Preliminary Draft
NA for Quotation

Testing for a Growth Slowdown Using a Model of Punctuated Technological Progress

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12–23–2024

Abstract

Disappointing productivity gains in recent decades have raised concerns that the trend rate of TFP improvement may have declined, and this paper examines that possibility. We construct a model with randomly-timed, punctuated technology advances (as from the arrivals of new general purpose technologies), technology-specific capital, and a 2-agent household sector. The model generates uneven, episodic short-run growth, though there is a stationary long-run average rate. Changes in the latter are difficult to establish due to the nature of the underlying stochastic process of shocks. We propose an indirect analytic approach, based upon the model's duality relations. With it, we find little evidence of a recent slowdown — though looking further into the past, arguably there was a decline at the close of the 1920s.

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1. Introduction

Disappointing productivity gains in recent decades have raised concerns that the trend rate of TFP improvement may have declined (Ramey [2020, 2021], Gordon [2019], Jorgenson [2019], and many others), and this paper examines that possibility. As a framework for analysis, we construct an aggregative model with randomly-timed punctuated technology advances. Conceptualizations of long-term growth often emphasize the importance of revolutionary inventions, and our setup features their role. We show the model can highlight potential challenges for detecting a long-run change in growth. The latter difficulties lead us to propose an indirect approach, based on duality relations, for identifying trend shifts. The new procedure does not, however, find evidence of a recent slowdown.

In the model, growth proceeds in episodes, each initiated by a discrete TFP improvement, which we call a "technology shock." Shock arrivals are Poisson events. The next shock truncates the current episode and starts another. Within an episode, the rate of progress surges at first but asymptotically approaches zero. The asymptotic behavior can cause a period of stagnation if the next shock is delayed, and we will want to avoid misconstruing short-run lulls as long-run changes.

Regardless of short-run unevenness, our model has a (stochastic) steady-state equilibrium with a constant average rate of advance. In principle, we can use direct observations of technology shocks (see below) to determine whether their underlying stochastic process has shifted. Unfortunately, given the wide spacing of the fundamental shocks, we may well need a very lengthy data set to estimate new parameters.

To obtain more timely results, we suggest an indirect approach, as follows. Suppose agents within the economy have more information about the pipeline of potential new technologies than external analysts. Assuming that major inventions can be quite disruptive, we expect agents to adjust their behavior if they learn, for example, that future shocks will be weaker and/or less frequent. A permanent slowdown may, therefore, lead to a reduction in investment risk, which, in turn, affects the financial market's equilibrium. Importantly, the latter effect can occur as agent beliefs change — waiting for new-regime technology-shock realizations may be unnecessary.

The idea of punctuated advances has long had adherents. Papers on general purpose technologies (GPTs), for instance, consider abrupt, large-scale technological advances (Helpman and Trajtenberg [1998], Howitt [1998]). Related work by economic historians describes growth as occurring in waves, with major inventions inaugurating periods of accelerated progress (Abramovitz and David [2014], Edquist and Henrekson [2007]).[1] Re-

[1] In general, the idea that seminal inventions are basic to long-run growth seems implicit in many analyses. For example, Gordon [2012] judges the post-1970 slowdown by asking whether the microprocessor and Internet can measure up to great inventions of the past;

cent studies of individual technological revolutions include David [1990], Hobijn and Jovanovic [2001], Mokyr [1990], Jovanovic and Rousseau [2005], Laitner and Stolyarov [2003], Pastor and Veronesi [2009], and Brynjolfsson and McAfee [2011]. In the present paper, each GPT arrival initiates an era of progress, and the procession of such eras determines the economy's steady-state equilibrium.

Our model has 2 key additional assumptions. The first is that capital is technology specific. That makes punctuated TFP improvements compatible with the relatively smooth GDP advances that we see in practice. We assume that as the potential advantages of a new technology become evident, the knowledge spreads rapidly. No firm has a particular advantage in adopting the new technology as knowledge is not privately owned.[2] With large technology shocks, the momentary rate of growth of the GDP can swing abruptly. Yet, effects on the level of the GDP are gradual, as they depend on the accumulation of new-vintage capital.

In contrast, financial-market reactions can be dramatic. As a new GPT arrives, investment in old-vintage capital will cease and the market price of existing plant and equipment will plummet due to obsolescence. A GPT's wide scope implies the drop, in turn, can cause a bear market in equities. The corresponding sequence of bull and bear markets plays a major role in our analysis.[3]

Technology-specific capital can assist in model implementation. In our framework, a GPT's "arrival" corresponds to the time when the new technology's significance is broadly acknowledged. In practice, a seminal invention's main applications may follow its first appearance with a significant lag (e.g., David [1990]). While historians' narratives can help us to establish the general time frame of a specific GPT's acceptance, we can date the arrival more precisely if it induces a bear market. (Moreover, the bear's depth can help to establish the magnitude of a technology shock's TFP contribution.)

Our second key additional assumption is limited participation (Kocherlakota [1996]). Specifically, our model has 2 types of households. According to the Survey of Consumer Finances (SCF), the top 5% of wealth holders own roughly half the U.S. economy's net worth, including a sizable share of the nonresidential capital stock (Kennickell [2009]). We pattern our type-I agents on that group. They finance the business sector's capital.[4] Type-II agents compose the remainder of the population. For simplicity, we assume they are hand-to-mouth consumers.

For tractability, we use a representative agent formulation to describe the sector of type-I households. We think of type-I households as having high earnings and/or inheritances. In practice, they would be prime candidates to leave intentional bequests, and our formulation's infinite time horizon may approximate a dynastic attention span.

Jorgenson [2019] emphasizes the singular role of the IT revolution in recent decades; and, news reports of potential breakthroughs in artificial intelligence, nuclear fusion, and genetic therapies attract a great deal of attention.

[2] For a different treatment, see Laitner and Stolyarov's [2003] use of intangible capital.

[3] Hobijn and Jovanovic [2001] provide direct evidence of this mechanism in the context of the IT revolution — see below. See also our use of stock-market data in our calibration process.

[4] For simplicity, we omit residential housing from our model.

The 2-agent specification can capture a potentially important relation between earnings and wealth. We have a production sector with capital and labor inputs. If labor earnings are imperfectly correlated with asset returns, they raise an agent's tolerance for investment risk. Though top wealth owners in the SCF have above-average wages, their wealth is disproportionately higher still. The same pattern can hold in our analysis because type-II households can account for most of the economy's labor earnings.[5] For a given CRRA, investors, that is to say, type-I households, can then manifest more risk aversion than otherwise.

Another implication of our specification pertains to consumption. In practice, aggregate consumption exhibits low volatility. In the model, however, the consumption response of portfolio holders to technology shocks may be sizable (see also the evidence in Ait-Sahalia, Parker, and Yogo [2004]). Our formulation allows these patterns to be mutually consistent: type-I households are few in number and save a high fraction of their income; hence, their expenditure changes need have only a limited effect on aggregate consumption.

Each technology shock simultaneously promotes future growth, causes immediate capital losses in financial markets, and creates exceptional opportunities for new investment. When a shock occurs, the latter 2 effects induce type-I agents to rein in their consumption immediately. Low overall portfolio returns (from capital losses) thus coincide with a high marginal utility of consumption, fueling an equity premium. A variant of that premium is the basis of our proposed indirect method for detecting growth-trend shifts.

The organization of this paper is as follows. Sections 2-4 describe our model and method for calculating its equilibrium. Section 5 explains our calibrations and simulations, and Section 6 uses them to illustrate the natural variability of growth and difficulties of detecting trend shifts. Section 7 proposes an indirect approach for identifying a permanent slowdown, and Section 8 concludes.

Relation to the literature. Our work attempts to bridge two strands of literature: macro-finance general equilibrium analyses where technological change is a source of aggregate risk, and frameworks of past or future growth slowdowns rooted in observations of real activity.

A number of current macroeconomic studies examine the possibility that the supply of promising new ideas may be running out (e.g., Fernald and Jones [2014], Bloom et al [2020]). Part of their argument is that the growth of expenditures on idea production observed during the last century may not be sustainable in the long run. We present a framework in which the absence of idea depletion is a maintained, but testable, hypothesis.

Statistical analyses show that labor productivity is indeed prone to cyclic swings, sometimes lasting for decades (e.g., Fritz, Gries, and Feng [2019], Pagano and Sbracia [2018], Antolin-Diaz, Drechsel, and Petrella [2017]).

[5] Our calibrated examples go further. Among the SCF's top wealth holders, some may be retired or living on inheritances, but others have very high earnings. Among the latter, an above-average share own their own business or have a corporate position with an annual bonus dependent upon the company's stock-market performance. Hence, we allow a portion of type-I agent earnings to be highly correlated with asset returns.

Recent structural approaches combining models and microeconomic data examine possible links between growth and impediments to diffusion of information technologies (e.g., Acemoglu, Autor, and Patterson [2023] and Andrews, Criscuolo, and Gal [2016]). Our focus is the long run: rather than studying a particular GPT's attributes, our attention centers on a steady-state equilibrium in which one GPT replaces another in a sequence of episodes of creative destruction.

One strand of literature — starting with Rietz [1988] and followed by Barro [2006], Gourio [2008] and Gabaix [2012] — studies the chance of major natural disasters and wars. Another, the real business cycle (RBC) literature (e.g., Jermann [1998]), incorporates technology shocks of business-cycle frequency. In contrast to the former, we focus on technology shocks; contrary to the later, we assume a constant labor supply. In general, our shocks are more frequent than natural disasters but one-third to one-half as common as those in RBC models; they always enhance the economy's TFP frontier; the changes they bring are permanent; and, their (immediate) impact falls upon asset prices rather than GDP, profits, or dividends.

Papers emphasizing punctuated change and medium-length cycles include Comin and Gertler [2006] and Garleanu et al [2012], with the latter also incorporating vintage capital. Kung and Schmid [2015] formulate an endogenous growth model where persistent changes in productivity growth are echoes of past R&D activity amplified through a positive knowledge externality. Kogan and Papanikolaou [2014] combine high frequency shocks with technology-specific capital, investigating the cross-section of stock returns. As stated, we seek to look past the unevenness of momentary growth rates to study their long-run mean, and that goal leads us to a highly aggregative modeling formulation.

A number of quantitative macro-finance models incorporate income distribution and risk-sharing. Guvenen [2009], Chien et al [2011], Danthine and Donaldson [2002], and Kogan, Papanikolaou, and Stoffman [2020] develop heterogeneous-agent models. Our 2-agent formulation is very basic in comparison, more closely resembling, for instance, Campbell and Mankiw [1989]. Nonetheless, it allows portfolio holders to manifest substantial risk aversion without a very high CRRA; applies an Euler-equation analysis only to the wealthiest segment of the population, who, in practice, may be the most financially sophisticated and least likely to be liquidity constrained (see, for example, Mankiw and Zeldes [1991], Vissing-Jorgensen [2002], and Attanasio et al [2002]); and, allows a time path of aggregate consumption that differs substantially from that of the consumption of major financial-market participants (Ait-Sahalia, Parker, and Yogo [2004]).

The underlying simplicity of our model may facilitate future extensions. Although the present paper includes only one shock type, earlier versions showed, for example, that we can easily incorporate multiple shock varieties (provided they are distributed independently of one another).

2. Model

We first describe our model's production sector and household preferences.

Production. Technology shock arrivals are Poisson events. The index of the best available technology at time t is J_t, where the frontier TFP level is $[\theta]^{J_t}$, with $\theta > 1$ a parameter. Thus, subsequent technology shocks multiply the frontier TFP level by θ. The hazard rate

for shocks is λ.

We have type-I and type-II households. Each household inelastically supplies 1 unit of labor at all times. The effectiveness of the labor of type-I and II households can differ (in practice, type-I agents have a higher wage rate, for example). The number of households of each type is fixed:

$$N_I = 1 \quad \text{and} \quad N_{II} = N\,. \tag{1}$$

We expect $N \gg 1$.

There is 1 output good at each t, its price is always 1, and output is homogeneously divisible into consumption and investment goods. Physical capital is technology specific. At t, $\mathcal{K}_{tj}$ is the quantity of capital specific to technology index j. The time-t production function for physical output $\mathcal{Y}_{jt}$ from labor and vintage $j \leq J_t$ capital is

$$\mathcal{Y}_{jt} = Z \cdot [\theta]^j \cdot [\mathcal{K}_{jt}]^{\alpha} \cdot [L_{I,j,t}]^{\beta\cdot(1-\alpha)} \cdot [L_{II,j,t}]^{(1-\beta)\cdot(1-\alpha)}\,, \quad \alpha\,,\,\beta \in (0,1)\,. \tag{2}$$

Investment is irreversible: once built, a unit of capital cannot be transformed to a different vintage or to consumption. Investment at time t theoretically can include any vintage $j \leq J_t$, with each unit of investment costing 1 dollar. The form of (2) shows, however, that higher TFP levels are unambiguously better. Thus, all new investment at t flows to capital of vintage J_t.

Physical capital depreciates at wear and tear rate $\delta \in (0,1)$. A second source of depreciation works as follows: when a technology shock occurs, the value of existing capital immediately declines through obsolescence. A key result is:

Prop. 1: *Let P_{jt} be the time-t price per physical unit of capital of vintage $j \leq J_t$. Then*

$$P_{jt} = [\theta]^{-(J_t - j)/\alpha}\,.$$

Proof: See Laitner/Stolyarov [2003].

In other words, as a new technology arrives, the value of each existing physical unit of capital drops to b times its value the moment before with

$$b \equiv [\theta]^{-1/\alpha}\,. \tag{3}$$

Using Proposition 1, the market value of the time-t capital stock is

$$K_t \equiv \sum_{j \leq J_t} [b]^{J_t - j} \cdot \mathcal{K}_{jt}\,. \tag{4}$$

Let I_t be the current flow of new investment. If there is no technology shock at t, we have

$$\dot{K}_t = I_t - \delta \cdot K_t\,. \tag{5}$$

If, however, a shock occurs at t, at that instant we have

$$J_t = J_{t-0} + 1 \text{ and } K_t = b \cdot K_{t-0}\,. \tag{6}$$

Let Y_t be aggregate output at t. Let L_{Ijt} be the labor allocation from type-I households to vintage j capital, and similarly for $L_{II,j,t}$. Given the labor supplies of (1), competitive behavior gives

$$Y_t = \max_{L_{Ijt}, L_{II,j,t}} \sum_{j \le J_t} \mathcal{Y}_{jt} \tag{7}$$

$$\text{subject to:} \quad \sum_{j \le J_t} L_{Ijt} = 1 \text{ and } \sum_{j \le J_t} L_{II,j,t} = N\,.$$

Fix N. Solow [1960] showed how to construct an aggregate production function $Y_t = F(K_t, J_t)$. We can generalize his methodology to encompass technological change in discrete steps. Let W_t be the wage per unit of type-I household labor and M_t be the marginal product of K_t net of wear and tear depreciation. Then

Prop. 2: *Set $J = J_t$ and, for convenience, set Z to satisfy*

$$Z \cdot [N]^{(1-\beta)\cdot(1-\alpha)} = 1\,.$$

Let K_t be as in (4). Then

$$Y_t = F(K_t, J) \equiv [\theta]^J \cdot [K_t]^\alpha\,, \quad W(K_t, J) = \beta \cdot (1-\alpha) \cdot [\theta]^J \cdot [K_t]^\alpha\,,$$
$$M_t \equiv M(K_t, J) = \alpha \cdot [\theta]^J \cdot [K_t]^{\alpha-1} - \delta\,.$$

Proof: See Laitner and Stolyarov [2003].[6]

Figure 1 (see the Appendix) illustrates the time paths of K_t and Y_t. Given Prop. 2, if $J = J_t$, the per capita gross income of type-I households — as well as their sector's aggregate gross income — is

$$\eta \cdot [\theta]^J \cdot [K_t]^\alpha\,, \quad \eta \equiv \alpha + \beta \cdot (1-\alpha)\,. \tag{8}$$

Let R_t be the return per dollar's worth of capital, net of depreciation. Let λ be as above. For an investment of \$1 and a short passage of time Δ, we have

$$1 + R_t \cdot \Delta = \begin{cases} 1 + M(K_t, J_t) \cdot \Delta\,, & \text{with prob. } 1 - \lambda \cdot \Delta, \\ b + M(b \cdot K_t, J_t + 1) \cdot \Delta\,, & \text{with prob. } \lambda \cdot \Delta, \end{cases} \tag{9}$$

where a new GPT arrives in the second instance but not the first. At time t, the return on capital in the ensuing instant is a random variable. From (9), the latter's mean is

$$\mathbb{E}_t[R_t] = M(K_t, J_t) - \lambda \cdot (1-b)\,. \tag{10}$$

[6] A detailed proof is on our website ....

Agent Preferences. We assume type-I agents constitute a set of measure 1, take prices as given, and are all alike — behaving, in equilibrium, alike as well. Let C_t be each's time-t consumption flow. Each has isoelastic preferences, $u(C_t) = [C_t]^\gamma/\gamma$, $\gamma < 1$, with subjective discount rate ρ. We employ a representative-agent formulation for the type-I agent sector. Call it an RAI specification.

3. Type-I Agent Behavior

Sections 3-4 introduce our concept of equilibrium and an algorithm for computing it.

Utility Maximization and Rational Expectations. Let $J = J_t$ be the initial "technology index" and society's initial frontier technology be $[\theta]^J$. As above, $\theta > 1$. The period over which a particular technology index prevails is an "era." While index J prevails, all type-I agents anticipate a time path for the aggregate capital stock of

$$\vec{K}(J) \equiv \{K_s\,, s \geq 0 \big|\, J\}\,,$$

where $s \geq 0$ measures time within the era. We model type-I households with a representative-agent specification. The RAI solves

$$V\big(A, \vec{K}(J), J\big) = \max_{C_t} \int_0^\infty \lambda \cdot e^{-\lambda \cdot T} \cdot \Big[\int_0^T e^{-\rho \cdot t} \cdot u(C_t)\, dt + e^{-\rho \cdot T} \cdot U\big(b \cdot A_T, \vec{K}(J+1), J+1\big)\Big]\, dT \tag{11a}$$

$$\text{subject to:} \quad \dot{A}_t = W(K_t, J) + M(K_t, J) \cdot A_t - C_t\,, \quad A_0 = A\,, \tag{11b}$$

where $V(.)$ is the value function, A_t the RAI's net worth, and C_t his/her consumption. The wage and marginal product of capital functions, $W(.)$ and $M(.)$, come from Prop. 2, and the RAI evaluates them taking as given values for K_s from $\vec{K}(J)$.

We want rational expectations. Hence, we derive first-order conditions for (11a-b), giving $(\dot{A}_t\,,\, \dot{C}_t)$ as a function of (A_t, C_t, K_t). Then we solve the 2 corresponding differential equations imposing, for all $s \geq 0$ during the current technology era,

$$K_s = A_s\,. \tag{11c}$$

Having obtained the rational expectations solution, we call the relation between the RAI's current consumption and its net worth the model's "consumption function," $\Phi(A_t, J) = C_t$. Our attention focuses on a long-run equilibrium in which the same consumption function holds for all eras and exhibits the homogeneity property

$$\Phi(A, J+1) = x \cdot \Phi(\frac{A}{x}, J) \quad \text{where} \quad x \equiv [\theta]^{\frac{1}{1-\alpha}}. \tag{12}$$

Auxiliary Problems. To solve the model, we numerically determine an equilibrium consumption function $\Phi(A, J) = C$. Because we can establish condition (12) theoretically, we need only construct $\Phi(A, 0)$. Furthermore, we show that when $J = 0$, we can restrict our attention to a bounded interval of values for A.

To compute Φ, we develop a sequence of auxiliary problems. The latter are indexed with $m = 0, 1, ...$, where m is the number of future technology shocks allowed. We are interested in the limit as $m \to \infty$. We solve the sequence of auxiliary problems numerically, recursively deriving consumption functions Φ^m. We numerically check for convergence $\Phi^m \to \Phi$. Provided uniform convergence obtains, Φ is the focus of our analysis. Because we have heterogeneous agents, we cannot apply the most standard representative-agent solution methodologies of, say, Cooley and Prescott [1995], to our auxiliary problems. Nevertheless, as above, we adapt related methods from Danthine and Donaldson [1995] and others.

To start the algorithm, set $m = 0$ — so that there are no future GPT arrivals. Let (C_s, A_s) be the consumption and net asset position of our RAI at time s. The RAI solves

$$v^0\big(A, \vec{K}(J), J\big) \equiv \max_{C_s} \int_0^\infty e^{-\rho \cdot s} \cdot u(C_s)\, ds \tag{13a}$$

$$\text{subject to:} \quad \dot{A}_s = W(K_s, J) + M(K_s, J) \cdot A_s - C_s\,, \quad A_0 = A\,. \tag{13b}$$

The K_s for $W(K_s, J)$ and $M(K_s, J)$ comes from $\vec{K}(J)$ — see above — and $W(.)$ and $M(.)$ come from Prop. 2.

Set up a current value Hamiltonian with costate μ_s:

$$\mathcal{H}^0 \equiv u(C_s) + \mu_s \cdot [W(K_s, J) + M(K_s, J) \cdot A_s - C_s]\,. \tag{14}$$

First-order conditions yield

$$\dot{C}_s = \frac{C_s}{1-\gamma} \cdot [M(K_s, J) - \rho]\,, \tag{15a}$$

$$\dot{A}_s = W(K_s, J) + M(K_s, J) \cdot A_s - C_s\,, \quad A_0 = A\,. \tag{15b}$$

The functions $W(.)$ and $M(.)$ are as in Prop. 2. Impose rational expectations, i.e., set $\vec{K} = \vec{A}$ as in (11c). We then have

$$\dot{C}_s = \frac{C_s}{1-\gamma} \cdot \left[\alpha \cdot [\theta]^J \cdot [A_s]^{\alpha-1} - (\delta + \rho)\right], \tag{16a}$$

$$\dot{A}_s = \eta \cdot [\theta]^J \cdot [A_s]^\alpha - \delta \cdot A_s - C_s\,, \quad A_0 = A\,. \tag{16b}$$

Figure 2 (see Appendix) presents the phase diagram. Suppose it has a stationary point (A^{*0}, C^{*0}) and a saddle path solution to (16a-b), as illustrated. First-order conditions (15a-b) hold by construction on the saddle path. The saddle path characterizes our $m = 0$ consumption function: $\Phi^0(A, J) = C$ for $A \in (0, A^U]$, If the RAI follows the saddle path solution, the conventional transversality condition for (13a-b) holds.

We assume the envelope theorem applies:

$$\frac{\partial v^0\big(A,\vec{K}(J),J\big)}{\partial A}\big|_{\vec{K}=\vec{A}} = u'\big(\Phi^0(A,J)\big)\,. \tag{17}$$

We often use the shorthand expression $V^0(A,J)$ for the maximized value of the criterion to (13) with rational expectations, but we must keep (17) in mind.

The following Lemma is basic to our analysis.

Lemma 1: *Let $x \equiv [\theta]^{\frac{1}{1-\alpha}}$ as in (12). Then*

$$\Phi^0(A,J+1) = x\cdot\Phi^0(\frac{A}{x},J)\,, \tag{i}$$

$$V^0(A,J+1) = [x]^{\gamma}\cdot V^0(\frac{A}{x},J)\,. \tag{ii}$$

Proof: See Appendix.

Proceed to $m=1$, i.e., the case with 1 GPT arrival in the future. Again, let $\vec{K}(J)$ denote the time path while the current technology index J predominates. The RAI takes $\vec{K}(J)$ as given. Let the era of technology J run from $s=0$ until the (single) future GPT arrives. Assume that at the latter's arrival, the RAI shifts to problem (13), with its rational expectations solution, consumption function Φ^0, and envelope condition (17).

The criterion for the $m=1$ problem eflects expected utility. With Poisson shocks, the density for arrival times is exponential. Thus, we have

$$\begin{aligned} &v^1\big(A,\vec{K}(J),J\big) \\ &\quad = \max_{C_t}\int_0^{\infty}\lambda\cdot e^{-\lambda\cdot T}\cdot\Big[\int_0^{T} e^{-\rho\cdot t}\cdot u(C_t)\,dt + e^{-\rho\cdot T}\cdot V^0(b\cdot A_T,J+1)\Big]\,dT. \end{aligned} \tag{18}$$

The right-hand side of (18) sums 2 terms. The first has a double integral. Changing the order of integration, noting that

$$\int_t^{\infty}\lambda\cdot e^{-\lambda\cdot s}\,ds = e^{-\lambda\cdot t}\,,$$

and changing the (dummy) variable of integration in the second term to t, we can rewrite the criterion's integral as

$$\int_0^{\infty} e^{-(\lambda+\rho)\cdot t}\cdot\big[u(C_t)+\lambda\cdot V^0(b\cdot A_t,J+1)\big]\,dt\,. \tag{19}$$

With the constraint remaining as in (13b), the Hamiltonian for the first era of the $m=1$ problem is

$$\mathcal{H}^1 = u(C_t)+\lambda\cdot V^0(b\cdot A_t,J+1)+\mu_t\cdot[W(K_t,J)+M(K_t,J)\cdot A_t - C_t]\,. \tag{20}$$

Imposing rational expectations (as above) and using (17) and Lemma 1, we have

$$\dot{C}_t = \frac{C_t}{1-\gamma} \cdot \left[\alpha \cdot [\theta]^J \cdot [A_t]^{\alpha-1} + \lambda \cdot b \cdot \frac{u'\big(x \cdot \Phi^0(\frac{b \cdot A_t}{x}, J)\big)}{u'\big(C_t\big)} - (\delta + \lambda + \rho)\right], \tag{21a}$$

$$\text{... same as (16b) ...} \tag{21b}$$

Figure 3 (see Appendix) illustrates the phase diagram. The A-isocline is the same as in Figure 2. The C-isocline is upward sloping. Provided we again have a saddle path solution, the transversality condition for the $m = 1$ problem holds and we can define Φ^1 and V^1 analogously to Φ^0 and V^0.

We have

Lemma 2: *If x is as in Lemma 1, then*

$$\Phi^1(A, J+1) = x \cdot \Phi^1(\frac{A}{x}, J)\,, \tag{i}$$

$$V^1(A, J+1) = [x]^\gamma \cdot V^1(\frac{A}{x}, J)\,. \tag{ii}$$

Proof: See Appendix.

Proceeding with $m = 2, 3, ...$, we can see that the proof of Lemma 2 carries over.

In Figures 2-3, notice that although $A = 0$ is not itself a value of interest, if we extend the saddle path to include the origin, Φ^m remains a continuous function. For convenience of exposition, define $\Phi^m(A, J)$ for $A \in [0, A^U]$, setting $\Phi^m(0, J) = 0$.

4. Steady-State Equilibrium

We solve our sequence of auxiliary problems numerically, finding the saddle path that satisfies the analogs (21a-b) at each $m = 0, 1, ...$ For each m, set $J = 0$ and compute, and save, the first era's stationary point (A^{*m}, C^{*m}) and the consumption function Φ^m, evaluated on the domain $[0, A^U]$. The $A - isocline$ (recall Figures 2-3) is identical for all $m = 0, 1, ...$ The $C - isocline$ has a positive slope, and a fixed intercept, all $m \geq 1$. The stationary points lie at the intersection of the isoclines; so,

$$\bar{C}(A^{*m}) = \Phi^m(A^{*m}, 0) \text{ where } \bar{C}(A) \equiv \eta \cdot [A]^\alpha - \delta \cdot A\,.$$

If we can determine $\Phi^m(A, 0)$, we can derive $\Phi^m(A, J)$ all $J > 0$ from Lemmas 1-2. Our analysis centers on $\Phi \equiv \lim_{m \to \infty} \Phi^m$, and we seek to establish the convergence numerically.

This paper focuses on outcomes obeying the following conditions:

Conditions: **(C1)** *For the first era with each m in our sequence of auxiliary problems, we have a phase diagram, as in Figs 2-3, with an upward sloping saddle path, a single positive stationary point (A^{*m}, C^{*m}), and implied motion toward the stationary point for every initial asset value in $(0, A^U]$. We have $C^U < \infty$ such that $\Phi^m(A^U, 0) \leq C^U$ all*

m. In addition, **(C2)** *as* $m \to \infty$*, we have uniform convergence* $\Phi^m(A,0) \to \Phi(A,0)$ *for* $A \in [0, A^U]$*;* **(C3)** *there is* $A^L > 0$ *such that* $A^L \leq A^{*m} < A^U$ *all* m*;* **(C4)** $A^{*m} \to A^*$ *with* $\bar{C}(A^*) = \Phi(A^*, 0)$ *and* $A < (>) A^*$ *implying* $\bar{C}(A) > (<) \Phi(A,0)$*; and,* **(C5)** *the sequence* $\partial\Phi^m(A,0)/\partial A$ *converges uniformly on any compact interval* $[\bar{A}, A^U]$ *with* $\bar{A} > 0$.[7] *Finally,* **(C6)** *the envelope theorem (see (17)) holds each* m*; and,* **(C7)** *if* $\gamma < 0$*, for some* $\xi > 0$ *and all* m *we have* $\Phi^m(A,0) \geq \xi \cdot A$ *all* $A \in [0, A^U]$.

This Section proceeds in 3 steps.

De-trending. Our basic de-trending formula is an extension of Lemmas 1-2.

Prop. 3A: *Let* $x > 1$ *be as in (12). Assume condition (C2). Then*

$$\Phi(A,J) \equiv \lim_{m\to\infty} \Phi^m(A,J) \text{ exists all } A \in \left[0, [x]^J \cdot a^U\right],\ J \geq 0, \tag{i}$$

$$\Phi(A,J) = x \cdot \Phi(\frac{A}{x}, J-1) = ... = [x]^J \cdot \Phi(\frac{A}{[x]^J}, 0)\,. \tag{ii}$$

Proof: see Appendix.

In textbook treatments of nonstochastic growth, given a rate of Hicks neutral technical change, we transform to a Harrod-neutral measure, divide the capital stock by the latter, and perform most of the analysis on the resulting ratio. Proposition 3A shows the analog applies here: if we transform our Hicks-neutral technology factor θ into a Harrod-neutral factor x (see (12)), we can focus our attention on

$$a_t \equiv A_t/[x]^{J_t} \quad \text{and} \quad c_t \equiv C_t/[x]^{J_t} \quad \text{and} \quad k_t \equiv K_t/[x]^{J_t}\,. \tag{22}$$

Note that (12) implies $\theta \cdot [x]^\alpha = x$, recall budget constraint (16b), and define

$$\Gamma(A, J, \Phi) \equiv \eta \cdot [\theta]^J \cdot [A]^\alpha - \delta \cdot A - \Phi(A,J)\,. \tag{23}$$

Then in an era with technology $J = J_t$, we have

$$\begin{aligned}
&\dot{A}_t = \Gamma(A_t, J, \Phi)\,, \quad A_0 = A\,,\\
\iff &\frac{\dot{A}_t}{[x]^J} = \frac{\eta\cdot[\theta]^J}{[x]^J}\cdot[A_t]^\alpha - \delta\cdot\frac{A_t}{[x]^J} - \frac{\Phi(A_t,J)}{[x]^J}\,, \quad A_0 = A\,,\\
\iff &\frac{\dot{A}_t}{[x]^J} = \eta\cdot[\frac{A_t}{[x]^J}]^\alpha - \delta\cdot\frac{A_t}{[x]^J} - \frac{[x]^J}{[x]^J}\cdot\Phi(\frac{A_t}{[x]^J},0)\,, \quad A_0/[x]^J = A/[x]^J\,,\\
\iff &\dot{a}_t = \Gamma(a_t, 0, \Phi)\,, \quad a_0 = A/[x]^J\,,
\end{aligned} \tag{24}$$

[7] Suppose (A_t^m, C_t^m) solve our differential equations for stage m. They are continuously differentiable by construction. Accordingly, Φ^m is continuously differentiable in A. We can see $\partial\Phi^m(A_t^m,0)/\partial A_t^m = \dot{C}_t^m/\dot{A}_t^m$ or, when $A_t^m = A^{*m}$, it is given by the slopes of the eigenvectors associated with the stationary point.

$$c_t \equiv \frac{C_t}{[x]^J} = \frac{\Phi(A_t, J)}{[x]^J} = \frac{[x]^J \cdot \Phi(a_t, 0)}{[x]^J} = \Phi(a_t, 0)\,. \tag{25}$$

And, rational expectations implies $K_t = A_t$ all t.

We can restrict our attention to a bounded set of values for a_t, as follows. Assume Conditions (C1)-(C7) and, letting A^* be as in (C4), define

$$a^* \equiv A^* \text{ and } \mathcal{A} \equiv (0, a^*]\,. \tag{26}$$

In the notation a_t, t refers to calendar time. It is often more convenient, however, to reset the time index to 0 at each GPT arrival. Suppose technology era J lasts for s with $0 \leq s < \bar{s}(J)$, with $a_{s,J}$ giving the value of a within the era. By (C5), $\Gamma(a, 0, \Phi)$ from (24) is continuously differentiable; thus, we expect to have a solution $a_{s,J} = \Psi(s, a_{0,J})$ from any initial condition $a_{0,J} \in \mathcal{A}$. Condition (C4) implies $\Psi(s, a_{0,J})$ increases in s, asymptotically approaching a^*. So, $\Psi(s, a_{0,J}) \in \mathcal{A}$ each $s \geq 0$. When the era ends, we jump to $a_{0,J+1} = b \cdot \Psi(\bar{s}(J), a_{0,J})/x$. The latter is clearly within $\mathcal{A}$. Hence, we have established

Prop. 3B: *Assume conditions (C1)-(C7). Then*

$$a_{s,J} \in \mathcal{A} \Longrightarrow a_{s',J'} \in \mathcal{A} \quad \forall s' \geq 0\,,\ J' \geq J\,. \tag{i}$$

Proof: See the preceding paragraph.

To summarize the notation, for any J and initial condition $a_{0,J}$,

$$a_{s,J} = \Psi(s, a_{0,J}) \text{ solves (24) all } s \geq 0 \text{ and } \lim_{s\to\infty} \Psi(s, a_{0,J}) = a^*\,. \tag{27}$$

Value Functions. Value functions are an important part of our equilibrium, and this section provides sufficient conditions for the existence of $V(A, J) = \lim_{m\to\infty} V^m(A, J)$. Since this paper's applications do not make explicit use of $V(.)$, the proofs are available on the authors' websites.

We have

Lemma 3: *Let Conditions (C1)-(C7) hold. Then sufficient conditions for $V^m(A, 0)$ any $A \in (0, A^U]$ to be finite all $m \geq 0$ are*

$$\lambda \cdot ([x]^\gamma - 1) - \rho < 0\,, \tag{i}$$

$$\lambda \cdot ([b]^\gamma - 1) - \rho < 0\,. \tag{ii}$$

Proof: See authors' website.

Conditions (i)-(ii) together require $\rho > 0$. If $\gamma > 0$, only condition (i) ever binds; if $\gamma < 0$, only condition (ii) matters.

Given Lemma 3 and the consumption function convergence established numerically in (C1-7), we can prove that a bounded $V(A, J)$ exists.

Prop. 3C: *Given the conditions of Lemma 3, $V(A,J) = \lim_{m\to\infty} V^m(A,J)$ exists each $A \in (0, [x]^J \cdot A^U]$; it is continuously differentiable in A with $\partial V(A,J)/\partial A = u'\big(\Phi(A,J)\big)$; it is increasing and concave at each saddle path point; and, it has the homogeneity property of Lemmas 1-2, part (ii).*

Proof: See authors' website.

For the purposes of this paper, we do not need to compute $V(A,J)$. Conceptually, its existence is important, however. Thus, we check conditions (i)-(ii) of Lemma 3 for our simulations below.

Saddle Path Growth. Most of the elements of the model's equilibrium are now in place. See Figure 4 in the Appendix.

Characteristics of the equilibrium evident so far are as follows. Referring to Fig. 4 in the Appendix, (i) the set $\mathcal{A}$ is invariant — if we start within it, Prop. 3B implies we stay within it. (ii) Within $\mathcal{A}$, our assumption that old-vintage capital cannot be transformed to new-vintage capital or consumption goods holds because net investment is non-negative between GPT arrivals (see (C4) above). (iii) Type-I households maximize their expected utility. Specifically, in each era they maximize a Hamiltonian, as in (20). It is continuously differentiable; nondecreasing in A_t; and, concave in (A_t, C_t). The corresponding transversality condition holds. And, (iv) our method of deriving a saddle path solution (see (16a-b) and (21a-b)) ensures that agents' expectations are rational.

Steady-State Distribution of Wealth. The final component of our equilibrium is a stationary distribution of wealth. We proceed in 2 steps.

In the first step, we define a discrete time, continuous state space Markov process. Suppose technology era J ends, and era $J+1$ begins, at calendar time $\bar{t}(J)$. Then $\bar{t}(J+1) = \bar{t}(J) + \bar{s}(J+1)$. Define $\bar{a}_{\bar{t}(J)} \equiv \Psi(\bar{s}(J), a_{0,J})$. Let $\mathcal{A}(J+1) \subseteq \mathcal{A}$ be a Borel set and μ be Lebesgue measure. Then the transition function

$$P\big(\bar{a}_{\bar{t}(J)}, \mathcal{A}(J+1)\big) = \int_{\{s\geq 0 | \Psi(s, b\cdot \bar{a}_{\bar{t}(J)}/x) \in \mathcal{A}(J+1)\}} \lambda \cdot e^{-\lambda \cdot s}\, \mu(ds) \tag{28}$$

defines a Markov process from $\bar{a}_{\bar{t}(J)} \in \mathcal{A}$ to points $\bar{a}_{\bar{t}(J+1)} \in \mathcal{A}(J+1)$. We have

Prop. 3D: *Suppose conditions (C1)-(C7) and (i)-(ii) from Lemma 3 hold and*

$$\frac{1}{1-\gamma} - \eta > 0\,. \tag{i}$$

Then P from (28) has a unique stationary distribution on $\mathcal{A}$, with corresponding stationary probability measure π; $\mathcal{A}$ is an ergodic set; and, $\mathcal{A}$ has no cyclic subsets.

Proof: See Appendix (including Lemma 4 therein).

Secondly, we show Prop. 3D implies there is a unique stationary distribution for a_t, where t is calendar time. In practice, we will want to compute moments of the stationary distribution — and we show the intuitive method for doing so will work.

Corollary 3D: *Suppose Prop. 3D holds. Then there is a unique stationary probability measure* Π *for* a_t *on* $\mathcal{A}$. *Choose any* $\bar{t}(0)$ *and* $\bar{a}_{\bar{t}(0)} \in \mathcal{A}$. *Let* $\bar{s}(J)\,,\ J = 1, 2, ...,$ *be random samplings from the exponential distribution with parameter* λ. *Let* $\bar{t}(J) = \bar{t}(J-1) + \bar{s}(J)$ *and* $\bar{a}_{\bar{t}(J)} = \Psi\big(\bar{s}(J), b \cdot \bar{a}_{\bar{t}(J-1)}/x\big)$ *all* $J = 1, 2, ...$ *Let* $h(.)$ *be measurable and*

$$\int_{\mathcal{A}} |h(a)|\, \Pi(da) < \infty\,. \tag{i}$$

Then

$$\mathbb{E}[h(a)] \equiv \int_{\mathcal{A}} h(a)\, \Pi(da) \overset{\text{P}}{=} \lim_{J\to\infty} \frac{\sum_{j=1}^{J} \int_0^{\bar{s}(j)} h\big(\Psi(s, b \cdot a_{\bar{t}(j-1)}/x)\big)\, ds}{\bar{t}(J) - \bar{t}(0)}\,. \tag{ii}$$

Proof: See Appendix.

To find the mean of the stationary distribution of a_t, for example, we set $h(a) = a$ and simulate a time path for a_t. Cor. 3D shows that the average value of a_t along the simulated path provides a consistent estimate of $\mathbb{E}[a]$.[8]

5. SIMULATIONS

Figures 1 and 4, for example, depict qualitative aspects our model's equilibrium. The remainder of this paper uses simulations to provide illustrative quantitative results.

Rates of Return. In the preceding sections, we solve for $\Phi(.)$ and the stationary distribution of a_t without the need to specify the structure of the financing of the economy's stock of plant and equipment. Now we assume fixed shares of bonds, υ, and common-stock, $1-\upsilon$. That enables us to determine the average equilibrium rates of return for bonds and stocks, from which the equity premium follows. As discussed in the introduction, we use a variant of the latter to detect changes in the long-run growth rate.

The model has an Euler equation relating the growth rate of type-I agent consumption to the return on (physical) investment (e.g., (16a), (21a), etc.), and we can derive the analog for the return on risk-free bonds. Suppose a risk-free bond issued at t costs \$1, promises an interest rate r_t, and expires at $t+h$, where h is small. If R_t^S and R_t are the realized returns per dollar of common stock and physical investment, respectively, then

$$R_t = (1-\upsilon) \cdot R_t^S + \upsilon \cdot r_t\,. \tag{29}$$

Without loss of generality, let the current technology index be 0. The sacrifice of current utility for a type-I household purchasing the bond is $u'(c_t)$. During the bond's lifespan, a new GPT arrives with probability $\lambda \cdot h$. If an arrival occurs, the household's marginal utility jumps to $u'(\tilde{c}_t)$ with $\tilde{c}_t \equiv \Phi(b \cdot a_t, 1) = x \cdot \Phi(b \cdot a_t/x, 0)$.[9] Absent an arrival, the time

[8] Verifying condition (i) in the corollary can be difficult if $h(.)$ is unbounded on $\mathcal{A}$ — for instance, if we evaluate the long-run average R_t (see below). In those cases, we compute the right-hand side of (ii) for J =1000, 2000, and 4000 and make sure it convergences.

[9] Suppose a GPT arrival occurs. As we let $h \to 0$ below, the timing of the shock within $(t, t+h)$, and consumption growth before and after the shock, become irrelevant.

path of marginal utility is continuously differentiable; so, $u'(c_{t+h}) \approx u'(c_t) + u''(c_t) \cdot \dot{c}_t \cdot h$. Thus,

$$u'(c_t) \approx (1 - \lambda \cdot h) \cdot \big(1 + (r_t - \rho) \cdot h\big) \cdot \Big(u'(c_t) + (\gamma - 1) \cdot u'(c_t) \cdot \frac{\dot{c}_t}{c_t} \cdot h\Big) + \lambda \cdot h \cdot \big(1 + (r_t - \rho) \cdot h\big) \cdot u'(\tilde{c}_t)\,.$$

Rearranging terms, letting $h \to 0$, and dropping terms with powers of h greater than 1,

$$r_t = \rho + (1 - \gamma) \cdot \frac{\dot{c}_t}{c_t} - \lambda \cdot \big[\frac{u'(\tilde{c}_t) - u'(c_t)}{u'(c_t)}\big]\,. \tag{30}$$

Thus, the risk-free rate reflects subjective discounting, within-technology-era consumption growth, and the possible arrival of a new GPT (which would raise the utility-value of the bond's principal).

The long-run average equity premium is[10]

$$\mathbb{E}[EP] = \mathbb{E}[R^S] - \mathbb{E}[r] = \frac{\mathbb{E}[R] - \upsilon \cdot \mathbb{E}[r]}{1 - \upsilon} - \mathbb{E}[r] = \frac{\mathbb{E}[R] - \mathbb{E}[r]}{1 - \upsilon}\,, \tag{31}$$

where the second equality uses (29). From Prop. 2, (10), and Cor. 3D, we have

$$\mathbb{E}[R] = \lim_{J \to \infty} \frac{\sum_{j=1}^{J} \int_0^{\bar{s}(j)} \big[\alpha \cdot [\Psi\big(s, b \cdot \bar{a}_{\bar{t}(j-1)}/x\big)]^{\alpha - 1} - \delta\big]\, ds}{\bar{t}(J) - \bar{t}(0)} - \lambda \cdot (1 - b)\,. \tag{32}$$

We can derive $r_t = r(a_t)$ from the detrended, $m \to \infty$ version of (21a), (25), and (30). So,

$$\mathbb{E}[r] = \lim_{J \to \infty} \frac{\sum_{j=1}^{J} \int_0^{\bar{s}(j)} r\big(\Psi\big(s, b \cdot \bar{a}_{\bar{t}(j-1)}/x\big)\big)\, ds}{\bar{t}(J) - \bar{t}(0)}\,. \tag{33}$$

Calibration Preliminaries. This subsection presents constants from the literature and additional model-based equations that we use in our calibrations.

As in Laitner and Stolyarov [2003], we assume $b = 0.40$.[11] Following Maddison [1991], we assume a long-run average growth rate of output per worker of $g = 0.0210$. For the share of bond financing, a number of papers use $\upsilon = 0.40$ (e.g., Boldrin et al [1995], Papanikolaou [2011], Barro [2006]) — which we take to be an upper bound.

[10] Note $\mathbb{E}[R]$ or $\mathbb{E}[R_t]$ refer to the mean of the stationary distribution — compared to the expected value of the momentary rate, $\mathbb{E}_t[R_t]$, in (10).

[11] Greenwood, Hurcowitz, and Krusell [1997] derive the average rate of decline in the relative price of investment goods 1954-1990 commensurate with an assessment of vintage-based quality gains. They attribute such gains only to equipment, but their average applied to all plant and equipment, over the average episode length derived below, yields an alternative estimate $b = 0.41$.

Maddison's g reflects the expansion of GDP over a long time period of years. Suppose we start at time $t = 0$ and end at $t = T$, with the frontier TFP indices being $J_0 = 0$ and $J_T = J$, and the corresponding values of k being k_0 and k_T. Algebraically,

$$\theta \cdot [x]^\alpha = x \quad \text{and} \quad \ln(x) = \frac{\alpha}{1-\alpha} \cdot \ln(1/b)\,, \tag{34}$$

where the second equality uses Prop. 1. Recall that the population and labor force are fixed. Using Prop. 2 and the first part of (34), we then have

$$e^{g \cdot T} = \frac{GDP_T}{GDP_0} = \frac{[\theta]^J \cdot [[x]^J \cdot k_T]^\alpha}{[k_0]^\alpha} = \frac{[x]^J \cdot [k_T]^\alpha}{[k_0]^\alpha} \Longleftrightarrow g = \lambda \cdot \ln(x) + \epsilon_1 + \epsilon_2\,,$$

$$\epsilon_1 \equiv [\frac{J}{T} - \lambda] \cdot \ln(x) \quad \text{and} \quad \epsilon_2 \equiv \frac{\alpha \cdot [\ln(k_T) - \ln(k_0)]}{T}\,.$$

As $T \to \infty$, $\operatorname{plim} \epsilon_1 = 0$ and $\lim \epsilon_2 = 0$.[12] Using the second part of (34), we thus set

$$g = \lambda \cdot \frac{\alpha}{1-\alpha} \cdot \ln(1/b)\,. \tag{35}$$

The parameter β is labor's share for type-I households. In Kennickell [2009], the 2007 SCF seems representative of 1989-2007. The income share of the top 5% of wealth holders was about 33%. Let $\mathcal{Y}$ measure 2007 GDP (excluding services of residential housing, investment in residential housing, and depreciation on the same). Let $\mathcal{D}$ measure depreciation (excluding residential housing). Then $\mathcal{Y} = \$12,308.3\ bil/yr$ and $\mathcal{D} = \$1458.3\ bil/yr$, and we derive β from

$$\frac{1}{3} \cdot [\mathcal{Y} - \mathcal{D}] = [\alpha \cdot \mathcal{Y} - \mathcal{D}] + \beta \cdot (1-\alpha) \cdot \mathcal{Y}\,. \tag{36}$$

Following Laitner and Stolyarov [2003], we set the overall annual depreciation rate for capital from NIPA data, $d = 0.075$. We assume d includes both wear and tear depreciation, say, at rate δ, and resale-price declines from obsolescence. Suppose d reflects year-to-year depreciation at different times for a wide selection of capital goods of different ages. Then the total depreciation per dollar's worth of capital from t to $t + h$ obeys

$$d \cdot h = \delta \cdot h + \begin{cases} 0\,, & \text{with prob. } (1-\lambda) \cdot h\,, \\ 1-b\,, & \text{with prob. } \lambda \cdot h\,. \end{cases}$$

Assuming d is estimated from the expected value of the right-hand side, we have

$$d = \delta + \lambda \cdot (1-b)\,. \tag{37}$$

[12] Use Prop. 3B for the second.

Table 1. Mean Annual Rates from U.S. Financial Market Data

Row	Time Span	Bull Market Years	No Excluded Years Within Time Span			Bull-Market Years Only
			Riskless Rate of Interest	Stock-Market Return	Equity Premium	Equity Premium
	Long Intervals					
1	1916-2020	... [a]	0.92%/yr	6.46%/yr	5.53%/yr	9.10%/yr
2	1929-2020	... [b]	0.90	6.03	5.13	8.44
	Individual Technology Eras					
3	1916-1928	1921-1928	1.05	9.44	8.39	15.95
4	1929-1972	1933-1972	0.67	5.99	5.32	8.71
5	1973-1999	1975-1999	2.51	7.59	5.08	7.84
6	2000-2020	2003-2020	-0.68	4.14	4.81	8.66

Source: See Appendix XX.

(a) Union from this column, rows 3-6, below.

(b) Union from this column, rows 4-6, below.

Stock-Market Data. Data from Table 1 provide the final inputs for our calibrations. We use Table 1 to determine utility parameters γ and ρ, and the hazard for technology shocks, λ.

According to our model, the stock market has episodes with a new GPT causing a short, severe bear market, followed by a bull.[13] Each bear-then-bull episode corresponds to a "technology era." For the 20th century, we think, for example, of development of a practical farm tractor about the end of WWI, which greatly reduced agricultural labor input (Dahlstrom [2022], Cox and Alm [1992]); methods of mass production for automobiles and electrical home appliances by the close of the 1920s (recall Landes [op cit]); the microprocessor about 1970 (Laitner and Stolyarov [2003], Hobijn and Jovanovic [2001]); and, the Internet about 2000 (Pastor and Veronesi [2009]). As noted in the Introduction, a bear market in stocks can help to pin down the date at which the economy accepts a seminal invention's viability. We use the cycles summarized in Albuquerque et al. [2015, tab. 2] — which are determined by bear and bull markets, and labeled according to their characteristic technology — choosing the following 4 to correspond to our shocks: 1916-1928, 1929-1972, 1973-1999, and 2000-2020. Although the last era has not yet ended, it has almost reached the average length of the others — and we include it to bring the analysis up to date.

Parameters Values. Table 2 presents our calibrations. To have the longest series available, we use data from the first 2 rows of Table 1.

Table 2, row 1, is based on Table 1, row 1. Set $b = 0.40$ and $\upsilon = 0.40$, as above.

[13] Note: this paper's "bull markets" consist of all times outside of bear markets.

Table 2. Parameter Values for Simulations Matching Data Averages from Table 1, Rows 1-2

Simulation Replicating:	ρ	γ	λ	b	υ	α	β	η	δ
1916-2020	0.0155	0.258	0.0357	0.400	0.400	0.391	0.0350	0.412	0.0538
1929-2020	0.0157	0.273	0.0359	0.400	0.350	0.390	0.0371	0.412	0.0537

Source: See text.

Eq. (9) shows that during bull markets, $R_t = M(K_t, J_t)$; (10) shows that in general, $\mathbb{E}_t[R_t] = M(K_t, J_t) - \lambda \cdot (1 - b)$. Hence, (31) shows the difference between the long-run bull-market and unrestricted equity premium is

$$\mathbb{E}[BMEP_t] - \mathbb{E}[EP_t] = \frac{\lambda \cdot (1 - b)}{1 - \upsilon}\,. \tag{38}$$

With the values for b and υ above, Table 1, cols 6-7, and (38) determine λ. Then Maddison's g and (35) determine α; (36) sets β; (8) sets η; and, (37) sets δ.

To complete our calibration, we adjust (ρ, γ) to jointly match the risk-free rate and the unrestricted equity premium. For any γ, we adjust ρ until the simulated $\mathbb{E}[r]$ — recall (33) — matches of Table 1, col 4. We adjust γ until $\mathbb{E}[EP]$ from (31)-(33) matches Table 1, col 6.

In the domain of the calibrations, a higher γ tends to lead to a higher $\mathbb{E}[EP]$. Consider the following interpretation. If we duplicate the derivation of r_t (see (30)) for R_t, from (31) we have

$$\mathbb{E}_t[EP_t] = \frac{1}{1 - \upsilon} \cdot \lambda \cdot (1 - b) \cdot \mathbb{E}_t[\frac{u'\big(\Phi(b \cdot a_t, 1)\big) - u'\big(\Phi(a_t, 0)\big)}{u'\big(\Phi(a_t, 0)\big)}]\,. \tag{39}$$

When a new GPT arrives, consumption transitions from $\Phi(a_t, 0)$ to $\Phi(b \cdot a_t, 1)$. The latter tends to be (i) lower due to the portfolio capital loss as the new GPT arrives; (ii) lower because the marginal product of new investment climbs, making saving more attractive; but, (iii) higher because the improvement in TFP boosts future incomes. With a higher γ, effect (ii) is strengthened and (iii) weakened; thus, $\Phi(b \cdot a_t, 1)$ tends to shrink in our computations relative to $\Phi(a_t, 0)$, causing (39) to rise.

For Table 2, row 2, we use Table 1, row 2. Adjusting υ downward slightly, we repeat the steps above. Because we worry (see below) that the economy may not have been in equilibrium during 1916-28, our simulation analysis focuses on the parameters of Table 2, row 2. In any case, Table 2, rows 1-2, are quite similar to one another.

Computations. Our simulations use a desktop computer. We work in fortran and make extensive use of the IMSL subroutine library. For given parameter values, the execution time for verifying C1-C7 and simulating the stationary moments is 1-2 seconds.

6. Model Overview

Before turning to analysis of changes in the long-run growth rate, we briefly highlight several features of our equilibrium outcomes.

Direct Evidence. We begin by noting direct evidence supporting our specification.

One of the most thoroughly studied examples of a GPT arrival is the microprocessor revolution of the early 1970s. Hobijn and Jovanovic [2001] provide detailed evidence of large-scale changes and technology-specific capital. They link the 1972-74 bear market to investor recognition of the significance of the microprocessor. Separating firms into those with old versus new-vintage capital, they document divergences after 1972, as follows: the value of firms with old-vintage capital fell by more than 50 percent, never fully recovering. Firms with new-vintage capital, on the other hand, accounted for a subsequent tripling of the market capitalization-to-GDP ratio. Notably, this vintage effect was more pronounced in the IT-intensive sectors of the economy (see their Fig. 10).

Landes' [1969] view of the early twentieth century also seems broadly consistent with our analysis. He writes:

> From 1870 on...British industry had exhausted the gains implicit in the original cluster of innovations that had constituted the Industrial Revolution.... Not until a series of major advances opened new areas of investment around the turn of the century was this deceleration reversed. These [later] years saw the lusty childhood, if not the birth, of electrical power and motors; organic chemistry and synthetics; the internal-combustion engine and automotive devices; precision manufacture and assembly-line production.... (pp. 234-5)

As in our stochastic steady-state equilibrium, Landes perceives waves of growth each followed by renewal: a GPTs arrival opens new investment options that can raise national output, but diminishing returns eventually set in and (momentary) growth slows. Nevertheless, when the next GPT follows, gains from investment rise again and productivity accelerates.

Pastor and Veronesi's [2009] analysis of the advent of the Internet has a related nature as well (though in contrast to our treatment, Pastor and Veronesi emphasize phases in agent recognition of the significance of new technologies).

Stock Market. A strength of our model is its ability to explain low-frequency economic "cycles" — in particular, in financial markets.

Tables 1-2 in Albuquerque et al. [2015] segment US stock market data 1869-2013 into bear-then-bull episodes. Their narrative identifies 5 bull markets associated with major technology diffusions, which is consistent with our model's interpretation of low-frequency cycles. The dated turning points show that the bear markets are relatively short; the bulls are long. The equity premium is positive in bulls but negative in bears. Our model's cycles have the same asymmetries, qualitatively resembling Albuquerque et al [2015, Tab 1].

Another pattern in practice is that price-earnings (PE) ratios tend to plunge during bears but rise in bulls, often more than doubling (e.g., Shiller [2001, Fig. 1.2]), and our model suggests an explanation, as follows.

In our model, the value of existing capital plummets due to obsolescence when a new GPT arrives. The GDP is not immediately affected, and capital's share is invariant; thus,

the PE ratio falls. Thereafter, a_t converges toward a^* (recall Fig. 4). The price of efficiency units of capital is constant as long as the current episode lasts; hence, the PE ratio rises because the the law of diminishing returns causes its denominator to fall.

A simulation can illustrate. Using detrended variables, GDP_t is $[a_t]^\alpha$. As described above, our α may reflect some human capital. Suppose the share of income for physical capital is $\bar{\alpha} = 0.33$ and for human capital $\alpha - \bar{\alpha}$. An equity investment of $1 - \upsilon$ dollars buys control of 1 unit of physical capital. If $r_t = r(a_t)$ is the riskless rate of return, let $\bar{\alpha} \cdot [a_t]^{\alpha-1} - \delta - \upsilon \cdot r(a_t)$ be our PE ratio's denominator. Then

$$PE_t = \frac{1 - \upsilon}{\bar{\alpha} \cdot [a_t]^{\alpha-1} - \delta - \upsilon \cdot r(a_t)} \,. \tag{40}$$

Using (40), our 1000-episode simulation from Table 2, row 2, yields an average PE ratio of 5.99 at the start of bull markets and 20.25 at the end.[14]

Aggregate Consumption. Ait-Sahalia, Parker, and Yogo [2004] construct a measure of luxury expenditures that, they argue, provides a proxy for stockholder consumption. They find that the growth rate of luxury expenditures has a standard deviation roughly an order of magnitude larger than that for aggregate consumption. They also show that stockholder's consumption growth is strongly positively correlated with stock returns. Their evidence is consistent with our mechanism for the equity premium, which relies on large declines in type-I agent consumption during the periods when stock returns are low. The timing of consumption changes in Ait-Sahalia et al's Fig 1B also seems consistent with our model's mechanism: in data covering 1961-2001, the largest changes in luxury consumption are dips, and the sharpest declines occur in the early to mid 1970s, exactly the period Hobijn and Jovanovic examine, and during 2000-2001, a period Pastor and Veronesi study.

In the simulation with Table 2 parameter values, the wealth of type-I agents falls 60% when a technology shock occurs, and the average corresponding drop in their consumption is 66.7%. However, the maximum shock-induced annual decline in aggregate consumption is only 9.6%. About 40% of the shocks cause (41) to drop 5% or more. As our hazard for GPT arrivals is 0.036, the implied average number per century of shock-induced declines in aggregate consumption exceeding 5% is thus 1.4. (By way of comparison, Mehra and Prescott [1988] propose benchmark upper bounds for credible annual changes in aggregate consumption based on a 100-year data set: the data displays an annual decline greater than 5% in 4 cases, with a maximum drop of 8.8%.)

Equity Premium. Our formulation uses isoelastic preferences. Their simplicity may facilitate future extensions of the analysis. Our calibrated CRRAs are moderate (Mankiw and Zeldes [1991, p.105], Mehra and Prescott [1988, p.135]). On the other hand, in the absence of additions or refinements, we would have difficulty matching significantly higher equity premium values than those of Table 1, rows 1-2.

[14] If a bull market reached $a_t = a^*$, the simulation's PE-ratio ceiling would be 35.25.

6. Short-Run Unevenness

Punctuated technological change makes our model's growth episodic. We now illustrate how inherently uneven that growth can be — and how difficult it may be to establish, from direct evidence, that a shift in the growth trend has occurred.

Uneven Growth. Cor. 3D shows how to compute features of the stationary distribution for k_t. Table 3 presents an illustration based upon a 1000-era (about 28,000-year) simulation with parameters from Table 2, row 2. (Parameters from Table 2, row 1, yield similar outcomes in all cases below.) Table 3 summarizes information on the deciles of the stationary distribution. For ease of interpretation, the size of the capital stock is characterized in terms of the capital-to-GDP ratio, $k_t/y_t = k_t/[k_t]^{\alpha}$ (recall Prop. 2).[15] Since the population and labor supply are constant, both k_t and y_t can be thought of as either constant-dollars per capita or per worker.

Table 3. Stationary Distribution for k_t: Capital-to-Output Ratio (k_t/y_t) = 4.01 at Mean k_t

Decile for k_t	Within Decile Range k_t/y_t	Fraction Time in Decile	Range Productivity Growth Rate	Average Productivity Growth Rate
1	0.00-1.38	2.42%	9.11%/yr - ...	12.96%/yr
2	1.38-2.11	7.72	5.01-9.11	6.50
3	2.11-2.70	11.38	3.28-5.01	4.05
4	2.70-3.22	10.72	2.28-3.28	2.75
5	3.22-3.69	10.09	1.61-2.28	1.93
6	3.69-4.12	9.75	1.13-1.61	1.36
7	4.12-4.53	9.77	0.75-1.13	0.93
8	4.53-4.91	10.29	0.45-0.75	0.59
9	4.91-5.28	11.28	0.21-0.45	0.32
10	5.28-5.66	16.59	0.00-0.21	0.09

Source: Model simulation for 1000 technology eras; parameters from Table 2, row 2.

Within any technology era J, the time-t productivity growth rate is $\dot{Y}_t/Y_t = \dot{y}_t/y_t$, where y_t is detrended but $Y_t = [x]^J \cdot y_t$ is not, and x is as in (12). We have $\dot{y}_t/y_t = \alpha \cdot \dot{k}_t/k_t$. As our phase diagrams illustrate, k_t increases during each technology era, asymptotically approaching k^*. The last means the momentary growth rate decreases to 0 as time spent within an era lengthens. In Table 3, in fact, $\dot{y}_t/y_t$ monotonically declines throughout the stationary distribution's support.

As the Introduction suggests, our model implies that intermittent periods of slow growth are likely to occur, and Table 3 bears that out. Although the long-run average

[15] Although the mean capital-to-output ratio, 4.01, seems high, 2 reasons are as follows: (i) our accounting identity relating growth in K_t to investment uses depreciation rate $\delta < d$ within each technology era (we take obsolescence into account only when the next GPT arrives); and, (ii) our K_t includes human capital (recall Sect. 5).

growth rate is $g = 2.10\%$/yr, the mean of the stationary distribution of k_t falls in the 6th decile, where $\dot{y}_t/y_t = 1.13$-1.61%/yr. A linearization of (21a-b) yields a half-life for movement along our phase diagram's saddle path of 14-15 years. A single half-life would carry us from the bottom of decile 6 to the interior of decile 10, where productivity grow is miniscule. Our shocks are Poisson; so, the average waiting time until the next GPT's arrival has an exponential distribution. The mean wait time is $1/\lambda$, which in our case is about 28 yrs. For a 90% chance of a new GPT arrival, the wait time is roughly 64 yrs. The chance of a long period of relative stagnation in the latter part of almost any episode is therefore high.

The last 2 rows of Table 3 show productivity growth rates under 0.5%/yr prevail about 30% of the time, and rates under 1.1%/yr about half the time. We also simulated for 1250 consecutive 20-year intervals, computing the average growth rate for each. Creating bins for 20-year average productivity rates 0-0.5%/yr, 0.5-1.0%/yr, etc., the fullest bin is the first, with 264 intervals. There are 580 20-yr intervals with an average growth rate 0-1.5%/yr.

Conversely, growth tends to be fast in the aftermath of a shock. Suppose k is near k^* when the next GPT arrives. At the arrival, our $b = 0.40$ then causes k to plunge to decile 3 in Table 3. In decile 3, the growth rate is 3.28-5.01%/yr. (Nonetheless, a single half-life will carry the economy all the way from decile 3 to decile 7.)

The spurt of growth can be stronger if several shocks follow in rapid succession. Suppose that at the end of technology era J we have $k_{\bar{t}(J)}$. Then technology $J+1$ begins with $k_{t(J+1)} = (b/x) \cdot k_{\bar{t}(J)}$. If a second shock arrives very quickly, it takes hold with $k_{t(J+2)} \approx (b/x)^2 \cdot k_{\bar{t}(J)}$. As $b < 1$ and $x > 1$, the second shock drives us further into the stationary distribution's support. The effect on output can then be stronger and longer lived.[16]

Detecting Changes in the Average Growth Rate. Provided the factor-share parameter α is constant, (35) shows that λ and b determine g. To emphasize the latter's endogeneity in our analysis, henceforth we write $g_t \equiv \dot{y}_t/y_t$ and refer to the long-run average g_t as $\mathbb{E}[g_t]$. A fall in λ lengthens the time between seminal inventions, reducing $\mathbb{E}[g_t]$; an increase in b lessens the TFP gain from each revolutionary change, again reducing $\mathbb{E}[g_t]$.

Though we can look for changes in λ and b by examining data on the magnitude of shocks and their spacing (recall the discussion in Sect. 1), the generally small magnitude of our hazard is likely to lead to difficulties in practice. As noted above, the average duration of a technology era is about 28 years in our calibrations. A single unusually long era could be due to random chance. If we want to establish with some assurance that the parameter λ has declined, we will need to observe a number of eras and average their lengths. If b were a random variable and its mean was our chief concern, the same problem would arise.

Let the null hypothesis be that λ remains unchanged and the alternative be that λ has declined. Consider Table 4. Start at time 0. The density for the arrival time, say, t, of

[16] Put another way, the TFP gain from a new GPT achieves its full expression when k_t has a chance to rise all the way to the stationary level k^*. When, however, a new shock interrupts the diffusion of its predecessor, the truncated portion of the predecessor's TFP gain is not lost to the economy. Rather, the residual is manifested jointly with the gains to future GPT arrivals.

Table 4. Duration of Single-Sided Test of Null, $\lambda = 0.0359$, and Power Against the Specific Alternatives $\lambda^{ALT} = 0.5 \times \lambda$ & $0.75 \times \lambda$

Reject Null, 10% Significance Level, If Number GPT Arrivals During Test is $< i$, where $i =$	Approximate Test Length	Power Against λ^{ALT}, where	
		$\lambda^{ALT} = 0.5 \times \lambda$	$\lambda^{ALT} = 0.75 \times \lambda$
1	64 yrs	32%	18%
3	149	50	24
9	364	79	36
18	661	94	50
54	1782	100	80

Source: See text.

the next GPT is $\lambda \cdot e^{-\lambda \cdot t}$. The delay in the next GPT's arrival which would imply that we can reject the null from a single Poisson realization with a 10% chance of a type-1 error is about 64 years (Table 4, row 1).

In the preceding test, even if the hazard has dropped by half (which (35) shows would correspond to a halving of $\mathbb{E}[g_t]$), the chance of a type-II error is about 68%. To increase the power of the test, we could monitor a larger sample of GPT arrivals. Switch to the gamma distribution. Suppose λ has declined by 50%. Then Table 4 shows that to reject the null at the 10% significance level while holding the probability of a type-II error to 50%, we need almost 150 years of observations.

Summary. In sum, the model, by nature, generates uneven growth, with episodes having random lengths and the effect of shocks depending upon the economy's state at their arrival time. Long periods of slow progress will not be surprising — and generally should not be assumed to indicate a permanent reduction in trend growth. If the periodicity and/or magnitude of technology shocks does shift, detecting the change and establishing its statistical significance from measures of annual productivity, or direct observations of the shocks themselves, may take many decades — or centuries.

7. AN INDIRECT APPROACH

An indirect test potentially able to deliver more timely results may, however, be possible. Suppose households within the economy have granular information (which is unavailable to external analysts). If the households begin to believe that λ has fallen or b risen, they will anticipate a reduction in the frequency and/or severity of future bear markets — which, in turn, should decrease the premium they demand for holding equities. A change in the equilibrium premium could then take place immediately — alerting us without the need for a long series of technology shock realizations.

Assuming the economy is in equilibrium, the measured equity premium for a given time frame reflects a combination of what agents think the stochastic process for technology shocks is, which determines their demand curve for assets, and how many shocks have occurred during the interval in question, which affects the realized average. The magnitude of what we call the bull-market equity premium (BMEP) — the equity premium computed

for bull-market years alone — depends almost exclusively, as we shall show, upon the former. Since the randomness of the number of shock realizations during any interval raises the low-hazard complications of Section 6, we focus on the BMEP.

Cor. 3D (see also fn XX) shows that in a steady-state equilibrium, the long-run average bull-market equity premium will be a constant, $\mathbb{E}[BMEP_t]$. Since the momentary premium $BMEP_t$ is observable, we will be interested in a regression formulation

$$BMEP_t = \mathbb{E}[BMEP_t] + \epsilon_t \,, \tag{42}$$

where the error ϵ_t has mean 0 by construction. Given Poisson shocks and agents with rational expectations, the deviation between $BMEP_t$ and $\mathbb{E}[BMEP_t]$ should depend on k_t alone. In fact, the latter dependence seems slight given our calibrations. In Table 3, for instance, $BMEP_t$ monotonically declines as k_t rises, but does so very slowly: it is 8.71%/yr in the first decile of the stationary distribution of k_t, 8.64 in the second, ..., and 8.32 in the tenth. The computed standard deviation for the simulated $BMEP_t$ is only 0.11%/yr. On that basis, we proceed without more regressors in (42).

For a fixed α, (35) shows that λ and b determine $\mathbb{E}[g_t]$. Table 5 presents simulation results showing the effect of these parameters on $\mathbb{E}[BMEP_t]$. Table 5 confirms the conjecture in our introductory paragraph above: a decrease in λ, or increase in b, lowers $\mathbb{E}[BMEP_t]$. In fact, the percentage changes in $\mathbb{E}[g_t]$ and $\mathbb{E}[BMEP_t]$ move together virtually 1-for-1 in all rows.

Table 5. Sensitivity of the Steady-State Equilibrium to Changes in λ and b

New Parameter Value	Growth Rate $\mathbb{E}[g]$ (Old Value 2.10%/yr)	Equity Premium $\mathbb{E}[EP]$ (Old Value 5.53%/yr)	Bull-Market Equity Premium $\mathbb{E}[BMEP]$ (Old Value 9.10%/yr)
Parameter Adjusted: λ. Old Value: $\lambda_0 = 0.0357$. New Value: λ_1.			
$\lambda_1 = 0.9 \times \lambda_0$	1.89%/yr	4.96%/yr	8.18%/yr
$\lambda_1 = 0.8 \times \lambda_0$	1.68	4.39	7.25
$\lambda_1 = 0.7 \times \lambda_0$	1.47	3.83	6.33
$\lambda_1 = 0.6 \times \lambda_0$	1.26	3.27	5.41
$\lambda_1 = 0.5 \times \lambda_0$	1.05	2.71	4.50
Parameter Adjusted: b. Old Value: $b_0 = 0.40$. New Value: b_1.			
$b_1 = 1.1 \times b_0$	1.88%/yr	4.45%/yr	7.78%/yr
$b_1 = 1.2 \times b_0$	1.68	3.57	6.67
$b_1 = 1.3 \times b_0$	1.50	2.86	5.72
$b_1 = 1.4 \times b_0$	1.33	2.28	4.89
$b_1 = 1.5 \times b_0$	1.17	1.80	4.18

Source: Simulation from Table 3 rerun for various values of λ and b. See text.

To detect a possible decline in $\mathbb{E}[g_t]$, we then use (42). Our specification is

$$BMEP_t = \beta_1 \cdot \iota(t \leq T) + \beta_2 \cdot \iota(t > T) + \epsilon_t \,, \tag{43}$$

$$\iota(t \leq T) \equiv \begin{cases} 1\,, & \text{for } t \leq T\,, \\ 0\,, & \text{otherwise,} \end{cases} \quad \text{and} \quad \iota(t > T) \equiv \begin{cases} 0\,, & \text{for } t \leq T\,, \\ 1\,, & \text{otherwise,} \end{cases}$$

with T the putative date at which the parameters changed, and ϵ_t a mean zero error reflecting noise outside the scope of the model.[17]

We specify a prospective date T — see below. Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be the corresponding parameter estimates. Under the null hypothesis of no parameter shift, we have $\hat{\beta}_1 - \hat{\beta}_2 = 0$. Our alternative hypothesis is a decline in trend growth at T, implying $\hat{\beta}_1 - \hat{\beta}_2 > 0$. We use a single-tailed test and the same data as Table 1.

Early 20th Century. Although we are primarily interested in the possibility of a recent growth rate decline, we first consider the early part of our sample. Specifically, we examine whether the bull market of the first episode in the bottom panel of Table 1 — the so-called "roaring 20s" — might have ended with a reduction in the economy's long-run rate of growth.

Estimating (43) for the bull-market years during 1916-2020 with $T = 1929$, we find

$$\begin{aligned} BMEP_t = \; & \underset{\{0.0400\}}{0.1595} \cdot \iota(t \leq 1929) + \underset{\{0.0170\}}{0.0844} \cdot \iota(t > 1929)\,, \quad df = 89\,,\; R^2 = 27.48\,, \end{aligned} \tag{44}$$

where the bracketed numbers are (White) standard errors. Our Wald statistic is $(\hat{\beta}_1 - \hat{\beta}_2)/\hat{\sigma} = 1.7305$, with $\hat{\sigma}$ the standard error. The critical value at the 5% significance level for a one-tailed test is $t_c(.05) = 1.66$, at the 10% level $t_c(.10) = 1.29$. Thus, we can reject the null at the 5% level.

If we check each individual year $T \in \{1925, 1926, ..., 1950\}$ and test with the supremum estimator (Andrews [1993]), the peak Wald statistic is 2.39, occurring at $T = 1937$. (The second highest Wald statistic is 2.26, for 1934; the third highest is the 1.73 above, for 1929.) Bootstrapped (Hanson [2000]) critical values for a single-tailed sup-estimator yield $t_c(.05) = 2.48$ and $t_c(.10) = 2.22$; hence, we can reject the null at the 10% though not the 5% level.

While the test results seem to point to a slowdown after 1929, we are wary in this case. Our null hypothesis implicitly assumes (i) the economy had attained a rational expectations stochastic steady-stated equilibrium by 1916 and (ii) underlying parameters remained constant 1916-2020. On the one hand, evidence outside this paper's scope suggests the economy's long-run growth rate increased about 1900 as a second, more dynamic, phase of the industrial revolution began (e.g., Landes [1969]). Conceivably agents were still learning about a new λ and/or b in the lead-up to 1929.[18] On the other hand, a modern stock

[17] In no cases do the regression errors exhibit significant autocorrelation.

[18] Indeed, data for year before 1916 seems to show a much lower equity premium (e.g., Siegel and Thaler [1997] and Siegel [1998]).

market was just emerging in the US in the 20th century's first quarter (e.g., Smith [2003]). Investor participation rose rapidly in the 1920s and agents' behavior, by some accounts (e.g., Galbraith [1961]), seemed to resemble unsophisticated momentum trading.

In sum, we suspect that our tests might be rejecting part (i) of the null hypothesis rather than part (ii).

We conduct the remainder of our analysis for 1929-2020.

Later Years. For the period 1929-2020, the null hypothesis remains that $\mathbb{E}[g_t]$ is constant. The alternative hypothesis is that agents believe a permanent reduction in $\mathbb{E}[g_t]$ took place at year $T \in \{1929, 1930, ..., 2020\}$.

Table 6. Supremum Estimator Wald Tests: Sample 1929-2020

Row	Domain for Change Year T	Year Peak Wald Statistic	Peak Wald Statistic Value	Critical Value 10% Test [a]
1	1935,...,2015	1937	1.71	3.67
2	1950,...,2000	1956	1.11	2.06
3	1970,...,1980	1976	0.40	1.55

Source: see text.

(a) Critical values from bootstraps of 1000 simulations — see text.

Table 6, row 1, considers a supremum estimator for possible change dates $T \in \{1935, 1936, ..., 2015\}$. We continue to use (43).

As Table 6 shows, we cannot reject the null hypothesis at the 10% significance level. We present rows 2-3 for illustration. They focus on narrower domains for T. Though the critical values noticeably decrease, the supremum Wald statistics decline as well. We find no evidence of a shift in $\mathbb{E}[BMEP_t]$.

Table 7. Illustrative Power Calculations: Sample 1929-2020

Domain for Sup Estimator	For each $\bar{T}$ (see text), percent of 1000 simulations of sup-regression procedure over given domain that avoid type 2 error [a]
1950,...,2000 [b]	$\bar{T} =$ 1955: 33%; 1965: 40%; 1975: 41%; 1985: 41%; 1995: 36%
1970,...,1980	$\bar{T} =$ 1975: 48%

Source: see text.

(a) Various $\bar{T}$, 10% significance level tests. Critical values from Table 6, column 4.

(b) Domains for sup regression as in Table 6.

Table 7 offers assessments of the power of the tests in Table 6. Let β_0 be the sample mean. For each listed $\bar{T}$, we run 1000 simulations (in the manner of Hansen [2000]) with

β_0 halved for $t \geq \bar{T}$. For each simulation, we determine the supremum Wald statistic over the domain from Table 6, row 2 or 3. Among the 1000 supremum-regression outcomes, we determine the fraction avoiding type-2 errors (for single-sided tests of the null at the 10% significance level).

A comparison of Tables 4 and 7 suggests that our financial-market-based tests can deliver about the same power as GPT arrival indicators but with a shorter wait time for results. The differences based on Table 7, row 2, are the strongest. In row 2, we can obtain approximately the same power with 50 years of new data (i.e., 1970-2020) as with Table 3's methodology in 150 years. For the results of Table 7, row 2, we must, of course, be willing to restrict our search for a change date T to a single decade.

8. CONCLUSION

This paper examines the trend rate of technological progress in the U.S., considering the possibility of a recent decline. We present a model in which the arrivals of new general purpose technologies are the fundamental driver of progress, the arrivals are Poisson events, and agents have rational expectations.

Our model is highly aggregative and, in its present form, includes only 1 type of shock. Nonetheless, we argue that it can illustrate (i) the potential volatility of the momentary growth rate even when its long-run average is stationary; (ii) the difficulty of establishing a change in the long-run growth rate from direct observations; and, (iii) the potential value of monitoring financial variables that can, with the aid of interpretations from an analytic framework, enable us to detect changes in market-participants' beliefs about the future.

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APPENDIX: FIGURES 1-4

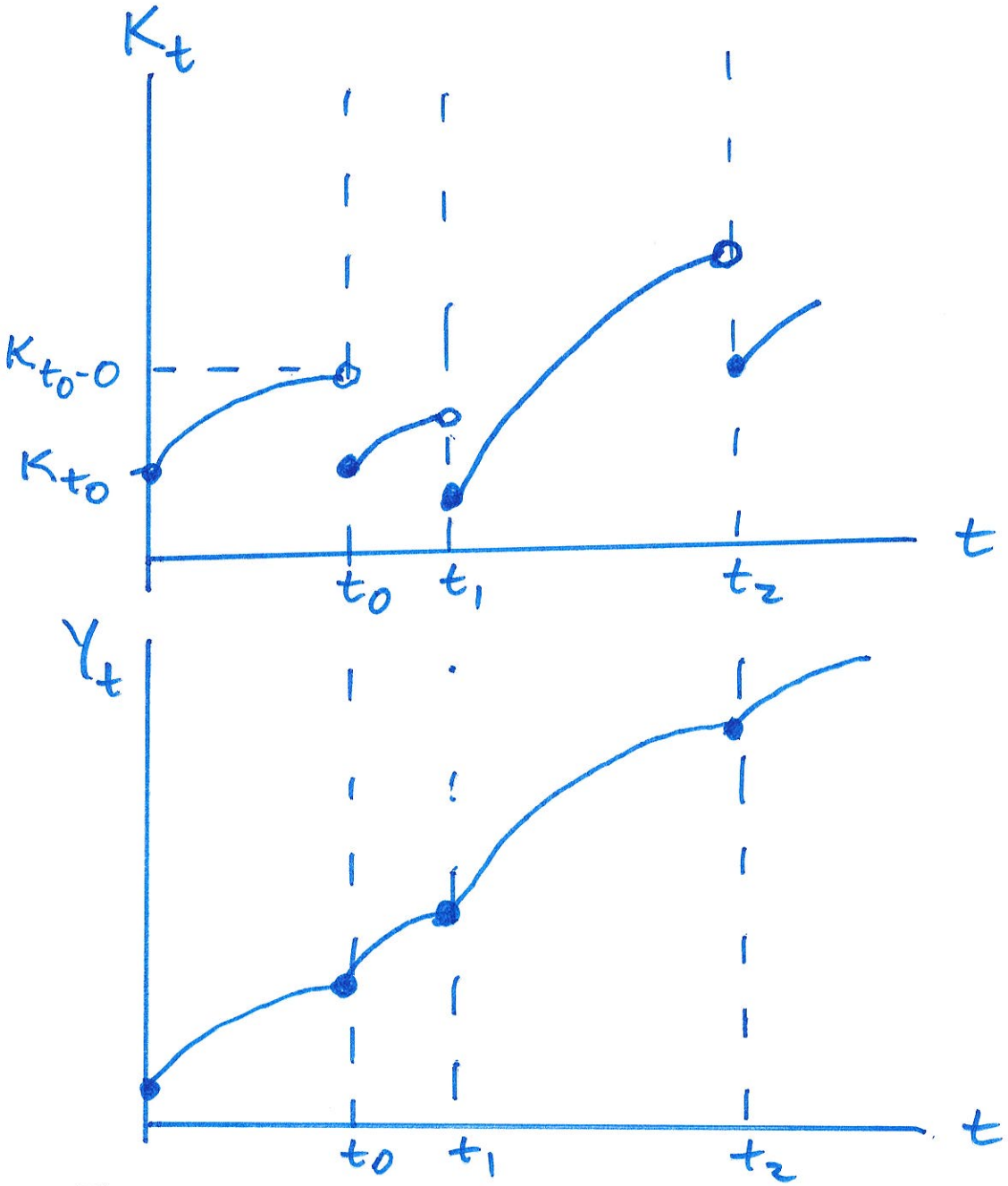


Figure 1: Time paths for K_t and Y_t, with GPT arrivals at $t_0, t_1, ...$, for technologies $J = 0, 1, ...$, respectively. Note that Prop. 1 shows $\theta \cdot [b]^\alpha = 1$. So, we have

$$K_{t_J-0} > b \cdot K_{t_J-0} = K_{t_J} ,$$

but

$$\begin{aligned} Y_{t_J-0} &= [\theta]^J \cdot [K_{t_J-0}]^\alpha = [\theta]^J \cdot \theta \cdot [b]^\alpha \cdot [K_{t_J-0}]^\alpha = [\theta]^{J+1} \cdot [b \cdot K_{t_J-0}]^\alpha \\ &= [\theta]^{J+1} \cdot [K_{t_J}]^\alpha = Y_{t_J} . \end{aligned}$$

In other words, the time path of GDP is continuous despite the abrupt revaluations of the existing capital stock as GPTs arrive.

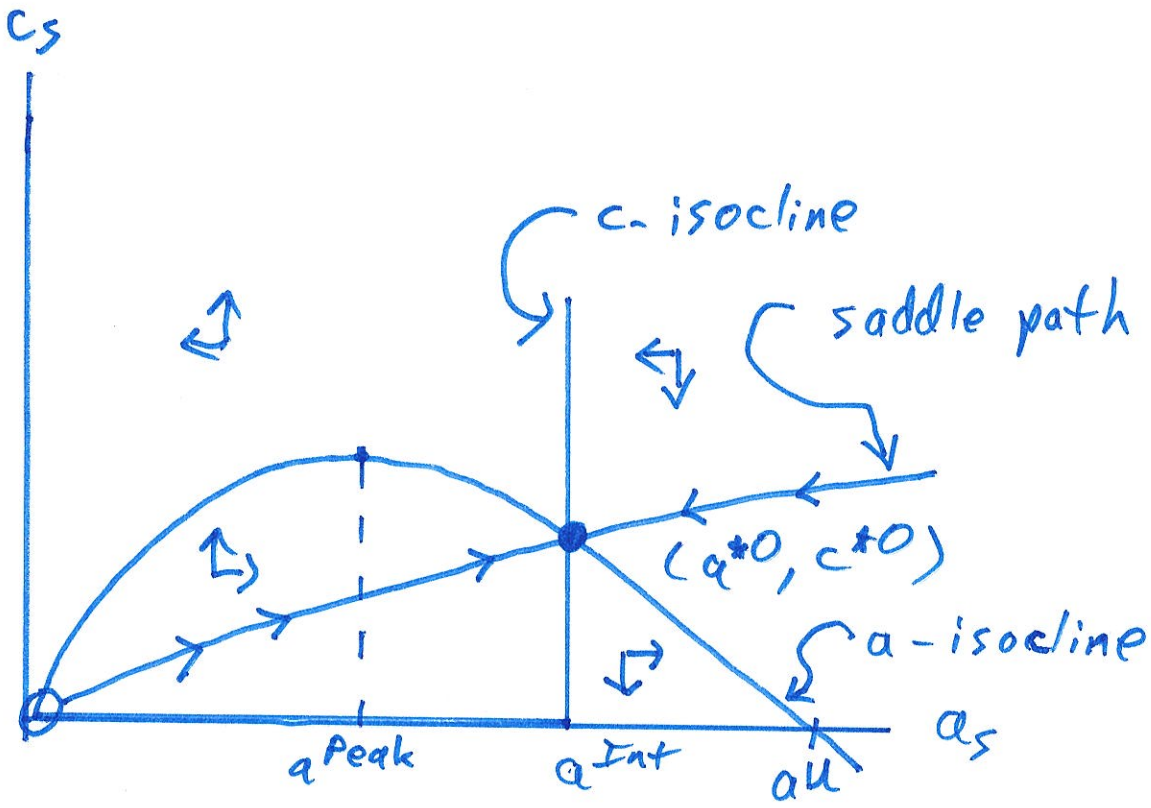


Figure 2: The $m = 0$ case. Assume $J = 0$, $a_s = A_s$, and $c_s = C_s$. We have $a^{Peak} = [\alpha \cdot \eta/\delta]^{1/(1-\alpha)}$ and $a^U = [\eta/\delta]^{1/(1-\alpha)}$. The c-isocline's vertical segment has intercept $a^{Int} = [\alpha/(\delta + \rho)]^{1/(1-\alpha)}$. A priori we have $a^{Int} \geq$ or $\leq a^{Peak}$. The saddle path is as shown. It approaches the origin but does not include it. It does include a stationary point (a^{*0}, c^{*0}).

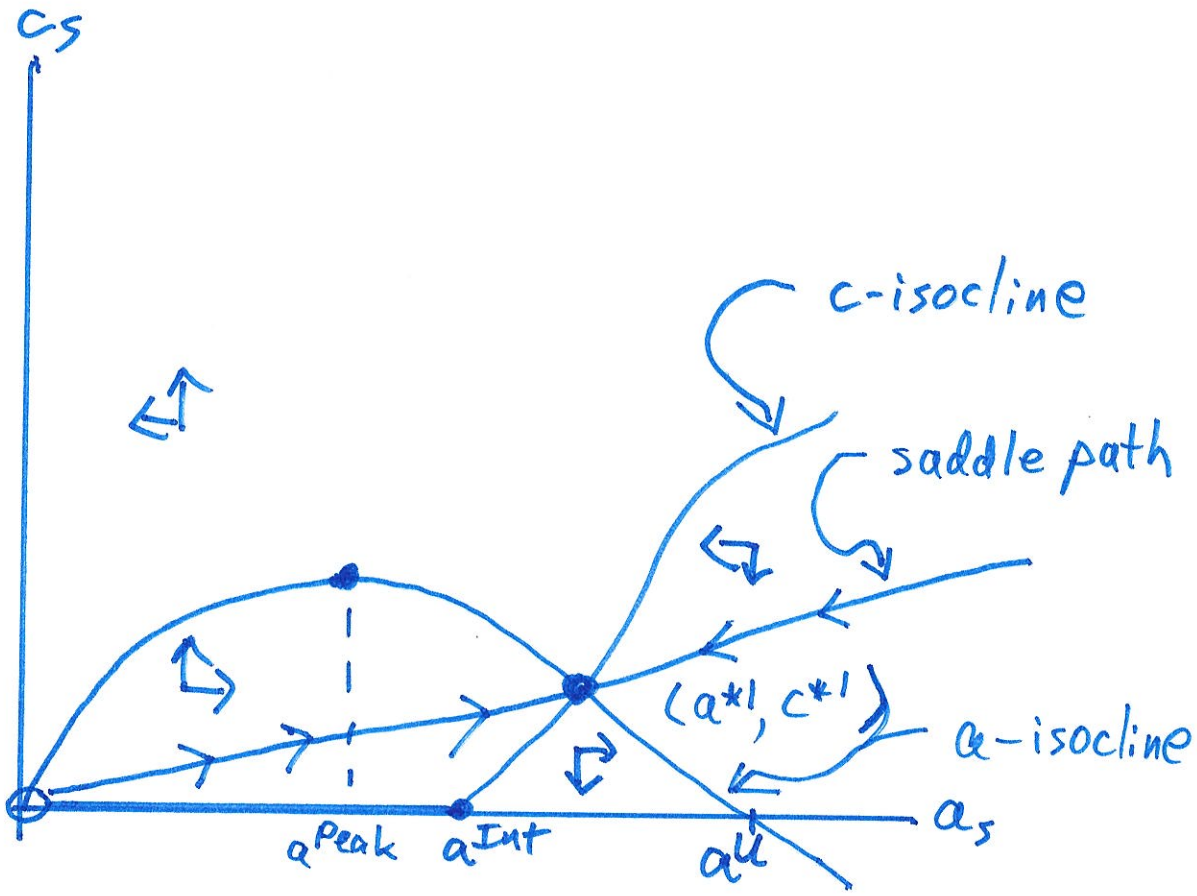


Figure 3: The $m = 1$ case. Assume $J = 0$, $a_s = A_s$, and $c_s = C_s$. The a-isocline is the same as Fig. 2; it remains unchanged for $m \geq 0$. The s-isocline's intercept is now $a^{Int} = [\alpha/(\delta+\rho+\lambda)]^{1/(1-\alpha)}$; it remains unchanged for $m \geq 1$. The part of the c-isocline with $c_s > 0$ must be upward sloping; the same is true all $m \geq 1$. The saddle path is as shown. It approaches the origin but does not include it. It does include a stationary point (a^{*1}, c^{*1}).

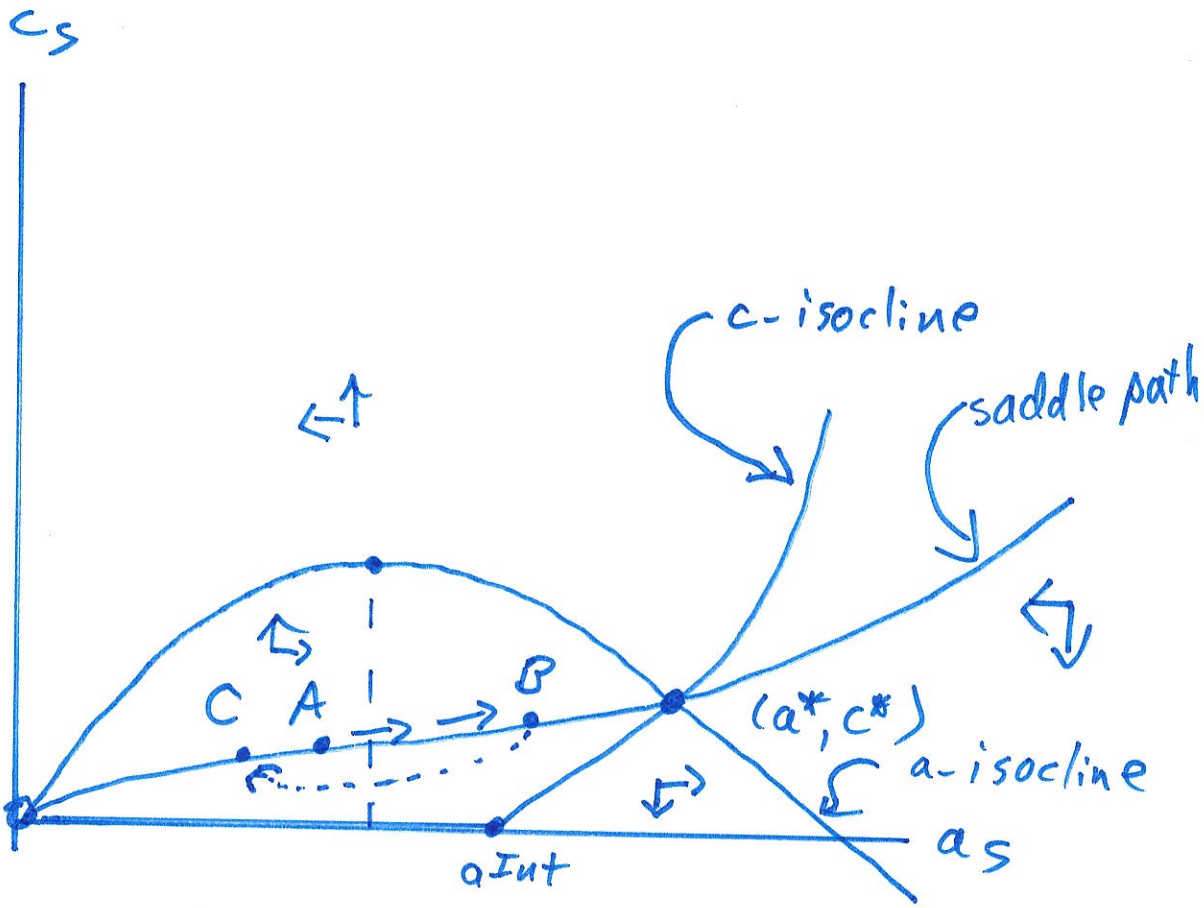


Figure 4: The limiting $m = \infty$ case. We work with the detrended variables $a_{s,J}$ and $c_{s,J}$. The a-isocline remains unchanged from Fig. 2. The c-isocline's intercept remains unchanged from Fig. 3. The part of the c-isocline with $c > 0$ must be upward sloping. The saddle path is as shown. It approaches the origin but does not include it. It has stationary point (a^*, c^*). We are only interested in values $a_{s,J} \in (0, a^*] \equiv \mathcal{A}$. Cor. 3D shows $\mathcal{A}$ is an ergodic set.

Suppose we start at A with $a = a_{0,J}$. We move northeast. Suppose as we are converging to B, a new GPT arrives at $s = s(J)$. So, we jump to $a_{0,J+1} = b \cdot a_{s(J),J}/x$ where $b \in (0, 1)$ and $x > 1$. The corresponding consumption level is $c_{0,J+1}$ that puts us on the saddle path at C. We begin moving northeast.... If a technology era lasts a long time, we will asymptotically approach (a^*, c^*). Conversely, if a large number of new GPTs arrive within a short time span, we could approach the origin.