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Incapacitation and the Optimal Use of Prison, Parole, and Probation

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Abstract: In this article we derive the mix of criminal sanctions—choosing among prison, parole, and probation—that maximizes social welfare when the sole purpose of the sanctions is to prevent potential offenders from committing crimes, that is, to incapacitate them. We assume that prison has a higher incapacitative benefit and a higher cost per unit of time than parole and probation and that the propensity of potential offenders to commit crimes declines with their age. We demonstrate that under plausible assumptions an optimal sentence uses out-of-prison supervision—in the form of parole or probation—notwithstanding that its incapacitative benefits are lower. We also show that it may be optimal to employ a prison term even if prison has a higher cost per crime prevented than does parole and probation because of prison's higher incapacitative power. Moreover, due to the aging-out-of-crime effect, whenever a sentence includes both a prison term and a parole term, the prison term should be used first. Additionally, we examine how optimal sentences depend on the level of harm from crime and on the age of the offender.

Key words: crime; imprisonment; parole; probation; prison costs; incapacitation; sanctions

JEL codes: H23; K14; K42

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1. Introduction

Virtually all popular and scholarly discussions of the incapacitation rationale for the criminal justice system focus on the sanction of imprisonment[1] for an obvious reason: imprisoned criminals are physically unable to commit crimes outside of prison. However, convicted offenders who are subject to state supervision outside of prison—those on parole or probation—also can be prevented from committing some crimes even though they are not physically isolated from society in the way that they would be if in prison. For example, an individual convicted of a domestic violence crime might be required to wear a GPS-tracking ankle bracelet while on parole and be prohibited from being closer than one mile from his ex-spouse's residence. If he violates this geographic restriction, a signal can be sent to the police officer on patrol nearest to him who may be able to detain him before he reaches his ex-spouse's home. Or a drug addict convicted of burglarizing homes might be required to attend a residential drug treatment program while on probation, during which he would have less freedom of movement.

As technology for monitoring and controlling potential offenders outside of prison improves, new opportunities will arise to incapacitate criminals through out-of-prison supervision. For instance, if facial or fingerprint recognition devices were sufficiently inexpensive to be required in all automobiles, individuals convicted of excessive speeding or drunk driving could be prevented from repeating those crimes without their having to be put in prison. In principle, debilitating, Taser-like shock devices could be incorporated into ankle

[1] For example, nine of the first ten articles returned by a Google search of the New York Times website for the words "incapacitation" and "criminal" mentioned prison as a means of incapacitation and none mentioned non-carceral sanctions in this regard. Similarly, all of the first ten works returned by a Google Scholar search for the word "incapacitation" discussed prison as a method of incapacitation and only one discussed non-carceral sanctions from this perspective. These searches were undertaken on November 26, 2024.

bracelets and triggered remotely based on information transmitted from wearable body cameras with microphones and motion sensors that detect physical aggression.[2] It seems inevitable that technological advances will increasingly favor the achievement of incapacitation goals through forms of out-of-prison supervision rather than through imprisonment.[3]

Although twice as many individuals are subject to parole or probationary supervision in the United States than are incarcerated,[4] no one has systematically considered the role that parole or probation can play in furthering the incapacitation goal of the criminal justice system.[5] The contribution of our article is to provide a theoretical analysis of the merits of parole and probation as means of achieving desirable incapacitation, together with, or as an alternative to, imprisonment. We undertake this task by deriving the mix of criminal sanctions—chosen from among prison, parole, and probation—that maximizes social welfare, defined here to be the crime prevention benefits from incapacitation less the costs of the sanctions employed to effectuate incapacitation.

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[2] Bagaric, Hunter, and Wolf (2018, pp. 98-110) describe the feasibility of this type of system, involving electronic monitoring of offenders' locations, computer surveillance of offenders' actions, and remote immobilization of offenders. They conclude (p. 110) that such a system "can achieve all of the benefits of conventional imprisonment . . ."

[3] While the incapacitative power of out-of-prison supervision undoubtedly will increase as a result of technological improvements, there is virtually no scope for the incapacitative power of prisons to increase because prisons already prevent essentially all crimes that might have been committed by prisoners if free. The conclusion in the text is consistent with the views of Bagaric, Hunter, and Wolf (2018), who advocate a broad set of proposals under the rubric of "technological incarceration" that would "see technology . . . pervasively incorporated into the punishment of criminals [outside of prison] and result in the closure of nearly all prisons in the United States" (p. 73).

[4] In 2022, there were 1.83 million individuals incarcerated in local, state, and federal correctional facilities in the United States, and 3.67 million individuals subject to parole or probationary supervision. See Buehler and Kluckow (2024, p. 5, Table 1). The pattern of having more offenders subject to out-of-prison supervision than in prison is also common in other developed countries. For instance, in 2021 in England and Wales the prison population totaled 78,789, while the number of persons under the supervision of probation agencies was 155,679. In France, these numbers were 62,991 and 175,306, respectively. See Fair and Walmsley (2021, pp. 12-13) and Aebi and Hashimoto (2022, pp. 31-32, table 8, column 1.0).

[5] See the last paragraph of this introduction for support for this claim.

Though parole and probation are both forms of out-of-prison state supervision of offenders, they differ in that parole follows a prison term, whereas probation is a stand-alone sanction. While the degree of state supervision need not be the same under parole and probation, nor their costs, we will treat parole and probation as equivalent sanctions per unit of time.[6] Thus, their difference in our analysis will be purely semantic—when a period of out-of-prison supervision is combined with a prison term, we will refer to this period as parole, but when such a period is used by itself, we will refer to it as probation.

In our model, prison accomplishes greater crime reduction through incapacitation and generates higher costs for the state per unit of time than do parole and probation. However, the cost of prison per crime prevented can be lower or higher than the cost of parole and probation per crime prevented.[7] An important assumption in our analysis, one that is widely supported empirically,[8] is that the propensity of potential offenders to commit crime declines with their age, so that the incapacitative benefit of a sanction declines with the age of the offender subject to it. For simplicity, we do not consider discounting by the state of the costs and incapacitative benefits of sanctions.

We demonstrate that under plausible assumptions an optimal sentence will use out-of-prison supervision—in the form of parole or probation—notwithstanding that its incapacitative benefits are lower than those of prison. This result occurs when out-of-prison supervision is more cost effective than prison in the sense of having a lower cost per crime prevented.

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[6] Although in practice parole is not usually referred to as a *sanction*, as prison and probation are, we will, for economy of language, refer to it as such here.

[7] We present some evidence relevant to assessing the cost effectiveness of prison relative to that of parole and probation in comment (a) in Section 6 below.

[8] See comment (b) in Section 6 below.

We also show that because of prison's greater incapacitative power it may be optimal to employ a prison term even if prison has a higher cost per crime prevented than do parole and probation. At each time in a sentence, the sanction that maximizes the net benefit from incapacitation—the crime reduction benefit of the sanction less the cost of the sanction—is optimal. Even if prison has a lower net benefit *per crime prevented* than out-of-prison supervision, it can produce higher *total* net benefits because it prevents a greater number of crimes.

Additionally, we prove that whenever an optimal sentence includes both a prison term and a parole term, the prison term should be used first. This result derives from the aging-out-of-crime assumption. Since prison prevents more crimes per unit of time than does parole, using prison first, when the offender is younger and more criminally active, and parole later, when the offender is older and commits fewer crimes, prevents more total crimes than using parole before prison.

Section 2 provides a two-period numerical example to illustrate some of the key intuitions behind our results. Section 3 presents the basic formal model. Section 4 derives the optimal sentence and shows how it depends on the relative cost-effectiveness of prison and out-of-prison supervision and on the level of harm from crime. Section 5 considers how the optimal sentence varies with the age of the offender. Section 6 concludes with several comments. Proofs of the results are contained in the Appendix.

Although this is the first article to analyze the optimal use of prison, parole, and probation with respect to the incapacitation goal of the criminal justice system, there are several articles of related interest. Shavell (1987; 2015) and Miceli (2010) derive the optimal sentence to

promote incapacitation, but they do so only considering the sanction of imprisonment.[9] Miceli (1994), Garoupa (1997), and Polinsky (2015) examine parole together with imprisonment, but they do so only in a model of deterrence. Polinsky and Riskind (2019) study the same sanctions as in the present article, but also do so exclusively from the perspective of deterrence.

2. Numerical Example

Consider the following example in which an individual lives for two periods, for simplicity each treated as one year long. Suppose that he would commit 10 crimes in the first period if he were free and, due to the aging-out-of-crime effect, 5 crimes in the second period if he were free. Prison would prevent all of these crimes, but out-of-prison supervision is assumed to be only 20 percent as effective, preventing 2 crimes in the first period and 1 crime in the second period. The harm from each crime is $5,000. The cost of prison is $35,000 per year, while the cost of out-of-prison supervision is $3,000 per year.

As noted in the introduction, social welfare is defined to be the crime prevention benefits from incapacitation less the costs of the sanctions employed to accomplish incapacitation. When discussing a particular sanction, we will refer to this difference as the net benefit of the sanction.

In the first period the cost per crime prevented is $3,500 if prison is used (= $35,000/10) and $1,500 if out-of-prison supervision is used (= $3,000/2). Thus, out-of-prison supervision is 2.3 times more cost effective in crime prevention than is imprisonment (= $3,500/$1,500). Yet, due to prison's greater incapacitative power, it results in higher net benefits in the first period and is therefore the superior sanction to employ in that period. Prison generates $50,000 of

[9] The Miceli article and the later Shavell article also evaluate the effects of prison sanctions on deterrence. An early contribution by Ehrlich (1981) first studied the effects of imprisonment on incapacitation; he too did not consider parole or probation.

incapacitative benefits in the first period (= 10 x $5,000) at a cost of $35,000, and therefore has a net benefit of $15,000. Out-of-prison supervision generates $10,000 of incapacitative benefits in the first period (= 2 x $5,000) at a cost of $3,000, resulting in a lower net benefit of $7,000.

The only difference in the second period is that the incapacitative power of both sanctions has declined by half due to the aging-out-of-crime effect. The cost per crime prevented therefore doubles for both sanctions, and out-of-prison supervision remains 2.3 times more cost effective than imprisonment. But now the net benefit of out-of-prison supervision exceeds that of imprisonment, making out-of-prison supervision the superior sanction in the second period. Specifically, the net benefit of imprisonment in the second period is -$10,000 (= (5 x $5,000) - $35,000), while the net benefit of out-of-prison supervision in the second period is $2,000 (= (1 x $5,000) - $3,000).

Although prison remains incapacitatively more powerful in the second period than out-of-prison supervision, its incapacitative advantage has declined considerably due to the aging-out-of-crime effect. In the first period, imprisonment prevented 8 more crimes than out-of-prison supervision (=10 – 2), while in the second period it only prevents 4 more crimes than out-of-prison supervision (= 5 – 1). Now that prison's incapacitative advantage has declined this much, out-of-prison supervision's superior cost effectiveness dominates prison's incapacitative advantage and makes out-of-prison supervision the optimal sanction in the second period.

This example illustrates how the optimal sentence to promote the incapacitation objective of the criminal justice system may consist of a term of imprisonment followed by a term of parole. In the formal analysis to follow, we show how the optimal sentence depends on which sanction is more cost effective and on how the level of harm and the rate of aging out of crime, among other factors, determine which sanction has the higher net benefit over time. The

preceding example also illustrates the result, mentioned in the introduction, that whenever imprisonment and out-of-prison supervision are used together, imprisonment should be used first due to the aging-out-of crime effect.

Additionally, we demonstrate that the optimal sentence for an older offender is the "tail end" of the optimal sentence for a younger offender. In the example above, an older offender would be someone who lives for just one period and whose propensity to commit crime would be the same as that of the younger offender in the second period of the younger offender's life. Hence, the optimal sentence for the older offender would be solely a term of out-of-prison supervision, that is, a term of probation.

We now turn to describing the formal model and deriving optimal sentences.

3. The State's Problem

An offender's sentence begins at time $t = 0$ and we assume for analytical convenience that individuals live forever.[10] At any time $t \geq 0$, the state can impose a prison sanction or an out-of-prison supervision sanction or no sanction at all. We will refer to the last choice as imposing the *null sanction*. The prison sanction prevents a greater number of crimes and costs the state more per unit of time than does the out-of-prison supervision sanction. The null sanction prevents no crimes and costs the state nothing. The three possible sanctions will be numbered in decreasing order of incapacitation—denoting prison with 1, out-of-prison supervision with 2, and the null sanction with 3.

We assume that the number of crimes the offender would commit per unit of time were he at liberty declines exponentially with his age and that the fraction of crimes prevented by each

sanction remains constant over the offender's lifetime. Hence, the numbers of crimes prevented by prison and out-of-prison supervision decline exponentially with the age of the offender. Let

n_i = number of crimes prevented by sanction i per unit of time at $t = 0$, where

$n_1 > n_2 > n_3 = 0$; and

α = rate at which the number of crimes that a free offender would commit declines exponentially with his age; $\alpha > 0$.

We will refer to α as the rate at which individuals "age out of crime."[11] Thus, the number of crimes prevented by sanction i at time $t \geq 0$ is $n_i e^{-\alpha t}$. For i and $j \in \{1, 2, 3\}$ such that $n_i > n_j$, we say that sanction i is *stronger* than sanction j.

The benefit of incapacitation is the harm from crime avoided as a result of incapacitation. Let

h = harm per crime; $h > 0$.[12]

Hence, the incapacitation benefit of sanction i at time $t \geq 0$ is $n_i e^{-\alpha t} h$. At any $t \geq 0$, prison produces the greatest benefit from incapacitation, out-of-prison supervision a smaller but positive benefit from incapacitation, and the null sanction no benefit from incapacitation.

Finally, let

c_i = cost to the state of sanction i per unit of time, where $c_1 > c_2 > c_3 = 0$;[13] and

$b_i(t)$ = *net benefit* of sanction i at time $t \geq 0 = n_i e^{-\alpha t} h - c_i$.

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[10] Since the optimal terms of prison and out-of-prison supervision will be seen to be finite, this assumption does not affect our results.

[11] We discuss some evidence about the rate at which individuals age out of crime in comment (b) in Section 6.

[12] The harm per crime h can be interpreted as being net of the gain to the criminal. See comment (d) in Section 6.

[13] The cost of a sanction c_i can be interpreted as including both the cost borne by the state to administer the sanction and the disutility borne by the offender subject to the sanction. See comment (d) in Section 6.

A *sentence* is a function $\sigma(t)$ mapping each non-negative time to a sanction, denoted by the corresponding element of the set $\{1, 2, 3\}$, such that the value of $\sigma(t)$ changes only a finite number of times and sanctions are imposed for strictly positive lengths of time. Let Σ be the set of sentences. A *term* of sentence $\sigma(t)$ is an interval of time such that $\sigma(t)$ is constant throughout the interior of the interval but changes at the lower bound of the interval if the lower bound is positive and at the upper bound of the interval if the interval is finite. Terms are thus the longest non-intersecting periods of time over which a sentence imposes a constant punishment, and this definition corresponds with the usual meaning of, for example, "a term of imprisonment." By definition, every sentence $\sigma(t)$ partitions the set of non-negative times into a finite number of terms. We specify a sentence by listing its terms in order from first to last, stating the length of each term and the sanction imposed.[14] If the sentence concludes with an infinite term of the null sanction following a term of imprisonment or out-of-prison supervision, then for brevity, we omit mentioning the final term.

The state's problem is to choose a sentence $\sigma(t)$ from the set of sentences Σ that maximizes the benefits of incapacitation less the costs of incapacitation. Thus, using the notation $n(i) = n_i$ and $c(i) = c_i$ for $i \in \{1, 2, 3\}$, the state's problem is:

$$\max_{\sigma(\cdot) \in \Sigma} W = \int_0^\infty [n(\sigma(t))e^{-\alpha t}h - c(\sigma(t))]dt. \tag{1}$$

The state's problem as just described does not impose any restrictions on the number of terms of each sanction (aside from ruling out an infinite number of terms). We prove in the Appendix that an optimal sentence employs a prison term and an out-of-prison supervision term

[14] Sentences with the same list of terms can only differ at one or more times at which the sentences switch from one sanction to another. Since these switching times are a set of measure zero, the sanctions imposed at these times do not affect our analysis. Accordingly, we do not distinguish between sentences that have the same list of terms.

at most once each and that any term of prison or out-of-prison supervision in an optimal sentence is finite. Accordingly, in discussing our results in the following sections, we will refer to a single prison term and a single out-of-prison supervision term. Let

s_1 = length of the prison term; $s_1 \geq 0$; and

s_2 = length of the out-of-prison supervision term; $s_2 \geq 0$.

As noted in the introduction, if an out-of-prison supervision term is employed in combination with a prison term it will be referred to as a parole term, whereas if it is used alone, it will be referred to as a probation term.

4. Optimal Sentences

In characterizing optimal sentences to promote incapacitation, we will say that for i and $j \in \{1, 2\}$, sanction i is more *cost-effective* than sanction j if sanction i has a lower cost per crime prevented than sanction j does, that is $c_i/n_i < c_j/n_j$.

Proposition 1: The optimal sentence to promote incapacitation is as follows:

(a) If prison is at least as cost-effective as out-of-prison supervision ($c_1/n_1 \leq c_2/n_2$), then

(i) if the level of harm is relatively low ($h \leq c_1/n_1$), the optimal sentence is an infinite term of the null sanction; and

(ii) if the level of harm is relatively high ($h > c_1/n_1$), the optimal sentence is a prison term ($s_1 = (1/\alpha)\ln(n_1 h/c_1)$).

(b) If out-of-prison supervision is more cost-effective than prison ($c_2/n_2 < c_1/n_1$), then

(i) if the level of harm is relatively low ($h \leq c_2/n_2$), the optimal sentence is an infinite term of the null sanction;

(ii) if the level of harm is in an intermediate range ($c_2/n_2 < h \leq (c_1 - c_2)/(n_1 - n_2)$), the

optimal sentence is a probation term ($s_2 = (1/\alpha)\ln(n_2 h/c_2)$); and

(iii) if the level of harm is relatively high ($h > (c_1 - c_2)/(n_1 - n_2)$), the optimal sentence is a prison term ($s_1 = (1/\alpha)\ln((n_1 - n_2)h/(c_1 - c_2))$), followed by a parole term ($s_2 = (1/\alpha)\ln((n_2/c_2)(c_1 - c_2)/(n_1 - n_2))$).

Proof: See the Appendix.

Comments: (i) We demonstrate in the Appendix that a sentence is optimal if and only if at each time $t \geq 0$ the sentence uses the sanction $i \in \{1, 2, 3\}$ that maximizes the benefit of incapacitation less the cost of the sanction, that is, its net benefit $b_i(t) = n_i e^{-\alpha t} h - c_i$.[15] For prison and out-of-prison supervision, the net benefit declines with the age of the offender because of his aging out of crime. Thus, if the net benefits of these sanctions are initially zero or negative—that is, if $b_i(0) = n_i h - c_i \leq 0$ for $i \in \{1, 2\}$—they will be weakly dominated by the null sanction at $t = 0$ and strictly dominated at all $t > 0$; an infinite term of the null sanction will therefore be optimal. The preceding observations explain part (a)(i) of the proposition since there $h \leq c_1/n_1 \leq c_2/n_2$. They similarly explain part (b)(i), where $h \leq c_2/n_2 < c_1/n_1$.

(ii) An optimal sentence that uses multiple sanctions will always use them in order from strongest to weakest. This result is due to individuals' aging out of crime. To understand this claim, suppose instead that a sentence were to use an out-of-prison supervision term before using an imprisonment term. Then reversing the order of the two terms would not change the total cost of the sentence but would increase the sentence's total benefits of incapacitation for the following reason. Since offenders age out of crime, using the imprisonment sanction earlier will increase the number of crimes it prevents, while using the out-of-prison supervision sanction

[15] We show in the Appendix that it is possible for multiple sanctions to maximize net benefit at a given time. Because the set of times at which multiple sanctions maximize net benefit has measure zero, this possibility does not affect our analysis and we ignore it.

later will decrease the number of crimes it prevents. But because imprisonment is stronger than out-of-prison supervision, the first effect will dominate the second effect, resulting in a net increase in crimes prevented.[16] Thus, the stronger sanction should be used first. This explains why in part (b)(iii) the prison term precedes the parole term.

(iii) Rewriting the net benefit functions for prison and out-of-prison supervision in the form $b_i(t) = n_i(e^{-\alpha t}h - c_i/n_i)$ for $i \in \{1, 2\}$ makes clear that the choice between prison and out-of-prison supervision depends on the incapacitative strength (determined by n_i) and the cost-effectiveness (determined by c_i/n_i) of the two sanctions. Since $n_1 > n_2$, prison generates greater incapacitation benefits at every time than does out-of-prison supervision. Thus, prison will always produce greater net benefit than out-of-prison supervision if prison is at least as cost-effective as out-of-prison supervision ($c_1/n_1 \leq c_2/n_2$). This observation explains part (a)(ii) of the proposition.

(iv) The greater incapacitative power of prison compared to that of out-of-prison supervision also explains why it might be desirable to use prison even if prison is less cost effective than out-of-prison supervision. Suppose, for example, that it would cost \$7,000 to prevent a crime if prison is used, but only \$3,500 if out-of-prison supervision is used, so prison is less cost effective. If the harm per crime is \$10,000, the net benefit *per crime prevented* is \$3,000 for prison (= \$10,000 – \$7,000) and \$6,500 for out-of-prison supervision (= \$10,000 – \$3,500). But if prison prevents five crimes per year and out-of-prison supervision prevents one crime per year, the *total* net benefit per year if prison is used is \$15,000, while it is only \$6,500 if out-of-

[16] For concreteness, suppose α, n_1, and n_2 were such that prison would prevent 10 crimes in year 1 and 5 crimes in year 2, while out-of-prison supervision would prevent 4 crimes in year 1 and 2 crimes in year 2. If a one-year term of out-of-prison supervision were used before a one-year term of prison, the number of crimes prevented during the two years would be 4 + 5 = 9. But if prison were used first, the number of crimes prevented would be 10 + 2 = 12.

prison supervision is employed. The larger scale of incapacitation from prison offsets in this example the lower cost effectiveness of prison. This observation provides the intuition behind the use of prison in part (b)(iii) of the proposition.

(v) Note, however, that even if it is optimal to use prison when it is less cost effective than out-of-prison supervision, it will eventually become preferable to use out-of-prison supervision as a prisoner ages. This is because the incapacitative advantage of prison over out-of-prison supervision declines over time as a prisoner ages out of crime. For instance, suppose α, n_1, and n_2 are such that prison would prevent 10 crimes in year 1 and 8 crimes in year 2, while out-of-prison supervision would prevent 5 crimes in year 1 and 4 crimes in year 2. Then the incapacitative advantage of prison is 5 crimes in year 1 (= 10 – 5) and 4 crimes in year 2 (= 8 – 4). At some point this advantage will become small enough to be dominated by the superior cost effectiveness of out-of-prison supervision (assumed in this comment). These observations explain why, in part (b)(iii), the initial term of imprisonment is followed by a term of parole.

(vi) The one remaining case is that in which out-of-prison supervision is more cost effective than prison and the level of harm is in the intermediate range ($c_2/n_2 < h \leq (c_1 - c_2)/(n_1 - n_2)$). For harm in this range, the net benefit of out-of-prison supervision initially is positive (since $h > c_2/n_2$) and at least as high as that of prison (because $h \leq (c_1 - c_2)/(n_1 - n_2)$). Since, as observed in the preceding comment, prison's incapacitative advantage declines as a prisoner ages, the net benefit of out-of-prison supervision will always exceed that of prison in the present case. This explains part (b)(ii) of the proposition, where the optimal sentence is a term of probation.

(vii) Intuitively, one would expect that the higher the level of harm from crime, the longer the optimal terms of prison and out-of-prison supervision because the benefit of incapacitation increases while the cost of incapacitation is unaffected. With one exception, the

formulas in the proposition for the lengths of optimal sanctions support this result. The exception occurs in part (b)(iii), where a parole term is included with a prison term. While the length of the prison term rises with the level of harm, the length of the parole term is independent of the level of harm.

This result follows from the equivalence, in a sense we will now describe, of an increase in harm and a reduction in the offender's age at sentencing. Reducing the age of the offender at sentencing by $\delta > 0$ increases the number of crimes prevented by sanction i at time t from $n_i e^{-\alpha t}$ to $n_i e^{-\alpha(t-\delta)}$, and therefore increases the incapacitation benefits of prison and out-of-prison supervision by a factor of $e^{\alpha\delta} > 1$. Similarly, increasing harm from h_0 to h_1 increases the incapacitation benefits of prison and out-of-prison supervision by a factor of $h_1/h_0 > 1$. Thus, for any increase in harm from h_0 to h_1, there is a corresponding reduction δ in the age of the offender at sentencing that yields the same net benefit functions and, therefore, the same optimal sentences.

Since s_1 and s_2 are the lengths of the prison and parole terms, respectively, the optimal sentence switches from prison to parole at $t = s_1$ and from parole to the null sanction at $t = s_1 + s_2$. Because the offender's age determines the net benefits of sanctions via the number of crimes prevented, reducing the offender's age at sentencing by δ merely delays the two switching times by δ. In turn, increasing the two switching times s_1 and $s_1 + s_2$ by δ increases the length of the prison term s_1 by δ, but does not affect the length of the parole term s_2. Since reducing the offender's age by δ is equivalent to increasing harm from h_0 to h_1, it follows that increasing harm also leaves the length of the parole term unchanged.

Impacts of changes in key parameters: The results in Proposition 1 can be employed to examine how optimal sentences are affected by changes in the propensity to commit crime, the

rate at which individuals age out of crime, and the costs of sanctions. Let $\lambda > 0$ be a factor multiplying n_i, the number of crimes prevented by sanction i per unit of time at $t = 0$. Thus, the higher is λ, the greater the propensity to commit crime. It is straightforward to observe that if the n_i ($i = 1, 2, 3$) in Proposition 1 are replaced with λn_i, the relative cost effectiveness of prison and out-of-prison supervision is not affected. But the threshold levels of harm at which prison and probation are imposed decline with λ. Additionally, the lengths of prison and probation terms increase with λ.[17]

It can be seen in Proposition 1 that the lengths of the optimal terms of prison, parole, and probation depend on the rate α at which individuals age out of crime. In particular, the length of each term includes the multiplicative factor $1/\alpha$. Thus, the higher is α, the shorter are the terms of prison and out-of-prison supervision in the optimal sentence. This makes sense because a rapid decline with age of the propensity to commit crime means that the benefits of incapacitation are rapidly declining with age, but the costs of incapacitation are not. Consequently, the net benefit of the sanction declines faster with higher α.[18]

Lastly, let $\gamma > 0$ be a factor multiplying c_i, the cost of sanction i. If the c_i in Proposition 1 are replaced with γc_i, the relative cost effectiveness of prison and out-of-prison supervision is not affected. The remaining effects of a higher γ are the mirror image of the effects of an increase in the propensity to commit crime due to a higher λ. Specifically, the imposition of prison or

[17] The length of the parole term does not depend on λ for the reason explained in comment (vii) above.

[18] Blumstein et al. (1986, p. 1) describe "career criminals" as those who "commit serious offenses with high frequency over extended periods of time." The discussion in this and the preceding paragraph can be used to compare the optimal sentence for a career criminal to that of an ordinary criminal. Let an ordinary criminal be characterized by $\lambda_O = 1$ and some $\alpha_O > 0$ and let a career criminal be characterized by some $\lambda_C > 1$ and some $\alpha_C < \alpha_O$. In other words, a career criminal commits more crimes at any $t \geq 0$ than an ordinary criminal, and he ages out of crime more slowly than an ordinary criminal. Thus, the optimal threshold levels of harm at which prison and probation are imposed are lower for career criminals and the optimal lengths of prison and out-of-prison supervision terms are longer for career criminals.

probation does not occur until higher levels of harm, and the lengths of prison and probation terms decline.

Example: To illustrate Proposition 1, suppose that the cost of prison is c_1 = \$35,000 per year, while the cost of out-of-prison supervision is c_2 = \$3,500 per year. Assume initially that prison prevents n_1 = 12 crimes per year, while out-of-prison supervision prevents n_2 = 1 crime per year. Finally, suppose that the harm from crime is h = \$5,000 and that the number of crimes that would be committed by potential offenders declines by 15 percent per year, which corresponds to $\alpha = .1625$.[19]

Given these values, prison is more cost effective than out-of-prison supervision: c_1/n_1 = \$2,917 < c_2/n_2 = \$3,500. Hence, part (a) of Proposition 1 applies. Because h = \$5,000 > c_1/n_1 = \$2,917, case (ii) of part (a) is relevant and the optimal sentence consists of a prison term of length s_1 = 3.3 years.

Holding everything else equal, suppose that the number of crimes prevented by prison declines to n_1 = 5 crimes per year. Now out-of-prison supervision is more cost effective than prison: c_2/n_2 = \$3,500 < c_1/n_1 = \$7,000. Thus, part (b) of the proposition governs. Because h = \$5,000 lies within the intermediate range of harm in part (b), which is c_2/n_2 = \$3,500 < $h \leq (c_1 - c_2)/(n_1 - n_2)$ = \$7,875, case (ii) of part (b) applies and the optimal sentence is a probation term of length s_2 = 2.2 years.

If instead harm were h = \$10,000, case (iii) of part (b) would be relevant and the optimal sentence would consist of s_1 = 1.5 years of prison followed by s_2 = 5.0 years of parole. In this example, 29 percent of the incapacitation benefits of the sentence are due to the parole term. If

[19] Given this last assumption, if n is the initial rate of crime commission, then after one year the rate of crime commission would be $.85n$. Using our continuous time formula for the rate of crime commission, $ne^{-\alpha t}$, this rate would be n at $t = 0$ and $ne^{-\alpha}$ at $t = 1$. Thus, if the number of crimes that would be committed by potential offenders declines by 15 percent per year, $ne^{-\alpha} = .85n$ and $\alpha = ln(1/.85) = .1625$.

harm were h = \$15,000, the optimal prison term would rise to s_1 = 4.0 years and the optimal parole term would remain at s_2 = 5.0 years for the reasons discussed in comment (vii) above.

5. How Optimal Sentences Depend on the Age of the Offender

In this section we examine how the optimal sentence to promote incapacitation depends on the age of the convicted offender. We will refer to a *younger offender* who is of age a_Y at the time of his sentencing and to an *older offender* who is $\delta > 0$ years older than the younger offender at the time of his sentencing, that is, who is of age $a_O = a_Y + \delta$ at sentencing. In all other respects the two types of offender are identical, including that each ages out of crime at the rate α starting at the benchmark age a_Y. Thus, at $t = 0$, the time of sentencing for both offenders, prison would initially prevent the younger offender from committing n_1 crimes and the older offender from committing $n_1 e^{-\alpha\delta} < n_1$ crimes per unit of time. A parallel statement would apply to crimes prevented during out-of-prison supervision.

The following proposition describes the relationship between the optimal sentences for the younger and older offenders.

Proposition 2: Let $\sigma_Y(t)$ be the optimal sentence for the younger offender and $\sigma_O(t)$ be the optimal sentence for the older offender who is δ years older than the younger offender. Then $\sigma_O(t) = \sigma_Y(t + \delta)$ for all $t \geq 0$. In other words, the optimal sentence for the older offender is just the part of the younger offender's optimal sentence that will remain when the younger offender reaches the older offender's age at sentencing.

Proof: See the Appendix.

Comments: (i) The explanation of this result follows directly from the point made in comment (i) following Proposition 1, that at each time $t \geq 0$ the optimal sentence uses the

sanction $i \in \{1, 2, 3\}$ that maximizes the net benefit $b_i(t) = n_i e^{-\alpha t} h - c_i$. Suppose that the younger offender is age 20 at sentencing and that $\delta = 25$, so the older offender is age 45 at sentencing. The claim of the proposition is that the optimal sanction for the older offender at, for example, time $t = 10$ will be the same as that for the younger offender at time $t + \delta = 35$. This makes sense because at time $t = 10$, the older offender will be age 55, and at time $t + \delta = 35$, the younger offender also will be age 55. The number of crimes that each would commit at that age if free would be the same and, thus, the same sanction will be optimal for both.

(ii) If the optimal sentence for the younger offender is a prison term alone, the optimal sentence for the older offender either will be a shorter prison term or an infinite term of the null sanction. Similarly, if the optimal sentence for the younger offender is a probation term, the optimal sentence for the older offender either will be a shorter probation term or an infinite term of the null sanction. And if the optimal sentence for the younger offender is a prison term followed by a parole term, the optimal sentence for the older offender will be a shorter prison term followed by the same parole term; or a probation term up to the length of the younger offender's parole term; or an infinite term of the null sanction. Obviously, the higher the age of the older offender, the less onerous will be the optimal sentence for the older offender.

Example: Reconsider the example in the previous section when out-of-prison supervision was more cost effective than prison and the optimal sentence was a prison term of length $s_1 = 1.5$ years followed by a parole term of length $s_2 = 5.0$ years.[20] If this sentence is optimal for a 20 year-old offender, the optimal sentence for a 21 year-old offender would be a prison term of 0.5 years and a parole term of 5.0 years, and that for a 22 year-old offender would be a probation term of 4.5 years. The length of the probation term then would decline until the age of the

[20] This corresponds to the case there in which $n_1 = 5$ and $h = \$10{,}000$.

offender at sentencing increased to 26.5, after which the optimal sentence would be an infinite term of the null sanction.

6. Concluding Comments

We conclude with several comments about our analysis.

(a) *The relative cost effectiveness of sanctions.* As observed in Proposition 1, the optimal sentence depends on the relative cost of sanctions per crime prevented, c_i/n_i. The average cost of incarcerating an offender in a state prison in the United States is $34,623 per year, as of 2017.[21] The corresponding average cost of out-of-prison supervision at the state level is $1,121 per year.[22] Since the cost of out-of-prison supervision is 3.2 percent of that of prison, out-of-prison supervision will be more (less) cost effective than prison if the number of crimes prevented by out-of-prison supervision per unit of time is greater (less) than 3.2 percent of that of prison.

We are aware of only one study that attempts to assess the extent to which crime declines during out-of-prison supervision compared to crime reduction during imprisonment. This study, by Petersilia and Turner (1986), examined 1,022 individuals convicted of assault, robbery, burglary, theft, or drug sale/possession, half of whom received a prison sentence and half of whom received a felony probation sentence. The authors attempted to match characteristics of the two groups relevant to their propensity to commit crime. Based on data from this study, we calculated that out-of-prison supervision is 16.9 percent as effective as prison in preventing

[21] The National Institute of Corrections (undated) reports the prison population and average annual cost per inmate for each state. The average annual cost of prison stated in the text is the weighted average of the states' costs per inmate, using as weights the states' shares of the total prison population. The majority of inmates are in state prisons, not federal prisons or local jails.

[22] We calculated the average annual cost of out-of-prison supervision by summing state expenditures in 2017 on parole and probation provided by the Justice Center of the Council of State Governments (2018, Part 3, Strategy 1) and dividing by the sum of states' populations subject to community supervision as of January 1, 2017, reported by Alper and Kaeble (2020, pp. 28-29, Appendix Table 9).

crime.[23] Thus, for the sample of criminals and crimes in this study, out-of-prison supervision is significantly more cost effective than prison.

The cost effectiveness of out-of-prison supervision relative to that of prison obviously will depend on the character of the crime in question. Counterintuitively, for crimes that require geographical movement, out-of-prison supervision often may be more cost effective than prison. Suppose a drug addict would burglarize ten houses a year if free but one less house a year if, while on parole or probation, his travel is constrained through use of a GPS-tracking ankle bracelet or required participation in a residential drug treatment program. Out-of-prison supervision would then be more cost effective than prison because it would prevent 10 percent of the crimes that would be prevented if the criminal were in prison. But for crimes involving spontaneous loss of emotional control, resulting, say, in physical aggression, out-of-prison supervision often may be less cost effective than prison. It might be virtually impossible to prevent a person from committing an assault while on parole or probation, whereas such a person could be more closely monitored and restrained within a prison (for example, by being put in a single-person cell). Then out-of-prison supervision would tend to be less cost effective than prison.

(b) *The rate at which individuals age out of crime.* There is extensive evidence that individuals age out of crime after their propensity to commit crime peaks, usually by their late

[23] Petersilia and Turner (1986, p. 28-32) determined that individuals on probation for two years committed an average of 9.8 crimes per year. They also found that prisoners committed an average of 10.8 crimes per year during the first two years after their release. Thus, individuals in the prisoner group committed on average 2 x 10.8 = 21.6 crimes over these two years. During the first year after their release they were on parole and during the second year they were free. If one assumes, as we have throughout this article, that parole and probation are equivalent with respect to crime reduction, this implies that individuals in the prisoner group committed an average of 21.6 – 9.8 = 11.8 crimes during the year that they were free. Hence, prison would prevent 11.8 crimes per year and parole and probation would prevent 11.8 – 9.8 = 2.0 crimes per year. In other words, in this study out-of-prison supervision is (2.0/11.8) x 100 = 16.9 percent as effective as prison in crime reduction.

teens or early twenties.[24] One prominent discussion of this evidence, by Blumstein et al. (1986), includes a figure (p. 23, Figure 1-2) showing the ages at which the number of arrests per 100,000 population for the crimes of burglary, robbery, and aggravated assault peak and the corresponding ages at which the number of arrests declines to 50 percent of the peak.[25] The former ages are 17, 17, and 21, while the latter ages are 21, 24, and 36, respectively. Thus, the arrest rates for burglary, robbery, and aggravated assault decline to half of their peak heights after 4, 7, and 15 years, respectively. Assuming that crime rates are proportional to arrest rates and that the propensity to commit crime declines exponentially, these numbers imply rates of aging out of crime—that is, values of α—for burglary, robbery, and aggravated assault of 17.3 percent, 9.9 percent, and 4.6 percent per year, respectively.

Differential rates at which individuals age out of crime for different offenses bear on the optimal sentences for the different offenses. As observed in Section 4, the lengths of the optimal terms of prison, parole, and probation include the multiplicative factor $1/\alpha$, the reciprocal of the rate at which individuals age out of crime. Hence, the faster that individuals age out of crime, the shorter are these terms. Given the rates in the preceding paragraph, this implies that any term of prison or out-of-prison supervision in the optimal sentence for burglary would be the shortest, everything else equal. The corresponding terms for robbery would be 1.7 times longer than those for burglary, and the terms for aggravated assault would be 3.8 times longer than those for burglary.[26]

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[24] See Laub and Sampson (2003, pp. 81-113) and Gottfredson and Hirschi (2020, pp. 51-74 and 154-57).

[25] For simplicity, we modeled the propensity to commit crime as steadily decreasing with age even though there usually is a relatively short period in offenders' lives—mainly the teenage years—during which the propensity is rising.

[26] The multiplicative factor $1/\alpha$ is 1/.173 = 5.78 for burglary, 1/.099 = 10.10 for robbery, and 1/.046 = 21.74 for aggravated assault. Hence, everything else equal, robbery terms would be 10.10/5.78 = 1.75 times longer than

(c) *Discounting the benefits of incapacitation and the costs of sanctions.* For simplicity we ignored discounting by the state of the benefits of incapacitation and of the costs of imposing sanctions. Taking discounting into account would not have changed our analysis in any fundamental way. It would still be the case that the optimal sanction at any time is the sanction that maximizes the net benefit at that time, but now the net benefit would be computed using the discounted incapacitation benefit and the discounted sanctioning cost. If the discount rates for incapacitation benefits and sanctioning costs are the same, the optimal sentence would not be affected.

If, however, the discount rates differ, optimal sentences will be affected. For instance, if the discount rate for incapacitation benefits exceeds that for sanctioning costs, the lengths of prison and out-of-prison supervision terms in an optimal sentence will be shorter than when the discount rates are the same. This is because raising the discount rate for incapacitation benefits above that for sanctioning costs is equivalent to raising the rate α at which individuals age out of crime; and, as noted previously, the lengths of the optimal terms of prison, parole, and probation include the multiplicative factor $1/\alpha$.

(d) *Incorporating the disutility of sanctions borne by convicted offenders and the gains obtained by criminals.* Assuming that the utilities of all individuals count in social welfare, a full welfare analysis of incapacitation should include the disutility of sanctions borne by convicted offenders and the gains obtained by criminals. Our analysis can be interpreted to include these components of utility. Specifically, the cost of sanction i per unit of time, c_i, can include both the cost to the state of imposing the sanction and the disutility borne by the person subject to the

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those for burglary; and aggravated assault terms would 21.74/5.78 = 3.76 times longer than those for burglary. The numbers reported in the text are rounded to the nearest tenth.

sanction. Similarly, the harm per crime, h, can be interpreted as the harm to the victim of the crime less the gain to the criminal.

(e) *Optimal sentences to promote incapacitation versus optimal sentences to promote deterrence.* As has been noted by others, the optimal sentence to promote incapacitation and the optimal sentence to promote deterrence often differ.[27] In the present analysis, this disparity can be illustrated most starkly by reconsidering the results in part (a) of Proposition 1. It was seen there that if the harm h exceeds the cost per crime prevented by prison c_1/n_1, the optimal sentence with respect to the incapacitation goal of the criminal justice system is a positive term of imprisonment. Moreover, as h increases, the length of the optimal term increases. But suppose, in the extreme, that potential offenders are impossible to deter. Then the optimal sentence with respect to the deterrence goal of the criminal justice system would be an infinite term of the null sanction since the use of any other sanction would be costly yet not produce any deterrence benefits.

[27] See Shavell (1987, pp. 109-10) and Miceli (2010, p. 207).

Appendix

In this appendix we provide proofs of Propositions 1 and 2. To facilitate the proof of Proposition 1, we first state and prove three lemmas.

Lemma 1: Let $M(t)$ be the set of sanctions that maximize net benefit at time $t \geq 0$:

$$M(t) = \underset{i \in \{1, 2, 3\}}{\text{argmax}}\ b_i(t).$$

Sentence $\sigma \in \Sigma$ solves the state's social welfare maximization problem (1) if and only if $\forall\ t \geq 0$, $\sigma(t) \in M(t)$.

Proof of Lemma 1: Observe that for each $i \in \{1, 2, 3\}$ and each $t \geq 0$, $n_i e^{-\alpha t} h - c_i$ is a finite number. Therefore, $M(t)$ is a well-defined, non-empty set for each $t \geq 0$.

Define

$$b(\sigma(t), t) = n(\sigma(t))e^{-\alpha t}h - c(\sigma(t)) \qquad \text{(A1)}$$

as the net benefit at time $t \geq 0$, conditional on using sentence σ.

We show first that σ is optimal *if* $\forall\ t \geq 0$, $\sigma(t) \in M(t)$, and then that σ is optimal *only if* $\forall\ t \geq 0$, $\sigma(t) \in M(t)$.

Step 1: σ is optimal *if* $\forall\ t \geq 0$, $\sigma(t) \in M(t)$.

Assume for the sake of contradiction that sentence σ is not optimal, but that $\forall\ t \geq 0$, $\sigma(t) \in M(t)$. Since σ is not optimal, there must be another sentence, s, that produces strictly higher social welfare than σ does.

For s to produce greater social welfare than σ, there must be some times $t \geq 0$ such that $b(s(t), t) > b(\sigma(t), t)$. However, a sanction cannot maximize the net benefit at time t if some other sanction produces a strictly greater net benefit at t, so at any time t at which $b(s(t), t) > b(\sigma(t), t)$,

it must be true that $\sigma(t) \notin M(t)$. We have thus contradicted our assumption that $\forall\ t \geq 0,\ \sigma(t) \in M(t)$. Sentence σ must therefore be optimal.

Step 2: σ is optimal *only if* $\forall\ t \geq 0,\ \sigma(t) \in M(t)$.

Let σ be an optimal sentence. Assume for the sake of contradiction that at some time $\tau \geq 0,\ \sigma(\tau) \notin M(\tau)$. For some $i \in \{1, 2, 3\}$, let $\sigma(\tau) = i$. Since $\sigma(\tau) = i \notin M(\tau)$ and by the observation at the beginning of this proof, $M(\tau)$ is not the empty set, there must be some $j \in \{1, 2, 3\}$, such that $j \neq i$ and $j \in M(\tau)$. By the definition of $M(\tau)$, it must be true that

$$b_j(\tau) > b_i(\tau). \tag{A2}$$

By our definitions of sentences and terms, τ must belong to a term of sanction i in sentence σ. Let t_1 and t_2 be endpoints of this term, and suppose $t_2 > t_1 \geq 0$.[28] We consider in turn the case in which τ is an interior point of this term ($\tau \in (t_1, t_2)$) and the cases in which it is an endpoint ($\tau \in \{t_1, t_2\}$).

Suppose first that $\tau \in (t_1, t_2)$, that is, τ is an interior point of the term that includes τ. Since for all $k \in \{1, 2, 3\}$, $b_k(t)$ is a continuous function, it follows from (A2) that for values of t sufficiently close to τ, $b_j(t) > b_i(t)$. Formally, we say that there exists some $\delta > 0$ such that $\tau - \delta > t_1$, $\tau + \delta < t_2$, and $\forall\ t \in (\tau - \delta,\ \tau + \delta)$, $b_j(t) > b_i(t)$. In this case, let $N = (\tau - \delta,\ \tau + \delta)$.

If instead $\tau = t_1$, then by an argument analogous to that of the previous paragraph, the continuity of the net benefit functions implies that there exists some $\delta > 0$ such that $\tau + \delta < t_2$ and $\forall\ t \in [\tau,\ \tau + \delta)$, $b_j(t) > b_i(t)$. In this case, let $N = [\tau,\ \tau + \delta)$.

If instead $\tau = t_2$, then the continuity of the net benefit functions implies that there exists some $\delta > 0$ such that $\tau - \delta > t_1$ and $\forall\ t \in (\tau - \delta,\ \tau]$, $b_j(t) > b_i(t)$. In this case, let $N = (\tau - \delta,\ \tau]$.

In all three cases above, by the construction of N, $\forall\ t \in N$, $b_j(t) > b_i(t)$. Let sentence s be identical to σ except that $\forall\ t \in N$, $s(t) = j$. Clearly, at every time s produces the same net benefit as σ except in the interval N, during which s produces a greater net benefit than σ does. Since in all three cases above N is an interval of positive length, s yields higher social welfare than σ does, so σ cannot have been optimal. We have obtained a contradiction, and conclude that if σ is an optimal sentence, then $\forall\ t \geq 0$, $\sigma(t) \in M(t)$. Both steps 1 and 2 of the proof are now complete. □

Lemma 2: Let $T_{12} = (1/\alpha)\ln((n_1 - n_2)h/(c_1 - c_2))$, $T_{13} = (1/\alpha)\ln(n_1 h/c_1)$, and $T_{23} = (1/\alpha)\ln(n_2 h/c_2)$. Then the following statements are true:

(a) $\forall\ t < T_{12}$, $b_1(t) > b_2(t)$; $b_1(T_{12}) = b_2(T_{12})$; and $\forall\ t > T_{12}$, $b_1(t) < b_2(t)$.

(b) $\forall\ t < T_{13}$, $b_1(t) > b_3(t)$; $b_1(T_{13}) = b_3(T_{13})$; and $\forall\ t > T_{13}$, $b_1(t) < b_3(t)$.

(c) $\forall\ t < T_{23}$, $b_2(t) > b_3(t)$; $b_2(T_{23}) = b_3(T_{23})$; and $\forall\ t > T_{23}$, $b_2(t) < b_3(t)$.

Proof of Lemma 2: This lemma establishes that each pair of net benefit functions intersects exactly once and states the times at which these intersections occur (which can be negative).

Suppose $i = 1$ or 2. The net benefit functions $b_i(t)$ and $b_3(t)$ intersect at time t if and only if

$$b_i(t) = b_3(t), \tag{A3}$$

that is, if and only if

$$n_i e^{-\alpha t} h - c_i = 0. \tag{A4}$$

Solving for t yields

[28] For convenience, we assume a finite term here. If the term of sanction i were infinite, we could choose any $t_2 > \tau$, and the proof would proceed essentially as in the text.

$$t = (1/\alpha)\ln(n_i h/c_i) = T_{i3}. \quad \text{(A5)}$$

Thus, T_{i3} is the unique time at which $b_i(t)$ and $b_3(t)$ intersect.

Similarly, the net benefit functions $b_1(t)$ and $b_2(t)$ intersect at time t if and only if

$$b_1(t) = b_2(t), \quad \text{(A6)}$$

that is, if and only if

$$n_1 e^{-\alpha t} h - c_1 = n_2 e^{-\alpha t} h - c_2, \quad \text{(A7)}$$

yielding

$$t = (1/\alpha)\ln((n_1 - n_2)h/(c_1 - c_2)) = T_{12}. \quad \text{(A8)}$$

Thus, T_{12} is the unique time at which $b_1(t)$ and $b_2(t)$ intersect.

We derived T_{12} from the condition $b_1(t) = b_2(t)$, T_{13} from the condition $b_1(t) = b_3(t)$, and T_{23} from the condition $b_2(t) = b_3(t)$, so the equalities in parts (a)-(c) of the lemma are true.

Replacing the equality in equation (A3) with an inequality and reasoning by analogy with equations (A3)-(A5), we conclude that for $i = 1$ and 2, $b_i(t) > b_3(t)$ if and only if $t < T_{i3}$, and $b_i(t) < b_3(t)$ if and only if $t > T_{i3}$. Thus, parts (b) and (c) of the lemma hold.

Similarly, replacing the equality in equation (A6) with an inequality and reasoning by analogy with equations (A6)-(A8), we conclude that $b_1(t) > b_2(t)$ if and only if $t < T_{12}$, and $b_1(t) < b_2(t)$ if and only if $t > T_{12}$. Thus, part (a) of the lemma holds. □

Lemma 3: Either

$(c_1 - c_2)/(n_1 - n_2) < c_1/n_1 < c_2/n_2$; or

$c_2/n_2 < c_1/n_1 < (c_1 - c_2)/(n_1 - n_2)$; or

$c_1/n_1 = c_2/n_2 = (c_1 - c_2)/(n_1 - n_2)$.

Proof of Lemma 3: Suppose $c_1/n_1 < c_2/n_2$. It follows that

$$c_1 n_2 < c_2 n_1, \quad \text{(A9)}$$

$$c_1(n_2 - n_1) < n_1(c_2 - c_1), \tag{A10}$$

$$c_1/n_1 > (c_1 - c_2)/(n_1 - n_2). \tag{A11}$$

Now suppose $c_2/n_2 < c_1/n_1$, which implies that

$$c_2 n_1 < c_1 n_2, \tag{A12}$$

$$n_1(c_2 - c_1) < c_1(n_2 - n_1), \tag{A13}$$

$$c_1/n_1 < (c_1 - c_2)/(n_1 - n_2). \tag{A14}$$

Last, suppose $c_1/n_1 = c_2/n_2$. Replacing the inequalities in (A9)-(A11) with equality establishes that $c_1/n_1 = (c_1 - c_2)/(n_1 - n_2)$.

These three cases are collectively exhaustive, so Lemma 3 follows. □

Proof of Proposition 1: Define T_{12}, T_{13}, and T_{23} as in Lemma 2.

We begin with a general observation about the choice of sanction at a time at which an optimal sentence switches from one sanction to another. Suppose that $i \in \{1, 2\}, j \in \{2, 3\}$ and $i < j$. We will see below that in each relevant subcase—$(i, j) = (1, 2)$, $(1, 3)$, or $(2, 3)$—if an optimal sentence switches from sanction i to sanction j at time $t > 0$, then $t = T_{ij}$. Therefore, by Lemma 2, $b_i(T_{ij}) = b_j(T_{ij})$. It follows that if an optimal sentence switches from sanction i to sanction j at $t = T_{ij}$, then $i, j \in M(T_{ij})$, so it does not matter which of the two sanctions the optimal sentence uses at T_{ij}. For this reason, we will not specify the sanction an optimal sentence uses at such a switching time.

Consider two cases, $c_1/n_1 \leq c_2/n_2$ and $c_2/n_2 < c_1/n_1$.

Case (a): $c_1/n_1 \leq c_2/n_2$

We begin by showing that when $c_1/n_1 \leq c_2/n_2$,

$$T_{13} \leq T_{12} \tag{A15}$$

and

$$T_{23} \leq T_{12}. \tag{A16}$$

We prove (A15) and (A16) by confirming that they hold for the subcases in which $c_1/n_1 < c_2/n_2$ and $c_1/n_1 = c_2/n_2$, respectively.

Suppose first that $c_1/n_1 < c_2/n_2$. Applying Lemma 3 yields

$$(c_1 - c_2)/(n_1 - n_2) < c_1/n_1 < c_2/n_2, \tag{A17}$$

which implies

$$n_2/c_2 < n_1/c_1 < (n_1 - n_2)/(c_1 - c_2). \tag{A18}$$

Condition (A18) and the definitions of T_{23}, T_{13}, and T_{12} require that

$$T_{23} < T_{13} < T_{12}, \tag{A19}$$

demonstrating that inequalities (A15) and (A16) hold in this subcase.

Next, suppose $c_1/n_1 = c_2/n_2$. Applying Lemma 3 yields

$$c_1/n_1 = c_2/n_2 = (c_1 - c_2)/(n_1 - n_2). \tag{A20}$$

Equation (A20) and the definitions of T_{13}, T_{23}, and T_{12} require that

$$T_{13} = T_{23} = T_{12}, \tag{A21}$$

demonstrating that (A15) and (A16) are true for this subcase. Since (A15) and (A16) hold for both subcases, we conclude that $c_1/n_1 \leq c_2/n_2$ implies (A15) and (A16).

We now show that if $c_1/n_1 \leq c_2/n_2$, condition (A16) restricts the form of an optimal sentence. By part (c) of Lemma 2, out-of-prison supervision beats the null sanction ($b_2(t) > b_3(t)$) only when $t < T_{23}$, and by part (a) of Lemma 2, $\forall\ t < T_{12}$, prison beats out-of-prison supervision ($b_1(t) > b_2(t)$). Therefore, inequality (A16) and parts (a) and (c) of Lemma 2 imply that for any time t at which out-of-prison supervision beats the null sanction, prison beats out-of-prison supervision. It follows from Lemma 1 that if $c_1/n_1 \leq c_2/n_2$, no optimal sentence can ever use out-of-prison supervision.

We now derive optimal sentences for two mutually exclusive and collectively exhaustive subcases.

Subcase (a)(i): $c_1/n_1 \leq c_2/n_2$ and $h \leq c_1/n_1$

If $h \leq c_1/n_1$, then by the definition of T_{13}, $T_{13} \leq 0$, and by part (b) of Lemma 2, the null sanction beats prison $\forall\ t > 0$. It follows from Lemma 1 that in this subcase, no optimal sentence can ever use prison. We proved above that if $c_1/n_1 \leq c_2/n_2$, no optimal sentence ever uses out-of-prison supervision. Accordingly, in this subcase, the optimal sentence is a perpetual term of the null sanction.

Subcase (a)(ii): $c_1/n_1 \leq c_2/n_2$ and *$h > c_1/n_1$*

If $h > c_1/n_1$ then $T_{13} > 0$, so $\forall\ t \in [0, T_{13})$, inequality (A15) and parts (a) and (b) of Lemma 2 require that $b_1(t) > b_2(t)$ and $b_1(t) > 0$. Therefore, by Lemma 1, an optimal sentence must use prison $\forall\ t \in [0, T_{13})$. When $t > T_{13}$, part (b) of Lemma 2 requires that $b_1(t) < 0$, so by Lemma 1, an optimal sentence cannot use prison for $t > T_{13}$. Our analysis above showed that if $c_1/n_1 \leq c_2/n_2$ no optimal sentence ever uses out-of-prison supervision, so Lemma 1 requires that an optimal sentence impose the null sanction $\forall\ t > T_{13}$. Thus, in this subcase, the optimal sentence consists of a prison term of length T_{13} followed by an infinite term of the null sanction.

Case (b): $c_2/n_2 < c_1/n_1$

Applying Lemma 3 yields

$$c_2/n_2 < c_1/n_1 < (c_1 - c_2)/(n_1 - n_2), \tag{A22}$$

and therefore

$$(n_1 - n_2)/(c_1 - c_2) < n_1/c_1 < n_2/c_2. \tag{A23}$$

Inequality (A23) and the definitions of T_{12}, T_{13}, and T_{23} require that

$$T_{12} < T_{13} < T_{23}. \tag{A24}$$

We now derive optimal sentences for three mutually exclusive and collectively exhaustive subcases.

Subcase (b)(i): $c_2/n_2 < c_1/n_1$ and $h \leq c_2/n_2$

The condition $h \leq c_2/n_2$ and the definition of T_{23} imply that $T_{23} \leq 0$, and therefore, using (A24),

$$T_{13} < T_{23} \leq 0. \tag{A25}$$

Inequality (A25) and parts (b) and (c) of Lemma 2 require that $\forall\ t > 0$, $b_1(t) < 0$ and $b_2(t) < 0$, so by Lemma 1, no optimal sentence can use either prison or out-of-prison supervision. Accordingly, in this subcase, the optimal sentence is an infinite term of the null sanction.

Subcase (b)(ii): $c_2/n_2 < c_1/n_1$ and $c_2/n_2 < h \leq (c_1 - c_2)/(n_1 - n_2)$

The condition $h \leq (c_1 - c_2)/(n_1 - n_2)$ implies that $T_{12} \leq 0$, so by part (a) of Lemma 2, out-of-prison supervision beats prison for all $t > 0$. Since $h > c_2/n_2$, $T_{23} > 0$, and part (c) of Lemma 2 requires that $\forall\ t \in [0, T_{23})$, $b_2(t) > 0$. Accordingly, by Lemma 1, an optimal sentence must use out-of-prison supervision $\forall\ t \in [0, T_{23})$. By (A24), $T_{23} > T_{13}$. Therefore $\forall\ t > T_{23}$, parts (b) and (c) of Lemma 2 require that $b_1(t) < 0$ and $b_2(t) < 0$, so by Lemma 1, any optimal sentence imposes the null sanction. Thus, in this subcase, the optimal sentence is a probation term of length T_{23}, followed by an infinite term of the null sanction.

Subcase (b)(iii): $c_2/n_2 < c_1/n_1$ and $h > (c_1 - c_2)/(n_1 - n_2)$

Since $h > (c_1 - c_2)/(n_1 - n_2)$, it follows that $(n_1 - n_2)h/(c_1 - c_2) > 1$, and therefore

$$T_{12} > 0. \tag{A26}$$

Inequalities (A24) and (A26) imply that $T_{13} > T_{12} > 0$. Applying parts (a) and (b) of Lemma 2, it follows that $\forall\ t \in [0, T_{12})$, $b_1(t) > b_2(t)$ and $b_1(t) > 0$, so by Lemma 1, an optimal sentence must use prison. By inequality (A24), $T_{23} > T_{12}$. $\forall\ t \in (T_{12}, T_{23})$, parts (a) and (c) of

Lemma 2 imply that $b_2(t) > b_1(t)$ and $b_2(t) > 0$, so by Lemma 1, an optimal sentence must use parole.

Since $T_{23} > T_{13}$ by inequality (A24), it follows that $\forall\ t > T_{23}$, parts (b) and (c) of Lemma 2 require that $b_1(t) < 0$ and $b_2(t) < 0$. Therefore, by Lemma 1, an optimal sentence must use the null sanction $\forall\ t > T_{23}$. Thus, in this subcase, the optimal sentence is a prison term of length T_{12}, followed by a parole term of length

$$T_{23} - T_{12} = (1/\alpha)\ln(n_2 h/c_2) - (1/\alpha)\ln((n_1 - n_2)h/(c_1 - c_2)) \tag{A27}$$

$$= (1/\alpha)\ln((n_2/c_2)(c_1 - c_2)/(n_1 - n_2)),$$

followed by an infinite term of the null sanction. □

Proof of Proposition 2: If the state imposes sentence $\sigma(t)$ on the younger offender, social welfare is

$$\int_0^\infty [n(\sigma(t))e^{-\alpha t}h - c(\sigma(t))]dt \tag{A28}$$

$$= \int_0^\delta [n(\sigma(t))e^{-\alpha t}h - c(\sigma(t))]dt + \int_\delta^\infty [n(\sigma(t))e^{-\alpha t}h - c(\sigma(t))]dt.$$

The last integral in (A28) is just the social welfare generated by the "tail end" of the younger offender's sentence, that is, the part of sentence $\sigma(t)$ remaining at $t = \delta$, when the younger offender reaches the older offender's age at sentencing. Substituting $\tau = t - \delta$ in the last integral in (A28) yields

$$\int_\delta^\infty [n(\sigma(t))e^{-\alpha t}h - c(\sigma(t))]dt = \int_0^\infty [n(\sigma(\tau + \delta))e^{-\alpha\tau}e^{-\alpha\delta}h - c(\sigma(\tau + \delta))]d\tau \tag{A29}$$

$$= \int_0^\infty [n(\sigma(t + \delta))e^{-\alpha t}e^{-\alpha\delta}h - c(\sigma(t + \delta))]dt.$$

The right-hand side of the second line of (A29) is just the social welfare produced by imposing sentence $\sigma(t + \delta)$ on the older offender. Thus, the welfare from imposing the tail end of the younger offender's sentence on the younger offender at $t = \delta$ is identical to the welfare from imposing this tail end on the older offender at $t = 0$.

In the context of our model, the principle of optimality in dynamic programming[29] requires that an optimal sentence have the property that at any future time, the remaining part of the optimal sentence maximizes welfare over the remainder of the offender's life. Therefore, the optimal sentence for the younger offender, $\sigma_Y(t)$, must maximize the left-hand side of (A29). It follows from (A29) that $\sigma_O(t) = \sigma_Y(t + \delta)$ is the optimal sentence for the older offender. □

[29] See Kamien and Schwartz (2012, p. 259).

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