

we show momentarily, the situation is different if the time horizon T shrinks: if a good deal can be realized over an asymptotically shrinking time interval, that is, if we can find portfolios whose (generalized) Sharpe ratios exceed a given positive threshold over shorter and shorter time horizons, then this implies arbitrage violations.

Definition 4. *Given $p \in (1,\infty)$, we say that pseudo-arbitrage of order p exists if for some $t \in [0,\mathcal{T})$ there is a sequence of portfolios with investment horizon T whose returns $R_{t,T}$ are priced according to (3.3) by a single risk-neutral measure $\mathbb{Q} \in \mathcal{Q}$ and whose generalized Sharpe ratios satisfy*

$$\liminf_{T \downarrow 0} GSR_{t,T}(p) > 0 \quad a.s. \tag{3.5}$$

If $p = 2$, the notion of pseudo-arbitrage is related to the concept of quasi-arbitrage by Huberman and Stanzl (2004). The latter refers to a sequence of portfolios whose Sharpe ratios diverge to infinity. Clearly, in our context, if there is quasi-arbitrage of order p in the sense that $GSR_{t,T}(p) \xrightarrow{\mathbb{P}} \infty$ as $T \downarrow 0$, we also have pseudo-arbitrage of order p. But pseudo-arbitrage is strictly weaker because it already occurs if $GSR_{t,T}(p)$ converges in probability to a strictly positive random variable, which is the case for some of the pseudo-arbitrage portfolios we analyze in Section 4. The next theorem is the main result of this section and explains why pseudo-arbitrage implies arbitrage violations.

Theorem 1. *If for a given risk-neutral measure $\mathbb{Q} \in \mathcal{Q}$, there is pseudo-arbitrage of order $p \in (1,\infty)$ at time $t \in [0,\mathcal{T})$ with a sequence of portfolios whose payoffs can all be priced by $\mathbb{Q}$, then either $\mathbb{Q} \notin \mathcal{Q}_e$ or $\mathbb{E}_t^{\mathbb{P}}(|M_s|^q) = \infty$ for all $s \in (t,\mathcal{T}]$, where $\frac{1}{p} + \frac{1}{q} = 1$ and M is the aggregate pricing kernel of $\mathbb{Q}$.*

Pseudo-arbitrage therefore implies that either there is no aggregate pricing kernel that can price the considered asset and options simultaneously, and the equity and options markets are segmented, or the qth moment of the aggregate pricing kernel must be infinite, implying highly unstable and oscillatory behavior. This gives us a powerful economic tool to test theoretical arbitrage conditions concerning market integration using practically feasible pseudo-arbitrage strategies.

It is crucial to Theorem 1 that we consider short horizons: In a setting where the time horizon of the considered portfolios does not shrink, there can be a substantial wedge between arbitrage-free and good-deal pricing bounds, as Cochrane and Saa-Requejo (2000) and Bernardo and Ledoit (2000) demonstrate. This wedge, however, disappears as the time horizon shrinks, and arbitrage-free and good-deal pricing bounds converge to each other in the limit as $T \downarrow 0$.

As Figure 1 shows, absence of pseudo-arbitrage, market integration and agreement about volatility in equity and options markets are essentially equivalent. The equivalence between market integration and agreement of asset and option prices about volatility follows from Girsanov's theorem (Theorem 3 in Appendix A). The main theoretical contribution of the paper is to establish that these two notions, which are central to the financial economics of equity and options markets, are equivalent to the absence of pseudo-arbitrage opportunities, an economically interpretable criterion that can be feasibly implemented in practice. One direction is shown in Theorem 1: market integration rules out pseudo-arbitrage opportunities as long as the aggregate pricing kernel has a certain number of finite moments. The other direction is the content of Theorem 2 below: a disagreement between $\mathbb{P}$ and $\mathbb{Q}$ about volatility leads to pseudo-arbitrage opportunities, so an absence of pseudo-arbitrage implies that $\mathbb{P}$ and $\mathbb{Q}$ must agree about volatility.

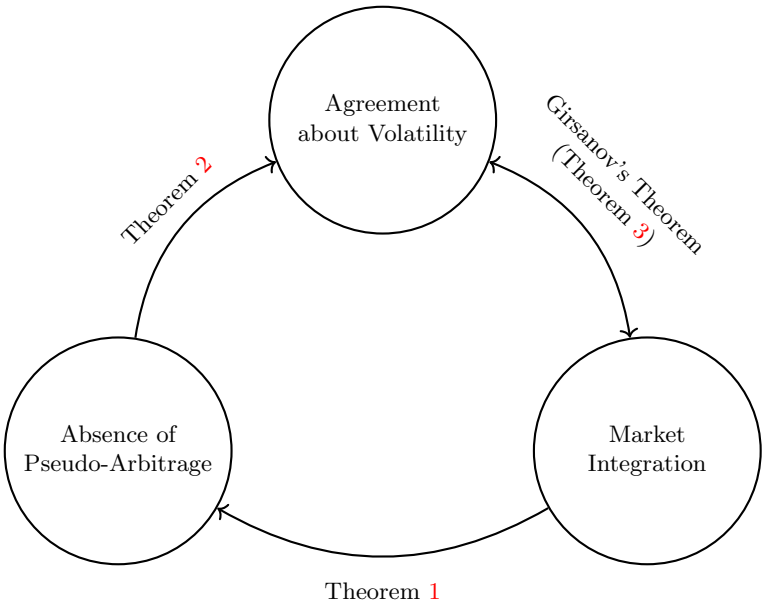


Figure 1: **Relationship between Market Integration, Absence of Pseudo-Arbitrage and Agreement about Volatility in Equity and Options Markets.**

4 Truncated Volatility and $\mathbb{P}$-$\mathbb{Q}$ Equivalence

The goal of this section is to construct a pseudo-arbitrage strategy (see Definition 4) if $\mathbb{P}$ and $\mathbb{Q}$ disagree about volatility (i.e., if condition (2.5) is violated) in a local neighborhood of time t. As already discussed in Section 2, this condition concerns the small and frequent moves in the asset price, which is why we aim for payoffs that capture exactly those. A natural candidate would be the log-price payoff which is behind the construction of the popular VIX index. That is, to consider $\log(X_{t+T}) - \log(X_t)$, for small T. An application of Itô's formula leads to

$$-\frac{2}{T}\mathbb{E}_t^{\mathbb{Q}}(\log(X_{t+T}) - \log(X_t)) \approx (\sigma_t^{\mathbb{Q}})^2 + 2\int_{\mathbb{R}}(e^z - 1 - z)\nu_t^{\mathbb{Q}}(dz), \quad \text{as } T \downarrow 0. \tag{4.1}$$

The problem for our purposes is that such a payoff depends on all jumps and not only the small ones. Indeed, note that the integral above is over the entire real line. Hence, due to the risk premium for big jumps, $\mathbb{E}_t^{\mathbb{Q}}(\log(X_{t+T}) - \log(X_t))$ and $\mathbb{E}_t^{\mathbb{P}}(\log(X_{t+T}) - \log(X_t))$ can differ even in the absence of (pseudo-)arbitrage and even if T is very small. We confirm that this is the case empirically later on. Therefore, we consider a different payoff which depends only on the small and frequent moves in the underlying asset price.

Definition 5. *We call* truncated volatility *the payoff* $f_T(X_{t+T}; X_t)$*, where*

$$f_T(x; x_0) = \phi\left(\eta_T \frac{\log(x) - \log(x_0)}{\sqrt{T}}\right)(\log(x) - \log(x_0))^2, \tag{4.2}$$

and ϕ *is a truncation function that decays exponentially fast for values of its argument away from zero and the truncation parameter satisfies* $\eta_T \downarrow 0$ *as* $T \downarrow 0$*.*

The truncation parameter η_T will be chosen to converge to 0 slowly, e.g., $\eta_T \sim 1/\sqrt{\log(1/T)}$. In this case, the conditional expectation $TV_{t,T}^{\mathbb{S}}(\eta_T) = \mathbb{E}_t^{\mathbb{S}}(f_T(X_{t+T}; X_t))$ of truncated volatility

under $\mathbb{S} \in \{\mathbb{P}, \mathbb{Q}\}$ satisfies the following expansion as $T \downarrow 0$:

$$\frac{1}{T} TV^{\mathbb{S}}_{t,T}(\eta_T) \approx (\sigma^{\mathbb{S}}_t)^2 + \int_{\mathbb{R}} \phi(\eta_T z/\sqrt{T}) z^2 \nu^{\mathbb{S}}_t(dz). \tag{4.3}$$

Because ϕ decreases exponentially, for any $a > 0$, the integral above restricted to $\{|z| \geq a\}$ converges to zero exponentially fast. Thus, unlike the case of the log-price payoff, which depends on the big jumps, the truncated volatility payoff eliminates the effect of the big jumps and they become negligible as $T \downarrow 0$. The condition in (2.5), therefore, implies

$$TV^{\mathbb{P}}_{t,T}(\eta_T) \approx TV^{\mathbb{Q}}_{t,T}(\eta_T) \quad \text{as } T \downarrow 0. \tag{4.4}$$

Conversely, as we show below, if $\sigma^{\mathbb{P}}_t$ and $\sigma^{\mathbb{Q}}_t$ are different and/or $\nu^{\mathbb{P}}_t$ and $\nu^{\mathbb{Q}}_t$ differ too much around zero, then there is η_T such that the above approximate equality does not hold. In this regard, (4.3) captures the volatility due to small and frequent moves in the asset price.

We can assess the quality of the approximation in (4.4) in a numerical example using the parametric model from our Monte Carlo experiment that we present in Section 7. In Figure 2, we plot the ratio $TV^{\mathbb{Q}}_{0,T}(\eta_T)/TV^{\mathbb{P}}_{0,T}(\eta_T)$ for different levels of the truncation parameter in the case of $\mathbb{P}$-$\mathbb{Q}$ equivalence. When we do not truncate ($\eta_T = 0$), then there is a significant gap between the two volatility measures, which is due to the compensation for the big negative jumps in the underlying asset price. As we increase the level of truncation, the gap between $TV^{\mathbb{P}}_{0,T}(\eta_T)$ and $TV^{\mathbb{Q}}_{0,T}(\eta_T)$ shrinks in relative terms. This is because truncated volatility depends less on the big jumps in the asset prices. Indeed, for the highest level of truncation considered here, $TV^{\mathbb{Q}}_{0,T}(\eta_T)$ is only around 3% higher than $TV^{\mathbb{P}}_{0,T}(\eta_T)$ for $T = 30$ minutes, which is a negligible difference. We also note that for every level of truncation $\eta_T > 0$, the upward bias in $TV^{\mathbb{Q}}_{0,T}(\eta_T)$ due to the compensation for big jumps shrinks in relative terms as T shrinks. For example, for the highest considered level of truncation, the ratio $TV^{\mathbb{Q}}_{0,T}(\eta_T)/TV^{\mathbb{P}}_{0,T}(\eta_T)$ shrinks from around 1.05 for T equal to 120 minutes to 1.03 for T equal to 30 minutes. We will consider the same levels of truncation as the ones used in Figure 2 in our applications.

5 Volatility Arbitrage Portfolios

We now show how to construct a pseudo-arbitrage strategy based on truncated volatility using short-dated options and the underlying asset if condition (2.5) is violated

5.1 Construction of Truncated Volatility Option Portfolios

According to the option spanning results of Bakshi and Madan (2000) and Carr and Madan (2001), payoffs that are nonlinear in the terminal price (such as truncated volatility) can be obtained via portfolios of options over a continuum of strikes and with the same expiration date. More precisely, using Taylor expansion with remainder for the function $f_T(x; x_0)$ in (4.2), we obtain

$$\begin{aligned} f_T(X_{t+T}; X_t) &= \phi(\eta_T(\log(X_{t+T}) - \log(X_t))/\sqrt{T})(\log(X_{t+T}) - \log(X_t))^2 \\ &= \int_0^{X_t} f''_T(K; X_t)(K - X_{t+T})^+ dK + \int_{X_t}^{\infty} f''_T(K; X_t)(X_{t+T} - K)^+ dK. \end{aligned} \tag{5.1}$$

Note that $(K - X_{t+T})^+$ and $(X_{t+T} - K)^+$ are the payoffs at time $t + T$ of European-style put and call options, respectively, written on the asset at time t with strike K and time to maturity

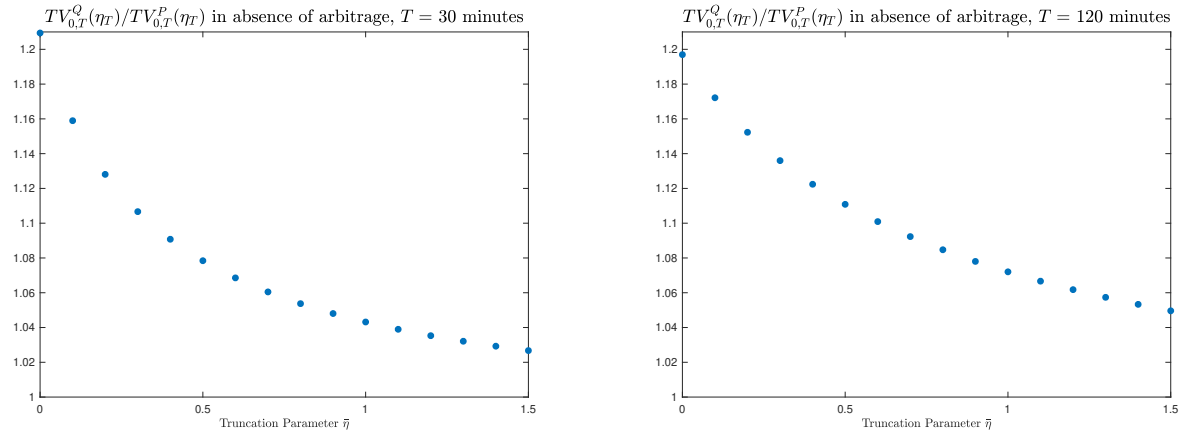


Figure 2: **Risk-Neutral versus True Conditional Expected Truncated Volatility in the Absence of Arbitrage**. The parameter settings for the no-arbitrage scenario are given in the Monte Carlo study. The conditional truncated volatilities are computed via simulation. The truncation function is $\phi(x) = e^{-x^2}$ and the truncation parameter is set to $\eta_T = \frac{\sqrt{\bar{\eta}}}{\sqrt{\log(1/T)}} \times \frac{1}{\sigma_0^{\mathbb{P}}}$.

T. This means that the nonlinear payoff $f_T(X_{t+T}; X_t)$ can be replicated by a static portfolio of options created at time t. By (2.2), the cost of this portfolio at time t is given by

$$\mathcal{C}_{t,T}(\eta_T) \equiv TV^{\mathbb{Q}}_{t,T}(\eta_T) = \int_0^\infty f''_T(K; X_t) O_{t,T}(K) dK, \tag{5.2}$$

where $O_{t,T}(K) = \min\{P_{t,T}(K), C_{t,T}(K)\}$ denotes the price of the out-of-the-money option for strike K.

We can compare the weights assigned to the options with different strikes in the above portfolio that generates the truncated volatility payoff for different levels of truncation. The case $\eta_T = 0$ (no truncation) corresponds to the second moment of the log-return that has been studied by Bakshi et al. (2003) (and which is very close to the VIX index discussed above). Higher truncation means shifting more weight to the smaller moves. As we can see from the left panel of Figure 3, this translates into putting higher weights in (5.2) on options with strikes in the vicinity of the current stock price. Interestingly, the truncated volatility option portfolio assigns negative weight to options which are slightly away from the money. Intuitively, this removes an upward bias of at-the-money options due to the larger asset price jumps. From a practical point of view, this also implies somewhat higher transaction costs for replicating the truncated volatility payoff ($\eta_T > 0$) versus replicating the total volatility payoff ($\eta_T = 0$).

As evident from the figure, the contribution of deep out-of-the-money puts and calls to the value of the option-based truncated volatility portfolio is negligible. In fact, one can show that due to the truncation in the payoff $f_T(X_{t+T}, X_t)$, one can consider only integration over a finite range of strikes covering the current stock price when computing the integral on the left-hand side of (5.2), without any change to the results that will follow. This is intuitive as deep out-of-the-money puts and calls carry less information about the small and frequent moves in the asset price as T shrinks. As Bryzgalova et al. (2023) report, retail investors are particularly active around the money. Therefore, focusing on options close to the money also makes sense for our intended application to 0DTEs.

The right panel of Figure 3 displays the payoff of truncated volatility portfolio for different

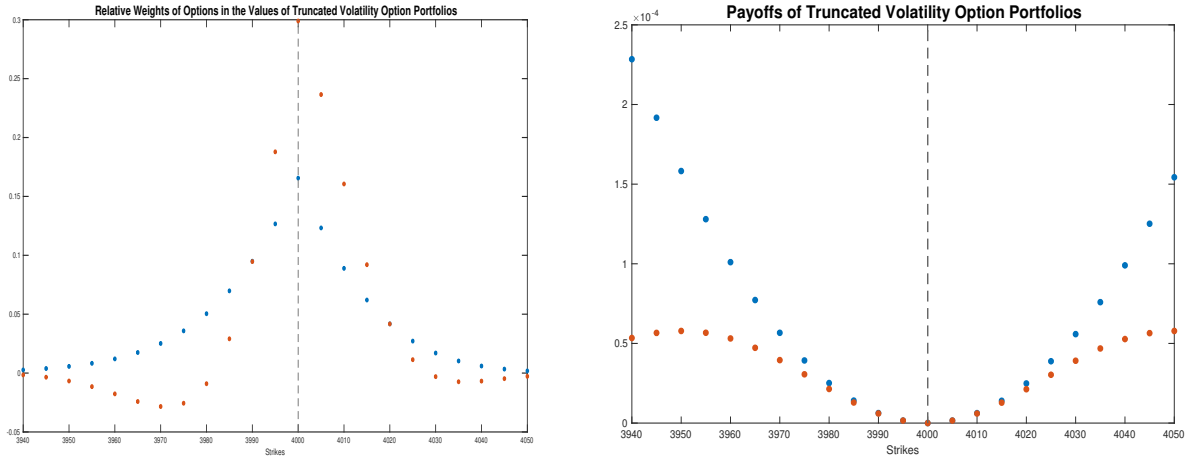


Figure 3: **Weights and Payoffs of Truncated Volatility Option Portfolios**. Each point on the left figure represents the fraction of wealth in the truncated volatility option portfolio invested in the option of the given strike. Options are generated from the no-arbitrage model for the risk-neutral dynamics given in the Monte Carlo study. The starting value of the underlying price and the spot volatility are 4,000 and 0.02, respectively. The time to maturity is 2 hours. The truncation function is $\phi(x) = e^{-x^2}$ and the truncation parameter is set as in (7.5) with $\overline{\eta} = 0$ (blue dots) and $\overline{\eta} = 1.5$ (red dots).

levels of truncation. When we truncate ($\eta_T > 0$), the payoff looks like a smooth version of an inverted butterfly spread. On the other hand, when we do not truncate ($\eta_T = 0$), the payoff looks like a smooth version of a straddle. Analyzing theoretically the properties of butterfly spreads and straddles is difficult due to the presence of jumps in the asset price, and this is the reason why we consider their smooth counterparts, i.e., the truncated volatility portfolios.

5.2 Hedging of Truncated Volatility Option Portfolios

We can now analyze the properties of the option portfolio replicating truncated volatility. Its simple return is given by

$$R_{t,T} = \frac{f_T(X_{t+T}; X_t)}{\mathcal{C}_{t,T}(\eta_T)} - 1 = \frac{f_T(X_{t+T}; X_t)}{\mathbb{E}_t^{\mathbb{Q}}(f_T(X_{t+T}; X_t))} - 1, \tag{5.3}$$

with conditional expectation

$$\mathbb{E}_t^{\mathbb{P}}(R_{t,T}) = \frac{\mathbb{E}_t^{\mathbb{P}}(f_T(X_{t+T}; X_t))}{\mathbb{E}_t^{\mathbb{Q}}(f_T(X_{t+T}; X_t))} - 1. \tag{5.4}$$

Given the discussion in the previous section, condition (2.5) implies that

$$\mathbb{E}_t^{\mathbb{P}}(R_{t,T}) \approx 0 \quad \text{as} \quad T \downarrow 0, \tag{5.5}$$

that is, there should be no reward for the risk of this portfolio asymptotically as $T \downarrow 0$. A violation of (2.5), on the other hand, means that either a long or a short position in the option portfolio should earn a non-vanishing positive return. To check if this is a pseudo-arbitrage

opportunity as $T \downarrow 0$, we need to analyze the generalized Sharpe ratio of the strategy, which we denote by

$$GSR_{t,T}(p) = \frac{\mathbb{E}_t^{\mathbb{P}}(R_{t,T})}{\left(\mathbb{E}_t^{\mathbb{P}}\left(\left|R_{t,T} - \mathbb{E}_t^{\mathbb{P}}(R_{t,T})\right|^p\right)\right)^{1/p}}, \qquad SR_{t,T} = GSR_{t,T}(2). \tag{5.6}$$

Even though the truncated volatility portfolio is already a pseudo-arbitrage if (2.5) is violated, one can achieve even higher generalized Sharpe ratios by eliminating the directional risk in the strategy. This can be done by adding a dynamic portfolio position in the underlying asset, where, at each point in time $s \in [t, t+T]$, the exposure to the position in the underlying asset is $-f_T'(X_{s-}; X_t)$. Altogether, the cumulative realized gains of the combined positions in the options and the underlying asset from t to $t+T$ are given by

$$\mathcal{RG}_{t,T} = f_T(X_{t+T}; X_t) - \int_t^{t+T} f_T'(X_{s-}; X_t) dX_s, \tag{5.7}$$

while the initial cost of the position is still $\mathcal{C}_{t,T}(\eta_T)$. Thus, the cumulative rate of return over the interval $[t, t+T]$ becomes

$$\mathcal{R}_{t,T} = \frac{\mathcal{RG}_{t,T}}{\mathcal{C}_{t,T}(\eta_T)} - 1. \tag{5.8}$$

We have that

$$\mathbb{E}_t^{\mathbb{P}}(\mathcal{R}_{t,T}) \approx \mathbb{E}_t^{\mathbb{P}}(R_{t,T}), \tag{5.9}$$

with the reason for the approximation sign above being the equity risk premium in the underlying asset over the interval $[t, t+T]$, which shrinks as $T \downarrow 0$.[9]

By contrast, adding the dynamic position in the underlying asset significantly reduces the risk in the option portfolio position and our calculations show that

$$\mathbb{E}_t^{\mathbb{P}}\left(\left|\mathcal{R}_{t,T} - \mathbb{E}_t^{\mathbb{P}}\left(\mathcal{R}_{t,T}\right)\right|^p\right) \ll \mathbb{E}_t^{\mathbb{P}}\left(\left|R_{t,T} - \mathbb{E}_t^{\mathbb{P}}\left(R_{t,T}\right)\right|^p\right), \tag{5.10}$$

for any $p \in [1, 2)$. For this reason, we refer to the combined option portfolio position with the dynamic position in the underlying asset described above as a *hedged truncated volatility portfolio*, to distinguish it from the portfolio with return (5.3) which we call a *naked truncated volatility portfolio*. We denote the (conditional) generalized Sharpe ratio of the hedged truncated volatility portfolio by

$$\mathcal{GSR}_{t,T}(p) = \frac{\mathbb{E}_t^{\mathbb{P}}\left(\mathcal{R}_{t,T}\right)}{\left(\mathbb{E}_t^{\mathbb{P}}\left(\left|\mathcal{R}_{t,T} - \mathbb{E}_t^{\mathbb{P}}\left(\mathcal{R}_{t,T}\right)\right|^p\right)\right)^{1/p}}, \qquad \mathcal{SR}_{t,T} = \mathcal{GSR}_{t,T}(2). \tag{5.11}$$

Note that the above strategies with returns $R_{t,T}$ and $\mathcal{R}_{t,T}$ are long positions in truncated volatility. The corresponding short strategies are obtained by flipping the signs of the positions in the long strategies.

5.3 Local Arbitrage When Diffusive Volatility is Different

The next result is the second main theorem of this paper and formally shows how pseudo-arbitrage arises if diffusive spot volatility is different under the physical and the risk-neutral probability measure. In the theorem, we let $m_p = (\mathbb{E}(|Z^2 - 1|^p))^{1/p}$ where Z is a standard normal variable.

[9] The conditional equity risk premium is given by $\mathbb{E}_t^{\mathbb{P}}(\int_t^{t+T} \alpha_s ds) = O_p(T)$.

Theorem 2. *Suppose that $\mathbb{Q} \in \mathcal{Q}$ is such that $\sigma_t^{\mathbb{P}} \neq \sigma_t^{\mathbb{Q}}$ for some $t \in [0, \mathcal{T})$. Further suppose that we have a continuous record of the asset price X over the interval $[t, t+T]$ as well as the prices $O_{t,T}(K)$ at time t of out-of-the-money options expiring at time $t+T$ over a continuum of strikes $K \in (0, \infty)$. Under Assumptions 1 and 2 detailed in Appendix A.3 and if $\eta_T \sim \eta/\sqrt{\log T^{-1}}$ for some $\eta > 0$, the (generalized) Sharpe ratios of the naked and hedged truncated volatility portfolios have the following asymptotic behavior as $T \downarrow 0$:*

$$GSR_{t,T}(p) \xrightarrow{\mathbb{P}} \frac{(\sigma_t^{\mathbb{P}})^2 - (\sigma_t^{\mathbb{Q}})^2}{2m_p(\sigma_t^{\mathbb{P}})^2}, \qquad |\mathcal{GSR}_{t,T}(p)| \xrightarrow{\mathbb{P}} \infty \tag{5.12}$$

for all $p \in [1, 2)$ and

$$SR_{t,T} \xrightarrow{\mathbb{P}} \frac{(\sigma_t^{\mathbb{P}})^2 - (\sigma_t^{\mathbb{Q}})^2}{2\sqrt{2}(\sigma_t^{\mathbb{P}})^2}, \qquad \mathcal{SR}_{t,T} \xrightarrow{\mathbb{P}} \frac{(\sigma_t^{\mathbb{P}})^2 - (\sigma_t^{\mathbb{Q}})^2}{\sigma_t^{\mathbb{P}}(\int_{\mathbb{R}} z^2 \nu_t^{\mathbb{P}}(dz))^{1/2}}. \tag{5.13}$$

Thus, if $\sigma_t^{\mathbb{P}} \neq \sigma_t^{\mathbb{Q}}$, both the hedged and the naked truncated volatility portfolio constitute pseudo-arbitrage of order p for any $p \in (1, 2]$.

The (generalized) Sharpe ratios of the naked truncated volatility portfolio never explode as $T \downarrow 0$. This is due to the directional risk in the strategy. For the truncated volatility portfolio, where this risk is removed, the generalized Sharpe ratio tends to infinity for all values of $p \in [1, 2)$, showing a significant reduction of L_p-risk through hedging. We note, however, that the Sharpe ratio (corresponding to $p = 2$) does not explode as T shrinks, even if we hedge. This is due to the jumps in the asset price, which are eliminated from the option portfolio but still impact the dynamic position in the underlying asset. As p increases, the jump contribution becomes more important and this is the reason for the different asymptotic behavior of $\mathcal{GSR}_{t,T}(p)$ for $p < 2$ and $p = 2$.

Nevertheless, as soon as $\sigma_t^{\mathbb{P}} \neq \sigma_t^{\mathbb{Q}}$, the (generalized) Sharpe ratios of both the naked and hedged truncated volatility portfolios remain bounded away from zero as $T \downarrow 0$, which is enough to constitute pseudo-arbitrage opportunities. This makes intuitive sense as other portfolios typically have a shrinking Sharpe ratio over such short time horizons. A long (or short) static position in the underlying asset is one such example. But also the volatility positions without truncation have shrinking Sharpe ratios as the next proposition shows.

Proposition 2. *In the setting of Theorem 2, we have*

$$\frac{SR_{t,T}^{\eta=0}}{\sqrt{T}} \xrightarrow{\mathbb{P}} \frac{(\sigma_t^{\mathbb{P}})^2 - (\sigma_t^{\mathbb{Q}})^2 + \int_{\mathbb{R}} z^2(\nu_t^{\mathbb{P}}(dz) - \nu_t^{\mathbb{Q}}(dz))}{(\int_{\mathbb{R}} z^4 \nu_t^{\mathbb{P}}(dz))^{1/2}}, \quad \text{as } T \to 0, \tag{5.14}$$

and

$$\frac{\mathcal{SR}_{t,T}^{\eta=0}}{\sqrt{T}} \xrightarrow{\mathbb{P}} \frac{(\sigma_t^{\mathbb{P}})^2 - (\sigma_t^{\mathbb{Q}})^2 + \int_{\mathbb{R}} z^2(\nu_t^{\mathbb{P}}(dz) - \nu_t^{\mathbb{Q}}(dz))}{(\int_{\mathbb{R}} z^4 \nu_t^{\mathbb{P}}(dz))^{1/2}}, \quad \text{as } T \to 0. \tag{5.15}$$

In words, the Sharpe ratios of the (hedged or naked) volatility strategies without truncation shrink at rate $\sqrt{T}$. In Figure 4, we plot the asymptotic approximations for the Sharpe ratios of the hedged long or short positions in truncated volatility, with and without truncation, when $\sigma_t^{\mathbb{P}} \neq \sigma_t^{\mathbb{Q}}$. We use the parametric model in the Monte Carlo to generate the figure. This model matches key features of real data and hence the numbers in Figure 4 are representative of the possible Sharpe ratios that can be achieved if $\sigma_t^{\mathbb{P}} \neq \sigma_t^{\mathbb{Q}}$ (without taking into account trading costs). As seen from the figure, even relatively small differences in $\sigma_t^{\mathbb{P}}$ and $\sigma_t^{\mathbb{Q}}$ can generate very

high Sharpe ratios using the hedged truncated volatility strategy with $\eta > 0$. The Sharpe ratio of the strategy without truncation ($\eta = 0$) is also large when T is relatively high but it quickly drops when T becomes small. The reason for this is that the payoff of an untruncated volatility position is significantly exposed to big jumps in the asset price.

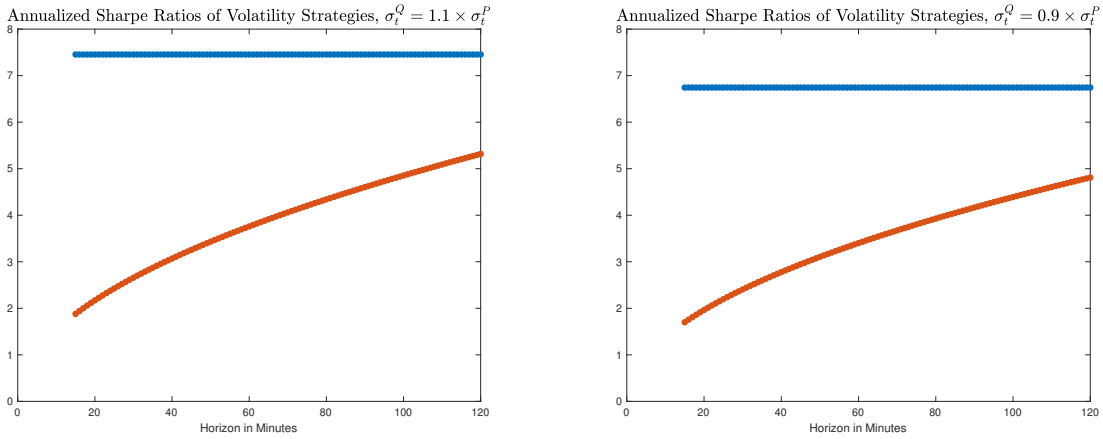


Figure 4: **Sharpe Ratios of Volatility Trading Strategies for Different Horizons**. The parameter settings for the different scenarios are given in the Monte Carlo study. The Sharpe ratios of the strategy with truncation ($\eta > 0$) and without truncation ($\eta = 0$) are computed using the asymptotic limits in (5.13) and (5.15). Blue line corresponds to $\eta > 0$ and red line to $\eta = 0$. The per-period Sharpe ratios are annualized by multiplying them by $\sqrt{252}$.

Pseudo-arbitrage can also arise if the physical and risk-neutral probability measures differ in the distribution of small asset price jumps. Note that this can occur only if assets have jumps of infinite variation. As this is rarely the case for most models considered in prior work, we defer the details to Appendix C. On the other hand, if $\mathbb{Q} \in \mathcal{Q}_e$, which implies that $\sigma_t^{\mathbb{P}} = \sigma_t^{\mathbb{Q}}$ and that $\nu_t^{\mathbb{P}}$ and $\nu_t^{\mathbb{Q}}$ agree about the small jumps for all $t \in [0, \mathcal{T}]$, then the truncated volatility strategies obviously do not lead to pseudo-arbitrage. This is formally established in the next proposition.

Proposition 3. *Suppose that the conditions of Theorem 2 are satisfied except that now $\mathbb{Q} \in \mathcal{Q}_e$ (which implies that $\sigma_t^{\mathbb{P}} = \sigma_t^{\mathbb{Q}}$). Then*

$$\mathcal{SR}_{t,T} \xrightarrow{\mathbb{P}} 0, \qquad SR_{t,T} \xrightarrow{\mathbb{P}} 0, \tag{5.16}$$

and neither the hedged nor the unhedged truncated volatility portfolio constitutes pseudo-arbitrage.

6 Implementing Volatility Arbitrage Strategies

In the previous section, we showed how to take advantage of any disagreement of the equity and option markets about volatility due to frequent and small asset moves. In reality, the underlying asset is traded discretely (even though this can happen at a very high frequency), options are observed on a discrete strike grid, and there are nontrivial bid–ask spreads when trading options and the underlying asset. All this has the potential to reduce the profitability of pseudo-arbitrage portfolios. In addition, the investor needs to use past information to decide if there are pseudo-arbitrage opportunities. In this section, we take care of all of these issues to propose volatility arbitrage trading strategies that are feasible in practice and which allow us to decide if such pseudo-arbitrage opportunities exist empirically.

6.1 Transaction Costs

We start with the option portfolio that replicates truncated volatility. We assume that at time t, we have access to options on the asset expiring at time $t+T$ on the discrete strike grid

$$K_{t,1} < \cdots < K_{t,N_t}, \tag{6.1}$$

for some positive integer N_t. By (5.1), our estimate of $TV^{\mathbb{Q}}_{t,T}(\eta_T)$ is then given by

$$\widehat{TV}^{\mathbb{Q}}_{t,T}(\eta_T) = \sum_{j=2}^{N_t} f''_T(K_{t,j-1}; X_t) O_{t,T}(K_{t,j-1})(K_{t,j} - K_{t,j-1}). \tag{6.2}$$

In the above, we assume that we have access to the true option price. In reality, investors face nontrivial bid–ask spreads when trading options. We can then form two estimates of $TV^{\mathbb{Q}}_{t,T}(\eta_T)$: one that corresponds to a long position in truncated volatility, denoted by $\widehat{TV}^{\mathbb{Q}}_{t,T}(\eta_T)^a$, and another one that corresponds to a short position in truncated volatility, denoted by $\widehat{TV}^{\mathbb{Q}}_{t,T}(\eta_T)^b$. We have

$$\widehat{TV}^{\mathbb{Q}}_{t,T}(\eta_T)^b < \widehat{TV}^{\mathbb{Q}}_{t,T}(\eta_T)^a,$$

with the gap determined by the bid–ask spreads in options markets.

For our hedged volatility strategies, we need to enter into a dynamic position in the underlying asset. Instead of trading continuously, investors trade at intervals of length $0 < \Delta_n < T$ in practice. This leads to the following feasible counterpart of $\mathcal{RG}_{t,T}$ from (5.7):

$$\begin{aligned} \widehat{\mathcal{RG}}_{t,T} = &\sum_{j=2}^{N_t} f''_T(K_{t,j-1}; X_t) \widehat{O}_{t+T}(K_{t,j-1})(K_{t,j} - K_{t,j-1}) \\ &- \sum_{j=1}^{k_n} f'_T(X_{t+(j-1)\Delta_n}; X_t)(X_{t+j\Delta_n} - X_{t+(j-1)\Delta_n}), \end{aligned} \tag{6.3}$$

where $k_n \sim T/\Delta_n$ and $\widehat{O}_{t+T}(K)$ is the payoff of the option at expiration. The first part of the realized gains is due to the payoffs of the static position in the portfolio of options. The second part is due to the cumulative gains from trading the underlying asset. This part can be represented equivalently as

$$-f'_T(X_{t+k_n\Delta_n}; X_t) X_{t+k_n\Delta_n} + \sum_{j=1}^{k_n} (f'_T(X_{t+j\Delta_n}; X_t) - f'_T(X_{t+(j-1)\Delta_n}; X_t)) X_{t+j\Delta_n}. \tag{6.4}$$

The first component in the above expression is the value of the position in the underlying asset at the terminal date $t+T$, while the second one contains the gains and losses from rebalancing the position in the underlying asset over the time window $[t, t+T]$. When trading the underlying asset, investors have to deal with bid–ask spreads. We account for that by using $X^b_{t+j\Delta_n}$ if the position at time $t+j\Delta_n$ is short and $X^a_{t+j\Delta_n}$ if the position at time $t+j\Delta_n$ is long, where $X^b_{t+j\Delta_n}$ and $X^a_{t+j\Delta_n}$ denote the best bid and best ask, respectively. For the amount held in the underlying asset at each point in time, we use the mid-quote $0.5 \times X^b_{t+j\Delta_n} + 0.5 \times X^a_{t+j\Delta_n}$.

We can derive an asymptotic approximation for the size of the trading costs due to the dynamic trading part of the strategy.

Proposition 4. *Let* $\widehat{DTC}_{t,T}$ *denote the trading costs due to the dynamic trading part of the above strategy, that is,*

$$\widehat{DTC}_{t,T} = |f_T'(X_{t+k_n\Delta_n}; X_t)|\epsilon_{t+k_n\Delta_n} + \sum_{j=1}^{k_n} |f_T'(X_{t+j\Delta_n}; X_t) - f_T'(X_{t+(j-1)\Delta_n}; X_t)|\epsilon_{t+j\Delta_n}, \quad (6.5)$$

where $\epsilon_{t+j\Delta_n} = X^a_{t+j\Delta_n} - X_t$ *if the position at time* $t + j\Delta_n$ *is short and* $\epsilon_{t+j\Delta_n} = X^b_{t+j\Delta_n} - X_t$ *if the position at time* $t + j\Delta_n$ *is long. If the true underlying asset price is the mid-quote and positive and the relative bid–ask spread is a constant denoted with* rba*, that is,*

$$X_s = \frac{1}{2}(X^a_s + X^b_s) \quad \textit{and} \quad \text{rba} = \frac{X^a_s - X^b_s}{X_s} \quad \textit{for all } s \in [t, t+T],$$

then

$$\frac{\sqrt{\Delta_n}}{T}\widehat{DTC}_{t,T} \xrightarrow{\mathbb{P}} \text{rba}\sqrt{\frac{2}{\pi}}\sigma^{\mathbb{P}}_t, \quad \textit{as } \Delta_n, T \to 0 \textit{ and } \Delta_n/T \to 0. \quad (6.6)$$

The proposition, which is proved in Appendix A.4, leads to the approximation

$$\widehat{DTC}_{t,T} \approx \frac{T}{\sqrt{\Delta_n}}\text{rba}\sqrt{\frac{2}{\pi}}\sigma^{\mathbb{P}}_t. \quad (6.7)$$

We note that the size of the transaction costs are governed by the diffusive volatility and do not depend on the level of truncation in the volatility arbitrage strategy. This is because the frequent moves in the asset price, which trigger most of the rebalancing of the position in the underlying asset, are dominated by the diffusive part of the price. Naturally, the transaction costs are higher for higher rebalancing frequency.

The above approximation of the trading costs due to trading in the underlying asset allows us to gauge their effect on the profitability of the volatility arbitrage strategies. For this, purpose, in Figure 5 we plot the relative transaction costs due to the dynamic part of the position. The latter are computed by dividing $\widehat{DTC}_{0,T}$ by $T \times (\sigma^{\mathbb{P}}_0)^2$. These numbers should be compared with the relative expected gains of the trading strategy without transaction costs which are given by $\max\{(\sigma^{\mathbb{Q}}_0/\sigma^{\mathbb{P}}_0)^2 - 1, 1 - (\sigma^{\mathbb{P}}_0/\sigma^{\mathbb{Q}}_0)^2\}$. For generating the numbers in the figure, we have set T to 30 minutes, Δ_n to 5 minutes, and rba to the one used in our Monte Carlo. As seen from the figure, even though the bid–ask spreads for trading in the underlying asset are relatively small, they nevertheless generate nontrivial transaction costs for the volatility arbitrage strategy. These costs are bigger for smaller values of volatility, which suggests that potential arbitrage violations will be harder to exploit for lower levels of volatility.

The transaction costs due to trading in the options are easier to account for as the position in the options is static. Assuming constant proportional bid–ask spreads for trading options with different strikes, the relative transaction costs due to trading options based on numbers calibrated to the real data used in the application are around 2–6%. These numbers can be added to those reported in Figure 5 to get the total transaction costs of implementing the volatility arbitrage strategies. They are nontrivial and determine the size of a potential gap between $(\sigma^{\mathbb{P}}_0)^2$ and $(\sigma^{\mathbb{Q}}_0)^2$ that cannot be acted upon in practice.

6.2 Determining the Direction of the Volatility Strategy

To implement the volatility arbitrage strategy, the investor needs to know if the options market over- or under-values volatility. For this, we need $\mathbb{P}$- and $\mathbb{Q}$-estimates of volatility due to the

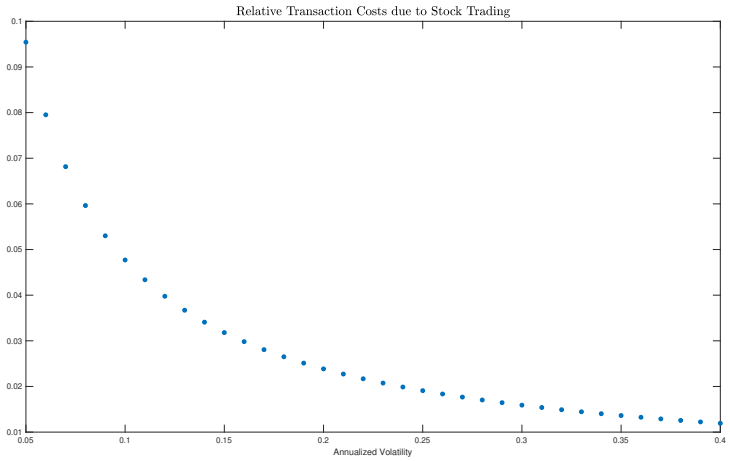


Figure 5: **Relative Transaction Costs due to Stock Trading**. The time window is $T = 30$ minutes and Δ_n corresponds to 5 minute frequency of rebalancing. The relative bid–ask spread (i.e., bid–ask spread over mid-quote) for trading in the underlying asset is set to $2 \times 2.1326e-05$. The transaction cost due to trading in the underlying asset is computed using the asymptotic result in (6.7). It is converted to a relative one by dividing by $T \times (\sigma_0^{\mathbb{P}})^2$.

small and frequent moves in the asset price at time t using information only up to that time. In general, conditional expectations under $\mathbb{P}$ are difficult to obtain as they require modeling the dynamics of the underlying asset. This is not the case, however, in our situation. The reason for this is that we need estimates of volatility that is due to the small and frequent asset price moves. In fact, if there was a continuous price record, then the investor should be able to infer exactly $\sigma_t^{\mathbb{P}}$ and $\nu_t^{\mathbb{P}}(dz)$, for small z, from the past trajectory of the asset price.

In reality, the econometrician, and possibly the investor as well, has a discrete price record only. In this case, the best forecast of future short-term volatility is an efficient estimator of spot volatility formed from high-frequency price records in a short time window prior to the time of forming the forecast. We denote this estimate by $\widehat{V}_{t,T}^{\mathbb{P}}$ and an estimate of its asymptotic standard error by $\widehat{\mathrm{Astd}}(\widehat{V}_{t,T}^{\mathbb{P}})$. The latter allows us to quantify the uncertainty in measuring volatility from discrete observations of the underlying price.

To account for the well-known and pronounced locally deterministic time-of-day volatility pattern, see e.g., Andersen et al. (2024), we use scaling ratios that compare volatility in the forecast window with that of the estimation window. These estimates are inferred from 0DTEs, following Todorov and Zhang (2024), in order to incorporate anticipated changes in this pattern. We form lower and upper bounds for the time-of-day scaling factors, denoted with $\widehat{\mathrm{tod}}_{t,T}^{\mathrm{low}}$ and $\widehat{\mathrm{tod}}_{t,T}^{\mathrm{high}}$, to account for the fact that true option prices are unobserved and the econometrician has access to bid and ask quotes only. Similarly, we form a lower and upper bound for $\mathbb{Q}$-volatility from the observed option quotes. We denote those by $\widehat{V}_{t,T}^{\mathbb{Q},\mathrm{low}}$ and $\widehat{V}_{t,T}^{\mathbb{Q},\mathrm{high}}$. All estimators are formally defined in Appendix B.

Given these estimates, the feasible strategy that exploits potential volatility arbitrage opportunities is given by:

- If $\left(\widehat{V}_{t,T}^{\mathbb{P}} + 2.3 \times \widehat{\mathrm{Astd}}(\widehat{V}_{t,T}^{\mathbb{P}})\right) \times \widehat{\mathrm{tod}}_{t,T}^{\mathrm{high}} < \widehat{V}_{t,T}^{\mathbb{Q},\mathrm{low}}$, we engage in a hedged short position in truncated volatility.

- If $\left(\widehat{V}^{\mathbb{P}}_{t,T} - 2.3 \times \widehat{\text{Astd}}(\widehat{V}^{\mathbb{P}}_{t,T})\right) \times \widehat{\text{tod}}^{\text{low}}_{t,T} > \widehat{V}^{\mathbb{Q},\text{high}}_{t,T}$, we engage in a hedged long position in truncated volatility.

We refer to the above strategy as the *volatility arbitrage strategy*. In words, the investor takes a short position in truncated volatility if a "plausible" upper bound for $\mathbb{P}$-volatility is below a lower bound for its $\mathbb{Q}$-counterpart. Similarly, the investor takes a long position in truncated volatility if a "plausible" lower bound for $\mathbb{P}$-volatility is above an upper bound for its $\mathbb{Q}$-counterpart. Here plausible bounds for $\mathbb{P}$-volatility are formed from 99% confidence interval.[10]

We compare the above feasible volatility arbitrage strategy with the popular one where one always sells total volatility, i.e., takes a short hedged position in truncated volatility with truncation parameter set to zero. We refer to this strategy as *selling volatility*. If we do not hedge the above short position in volatility, as has been commonly done in earlier work, then we refer to this strategy as *naked selling of volatility*. Note that these strategies are not pseudo-arbitrage opportunities as they are exposed (lose money) to the big jumps in the asset price (cf. Proposition 2).

7 Monte Carlo Study

We now evaluate the performance of the volatility strategies discussed in the previous section in a simulation study that is designed to mimic the key properties of the real data that we are going to use later on.

7.1 Model

The dynamics of the underlying stock price under $\mathbb{P}$ in the Monte Carlo is given by (2.1) with

$$\alpha_t = 0.05, \quad \sigma^{\mathbb{P}}_t = \sigma_t, \quad d\sigma^2_t = 8(0.02 - \sigma^2_t)dt + 0.2\sigma_t dB^{\mathbb{P}}_t, \tag{7.1}$$

where $B^{\mathbb{P}}_t$ is a $\mathbb{P}$-Brownian motion with $\text{corr}(W^{\mathbb{P}}_t, B^{\mathbb{P}}_t) = -0.9$. Our unit of time is one year, so the above parameters imply a half-life of a volatility shock of approximately one month. The jump compensator under $\mathbb{P}$ is given by

$$\nu^{\mathbb{P}}(dt, dz) = \nu^{ts}(dz)\, \sigma^2_t dt, \tag{7.2}$$

where ν^{ts} is the compensator of a tempered stable process given by

$$\nu^{ts}(dz) = c_- \frac{e^{-\lambda_-|z|}}{|z|^{\alpha+1}} 1_{\{z<0\}} dz + c_+ \frac{e^{-\lambda_+|z|}}{|z|^{\alpha+1}} 1_{\{z>0\}} dz, \quad c_\pm \geq 0,\ \lambda_\pm > 0,\ \alpha < 2, \tag{7.3}$$

with parameters set to

$$\alpha = -1, \quad c_- = 0.1 \times \frac{(\lambda^{\mathbb{P}}_-)^{2-\alpha}}{\Gamma(2-\alpha)}, \quad c_+ = 0.1 \times \frac{(\lambda^{\mathbb{P}}_+)^{2-\alpha}}{\Gamma(2-\alpha)}, \quad \lambda^{\mathbb{P}}_\pm = 500. \tag{7.4}$$

This specification of the jumps corresponds to a time-changed double-exponential jump model that has been commonly used in existing parametric option pricing work.

[10] Similarly, one can use the asymptotic approximations for the transaction costs derived above to account for their effect and determine the rebalancing frequency. To keep things simple, we do not do this here and leave such analysis for future work.

Turning next to the risk-neutral probability, we assume that $\sigma_t^{\mathbb{Q}} = c\sigma_t$ where we set $c = 1$ in the absence of arbitrage violations and $c \neq 1$ to generate pseudo-arbitrage opportunities. We consider situations in which the option market either overestimates ($c > 1$) or underestimates ($c < 1$) spot volatility. The jump compensator under $\mathbb{Q}$ is the same as its $\mathbb{P}$-counterpart, with only the parameter $\lambda_-^{\mathbb{P}}$ changing to $\lambda_-^{\mathbb{Q}}$. For the specifications for which there is pseudo-arbitrage, we keep $\lambda_-^{\mathbb{Q}} = \lambda_-^{\mathbb{P}}$ while for the ones without such arbitrage opportunities, we determine $\lambda_-^{\mathbb{Q}}$ from the value of $\lambda_-^{\mathbb{P}}$ and

$$\left(\frac{\lambda_-^{\mathbb{P}}}{\lambda_-^{\mathbb{Q}}}\right)^{2-\alpha} = \frac{0.1 + 0.2 \times 1.2}{0.1}.$$

This choice of $\lambda_-^{\mathbb{Q}}$ implies a variance risk premium of 20%, which is similar to that observed in the real data. Thus, for all model specifications considered in our analysis, there is a variance risk premium (defined as difference between $\mathbb{Q}$- and $\mathbb{P}$-conditional expectations of future total return volatility) but its source differs. For the specifications with no arbitrage, the source of the variance risk premium is the pricing of negative jump risk in the asset price. For the specifications with pseudo-arbitrage opportunities, the variance risk premium is solely due to difference in diffusive spot volatility. The performance of the different volatility strategies designed in the previous sections should be able to identify the source of the variance risk premium.

7.2 Sampling Scheme and Choice of Truncation

The sampling scheme in the Monte Carlo mimics that of the real data we use. More specifically, we sample the underlying stock price at five second frequency. This corresponds to $\Delta_n = (1/252) \times (1/4680)$ in a 6.5 hour trading day when using business time convention. The starting values of the stock price and the volatility are $X_0 = 4000$ and $\sigma_0 = 0.02$, and we simulate price and volatility paths over 10,000 days. We generate zero-date option prices given the $\mathbb{Q}$-dynamics of X_t described above at every minute during the trading day. We set the bid/ask quotes to be 2.5% below/above the true option price. At each point in time, the available strike grid is equidistant with mesh of \$5. The highest and lowest strikes are determined as the lowest and highest multiples of \$5, respectively, for which the corresponding out-of-the-money option mid-quote falls below \$0.075. Similarly, for the underlying asset, we set the bid and ask to be 2.1326e-03% below and above the true price, respectively. The size of the relative bid–ask spread in the underlying asset is slightly higher than the average one for the data used in our empirical application.

We use the highest frequency of five seconds when forming our estimates of $\mathbb{P}$-volatility. We use a coarser frequency of five minutes for the dynamic position in the underlying asset. As discussed in the previous section, when deciding at what frequency to rebalance the position in the underlying asset, we face a trade-off. A higher frequency means better hedging of the risks in the option portfolio but this comes at the expense of higher transaction costs. Our choice of frequency to update the position in the underlying asset aims to strike a balance between these two effects.

Finally, we use the truncation function $\phi(x) = e^{-x^2}$ and set the truncation parameter η_T in the following data-adaptive way:

$$\widehat{\eta}_{t,T} = \sqrt{\frac{\overline{\eta}}{\log(1/T)}} \times \sqrt{\frac{1}{\widehat{TV}_{t,T}^{\mathbb{Q}}(0)}}, \tag{7.5}$$

for some positive constant $\overline{\eta}$. We experiment with several different values of $\overline{\eta}$, with higher values

corresponding to more aggressive truncation. The above choice of the truncation parameter means that the truncation is relative to the overall level of volatility.

7.3 Results

In Table 1, we report the percentage of times our estimates of $\mathbb{P}$- and $\mathbb{Q}$-volatility trigger activation of the volatility arbitrage strategy. In the case of no arbitrage and when the time horizon T is relatively long (2 hours), the strategy is erroneously triggered on around 15% of the days. The reason for this is that for such relatively long horizon T, it is difficult to infer from the data whether the variance risk premium embedded in the options is due to compensation for big jumps in the asset price or due to the existence of pseudo-arbitrage opportunities. When the horizon T shrinks, this separation is much easier. As a result for T equal to 30 minutes, the activation ratio drops to 3.9% only when markets are free of arbitrage. On the other hand, when there are large pseudo-arbitrage opportunities ($\sigma_t^{\mathbb{Q}}$ deviates from $\sigma_t^{\mathbb{P}}$ by more than 20%), the arbitrage volatility strategy is triggered very frequently. The activation ratio naturally drops when the size of the arbitrage violation gets smaller. This is particularly true for the case $\sigma_t^{\mathbb{Q}} = 0.9 \times \sigma_t^{\mathbb{P}}$ and illustrates the difficulties in identifying the existence of small pseudo-arbitrage opportunities from the data. Nevertheless, the activation ratios of the volatility arbitrage strategy are nontrivial for a horizon of 30 minutes even if there is only a 10% gap between $\sigma_t^{\mathbb{P}}$ and $\sigma_t^{\mathbb{Q}}$.

Table 1: **Monte Carlo Results: Activation Ratios for the Vol. Arb. Strategy**

Scenario	TTM	Activation Ratio	Scenario	TTM	Activation Ratio
no arbitrage	30 minutes	3.9%			
	60 minutes	5.3%			
	90 minutes	9.4%			
	120 minutes	14.5%			
$\sigma_t^{\mathbb{Q}} = 1.1 \times \sigma_t^{\mathbb{P}}$	30 minutes	36.2%	$\sigma_t^{\mathbb{Q}} = 1.2 \times \sigma_t^{\mathbb{P}}$	30 minutes	69.9%
	60 minutes	37.6%		60 minutes	67.0%
	90 minutes	39.7%		90 minutes	67.8%
	120 minutes	37.2%		120 minutes	67.6%
$\sigma_t^{\mathbb{Q}} = 0.9 \times \sigma_t^{\mathbb{P}}$	30 minutes	41.1%	$\sigma_t^{\mathbb{Q}} = 0.8 \times \sigma_t^{\mathbb{P}}$	30 minutes	87.1%
	60 minutes	5.2%		60 minutes	75.5%
	90 minutes	1.5%		90 minutes	66.6%
	120 minutes	10.58%		120 minutes	58.3%

Note: TTM stands for time to maturity. The activation ratio for each feasible volatility arbitrage strategy corresponds to the percentage of days that the strategy is implemented from 10,000 days.

In Table 2, we report the performance of the various volatility strategies in terms of annualized Sharpe ratios in the Monte Carlo. The performance of the volatility arbitrage strategy is computed only on the days when the strategy is triggered. For simplicity, we report only unconditional Sharpe ratios. For computing conditional Sharpe ratios, one would need to estimate the conditional jump distribution which is significantly more difficult. Moreover, in presence of pseudo-arbitrage, the unconditional Sharpe ratio, just like the conditional one, should also remain high as T shrinks. For brevity, we also do not report estimates of $GSR(p)$, for different values of $p \neq 2$. To compare the performance of the different volatility strategies, with and without pseudo-arbitrage, the unconditional Sharpe ratio suffices.

We can draw several conclusions from the reported simulation results. First, when the market

is free of pseudo-arbitrage, then the feasible volatility arbitrage strategy always looses money on average. This is due to the transaction costs. Recall, however, that in this case the strategies are triggered with low probability. In contrast, the strategy which sells volatility generates positive and nontrivial Sharpe ratios in the arbitrage-free case. Consistent with our theoretical analysis, the Sharpe ratios of selling volatility shrink as T shrinks and do not indicate arbitrage violations.

Second, when $\sigma_t^{\mathbb{P}}$ and $\sigma_t^{\mathbb{Q}}$ differ by around 10%, then the infeasible volatility arbitrage strategy, which knows (and does not need to estimate) whether $\sigma_t^{\mathbb{P}}$ is bigger or smaller than $\sigma_t^{\mathbb{Q}}$, generates positive Sharpe ratios. The feasible volatility strategies also generate positive Sharpe ratios for any considered T below 120 minutes. Both the feasible and infeasible volatility strategies have similar unconditional Sharpe ratios. This is because the mispricing is correctly detected from observing past data.

Third, comparing the Sharpe ratios displayed in Figure 4, which do not take into account transaction costs, with the ones reported in Table 2, which do, reveals the nontrivial impact transaction costs have on the profitability of the volatility strategies. For example, with transaction costs, the volatility arbitrage strategies that exploit a 10% gap in $\sigma_t^{\mathbb{P}}$ and $\sigma_t^{\mathbb{Q}}$ and use $\overline{\eta} = 1.5$ have unconditional Sharpe ratios that are only slightly higher than that of holding the market for a day.

Table 2: **Monte Carlo Results: Sharpe Ratios**

Scenario	**TTM**	**Strategy**							
		infeasible vol. arb.			**feasible vol. arb.**			**selling**	**naked short**
		$\overline{\eta} = 0.5$	$\overline{\eta} = 1.0$	$\overline{\eta} = 1.5$	$\overline{\eta} = 0.5$	$\overline{\eta} = 1.0$	$\overline{\eta} = 1.5$	**volatility**	**volatility**
no arbitrage	30 minutes				−0.56	−1.19	−1.45	1.81	1.61
	60 minutes				0.80	0.17	−0.16	2.14	1.38
	90 minutes				1.37	0.24	−0.41	2.82	1.93
	120 minutes				2.01	0.94	0.28	3.35	1.82
$\sigma_t^{\mathbb{Q}} = 1.1 \times \sigma_t^{\mathbb{P}}$	30 minutes	1.77	1.45	0.98	1.53	1.26	0.86	1.68	1.49
	60 minutes	1.52	1.02	0.50	1.45	1.02	0.60	1.80	1.42
	90 minutes	2.16	1.37	0.62	2.00	1.33	0.61	2.53	1.58
	120 minutes	2.37	1.42	0.65	2.66	1.65	0.86	2.94	1.50
$\sigma_t^{\mathbb{Q}} = 0.9 \times \sigma_t^{\mathbb{P}}$	30 minutes	1.42	1.15	0.75	1.29	1.07	0.70	−3.32	−1.93
	60 minutes	1.64	1.02	0.45	1.39	0.69	0.25	−4.26	−1.61
	90 minutes	1.76	0.91	0.25	2.19	1.42	0.73	−4.76	−1.93
	120 minutes	2.41	1.46	0.71	−0.50	−0.88	−1.22	−5.62	−2.02
$\sigma_t^{\mathbb{Q}} = 1.2 \times \sigma_t^{\mathbb{P}}$	30 minutes	4.92	4.39	3.47	5.16	4.61	3.61	4.58	3.37
	60 minutes	5.27	4.25	3.07	5.14	4.16	3.00	5.58	3.69
	90 minutes	6.30	5.12	3.69	6.08	5.02	3.67	6.41	3.38
	120 minutes	6.61	4.74	3.00	6.52	4.64	2.90	7.54	3.68
$\sigma_t^{\mathbb{Q}} = 0.8 \times \sigma_t^{\mathbb{P}}$	30 minutes	3.70	3.16	2.28	3.70	3.19	2.31	−5.34	−3.20
	60 minutes	4.83	3.59	2.40	4.87	3.66	2.49	−7.21	−3.26
	90 minutes	5.46	3.46	1.94	5.46	3.55	2.00	−8.21	−3.35
	120 minutes	5.50	3.30	1.76	5.72	3.50	1.97	−9.10	−3.61

Note: TTM stands for time to maturity. The reported numbers for each strategy correspond to annualized Sharpe ratios estimated from 10,000 days of implementing the strategies. The per-period Sharpe ratios are annualized by multiplying them by $\sqrt{252}$.

Overall, the Monte Carlo analysis shows two distinguishing features of the volatility arbitrage strategies in the presence versus absence of pseudo-arbitrage opportunities. One is the fact that,

in the presence of arbitrage, the volatility arbitrage strategies deliver nontrivial Sharpe ratios even for moderate levels of truncation ($\overline{\eta} \leq 1.5$). The other is the fact that when there is a pseudo-arbitrage opportunity, the performance of the volatility arbitrage strategies does not deteriorate as the horizon T shrinks.

8 Do Equity and Options Markets Agree about Volatility?

8.1 Data

We use three sets of data in the empirical analysis. The first one consists of intraday best bid and best ask quotes for SPY, an ETF tracking the S&P 500 index. This dataset is extracted from the TAQ database. We sample the SPY every 5 seconds during the trading hours range of 9:30–16:00 EST. The second dataset consists of prices of zero-date S&P 500 index (SPX) options. The source of this dataset is the CBOE DataShop. We sample the option prices at one minute frequency and keep only out-of-the-money options with positive bid quotes. We perform the analysis at a point in time during the day only if there are at least five strikes with non-zero bid quotes for out-of-the-money options. The moneyness of the options is determined by implied forward, which in turn is recovered via put–call parity from pairs of call and put mid-quotes with the same strike for which the difference in the put–call premium is the smallest in absolute value. We use three such pairs with smallest put–call premium gap and take the median as our implied forward estimate. Finally, we obtain closing prices for the S&P 500 index from the CRSP database.

The sample period for the study is 2020–2024. Prior to Spring 2022, the available SPX option expiration dates were Monday, Wednesday and Friday. After Spring 2022, there are expiration dates on every trading day of the week. We remove from the analysis days with partial trading around holidays. We further remove days with FOMC announcements as the zero-date options on these days can be partially affected (depending on the time of the day) by the event risk contained in these announcements.

8.2 Empirical Evidence

We implement the feasible volatility arbitrage strategies in exactly the same way as in the Monte Carlo. Our estimates for $\mathbb{P}$- and $\mathbb{Q}$-volatility which determine the activation of these strategies are plotted in Figure 6. As seen from the figure, for all considered values of T, the estimates $\widehat{V}^{\mathbb{P}}_{t,T}$ and $\widehat{V}^{\mathbb{Q}}_{t,T}$ appear close to each other both in periods of low and high volatility. Indeed, the median absolute percentage difference between the $\mathbb{P}$- and $\mathbb{Q}$-volatility is in the range 12%–15.5%.

The performance of the various volatility strategies on the real data is reported in Table 3. The volatility arbitrage strategies yield negative Sharpe ratios for all considered levels of $\overline{\eta}$ and almost all horizons T. This performance is in sharp contrast to that of the hedged selling of volatility strategy which generates high Sharpe ratios. The magnitudes of the Sharpe ratios of this strategy in the data are similar to those in the arbitrage-free Monte Carlo case. The naked short volatility positions, on the other hand, do not do so well. This is in line with the empirical findings of Vilkov (2023) for modest and only marginally significant variance risk premiums (before accounting for transaction costs). Our results show, however, that the evidence for the latter becomes much more significant once one hedges the directional risk in the strategy.

In Figure 7, we display the returns from the different volatility strategies for a horizon of one hour. We can see from the figure that the volatility arbitrage strategies have nearly symmetric returns. Also, there is no apparent clustering of the periods when the strategies are inactive.

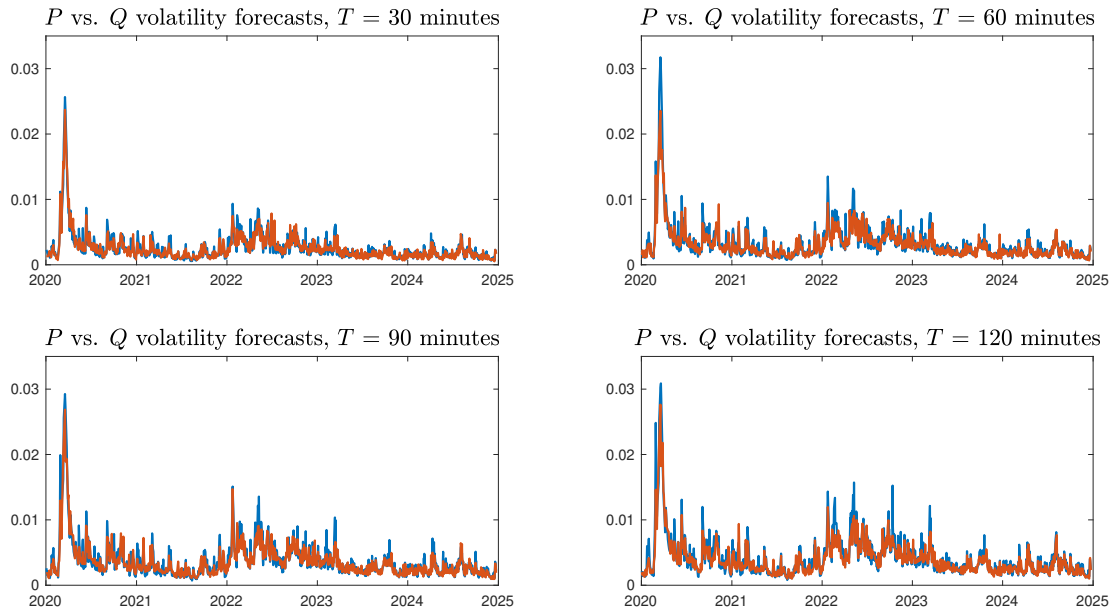


Figure 6: **Risk-Neutral versus True Conditional Expected Truncated Volatility**. The estimates of the conditional expected volatility under $\mathbb{P}$ and $\mathbb{Q}$ are $\sqrt{\widehat{V}^{\mathbb{P}}_{t,T}}$ (blue line) and $\sqrt{\widehat{V}^{\mathbb{Q}}_{t,T}}$ (red line), respectively.

Finally, the returns from these strategies do not appear significantly fat-tailed. That said, jump risk is apparently still affecting these strategies, which is in line with our theoretical analysis. On the other hand, the strategies involving selling of total volatility (the bottom two panels of the figure) are extremely heavy-tailed and highly negatively skewed. The reason for this is that these strategies are significantly more exposed to the big jumps in the underlying asset price than the truncated volatility strategies. This is further confirmed by comparing the skewness and kurtosis of the various strategies reported in Table 3. Thus, in particular, the high Sharpe ratio that the hedged short volatility strategy generates comes at the cost of very fat-tailed and skewed return distributions.

Overall, the reported results indicate no economic gains from implementing the volatility arbitrage strategies. This means that even if there is disagreement between the equity and option markets about volatility, such disagreements must be small and/or hard to detect. To better asses this, we compare in Table 4 the performance of the volatility arbitrage strategies with and without including transaction costs. As seen from the reported results, when transaction costs are not accounted for, the volatility arbitrage strategies generate nontrivial unconditional Sharpe ratios. Thus, even for the highly liquid S&P 500 index equity and options markets with tight bid–ask spreads, the transaction costs can provide limits to arbitrage.

In addition to transaction costs, estimation risk and short-term volatility risk provide another limit to exploiting pseudo-arbitrage opportunities. As we discussed earlier, volatility prediction in our case is relatively easy conceptually because the best predictor of future ultra short-term volatility is an estimator of current spot volatility. Nevertheless, as seen in our Monte Carlo, estimation risk can limit the detection of potential smaller in magnitude disagreements between the equity and option markets. In the real data, we notice slightly higher activation rates of the volatility arbitrage strategy than in the arbitrage-free Monte Carlo scenario and yet these higher

Table 3: **Empirical Results with Transaction Costs**

Strategy	TTM	SR	% active	Skewness	Kurtosis
vol. arb., $\overline{\eta} = 0.5$	30 minutes	-0.05 (1.50)	51%	-0.09	6.43
	60 minutes	-1.37 (1.71)	48%	0.16	6.92
	90 minutes	0.19 (1.45)	45%	0.38	5.31
	120 minutes	-1.17 (1.52)	42%	0.36	5.30
vol. arb., $\overline{\eta} = 1.0$	30 minutes	-0.41 (1.50)	51%	0.25	7.10
	60 minutes	-1.23 (1.43)	48%	0.21	5.90
	90 minutes	0.63 (1.59)	45%	0.51	8.69
	120 minutes	-1.49 (2.43)	42%	-1.63	11.82
vol. arb., $\overline{\eta} = 1.5$	30 minutes	-0.75 (1.47)	51%	0.30	8.14
	60 minutes	-1.12 (1.19)	48%	0.22	4.76
	90 minutes	0.77 (1.45)	45%	0.48	9.82
	120 minutes	-1.53 (2.24)	42%	-1.93	12.47
selling volatility	30 minutes	1.57 (1.77)	100%	-2.78	17.46
	60 minutes	3.43 (2.49)	100%	-3.76	30.14
	90 minutes	3.59 (3.70)	100%	-5.83	82.65
	120 minutes	3.11 (3.70)	100%	-6.84	90.71
naked short volatility	30 minutes	0.46 (1.73)	100%	-7.58	108.44
	60 minutes	0.77 (1.60)	100%	-7.21	83.69
	90 minutes	0.57 (2.00)	100%	-11.01	202.04
	120 minutes	0.07 (1.69)	100%	-10.99	178.25

Note: TTM stands for time to maturity and SR for annualized Sharpe Ratio. The per-period Sharpe ratios are annualized by multiplying them by $\sqrt{252}$. The numbers in brackets are Newey–West HAC standard errors.

Table 4: **Empirical Results with and without Transaction Costs**

Strategy	TTM	SR with transaction costs	% SR without transaction costs
vol. arb., $\overline{\eta} = 0.5$	30 minutes	-0.05 (1.50)	1.62 (1.35)
	60 minutes	-1.37 (1.71)	0.61 (1.65)
	90 minutes	0.19 (1.45)	2.23 (1.41)
	120 minutes	-1.17 (1.52)	0.98 (1.41)
vol. arb., $\overline{\eta} = 1.0$	30 minutes	-0.41 (1.50)	1.29 (1.34)
	60 minutes	-1.23 (1.43)	0.74 (1.38)
	90 minutes	0.63 (1.59)	2.58 (1.48)
	120 minutes	-1.49 (2.43)	0.37 (2.18)
vol. arb., $\overline{\eta} = 1.5$	30 minutes	-0.75 (1.47)	0.90 (1.32)
	60 minutes	-1.12 (1.19)	0.78 (1.11)
	90 minutes	0.77 (1.45)	2.60 (1.34)
	120 minutes	-1.53 (2.24)	0.15 (2.03)

Note: TTM stands for time to maturity and SR for annualized Sharpe Ratio. The per-period Sharpe ratios are annualized by multiplying them by $\sqrt{252}$. The numbers in brackets are Newey–West HAC standard errors.

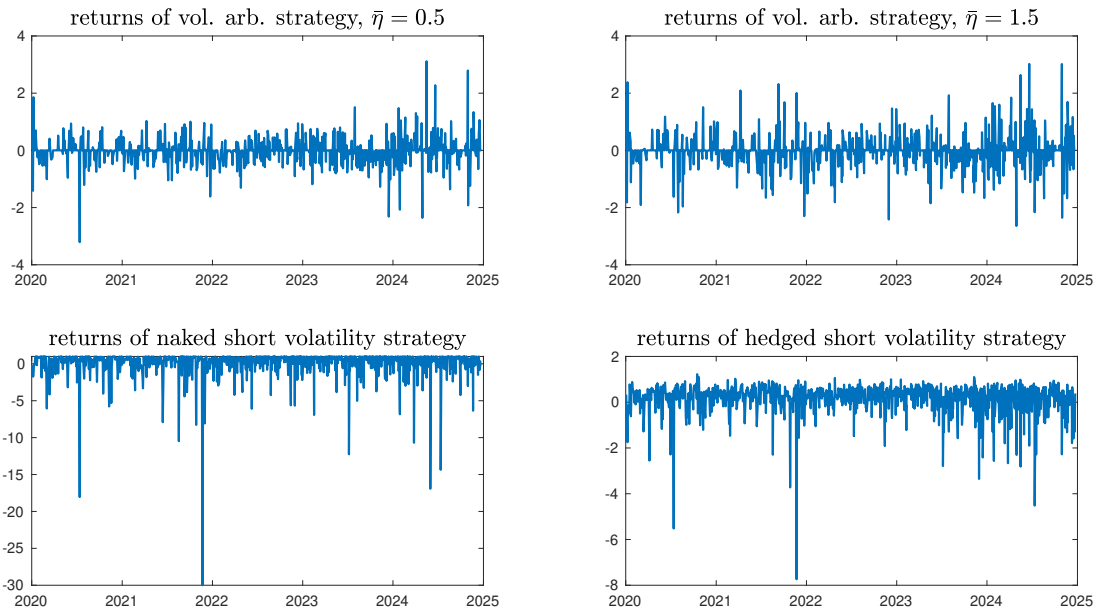


Figure 7: **Returns of Volatility Arbitrage Strategies**. If a volatility strategy is not activated on a given day, then we put a return of zero on that day. The time horizon is $T = 60$ minutes.

activation ratios do not lead to good performance of the volatility arbitrage strategies. This is likely due to short-term volatility risk in the form of volatility jumps. This risk is of higher asymptotic order as the horizon of the strategy shrinks, and this is why it does not appear in the limit of the conditional Sharpe ratios in Theorem 2.

To get an idea of the role of this source of risk, we compute the ex-ante conditional Sharpe ratios of the volatility arbitrage strategies based on the limit results in Theorem 2. These limits do not take into account transaction costs and treat $\mathbb{P}$-diffusive volatility as known. We do this calculation for the volatility arbitrage strategy without hedging as its limit depends only on $\mathbb{P}$- and $\mathbb{Q}$-volatility estimates which are readily available. The average and median ex-ante conditional annualized Sharpe ratio for a horizon of 30 minutes and truncation parameter $\overline{\eta} = 1.5$, taken over days on which the volatility strategy is triggered, are 1.76 and 1.16, respectively. These are somewhat higher than their realized counterpart for the corresponding setting without transaction costs, which equals 0.90 (reported in the last column of Table 4). This shows that in practice short-term volatility risk and estimation risk can also limit the incentives to implement volatility arbitrage strategies that exploit a potential disagreement between equity and option markets about volatility.

Overall, the reported results for the S&P 500 index equity and option markets suggest possible mild disagreements about volatility. But these are hard to detect from past discrete price records and not big enough in size to be exploited given costs for trading in these markets and the short-term volatility risk. We leave for future work an extension of the current analysis to different assets, particularly to ones which have significantly higher share of retail option traders that can cause the existence of local pseudo-arbitrage opportunities.

9 Conclusion

In this paper we consider the question whether equity and option markets are integrated, i.e., whether there is an aggregate pricing kernel that can jointly rationalize observed equity and option prices. We show that such market integration is equivalent to an agreement between the two markets about the volatility due to small and frequent moves in the asset price. Absence of such an agreement leads to pseudo-arbitrage opportunities, which can be exploited via a portfolio of short-dated options together with a dynamic position in the underlying asset that aims to hedge some of the risk in the options.

Empirically, we find that, in spite of the large premium embedded in short-dated S&P 500 index options, their prices correctly reflect the volatility due to small and frequent asset price moves, with only mild differences between $\mathbb{P}$- and $\mathbb{Q}$-volatility that cannot be acted upon due to transaction costs, estimation risk and short-term volatility risk. Thus, zero-date options are expensive but do not appear mis-priced.

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Appendix

A Technical Results and Proofs

A.1 A Refinement of Girsanov's Theorem

We first give a version of Girsanov's theorem that contains necessary and sufficient conditions for market integration. As we are only interested in equivalence of $\mathbb{P}$ and $\mathbb{Q}$ in restriction to $\mathcal{F}_{\mathcal{T}}$, there is no loss of generality to assume that $\mathcal{F} = \mathcal{F}_{\mathcal{T}}$ and $\mathbb{P} = \mathbb{P}|_{\mathcal{F}_{\mathcal{T}}}$.

Theorem 3. *Suppose that $\mathcal{G}_0$ is a sub-σ-field of $\mathcal{F}_0$, equipped with a probability measure P_0 (initial condition). Further suppose that $\mathbb{P}$ is the locally unique probability measure (see Definition III.2.37 in Jacod and Shiryaev (2003)) that satisfies $\mathbb{P}|_{\mathcal{G}_0} = P_0$ and that renders X a special semimartingale[11] with local characteristics $(B^{X,\mathbb{P}}, C^{X,\mathbb{P}}, \nu^{X,\mathbb{P}})$, where for all $t \in [0, \mathcal{T}]$,*

$$B_t^{X,\mathbb{P}} = \int_0^t X_s \alpha_s ds, \quad C_t^{X,\mathbb{P}} = \int_0^t (X_s \sigma_s^{\mathbb{P}})^2 ds, \quad \nu^{X,\mathbb{P}}(dt, dz) = \left[\nu_t^{\mathbb{P}} \circ (X_{t-}(e^z - 1))^{-1}\right](dz)dt. \tag{A.1}$$

Then $\mathcal{Q}_e$ is equal to the family of risk-neutral measures $\mathbb{Q} \in \mathcal{Q}$ for which $\mathbb{Q}|_{\mathcal{G}_0}$ is equivalent to P_0 and X is a martingale under $\mathbb{Q}$ with local characteristics $(B^{X,\mathbb{Q}}, C^{X,\mathbb{Q}}, \nu^{X,\mathbb{Q}})$ given by

$$B_t^{X,\mathbb{Q}} = 0, \quad C_t^{X,\mathbb{Q}} = \int_0^t (X_s \sigma_s^{\mathbb{Q}})^2 ds, \quad \nu^{X,\mathbb{Q}}(dt, dz) = \left[\nu_t^{\mathbb{Q}} \circ (X_{t-}(e^z - 1))^{-1}\right](dz)dt \tag{A.2}$$

for $t \in [0, \mathcal{T}]$, where

$$\sigma_t^{\mathbb{Q}} = \sigma_t^{\mathbb{P}} \quad \text{and} \quad \nu_t^{\mathbb{Q}}(dz) = Y_t(z)\nu_t^{\mathbb{P}}(dz), \tag{A.3}$$

and $Y_t(z)$ is a strictly positive predictable function such that $\mathbb{P}$- and $\mathbb{Q}$-almost surely,

$$\begin{aligned} &\int_0^{\mathcal{T}} \int_{\mathbb{R}} \left(\frac{\alpha_t + \int_{\mathbb{R}} (e^z - 1)(Y_t(z) - 1)\nu_t^{\mathbb{P}}(dz)}{\sigma_t^{\mathbb{P}}} \right)^2 dt < \infty, \\ &\int_0^{\mathcal{T}} \int_{\mathbb{R}} (|Y_t(z) - 1|^2 \wedge |Y_t(z) - 1|)\nu_t^{\mathbb{P}}(dz)dt < \infty. \end{aligned} \tag{A.4}$$

Proof. Suppose that $\mathbb{Q} \in \mathcal{Q}_e$. By definition, we have $\mathbb{P} \sim \mathbb{Q}$ ("$\mathbb{P}$ is equivalent to $\mathbb{Q}$"), which implies $\mathbb{Q}|_{\mathcal{G}_0} \sim P_0$. Moreover, by Girsanov's theorem (see Theorem III.3.24 in Jacod and Shiryaev (2003)), X is a $\mathbb{Q}$-semimartingale with characteristics $(B^{X,\mathbb{Q}}, C^{X,\mathbb{Q}}, \nu^{X,\mathbb{Q}})$ where $C^{X,\mathbb{Q}}$ and $\nu^{X,\mathbb{Q}}$ are given by (A.2) and satisfy (A.3) for some nonnegative measurable $Y_t(z)$, and

$$B_t^{X,\mathbb{Q}} = \int_0^t X_s \left(\alpha_s + \beta_s X_s (\sigma_s^{\mathbb{P}})^2 + \int_{\mathbb{R}} (e^z - 1)(Y_s(z) - 1)\nu_s^{\mathbb{P}}(dz) \right) ds \tag{A.5}$$

for some predictable process β_t. Because $\mathbb{Q} \in \mathcal{Q}_e \subseteq \mathcal{Q}$, we know that X is a $\mathbb{Q}$-martingale, hence $B^{X,\mathbb{Q}} \equiv 0$, which implies that β_t can be chosen such that

$$\beta_t X_t = -\frac{\alpha_t + \int_{\mathbb{R}} (e^z - 1)(Y_t(z) - 1)\nu_t^{\mathbb{P}}(dz)}{(\sigma_t^{\mathbb{P}})^2}, \quad t \in [0, \mathcal{T}]. \tag{A.6}$$

[11] A semimartingale X is special if it is locally integrable. In this case, it can be decomposed into the sum of a local martingale and a predictable process of finite variation (called the drift of X). Furthermore, in writing the semimartingale characteristics of X, one can then take the identity as truncation function (i.e., no truncation), which is implicitly assumed in (A.1) and (A.2).

Together with the assumption $\mathbb{P} \sim \mathbb{Q}$, we can use Théorème 4.1 in Jacod and Mémin (1976) to deduce that $Y_t(z)$ can be chosen as strictly positive and must satisfy (A.4) $\mathbb{P}$- and $\mathbb{Q}$-almost surely.

Conversely, suppose now that $\mathbb{Q} \in \mathcal{Q}$ satisfies the properties listed in the proposition. By assumption, $\mathbb{P}$ is the locally unique probability measure that equals P_0 if restricted on $\mathcal{G}_0$ and makes X a semimartingale with local characteristics given in (A.1). Therefore, by Théorème 4.3 in Jacod and Mémin (1976), we have $\mathbb{P} \sim \mathbb{Q}$, which means $\mathbb{Q} \in \mathcal{Q}_e$. □

A.2 Proofs for Section 3

Proof of Proposition 1. By the definition of the stochastic discount factor, we have

$$\mathbb{E}_t^{\mathbb{Q}}(R_{t,T}) = \mathbb{E}_t^{\mathbb{P}}(D_{t,T}R_{t,T}),$$

and therefore,

$$\mathbb{E}_t^{\mathbb{Q}}(R_{t,T}) - \mathbb{E}_t^{\mathbb{P}}(R_{t,T}) = \mathbb{E}_t^{\mathbb{P}}\left[(D_{t,T}-1)R_{t,T}\right] = \mathbb{E}_t^{\mathbb{P}}\left[(D_{t,T}-1)\left(R_{t,T} - \mathbb{E}_t^{\mathbb{P}}(R_{t,T})\right)\right],$$

because $\mathbb{E}_t^{\mathbb{P}}(D_{t,T}-1) = 0$. Applying Hölder's inequality, we derive the estimate

$$\begin{aligned}&\left|\mathbb{E}_t^{\mathbb{P}}\left[(D_{t,T}-1)\left(R_{t,T} - \mathbb{E}_t^{\mathbb{P}}(R_{t,T})\right)\right]\right| \\ &\quad\leq \left(\mathbb{E}_t^{\mathbb{P}}\left(\left|R_{t,T} - \mathbb{E}_t^{\mathbb{P}}(R_{t,T})\right|^p\right)\right)^{\frac{1}{p}} \left(\mathbb{E}_t^{\mathbb{P}}\left(|D_{t,T}-1|^{\frac{p}{p-1}}\right)\right)^{1-\frac{1}{p}}.\end{aligned} \tag{A.7}$$

As $\mathbb{E}_t^{\mathbb{Q}}(R_{t,T}) = 0$ by (3.3), we obtain

$$\left(\mathbb{E}_t^{\mathbb{P}}\left(|D_{t,T}-1|^{\frac{p}{p-1}}\right)\right)^{1-\frac{1}{p}} \geq \frac{\left|\mathbb{E}_t^{\mathbb{Q}}(R_{t,T}) - \mathbb{E}_t^{\mathbb{P}}(R_{t,T})\right|}{\left(\mathbb{E}_t^{\mathbb{P}}\left(\left|R_{t,T} - \mathbb{E}_t^{\mathbb{P}}(R_{t,T})\right|^p\right)\right)^{\frac{1}{p}}} = |GSR_{t,T}(p)|, \tag{A.8}$$

proving the proposition. □

For the proof of Theorem 1, we need an auxiliary result.

Lemma 1. *If $\mathbb{Q} \in \mathcal{Q}_e$ and $t \in [0, \mathcal{T})$, the stochastic discount factor satisfies*

$$D_{t,T} \to 1 \quad \text{a.s. as } T \downarrow 0. \tag{A.9}$$

Moreover, if $\mathbb{E}_t^{\mathbb{P}}(|M_s|^q) < \infty$ for some $s \in (t, \mathcal{T}]$ and $q \in (1, \infty)$, then

$$\mathbb{E}_t^{\mathbb{P}}(|D_{t,T}-1|^q) \to 0 \quad \text{a.s. as } T \downarrow 0. \tag{A.10}$$

Proof. Because $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we have $M_t > 0$ almost surely for all $t \in [0, \mathcal{T}]$. As M is a $\mathbb{P}$-martingale, by right-continuity, we obtain (A.9). To upgrade this to (A.10), by dominated convergence, it suffices to note that $\mathbb{E}_t^{\mathbb{P}}(\sup_{r\in[t,s]}|M_r|^q) < \infty$ by Doob's martingale inequality and our assumption that $\mathbb{E}_t^{\mathbb{P}}(|M_s|^q) < \infty$. □

Proof of Theorem 1. Suppose that $\mathbb{Q} \in \mathcal{Q}_e$. By Proposition 1 and Definition 4, pseudo-arbitrage implies that for a subsequence of time horizons, which we still denote by T,

$$\left(\mathbb{E}_t^{\mathbb{P}}\left(|D_{t,T}-1|^q\right)\right)^{1/q} \geq |GSR_{t,T}(p)| \to G_t(p) \quad \text{a.s. as } T \downarrow 0,$$

where $G_t(p)$ denotes the left-hand side of (3.5) and is therefore a strictly positive random variable. As this contradicts (A.10), the assumptions behind (A.10) must be violated, which means that we must have $\mathbb{E}_t^{\mathbb{P}}(|M_s|^q) = \infty$ for all $s \in (t, \mathcal{T}]$. □

A.3 Proofs for Section 5

For the proof of Theorems 2, we can assume without loss of generality that $t = 0$ and that $X_0 = 1$. We also introduce

$$\theta(x) = \phi(x)x^2 \qquad \text{and} \qquad \Phi(\eta, x)_T = \phi(\eta x/\sqrt{T})x^2 = \frac{T}{\eta^2}\theta(\eta x/\sqrt{T}). \tag{A.11}$$

Then

$$\begin{aligned} \theta'(x) &= \phi'(x)x^2 + 2x\phi(x), & \theta''(x) &= \phi''(x)x^2 + 4\phi'(x)x + 2\phi(x), \\ \theta'''(x) &= \phi'''(x)x^2 + 6\phi''(x)x + 6\phi'(x), & \theta^{(4)}(x) &= \phi^{(4)}(x)x^2 + 8\phi'''(x)x + 12\phi''(x) \end{aligned} \tag{A.12}$$

and

$$\frac{\partial}{\partial x}\Phi(\eta, x)_T = \frac{\sqrt{T}}{\eta}\theta'(\eta x/\sqrt{T}), \qquad \frac{\partial^2}{\partial x^2}\Phi(\eta, x)_T = \theta''(\eta x/\sqrt{T}). \tag{A.13}$$

By Itô's formula, the log-price $x_t = \log X_t$ satisfies

$$dx_t = a_t^{\mathbb{S}}dt + \sigma_t^{\mathbb{S}}dW_t^{\mathbb{S}} + \int_{\mathbb{R}} z\widetilde{\mu}^{\mathbb{S}}(dt, dz), \quad x_0 = \log X_0,$$

where $a_t^{\mathbb{S}} = \alpha_t^{\mathbb{S}} - \frac{1}{2}(\sigma_t^{\mathbb{S}})^2 - \int_{\mathbb{R}}(e^z - 1 - z)\nu_t^{\mathbb{S}}(dz)$ with $\alpha_t^{\mathbb{S}} = \alpha_t$ if $\mathbb{S} = \mathbb{P}$ and $\alpha_t^{\mathbb{S}} = 0$ if $\mathbb{S} = \mathbb{Q}$. For Theorem 2, we assume the following conditions on ϕ and the coefficients of x:

Assumption 1. *The function $\phi(x)$ is nonnegative, symmetric, equal to unity at zero and twice continuously differentiable with all its derivatives (including ϕ itself) decreasing exponentially fast as $|x| \to \infty$.*

Assumption 2. *For both $\mathbb{S} = \mathbb{P}$ and $\mathbb{S} = \mathbb{Q}$, we have the following:*

1. *There is $r^{\mathbb{S}} \in (0, 2)$ such that for all $0 \le s \le t$,*

$$\mathbb{E}_s^{\mathbb{P}}\left(\int_{\mathbb{R}} |z|^{r^{\mathbb{S}}}\nu_t^{\mathbb{S}}(dz)\right) < \infty. \tag{A.14}$$

2. *There are positive $\mathcal{F}_t$-measurable random variables C_t, only depending on t, such that*

$$\sup_{0\le s\le t\le s+1} \mathbb{E}_s^{\mathbb{S}}(z_t^4) < C_s \quad \text{for } z \in \{a^{\mathbb{S}}, (\sigma^{\mathbb{S}})^{-4}\} \tag{A.15}$$

and

$$\sup_{0\le s\le t\le s+1} \mathbb{E}_s^{\mathbb{S}}((z_t - z_s)^2) \le C_s(t - s) \quad \text{for } z \in \{\sigma^{\mathbb{S}}, (\sigma^{\mathbb{S}})^2\} \tag{A.16}$$

and

$$\sup_{0\le s\le t\le s+1} \sup_{\psi} \mathbb{E}_s^{\mathbb{S}}\left(\left(\int_{\mathbb{R}} \psi_t(z)(|z|^{r^{\mathbb{S}}} \vee |z|^4)(\nu_t^{\mathbb{S}} - \nu_s^{\mathbb{S}})(dz)\right)^2\right) \le C_s(t - s), \tag{A.17}$$

where in the last line the second supremum is taken over all adapted processes $\psi = \psi_t(z)$ that are uniformly bounded by 1.

There are many choices of the truncation function ϕ that satisfy Assumption 1. For example, the one that we use in our application is $\phi(x) = e^{-x^2}$. Assumption 2 is a weak restriction on the asset price dynamics. This assumption is satisfied by most models considered in finance. In particular, we do not restrict the activity of the jumps (e.g., jumps of infinite variation are allowed).

Proof of Theorem 2. We compute the expected return of the hedged truncated volatility portfolio: By (5.2), (5.7) and (5.8), we have

$$\begin{aligned}\mathbb{E}_0^{\mathbb{P}}(\mathcal{R}_{0,T}) &= \frac{\mathbb{E}_0^{\mathbb{P}}(\mathcal{RG}_{0,T}) - \mathcal{C}_{0,T}(\eta_T)}{\mathcal{C}_{0,T}(\eta_T)} \\ &= \frac{TV_{0,T}^{\mathbb{P}}(\eta_T) - TV_{0,T}^{\mathbb{Q}}(\eta_T)}{TV_{0,T}^{\mathbb{Q}}(\eta_T)} - \frac{\int_0^T \mathbb{E}_0^{\mathbb{P}}(f_T'(X_t;X_0)\alpha_t X_t)dt}{TV_{0,T}^{\mathbb{Q}}(\eta_T)}.\end{aligned} \tag{A.18}$$

By Itô's formula, for both $\mathbb{S} = \mathbb{P}$ and $\mathbb{S} = \mathbb{Q}$,

$$\begin{aligned}TV_{0,T}^{\mathbb{S}}(\eta_T) &= \mathbb{E}_0^{\mathbb{S}}(\Phi(\eta_T, x_T)_T) \\ &= \mathbb{E}_0^{\mathbb{S}}\Bigg(\frac{1}{2}\int_0^T \theta''(\tfrac{\eta_T}{\sqrt{T}}x_t)(\sigma_t^{\mathbb{S}})^2 dt + \frac{\sqrt{T}}{\eta_T}\int_0^T \theta'(\tfrac{\eta_T}{\sqrt{T}}x_t)a_t^{\mathbb{S}}dt \\ &\quad + \int_0^T\int_{\mathbb{R}}\left[\Phi(\eta_T, x_t+z) - \Phi(\eta_T,x_t) - \frac{\partial}{\partial x}\Phi(\eta_T,x_t)_T z\right]\nu_t^{\mathbb{S}}(dz)dt\Bigg) \\ &= \mathbb{E}_0^{\mathbb{S}}\Bigg(\frac{1}{2}\int_0^T \theta''(\tfrac{\eta_T}{\sqrt{T}}x_t)(\sigma_t^{\mathbb{S}})^2 dt + \frac{\sqrt{T}}{\eta_T}\int_0^T \theta'(\tfrac{\eta_T}{\sqrt{T}}x_t)a_t^{\mathbb{S}}dt \\ &\quad + \int_0^T\int_{\mathbb{R}}\int_0^1 \theta''(\tfrac{\eta_T}{\sqrt{T}}(x_t+vz))(1-v)dv\, z^2\nu_t^{\mathbb{S}}(dz)dt\Bigg).\end{aligned} \tag{A.19}$$

Note that θ and its derivatives are bounded functions and $|\theta'(x)| \le C|x|$. Therefore, by (A.16), (A.17) and the fact that $\mathbb{E}_0^{\mathbb{S}}(|x_t/\sqrt{T}|) \le (\mathbb{E}_0^{\mathbb{S}}(|x_t/\sqrt{T}|^2))^{1/2} \le C$ for all $t \in [0,T]$, we have

$$\begin{aligned}\frac{1}{T}TV_{0,T}^{\mathbb{S}}(\eta_T) &= \frac{(\sigma_0^{\mathbb{S}})^2}{2T}\int_0^T \mathbb{E}_0^{\mathbb{S}}\left(\theta''(\tfrac{\eta_T}{\sqrt{T}}x_t)\right)dt \\ &\quad + \frac{1}{T}\int_0^T\int_{\mathbb{R}}\int_0^1 \mathbb{E}_0^{\mathbb{S}}\left(\theta''(\tfrac{\eta_T}{\sqrt{T}}(x_t+vz))\right)(1-v)dv\, z^2\nu_0^{\mathbb{S}}(dz)dt + O_p(\sqrt{T}).\end{aligned} \tag{A.20}$$

As $\theta''(0) = 2$, using the mean-value theorem, we further obtain

$$\frac{1}{T}TV_{0,T}^{\mathbb{S}}(\eta_T) = (\sigma_0^{\mathbb{S}})^2 + \int_{\mathbb{R}}\int_0^1 \theta''(\tfrac{\eta_T}{\sqrt{T}}vz)(1-v)dv\, z^2\nu_0^{\mathbb{S}}(dz) + O_p(\eta_T).$$

We have $\int_0^1 \theta''(\frac{\eta_T}{\sqrt{T}}vz)(1-v)dv = \phi(\frac{\eta_T}{\sqrt{T}}z)$ and $\phi(x) \to 0$ as $|x| \to \infty$, so the dominated convergence theorem implies that

$$\int_{\mathbb{R}}\int_0^1 \theta''(\tfrac{\eta_T}{\sqrt{T}}vz)(1-v)dv\, z^2\nu_0^{\mathbb{S}}(dz) = \int_{\mathbb{R}}\phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2\nu_0^{\mathbb{S}}(dz) = o_p(1).$$

In summary, we have shown that

$$\frac{1}{T}TV_{0,T}^{\mathbb{S}}(\eta_T) = (\sigma_0^{\mathbb{S}})^2 + o_p(1). \tag{A.21}$$

Since $\sigma_0^{\mathbb{P}} \neq \sigma_0^{\mathbb{Q}}$ by assumption, it follows that

$$\mathbb{E}_0^{\mathbb{P}}(\mathcal{R}_{0,T}) = \frac{(\sigma_0^{\mathbb{P}})^2 - (\sigma_0^{\mathbb{Q}})^2}{(\sigma_0^{\mathbb{Q}})^2} + o_p(1). \tag{A.22}$$

Next, we turn to the L_p-risk of the hedged volatility portfolio, which is given by

$$\left(\mathbb{E}_0^{\mathbb{P}}\left(|\mathcal{R}_{0,T}-\mathbb{E}_0^{\mathbb{P}}(\mathcal{R}_{0,T})|^p\right)\right)^{1/p}=\frac{\left(\mathbb{E}_0^{\mathbb{P}}\left(|\mathcal{RG}_{0,T}-\mathbb{E}_0^{\mathbb{P}}(\mathcal{RG}_{0,T})|^p\right)\right)^{1/p}}{TV_{0,T}^{\mathbb{Q}}(\eta_T)}. \tag{A.23}$$

By (A.12), we have $\Phi(\eta_T,x_{t-}+z)-\Phi(\eta_T,x_{t-})-\frac{\partial}{\partial x}\Phi(\eta_T,x_{t-})z=\int_0^1\theta''(\frac{\eta_T}{\sqrt{T}}(x_t+vz))(1-v)dv$. Thus, Itô's formula implies that

$$\begin{aligned}
&\mathcal{RG}_{0,T}-\mathbb{E}_0^{\mathbb{P}}(\mathcal{RG}_{0,T})\\
&\quad=\frac{1}{2}\int_0^T\left[\theta''(\tfrac{\eta_T}{\sqrt{T}}x_t)(\sigma_t^{\mathbb{P}})^2-\mathbb{E}_0^{\mathbb{P}}(\theta''(\tfrac{\eta_T}{\sqrt{T}}x_t)(\sigma_t^{\mathbb{P}})^2)\right]dt\\
&\qquad+\int_0^T\int_{\mathbb{R}}\left[\Phi(\eta_T,x_{t-}+z)-\Phi(\eta_T,x_{t-})-\frac{\partial}{\partial x}\Phi(\eta_T,x_{t-})z\right]\widetilde{\mu}^{\mathbb{P}}(dt,dz)\\
&\qquad+\int_0^T\left[\int_{\mathbb{R}}\int_0^1\theta''(\tfrac{\eta_T}{\sqrt{T}}(x_t+vz))(1-v)dv\,z^2\nu_t^{\mathbb{P}}(dz)\right.\\
&\qquad\qquad\left.-\mathbb{E}_0^{\mathbb{P}}\left(\int_{\mathbb{R}}\int_0^1\theta''(\tfrac{\eta_T}{\sqrt{T}}(x_t+vz))(1-v)dv\,z^2\nu_t^{\mathbb{P}}(dz)\right)\right]dt.
\end{aligned} \tag{A.24}$$

We denote the three terms on the right-hand side by $I_{0,T}^{(1)}(\eta_T)$, $I_{0,T}^{(2)}(\eta_T)$ and $I_{0,T}^{(3)}(\eta_T)$, respectively. By the mean-value theorem, $|\theta''(\frac{\eta_T}{\sqrt{T}}x_t)-\theta''(0)|\le C\frac{\eta_T}{\sqrt{T}}|x_t|$, so the first and the third term satisfy

$$\left(\mathbb{E}_0^{\mathbb{P}}\left(|I_{0,T}^{(1)}(\eta_T)|^p\right)\right)^{1/p}+\left(\mathbb{E}_0^{\mathbb{P}}\left(|I_{0,T}^{(3)}(\eta_T)|^p\right)\right)^{1/p}=O_p(T\eta_T). \tag{A.25}$$

To analyze the second term, we further split it into the sum of

$$\begin{aligned}
I_{0,T}^{(2,1)}(\eta_T)&=\int_0^T\int_{\mathbb{R}}\left[\Phi(\eta_T,x_{t-}+z)-\Phi(\eta_T,x_{t-})\right]\widetilde{\mu}^{\mathbb{P}}(dt,dz)\\
&=\frac{\sqrt{T}}{\eta_T}\int_0^T\int_{\mathbb{R}}\int_0^1\theta'(\tfrac{\eta_T}{\sqrt{T}}(x_{t-}+vz)dv\,z\widetilde{\mu}^{\mathbb{P}}(dt,dz),\\
I_{0,T}^{(2,2)}(\eta_T)&=-\int_0^T\int_{\mathbb{R}}\frac{\partial}{\partial x}\Phi(\eta_T,x_{t-})z\widetilde{\mu}^{\mathbb{P}}(dt,dz)=-\frac{\sqrt{T}}{\eta_T}\int_0^T\int_{\mathbb{R}}\theta'(\tfrac{\eta_T}{\sqrt{T}}x_{t-})z\widetilde{\mu}^{\mathbb{P}}(dt,dz).
\end{aligned} \tag{A.26}$$

By the Burkholder–Davis–Gundy inequality and (A.17), we have

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{P}}\left(|I_{0,T}^{(2,1)}(\eta_T)|^p\right)&\le\frac{CT^{p/2}}{\eta_T^p}\mathbb{E}_0^{\mathbb{P}}\left(\int_0^T\int_{\mathbb{R}}\left|\int_0^1\theta'(\tfrac{\eta_T}{\sqrt{T}}(x_t+vz))dv\right|^p|z|^p\nu_t^{\mathbb{P}}(dz)dt\right)\\
&\le\frac{CT^{p/2}}{\eta_T^p}\mathbb{E}_0^{\mathbb{P}}\left(\int_0^T\int_{\mathbb{R}}\left(\int_0^{\frac{\eta_T|z|}{\sqrt{T}}}|\theta'(\tfrac{\eta_T}{\sqrt{T}}x_t+\mathrm{sgn}(z)u)|du\frac{\sqrt{T}}{\eta_T|z|}\right)^p|z|^p\nu_t^{\mathbb{P}}(dz)dt\right)\\
&\le\frac{CT^{p/2}}{\eta_T^p}\mathbb{E}_0^{\mathbb{P}}\left(\int_0^T\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\theta'(u)|du\frac{\sqrt{T}}{\eta_T|z|}\right)^{p-r^{\mathbb{P}}}\left(\max_{u\in\mathbb{R}}|\theta'(u)|^{r^{\mathbb{P}}}\right)|z|^p\nu_t^{\mathbb{P}}(dz)dt\right)\\
&\le\frac{CT^{p-r^{\mathbb{P}}/2}}{\eta_T^{2p-r^{\mathbb{P}}}}\mathbb{E}_0^{\mathbb{P}}\left(\int_0^T\int_{\mathbb{R}}|z|^{r^{\mathbb{P}}}\nu_t^{\mathbb{P}}(dz)dt\right)=O_p(T^{1+p-r^{\mathbb{P}}/2}/\eta_T^{2p-r^{\mathbb{P}}})
\end{aligned}$$

for any $p \in (r^{\mathbb{P}}, 2]$. In particular, Jensen's inequality shows that for all $p \in [1,2]$,

$$\left(\mathbb{E}_0^{\mathbb{P}}\left(|I_{0,T}^{(2,1)}(\eta_T)|^p\right)\right)^{1/p} \leq \left(\mathbb{E}_0^{\mathbb{P}}\left(|I_{0,T}^{(2,1)}(\eta_T)|^2\right)\right)^{1/2} = O_p(T^{3/2-r^{\mathbb{P}}/4}/\eta_T^{2-r^{\mathbb{P}}/2}) = o_p(T). \quad \text{(A.27)}$$

Next, using the Burkholder–Davis–Gundy inequality, (A.17) and the bounds $|\theta'(x)| \leq C|x|$ and $\mathbb{E}_0^{\mathbb{P}}(|x_t/\sqrt{T}|^p) \leq C$, we derive the estimate

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{P}}\left(|I_{0,T}^{(2,2)}(\eta_T)|^p\right) &\leq \frac{CT^{p/2}}{\eta_T^p}\mathbb{E}_0^{\mathbb{P}}\left(\int_0^T\int_{\mathbb{R}}|\theta'(\tfrac{\eta_T}{\sqrt{T}}x_t)|^p|z|^p\nu_t^{\mathbb{P}}(dz)dt\right) \\
&\leq \frac{CT^{p/2}}{\eta_T^p}\mathbb{E}_0^{\mathbb{P}}\left(\int_0^T\int_{\mathbb{R}}|\theta'(\tfrac{\eta_T}{\sqrt{T}}x_t)|^p|z|^p\nu_0^{\mathbb{P}}(dz)dt\right) + O_p(T^{(p+3)/2}/\eta_T^p) \\
&\leq CT^{1+p/2}\int_{\mathbb{R}}|z|^p\nu_0^{\mathbb{P}}(dz) + O_p(T^{(p+3)/2}/\eta_T^p) = O_p(T^{1+p/2})
\end{aligned}$$

for all $p \in (r^{\mathbb{P}}, 2]$. Therefore, for any $p \in (r^{\mathbb{P}}, 2)$,

$$\left(\mathbb{E}_0^{\mathbb{P}}\left(|I_{0,T}^{(2,2)}(\eta_T)|^p\right)\right)^{1/p} = O_p(T^{1/p+1/2}) = o_p(T). \quad \text{(A.28)}$$

By Jensen's inequality, the final bound of the last line extends to all $p \in [1,2)$. For $p = 2$, we can use (A.12) to explicitly compute the limit as

$$\begin{aligned}
\mathbb{E}_0^{\mathbb{P}}\left(|I_{0,T}^{(2,2)}(\eta_T)|^2\right) &= \frac{T}{\eta_T^2}\mathbb{E}_0^{\mathbb{P}}\left(\int_0^T\int_{\mathbb{R}}|\theta'(\tfrac{\eta_T}{\sqrt{T}}x_t)|^2z^2\nu_0^{\mathbb{P}}(dz)dt\right) + O_p(T^{5/2}/\eta_T^2) \\
&= 2\int_0^T\int_{\mathbb{R}}\mathbb{E}_0^{\mathbb{P}}((x_t^c)^2)z^2\nu_0^{\mathbb{P}}(dz)dt + o_p(T^2) \\
&= T^2(\sigma_0^{\mathbb{P}})^2\int_{\mathbb{R}}z^2\nu_0^{\mathbb{P}}(dz) + o_p(T^2).
\end{aligned} \quad \text{(A.29)}$$

Combining (A.21)–(A.23), (A.25) and (A.27)–(A.29) shows the statements in (5.12) and (5.13) concerning the hedged truncated volatility portfolio.

Next, we analyze the generalized Sharpe ratio of the unhedged truncated volatility portfolio. We write

$$\int_0^T f_T'(X_t;X_0)dX_t = J_{0,T}^{(1)}(\eta_T) + J_{0,T}^{(2)}(\eta_T) + J_{0,T}^{(3)}(\eta_T), \quad \text{(A.30)}$$

where

$$\begin{aligned}
J_{0,T}^{(1)}(\eta_T) &= \int_0^T \frac{\sqrt{T}}{\eta_T}\theta'(\tfrac{\eta_T}{\sqrt{T}}x_t)\alpha_t dt, \qquad J_{0,T}^{(2)}(\eta_T) = \int_0^T \frac{\sqrt{T}}{\eta_T}\theta'(\tfrac{\eta_T}{\sqrt{T}}x_t)\sigma_t^{\mathbb{P}}dW_t, \\
J_{0,T}^{(3)}(\eta_T) &= \int_0^T\int_{\mathbb{R}} \frac{\sqrt{T}}{\eta_T}\theta'(\tfrac{\eta_T}{\sqrt{T}}x_{t-})z\widetilde{\mu}^{\mathbb{P}}(dt,dz).
\end{aligned}$$

As θ' is a bounded function and α has bounded second moments, we have for any $p \in [1,2]$ that

$$\left(\mathbb{E}_0^{\mathbb{P}}\left(|J_{0,T}^{(1)}(\eta_T)|^p\right)\right)^{1/p} \leq \left(\mathbb{E}_0^{\mathbb{P}}\left(|J_{0,T}^{(1)}(\eta_T)|^2\right)\right)^{1/2} \leq \frac{C\sqrt{T}}{\eta_T}\int_0^T\left(\mathbb{E}_0^{\mathbb{P}}(|\alpha_t|^2)\right)^{1/2}dt = O_p(T^{3/2}/\eta_T).$$

Since $J_{0,T}^{(3)}(\eta_T) = -I_{0,T}^{(2,2)}(\eta_T)$, both terms annihilate each other. Using assumption (A.16) in the first step, a similar calculation as in (A.29) for the second step, the mean value theorem for the third step (note that $\theta''(0) = 2$) and a well-known identity for the last step, we have

$$J_{0,T}^{(2)}(\eta_T) = \sigma_0^{\mathbb{P}}\int_0^T \frac{\sqrt{T}}{\eta_T}\theta'(\tfrac{\eta_T}{\sqrt{T}}x_t)dW_t + O_p(T^{3/2}/\eta_T) = \sigma_0^{\mathbb{P}}\int_0^T \frac{\sqrt{T}}{\eta_T}\theta'(\tfrac{\eta_T}{\sqrt{T}}\sigma_0^{\mathbb{P}}W_t)dW_t + o_p(T)$$

$$= 2(\sigma_0^{\mathbb{P}})^2 \int_0^T W_t dW_t + o_p(T) = (\sigma_0^{\mathbb{P}})^2 (W_T^2 - T) + o_p(T).$$

This shows that $(\mathbb{E}_0^{\mathbb{P}}(|J_{0,T}^{(2)}(\eta_T)|^p))^{1/p} = (\sigma_0^{\mathbb{P}})^2 m_p T + o_p(T)$, which yields the remaining statements in (5.12) and (5.13). □

Proof of Proposition 2. We further assume $x_0 = 0$ and $t = 0$. In the absence of truncation (i.e., if $\eta = 0$), we have

$$\mathcal{RG}_{0,T} = x_T^2 - 2\int_0^T x_{s-} dx_s = \int_0^T (\sigma_t^{\mathbb{P}})^2 dt + \int_0^T \int_{\mathbb{R}} z^2 \mu(dt, dz)$$

by Itô's formula. Therefore,

$$\begin{aligned} \mathbb{E}_0^{\mathbb{P}}(\mathcal{R}_{0,T}) &= \frac{\mathbb{E}_0^{\mathbb{P}}(\mathcal{RG}_{0,T})}{\mathcal{C}_{0,T}(0)} - 1 = \frac{[(\sigma_0^{\mathbb{P}})^2 + \int_{\mathbb{R}} z^2 \nu_0^{\mathbb{P}}(dz)]T + o_p(T)}{[(\sigma_0^{\mathbb{Q}})^2 + \int_{\mathbb{R}} z^2 \nu_0^{\mathbb{Q}}(dz)]T + o_p(T)} - 1 \\ &= \frac{(\sigma_0^{\mathbb{P}})^2 - (\sigma_0^{\mathbb{Q}})^2 + \int_{\mathbb{R}} z^2 (\nu_0^{\mathbb{P}}(dz) - \nu_0^{\mathbb{Q}}(dz))}{(\sigma_0^{\mathbb{Q}})^2 + \int_{\mathbb{R}} z^2 \nu_0^{\mathbb{Q}}(dz)} + o_p(1). \end{aligned} \tag{A.31}$$

At the same time, we have

$$\begin{aligned} \mathcal{RG}_{0,T} - \mathbb{E}_0^{\mathbb{P}}(\mathcal{RG}_{0,T}) = &\int_0^T \left[(\sigma_t^{\mathbb{P}})^2 - \mathbb{E}_0^{\mathbb{P}}((\sigma_t^{\mathbb{P}})^2)\right] dt + \int_0^T \int_{\mathbb{R}} z^2 \widetilde{\mu}^{\mathbb{P}}(dt, dz) \\ &+ \int_0^T \int_{\mathbb{R}} z^2 (\nu_t^{\mathbb{P}}(dz) - \mathbb{E}_0^{\mathbb{P}}(\nu_t^{\mathbb{P}}(dz)) dt, \end{aligned}$$

which implies that

$$\mathbb{E}_0^{\mathbb{P}}\left(|\mathcal{RG}_{0,T} - \mathbb{E}_0^{\mathbb{P}}(\mathcal{RG}_{0,T})|^2\right) = T\int_{\mathbb{R}} z^4 \nu_0^{\mathbb{P}}(dz) + o_p(T).$$

We obtain

$$\mathrm{Var}_0^{\mathbb{P}}(\mathcal{R}_{0,T}) = \frac{\mathbb{E}_0^{\mathbb{P}}\left(|\mathcal{RG}_{0,T} - \mathbb{E}_0^{\mathbb{P}}(\mathcal{RG}_{0,T})|^2\right)}{\mathcal{C}_{0,T}(0)^2} = \frac{\int_{\mathbb{R}} z^4 \nu_0^{\mathbb{P}}(dz)}{[(\sigma_0^{\mathbb{Q}})^2 + \int_{\mathbb{R}} z^2 \nu_0^{\mathbb{Q}}(dz)]^2 T} + o_p(T^{-1}). \tag{A.32}$$

Dividing (A.31) by the square root of (A.32) yields (5.15). The proof of (5.14) follows along the same lines except that now one has

$$\mathrm{Var}_0^{\mathbb{P}}(x_T^2) = \mathbb{E}_0^{\mathbb{P}}(x_T^4) - \left(\mathbb{E}_0^{\mathbb{P}}(x_T^2)\right)^2 = \mathbb{E}_0^{\mathbb{P}}(x_T^4) + o_p(T^2).$$

Using Itô's formula and (A.17), we obtain

$$\begin{aligned} \mathbb{E}_0^{\mathbb{P}}(x_T^4) &= \mathbb{E}_0^{\mathbb{P}}\left(\int_0^T \int_{\mathbb{R}} [(x_{t-} + z)^4 - x_{t-}^4 - 4x_{t-}^3 z]\mu(dt, dz)\right) + o_p(T) \\ &= \int_0^T \int_{\mathbb{R}} [6z^2 \mathbb{E}_0^{\mathbb{P}}(x_t^2) + 4z^3 \mathbb{E}_0^{\mathbb{P}}(x_t) + z^4] \nu_0^{\mathbb{P}}(dz) dt + o_p(T) \\ &= T\int_{\mathbb{R}} z^4 \nu_0^{\mathbb{P}}(dz) + o_p(1), \end{aligned}$$

which completes the proof of (5.14). □

Proof of Proposition 3. We can consider $t = 0$ without loss of generality. By (A.22), we have $\mathbb{E}_0^{\mathbb{P}}(\mathcal{R}_{0,T}) = o_p(1)$ if $\sigma_0^{\mathbb{P}} = \sigma_0^{\mathbb{Q}}$. At the same time, we have

$$\mathrm{Var}_0^{\mathbb{P}}(\mathcal{R}_{0,T})^{1/2} = \frac{\sigma_0^{\mathbb{P}}(\int_{\mathbb{R}} z^2 \nu_0^{\mathbb{P}}(dz))^{1/2}}{(\sigma_0^{\mathbb{Q}})^2} + o_p(1)$$

by combining (A.21), (A.23), (A.25) and (A.29). This shows $\mathcal{SR}_{0,T} \xrightarrow{\mathbb{P}} 0$. The proof of $SR_{0,T} \xrightarrow{\mathbb{P}} 0$ is analogous. □

A.4 Proof of Proposition 4

Proof of Propostion 4. We suppress the dependence of f_T on the second argument (which is always X_t) in this proof. By assumption, we have $\epsilon_{t+j\Delta_n} = \frac{1}{2}\mathrm{rba} X_{t+j\Delta_n}$, so

$$\begin{aligned}
\widehat{DTC}_{t,T} &= \frac{1}{2}\mathrm{rba}\left(|f_T'(X_{t+k_n\Delta_n})|X_{t+k_n\Delta_n} + \sum_{j=1}^{k_n}|f_T'(X_{t+j\Delta_n}) - f_T'(X_{t+(j-1)\Delta_n})|X_{t+j\Delta_n}\right) \\
&= \frac{1}{2}\mathrm{rba} X_t\left(|f_T'(X_{t+k_n\Delta_n})|X_{t+k_n\Delta_n} + \sum_{j=1}^{k_n}|f_T''(X_{t+(j-1)\Delta_n})||X_{t+j\Delta_n} - X_{t+(j-1)\Delta_n}|\right) \\
&\quad + o_p(k_n\sqrt{\Delta_n}).
\end{aligned}$$

Because $f_T'(X_{t+k_n\Delta_n}) = o_p(1)$ and $f_T''(X_{t+(j-1)\Delta_n}) = 2X_t^{-2} + o_p(1)$, we can use Theorem 7.2.2 in Jacod and Protter (2012) to show that

$$\begin{aligned}
\widehat{DTC}_{t,T} &= \frac{\mathrm{rba}}{X_t}\sum_{j=1}^{k_n}\mathbb{E}^{\mathbb{P}}_{t+(j-1)\Delta_n}\left[|X_{t+j\Delta_n} - X_{t+(j-1)\Delta_n}|\right] + o_p(k_n\sqrt{\Delta_n}) \\
&= \sqrt{\frac{2}{\pi}}\sigma_t^{\mathbb{P}} k_n\sqrt{\Delta_n}\mathrm{rba} + o_p(k_n\sqrt{\Delta_n}).
\end{aligned}$$

As $k_n \sim T/\Delta_n$, the proposition is proved. □

B Estimating Volatility Under $\mathbb{P}$ and $\mathbb{Q}$

In this section, we provide the volatility estimators that we use to determine if there is an arbitrage opportunity. These estimators are based on the characteristic function of the price increments, which, similarly to the truncated volatility, can separate small and frequent moves from the big and infrequent ones, see Jacod and Todorov (2014) and Todorov (2019). We start with $\widehat{V}_{t,T}^{\mathbb{Q}}$. It is given by

$$\widehat{V}_{t,T}^{\mathbb{Q}} = -\frac{2T}{\widehat{u}_t^2}\log\left|\widehat{\mathcal{L}}_{t,T}^{\mathbb{Q}}(\widehat{u}_t)\right|, \tag{B.1}$$

where the risk-neutral characteristic function of the price increments is inferred from the options via

$$\widehat{\mathcal{L}}_{t,T}^{\mathbb{Q}}(u) = 1 - (u^2 + iu)\sum_{j=2}^{N_t}\frac{e^{iu(\log(K_{t,j-1}) - \log(X_t))}}{K_{t,j-1}^2}O_{t,T}(K_{t,j-1})(K_{t,j} - K_{t,j-1}), \tag{B.2}$$

and the characteristic exponent is set to

$$\widehat{u}_t = \sqrt{\frac{-2\log(0.3)}{\widehat{TV}_{t,T}(0)}}. \tag{B.3}$$

This estimator was proposed by Todorov (2019) and the above construction is behind the spot volatility index reported by the CBOE options exchange. We denote the counterparts of $\widehat{V}^{\mathbb{Q}}_{t,T}$ formed using bid and ask quotes, respectively, with $\widehat{V}^{\mathbb{Q},\text{low}}_{t,T}$ and $\widehat{V}^{\mathbb{Q},\text{high}}_{t,T}$.

We continue next with $\widehat{V}^{\mathbb{P}}_{t,T}$. It is constructed as the return counterpart of the option-based estimator $\widehat{V}^{\mathbb{Q}}_{t,T}$. More specifically, we set

$$\widehat{V}^{\mathbb{P}}_{t,T} = -\frac{2k_n}{\widehat{u}_t^2}\log\left|\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t)\right|, \quad \widehat{\mathcal{L}}^{\mathbb{P}}_t(u) = \frac{1}{m_n}\sum_{j=1}^{m_n} e^{iu\Delta^n_{t,j}x}, \quad \Delta^n_{t,j}x = x_{t-(j-1)\Delta_n} - x_{t-j\Delta_n}, \tag{B.4}$$

where $k_n = \lfloor T/\Delta_n \rfloor$ and $m_n = 12\times 180$, which corresponds to a window for volatility estimation of 3 hours. The theoretical properties of $\widehat{V}^{\mathbb{P}}_{t,T}$ are investigated in Jacod and Todorov (2014). The estimator for the asymptotic standard deviation of $\widehat{V}^{\mathbb{P}}_{t,T}$ is given by

$$\widehat{\text{Astd}}(\widehat{V}^{\mathbb{P}}_{t,T}) = \frac{2k_n}{\widehat{u}_t^2\left|\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t)\right|^2}\sqrt{\Re(\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t))^2\widehat{\mathcal{V}}^{rr}_t + \Im(\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t))^2\widehat{\mathcal{V}}^{ii}_t + 2\Re(\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t))\Im(\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t))\widehat{\mathcal{V}}^{ri}_t}, \tag{B.5}$$

$$\widehat{\mathcal{V}}^{rr}_t = \frac{1}{m_n}\sum_{j=1}^{m_n}\left(\cos(\widehat{u}_t\Delta^n_{t,j}x) - \Re(\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t))\right)^2, \tag{B.6}$$

$$\widehat{\mathcal{V}}^{ii}_t = \frac{1}{m_n}\sum_{j=1}^{m_n}\left(\sin(\widehat{u}_t\Delta^n_{t,j}x) - \Im(\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t))\right)^2, \tag{B.7}$$

$$\widehat{\mathcal{V}}^{ri}_t = \frac{1}{m_n}\sum_{j=1}^{m_n}\left[\left(\cos(\widehat{u}_t\Delta^n_{t,j}x) - \Re(\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t))\right)\left(\sin(\widehat{u}_t\Delta^n_{t,j}x) - \Im(\widehat{\mathcal{L}}^{\mathbb{P}}_t(\widehat{u}_t))\right)\right]. \tag{B.8}$$

Finally, following Todorov and Zhang (2024), we use the time-of-day scaling factor

$$\widehat{\text{tod}}_{t,T} = \frac{m_n\Delta_n}{T}\frac{\sum_{j=1}^5 \widehat{TV}_{t_j,T}(0)}{\sum_{j=1}^5 \widehat{TV}_{t_j-m_n\Delta_n,T+m_n\Delta_n}(0) - \sum_{j=1}^5 \widehat{TV}_{t_j,T}(0)}, \tag{B.9}$$

where $t_5 \equiv t$ and $t_2 - t_1 = \cdots = t_5 - t_4$ equals one business day. The lower bound for it, $\widehat{\text{tod}}^{\text{low}}_{t,T}$, is formed by using the counterpart of $\widehat{TV}_{t_j,T}(0)$ based on the bid quotes and the counterpart of $\widehat{TV}_{t_j-m_n\Delta_n,T+m_n\Delta_n}(0)$ formed from the ask quotes. The upper bound for it, $\widehat{\text{tod}}^{\text{high}}_{t,T}$, is formed analogously.

C Local Arbitrage When Small Jumps are Different

In this section, we show how to construct local arbitrage portfolios if the physical and risk-neutral probability measures differ in the distribution of small asset price jumps. To this end, we set up a multi-period version of the volatility arbitrage portfolio from Section 5.3 and consider k_n time points $\tau_i = t + (i-1)\tau$, $i = 1,\ldots,k_n$, where $\tau \geq T$. For example, if T is the time to maturity of zero-date options, τ could be the length of one trading day. Now, at each time

point τ_i, we invest in the volatility arbitrage portfolio from Section 5 (without compounding, to simplify the analysis), so that the return of this strategy after k_n periods is

$$\mathcal{R}_T^{(k_n)} = \sum_{i=1}^{k_n} \mathcal{R}_{\tau_i,T}, \qquad \mathcal{R}_{\tau_i,T} = \frac{\mathcal{RG}_{\tau_i,T}}{\mathcal{C}_{\tau_i,T}(\eta_T)} - 1, \tag{C.1}$$

where

$$\mathcal{RG}_{\tau_i,T} = f_T(X_{\tau_i+T}; X_{\tau_i}) - \int_{\tau_i}^{\tau_i+T} f_T'(X_{t-}; X_{\tau_i}) dX_t \tag{C.2}$$

and

$$\mathcal{C}_{\tau_i,T}(\eta_T) = \int_0^\infty f_T''(K; X_{\tau_i}) O_{\tau_i,T}(K) dK. \tag{C.3}$$

While the number of periods k_n increases to infinity asymptotically, we assume that $k_n\tau \to 0$, which means that we are still looking to detect pseudo-arbitrage violations with $\mathcal{R}_T^{(k_n)}$.

We impose the following conditions on the truncation function ϕ and the asset dynamics:

Assumption 3. *The function $\phi(x)$ is nonnegative, symmetric, equal to unity at zero, decreasing in $|x|$ and six times continuously differentiable with all its derivatives (including ϕ itself) decreasing exponentially fast as $|x| \to \infty$.*

Assumption 4. *In addition to Assumption 2, assume that under both $\mathbb{S} = \mathbb{P}$ and $\mathbb{S} = \mathbb{Q}$,*

$$|\nu_t^{\mathbb{S}}(z : |z| > u) - c_t^{\mathbb{S}} u^{-\beta^{\mathbb{S}}} \mathbf{1}_{(0,1)}(u)| \leq c_t'^{\mathbb{S}}(u^{-r^{\mathbb{S}}} \wedge u^{-r'^{\mathbb{S}}}), \quad t \in [0, \mathcal{T}], \quad u > 0, \tag{C.4}$$

where $\beta^{\mathbb{S}} \in (1,2)$, $r^{\mathbb{S}} \in (0, \beta^{\mathbb{S}})$, $r'^{\mathbb{S}} \in (4, \infty)$ and $c^{\mathbb{S}}$ and $c'^{\mathbb{S}}$ are predictable processes such that (A.15) and (A.16) also hold for $z \in \{c^{\mathbb{S}}, c'^{\mathbb{S}}\}$.

Theorem 4. *Suppose that $\sigma^{\mathbb{P}} = \sigma^{\mathbb{Q}}$ for some $\mathbb{Q} \in \mathcal{Q}$ but $\nu_t^{\mathbb{P}}$ and $\nu_t^{\mathbb{Q}}$ differ at time $t \in [0, \mathcal{T})$ in such a way that either $\beta^{\mathbb{P}} \neq \beta^{\mathbb{Q}}$ or $c_t^{\mathbb{P}} \neq c_t^{\mathbb{Q}}$ for the jump measure parameters introduced in Assumption 4. Further suppose that we have a continuous record of the asset price X over the time intervals $[\tau_i, \tau_i + T]$ and a record of the out-of-the-money option prices $O_{\tau_i,T}(K)$ at time τ_i over a continuum of strikes $K \in (0, \infty)$ for all $i = 1, \ldots, k_n$, so that we can form the multi-period truncated volatility portfolio whose return is given by (C.1). Under Assumptions 3 and 4 detailed in Appendix A.3 and if $\eta_T \sim \eta T^{1/6}$ for some $\eta > 0$, $k_n \sim \theta T^{-\kappa}$ for some $\theta > 0$ and $\kappa \in [\frac{2}{3}, 1)$ and $\tau = O(T)$, then the generalized Sharpe ratio of the multi-period portfolio satisfies*

$$|\mathcal{GSR}_{t,T}^{(k_n)}(p)| = \frac{|\mathbb{E}_t^{\mathbb{P}}(\mathcal{R}_T^{(k_n)})|}{(\mathbb{E}_t^{\mathbb{P}}(|\mathcal{R}_T^{(k_n)} - \mathbb{E}_t^{\mathbb{P}}(\mathcal{R}_T^{(k_n)})|^p))^{1/p}} \xrightarrow{\mathbb{P}} \infty, \tag{C.5}$$

for all $p \in [1, \beta^{\mathbb{P}}]$, implying the existence of pseudo-arbitrage of any order $p \in (1, \beta^{\mathbb{P}}]$.

Remark 1. In the situation of Theorem 4, one can show that if $\sigma_t^{\mathbb{P}} \neq \sigma_t^{\mathbb{Q}}$, then (C.5) holds irrespective of whether $(\beta^{\mathbb{P}}, c_t^{\mathbb{P}}) \neq (\beta^{\mathbb{Q}}, c_t^{\mathbb{Q}})$ or not. Therefore, the multi-period hedged volatility portfolio is able to exploit pseudo-arbitrage violations no matter whether they are due to differences in volatility or small jumps.

Remark 2. Our proof of Theorem 4 shows that it suffices to choose $\kappa \in (1 - \frac{\beta}{3}, 1)$. The proof can also be refined to show that if ϕ satisfies $|\phi(x) - 1| \leq C_1 \exp(-C_2/|x|^2)$ for $|x| \leq 1$ and some $C_1, C_2 > 0$, then the assertion of Theorem 4 continues to hold for $\eta_T \sim \eta/\sqrt{\log T^{-1}}$ (as in Theorem 2) and $\kappa \in (1 - \frac{\beta}{2}, 1)$.

Proof of Theorem 4. Clearly, we have

$$\mathbb{E}_0^{\mathbb{P}}(\mathcal{R}_T^{(k_n)}) = \sum_{i=1}^{k_n} \mathbb{E}_0^{\mathbb{P}}\left(\mathbb{E}_{\tau_i}^{\mathbb{P}}(\mathcal{R}_{\tau_i,T})\right) \tag{C.6}$$

and, for $p \geq 1$,

$$\begin{aligned}\mathbb{E}_0^{\mathbb{P}}\left(\left|\mathcal{R}_T^{(k_n)} - \mathbb{E}_0^{\mathbb{P}}(\mathcal{R}_T^{(k_n)})\right|^p\right)^{1/p} &\leq \mathbb{E}_0^{\mathbb{P}}\left(\left|\sum_{i=1}^{k_n}\left[\mathcal{R}_{\tau_i,T} - \mathbb{E}_{\tau_i}^{\mathbb{P}}(\mathcal{R}_{\tau_i,T})\right]\right|^p\right)^{1/p} \\ &\quad + \mathbb{E}_0^{\mathbb{P}}\left(\left|\sum_{i=1}^{k_n}\left[\mathbb{E}_{\tau_i}^{\mathbb{P}}(\mathcal{R}_{\tau_i,T}) - \mathbb{E}_0^{\mathbb{P}}(\mathcal{R}_{\tau_i,T})\right]\right|^p\right)^{1/p}.\end{aligned} \tag{C.7}$$

We first derive an asymptotic expansion of $\mathbb{E}_{\tau_i}^{\mathbb{P}}(\mathcal{R}_{\tau_i,T})$ and note that (A.18)–(A.20) remain valid (with obvious modifications to accommodate a general τ_i). Writing $x_t^{(i)} = x_{\tau_i+t} - x_{\tau_i}$ and using Itô's formula as well as (A.16) and (A.17), we further find that

$$\begin{aligned}&\mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta''(\tfrac{\eta_T}{\sqrt{T}}(x_t^{(i)} + vz))\right) \\ &= \theta''(\tfrac{\eta_T}{\sqrt{T}}vz) + \frac{\eta_T^2(\sigma_{\tau_i}^{\mathbb{S}})^2}{2T}\int_{\tau_i}^{\tau_i+t}\mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i)} + vz))\right)ds \\ &\quad + \frac{\eta_T^2}{T}\iint_{\tau_i}^{\tau_i+t}\int_0^1 \mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i)} + vz + wy))\right)(1-w)dw\, y^2\nu_{\tau_i}^{\mathbb{S}}(dy)ds + O_p(\sqrt{T})\end{aligned} \tag{C.8}$$

for every $v \in [0,1]$. In the last line, and for the remainder of this proof, we use $O_p(g(T))$ to denote a random variable whose $\mathcal{F}_0$-conditional L_1-norm is $O(g(T))$. Clearly, we have

$$\mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i)} + vz + wy))\right) = \mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),m} + vz + wy))\right) + O_p(\eta_T\sqrt{T}), \tag{C.9}$$

where $x_t^{(i),m} = x_t^{(i)} - \int_{\tau_i}^{\tau_i+t} a_s^{\mathbb{S}}ds$ is the martingale component of $x^{(i)}$. Next, writing $x_t^{(i),c} = \int_{\tau_i}^{\tau_i+t}\sigma_s^{\mathbb{S}}dW_s^{\mathbb{S}}$ for the continuous martingale part and $x_t^{(i),d} = \int_{\tau_i}^{\tau_i+t}\int_{\mathbb{R}} z\widetilde{\mu}^{\mathbb{S}}(ds,dz)$ for the discontinuous martingale part of x, we represent $\mathbb{E}_{\tau_i}^{\mathbb{S}}(\theta^{(4)}(\frac{\eta_T}{\sqrt{T}}(x_s^{(i),m} + vz + wy))$ as a sum of three terms, which we analyze separately:

$$\begin{aligned}&\mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),m} + vz + wy))\right) \\ &= \mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),c} + vz + wy))\right) + \mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(5)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),c} + vz + wy))\frac{\eta_T x_s^{(i),d}}{\sqrt{T}}\right) \\ &\quad + \mathbb{E}_{\tau_i}^{\mathbb{S}}\Bigg(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),m} + vz + wy)) - \theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),c} + vz + wy)) \\ &\qquad\qquad - \theta^{(5)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),c} + vz + wy))\frac{\eta_T x_s^{(i),d}}{\sqrt{T}}\Bigg).\end{aligned} \tag{C.10}$$

By (A.16), the first term on the right-hand side of (C.10) satisfies

$$\mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),c} + vz + wy))\right) = \mathbb{E}_{\tau_i}^{\mathbb{S}}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma_{\tau_i}^{\mathbb{S}}W_s^{(i),\mathbb{S}} + vz + wy))\right) + O_p(\eta_T\sqrt{T}),$$

where $W_t^{(i),\mathbb{S}} = W^{\mathbb{S}}_{\tau_i+t} - W^{\mathbb{S}}_{\tau_i}$. The second term on the right-hand side of (C.10) satisfies

$$\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(5)}(\tfrac{\eta_T}{\sqrt{T}}(x_s^{(i),c}+vz+wy))\frac{\eta_T x_s^{(i),d}}{\sqrt{T}}\right) = \mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(5)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W_s^{(i),\mathbb{S}}+vz+wy))\frac{\eta_T x_s^{(i),d}}{\sqrt{T}}\right) + O_p(\eta_T\sqrt{T}).$$

By the Grigelionis representation theorem for Itô semimartingales (see e.g., Theorem 2.1.2 in Jacod and Protter (2012)), one can assume that $x^{(i),d}$ is a compensated Poisson integral where the Poisson measure is independent of $W^{(i),\mathbb{S}}$. So, if we condition on $W^{(i),\mathbb{S}}$, we realize that the conditional expectation on the right-hand side of the last display is identically zero. We turn to the third term on the right-hand side of (C.10). By Assumption 3, both $\theta^{(5)}$ and $\theta^{(6)}$ are bounded functions, so it is easily established using Taylor's theorem that the absolute value of this term is bounded by

$$C\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\left|\frac{\eta_T x_s^{(i),d}}{\sqrt{T}}\right| \wedge \left|\frac{\eta_T x_s^{(i),d}}{\sqrt{T}}\right|^2\right) \leq C\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\left|\frac{\eta_T x_s^{(i),d}}{\sqrt{T}}\right|^p\right)$$

for any choice of $p \in [1,2]$. Applying the Burkholder–Davis–Gundy inequality in the first step and Jensen's inequality and the elementary inequality $(a+b)^p \leq a^p + b^p$ for $p \in [0,1]$ and $a, b \geq 0$ in the second step, we derive the estimates

$$\begin{aligned}
\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\left|\frac{\eta_T x_s^{(i),d}}{\sqrt{T}}\right|^p\right) &\leq \frac{C\eta_T^p}{T^{p/2}}\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\left(\int_{\tau_i}^{\tau_i+s}\int_{\mathbb{R}} z^2\mu(dr,dz)\right)^{p/2}\right) \\
&\leq \frac{C\eta_T^p}{T^{p/2}}\left[\left(\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\int_{\tau_i}^{\tau_i+s}\int_{\mathbb{R}}|z|^2\mathbf{1}_{\{|z|\leq T^{1/\beta^{\mathbb{S}}}\}}\nu_r^{\mathbb{S}}(dz)dr\right)\right)^{p/2}\right. \\
&\quad \left. + \mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\int_{\tau_i}^{\tau_i+s}\int_{\mathbb{R}}|z|^p\mathbf{1}_{\{|z|>T^{1/\beta^{\mathbb{S}}}\}}\nu_r^{\mathbb{S}}(dz)dr\right)\right].
\end{aligned}$$

Using (C.4), we have for $p = \beta^{\mathbb{S}}$ that

$$\begin{aligned}
\int_{\mathbb{R}}|z|^{\beta^{\mathbb{S}}}\mathbf{1}_{\{|z|>T^{1/\beta^{\mathbb{S}}}\}}\nu_r^{\mathbb{S}}(dz) &= \beta^{\mathbb{S}}\int_{\mathbb{R}}\int_0^{|z|}u^{\beta^{\mathbb{S}}-1}du\,\mathbf{1}_{\{|z|>T^{1/\beta^{\mathbb{S}}}\}}\nu_r^{\mathbb{S}}(dz) \\
&= \beta^{\mathbb{S}}\int_0^{\infty}u^{\beta^{\mathbb{S}}-1}\nu_r^{\mathbb{S}}(z:|z|>u\vee T^{1/\beta^{\mathbb{S}}})du \\
&= T\nu_r^{\mathbb{S}}(z:|z|>T^{1/\beta^{\mathbb{S}}}) + \beta^{\mathbb{S}}\int_{T^{1/\beta^{\mathbb{S}}}}^{\infty}u^{\beta^{\mathbb{S}}-1}\nu_r^{\mathbb{S}}(z:|z|>u)du \\
&= c_r^{\mathbb{S}}(1+\log T^{-1}) + O_p(1).
\end{aligned} \tag{C.11}$$

Similarly, we have

$$\begin{aligned}
\int_{\mathbb{R}}|z|^2\mathbf{1}_{\{|z|\leq T^{1/\beta^{\mathbb{S}}}\}}\nu_r^{\mathbb{S}}(dz) &= 2\int_0^{T^{1/\beta^{\mathbb{S}}}}u\nu_r^{\mathbb{S}}(z:u\leq|z|\leq T^{1/\beta})du \\
&= 2c_r^{\mathbb{S}}\int_0^{T^{1/\beta^{\mathbb{S}}}}u(u^{-\beta^{\mathbb{S}}}-T^{-1})du + O_p(T^{2/r^{\mathbb{S}}-1}) \\
&= \frac{2\beta^{\mathbb{S}}c_r^{\mathbb{S}}}{4-2\beta^{\mathbb{S}}}T^{2/\beta^{\mathbb{S}}-1} + O_p(T^{2/r^{\mathbb{S}}-1}).
\end{aligned} \tag{C.12}$$

We have thus proved the bound

$$\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\left|\frac{\eta_T x^{(i),d}_s}{\sqrt{T}}\right|^{\beta^{\mathbb{S}}}\right) = O_p(T^{1-\beta^{\mathbb{S}}/2}\eta_T^{\beta^{\mathbb{S}}}\log T^{-1}),$$

which in turn shows that

$$\begin{aligned}\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(x^{(i)}_s + vz + wy))\right) &= \mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W^{(i),\mathbb{S}}_s + vz + wy))\right)\\ &\quad + O_p(T^{1-\beta^{\mathbb{S}}/2}\eta_T^{\beta^{\mathbb{S}}}\log T^{-1}).\end{aligned} \tag{C.13}$$

Inserting this result in (C.8), we obtain

$$\begin{aligned}&\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta''(\tfrac{\eta_T}{\sqrt{T}}(x^{(i)}_t + vz))\right)\\ &\quad= \theta''(\tfrac{\eta_T}{\sqrt{T}}vz) + \frac{\eta_T^2(\sigma^{\mathbb{S}}_{\tau_i})^2}{2T}\int_{\tau_i}^{\tau_i+t}\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W^{(i),\mathbb{S}}_s + vz))\right)ds\\ &\qquad+ \frac{\eta_T^2}{T}\int_{\tau_i}^{\tau_i+t}\int_{\mathbb{R}}\int_0^1\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W^{(i),\mathbb{S}}_s + vz + wy))\right)(1-w)dw\,y^2\nu^{\mathbb{S}}_{\tau_i}(dy)ds\\ &\qquad+ O_p(T^{1-\beta^{\mathbb{S}}/2}\eta_T^{\beta^{\mathbb{S}}+2}\log T^{-1}).\end{aligned}$$

By (A.20), it follows that

$$\begin{aligned}\frac{1}{T}TV^{\mathbb{S}}_{\tau_i,T}(\eta_T) &= \frac{(\sigma^{\mathbb{S}}_{\tau_i})^2}{2T}\int_{\tau_i}^{\tau_i+T}\left(2 + \frac{\eta_T^2(\sigma^{\mathbb{S}}_{\tau_i})^2}{2T}\int_{\tau_i}^{\tau_i+t}\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W^{(i),\mathbb{S}}_s))\right)ds\right)dt\\ &\quad+ \frac{\eta_T^2(\sigma^{\mathbb{S}}_{\tau_i})^2}{2T^2}\int_{\tau_i}^{\tau_i+T}\int_{\tau_i}^{\tau_i+t}\int_{\mathbb{R}}\int_0^1\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W^{(i),\mathbb{S}}_s + wy))\right)(1-w)dw\,y^2\nu^{\mathbb{S}}_{\tau_i}(dy)dsdt\\ &\quad+ \frac{1}{T}\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\int_0^1\theta''(\tfrac{\eta_T}{\sqrt{T}}vz)(1-v)dv\,z^2\nu^{\mathbb{S}}_{\tau_i}(dz)dt\\ &\quad+ \frac{\eta_T^2(\sigma^{\mathbb{S}}_{\tau_i})^2}{2T^2}\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\int_0^1\int_{\tau_i}^{\tau_i+t}\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W^{(i),\mathbb{S}}_s + vz))\right)ds\,(1-v)dv\,z^2\nu^{\mathbb{S}}_{\tau_i}(dz)dt\\ &\quad+ \frac{\eta_T^2}{T^2}\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\int_0^1\int_{\tau_i}^{\tau_i+t}\int_{\mathbb{R}}\int_0^1\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W^{(i),\mathbb{S}}_s + vz + wy))\right)\\ &\qquad\qquad\times(1-w)dw\,y^2\nu^{\mathbb{S}}_{\tau_i}(dy)ds\,(1-v)dv\,z^2\nu^{\mathbb{S}}_{\tau_i}(dz)dt\\ &\quad+ O_p(T^{1-\beta^{\mathbb{S}}/2}\eta_T^{\beta^{\mathbb{S}}+2}\log T^{-1}).\end{aligned}$$

We can now determine the leading order terms of $\frac{1}{T}(TV^{\mathbb{P}}_{\tau_i,T}(\eta_T) - \frac{1}{T}TV^{\mathbb{Q}}_{\tau_i,T}(\eta_T)$. Because $\sigma^{\mathbb{P}}_{\tau_i} = \sigma^{\mathbb{Q}}_{\tau_i}$ by assumption, the first term on the right-hand side of the previous display cancels out. The second and the fourth term are identical and satisfy

$$\begin{aligned}&\left|\frac{\eta_T^2(\sigma^{\mathbb{S}}_{\tau_i})^2}{2T^2}\int_{\tau_i}^{\tau_i+T}\int_{\tau_i}^{\tau_i+t}\int_{\mathbb{R}}\int_0^1\mathbb{E}^{\mathbb{S}}_{\tau_i}\left(\theta^{(4)}(\tfrac{\eta_T}{\sqrt{T}}(\sigma^{\mathbb{S}}_{\tau_i}W^{(i),\mathbb{S}}_s + wy))\right)(1-w)dw\,y^2\nu^{\mathbb{S}}_{\tau_i}(dy)dsdt\right|\\ &\quad\le \frac{\eta_T^2(\sigma^{\mathbb{S}}_{\tau_i})^2}{4}\max_{x\in\mathbb{R}}|\theta^{(4)}(x)|\int_{\mathbb{R}}y^2\mathbf{1}_{\{|y|\le\sqrt{T}/\eta_T\}}\nu^{\mathbb{S}}_{\tau_i}(dy)\\ &\qquad+ \frac{\eta_T(\sigma^{\mathbb{S}}_{\tau_i})^2T^{1/2}}{4}\int_{\mathbb{R}}|\theta^{(4)}(x)|dx\int_{\mathbb{R}}|y|\mathbf{1}_{\{|y>\sqrt{T}/\eta_T\}}\nu^{\mathbb{S}}_{\tau_i}(dy)\end{aligned}$$

$$= O_p(T^{1-\beta^{\mathbb{S}}/2}\eta_T^{\beta^{\mathbb{S}}}).$$

Using similar arguments, one can derive the same bound for the fifth term on the right-hand side of the penultimate display. The leading term is the third one. Because $\int_0^1 \theta''(\frac{\eta_T}{\sqrt{T}}vz)(1-v)dv = \phi(\frac{\eta_T}{\sqrt{T}}z)$ and ϕ is symmetric and exponentially decreasing by assumption, it can be written as

$$\begin{aligned}
\int_{\mathbb{R}} \phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2\nu^{\mathbb{S}}_{\tau_i}(dz) &= \int_{\mathbb{R}}\int_0^{|z|} \tfrac{d}{dz}[\phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2]\Big|_{z=y} dy\nu^{\mathbb{S}}_{\tau_i}(dz) \\
&= \int_0^\infty \nu^{\mathbb{S}}_{\tau_i}(z:|z|\geq y)\tfrac{d}{dz}[\phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2]\Big|_{z=y} dy \\
&= c^{\mathbb{S}}_{\tau_i}\int_0^1 y^{-\beta^{\mathbb{S}}}\tfrac{d}{dz}[\phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2]\Big|_{z=y} dy \\
&\quad + \int_1^\infty \nu^{\mathbb{S}}_{\tau_i}(z:|z|\geq y)\tfrac{d}{dz}[\phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2]\Big|_{z=y} dy \\
&\quad + \int_0^1 \left(\nu^{\mathbb{S}}_{\tau_i}(z:|z|\geq y) - c^{\mathbb{S}}_{\tau_i}y^{-\beta^{\mathbb{S}}}\right)\tfrac{d}{dz}[\phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2]\Big|_{z=y} dy \\
&= c^{\mathbb{S}}_{\tau_i}\phi(\tfrac{\eta_T}{\sqrt{T}}) + \beta^{\mathbb{S}}c^{\mathbb{S}}_{\tau_i}\int_0^1 \phi(\tfrac{\eta_T}{\sqrt{T}}z)z^{1-\beta^{\mathbb{S}}}dz \\
&\quad + O_p\left(\int_0^\infty (y^{-r^{\mathbb{S}}}\wedge y^{-r'^{\mathbb{S}}})\left|\phi'(\tfrac{\eta_T}{\sqrt{T}}y)\tfrac{\eta_T}{\sqrt{T}}y^2 + 2\phi(\tfrac{\eta_T}{\sqrt{T}}y)y\right|dy\right) \\
&= \frac{\beta^{\mathbb{S}}c^{\mathbb{S}}_{\tau_i}T^{1-\beta^{\mathbb{S}}/2}}{\eta_T^{2-\beta^{\mathbb{S}}}}\int_0^\infty \phi(w)w^{1-\beta^{\mathbb{S}}}dw + O_p(T^{1-r^{\mathbb{S}}/2}/\eta_T^{2-r^{\mathbb{S}}}).
\end{aligned} \tag{C.14}$$

Therefore, using the notation $\beta = \beta^{\mathbb{P}}\vee\beta^{\mathbb{Q}}$ and $c_{\tau_i} = c^{\mathbb{P}}_{\tau_i}\mathbf{1}_{\{\beta^{\mathbb{P}}=\beta\}} - c^{\mathbb{Q}}_{\tau_i}\mathbf{1}_{\{\beta^{\mathbb{Q}}=\beta\}}$ and recalling (A.21), we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}_{\tau_i}(\mathcal{R}_{\tau_i,T}) &= \frac{1}{(\sigma^{\mathbb{Q}}_{\tau_i})^2}\int_{\mathbb{R}} \phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2(\nu^{\mathbb{P}}_{\tau_i} - \nu^{\mathbb{Q}}_{\tau_i})(dz) + o_p(T^{1-\beta/2}/\eta_T^{2-\beta}) \\
&= \frac{T^{1-\beta/2}}{\eta_T^{2-\beta}}\frac{\beta c_{\tau_i}}{(\sigma^{\mathbb{Q}}_{\tau_i})^2}\int_0^\infty \phi(w)w^{1-\beta}dw + o_p(T^{1-\beta/2}/\eta_T^{2-\beta}).
\end{aligned} \tag{C.15}$$

By (C.6) and (A.17), it follows that

$$\frac{1}{k_n}\mathbb{E}^{\mathbb{P}}_0(\mathcal{R}^{(k_n)}_T) = \frac{T^{1-\beta/2}}{\eta_T^{2-\beta}}\frac{\beta c_0}{(\sigma^{\mathbb{Q}}_0)^2}\int_0^\infty \phi(w)w^{1-\beta}dw + o_p(T^{1-\beta/2}/\eta_T^{2-\beta}). \tag{C.16}$$

Next, we estimate the L_p-risk of the multi-period hedged volatility portfolio. By (A.16) and (C.15), the second term on the right-hand side of (C.7) satisfies

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}_0\left(\left|\sum_{i=1}^{k_n}\left[\mathbb{E}^{\mathbb{P}}_{\tau_i}(\mathcal{R}_{\tau_i,T}) - \mathbb{E}^{\mathbb{P}}_0(\mathcal{R}_{\tau_i,T})\right]\right|^p\right)^{1/p} &= O_p(k_n(k_nT)^{1/2}T^{1-\beta/2}/\eta_T^{2-\beta}) + o_p(k_nT^{1-\beta/2}/\eta_T^{2-\beta}) \\
&= o_p(k_nT^{1-\beta/2}/\eta_T^{2-\beta}).
\end{aligned} \tag{C.17}$$

The first term on the right-hand side of (C.7) is a martingale sum, so we can apply the Burkholder–Davis–Gundy inequality to obtain

$$\mathbb{E}^{\mathbb{P}}_0\left(\left|\sum_{i=1}^{k_n}\left[\mathcal{R}_{\tau_i,T} - \mathbb{E}^{\mathbb{P}}_{\tau_i}(\mathcal{R}_{\tau_i,T})\right]\right|^p\right)^{1/p} \leq C\left(\sum_{i=1}^{k_n}\mathbb{E}^{\mathbb{P}}_0\left(\left|\mathcal{R}_{\tau_i,T} - \mathbb{E}^{\mathbb{P}}_{\tau_i}(\mathcal{R}_{\tau_i,T})\right|^p\right)\right)^{1/p}. \tag{C.18}$$

Clearly, we have

$$\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(\left|\mathcal{R}_{\tau_i,T}-\mathbb{E}^{\mathbb{P}}_{\tau_i}(\mathcal{R}_{\tau_i,T})\right|^p\right)\right)^{1/p}=\frac{\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(|\mathcal{RG}_{\tau_i,T}-\mathbb{E}^{\mathbb{P}}_{\tau_i}(\mathcal{RG}_{\tau_i,T})|^p\right)\right)^{1/p}}{TV^{\mathbb{Q}}_{\tau_i,T}(\eta_T)}. \tag{C.19}$$

By Itô's formula and (A.12),

$$\begin{aligned}
&\mathcal{RG}_{\tau_i,T}-\mathbb{E}^{\mathbb{P}}_{\tau_i}(\mathcal{RG}_{\tau_i,T})\\
&=\frac{1}{2}\int_{\tau_i}^{\tau_i+T}\left[\theta''(\tfrac{\eta_T}{\sqrt{T}}x_t^{(i)})(\sigma_t^{\mathbb{P}})^2-\mathbb{E}^{\mathbb{P}}_{\tau_i}(\theta''(\tfrac{\eta_T}{\sqrt{T}}x_t^{(i)})(\sigma_t^{\mathbb{P}})^2)\right]dt\\
&\quad+\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\left[\Phi(\eta_T,x_t^{(i)}+z)_T-\Phi(\eta_T,x_t^{(i)})_T\right]\widetilde{\mu}^{\mathbb{P}}(dt,dz)\\
&\quad+\int_{\tau_j}^{\tau_i+T}\int_{\mathbb{R}}\left[-\frac{\partial}{\partial x}\Phi(\eta_T,x_t^{(i)})_T z\right]\widetilde{\mu}^{\mathbb{P}}(dt,dz)\\
&\quad+\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\int_0^1\left[\theta''(\tfrac{\eta_T}{\sqrt{T}}(x_t^{(i)}+vz))-\mathbb{E}^{\mathbb{P}}_{\tau_i}(\theta''(\tfrac{\eta_T}{\sqrt{T}}(x_t^{(i)}+vz)))\right](1-v)dv\,z^2\nu_t^{\mathbb{P}}(dz)dt.
\end{aligned}$$

We denote the four terms on the right-hand side by $I^{(1)}_{\tau_i,T}(\eta_T)$, $I^{(2)}_{\tau_i,T}(\eta_T)$, $I^{(3)}_{\tau_i,T}(\eta_T)$ and $I^{(4)}_{\tau_i,T}(\eta_T)$.

Because

$$\begin{aligned}
\Phi(\eta_T,x_t^{(i)}+z)_T-\Phi(\eta_T,x_t^{(i)})_T&=\frac{\sqrt{T}}{\eta_T}z\int_0^1\theta'(\tfrac{\eta_T}{\sqrt{T}}(x_t^{(i)}+vz)dv\\
&=\frac{\sqrt{T}}{\eta_T}z\int_0^1\theta'(\tfrac{\eta_T}{\sqrt{T}}vz)dv+x_t^{(i)}z\int_0^1\int_0^1\theta''(\tfrac{\eta_T}{\sqrt{T}}(vz+ux_t^{(i)})dudv\\
&=\phi(\tfrac{\eta_T}{\sqrt{T}}z)z^2+x_t^{(i)}z\int_0^1\int_0^1\theta''(\tfrac{\eta_T}{\sqrt{T}}(vz+ux_t^{(i)})dudv,
\end{aligned}$$

the Burkholder–Davis–Gundy inequality implies that for $p\in[1,2]$,

$$\begin{aligned}
&\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(|I^{(2)}_{\tau_i,T}(\eta_T)|^p\right)\right)^{1/p}\\
&\le C\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\phi(\tfrac{\eta_T}{\sqrt{T}}z)^p|z|^{2p}\nu_t^{\mathbb{P}}(dz)dt\right)\right)^{1/p}\\
&\quad+C\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\left|\int_0^1\int_0^1\theta''(\tfrac{\eta_T}{\sqrt{T}}(vz+ux_t^{(i)}))dudv\,zx_t^{(i)}\right|^p\mathbf{1}_{\{|z|>T^{1/\beta^{\mathbb{P}}}\}}\nu_t^{\mathbb{P}}(dz)dt\right)\right)^{1/p}\\
&\quad+C\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\left|\int_0^1\int_0^1\theta''(\tfrac{\eta_T}{\sqrt{T}}(vz+ux_t^{(i)}))dudv\,zx_t^{(i)}\right|^2\mathbf{1}_{\{|z|\le T^{1/\beta^{\mathbb{P}}}\}}\nu_t^{\mathbb{P}}(dz)dt\right)\right)^{1/2}.
\end{aligned}$$

Analogously to (C.15), one can show that

$$\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\phi(\tfrac{\eta_T}{\sqrt{T}}z)^p|z|^{2p}\nu_t^{\mathbb{P}}(dz)dt\right)\right)^{1/p}=O_p(T^{1+(1-\beta^{\mathbb{P}}/2)/p}/\eta_T^{2-\beta^{\mathbb{P}}/p})$$

for all $p\in[1,2]$. Regarding the other two terms term, we first note that a refinement of (C.11) and (C.12) leads to the bounds

$$\int_{\mathbb{R}}|z|^{\beta^{\mathbb{P}}}\mathbf{1}_{\{|z|>T^{1/\beta^{\mathbb{P}}}\}}\nu_t^{\mathbb{P}}(dz)\le c_t^{\mathbb{P}}(1+\log T^{-1})+c_t'^{\mathbb{P}}\left(\frac{\beta^{\mathbb{P}}}{\beta^{\mathbb{P}}-r^{\mathbb{P}}}+\frac{\beta^{\mathbb{P}}}{r'^{\mathbb{P}}-\beta^{\mathbb{P}}}+T^{1-r'^{\mathbb{P}}/\beta^{\mathbb{P}}}\right)$$

$$\leq 2c_t^{\mathbb{P}} \log T^{-1} + 2\beta^{\mathbb{P}} \left(\frac{1}{\beta^{\mathbb{P}} - r^{\mathbb{P}}} + \frac{1}{r'^{\mathbb{P}} - \beta^{\mathbb{P}}} \right) c_t'^{\mathbb{P}},$$

$$\int_{\mathbb{R}} |z|^2 \mathbf{1}_{\{|z| \leq T^{1/\beta^{\mathbb{P}}}\}} \nu_t^{\mathbb{P}}(dz) = \frac{2\beta^{\mathbb{P}} c_t^{\mathbb{P}}}{4 - 2\beta^{\mathbb{P}}} T^{2/\beta^{\mathbb{P}} - 1} + \frac{2c_t'^{\mathbb{P}}}{2 - r^{\mathbb{P}}} T^{(2-r^{\mathbb{P}})/\beta^{\mathbb{P}}}.$$

Therefore, for $p = \beta^{\mathbb{P}}$,

$$\begin{aligned}
&\left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(\int_{\tau_i}^{\tau_i + T} \int_{\mathbb{R}} \left| \int_0^1 \int_0^1 \theta''\left(\tfrac{\eta_T}{\sqrt{T}}(vz + ux_t^{(i)})\right) du dv \, z x_t^{(i)} \right|^{\beta^{\mathbb{P}}} \mathbf{1}_{\{|z| > T^{1/\beta^{\mathbb{P}}}\}} \nu_t^{\mathbb{P}}(dz) dt \right) \right)^{1/\beta^{\mathbb{P}}} \\
&\leq C \left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(\int_{\tau_i}^{\tau_i + T} |x_t^{(i)}|^{\beta^{\mathbb{P}}} \int_{\mathbb{R}} |z|^{\beta^{\mathbb{P}}} \mathbf{1}_{\{|z| > T^{1/\beta^{\mathbb{P}}}\}} \nu_t^{\mathbb{P}}(dz) dt \right) \right)^{1/\beta^{\mathbb{P}}} \\
&\leq C \left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(\int_{\tau_i}^{\tau_i + T} |x_t^{(i)}|^{\beta^{\mathbb{P}}} (c_t^{\mathbb{P}} \log T^{-1} + c_t'^{\mathbb{P}}) dt \right) \right)^{1/\beta^{\mathbb{P}}} \\
&\leq C \Bigg[\left((c_{\tau_i}^{\mathbb{P}} \log T^{-1} + c_{\tau_i}'^{\mathbb{P}}) \int_{\tau_i}^{\tau_i + T} \mathbb{E}_{\tau_i}^{\mathbb{P}}(|x_t^{(i)}|^{\beta^{\mathbb{P}}}) dt \right)^{1/\beta^{\mathbb{P}}} \\
&\quad + \left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(\int_{\tau_i}^{\tau_i + T} |x_t^{(i)}|^{\beta^{\mathbb{P}}} \left[(c_t^{\mathbb{P}} - c_{\tau_i}^{\mathbb{P}}) \log T^{-1} + (c_t'^{\mathbb{P}} - c_{\tau_i}'^{\mathbb{P}}) \right] dt \right) \right)^{1/\beta^{\mathbb{P}}} \Bigg].
\end{aligned}$$

As $\mathbb{E}_{\tau_i}^{\mathbb{P}}(|x_t^{(i)}|^{\beta^{\mathbb{P}}}) \leq C_{\tau_i} t^{\beta^{\mathbb{P}}/2}$ and $\mathbb{E}_{\tau_i}^{\mathbb{P}}(|x_t^{(i)}|^{2\beta^{\mathbb{P}}}) \leq C_{\tau_i} t$, applying the Cauchy–Schwarz inequality to the second term and using (A.16) for $z = c^{\mathbb{P}}$ and $z = c'^{\mathbb{P}}$, we obtain

$$\begin{aligned}
&\left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(\int_{\tau_i}^{\tau_i + T} \int_{\mathbb{R}} \left| \int_0^1 \int_0^1 \theta''\left(\tfrac{\eta_T}{\sqrt{T}}(vz + ux_t^{(i)})\right) du dv \, z x_t^{(i)} \right|^{\beta^{\mathbb{P}}} \mathbf{1}_{\{|z| > T^{1/\beta^{\mathbb{P}}}\}} \nu_t^{\mathbb{P}}(dz) dt \right) \right)^{1/\beta^{\mathbb{P}}} \\
&= O_p(T^{1/2 + 1/\beta^{\mathbb{P}}} \log T^{-1}).
\end{aligned}$$

A similar argument shows that

$$\begin{aligned}
&\left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(\int_{\tau_i}^{\tau_i + T} \int_{\mathbb{R}} \left| \int_0^1 \int_0^1 \theta''\left(\tfrac{\eta_T}{\sqrt{T}}(vz + ux_t^{(i)})\right) du dv \, z x_t^{(i)} \right|^{2} \mathbf{1}_{\{|z| \leq T^{1/\beta^{\mathbb{P}}}\}} \nu_t^{\mathbb{P}}(dz) dt \right) \right)^{1/2} \\
&= O_p(T^{1/2 + 1/\beta^{\mathbb{P}}}).
\end{aligned}$$

Altogether, we have shown that

$$\left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(|I_{\tau_i, T}^{(2)}(\eta_T)|^{\beta^{\mathbb{P}}} \right) \right)^{1/\beta^{\mathbb{P}}} = O_p(T^{1/2 + 1/\beta^{\mathbb{P}}}/\eta_T + T^{1/2 + 1/\beta^{\mathbb{P}}} \log T^{-1}). \tag{C.20}$$

Because $\frac{\partial}{\partial x} \Phi(\eta_T, x_t^{(i)})_T = \frac{\sqrt{T}}{\eta_T} \theta'(\frac{\eta_T}{\sqrt{T}} x_t^{(i)})$, a similar calculation shows that

$$\left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(|I_{\tau_i, T}^{(3)}(\eta_T)|^{\beta^{\mathbb{P}}} \right) \right)^{1/\beta^{\mathbb{P}}} = O_p(T^{1/2 + 1/\beta^{\mathbb{P}}} \log T^{-1}). \tag{C.21}$$

Next, we turn to $I_{\tau_i, T}^{(1)}(\eta_T)$ and $I_{\tau_i, T}^{(4)}(\eta_T)$. Because θ'' is bounded, (A.16) for $(\sigma^{\mathbb{P}})^2$ implies that for $p \in [1, 2]$,

$$\left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(|I_{\tau_i, T}^{(1)}(\eta_T)|^{p} \right) \right)^{1/p} = \frac{(\sigma_{\tau_i}^{\mathbb{P}})^2}{2} \left(\mathbb{E}_{\tau_i}^{\mathbb{P}} \left(\left| \frac{1}{2} \int_{\tau_i}^{\tau_i + T} \left[\theta''(\tfrac{\eta_T}{\sqrt{T}} x_t^{(i)}) - \mathbb{E}_{\tau_i}^{\mathbb{P}}(\theta''(\tfrac{\eta_T}{\sqrt{T}} x_t^{(i)})) \right] dt \right|^{p} \right) \right)^{1/p}$$

$$+ O_p(T^{3/2}).$$

Since $\theta'''(0) = 0$ and $\theta^{(4)}$ is bounded, Taylor's theorem implies that $|\theta''(\frac{\eta_T}{\sqrt{T}}x_t^{(i)}) - 2| \leq C\frac{\eta_T^2}{T}(x_t^{(i)})^2$. As $\mathbb{E}^{\mathbb{P}}_{\tau_i}(|x_t^{(i)}|^{2p}) \leq C_{\tau_i} t$, it follows that

$$\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(|I^{(1)}_{\tau_i,T}(\eta_T)|^p\right)\right)^{1/p} = O_p(T^{1/p}\eta_T^2). \tag{C.22}$$

With the same arguments, one can show that

$$\begin{aligned}
&\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(|I^{(4)}_{\tau_i,T}(\eta_T)|^p\right)\right)^{1/p} \\
&\leq C\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(\left|\int_{\tau_i}^{\tau_i+T}\int_{\mathbb{R}}\int_0^1 \theta'''(\tfrac{\eta_T}{\sqrt{T}}vz)\frac{\eta_T}{\sqrt{T}}\left[x_t^{(i)} - \mathbb{E}^{\mathbb{P}}_{\tau_i}(x_t^{(i)})\right](1-v)dv\, z^2\nu^{\mathbb{P}}_{\tau_i}(dz)dt\right|^p\right)\right)^{1/p} \\
&\quad + O_p(T^{1/p}\eta_T^2).
\end{aligned}$$

Since $\int_0^1 \theta'''(\frac{\eta_T}{\sqrt{T}}vz)(1-v)dv = \phi'(\frac{\eta_T}{\sqrt{T}}z) + 2(\phi(\frac{\eta_T}{\sqrt{T}}z) - 1)\frac{\sqrt{T}}{\eta_T z}$, a straightforward computation along the lines of (C.14) yields

$$\int_{\mathbb{R}}\int_0^1 \theta'''(\tfrac{\eta_T}{\sqrt{T}}vz)(1-v)dv\, z^2\nu^{\mathbb{P}}_{\tau_i}(dz) = O_p(T^{1-\beta^{\mathbb{P}}/2}/\eta_T^{2-\beta^{\mathbb{P}}}).$$

As a result,

$$\left(\mathbb{E}^{\mathbb{P}}_{\tau_i}\left(|I^{(4)}_{\tau_i,T}(\eta_T)|^p\right)\right)^{1/p} = O_p(T^{2-\beta^{\mathbb{P}}/2}\eta_T^{\beta^{\mathbb{P}}-1} + T^{1/p}\eta_T^2). \tag{C.23}$$

Combining (C.20)–(C.23) and recalling (A.21), (C.18) and (C.19), we conclude that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}_0\left(\left|\sum_{i=1}^{k_n}\left[\mathcal{R}_{\tau_i,T} - \mathbb{E}^{\mathbb{P}}_{\tau_i}(\mathcal{R}_{\tau_i,T})\right]\right|^{\beta^{\mathbb{P}}}\right)^{1/\beta^{\mathbb{P}}} \\
&= O_p(k_n^{1/\beta^{\mathbb{P}}}[T^{1/\beta^{\mathbb{P}}-1/2}(1/\eta_T + \log T^{-1}) + T^{-(1-1/\beta^{\mathbb{P}})}\eta_T^2 + T^{1-\beta^{\mathbb{P}}/2}\eta_T^{\beta^{\mathbb{P}}-1}]).
\end{aligned} \tag{C.24}$$

In conjunction with (C.17), we have shown that for $p \in [1, \beta^{\mathbb{P}}]$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}_0\left(\left|\mathcal{R}_T^{(k_n)} - \mathbb{E}^{\mathbb{P}}_0(\mathcal{R}_T^{(k_n)})\right|^p\right)^{1/p} &\leq \mathbb{E}^{\mathbb{P}}_0\left(\left|\mathcal{R}_T^{(k_n)} - \mathbb{E}^{\mathbb{P}}_0(\mathcal{R}_T^{(k_n)})\right|^{\beta^{\mathbb{P}}}\right)^{1/\beta^{\mathbb{P}}} \\
&= O_p(k_n^{1/\beta^{\mathbb{P}}}T^{1/\beta^{\mathbb{P}}-1/2}(1/\eta_T + \log T^{-1}) + k_n^{1/\beta^{\mathbb{P}}}T^{-(1-1/\beta^{\mathbb{P}})}\eta_T^2 \\
&\quad + k_n^{1/\beta^{\mathbb{P}}}T^{1-\beta^{\mathbb{P}}/2}\eta_T^{\beta^{\mathbb{P}}-1}) + o_p(k_nT^{1-\beta/2}/\eta_T^{2-\beta}).
\end{aligned}$$

Up to now, all the derivations are valid for general k_n and η_T (as long as $\sqrt{T}/\eta_T \to 0$). By specializing the last display to the case where $\eta_T \sim \eta T^{1/6}$ for some $\eta > 0$, we obtain

$$\mathbb{E}^{\mathbb{P}}_0\left(\left|\mathcal{R}_T^{(k_n)} - \mathbb{E}^{\mathbb{P}}_0(\mathcal{R}_T^{(k_n)})\right|^p\right)^{1/p} = O_p(k_n^{1/\beta^{\mathbb{P}}}T^{1/\beta^{\mathbb{P}}-1/2}/\eta_T) + o_p(k_nT^{1-\beta/2}/\eta_T^{2-\beta}).$$

Further taking $k_n \sim \theta T^{-\kappa}$ for some $\theta > 0$ and $\kappa \in [\frac{2}{3}, 1)$, we have

$$\mathbb{E}^{\mathbb{P}}_0\left(\left|\mathcal{R}_T^{(k_n)} - \mathbb{E}^{\mathbb{P}}_0(\mathcal{R}_T^{(k_n)})\right|^p\right)^{1/p} = o_p(k_nT^{1-\beta/2}/\eta_T^{2-\beta}),$$

which by (C.16) shows that the L_p-risk of the multi-period hedged volatility portfolio is much smaller asymptotically than the absolute value of the expected gain. □