Frictional Capital Markets and Economic Growth*

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Abstract

This paper develops a model of endogenous growth where heterogeneous firms experience idiosyncratic productivity shocks, necessitating continuous capital reallocation among firms, while the market for capital goods operates as a frictional dealer market. The real interest rate incorporates an illiquidity discount influenced by idiosyncratic risk, brokers' market power, and trading frictions. Knowledge spillovers a la Arrow (1962) create a nonmonotone relationship between market frictions and economic growth. Counter-intuitively, eliminating frictions altogether may harm growth and welfare. By endogenizing market frictions through dealer entry, the analysis reveals a nonmonotone link between growth and dealers' market power.

JEL Classification:

Keywords: Growth, capital, search, bargaining

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1 Introduction

The accumulation and allocation of capital are critical to economic growth. In textbook models, capital goods are traded seamlessly in Walrasian markets and optimally allocated among firms. In reality, capital must be continuously reallocated as firms enter, grow, downsize, or exit—capital reallocation represents one third of total capital expenditures (Cui, 2022)—and this process is far from seamless. Secondary markets for capital goods play a crucial role in this reallocation, redistributing machinery and equipment among firms—a role highlighted by the supply-chain disruptions during the 2020-21 pandemic (Darmouni and Sutherland, 2023).[1] This role makes the market structure of the markets for capital goods paramount.

Markets for capital goods share many characteristics with over-the-counter (OTC) financial asset markets: they are largely decentralized and segmented. This segmentation is evident in the multitude of trading venues. Additionally, capital goods' characteristics and firms' equipment needs can be highly specific, making the process of reallocating capital time consuming.[2] This process is often facilitated by dealers and brokers, who typically have significant market power to set prices.[3] This paper explores how the microstructure of the capital market affects the allocation of capital across heterogeneous firms, the real interest rate, and the rate of economic growth.

More specifically, it tackles the following questions. What is the relationship between market structure, capital misallocation, and the long-run growth of an economy? Do the market frictions that cause inefficient capital distribution across firms necessarily hurt long-run growth? Relatedly, do recent innovations in secondary capital markets, such as online platforms, promote economic growth and enhance social welfare? How does the market power of brokers and dealers affect firms' capital choices and growth?

In order to answer these questions, I develop a parsimonious and tractable model of endogenous growth with firm heterogeneity that explicitly incorporates the frictional nature of the capital goods market. I will seek to disentangle frictions in reallocating capital that are

[1] This point was conveyed in a Forbes article of January 19, 2022, titled "Why the growth of a second market for industrial assets is critical for the supply chain."

[2] Search frictions in the market for commercial office properties are documented, for instance, in D'Lima and Smith (2024). Gavazza (2011) documents frictions in the context of commercial aircraft markets. According to Ottonello (2024), the share of capital (non-residential structures) that is unemployed is larger than 10%.

[3] Gavazza (2016) mentions the presence of independent brokers to match buyers and sellers in the market for commercial aircrafts and argues that "intermediaries play an important role in mediating transactions. Most intermediaries operate as brokers who match buyers and sellers." On the role of brokers in the machinery market, see https://www.energybusinessreview.com/news/reasons-to-engage-industrial-machinery-brokers-nwid-1368.html.

inherent to the process of matching buyers and sellers from those that arise from the pricing mechanism and the market power of intermediaries. I will distinguish the implications of each type of friction for dynamic and static misallocations where the former refers to the rate of accumulation of capital while the latter is about the level of production taking the capital stock as given.

The benchmark model depicts an economy with heterogeneous firms utilizing a technology that features a constant marginal product of capital. Firms differ in their idiosyncratic productivities, which evolve stochastically over time. Households own equity shares, while firms own and manage the capital they operate with the goal of maximizing their market value. Capital is traded in a frictional dealer market, as formalized in the literature on over-the-counter markets (e.g., Lagos and Rocheteau, 2009). In this setup, firms negotiate with brokers the terms at which they buy or sell capital. These brokers form a frictionless network and trade competitively among themselves. An equivalent interpretation is that brokers give firms access to a competitive platform in exchange for a fee.

The model features a unique balanced-growth path equilibrium that can be solved analytically, including its transitional dynamics. Although it is optimal for all capital to be operated by high-productivity firms, in equilibrium, a portion of the capital stock remains with low-productivity firms. The degree of capital misallocation depends on search frictions, the endogenous growth rate of the capital stock, and the frequency of idiosyncratic productivity shocks. Consequently, the equilibrium real interest rate is lower than its frictionless counterpart. The *misallocation discount* is influenced by search frictions, brokers' bargaining power, productivity differences across firms, and idiosyncratic productivity risk.

If brokers do not have full bargaining power, the equilibrium converges to the Walrasian equilibrium of the textbook AK model—where "A" represents the productivity of the most efficient firms—as search frictions vanish. However, when brokers are perfect monopolists with full information and bargaining power, the real interest rate remains inefficiently low regardless of search frictions. The solution to a social planner's problem, under the same technological constraints to reallocate capital as those faced by brokers, aligns with the decentralized equilibrium when brokers have zero bargaining power.

While the benchmark model is analytically tractable, it overlooks a core aspect of economic growth: knowledge creation. Since the focus is on the capital market, I link knowledge creation to capital accumulation, as in Arrow (1962), Frankel (1962), and Romer (1986). This knowledge is nonrival, leading to spillovers across firms. Each firm's technology exhibits diminishing marginal returns with respect to its own capital stock but constant returns to scale with respect to both its own capital and the economy-wide capital. As in Romer (1986), the

frictionless competitive equilibrium is inefficient because the rate of capital accumulation is too low compared to the first-best scenario due to the spillover externality. However, taking as given the aggregate capital stock at a point in time, capital is allocated efficiently across firms, thereby maximizing aggregate output.

The decentralized equilibrium with frictions in the capital market exhibits both dynamic and static misallocation. Dynamic misallocation arises because the rate of capital accumulation in the decentralized equilibrium is lower than in the first-best scenario. Frictions in the capital market can either exacerbate or mitigate this type of misallocation. Specifically, the real interest rate and the economy's growth rate can vary non-monotonically with the frequency at which firms access the capital market and the bargaining power of brokers.

In some cases, increased trading frictions in the capital market can enhance the economy's growth rate, even surpassing that of the Walrasian equilibrium. This occurs because firms with infrequent access to capital markets tend to frontload their capital investments to capitalize on future productivity growth. Similarly, when brokers wield significant bargaining power, firms can mitigate the holdup problem associated with the need to raise their capital over time. This strategy partially offsets the spillover externality, resulting in a growth rate that exceeds the frictional competitive equilibrium.

Static misallocation refers to the observation that, for a given capital stock, its distribution across firms deviates from the first-best allocation, leading to inefficiently low aggregate output. Trading frictions unambiguously exacerbate this form of misallocation. Specifically, high-productivity firms reduce their capital holdings as frictions increase, while low-productivity firms expand theirs. As a result, in equilibrium, too much capital is held by the least productive firms, and insufficient capital is allocated to the most productive firms. Despite this, market frictions can lead to a higher growth rate of output, making the overall effect on household welfare theoretically ambiguous. In a calibrated version of the model, both higher search frictions and greater bargaining power for brokers increase welfare across balanced-growth equilibria.

I conclude the paper by endogenizing the market structure of the capital market by providing microfoundations for search frictions.[4] I endogenize the time it takes for a firm to contact a broker by allowing for broker entry. As more brokers enter, it becomes easier for firms to purchase or resell the capital they need. Broker entry depends on the share of misallocated capital. As brokers' entry costs decrease, both the real interest rate and the

[4]In the Appendix, I endogenize the bargaining power of brokers by assuming they can invest resources to enhance their bargaining strength. I demonstrate the possibility of multiple balanced-growth path equilibria, which vary in terms of brokers' market power, the real interest rate, and the growth rate of the economy. Furthermore, this multiplicity can persist even when trading frictions disappear.

growth rate of the economy increase. However, these variables can vary non-monotonically with brokers' bargaining power. In general, the growth rate is maximized when brokers have some, but not all, bargaining power. Additionally, the higher the contribution of brokers to the market's functioning, the greater the growth-maximizing bargaining power.

1.1 Literature

Eberly and Wang (2009) study capital reallocation and growth in an AK model a la Romer (1986). They consider a two-sector economy with capital adjustment costs. While the two sectors can have different productivities, it takes time and real resources for capital to be allocated to the high productivity sector. Eisfeldt and Rampini (2006) adopt quadratic adjustment costs in a two-sector model with diminishing marginal product of capital and aggregate shocks in order to focus on the business cycle properties of capital reallocation. Relative to this literature, my model provides microfoundations for these adjustment costs by diving into the micro-structure of the capital market. The costs of adjusting capital take different form. There is the time it takes to find a counterpart to a trade, which is formalized as search frictions. There is also a cost in terms of market power and rent seeking by the intermediaries in the market. These different costs have differentiated effects on the pricing of capital goods and their allocation across firms. The role of market power in neoclassical growth models has been studied recently by Ball and Mankiw (2023). They show that firms' market power in the goods market reduces the real interest rate. In our model, it is the market power of intermediaries in the capital market that pushes the real interest rate down.

My model emphasizes the role of brokers/dealers in the market for capital goods. Relatedly, Greenwood and Jovanovic (1990) formalizes the role of financial intermediaries to promote growth in an AK model. Intermediaries emerge endogenously through trading arrangements that are costly to establish and allow households to obtain a higher and safer return. Mehra, Piguillem, and Prescott (2011) also formalize costly financial intermediation in a neoclassical growth model. The intermediation in my model is not between households and firms: it is between firms in the market for capital goods.

The description of the market structure follows the pioneering work of Duffie, Garleanu, and Pedersen (2007) for over-the-counter markets. I adopt the version with divisible and unrestricted asset holdings of Lagos and Rocheteau (2007, 2009). Nosal and Rocheteau (2011, Chapter 12) are the first to propose a re-interpretation of the model of OTC market of Lagos and Rocheteau (2009) to the market for physical capital. Kurmann (2014) and Kurmann and Rabonovich (2018) allow capital to be produced under linear preferences and an exogenous

real interest rate and show that the decentralized is constrained inefficient. Gavazza (2011) proposes a model based on Diamond (1982) and Duffie, Garleanu, and Pedersen (2007) where the supply of real assets is fixed and agents own either zero or one unit of asset in order to formalize the market for commercial aircrafts. Gavazza (2016) considers an extension with dealers as in Rubinstein and Wolinsky (1987) and Duffie, Garleanu, and Pedersen (2005). There is an exogenous constant flow of new aircrafts that enter the market and that become obsolete at a finite horizon. Ottonello (2024) proposes a market structure following the one in the labor market literature, e.g. Mortensen and Pissarides (1994), to study capital unemployment with a focus on explaining business cycles facts. The matching technology has unemployed units of capital and search effort of buyers as inputs and the equilibrium notion is competitive search. Instead, our market structure corresponds to the one in the over-the-counter market literature. We are assuming random matching between agents, firms and intermediaries, and we keep the capital market competitive.

There is a vast literature on capital reallocation. From a theory standpoint, existing models of capital incorporate different features, including adverse selection (e.g., Eistfeld, 2004), agency costs (e.g., Eistfeld and Rampini, 2008), and search frictions.[5] David and Venkateswaran (2019) provide a methodology to disentangle the different sources of capital misallocation, including adjustment costs, informational frictions, unobserved heterogeneity in markups and technologies, within a general equilibrium model of firm dynamics. In my model, capital reallocation stems from two sources, search frictions and brokers' bargaining power, and I show how these frictions are combined in different ways to explain firms' valuations, the real interest rate, and capital reallocation. An overview of the stylized facts and the different theoretical approaches is provided by Eisfeldt and Shi (2018). Dou, Ji, Tian, and Wang (2024) study misallocation in a Schumpeterian growth model with heterogeneous firms facing financial frictions due to agency problems. From a quantitative viewpoint, Hsieh and Klenow (2009) provide a methodology to measure the effects of misallocation on TFP based on a model of monopolistic competition with heterogeneous firms. The authors make a link between reforms to reduce misallocation and economic growth for China and India. Jones (2013) discusses the link between misallocation and income differences across countries and emphasizes the potential role of intermediate goods and input-output linkages.

[5] For instance, Cui (2022) studies capital reallocation in the presence of time-varying, idiosyncratic costs to liquidate capital and borrowing constraints.

2 Environment

Time is continuous and indexed by $t \in \mathbb{R}_+$. There is a single consumption good and a unit measure of identical, infinitely-lived households whose preferences are represented by the following lifetime utility function,

$$\int_0^{+\infty} e^{-\rho t} u(c_t) dt, \tag{1}$$

where $\rho > 0$ is the rate of time preference, and c_t is the time-path for consumption. The instantaneous utility function is of the CRRA form,

$$\begin{aligned} u(c) &= \frac{c^{1-\theta}}{1-\theta} \text{ if } \theta \neq 1 \\ &= \ln(c) \text{ if } \theta = 1. \end{aligned} \tag{2}$$

Since the focus is on economic growth, I omit the choice between consumption and leisure.

Households have the possibility to acquire financial assets, such as equity shares of firms. There is no aggregate uncertainty and the idiosyncratic risk associated with financial assets can be diversified away. The endogenous real rate of return on financial wealth is denoted r_t.

There are three category of firms. There is a unit measure of *capital producers* that have the technology to transform the perishable consumption good into a durable capital good, one for one. Capital does not depreciate. The aggregate capital stock is denoted $K_t \in \mathbb{R}_+$ and the production of capital goods is denoted $\dot{K}_t$, where a dot over a variable denotes a time derivative. There is no rental market for physical capital, i.e., firms own the capital they operate.[6] Capital goods are traded in an frictional broker market, the micro-structure of which is detailed below.

There is a unit measure of final good producers, simply labeled *firms*. Each firm is endowed with a technology represented by the production function, $f_j(k)$, where $k \in \mathbb{R}_+$ is the capital held by the firm, and $j \in \{L, H\}$ indicates the idiosyncratic and time-varying productivity of the firm.[7] I assume for now, that the technology is linear,

$$f_j(k) = A_j k \text{ with } 0 \leq A_L < A_H. \tag{3}$$

[6] Alternatively, one could interpret the firms in our model as rental companies that aim at renting their capital to the most productive firms. When the capital is not rented, those companies receive a low (possibly negative) return. The return turns positive when the idle capital finds a renting opportunity.

[7] If productivity shocks are i.i.d., there is no loss in generality in assuming two productivity levels. See the Appendix for a description of a general Markov chain for the idiosyncratic productivity shocks. If $A_L = A_H$, our model is a standard AK model. As indicated by Jones (2019), early versions of this model were developed by Von Neumann (1945), Frankel (1962), and Solow (1956). For a textbook treatment of this model and its various extensions, see, e.g., Barro and Sala-i-Martin (2004).

Later, in Section 5, I introduce diminishing private returns to capital and I consider a richer set of productivity realizations. The idiosyncratic productivity shocks are formalized as follows. At Poisson arrival rate $\lambda > 0$, each firm draws a new productivity, A_j with $j \in \{L, H\}$, with probability $\pi_j > 0$ and $\pi_L + \pi_H = 1$. The average productivity is denoted by $\bar{A} = \pi_L A_L + \pi_H A_H$. The objective of each firm is to maximize its market value.[8]

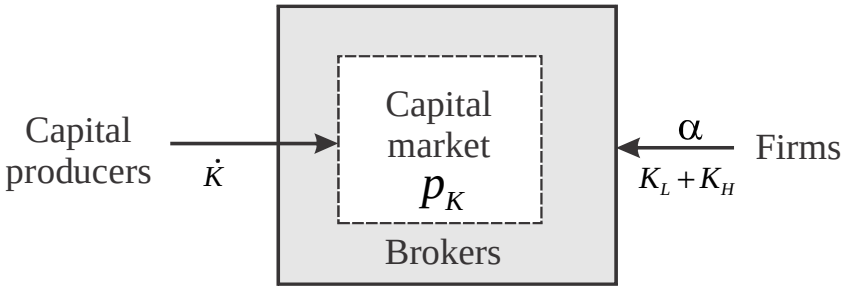


Figure 1: Description of the capital market structure

Finally, there is a unit measure of *brokers* who provide intermediation services in the capital market. The market structure is represented graphically in Figure 1.[9] Brokers, who can contact each other instantly, trade capital goods competitively among themselves.[10] The competitive price of capital among brokers, expressed in terms of the consumption good, is p_K. Firms are matched bilaterally with brokers at Poisson arrival rate $\alpha > 0$. Brokers can execute trades on belhaf of firms instantly. The terms of trade between brokers and firms are negotiated.

Capital producers can trade competitively with brokers and sell the newly produced capital goods instantly at price p_K. Therefore, the new capital is allocated to firms instantly. This assumption captures the idea that the new capital is easier to trade for various reasons, including the fact that its quality is more easily established. I relax this assumption in Section 4.

[8] Similarly, Jermann (1998) describes a model where the manager of the firm makes investment decisions in order to maximize the value of the firm to its owners (the representative household) under adjustment costs. For a discussion of the assumption that households are shareholders rather than owners of the capital stock in the context of incomplete-market models, see Poveday and Coen-Pirani (2010).

[9] The description is analogous to the one in Duffie, Garleanu, and Pedersen (2005) and Lagos and Rocheteau (2009) for over-the-counter market. While intended for financial assets (e.g., corporate bonds), this description also captures some of the key features of markets for real assets (e.g., Gavazza, 2011).

[10] Farboodi, Jarosch, and Shimer (2023) construct a model of an OTC market in which traders invest in a contact technology and show that middlemen with unboundedly high contact rates emerge endogenously.

Interpretation of the market microstructure.

This semi-centralized over-the-counter market structure captures the observation that dealers of capital goods can contact each other quickly to match their clients' supplies and demands. For instance, according to the Energy Business Review (2024), machinery brokers "*have an extensive network of contacts, including manufacturers, sellers, and other brokers*". Similarly, the Machinery Dealer National Association advertise that "*MDNA dealers can communicate with each other through various means*" thereby providing machinery availability.[11] In the secondary patent market, according to Richardson, Costa, and Oliver (2024), "*brokers leverage their networks of connections to find interesting patents to sell and to find willing buyers.*" Regarding the use of bargaining to determine terms of trade between brokers and firms, the Energy Business Review (2024) mentions, among the reasons to engage industrial machinery brokers, the fact that "*brokers are expert negotiators.*" In the secondary patent market, brokers "*sell, buy, market, negotiate, evangelise, and promote patents in the market*" (Richardson, Costa, and Oliver, 2024).

Instead of having dealers trading on behalf of firms, an alternative interpretation is that brokers give firms access to a competitive market where they can trade among themselves in exchange for a fee. This market can be, for instance, an online platform. The observation that the market is segmented is captured by the parameter, α, i.e., firms search across platforms.[12]

While I assumed that all trades go through brokers, one can also allow for business to business (B2B) trades. Indeed, an equivalent reformulation consists in assuming that firms access the competitive market directly at rate α_d and indirectly through brokers with bargaining power η_b at rate α_b. It will be clear soon that this market structure is equivalent to the one described above when

$$\begin{aligned} \alpha &= \alpha_m + \alpha_b \\ 1-\eta &= \frac{\alpha_m + \alpha_b(1-\eta_b)}{\alpha_m + \alpha_b}. \end{aligned}$$

So the meeting rate, α, can be be interpreted as the overall access rate to the capital market, via dealers or directly. The firm's bargaining power, $1-\eta$, is a weighted average of its bargaining power when accessing the market directly, one, and indirectly via brokers, $1-\eta_b$.

Finally, from a methology standpoint, the competitive (interdealer) market is a tractable formalization device to bypass the inventory problem of dealers in order to focus on their

[11] https://www.mdna.org/

[12] Here are a few examples: indiamart.com, machineadda.com, marutimachinery.com, exapro.com, euromachinery.com, machineseeker.com.

market power when trading with firms, which is captured by η, and their role in providing immediacy, which is captured by α. As we will see, this formalization keeps the model analytically tractable even when there is rich heterogeneity across firms and holdings of capital goods are perfectly divisible.

3 The benchmark model

In this section, I characterize balanced-growth equilibria where aggregate output and capital stock grow at some endogenous constant rate, $g > 0$. I first describe the household's problem and then I turn to the valuation of firms and brokers. Last, I endogenize the real interest rate and the growth rate of the economy.

3.1 Households

The representative household maximizes her lifetime utility given some initial financial wealth, ω_0, and a time-path for the real interest rate, r_t, i.e.,

$$\max_{c_t,\omega_t} \int_0^{+\infty} e^{-\rho t} u(c_t) dt \text{ s.t. } \dot{\omega}_t = r_t \omega_t - c_t \text{ and } \omega_0 \text{ given.} \tag{4}$$

It chooses its consumption, c_t, and its investment in financial wealth, $\dot{\omega}_t$, subjet to the budget identity in (4) and taking as given the rate of return of financial wealth, r_t. The portfolio of financial assets is perfectly diversified so that r_t is deterministic. The current-value Hamiltonian is $H(c, \omega, \xi) \equiv u(c) + \xi (r\omega - c)$, where ξ is the co-state variable associated with ω. By the Maximum Principle, the optimal choice for consumption satisfies

$$u'(c_t) = \xi_t \text{ for all } t, \tag{5}$$

where ξ_t solves

$$\rho \xi_t = \xi_t r_t + \dot{\xi}_t. \tag{6}$$

From (5) the marginal utility of consumption is equalized to the marginal utility of wealth at all dates. The ODE, (6), determines the household's marginal utility of wealth, ξ_t, given the real interest rate, r_t, and the rate of time preference, ρ. From (6), $\dot{\xi}_t / \xi_t = \rho - r_t$, which, from (5), gives the standard Euler equation:

$$g_t \equiv \frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}, \tag{7}$$

where $-u''(c_t)c_t/u'(c_t) = \theta$ and g_t denotes the growth rate of consumption. From (7), there is a one-to-one relationship between the growth rate of the economy and the real interest rate.

3.2 Capital good producers

Each capital producer chooses the quantity, i_t, of physical capital to produce at a unit cost in terms of the consumption good. The newly-formed capital is sold instantly in the capital market at price, p_K, also expressed in terms of the final good. Hence, the problem of the capital producer at time t is

$$\max_{i_t \geq 0} \{i_t(p_{K,t} - 1)\}.$$

For a solution with $i_t > 0$ to exist, $p_{K,t} = 1$. Capital good producers generate no profit, $i_t(p_K - 1) = 0$.

3.3 Valuation of firms

I denote the value of a firm with productivity $j \in \{L, H\}$ and capital stock k as $\upsilon_j(k)$. In order to determine υ_j, one needs to know the terms at which a firm trades with brokers in the capital market. The post-trade capital stock, k', and the payment to the broker, ϕ, are determined by the generalized Nash bargaining solution where the broker's bargaining power is $\eta \in [0, 1]$. The firm's surplus from the negotiated outcome, (k', ϕ), is equal to its market value after adjusting its capital stock, $\upsilon_j(k')$, net of capital expenditure, $k' - k$, and payment to the broker, ϕ, minus the value of the firm if it does not adjust its capital stock, $\upsilon_j(k)$. The firm is not financially constrained when paying $k' - k + \phi$, e.g., it can borrow at rate r_t. The surplus of the broker is simply ϕ. Hence:

$$(k', \phi) = \arg\max \left[\upsilon_j(k') - (k' - k) - \phi - \upsilon_j(k)\right]^{1-\eta} [\phi]^{\eta}. \tag{8}$$

The solution to (8) is

$$k' = \arg\max \{\upsilon_j(k') - k'\} \tag{9}$$

$$\phi_j(k) = \eta \max_{k'} \left[\upsilon_j(k') - (k' - k) - \upsilon_j(k)\right], \quad j \in \{L, H\}. \tag{10}$$

From (9), the post-trade capital stock maximizes the value of the firm net of the investment in physical capital. It is the same choice of capital that the firm would make if it had access to the capital market without a broker. In that sense, the negotiation between a firm and a broker is efficient and does not distort individual choices relative to a static Walrasian outcome where $\upsilon_j(k)$ is taken as given. From (10), the payment to the intermediary is a fraction, η, of the increase in the value of the firm net of the investment. One interpretation from (9)-(10) is that the broker and the firm only negotiate the fee to participate in the market while the firm takes the decision of how much capital to buy or sell. Using the expression for ϕ in (10), the surplus of the firm can be reexpressed as

$$\upsilon_j(k') - (k' - k) - \upsilon_j(k) - \phi = (1 - \eta) \max_{k'} \left[\upsilon_j(k') - (k' - k) - \upsilon_j(k)\right].$$

The firm receives a share $1-\eta$ of the surplus it would obtain if it had direct access to the capital market. Hence, from the standpoint of the firm, the economy where brokers have bargaining power, η, is equivalent to an economy where the firm accesses the capital market at rate $\alpha(1-\eta)$ with full bargaining power. I make use of this equivalence in the following to write the Hamilton-Jacobi-Bellman (HJB) equation for υ_j.

The value of a firm with productivity state $j \in \{L, H\}$ and capital $k \in \mathbb{R}^+$ solves:

$$\begin{aligned} r_t \upsilon_j(k) &= A_j f(k) + \lambda \pi_{-j} \left[\upsilon_{-j}(k) - \upsilon_j(k)\right] \\ &\quad + \alpha(1-\eta) \max_{k' \geq 0} \left[\upsilon_j(k') - (k'-k) - \upsilon_j(k)\right] + \dot{\upsilon}_j(k), \end{aligned} \tag{11}$$

where $-j \in \{L, H\} \backslash \{j\}$. The rate at which the firm discounts its profits is the rate of return of financial wealth in the household's problem, (6).[13] The left side of (11) is the opportunity cost from investing in the firm. On the right side, the firm generates an output flow equal to $A_j f(k)$. At Poisson arrival rate, $\lambda \pi_{-j}$, it draws a new productivity, $-j \neq j$. The capital gain is $\upsilon_{-j}(k) - \upsilon_j(k)$. At Poisson arrival rate, α, the firm has access to the capital market via a broker. The change in the value of the firm net of the investment is $\upsilon_j(k') - (k'-k) - \upsilon_j(k)$ and, after payment to the broker, the firm keeps a fraction $1-\eta$ of that gain. The last term is the change in the value of the firm over time.

From the assumption that $f(k) = k$, I guess that $\upsilon_j(k) = q_j k$, where $q_j \in \mathbb{R}$ is interpreted as the Tobin's q of the firm. It is the market value of a unit of physical capital invested in a firm with productivity j. Using this guess, the optimal choice of capital, from (9), can be reexpressed as

$$k' \in \arg\max \left\{ (q_j - 1) k' \right\}.$$

For this problem to have a solution, $q_j \leq 1$ for all $j \in \{L, H\}$. Firms might invest if $q_j = 1$ and they disinvest if $q_j < 1$. Substitute $\upsilon_j(k) = q_j k$ into (11) and divide both sides by k, the Tobin's q obeys the following dynamic equations:

$$r q_j = A_j + \lambda \pi_{-j} \left(q_{-j} - q_j\right) + \alpha(1-\eta)\left(1 - q_j\right) \quad \forall j \in \{L, H\}, \tag{12}$$

where $-j \in \{L, H\} \backslash \{j\}$. The left side is the opportunity cost of investing in the firm. The right side is the return from that investment, which is composed of the productivity, A_j, the expected capital gain as idiosyncratic productivity changes over time, and the gain from reselling the capital when the firm has access to the capital market. Summing across all

[13] It would be equivalent to measure the output flow, $A_j f(k)$, in terms of the household marginal utility of wealth by multiplying them by ξ_t. In that case, the discount rate would be the rate of time preference of the household, ρ. The equivalence between the two formulations comes from the ODE for ξ_t, (6).

productivity states, the average Tobin's q is equal to

$$\sum_{j\in\{L,H\}} \pi_j q_j = \frac{\bar{A} + \alpha(1-\eta)}{r + \alpha(1-\eta)}.$$

It is equal to the discounted sum of the average productivity of a firm, $\bar{A}$, plus the resale value of capital at the time the firm accesses the capital market, $\alpha(1-\eta)$. Substitute $\pi_{-j}(q_{-j} - q_j) = \sum_{i\in\{L,H\}} \pi_i q_i - q_j$ into (12) to obtain closed-form solutions:

$$q_j = \frac{[r + \alpha(1-\eta)]\, A_j + \lambda \bar{A}}{[r + \lambda + \alpha(1-\eta)]\,[r + \alpha(1-\eta)]} + \frac{\alpha(1-\eta)}{r + \alpha(1-\eta)} \quad \forall j \in \{L, H\}. \tag{13}$$

The Tobin's q has two components. The first term corresponds to a weighted average of the firm's current productivity, A_j, and its future expected productivity, $\bar{A}$, if it experiences a productivity shock. The second term is the expected discounted resale value of the firm's capital at the time the firm can access the capital market. If the capital market is active, there is a positive demand for capital, which requires $q_H = 1$. The value of the capital invested in the most productive firm is equal to the cost of one unit of capital, $p_K = 1$. A manifestation of the frictions in the capital market is in terms of the dispersion of the Tobin's q across firms. From (13) the following corollary provides of this dispersion.

Corollary 1 *A measure of the dispersion across Tobin's q is*

$$q_H - q_L = \frac{A_H - A_L}{r + \lambda + \alpha(1-\eta)}.$$

The dispersion of firm's valuations is proportional to the dispersion of idiosyncratic productivities. For given r, the passthrough from the dispersion in idiosyncratic productivities, as measured by $A_H - A_L$, to the dispersion in Tobin's q increases as search frictions become more severe, brokers have more bargaining power, and idiosyncratic productivity shocks are more persistent.

3.4 Valuation of brokers

Let w_t denote the expected discounted sum of the profits of a broker. It solves:

$$r w_t = \alpha \int_{\{L,H\}\times\mathbb{R}^+} \phi_j(k) d\Upsilon_t(j,k) + \dot{w}_t, \tag{14}$$

where $\phi_j(k)$ is given by (10) and where $\Upsilon_t(j,k)$ is the Lebesgue-Stieltjes measure of firms across productivity types and holdings of capital. Brokerage firms are owned by households and their profits are discounted at rate r_t. According to (14), a broker meets a firm at

random at rate α. From (10), the revenue received by the intermediary from this encounter is $\phi_j(k)$, which depends on the capital stock held by the firm and its productivity type. From the solution to the bargaining problem, (10), in an equilibrium where $q_H = 1$,

$$\begin{aligned} \phi_H(k) &= 0, \\ \phi_L(k) &= \eta(1-q_L)k. \end{aligned}$$

A meeting with a high-productivity firm generates no revenue to brokers because H-firms have no strict incentive to buy or sell capital. In contrast, brokers make profits when they meet firms that want to liquidate their capital because their Tobin's q is less than one and the gains from trade, $(1-q_L)k$, are positive. The profits are proportional to the capital stock of the firm where the coefficient of proportionality is the product of the bargaining power, η, and the per unit gain from liquidating capital, $1-q_L$. Using that $1-q_L = q_H - q_L$ and the closed-form solution for q_j, (13),

$$\phi_L(k) = \frac{\eta\left(A_H - A_L\right)}{r+\lambda+\alpha(1-\eta)}k. \tag{15}$$

The profits of the broker increase with the distance between the current productivity of the firm, A_L, and the highest productivity, A_H. This difference is discounted at rate $r+\lambda+\alpha(1-\eta)$ because the productivity is redrawn at rate λ and the firm has effective access to the capital market at rate $\alpha(1-\eta)$. I substitute the expression for $\phi_j(k)$ from (15) into (14) to obtain

$$rw_t = \alpha\int_{\mathbb{R}^+}\phi_L(k)d\Upsilon_t(L,k) + \dot{w} = \frac{\alpha\eta\left(A_H - A_L\right)K_{L,t}}{r+\lambda+\alpha(1-\eta)} + \dot{w}_t, \tag{16}$$

where $K_{L,t} \equiv \int_{\mathbb{R}^+} k d\Upsilon_t(L,k)$ is the aggregate capital stock held by low-productivity firms.

3.5 Distribution of capital

The productivity shocks are independent across firms and identically distributed. Hence, the share of firms in a long-run equilibrium with productivity type j is π_j. I focus on equilibria where all type-H firms leave the capital market with the same amount of capital, $\alpha K_t/\pi_H$. Moreover, I assume that the initial distribution of firms across productivity levels is given by the invariant distribution, $(\pi_j)_{j\in\{L,H\}}$.

Under the assumption that the technologies, $f_j(k)$, are linear, the distribution of capital across firms of a given productivity type does not matter for allocative efficiency. However, the distribution of capital across firms of different productivities is critical. Let denote $K_H \equiv \int_{\mathbb{R}^+} k d\Upsilon(H,k)$ the aggregate capital stock held by the most productive firms. It obeys the following law of motion:

$$\dot{K}_H = \alpha K_L + \lambda\pi_H K_L - \lambda\pi_L K_H + \dot{K}. \tag{17}$$

Over a small time interval of length dt, αdt firms access the capital market. A fraction π_L of those firms are type-L firms. Each low-productivity firms holds on average K_L/π_L. Hence, by the law of large number, a quantity $\alpha dt K_L$ of capital is reallocated from low-productivity firms to firms of type H.[14] According to the second term on the right side of (17), a measure λdt of firms receive a productivity shock and the realization is H with probability π_H. So, a quantity $\lambda dt \pi_H K_L$ of capital that was held by low-productivity firms is now held by firms of type H. Conversely, a measure $\lambda dt \pi_L$ of firms of type H receive a productivity shock with realization L. Hence, a quantity $\lambda dt \pi_L K_H$ of capital moves from firms H to firms L. The last term on the right side of (17) represents the accumulation of new capital that is allocated to the most productive firms, H.

I denote $\hat{k}_j = K_j/K$, $j \in \{L, H\}$, the share of the aggregate capital stock that is allocated to firms of type j. On a balanced-growth path, K_j grows at a constant rate g. Hence, from (17), using that $K_L = K - K_H$,

$$g\hat{k}_H = \alpha(1 - \hat{k}_H) + \lambda\pi_H(1 - \hat{k}_H) - \lambda(1 - \pi_H)\hat{k}_H + g.$$

Solving for $\hat{k}_H$,

$$\hat{k}_H = 1 - \hat{k}_L = 1 - \frac{\lambda\pi_L}{\lambda + \alpha + g}. \tag{18}$$

In a frictionless economy, the share of the capital stock allocated to the most productive firms is equal to one, which corresponds to the first term on the right side. The second term, which is equal to the share of the capital stock operated by the least productive firms, captures the misallocation of capital. It depends positively on the frequency of idiosyncratic productivity shocks, and negatively on the meeting rate with brokers and and the growth rate of the economy. One can use (18) to obtain the following corollary.

Corollary 2 *The amount of capital reallocation rekative to the aggregate capital stock is*

$$\textit{Reallocation} = \frac{\alpha K_L}{K} = \frac{\alpha\lambda\pi_L}{\lambda + \alpha + g}.$$

Reallocation increases with α and λ but it decreases with g. In a frictionless economy, $\alpha \to +\infty$, the amount of reallocation is $\lambda\pi_L$. Hence, the presence of frictions scales down reallocation by a factor equal to $\alpha/(\lambda + \alpha + g)$. In particular, the rate of production of new capital, g, tends to reduce reallocation.

I now make use of (16), the expression for $\hat{k}_L$ above, and the restriction to balanced-growth equilibria where $\dot{w} = gw$, to compute the market capitalization of brokers as

$$(r - g)\, w_t = \frac{\alpha\eta(A_H - A_L)\lambda\pi_L}{(\lambda + \alpha + g)\,[r + \lambda + \alpha(1 - \eta)]} K_t. \tag{19}$$

[14] For some difficulties with the law of large numbers with a continuum of random variables, see Judd (1985) and Uhlig (1996) for a resolution.

It is nonnegative and bounded if $r - g > 0$.

3.6 Taking stock

I define a balanced-growth equilibrium as a time-path for aggregate output, Y_t, aggregate consumption, C_t, aggregate capital, K_t, shares of total capital operated by firms per productivity level, $\{\hat{k}_j\}_{j\in\{L,H\}}$, Tobin's q, $\{q_j\}_{j\in\{L,H\}}$, the market capitalization of brokers, w_t, and the real interest rate, r, that satisfy the conditions below.

Aggregate output and TFP. The total output of the economy is equal to

$$Y_t = \overbrace{K_t \sum_{j\in\{L,H\}} A_j \hat{k}_j}^{\text{value added}} = \overbrace{C_t + \dot{K}_t}^{\text{expenditure}}, \tag{20}$$

where $\hat{k}_j$ is given by (18). According to the right side of (20), total output is divided between aggregate consumption, C_t, and investment, $\dot{K}_t$. The term, $\sum_{j\in\{L,H\}} A_j \hat{k}_j$, is the endogenous total factor productivity of the aggregate production function that has capital as the sole input. By substituting $\hat{k}_j$ by its value given by (18), I obtain the following corollary.

Corollary 3 *Total factor productivity is equal to*

$$TFP = A_H - \frac{\lambda \pi_L (A_H - A_L)}{\lambda + \alpha + g}. \tag{21}$$

The endogenous TFP depends on the market frictions, the idiosyncratic risk, and the endogenous growth rate of the economy through $\hat{k}_j$.

Balanced growth path. Along a balanced growth path,

$$C_t = C_0 e^{gt} \tag{22}$$

$$K_t = K_0 e^{gt}, \tag{23}$$

where, from the Euler equation, (7), the growth rate of the economy is $g = (r - \rho)/\theta$, where $K_0 \in \mathbb{R}_+$ is given, and from (20),

$$C_0 = \left(\sum_{j\in\{L,H\}} A_j \hat{k}_j - g \right) K_0. \tag{24}$$

Aggregate financial wealth. Households' total financial wealth is

$$\Omega_t = w_t + \left(q_L \hat{k}_L + q_H \hat{k}_H\right) K_t. \tag{25}$$

The first term on the right side of (25) is the market capitalization of brokers while the second term is the market capitalization of firms. From the budget identity of the household problem, (4), using that $\dot{\omega}_t/\omega_t = g$, one can also express aggregate consumption at time 0 as

$$C_0 = (r - g)\Omega_0. \tag{26}$$

For consumption to be positive, $r > g$.

Rate of return of capital. Finally, the capital market clears if H-firms are willing to invest. Setting $q_H = 1$ into (13), the real interest rate solves

$$r = A_H - \frac{\lambda \pi_L (A_H - A_L)}{r + \lambda + \alpha(1 - \eta)}. \tag{27}$$

The equilibrium real interest rate is determined uniquely from (27) and is represented graphically in Figure 2. It is equal to the productivity of the type-H firms minus a *misallocation discount*. This discount increases with the idiosyncratic productivity risk, λ, the gap between high and low idiosyncratic productivities, $A_H - A_L$, and brokers' bargaining power, η. It decreases with the rate at which firms meet brokers, α. As frictions vanish, or as productivity becomes persistent, r approaches A_H. The comparison of (21) and (27) shows that r is generically distinct from TFP.

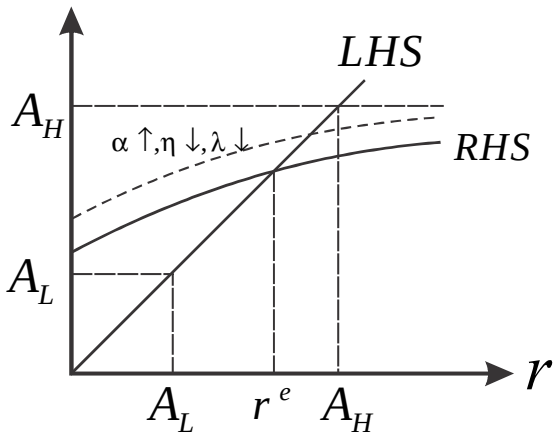


Figure 2: Determination of the equilibrium real interest rate.

Recursive equilibrium. An equilibrium has a recursive structure. The real interest rate, r, is determined from (27). Given r, the growth rate of the economy is given by $g = (r - \rho)/\theta$ and the Tobin's q are given by (13). Given g, the capital shares, $\hat{k}_L$ and $\hat{k}_H$, and the aggregate capital stock, K_t, are given by (18) and (23) and aggregate output is given by (20). Given r, g, and K_t, the market capitalization of brokers, w_t, is given by (19). Given w_t, q_j, $\hat{k}_j$ and K_t, the total financial wealth, Ω_t, is given by (25). Given r, g, and Ω, the initial aggregate consumption is given by (26) and future consumption is given by (22).

Proposition 1 ***(Existence and uniqueness of balanced-growth equilibrium.)*** *Suppose $K_{L,0} = \hat{k}_L K_0$. There exists a balanced-growth path equilibrium featuring $0 < g < r$ if and only if*

$$[\rho + \alpha(1-\eta)](A_H - \rho) + \lambda(\bar{A} - \rho) > 0, \tag{28}$$

and either $\theta \geq 1$ or

$$\frac{(1-\theta)A_H - \rho}{1-\theta} < \frac{(1-\theta)\lambda\pi_L(A_H - A_L)}{\rho + (1-\theta)[\lambda + \alpha(1-\eta)]}. \tag{29}$$

The equilibrium is unique and such that $r \in [A_L, A_H]$. Moreover, $\partial r/\partial\alpha > 0$, $\partial r/\partial\eta < 0$, and $\partial r/\partial\lambda < 0$.

According to (28), a necessary condition for the existence of an equilibrium with positive growth is $A_H > \rho$ and a sufficient condition is $\bar{A} > \rho$. If the latter condition does not hold, because the productivity in the low state is low, then positive growth necessitates that the productivity in the high state is sufficiently high and frictions are not too severe. The lifetime utility of households is bounded if either $\theta \geq 1$ or if (29) holds, i.e., the difference between the productivity levels across firms is sufficiently large.

Transitional dynamics. I now describe tranditional dynamics from arbitrary initial conditions, $(K_{L,0}, K_{H,0})$. First, the asset pricing equations defining the Tobin's q, (12), do not depend on the composition of the capital. Hence, q_j jumps instantly to its new balanced-growth path value following any unanticipated change in the environment. Similarly, the real interest rate, r, which is determined from $q_H = 1$, jumps instantly to its balanced-growth-path value. The growth rate of consumption, $g = (r - \rho)/\theta$, is also constant on the transitional path.

Consider next the time-paths of the capital stocks. Using that, $\dot{K}_t = Y_t - C_t$, where $C_t = C_0 e^{gt}$, and removing the long-run trend by denoting $k_t \equiv K_t/(e^{gt}K_0)$, $k_{L,t} \equiv K_{L,t}/(e^{gt}K_0)$,

and $c_0 \equiv C_0/K_0$, the aggregate capital and the capital held by L-firms evolve according to:

$$\dot{k}_t = (A_H - g)k_t - (A_H - A_L)k_{L,t} - c_0 \tag{30}$$

$$\dot{k}_{L,t} = \lambda\pi_L k_t - (\alpha + g + \lambda)\, k_{L,t}. \tag{31}$$

The initial conditions are $k_0 = 1$ and $k_{L,0} = K_{L,0}/K_0$.

Lemma 1 *Suppose there exists a balanced-growth path equilibrium. The stationary solution to (30)-(31) is a saddle point.*

From (31), the k_L-isocline given by $\dot{k}_{L,t} = 0$ is such that

$$\frac{k_{L,t}}{k_t} = \frac{\lambda\pi_L}{\alpha + g + \lambda} = \hat{k}_L.$$

From (30), the k-isocline given by $\dot{k}_t = 0$ is such that

$$k_{L,t} = \frac{(A_H - g)k_t - c_0}{(A_H - A_L)}.$$

So, both isoclines have a positive slope but, provided a balanced-growth path exists, the k-isocline is steeper. The phase diagram is represented in Figure 3. The k-isocline depends on the endogenous variable, c_0. It is chosen so that the initial condition, $(1, k_{L,0})$, lies on the linear saddle path. In Figure 3, the initial condition is such that $k_{L,0} > \hat{k}_L$. Along the transition, the growth rates of K_t and $K_{L,t}$, which are lower than g, increase so that K_t and $K_{L,t}$ approach their balanced growth path asymptotically. The ratio, $K_{L,t}/K_t$, decreases and tends to $\hat{k}_L$.

3.7 Misallocation

I now describe the extent to which capital is misallocated by comparing the equilibrium allocations to first-best allocations and constrained-efficient allocations. I will also characterize allocations at the limit when trading frictions vanish. Finally, I will consider an economy with three productivity types in order to illustrate an additional channel for misallocation.

Misallocation relative to the first best If a benevolent planner who wishes to maximize the lifetime utility of the household could operate the L and H technologies, it would only operate the H-technology and it would choose to accumulate capital at the rate $(A_H - \rho)/\theta$. The misallocation of capital relative to that first-best benchmark has both a *static* and a *dynamic* dimension.

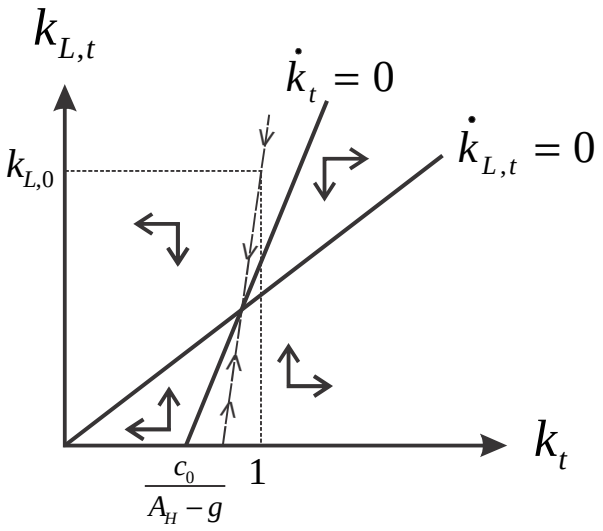


Figure 3: Transition dynamics

The aggregate capital stock is misallocated in a static sense because its distribution across heterogenous firms fails to maximize aggregate output. Under linear technologies, output is maximum if all the capital is held by type-H firms. The share of the aggregate capital stock that is not allocated to the most productive firms is equal to

$$\hat{k}_L = \frac{\lambda \pi_L}{\lambda + \alpha + g}.$$

It increases with the idiosyncratic risk, λ, and with the search frictions, $1/\alpha$. The bargaining power of brokers has no direct effect on this measure of misallocation.[15] However, it has an indirect effect through the rate of growth of the economy. As η increases, g decreases and $\hat{k}_L$ increases. So, while market frictions can be summarized by one sufficient statistic, $\alpha(1-\eta)$, when studying firms' valuations and Tobin's q, search frictions and broker's bargaining power must be distinguished when considering the (mis-)allocation of capital.

The dynamic misallocation refers to the rate of capital accumulation that is inefficiently low. I measure this misallocation by the difference in the growth rates between a frictionless economy and a frictional economy, i.e., $g^* - g$. It is equal to

$$g^* - g = \frac{r^* - r}{\theta},$$

where $r^* = A_H$. Using the expression for r, it can be rearranged as:

$$g^* - g = \frac{\lambda \pi_L \left(A_H - A_L\right)}{\theta \left[r + \lambda + \alpha(1-\eta)\right]}.$$

[15] If there are more than two realizations for the productivity shocks and if the stochastic process for these shocks obeys a continous-time Markov chain, then it is possible for brokers' bargaining power to affect the static misallocation of capital. See the Appendix.

This *growth deficit* increases with η and λ but it decreases with α.

Misallocation relative to constrained-efficient allocations I now consider the problem of a social planner who is subject to the same trading frictions as the ones agents face. More specifically, over a time interval of length dt, the amount of capital that can be reallocated between low- and high-productivity firms is $\alpha dt K_t$. The planner chooses how to divide total output between consumption and capital formation and how to allocate the flow of capital that reaches the market across different firms.

Proposition 2 ***(Optimal growth.)*** *Suppose the intial condition is such that* $K_{L,0}/K_0 = \lambda\pi_L/(g+\alpha+\lambda)$. *The decentralized equilibrium is constrained-efficient if and only if* $\eta = 0$.

The presence of search frictions in the capital market is a technological constraint that cannot be undone by the social planner. However, the planner would like to eliminate the bargaining power of brokers. For a given K, the bargaining power of dealers does not generate static misallocation of capital across firms. The capital that reaches the market flows toward the most productive firms. However, by capturing a fraction of the surplus generated by the reallocation of capital, brokers lower the expected rate of return of financial assets, the accumulation of capital by households, and hence the rate of growth of the economy.[16]

Misallocation at the frictionless limit I consider the limit of the decentralized equilibrium as trading frictions in the capital market vanish, $\alpha \to +\infty$, and ask whether the equilibrium allocation converges to a first-best allocation. I distinguish two cases: firms have some bargaining power, $\eta < 1$, and brokers have all the bargaining power, $\eta = 1$.

Proposition 3 ***(Growth at the frictionless limit.)***

1. *Nonmonopolistic dealers,* $\eta < 1$. *As* $\alpha \to +\infty$, $r \to A_H$ *and* $g \to (A_H - \rho)/\theta$.
   - *The share of capital allocated to H-firms,* $\hat{k}_H \to 1$,
   - *Intermediation fees,* $\phi_j(k) \to 0$ *for all* j.

[16] This result is different from the portfolio inefficiency in Lagos and Rocheteau (2009) when $\eta > 0$. In their model, both the real interest rate and the asset supply are exogenous. If the payoff from holding the asset is linear in asset holdings, then there is no distortion associated with dealer's bargaining power. In contrast, in this paper, brokers' bargaining power distorts the rate of return of capital and the rate at which it is accumulated.

2. *Monopolistic dealers, $\eta = 1$. For all α, $r < A_H$ is the unique solution to*

$$r = A_H - \frac{\lambda}{r+\lambda}(A_H - \bar{A}), \tag{32}$$

and $g = (r-\rho)/\theta < (A_H - \rho)/\theta$.

- *The share of capital allocated to H-firms, $\hat{k}_H \to 1$.*
- *Intermediation fees, $\phi_L(k) = (A_H - A_L)\, k/(r+\lambda) > 0$ for all $k > 0$.*

When firms have some bargaining power, the frictionless limit of the decentralized equilibrium coincides with the textbook AK model. The real interest rate approaches the highest productivity, which then determines the rate of growth of the economy. All the capital is held by the most productive firms and payments to brokers vanish.

If brokers have all the bargaining power, then the equilibrium allocation fails to converge to a first best. This result is a version of the Diamond (1971) paradox whereby competitive pressures are inexistant, irrespective of the frequency at which agents meet each other, when one side of the market has all the bargaining power. Indeed, the effective rate at which firms meet brokers is $\alpha(1-\eta) = 0$ for all $\alpha > 0$. From (27), the real interest rate is the solution to (32). It is easy to check that $r < A_H$ with $r \to A_H$ as $\lambda \to 0$ and $r \to \bar{A}$ as $\lambda \to +\infty$. So the economy can grow even when brokers extract all the surplus associated with firms' capital reallocation. It is only when the idiosyncratic productivity risk disappears that the economy converges to the competitive benchmark. Even though the economy grows at a lower rate than the frictionless benchmark, the capital is not misallocated in a static sense since it is held by the most productive firms.

Misallocation with three productivity types In the version of the model studied so far, brokers reallocate the capital toward the most productive firms, those with idiosyncratic productivity A_H. Hence, the static misallocation of the capital is only due to the idiosyncratic productivity shocks that hit firms in-between two consecutive accesses to the market. This result holds true if there are more than two realizations for the productivity shocks provided that the shocks are i.i.d. across time. However, if the shocks are not independent, then there can exist equilibria where the capital is not reallocated toward the most productive firms.

In order to illustrate this point, I consider a generalization of the model with $J = 3$ distinct productivity levels, $A_1 < A_2 < A_3$. Firms of type-j transition to type-$(j+1)$ (modulo 3) at rate $\lambda_{j(j+1)} > 0$. There are no other transition, i.e., $\lambda_{ij} = 0$ if i and j are not consecutive integers (modulo 3). The Tobin's q of a firm with idiosyncratic productivity j

solves

$$rq_j = A_j + \lambda_{j(j+1)}(q_{j+1} - q_j) + \alpha(1-\eta)\,(1-q_j)\,. \tag{33}$$

I consider equilibria where $\arg\max\{q_j\} = 2$. Firms with the highest market value are type-2 firms even though they do not have the highest instantaneous productivity. In order for the asset market to clear, the highest q_j must be equal to $p_K = 1$. Hence, $q_2 = 1$. For an equilibria, one needs to verify that both $q_1 < 1$ and $q_2 < 1$.

Proposition 4 *Suppose $A_1 > \rho$ and $\theta > 1$. For any $\alpha < +\infty$, if η is sufficiently close to 1 and λ_{31} is sufficiently large, then there exist an equilibrium with $q_1 < 1$, $q_3 < 1$, and $q_2 = 1$.*

If brokers' bargaining power is close to 1 and the transition from type 3 to type 1 is fast, then capital operated by type-2 firms has the highest market value. Moreover, if α is large, most of the capital will be held by type-2 firms even though both the first-best and the constrained-efficient allocations prescribe that capital should be held by type-3 firms. So, brokers' bargaining power can affect directly both the static misallocation of capital and the dynamic misallocation.

4 Frictions in the primary capital market

So far, I have assumed that the newly produced capital was sold to brokers instantly at the competitive price, $p_K = 1$. In reality, even though capital producers have "*embraced a lean manufacturing model focused on matching production to demand*" (Darmouni and Sutherland, 2024), there are delays for new equipment to reach frms.[17] Therefore, I now relax the assumption that new capital is sold instantly and I make the capital market a two-sided brokered market as shown in Figure 4. There is a unit measure of *primary brokers* who purchase capital goods from capital producers. The meeting rate between capital producers and primary brokers is α_0. The bargaining power of primary brokers is $\eta_0 \in [0,1]$. In the previous section, $\eta_0 = 0$ and $\alpha_0 = +\infty$. The secondary capital market is as described in the previous section. Firms meet *secondary brokers*, who have bargaining power η, at rate α.

4.1 The value of idle and employed capital

In order to determine the value of idle capital, I conjecture that the expected discounted profits of a capital producer with $k \geq 0$ units of idle capital can be expressed as $q_0 k$ where $q_0 \in \mathbb{R}^+$ is independent of k. Consider first the determination of the terms of trade in

[17] Darmouni and Sutherland (2024) use the COVID episode to illustrate how supply chain disruptions can make new capital hard to find.

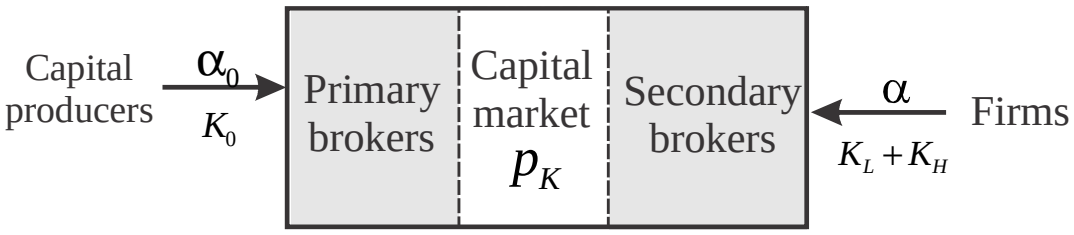


Figure 4: Capital market structure

a meeting between a capital producer with k units of capital and a primary broker. The outcome of the negotiation is a pair, $(k^s, \phi) \in [0, k] \times \mathbb{R}_+$, that specifies the quantity sold, $k^s \leq k$, and the payment to the intermediary. The surplus of the capital producer is

$$q_0(k - k^s) + p_K k^s - \phi - q_0 k = (p_K - q_0)k^s - \phi,$$

where p_K is the price at which brokers trade capital and q_0 is the value to the capital producer of one unit of unemployed capital. The first term on the left side is the value of $k - k^s$ units of unsold capital. The following two terms, $p_K k^s - \phi$, represent the revenue from the sale to the broker. The last term is the disagreement point if no trade takes place, in which case the value of the capital producer is $q_0 k$. The surplus of the broker is ϕ. As long as $p_K \geq q_0$, $k^s = k$ is Pareto optimal. From the generalized Nash solution, $\phi = \eta_0(p_K - q_0)k$. Hence, the capital producer's total surplus is $(1 - \eta_0)(p_K - q_0)k$.

The value to the capital producer of one unit of capital is q_0 solution to:

$$rq_0 = \alpha_0(1 - \eta_0)(p_K - q_0) + \dot{q}_0. \tag{34}$$

The opportunity cost of holding a unit of idle capital is given by the left side of (34). According to the right side, the unit of capital is sold to a broker at rate α_0 at which point the capital producer makes a profit equal to $(1 - \eta_0)(p_K - q_0)$. Capital producers choose the level of investment to maximize $i_t(q_0 - 1)$. Hence, in any equilibrium with $i_t > 0$, $q_0 = 1$. It follows from (34) that the price of capital solves

$$p_K = 1 + \frac{r}{\alpha_0(1 - \eta_0)}. \tag{35}$$

It is equal to its production cost, one, plus the holding cost until this unit can be sold. The bargaining power of the broker gives rise to a multiplier, $1/(1-\eta_0)$, of the holding cost of idle capital, r/α_0. The logic is as follows. The profit that the capital producer makes by selling one unit of capital, $(1 - \eta_0)(p_K - 1)$, must cover the holding cost, r/α_0. As the holding cost increases by one, the price of capital, p_K, must increases but by more than one because brokers captures a fraction of the gains that the sale of the capital goods generates.

The value of a unit of capital employed by a firm of type $j \in \{L, H\}$, q_j, solves an equation analogous to (12) where the price of capital, p_K, is now given by (35). Using the same steps as before, the Tobin's q is:

$$q_j = \frac{[r + \alpha(1-\eta)]\, A_j + \lambda \bar{A}}{[r + \alpha(1-\eta)]\,[r + \lambda + \alpha(1-\eta)]} + \frac{\alpha(1-\eta) p_K}{r + \alpha(1-\eta)} \quad \forall j \in \{L, H\}. \tag{36}$$

Each firm chooses its capital stock to maximize $(q_j - p_K)k$. Hence, in order for the capital market to clear, $q_L < p_K$ and

$$q_H = p_K. \tag{37}$$

An equilibrium can be reduced to a 4-tuple, $(p_K, q_L, q_H, r) \in \mathbb{R}_+ \times [0,1] \times \{1\} \times (\rho, +\infty)$, solution to (35), (36), (37) and $r < \rho/(1-\theta)$ if $\theta < 1$. I substitute p_K by its expression given by (35) into (36) for $j = H$ to obtain the following proposition.

Proposition 5 ***(Real interest rate with frictional primary capital market.)*** *Assume*

$$\rho \left[1 + \frac{\rho}{\alpha_0(1-\eta_0)}\right] < \frac{[\rho + \alpha(1-\eta)]\, A_H + \lambda \bar{A}}{\rho + \lambda + \alpha(1-\eta)}. \tag{38}$$

Moreover, $\theta \geq 1$ or, if $\theta < 1$,

$$\frac{\rho}{(1-\theta)} \left[1 + \frac{\rho}{\alpha_0(1-\eta_0)(1-\theta)}\right] > \frac{[\rho + \alpha(1-\eta)(1-\theta)]\, A_H + \lambda \bar{A}}{\rho + (1-\theta)\,[\lambda + \alpha(1-\eta)]}. \tag{39}$$

The equilibrium real interest rate along a balanced-growth path with $g > 0$ is uniquely determined as the solution to

$$r \left[1 + \frac{r}{\alpha_0(1-\eta_0)}\right] = \frac{[r + \alpha(1-\eta)]\, A_H + \lambda \bar{A}}{r + \lambda + \alpha(1-\eta)}. \tag{40}$$

It is such that $r \in (\rho, A_H)$, $\partial r/\partial \alpha_0 > 0$, and $\partial r/\partial \eta_0 < 0$.

Equation (40), which is represented graphically in Figure 5, determines a unique $r > 0$. The right side depends on the frictions in the secondary market, α and η, while the left side depends on the frictions in the primary market, α_0 and η_0. If α_0 increases or η_0 decreases, then the left side falls and r increases. If α increases or η decreases, then the right side increases and r increases. Under condition (38), r is larger than ρ, which guarantees that the growth rate of the economy, $g = (r - \rho)/\theta$, is positive. For (38) to be satisfied, the term $\alpha_0(1 - \eta_0)$ must be sufficiently large. Equivalently, the frictions in the primary market cannot be too large.

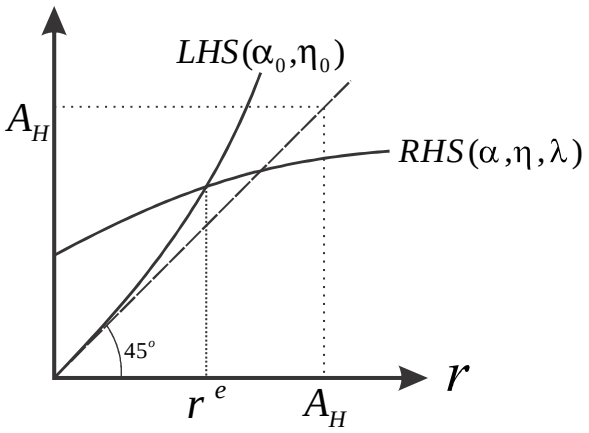


Figure 5: Determination of the real interest rate when the primary and secondary capital markets are frictional.

4.2 Capital misallocation

As in the previous section, there is dynamic misallocation in that the rate of capital accumulation is inefficiently low and there is static misallocation in that a fraction of the capital is not operated by type-H firms.

Dynamic misallocation In order to understand the sources of the dynamic misallocation, it can be instructive to rewrite (40) as follows:

$$r = \left[A_H - \overbrace{\frac{\lambda \pi_L (A_H - A_L)}{r + \lambda + \alpha(1-\eta)}}^{\text{misallocation discount}} \right] \overbrace{\left[1 + \frac{r}{\alpha_0 (1-\eta_0)} \right]^{-1}}^{\text{primary markdown}}.$$

It shows the different forces that push the real interest rate below its first-best value, A_H. There is the misallocation discount described earlier according to which the real interest rate falls below its frictionless value in the presence of idiosyncratic productivity shocks due to the frictions to reallocate capital. The novelty is the second term on the right side, labelled *primary markdown*, which corresponds to the markdown of the wholesale price of newly-produced capital goods that results from the market power of primary brokers.

I now investigate how the reduction of the frictions on each side of the market affects misallocation. Consider the limit as the frictions in primary market vanish. Since $\eta_0 < 1$, then as $\alpha_0 \to +\infty$, r approaches to the solution given by (27). Graphically, the left side approaches the 45^o line. Suppose next that the frictions in the secondary market vanish, $\alpha(1-\eta) \to +\infty$, while the primary market remains frictional. Then r converges to the

solution to

$$r\left[1+\frac{r}{\alpha_0(1-\eta_0)}\right]=A_H.$$

Even though all units of employed capital are allocated to firms of type H, there is some idle or unemployed capital. As a result, the real interest rate is below A_H.

Static misallocation I turn to the allocation of the capital stock. In a first-best allocation, all the capital should be held by type-H firms. In the presence of frictions, some capital will be idle and some capital will be held by type-L firms. Let K_I denote the aggregate stock of idle capital, that is the newly produced capital that has not reached the market and is not employed by any firm. It obeys the following law of motion:

$$\dot{K}_{I,t}=\dot{K}_t-\alpha_0 K_{I,t}. \tag{41}$$

The stock of idle capital increases with investment, $\dot{K}_t$, but it decreases with the flow of idle capital that is sold to brokers and allocated to firms at rate α_0. In the following proposition, I denote $\hat{k}_j \equiv K_j/K$ the share ot total capital in state $j \in \{I, L, H\}$.

Proposition 6 *On a balanced-growth path, the distribution of the capital stock across states is:*

$$\hat{k}_I = \frac{g}{g+\alpha_0} \tag{42}$$

$$\hat{k}_L = \frac{\lambda\pi_L\alpha_0}{(g+\lambda+\alpha)(g+\alpha_0)} \tag{43}$$

$$\hat{k}_H = \frac{\alpha_0(g+\alpha+\lambda\pi_H)}{(g+\alpha+\lambda)(g+\alpha_0)}. \tag{44}$$

The unemployment rate of the capital, $K_{I,t}/K_t$, increases with the growth rate of the economy but decreases with the meeting rate in the primary capital market, α_0. The share of capital employed by firms of type L, $\hat{k}_L$, increases with α_0 and decreases with g. The share of capital held by the most productive firms, $\hat{k}_H = K_H/K$, increases with α_0 while the effect of g is ambiguous. In order to measure the static misallocation, I express the aggregate output as

$$Y_t = K_t \sum_{j\in\{L,H\}} \hat{k}_j A_j = K_t\left(\frac{\alpha_0}{g+\alpha_0}\right)\left[A_H-\frac{\lambda\pi_L(A_H-A_L)}{\lambda+\alpha+g}\right]. \tag{45}$$

The TFP is equal to the one in (21) scaled down be the employment rate of the capital stock, $\alpha_0/(g+\alpha_0)$. While an increase in g reduces the misallocation across the employed capital, it also reduces the employment rate of capital.

5 Frictional capital market with knowledge spillovers

The linear technology assumed so far provides tractability but it fails to capture realistic diminishing returns of capital at the firm level. Moreover, it omits a key insight from the literature according to which growth arises from the production of knowledge. In the following, I generalize the technology to address both limitations. Since the focus of this paper is on the capital market, I follow Arrow (1962), Frankel (1962), and Romer (1986), and assume that knowledge is created by learning how to operate capital. As firms install new machines, they learn the most efficient ways to operate them in order to increase production.[18] This new knowledge, which is nonrival, is accessible to all firms. In Arrow's words:

> "I advance the hypothesis here that technical change in general can be ascribed to experience, that is the very activity of production which gives rise to problems for which favorable responses are selected over time. (...) [I take] cumulative gross investment (cumulative production of capital goods) as an index of experience. Each new machine produced and put into use is capable of changing the environment in which production takes place, so that learning is taking place with continually new stimuli."

Formally, the technology of a firm is of the form $A_j K^{1-\beta} k^{\beta}$ where $\beta \in (0,1)$ and K is the cumulative production of capital goods.[19] While the technology features diminishing marginal returns at the firm level, it has constant returns to scale with respect to both K and k. Following Arrow (1962), K is also a measure of the stock of knowledge or experience. When individual firms accumulate k, they do not take into account how it affects the stock of knowlege and other firms' productivity. This knowledge externality is stronger as β decreases. In the previous section, $\beta = 1$ and the externality is mute. I generalize the set of idiosyncratic productivities to any arbitrary finite set, $\{A_1, ..., A_j, ...A_J\}$, with $0 < A_1 < ... < A_j < ... < A_J$. The average idiosyncratic productivity is $\bar{A} \equiv \sum_{j=1}^{J} \pi_j A_j$.

I start with the characterization of the first-best allocation chosen by a planner who is not subject to any friction to allocate the capital among firms and who understands the knowledge externality associated with capital accumulation.

Lemma 2 ***(First-best allocations.)*** *Assume*

$$1 - \theta < \frac{\rho}{\left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}} < 1. \tag{46}$$

[18] Irwin and Klenow (1994) provide evidence of learning-by-doing spillovers in the semiconductor industry.
[19] See also the review of the literature in Barro and Sala-i-Martin (2004), Chapter 4.3.

At the first best, the economy grows at rate $g^* = (r^* - \rho)/\theta$ *where*

$$r^* = \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}} \right)^{1-\beta}. \tag{47}$$

The share of capital allocated to firms of type j *is*

$$\Xi_j^* \equiv \frac{\pi_j k_{j,t}}{K_t} = \frac{\pi_j A_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}} = \pi_j \left(\frac{A_j}{r^*} \right)^{\frac{1}{1-\beta}} \quad \forall j \in \{1, ..., J\}. \tag{48}$$

The first-best rate-of-return of capital, r^*, is a function homogeneous of degree 1 of individual productivities, A_j. As shown from (48), and in contrast to the linear technology studied earlier, provided $\beta < 1$, it is optimal to allocate a positive amount of capital to all firms such that $A_j > 0$.

5.1 Firms' valuations and Tobin's q

The value of a firm solves

$$\begin{aligned} r v_{j,t}(k) &= A_j K_t^{1-\beta} k^\beta + \lambda \sum_{x=1}^{J} \pi_x \left[v_{x,t}(k) - v_{j,t}(k) \right] \\ &\quad + \alpha(1-\eta) \max_{k' \geq 0} \left[v_{j,t}(k') - (k' - k) - v_{j,t}(k) \right] + \dot{v}_{j,t}(k), \end{aligned} \tag{49}$$

for $j \in \{1, ..., J\}$. Relative to the previous section, $v_{j,t}(k)$ is not time-invariant because productivity, which depends on K_t, grows over time. Indeed, on a balanced-growth path, $\dot{K}_t / K_t = g$. Hence, the output flow, $A_j K_t^{1-\beta} k^\beta = A_j e^{(1-\beta)gt} K_0^{1-\beta} k^\beta$, grows at rate $(1-\beta)g$. The following lemma provides a closed-form solution for the value function.

Lemma 3 ***(Firms' valuations.)*** *The value function of the firm is*

$$v_{j,t}(k) = \frac{\hat{A}_j K_t^{1-\beta} k^\beta}{r - (1-\beta)g + \alpha(1-\eta)} + \frac{\alpha(1-\eta)}{\alpha(1-\eta) + r} k + \Lambda_{j,t}, \tag{50}$$

where $\Lambda_{j,t} \in \mathbb{R}$ *is independent of* k *and*

$$\hat{A}_j \equiv \frac{\left[r - (1-\beta)g + \alpha(1-\eta) \right] A_j + \lambda \bar{A}}{r - (1-\beta)g + \alpha(1-\eta) + \lambda}. \tag{51}$$

The first term on the right side of (50) is the expected discounted sum of output flows from t until the next effective access to the capital market at Poisson arrival rate $\alpha(1-\eta)$. The effective discount rate on the denominator, $r - (1-\beta)g + \alpha(1-\eta)$, which corresponds to the Gordon (1959) growth formula, incorporates a capitalization effect that accounts for

the growth rate of productivity, $(1-\beta)g$. The second term on the right side is the expected discounted resale value of the capital at the time of the next effective access to the market. The effective discount rate of that second term, $\alpha(1-\eta)+r$, does not incorporate the growth rate of productivity. The reason is that while the output flow of the firm is growing over time, the price of capital remains constant and equal to $p_K = 1$. The third term is the continuation value that is independent of k.

The quantity $\hat{A}_j$ given by (51) is the average discounted idiosyncratic productivity until the next effective access to the market conditional on the current productivity, j. It is equal to a weighted average of the current productivity and the future expected productivity. As the growth rate increases, the weight on current productivity decreases. Hence, as g increases, $\hat{A}_j$ falls if $A_j > \bar{A}$ and it increases if $A_j < \bar{A}$. Thus, productivity growth reduces the dispersion of effective productivities.

The problem of the firm of type $j \in \{1, ..., J\}$ who has access to the capital market at time t is $\max_{k\geq 0}\{-k + \upsilon_{j,t}(k)\}$. By substituting $\upsilon_{j,t}(k)$ by its expression given by (50), it can be rewritten as:

$$k_{j,t} = \arg\max_{k\geq 0}\left\{-rk + \left[\frac{r+\alpha(1-\eta)}{r+\alpha(1-\eta)-(1-\beta)g}\right]\hat{A}_j K_t^{1-\beta}k^{\beta}\right\}. \tag{52}$$

The novelty relative to the problem of a firm in a frictionless market is the term between squared brackets on the right side, which is larger than one if $g > 0$. This term captures the fact that the firm can only readjust its capital infrequently, at effective rate $\alpha(1-\eta)$, while productivity is growing at rate $(1-\beta)g$. Thus, when determining its demand for capital, the firm scales up its current outplut flow by a factor greater than one. The first-order condition for an optimal capital choice is

$$\left(\frac{k_{j,t}}{K_t}\right)^{1-\beta} = \frac{\beta\left[r+\alpha(1-\eta)\right]}{r\left[r+\alpha(1-\eta)-(1-\beta)g\right]}\hat{A}_j, \tag{53}$$

for $j \in \{1, ..., J\}$. Provided $r > (1-\beta)g - \alpha(1-\eta)$, the demand for capital is positive. It increases with the effective productivity of the firm, $\hat{A}_j$, which itself depends on market structure via α and η. Productivity growth, $(1-\beta)g$, affects the demand for capital through two channels. First, there is the Gordon effect that reduces the effective discount rate and tends to increase the demand for capital. Second, there is an effect operating through the effective idiosyncratic productivity, $\hat{A}_j$, that increases the demand for capital if A_j is less than $\bar{A}$ and reduces it otherwise.

One can use Lemma 3 to obtain the value of a marginal unit of capital at a firm, the Tobin's marginal q defined as $\upsilon'_{j,t}(k)$, along the equilibrium path.

Lemma 4 ***(Tobin's marginal*** *q)The Tobin's marginal q of a firm at time t with current productivity type $j \in \{1, ..., J\}$ who accessed the market for the last time at $t - \tau$ with productivity type $i \in \{1, ..., J\}$ is equal to:*

$$q_{ij}(\tau) = 1 + \frac{r}{r + \alpha(1 - \eta)} \left[e^{(1-\beta)g\tau} \frac{\hat{A}_j}{\hat{A}_i} - 1 \right], \ \forall (i, j) \in \{1, ..., J\}^2, \forall \tau \in \mathbb{R}^+. \tag{54}$$

According to (54), the Tobin's q is exactly equal to one when $\tau = 0$ and $A_j = A_i$, which corresponds to the time at which a firm exits the capital market. Once a firm has readjusted its capital, its Tobin's q, $q_{jj}(\tau)$, increases over time, due to aggregate productivity rising, until the next idiosyncratic shock hits. At that point, the Tobin's q can increase or decrease. The term between squared brackets, $e^{(1-\beta)g\tau}\hat{A}_j/\hat{A}_i - 1$, is a measure of the mismatch between the firm's current capital and its current productivity. It follows that

$$q_{ij}(\tau) < 1 \Leftrightarrow \tau < \frac{\ln(\hat{A}_i) - \ln(\hat{A}_j)}{(1 - \beta)g}.$$

The Tobin's q is less than one if the firm's current idiosyncratic productivity is lower than its value at the time the firm accessed the market, $A_j < A_i$, and that time is not too far away. If $A_j > A_i$, the Tobin's q is greater than one irrespective of τ. In the following, I characterize the distribution of firms across the dimension (i, j, τ).

5.2 Distribution of capital

The joint distribution of capital holdings and productivity types across firms is determined as follows. The capital of a firm is entirely determined by the last time, $t - \tau$, it had access to the capital market, and its productivity at that time, $i \in \{1, ..., J\}$. It is equal to $k_{i,t-\tau} = e^{-g\tau}k_{i,t}$ where $k_{i,t}$ is given by (52). Therefore, the type of a firm at some time t, its capital, and its current productivity can be reexpressed as a triple, (i, j, τ), composed of its current productivity, $j \in \{1, ..., J\}$, and the pair $(\tau, i) \in \{1, ..., J\} \times \mathbb{R}^+$.

Let $\gamma(i, j, \tau)$ denote the density measure of firms of type $(i, j, \tau) \in \{1, ..., J\}^2 \times \mathbb{R}^+$. It is equal to

$$\gamma(i, j, \tau) = \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau} \mathbb{I}_{\{i=j\}} + (1 - e^{-\lambda\tau})\pi_j \right]. \tag{55}$$

The density measure of firms who contacted the market at time $t - \tau$ and were of type i at that time is $\pi_i \alpha e^{-\alpha\tau}$, where I used the independence between the firm-broker matching process and the process for idiosyncratic productivity shocks. The density of the former is an exponential distribution with parameter α while the distribution of the idiosyncratic productivities is $\{\pi_i\}_{i=1}^J$. If the current type of a firm is $j \neq i$, then the firm had to experience

at least one productivity shock over the time interval $(t-\tau, t)$, with probability $1-e^{-\lambda\tau}$, and the realization of the last shock was j with probability π_j. If the current type is $j=i$, it is also possible that the firm did not experience any shock with probability $e^{-\lambda\tau}$.

The total capital held by firms with idiosyncratic productivity j is

$$K_{j,t} \equiv \sum_{i\in\{1,...,J\}} \int_{\mathbb{R}^+} k_{i,t-\tau}\gamma(i,j,\tau)d\tau, \quad \forall j \in \{1,...,J\}. \tag{56}$$

In the following lemma, $K_{j,t}$ is computed in closed form.

Lemma 5 ***(Distribution of capital.)****In a balanced-growth path equilibrium, the share of the aggregate capital stock held by type-j firms is equal to*

$$\frac{K_j}{K} = \frac{\pi_j}{g+\alpha+\lambda}\left[\alpha\left\{\frac{\beta\left[\alpha(1-\eta)+r\right]}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]}\right\}^{\frac{1}{1-\beta}}\hat{A}_j^{\frac{1}{1-\beta}}+\lambda\right]. \tag{57}$$

5.3 Value of intermediaries

The market value of a broker, w_t, solves

$$rw_t = \alpha\Phi_t + \dot{w}_t \quad \text{where } \Phi_t \equiv \sum_{(i,j)\in\{1,...,J\}^2}\int_{\mathbb{R}^+}\phi_t(i,j,\tau)\gamma(i,j,\tau)d\tau \tag{58}$$

is the broker's expected revenue from a random meeting with a firm at rate α and $\phi_t(i,j,\tau)$ denotes the broker's revenue in a meeting with a firm of type (i,j,τ). From the bargaining solution, (10),

$$\phi_t(i,j,\tau) = \eta\left[\upsilon_{j,t}(k_{j,t}) - (k_{j,t}-k_{i,t-\tau}) - \upsilon_{j,t}(k_{i,t-\tau})\right]. \tag{59}$$

The broker appropriates a fraction, η, of the increase in the firm's market value from adjusting its capital stock from $k_{i,t-\tau}$ to $k_{j,t}$. Note that there are two reasons for the firm to readjust its capital. First, its idiosyncratic productivity might have changed since its last access to the market, $i \neq j$. Second, the steady increase in aggregate productivity prompts firms to demand more capital over time.

Lemma 6 *The value of a broker is*

$$w_t = \frac{\alpha\Phi_t}{r-g}, \tag{60}$$

where

$$\Phi_t = \eta\left\{\frac{\sum_j\left[(\lambda+\beta g)\hat{A}_j - \frac{\alpha\lambda}{\alpha+\beta g}\bar{A}\right]\pi_j\tilde{k}_j^{\beta}}{\left[r-(1-\beta)g+\alpha(1-\eta)\right](\alpha+\lambda+\beta g)} - \frac{rg}{\alpha\left[\alpha(1-\eta)+r\right]}\right\}K_t. \tag{61}$$

In order to get some intuition for the expression for Φ_t in (61), suppose there is no productivity growth, $g = 0$, and $J = 2$. The average profits of the broker reduce to

$$\Phi_t = \eta \left(\frac{\lambda \pi_1 \pi_2}{\alpha + \lambda} \right) \frac{\left(\hat{A}_2 - \hat{A}_1 \right) \left(\tilde{k}_2^{\beta} - \tilde{k}_1^{\beta} \right)}{r + \alpha(1 - \eta)} K_t.$$

In the capital market, brokers reallocate $\tilde{k}_2 - \tilde{k}_1$ units of capital from the type-1 firms who hold $\tilde{k}_2$ to the type-2 firms who hold $\tilde{k}_1$. The steady-state measure of firms with productivity type $j = 1$ holding k_2 is equal to $\lambda \pi_1 \pi_2 / (\alpha + \lambda)$, which is also the measure of type-2 firms holding k_1. Thus, $\lambda \pi_1 \pi_2 / (\alpha + \lambda)$ can be interpreted as the flow of low- and high-productivity firms that are matched to reallocate their capital. The instantaneous output gain from this reallocation, normalized by the aggregate capital stock, is $\left(\hat{A}_2 - \hat{A}_1 \right) \left(\tilde{k}_2^{\beta} - \tilde{k}_1^{\beta} \right)$. Finally, the effective discount rate is $r + \alpha(1 - \eta)$.

5.4 Equilibrium real interest rate

I close the characterization of the equilibrium with the determination of r. There are two equivalent approaches to market clearing, one in terms of flow and one in terms of stock. Market clearing in flows requires that

$$\alpha \sum_{j=1}^{J} \pi_j k_{j,t} = (\alpha + g) K_t. \tag{62}$$

The left side represents the demand of capital emanating from firms accessing the market at rate α. The measure π_j of firms with productivity A_j demand $k_{j,t}$. The right side corresponds to the supply of capital. By the law of large numbers, a firm who accesses the market holds on average the mean of the distribution of capital across firms, K_t. In addition, capital producers supply $\dot{K}_t = g K_t$ units of new capital.

The market-clearing condition in terms of stocks is:

$$\sum_{(i,j) \in \{1,\ldots,J\}^2} \int_{\mathbb{R}^+} e^{-g\tau} k_{i,t} \gamma(i, j, \tau) d\tau = 1, \tag{63}$$

where $e^{-g\tau} k_{i,t}$ is the capital held by a firm who had access to the market at time $t - \tau$ and had productivity type i at that time. The right side corresponds to the aggregate capital held by all firms normalized by K_t, which is equal to one. Substituting $\gamma(i, j, \tau)$ by its expression given by (55), it can be shown after some calculation that (63) reduces to (62).

Substitute $k_{j,t}$ by its value in (62) to show that the real interest rate solves

$$r = \underbrace{\frac{[\alpha(1 - \eta) + r]}{r - (1 - \beta)g + \alpha(1 - \eta)}}_{\text{capitalization effect}} \times \underbrace{\left(\frac{\alpha}{\alpha + g} \right)^{1 - \beta}}_{\text{congestion effect}} \times \underbrace{\beta}_{\text{knowledge externality}} \times \hat{r}^*, \tag{64}$$

where

$$\hat{r}^* \equiv \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{1-\beta}.$$

It is instructive to compare this expression for r with the first-best value, r^*, given by (47).

The last term on the right side of (64), $\hat{r}^*$, is a CES aggregator of individual productivities similar to r^*. In the decentralized equilibrium, $\hat{r}^*$ is constructed from the effective productivities, $\hat{A}_j$, instead of the current ones, A_j. These effective productivities are endogenous and depend on r. Using that $\{(\hat{A}_j, \pi_j)\}_{j=1}^J$ is a mean-preserving reduction of the spread of $\{(A_j, \pi_j)\}_{j=1}^J$, together with the observation that $x^{\frac{1}{1-\beta}}$ is a convex function, $\hat{r}^* < r^*$. In addition, the term, β, scales down $\hat{r}^*$ due to the fact that firms do not internalize the effect of their capital accumulation decision on aggregate productivity.

The second term on the right side of (64), $[\alpha/(\alpha+g)]^{1-\beta}$, is a congestion effect that captures the negative pressure that the flow of new capital, g, exerts on the real interest rate. Indeed, in the presence of trading frictions, only a flow of firms of measure αdt access the market over an infinitesimal time inverval of length dt, which makes the market thin. As a result, the flow of new capital goods that are produced during that same time interval, gdt, raises the supply relative to the demand, which tends to reduce r.

Finally, the first term on the right side of (64) is a capitalization effect, akin to the one in Gordon's formula, that scales the real interest rate up. It accounts for the fact that the productivity of capital is growing between two consecutive accesses to the market. For this effect to operate, it is important that the firm can only adjust its capital stock infrequently, at the effective rate $\alpha(1-\eta)$, thereby preventing it from equalizing its marginal product and the user cost of capital at all points in time.

By substituting $g = (r-\rho)/\theta$ into (64) one obtains an equation in r:

$$r = \frac{\theta\beta\left[\alpha(1-\eta)+r\right]}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)} \left\{ \frac{\alpha\theta}{\alpha\theta+r-\rho} \sum_{j=1}^{J} \pi_j \left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}} \right\}^{1-\beta}, \qquad (65)$$

where, from (51), with a slight abuse of notation,

$$\hat{A}_j(r) \equiv \frac{\left\{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)\right\} A_j + \theta\lambda\bar{A}}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)+\theta\lambda}. \qquad (66)$$

In any equilibrium, r solution to (65) must satisfy $r > \rho$, so that the growth rate is positive, and $r > g$, i.e., $r(1-\theta) < \rho$, so that lifetime utilities are finite. The equilibrium has a simple recursive structure. The real interest rate is determined by (65), which gives the growth rate of the capital stock, g. Given r, (53) determines the demand for capital and (55) gives the distribution of capital across firms. The following proposition provides sufficient conditions for the existence of an equilibrium.

Proposition 7 ***(Existence of equilibria with capital market frictions and learning externality.)*** *If*

$$\rho < \beta \left\{ \sum_{j=1}^{J} \pi_j \left\{ \frac{[\rho + \alpha(1-\eta)] A_j + \lambda \bar{A}}{\rho + \alpha(1-\eta) + \lambda} \right\}^{\frac{1}{1-\beta}} \right\}^{1-\beta}, \tag{67}$$

and either $\theta \geq 1$ *or* $\theta < 1$ *and*

$$\frac{\rho}{1-\theta} > \left[1 - \frac{(1-\beta)\alpha(1-\theta)(1-\eta)}{\beta\rho + \alpha(1-\theta)(1-\eta)} \right] \left\{ \frac{\alpha(1-\theta)}{\alpha(1-\theta) + \rho} \sum_{j=1}^{J} \pi_j \left[\hat{A}_j \left(\frac{\rho}{1-\theta} \right) \right]^{\frac{1}{1-\beta}} \right\}^{1-\beta}, \tag{68}$$

then there exists a balanced-growth equilibrium. The equilibrium real interest rate is bounded above by r^* *given by (47).*

According to (67), if $\theta \geq 1$, an equilibrium with positive growth exists provided that a CES transformation of the effective productivities is sufficiently large relative to the rate of time preference. If $\theta < 1$, an additional condition, (68), guarantees that the lifetime utility of households is bounded for all $t < +\infty$.

5.5 Limiting cases

I study three limiting cases in order to disentangle the roles played by the knowledge externality, market frictions, and the idiosyncratic productivity risk in the determination of the real interest rate and the allocation of capital across heterogenous firms.

Lemma 7 *Consider the following limiting cases.*

1. *Vanishing learning externality. As* $\beta \to 1$,

$$r \to \lim_{\beta \to 1} \hat{A}_J = \frac{\{r + \alpha(1-\eta)\} A_J + \lambda \bar{A}}{r + \alpha(1-\eta) + \lambda}. \tag{69}$$

   The share of capital allocated to firms of type $j \in \{1, ..., J\}$ *is*

$$\frac{K_j}{K} = \frac{g + \alpha}{g + \alpha + \lambda} \mathbb{I}_{\{j=J\}} + \frac{\lambda \pi_j}{g + \alpha + \lambda}. \tag{70}$$

2. *Frictionless limit. Suppose* $\eta < 1$. *As* $\alpha \to +\infty$,

$$r \to \beta \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}} \right)^{1-\beta} = \beta r^*. \tag{71}$$

The share of capital allocated to firms of type $j \in \{1, ..., J\}$ is

$$\frac{K_j}{K} = \frac{\pi_j A_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}} = \Xi_j^*. \tag{72}$$

3. *Vanishing idiosyncratic risk. As $A_j \to \bar{A}$ for all $j \in \{1, ..., J\}$,*

$$r \to \frac{\alpha(1-\eta)+r}{\alpha(1-\eta)+r-(1-\beta)g} \left(\frac{\alpha}{\alpha+g}\right)^{1-\beta} \beta\bar{A}. \tag{73}$$

The first part of Lemma 7 shows that the equilibrium when $\beta \to 1$ coincides with the one from Section 3. As the knowledge externality vanishes, the real interest rate approaches $\max_j\{\hat{A}_j\}$. According to (70), some capital is misallocated in that it is held by less productive firms. The first term on the right side represents the (re)allocation of capital towards the most productive firms – those of type J. It is composed of the new capital produced at rate g and the existing capital that returns to the market at rate α. The second term represents the idiosyncratic shocks that redistribute the capital randomly across productivity types at rate λ.

According to the second part of Lemma 7, as trading frictions vanish, the real interest rate approaches the product of $r^*$and β. This result is a generalization of Romer (1986) in an environment with heterogenous firms. Firms do not internalize the effect of their capital choice on the creation of knowledge. As a result, firms underestimate the rate of return of their investment in capital goods. However, from (72), the share of K_t that is allocated to firms of type j coincides with the capital share in the first-best allocation.

The third part of Lemma 7 gives the expression for r in the absence of idiosyncratic productivity shocks, when $A_j \to \bar{A}$ for all j. Frictions in the capital market are still relevant because, in the presence of a growing TFP, firms would like to readjust their capital stock continuously over time. The competitive real interest rate, $\beta\bar{A}$, is multiplied by two factors: a front-loading term that is greater than one because productivity growth induces firms to demand more capital, and a congestion term that is less than one because limited participation of firms in the market tends to depress demand. As will be shown later, r and g can be larger than their values in a frictionless competitive economy.

5.6 Capital misallocation

Misallocation occurs along two dimensions. There is *dynamic misallocation* due to the rate of capital accumulation that is too low relative to the first best. There is *static misallocation* when for a given aggregate capital stock, K_t, the distribution of that capital across firms fails to maximize aggregate output.

Dynamic misallocation From Proposition 7, $r < r^*$ in all equilibria. As a result, the rate of capital accumulation in equilibrium is lower than the one at the first best. From Lemma 7, the equilibrium real interest rate in a frictionless Walrasian economy is $\beta r^* < r^*$ due to the knowledge externality. What remains to be determined is whether r can be between βr^* and r^* when the capital market is frictional. In order to address this point, I consider the case where α is large and I rewrite (65) as

$$r \approx \left\{1 - \frac{(1-\beta)(r-\rho)}{\theta\left[\alpha(1-\eta)+r\right]}\right\}^{-1} \beta \left\{\sum_{j=1}^{J} \pi_j \left[\hat{A}_j(r;\alpha,\eta)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta}, \tag{74}$$

where I used that $\alpha\theta/(\alpha\theta + r - \rho) \to 1$. Consider an increase in η. There are two distinct holdup problems working in opposite direction.

First, firms hedge against future idiosyncratic shocks by taking a position that is closer to the average value for A_j. Formally, as η increases, $\hat{A}_j$ increases if $A_j < \bar{A}$ and it decreases if $A_j > \bar{A}$. Thus, an increase in η generates a mean-preserving reduction in the distribution of effective productivities, $\hat{A}_j$. Using that $\hat{A}_j^{\frac{1}{1-\beta}}$ is a convex function of $\hat{A}_j$, the reduction in the dispersion of effective productivities pushes the real interest rate downward.

Second, firms anticipate that they will demand more capital in the future, for a given A_j, due to a growing aggregate productivity. They mitigate the associated holdup problem by frontloading their demand for capital. This effect, which is captured by the first term on the right side of (74), contributes to a higher real interest rate.

When the idiosyncratic productivities, A_j, are very concentrated, then the second effect can dominate. The following proposition establishes that r can be larger than its competitive value, obtained when $\alpha(1-\eta) \to +\infty$.

Proposition 8 ***(Frictional vs frictionless real interest rate.)*** *Suppose $A_j = \bar{A}$ for all $j \in \{1, ..., J\}$. Moreover, $\eta = 1$ and $\beta \in (1-\theta, 1)$. There exists a $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$, the equilibrium real interest rate is greater than the Walrasian real interest rate, i.e., $r > \beta\bar{A}$.*

If brokers have all the bargaining power and the capital market is not too illiquid, then the real interest rate is greater than the one in a Walrasian economy. So, a high bargaining power reduces the dynamic misallocation. There is a cost, however, when the distribution of A_j is nondegenerate: capital is not distributed so as to maximize output.

Static misallocation. I now turn to the static misallocation of the capital stock. I start by characterizing the distribution of the aggregate capital stock across firms of different

productivity levels. From (57) and (65), the share of K_t that is allocated to firms of type j is

$$\frac{K_j}{K} = \underbrace{\hat{\Xi}_j}_{\text{share in the } K \text{ market}} + \underbrace{\frac{\lambda}{g+\alpha+\lambda}\left(\pi_j - \hat{\Xi}_j\right)}_{\text{productivity shocks}}, \tag{75}$$

where

$$\hat{\Xi}_j \equiv \frac{\pi_j \hat{A}_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}}.$$

The first term on the right side, $\hat{\Xi}_j$, describes the allocation of the capital among the firms accessing the market at a given point in time. The share of capital received by type-j firms is analogous to the first-best share, Ξ_j^*, defined in (72), where current productivity, A_j, is replaced with effective productivity, $\hat{A}_j$. The presence of frictions reduces the denominator, $\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}$, and raises the numerator, $\pi_j \hat{A}_j^{\frac{1}{1-\beta}}$, if j is small. The second term on the right side of (75) corresponds to the misallocation generated by the occurrence of idiosyncratic productivity shocks. Those shocks allocate the capital to type-j firms according to the shares, π_j.

Based on (75), a measure of misallocation is

$$MIS_K^j = \hat{\Xi}_j - \Xi_j^* + \frac{\lambda}{g+\alpha+\lambda}\left(\pi_j - \hat{\Xi}_j\right). \tag{76}$$

It is equal to the difference between K_j/K and its first-best value. If $MIS_K^j > 0$, too much capital is allocated to firms of type j whereas if $MIS_K^j < 0$ too little capital is allocated to those firms.

Proposition 9 ***(Capital misallocation.)*** *Consider an equilibrium featuring $r - g > 0$. The share of capital allocated to the least productive firms is too high, $MIS_K^1 > 0$, while the share of capital allocated to the most productive firms is too low, $MIS_K^J < 0$.*

The quantity, MIS_K^j, is an incomplete measure of misallocation. It does not take into account the distribution of capital among firms with the same productivity level. I now measure the degree of overall capital misallocation by the loss of output for a given capital stock, K_t, relative to the first best. At the first-best allocation, the average output per unit of capital is

$$y_K^* \equiv \frac{Y_t}{K_t} = \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}. \tag{77}$$

It is the maximum output that can be obtained from a given capital stock, K. In equilibrium, the output per unit of capital is

$$y_K = \frac{Y_t}{K_t} \equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,...,J\}^2} e^{-\beta g \tau} A_j \tilde{k}_i^{\beta} \gamma(i,j,\tau) d\tau, \tag{78}$$

where $\tilde{k}_i \equiv k_i/K$ is given by (53). The right side is summing the output of all firms. The output of a firm in state (i,j,τ) normalized by K_t is $A_j \left(k_{i,t-\tau}/K_t\right)^{\beta} = e^{-\beta g \tau} A_j \tilde{k}_i^{\beta}$. The following lemma provides a closed-form solution for y_K.

Lemma 8 *The average output per unit of capital in the decentralized equilibrium is equal to*

$$y_K = \left(\frac{\alpha+g}{\alpha}\right)^{\beta} \left(\frac{\alpha}{\alpha+\beta g}\right) \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}\right)^{-\beta} \sum_{j=1}^{J} \pi_j \tilde{A}_j \hat{A}_j^{\frac{\beta}{1-\beta}} \tag{79}$$

where

$$\tilde{A}_j \equiv \frac{(\alpha+\beta g) A_j + \lambda \bar{A}}{\alpha+\beta g+\lambda}.$$

The term, $\tilde{A}_j$, in the definition of y_K, resembles $\hat{A}_j$ in that it is a weighted average of A_j and $\bar{A}$. The weights, however, are different. In the definition of $\hat{A}_j$, the weights involve the discount rate and firms' bargaining power. In the definition of $\tilde{A}_j$, the weights involve the rate of productivity growth, βg. Suppose $\eta = 0$, and α and λ are large relative to r^*. Then, $\hat{A}_j \approx \tilde{A}_j \approx \left(\alpha A_j + \lambda \bar{A}\right)/(\alpha+\lambda)$ and

$$y_K \approx \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}\right)^{1-\beta}. \tag{80}$$

This expression corresponds to the one for y_K^* where A_j has been replaced with $\hat{A}_j$. For given K_t, the misallocation of capital can be measured by the percentage deviation of y_K relative to the first best,

$$MIS_Y \equiv \frac{y_K^* - y_K}{y_K^*}. \tag{81}$$

Proposition 10 ***(Static misallocation.)*** *Suppose $\eta = 0$, and α and λ are large relative to r^*. A decrease in α or an increase in λ generate an increase in MIS_Y.*

6 A calibrated example

The unit of time is a year. The rate of time preference is set to $\rho = 0.01$ and the intertemporal elasticity of substitution to $\theta^{-1} = 0.5$. I use average productivity, $\bar{A}$, to target a growth rate

of GDP of $g = 2\%$. As a result, the real interest rate is $r = \theta g + \rho = 5\%$. I choose β to be approximately equal to the capital share, i.e., $\beta = 0.4$.[20]

The process for the idiosyncratic productivity shocks is analogous to the one in Cui (2022). There are two levels of productivity, $A_L < A_H$, such that $A_H/A_L = 1.64$, drawn with equal probabilities, $\pi_L = \pi_H = 0.5$. I set $\lambda = 0.5$. Hence, on average, a firm changes productivity every $(\lambda\pi_j)^{-1} = 4$ years. The probability that a firm does not draw a new productivity value within a year is $e^{-\lambda\pi_H} = e^{-\lambda\pi_L} = 0.78$, which is in the same ballpark as the measure of yearly persistence for plant-level productivity shocks in the US reported by Cui (2022) and references therein.

Capital reallocation is defined as the sales of existing capital across all firms. In the model, it is ϱK where

$$\varrho \equiv \alpha \sum_{(i,j)\in\{L,H\}^2} \int_0^{+\infty} \gamma(i,j,\tau) \max\{e^{-g\tau}k_i - k_j, 0\} d\tau.$$

The flow of firms who access the market per unit of time is α. The capital sale generated by a firm of type j who last accessed the market at time $t-\tau$ with productivity i is equal to $e^{-g\tau}k_i - k_j$ if this quantity is positive and zero otherwise. The term $e^{-g\tau}k_i$ corresponds to the capital held by the firm (normalized by the aggregate capital stock, K_t) while k_j is its desired capital stock. Only type-L firms who were previously type-H when they accessed the market at time $\tau < \hat{\tau} \equiv \ln(k_H/k_L)/g$ sell some capital. Hence,

$$\varrho = \alpha \int_0^{\hat{\tau}} \gamma(H,L,\tau) \left(e^{-g\tau}k_H - k_L\right) d\tau.$$

Substitute $\gamma(H,L,\tau)$ by its expression given by (55), i.e., $\alpha e^{-\alpha\tau}\pi_H\pi_L(1-e^{-\lambda\tau})$, and use that $k_H = e^{\hat{\tau}g}k_L$ to obtain

$$\varrho = \alpha\pi_H\pi_L k_L \int_0^{\hat{\tau}} \alpha e^{-\alpha\tau}(1-e^{-\lambda\tau}) \left[e^{g(\hat{\tau}-\tau)} - 1\right] d\tau.$$

According to Eistfeldt and Shi (2018), capital reallocation represents about 2% of firms total assets annually. I interpret total assets as K_t and I target $\varrho = 0.02$. The instrument to achieve this target will be broker's bargaining power. Indeed, η affects the dispersion of asset demands, k_H/k_L, and hence $\hat{\tau}$ and the measure of reallocation above.

In order to capture realistic search frictions in the second-hand capital market, I resort to different sources. Darmouni and Sutherland (2024) study secondary markets for physical

[20] In the model, there is no labor so that all the income is capital income. However, the production function could be generalized so that the output flow of a firm is equal $AK^{1-\beta}k^{\beta}h^{1-\beta}$ where h represents hours of work. The production of capital and brokerage services do not require labor. Households supply $\bar{h}$ inelastically and the labor market is competitive. The real wage is w. For each firm, the payment to labor is $wh = (1-\beta)AK^{1-\beta}k^{\beta}h^{1-\beta}$. By summing across all firms, the aggregate labor share is $1-\beta$.

capital in 2021–22 during the disruption caused by the COVID-19 pandemic. They report that one of the world's largest equipment manufacturers describing the episode as "*the worst supply chain I've seen in my career... Dealers had to tell customers they had to wait, sometimes as long as 12 months, to get a new machine*". Therefore, I will think of one year as an usually long delay. Exapro, an online marketplace for used industrial equipment, reports that "*on average machines were sold on Exapro after 8 months since being published in the catalog.*"[21] Southern Fabricating Machinery Sales, a machinery broker, reports that the time to sell varies from 0-30 days for newer machinery and 60-90 days for older machinery.[22] Finally, Gavazza (2016) in his study of the market for business aircrafts estimates that an aircraft stays on the market approximately eight months before a low-valuation owner is able to sell it and that, on average, it takes slightly more than four months for a dealer to find a buyer. In order to obtain search frictions in the ballpart of these numbers, I set $\alpha = 2$. On average it takes 6 months for a firm to readjust its capital stock.

Parameter	Explanation/Target	Value
ρ	rate of time preference	0.01
θ	intertemporal elasticity of substitution	2
β	capital share	0.4
(A_L, A_H)	Dispersion of firm productivities and growth rate	$(0.089, 0.146)$
(π_H, π_L)	Representation of US plant-level productivity as AR(1)	(0.5,0.5)
λ	Persistence of US plant-level productivity	0.5
α	Time to sell machinery equipments	2
η	Capital reallocation	0.926

Table 1. Calibrated parameter values

Table 1 reports the parameter values resulting from this calibration strategy. A key parameter is broker's bargaining power which is equal to 0.93. So brokers have almost all the bargaining power when interacting with firms. A high dealer's bargaining power is consistent with Gavazza's (2016) study of the market for business aircrafts. He estimates dealers' bargaining to be 0.9732. One justification is that intermediaries, who trade frequently, are more sophisticated in negotiations than regular firms. In order to interpret this value, I compute the net margin of the brokers as follows:

$$\text{Net margin} = \frac{\Phi}{(2\varrho + g)\, K}.$$

The numerator is the average revenue per trade. The denominator is the total amount of capital that is purchased, $(\varrho + g)\, K$, or sold, ϱK, by firms. The net margin is equal to 0.6%.

[21] https://hub.exapro.com/how-long-does-it-take-to-sell-used-machinery

[22] https://www.southernfabsales.com/blog/the-art-of-valuing-used-machinery

So, despite having a large bargaining power, brokers are generating small profits per unit of capital traded.

Finally, for these parameter values, the competitive real interest rate is

$$r^{ce} = \beta \left(\sum_{j \in \{L,H\}} \pi_j A_j^{\frac{1}{1-\beta}} \right)^{1-\beta} = 4.79\%,$$

while the first-best rate of return of capital is $r^* = 12\,\%$. So, the equilibrium real interest rate, 5%, is great than the frictionless, competitive value. The growth rate, $g = 2\%$, is also larger than the one in the frictionless economy, $g^{ce} = 1.89\%$.

6.1 Real interest rate and market frictions

I first measure the effects of market frictions on the real interest rate and hence on the growth rate of the economy. The left panel of Figure 6 plots r as a function of α. The relation between r and α is monotone increasing when $\eta = 0$ but it is decreasing over some range when the broker's bargaining power is high. Thus, reducing the frictions (by increasing α) in the capital market is not necessarily growth enhancing. In fact, for the calibrated values in Table 1, an increase in α reduces r and g. If search frictions are completely removed, r falls by about 0.2 percentage points and g falls by about 0.4 percentage points. Conversely, if α is reduced by a factor of 10, from 2 to 0.2, r increases by 0.45 percentage points and g increases by 0.9 percentage points. So, maybe surprisingly, a very illiquid market can be beneficial for growth.

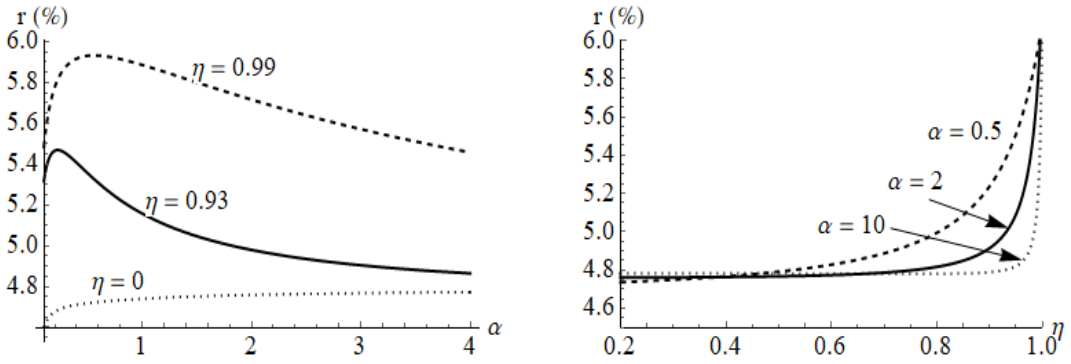


Figure 6: Real interest and capital market frictions

The right panel of Figure 6 conveys a similar idea by ploting r as a function of η. For a large range of values for η, r increases with η. Moreover, the increase is very steep as η approaches one. For instance, for the benchmark calibration, increasing η from 0.93 to 0.99

raises r by more than one percentage point. This observation suggests that the knowledge externality associated with capital accumulation can be partially mitigated by giving brokers monopoly power.

6.2 Misallocation

Figure 7 plots the allocation of capital across firms of different productivities. The top panels represent the demand for capital from firms accessing the market. The bottom panels plot the shares of the total capital stock operated by firms of different types. The patterns are similar. As trading frictions are reduced, the demands and the shares become more disperse. The more productive firms demand more capital, except when α is very low, while the less productive ones demand less. For instance, while the capital is divided evenly between L and H firms when α is close to 0, more than 60% of the capital is held by type-H firms when α rises above 8. At the opposite, as brokers' bargaining power increases, the demands and shares become more concentraded. If $\eta = 0$ more than 60% of the capital is operated by H-firms and this share falls to 50% as η approaches one.

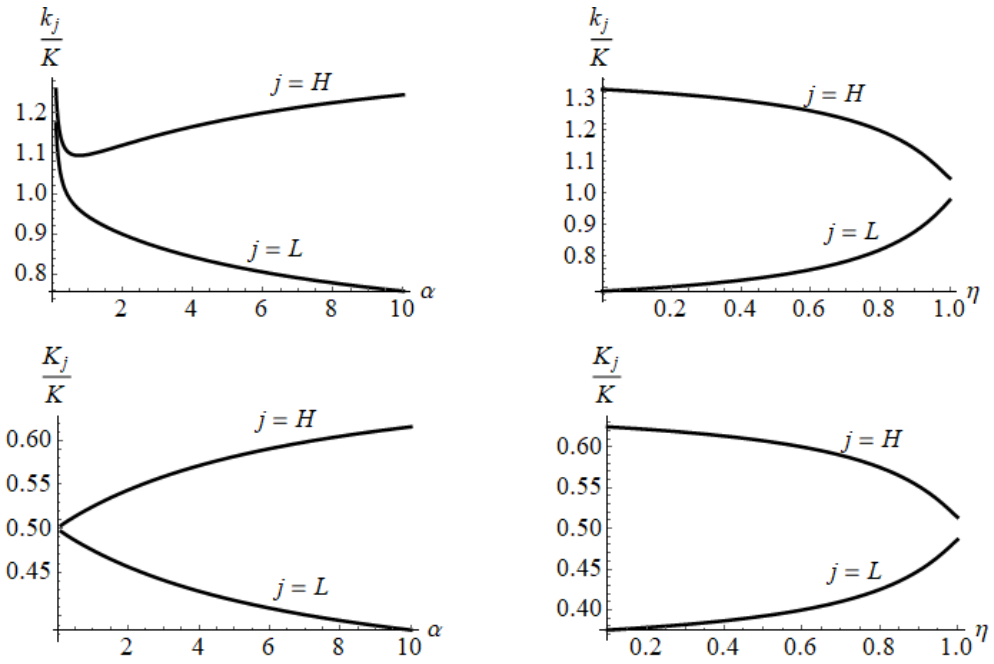


Figure 7: Top panels: demand for capital for firms of productivity type j; Bottom panels: share of total capital held by firms of productivity type j.

Figure 8 plots the measure of misallocation, MIS_Y, in terms of lost output relative to the first-best allocation taking K_t as given. For the benchmark calibration, the loss of output is about 1.2%. It increases to about 1.5% if α is divided by 2 and it falls to 0.85% if α is

multiplied by 2. If η increases to 1, then the misallocation loss rises to 1.7%. Thus, controling for the aggregate capital stock and its effect on aggregate productivity, the output loss due to the misallocation of capital is limited.

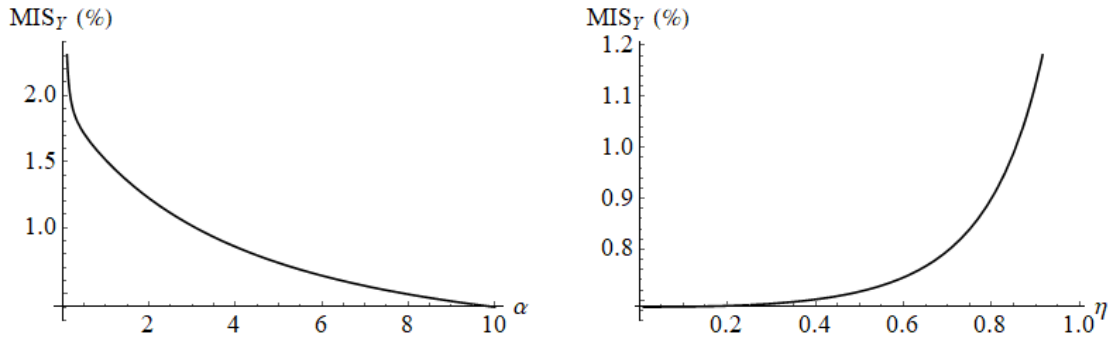


Figure 8: Misallocation measured by the loss of output relative to the first best.

6.3 Welfare

Household's welfare is affected by both the rate at which capital is accumulated and the allocation of a given stock of capital across heterogenous firms at a point in time. In order to determine which effect dominates for household's lifetime utility, I now measure the net welfare loss associated with market frictions. Along a balanced growth path, consumption is $C_t = Y_t - \dot{K}_t = (y_K - g) K_t$. Hence, the household lifetime utility is

$$U(C_t) \equiv \int_0^{+\infty} e^{-\rho t} \frac{(C_t)^{1-\theta}}{1-\theta} dt = \frac{(y_K - g)^{1-\theta} K_0^{1-\theta}}{(1-\theta)\left[\rho - (1-\theta)g\right]},$$

where I used that $C_t = C_0 e^{gt}$ with $C_0 = (y_K - g) K_0$. The term $\rho - (1-\theta)g$ on the denominator is the effective discount rate that takes into account the growth rate of consumption. I measure the welfare loss in consumption-equivalent units, Δ, relative to the frictionless allocation obtained as $\alpha(1-\eta) \to +\infty$. It solves $U(C_t) = U\left[C_t^{ce}(1-\Delta)\right]$, where C_t^{ce} is the time-path for consumption at the competitive equilibrium. Hence,

$$1 - \Delta = \left[\frac{\rho - (1-\theta)g^{ce}}{\rho - (1-\theta)g}\right]^{\frac{1}{1-\theta}} \frac{y_K - g}{y_K^* - g^{ce}},$$

where $g^{ce} = (r^{ce} - \rho)/\theta$ is the growth rate at the competitive equilibrium.

The left panel of Figure 9 shows that the welfare loss is nonmonotone in α. It is negative when α is sufficiently low (but not too close to 0) and it is positive when α is above 4. Thus, welfare can be higher than the one in a frictionless economy. Given η fixed at its calibrated

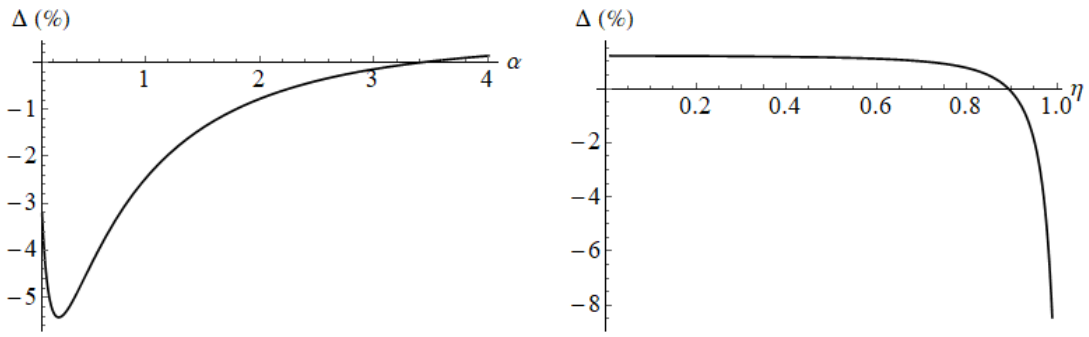


Figure 9: Welfare loss relative to frictionless equilibrium

value, the value of α that maximizes household's lifetime utility is slightly above 0.2. In the right panel, the welfare loss is decreasing in η and it is minimum at $\eta = 1$. It should be emphasized that these welfare calculations focus on economies along a balanced growth path and do not incorporate transitional paths. However, because a lower α and a higher η are associated with larger static misallocation, the true welfare gains from reducing α or raising η are likely to be larger than those reported in Figure 9.

7 Entry of brokers

Up to this point, the micro-structure of the capital market was captured by two exogenous parameters, α and η. Realistically, both parameters are endogenous and depend on the economy's needs for capital reallocation. In order to show how the micro-structure of the capital market and economic growth can be intertwined, I endogenize α, the speed at which firms can access brokers.[23] I introduce a general matching function between firms and brokers and I allow for free entry of the latter. This extension will make the technological role played by brokers to reallocate capital explicit and it will provide a rational for their positive bargaining power.

The measure of brokers per firm, b, is endogenous and determined by free entry. The rate at which a firm meets a broker is given by an increasing and strictly concave function, $\alpha(b)$, such that $\alpha(0) = 0$, $\alpha(+\infty) = +\infty$ $\alpha'(0) = +\infty$, and $\alpha'(+\infty) = 0$. Since meetings between firms and brokers are bilateral, the rate at which a broker meets a firm is $\alpha(b)/b$. Brokers who participate in the market incur a flow entry cost. In order to obtain a balanced growth path with a constant α, the entry cost, κK, is proportional to the aggregate capital stock

[23] In the Appendix, I endogenize the bargaining power of brokers by allowing them to invest in their bargaining strength.

with $\kappa > 0$.[24]

I start by adding firm entry to the model of Section 3 where the firm's technology is linear in k. From (16), the value of a broker who participates in the market solves

$$rw = -\kappa K + \frac{\alpha(b)}{b} \frac{\eta \left(A_H - A_L\right) K_L}{r + \lambda + \alpha(b)(1-\eta)} + \dot{w}. \tag{82}$$

The broker meets a firm at rate $\alpha(b)/b$. The firm has low productivity, with probability π_L, and its average capital is K_L/π_L. The gain from reallocating capital from low-productivity firms to high-productivity ones is $\left(A_H - A_L\right) / \left[r + \lambda + \alpha(b)(1-\eta)\right]$. The broker captures a share η of that gain. The last term on the right side of (82) is the increase in the value of the broker over time as the economy grows and more capital needs to be reallocated. From free entry, $w_t = 0$ for all t, allowing us to rewrite (82) as

$$\kappa = \frac{\alpha(b)}{b} \frac{\eta \left(A_H - A_L\right) \hat{k}_L}{r + \lambda + \alpha(b)(1-\eta)}. \tag{83}$$

On the right side, the broker's expected profits are proportional to the share of capital that is misallocated, $\hat{k}_L$. Substitute $\hat{k}_L$ by its expression to obtain:

$$\frac{b}{\alpha(b)} = \frac{\theta\eta \left(A_H - A_L\right) \lambda \pi_L}{\left[r + \lambda + \alpha(b)(1-\eta)\right] \left\{\left[\lambda + \alpha(b)\right] \theta + r - \rho\right\} \kappa}. \tag{84}$$

Equation (84) determines the measure of brokers, b, as a function of r. The left side is increasing in b from 0 to $+\infty$ as b varies over $\mathbb{R}^+$. The right side is decreasing in b because of a competition effect and because the share of misallocated capital decreases as b increases. Hence, for all r such that $r - \rho > 0$, there is a unique b solution to (84). Moreover, the right side of (84) is decreasing in r for all $r > \rho$. Hence, b solution to (84) is a decreasing function of r.

From (27), the real interest is determined by

$$r = \frac{\left[r + \alpha(b)(1-\eta)\right] A_H + \lambda \bar{A}}{r + \lambda + \alpha(b)(1-\eta)}. \tag{85}$$

The real interest rate solution to (85) is an increasing function of b. It is positive when $b = 0$ and it approaches A_H as b tends to $+\infty$. An equilibrium is a pair, $(b, r) \in \mathbb{R}^+ \times (\rho, +\infty)$, solution to (84)-(85) and such that $r < \rho/(1-\theta)$ if $\theta < 1$.

Proposition 11 ***(Endogenous intermediation in the capital market.)*** *Suppose $\rho \approx 0$, and $\eta \in (0, 1)$. There exits a unique balanced-growth path equilibrium and it is such that $r \in (0, A_H)$.*

[24] Alternatively, one could assume that a broker who enters incurs a one-time setup cost. The presence of such fixed cost can give rise to multiple equilibria because the effective discount rate of the broker's profits, $r - g$, is endogenous and jointly determined with $\alpha(b)$.

1. *A decrease in the entry cost, κ, raises the measure of brokers, the real interest rate, and the growth rate.*

2. *Suppose $A_L < 0$ and $\bar{A} = 0$. As $\eta \to 0$ and $\eta \to 1$, $r \to 0$.*

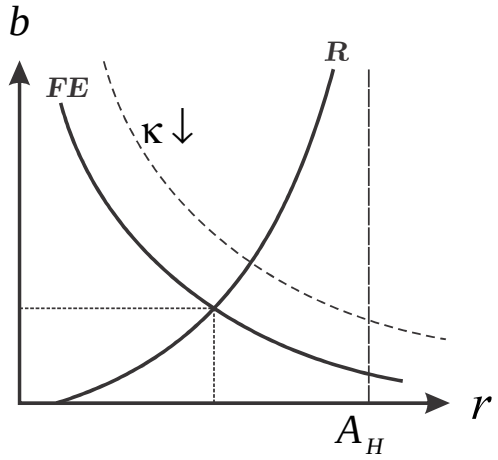


Figure 10: Equilibrium with free-entry of capital market intermediaries.

Under the assumption $\rho \approx 0$, the growth rate of the economy is positive whenever $r > 0$. The determination of the equilibrium is represented graphically in Figure 10. The FE curve corresponds to (10). As r increases, the discounted profits of the brokers fall, and hence b falls. The R-curve corresponds to (85). As the measure of brokers increases, the misallocation of the capital decreases, and its rate of return increases. A decrease of the entry cost, κ, shifts the FE-curve upward. The measure of brokers and the real interest rate increase.

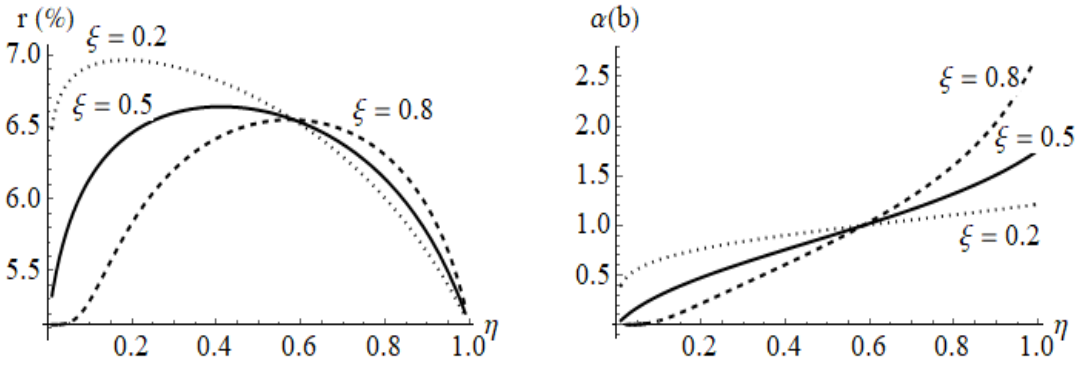


Figure 11: Effects of brokers' bargaining power on the real interest rate and firms' access to the capital market. The matching function is $\alpha(b) = 2b^{\xi}$. The parameters are $\rho = 0.01$, $\theta = 2$, $\gamma = 0.01$, $\lambda = 2$, $A_H = 0.1$, $A_L = 0$, $\pi_H = 0.5$.

The second part of Proposition 11 describes the effects of a change in brokers' bargaining power, η, on r and g. Suppose $\bar{A} = 0$, which requires that productivity in the low state is negative, $A_L < 0$, i.e., firms incur losses and capital cannot be disposed of. As α goes to 0, r given by (85) tends to 0. Graphically, as η increases, the FE-curve shifts upward while the R-curve shifts to the left. So, the effect on r is a priori ambiguous. If $\eta = 0$, brokers do not enter, the capital is not reallocated, and $r = 0$. If $\eta = 1$, the holdup problem is so severe that r tends to 0. So, r is a nonmonotone function of η that achieves a maximum for some intermediate value of brokers' bargaining power.

Figure 11 illustrates this result with a numerical example featuring $\bar{A} > 0$. The matching technology is $\alpha(b) = b^{\xi}$ where $\xi \in (0, 1)$ reflects the contribution of brokers to the matching process in the capital market. The relation between r and ξ is hump-shaped. So, the real interest rate and the rate of growth of the economy are maximum for a value of brokers' bargaining power between 0 and 1. The reason is that brokers play a useful role to reallocate capital across firms of different productivities and their incentive to play that role depends on their bargaining power, as shown by the right panel of Figure 11. The left panel shows that the bargaining power that maximizes r increases with ξ. So, the larger the contribution of brokers to the matching process, the larger their bargaining power in order to maximize economic growth.

8 Conclusion

I developed a model of endogenous growth with heterogeneous firms, where the market for capital goods features search frictions and is intermediated by brokers with some market power. In the absence of knowledge externalities, market frictions lower both the real interest rate and the economy's growth rate. I demonstrated that search and bargaining frictions play distinct roles in explaining the dynamic and static misallocation of capital. When idiosyncratic productivity shocks are independent over time, search frictions directly influence the degree of capital misallocation, while brokers' bargaining power indirectly affects it through the rate of new capital creation. If brokers lack full bargaining power, as search frictions vanish the equilibrium converges to a Walrasian equilibrium, where the real interest rate equals the TFP of the most productive firms. Furthermore, the decentralized equilibrium coincides with the constrained-efficient allocation when brokers have no bargaining power.

In the presence of a knowledge externality associated with the operation of the capital stock, the real interest rate varies non-monotonically with search and bargaining frictions.

An increase in brokers' bargaining power can raise the real interest rate and the economy's growth above their frictionless, Walrasian levels. However, both search and bargaining frictions exacerbate the static misallocation of capital across firms with different productivities. Households' lifetime utility can also be a non-monotonic function of brokers' bargaining power.

Finally, I endogenized the market structure by allowing the entry of brokers. I showed that the real interest rate is a non-monotonic function of brokers' bargaining power, even in the absence of knowledge externalities. I also demonstrated the possibility of multiple balanced-growth path equilibria, distinguished by the degree of intermediation in the capital market and the real interest rate.

Proofs of lemmas and propositions

Proof of Proposition 1. The equilibrium real interest rate along a balanced-growth path with positive growth is a $r \in (\rho, +\infty)$ solution to (27), i.e.,

$$r = RHS(r) \equiv \frac{[r + \alpha(1-\eta)]\, A_H + \lambda \bar{A}}{r + \lambda + \alpha(1-\eta)}.$$

The derivative of the right side is

$$\frac{\partial RHS}{\partial r} = \frac{\lambda \left(A_H - \bar{A}\right)}{\left[r + \lambda + \alpha(1-\eta)\right]^2} > 0.$$

The slope is positive and it is decreasing in r, i.e., $RHS''(r) < 0$. So RHS is an increasing and concave function of r. Moreover,

$$\begin{aligned} RHS|_{r=0} &= \frac{\alpha(1-\eta)A_H + \lambda \bar{A}}{\alpha(1-\eta) + \lambda} \in (A_L, A_H) \\ RHS|_{r=+\infty} &= A_H. \end{aligned}$$

Hence, there is a unique $r > 0$ solution to (27) and it belongs to (A_L, A_H). The solution is such that $r > \rho$ if and only if $RHS|_{r=\rho} > \rho$, i.e.,

$$\frac{[\rho + \alpha(1-\eta)]\, A_H + \lambda \bar{A}}{\rho + \lambda + \alpha(1-\eta)} > \rho.$$

This condition can be rewritten as (28).

I now check that $r - g > 0$ so that the lifetime utility of the household is bounded. Using the Euler equation according to which $g = (r - \rho)/\theta$, the condition $r - g > 0$ can be rewritten as $(\theta - 1)r + \rho > 0$. If $\theta \geq 1$, this inequality holds for all $r > 0$. Suppose $\theta < 1$. The condition $r - g > 0$ can be rewritten as $r < \rho/(1-\theta)$. This inequality holds if and only if $RHS|_{r=\rho/(1-\theta)} < \rho/(1-\theta)$, i.e., (29) holds.

In order to obtain the comparative statics, I make use of the graphical determination of r, denoted r^e, in Figure 2. The RHS increases as α increases, η decreases, or λ decreases. Hence, $\partial r/\partial \alpha > 0$, $\partial r/\partial \eta < 0$, and $\partial r/\partial \lambda < 0$. ■

Proof of Lemma 1. The determinant of the Jacobian matrix associated with (30)-(31) is

$$\det J_k = -\left(A_H - g\right)\left(\alpha + g + \lambda\right) + \lambda \pi_L \left(A_H - A_L\right).$$

In any balanced-growth-path equilibrium,

$$\begin{aligned}\frac{C_0}{K_0} &= A_H - (A_H - A_L)\,\hat{k}_L - g \\ &= A_H - g - (A_H - A_L)\,\frac{\lambda \pi_L}{\lambda + \alpha + g} > 0.\end{aligned}$$

It follows that $\det J_k < 0$. Hence, the solution to $\dot{k}_t = \dot{k}_{L,t} = 0$ is a sadle point. ■

Proof of Proposition 3. Part 1: $\eta < 1$. The result that $\lim_{\alpha \to +\infty} r = A_H$ follows directly from (27), i.e.,

$$\lim_{\alpha \to +\infty} r = \lim_{\alpha \to +\infty} \left\{ A_H - \frac{\lambda \pi_L (A_H - A_L)}{r + \lambda + \alpha(1-\eta)} \right\} = A_H.$$

Given r, g is determined by (7). The share of capital allocated to high productivity firms, $\hat{k}_H$, is given by (18). It approaches one as $\alpha \to +\infty$. From (10),

$$\phi_j(k) = \eta\,(q_j - 1) \max_{k'}(k' - k), \quad j \in \{L, H\}.$$

Since $p_K = q_H = 1$, intermediation fees for high-productivity firms are always 0, $\phi_H(k) = 0$ for all k. Intermediation fees for low-productivity firms are given by (15). Thus,

$$\lim_{\alpha \to +\infty} \phi_L(k) = \frac{\eta\,(A_H - A_L)\,k}{\lim_{\alpha \to +\infty} [r + \lambda + \alpha(1-\eta)]} = 0,$$

provided that $\eta < 1$.

Part 2: $\eta = 1$. From (27), if $\eta = 1$, the real interest rate is determined by the following quadratic equation,

$$r^2 + (\lambda - A_H) r - \lambda \bar{A} = 0.$$

The discriminant is positive, so the equation admits two roots. The product of the roots is negative, so the two roots have opposite sign. I focus on the positive root since $r > 0$ is a necessary condition for an equilibrium with positive growth. When evaluated at $r = A_H$, the left side is equal to $\lambda(A_H - \bar{A}) > 0$. Hence, $r < A_H$. In closed form, the positive solution is

$$r = \frac{A_H - \lambda + \sqrt{(A_H - \lambda)^2 + 4\lambda \bar{A}}}{2} < A_H.$$

The growth rate, g, and the share of capital allocated to high productivity firms, $\hat{k}_H$, are determined as in Part 1. Finally, the intermediation fees for low-productivity firms are determined by (15),

$$\phi_L(k) = \frac{(A_H - A_L)\,k}{r + \lambda}.$$

For all $k > 0$, $\phi_L(k)$ is independent of α and bounded away from 0. ■

Proof of Proposition 2. The planner's problem is:

$$\max_{C_t, \mu_t, K_{H,t}, K_{L,t}} \int_0^{+\infty} e^{-\rho t} u(C_t) dt \tag{86}$$

$$\text{s.t. } \dot{K}_{H,t} = \mu \Upsilon (C, K_H, K_L) + \lambda \pi_H K_L - (\lambda \pi_L + \alpha) K_H \tag{87}$$

$$\dot{K}_{L,t} = (1-\mu) \Upsilon (C, K_H, K_L) + \lambda \pi_L K_H - (\lambda \pi_H + \alpha) K_L, \tag{88}$$

where

$$\Upsilon (C, K_H, K_L) \equiv A_H K_H + A_L K_L - C + \alpha (K_H + K_L).$$

According to (86), the planner maximizes the lifetime discounted utility of a representative household by choosing the time-paths for consumption, C_t, the share of the capital flow, Υ, that is allocated to high-productivity firms, μ_t, and the aggregate capital stocks of high and low productivity firms, $K_{H,t}$ and $K_{L,t}$. According to the definition of Υ, the flow of capital that can be allocated to different firms has two components. First, there is the new capital, which is equal to total output net of consumption, $A_H K_H + A_L K_L - C$. Second, there is the existing capital that becomes available to be reallocated, $\alpha(K_H + K_L)$. The terms $\lambda \pi_H K_L$ and $\lambda \pi_L K_H$ in (87) and (88) correspond to the capital owned by firms that are subject to idiosyncratic productivity shocks.

Necessary conditions. We start by writing the necessary conditions for an optimum according to the Maximum Principle. The current-value Hamiltonian is $H \equiv u(C) + \xi_H \dot{K}_H + \xi_L \dot{K}_L$, where ξ_H and ξ_L denote the co-state variables. The optimal consumption satisfies

$$u'(C) = \mu \xi_H + (1-\mu) \xi_L.$$

The first-order condition for μ is

$$\Upsilon (C, K_H, K_L) (\xi_H - \xi_L) \begin{array}{l} \geq \\ = 0 \\ \leq \end{array} \begin{array}{l} = 1 \\ \text{if } \mu \in (0,1) \\ = 0 \end{array} .$$

The co-state variables solve

$$\rho \xi_H = \xi_H [\mu (A_H + \alpha) - (\lambda \pi_L + \alpha)] + \xi_L [(1-\mu)(A_H + \alpha) + \lambda \pi_L] + \dot{\xi}_H$$

$$\rho \xi_L = \xi_H [\mu (A_L + \alpha) + \lambda \pi_H] + \xi_L [(1-\mu)(A_L + \alpha) - (\lambda \pi_H + \alpha)] + \dot{\xi}_L.$$

Conjecture. We conjecture the solution is such that $\Upsilon (C_t, K_{H,t}, K_{L,t}) > 0$ and $\xi_{H,t} > \xi_{L,t}$ for all t. The condition $\Upsilon (C_t, K_{H,t}, K_{L,t}) > 0$ will be satisfied as long as the economy grows,

$\dot{K}_t > 0$. Hence, $\mu_t = 1$, i.e., all the capital is allocated to high-productivity firms. Under this conjecture, the ODEs for $\xi_{H,t}$ and $\xi_{L,t}$ are given by:

$$\rho\xi_H = \xi_H A_H + \lambda\pi_L(\xi_L - \xi_H) + \dot{\xi}_H \tag{89}$$

$$\rho\xi_L = \xi_H A_L + (\lambda\pi_H + \alpha)(\xi_H - \xi_L) + \dot{\xi}_L. \tag{90}$$

The co-state ξ_H is the social value of a unit of capital invested in high-productivity firms and it is also the marginal utility of consumption. From the right side of (89), a unit of capital in type-H firms yields A_H units of consumption valued at ξ_H. At Poisson arrival rate $\lambda\pi_L$, the firm becomes low productivity and the social value of capital switches from ξ_H to ξ_L. Equation (90) has a similar interpretation with an additional term $\alpha(\xi_H - \xi_L)$ corresponding to the social capital gain when the unit of capital held by a low-productivity firm can be reallocated to a high-productivity firm at Poisson arrival rate α. We look at a solution such that $\dot{\xi}_H/\xi_H = \dot{\xi}_L/\xi_L = \nu$ is constant. From the first-oder condition for C, $u'(C) = \xi_H$, it follows that $\dot{C}/C = g^* \equiv -\nu/\theta$. We conjecture a balance-growth path solution such that $\dot{K}_H/K_H = \dot{K}_L/K_L = g^*$. From (87)-(88),

$$C = A_H K_H + A_L K_L + \alpha K + \lambda\pi_H K_L - (\lambda\pi_L + \alpha + g^*) K_H \tag{91}$$

$$\frac{K_L}{K} = \frac{\lambda\pi_L}{\lambda + \alpha + g^*}. \tag{92}$$

So, (91) determines C_0 given $(K_{H,0}, K_{L,0})$. Equation (92) shows that a balanced-growth path necessitates that the initial share of the capital allocated to low-productivity firms is $\lambda\pi_L/(\lambda + \alpha + g^*)$. Combining (91) and (92),

$$C_0 = A_H K_{H,0} + A_L K_{L,0} - g^* K_0.$$

Denote $\varrho \equiv \rho - \nu$, which is the analog of r in the decentralized equilibrium. We can rewrite (89) and (90) as follows:

$$\varrho\xi_H = \xi_H A_H + \lambda\pi_L(\xi_L - \xi_H) \tag{93}$$

$$\varrho\xi_L = \xi_H A_L + (\lambda\pi_H + \alpha)(\xi_H - \xi_L). \tag{94}$$

Solving (93)-(94) with respect to ϱ and ξ_L/ξ_H, we obtain:

$$\varrho = \frac{(\varrho + \lambda\pi_H + \alpha) A_H + \lambda\pi_L A_L}{\varrho + \lambda + \alpha} \tag{95}$$

$$\frac{\xi_L}{\xi_H} = \frac{A_L + \lambda\pi_H + \alpha}{\varrho + \lambda\pi_H + \alpha}. \tag{96}$$

Equation (95) pins down ϱ uniquely. Indeed, it can be rewritten as:

$$\varrho(\varrho + \lambda + \alpha) - (\varrho + \lambda\pi_H + \alpha) A_H - \lambda\pi_L A_L = 0.$$

It is a quadratic equation with two roots. Since the left side is negative at $\varrho = 0$ and it goes to $+\infty$ as $\varrho \to \pm\infty$, there is a single positive root. Finally, using that $\xi_{H,0} = u'(C_0)$, (96) determines $\xi_{L,0}$ as follows:

$$\xi_{L,0} = \frac{A_L + \lambda\pi_H + \alpha}{\varrho + \lambda\pi_H + \alpha} u'(C_0).$$

Comparaison to decentralized equilibrium. We can compare ϱ given by (95) to r given by (27). They only coincide if and only if $\eta = 0$. It follows that $g^* = g$ if and only if $\eta = 0$. Similarly, ξ_L/ξ_H given by (96) coincides with q_L given by (12),

$$q_L = \frac{A_L + \lambda\pi_H + \alpha(1-\eta)}{r + \lambda\pi_H + \alpha(1-\eta)},$$

if and only if $\eta = 0$. Finally, C_0 is determined as in the decentralized equilibrium. Under our assumption on A_H and A_L, we have shown that $C_0 > 0$.

Sufficiency condition. Now that we have a candidate solution that satisfies the necessary conditions for optimality, we turn to the Arrow sufficiency condition for optimality. The Hamiltonian maximized with respect to μ and c is

$$\begin{aligned}\hat{H}(K_H, K_L, \xi_H, \xi_L) \equiv & \max_{C\geq 0} \{u(C) - \xi_H C\} + \xi_H \left[A_H K_H + A_L K_L + \alpha(K_H + K_L) + \lambda\pi_H K_L - (\lambda\pi_L + \alpha) K\right. \\ & \left. + \xi_L \left[\lambda\pi_L K_H - (\lambda\pi_H + \alpha) K_L\right].\right.\end{aligned}$$

It is linear in K_H and K_L and hence it is concave in (K_H, K_L). Moreover,

$$\lim_{t\to+\infty} e^{-\rho t}\xi_{H,t}K_{H,t} = \lim_{t\to+\infty} e^{-\left[\frac{\rho-(1-\theta)\varrho}{\theta}\right]t}\xi_{H,0}K_{H,0}$$

where I used that

$$\dot{K}_H/K_H = \dot{C}/C = -\nu/\theta.$$

By assumption, $\rho > A_H(1-\theta)$. Hence, $\rho - (1-\theta)\varrho > 0$ and $\lim_{t\to+\infty} e^{-\rho t}\xi_{H,t}K_{H,t} = 0$. By a similar reasoning, $\lim_{t\to+\infty} e^{-\rho t}\xi_{L,t}K_{L,t} = 0$. So the proposed solution is an optimum. ■

Proof of Proposition 4. Using that $q_2 = 1$, the HJB equations, (33), can be rewritten as:

$$\begin{aligned} rq_1 &= A_1 + \lambda_{12}(1-q_1) + \alpha(1-\eta)(1-q_1) \\ r &= A_2 + \lambda_{23}(q_3 - 1) \\ rq_3 &= A_3 + \lambda_{31}(q_1 - q_3) + \alpha(1-\eta)(1-q_3) \end{aligned}$$

It can be solved for q_1 and q_3 to obtain:

$$\begin{aligned} q_1 &= 1 - \frac{r - A_1}{\lambda_{12} + \alpha(1-\eta) + r} \\ q_3 &= [r + \alpha(1-\eta)]^{-1}\left\{A_3 + \lambda_{31}\frac{(A_1 - A_3)[\lambda_{12} + \alpha(1-\eta) + r] + \lambda_{12}(r - A_1)}{[r + \lambda_{31} + \alpha(1-\eta)][\lambda_{12} + \alpha(1-\eta) + r]} + \alpha(1-\eta)\right\} \end{aligned}$$

Suppose $\eta \approx 1$ and $\lambda_{31} \to +\infty$. Then,

$$\begin{aligned} q_1 &= q_3 = \frac{\lambda_{12} + A_1}{\lambda_{12} + r} \\ r &= A_2 + \lambda_{23} \frac{A_1 - r}{\lambda_{12} + r} \end{aligned}$$

It can be checked that $r \in (A_1, A_2)$ and hence $q_1 = q_3 < 1$. Since $r > A_1$, $g = (r - \rho)/\theta > 0$. Moreover, since $\theta > 1$, $r - g > 0$. ■

Proof of Proposition 5. From (36) for $j = H$ and using that $q_H = p_K$,

$$r p_K = \frac{[r + \alpha(1 - \eta)] A_j + \lambda \bar{A}}{r + \lambda + \alpha(1 - \eta)}.$$

Substituting p_K by its expression given by (35) one obtain (40). The left side is

$$LHS(r) \equiv r \left[1 + \frac{r}{\alpha_0(1 - \eta_0)} \right].$$

It is a quadratic function with two roots, $r = 0$ and $r = -\alpha_0(1 - \eta_0)$. It is increasing and convex for all $r > 0$. The right side can be rewritten as

$$RHS(r) \equiv A_H - \frac{\lambda(A_H - \bar{A})}{r + \lambda + \alpha(1 - \eta)}.$$

It is increasing, concave, strictly positive at $r = 0$, and it approaches A_H as $r \to +\infty$. Hence, as depicted in Figure 5, there is a unique $r \in (0, A_H)$ solution to $LHS(r) = RHS(r)$. The real interest rate is larger than ρ if and only if $RHS(\rho) > LHS(\rho)$, i.e., (38) holds. The condition $r > g$ can be reexpressed as $r(1 - \theta) < \rho$. It holds if $\theta \geq 1$. Suppose $\theta < 1$. Then $r - g > 0$ can be rewritten as $r < \rho/(1 - \theta)$. It holds if and only if $LHS\left[\rho/(1 - \theta)\right] > RHS\left[\rho/(1 - \theta)\right]$, i.e., (39) holds. ■

Proof of Proposition 6.

I focus on a balanced growth path where K_I and K grow at the same rate, g. Hence, $\dot{K}_{I,t} = g K_{I,t}$ and $\dot{K}_t = g K_t$ so that (41) can be rewritten as

$$K_{I,t} = \frac{g}{g + \alpha_0} K_t, \tag{97}$$

which gives (42). The capital held by type-H firms evolves according to

$$\dot{K}_H = \alpha(K - K_I - K_H) + \lambda \pi_H (K - K_I - K_H) - \lambda \pi_L K_H + \alpha_0 K_{I,t}. \tag{98}$$

The first term on the right side of (98) corresponds to the stock of capital that is employed by firms of type L, $K_L = K - K_I - K_H$, and that reaches the secondary capital market at rate α to be reallocated to the highest productivity firm. The second term corresponds to the capital employed by firms of type L who receive a positive productivity shock with probability π_H so that the capital is now owned by firms of type H. The third term corresponds to firms of type H that receive a negative productivity shock at rate $\lambda(1-\pi_H)$. The last term on the right side corresponds to the unemployed capital that reaches the market at rate α_0 and is allocated to the most productive firms. Using that $\dot{K}_H = gK_H$ and substituting K_I by its expression, $K_H/K = \hat{k}_H$ given by (44). The capital held by type-L firms evolves according to

$$\dot{K}_L = \lambda\pi_L(K - K_I - K_L) - \lambda\pi_H K_L - \alpha K_L. \tag{99}$$

The interpretation is similar to the one of (98). Using that $\dot{K}_L = gK_L$, $\hat{k}_L \equiv K_L/K$ is given by (43). ■

Proof of Lemma 2.

The planner's problem is:

$$\max_{\{c_t, k_{j,t}, K_t\}} \int_0^{+\infty} e^{-\rho t} \frac{(c_t)^{1-\theta}}{1-\theta} dt \text{ s.t. } c_t = Y_t - \dot{K}_t$$
$$\text{where } Y_t = \sum_{j=1}^{J} \pi_j A_j K_t^{1-\beta} k_{j,t}^{\beta} \text{ and } K_t = \sum_{j=1}^{J} \pi_j k_{j,t}.$$

The variable, $k_{j,t}$, denotes the capital allocated to type-j firms. In the absence of search frictions, it is a control variable. The current-value Hamiltonian is

$$\mathcal{H}(c, K, k_j, \xi, \mu) \equiv \frac{c^{1-\theta}}{1-\theta} + \xi\left(\sum_{j=1}^{J} \pi_j A_j K^{1-\beta} k_j^{\beta} - c\right) + \mu\left(K - \sum_{j=1}^{J} \pi_j k_j\right),$$

where ξ is the costate variable associated with the law of motion for K_t and μ is the Lagrange multiplier associated with the definition of K_t. By the Maximum Principle,

$$(c_t, \{k_{j,t}\}) \in \arg\max \mathcal{H}(c_t, K_t, k_{j,t}, \xi_t, \mu_t) \text{ for all } t.$$

The first-order conditions with respect to c_t and $k_{j,t}$ are:

$$\begin{aligned} (c_t)^{-\theta} &= \xi_t, \, \forall t \in \mathbb{R}_+ \\ \beta A_j \left(\frac{K_t}{k_{j,t}}\right)^{1-\beta} &= \frac{\mu_t}{\xi_t} \, \forall j \in \{1, ..., J\}, \, \forall t \in \mathbb{R}_+. \end{aligned}$$

The second first-order condition can be rexpressed as

$$\frac{k_{j,t}}{K_t} = \left(\beta \frac{\xi_t}{\mu_t} A_j\right)^{\frac{1}{1-\beta}} \quad \forall j \in \{1, ..., J\}.$$

Multiply both sides by π_j and sum across all j's to obtain:

$$1 = \left(\beta \frac{\xi_t}{\mu_t}\right)^{\frac{1}{1-\beta}} \sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}.$$

Solve for the ratio μ_t/ξ_t:

$$\frac{\mu_t}{\xi_t} = \beta \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

Substitute μ_t/ξ_t into the expression for $k_{j,t}/K_t$ to obtain the share of the capital allocated to firms of type j:

$$\frac{\pi_j k_{j,t}}{K_t} = \frac{\pi_j A_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}} \quad \forall j \in \{1, ..., J\}.$$

It follows that aggregate output can be rewritten as:

$$Y_t = K_t \sum_{j=1}^{J} \pi_j A_j \left(\frac{k_{j,t}}{K_t}\right)^{\beta} = K_t \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

From the Maximum Principle, the law of motion of the co-state variable, ξ_t, are given by:

$$\rho \xi_t = \xi_t r_t^* + \dot{\xi}_t,$$

where

$$\begin{aligned} r_t^* &= (1-\beta) \sum_{j=1}^{J} \pi_j A_j \left(\frac{k_{j,t}}{K_t}\right)^{\beta} + \frac{\mu_t}{\xi_t} \\ &= \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}. \end{aligned}$$

The second equality was obtained by substituting $k_{j,t}/K_t$ and μ_t/ξ_t by their expressions. It follows that

$$\frac{\dot{\xi}_t}{\xi_t} = \rho - \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}.$$

Using that $(c_t)^{-\theta} = \xi_t$, $-\theta \dot{c}_t/c_t = \dot{\xi}_t/\xi_t$, and hence the growth rate of consumption is

$$g^* = \frac{\dot{c}_t}{c_t} = \frac{\left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta} - \rho}{\theta}.$$

In order for $g^* > 0$, $\rho < \left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}$, which holds from (46). I consider the solution to the necessary conditions such that c_t/K_t is constant. Hence, $\dot{K}_t/K_t = g^*$ and the initial consumption is $c_0 = Y_0 - g^* K_0$. Using that $Y_t = r^* K_t$,

$$\frac{c_0}{K_0} = r^* - g^*.$$

It is positive if

$$\rho > (1-\theta)\left(\sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta},$$

which holds from (46).

For sufficiency, I check the Mangasarian condition according to which $\mathcal{H}$ is concave in (c, K, k_j) and

$$\lim_{t\to+\infty} e^{-\rho t}\xi_t K_t = 0.$$

Using that $\dot{\xi}_t/\xi_t = \rho - r^*$,

$$e^{-\rho t}\xi_t K_t = \xi_0 K_0 e^{(g^*-r^*)t}.$$

Since $r^* - g^* > 0$, the transversality condition holds. ■

Proof of Lemma 3. I decompose the value function as illustrated in Figure 12:

$$\upsilon_{j,t}(k) = F_{j,t}(k) + \mathbb{E}\left\{e^{-r(T-t)} \max_{k'\geq 0} \left[\upsilon_{j',T}(k') - (k'-k)\right]\right\}, \tag{100}$$

where $F_{j,t}(k) \equiv \mathbb{E}\left[\int_t^T e^{-r(s-t)} K_s^{1-\beta} A_{x(s)} k^\beta ds\right]$ is the expected discounted sum of output flows until the next effective meeting with a broker at time T, where $T-t$ is exponentially distributed with mean $1/\alpha(1-\eta)$, and $x(s)$ is the idiosyncratic productivity type of the firm at time $s \in [t,T]$ with $x(t) = j$. The second term on the right side of (100) is the continuation value the firm at time T. The firm invests $k'-k$ and its value is $\upsilon_{j',T}(k')$ where $j' = x(T) \in \{1,...,J\}$ is the idiosyncratic productivity type at time T. The expectation is taken with respect to (T, j').

The function, $F_{j,t}(k)$, can be characterized recursively as follows:

$$\begin{aligned} rF_{j,t}(k) &= A_j K_0^{1-\beta} e^{(1-\beta)gt} k^\beta + \lambda \sum_{x=1}^{J} \pi_x \left[F_{x,t}(k) - F_{j,t}(k)\right] \\ &\quad -\alpha(1-\eta)F_{j,t}(k) + \dot{F}_{j,t}(k), \end{aligned} \tag{101}$$

for all $j \in \{1,...,J\}$. The first term on the right side is the firm's output flow at time t where $K_t = K_0 e^{gt}$. The second term corresponds to the arrival of idiosyncratic productivity shocks

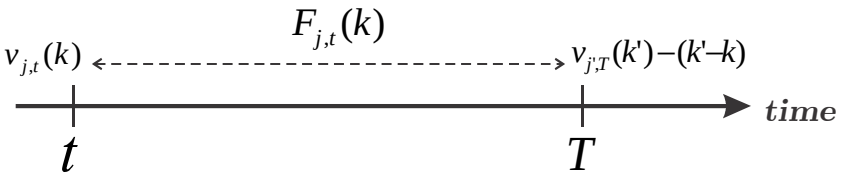


Figure 12: Recursive dermination of the value of a firm

at rate λ. The third term corresponds to the effective access to the market at rate $\alpha(1-\eta)$, which is the terminal point in the calculation of F_j. The last term is the change in $F_{j,t}$ over time due to aggregate productivity growth.

Since aggregate productivity grows at rate $(1-\beta)g$, I detrend cumulative output by defining

$$\begin{aligned}\tilde{F}_j(k) &\equiv e^{-(1-\beta)gt}F_{j,t}(k) \\ &= e^{-(1-\beta)gt}\mathbb{E}\left[\int_t^T e^{-r(s-t)}e^{(1-\beta)gs}K_0^{1-\beta}A_{x(s)}k^\beta ds\right] \\ &= \mathbb{E}\left[\int_t^T e^{-[r-(1-\beta)g](s-t)}K_0^{1-\beta}A_{x(s)}k^\beta ds\right],\end{aligned}$$

where to obtain the second equality I used that $K_s = e^{gs}K_0$ and to obtain the third equality I rewrote $e^{(1-\beta)gs} = e^{(1-\beta)g(s-t)}e^{(1-\beta)gt}$. Using the change of variable, $u = s-t$, and the fact that the distribution of $T-t$ is exponential,

$$\tilde{F}_j(k) = \int_0^{+\infty}\alpha(1-\eta)e^{-\alpha(1-\eta)z}\int_0^z e^{-[r-(1-\beta)g]u}K_0^{1-\beta}A_{x(t+u)}k^\beta du dz.$$

Hence, $\tilde{F}_j(k)$ only depends on t through the initial idiosyncratic productivity state, $x(t) = j$. By changing the order of integration, it can be rewritten as follows:

$$\tilde{F}_j(k) = \int_0^{+\infty} e^{-[r-(1-\beta)g+\alpha(1-\eta)]u}K_0^{1-\beta}A_{x(t+u)}k^\beta du.$$

Hence, the effective discount rate is $r-(1-\beta)g+\alpha(1-\eta)$. It can then be written recursively as follows:

$$\left[r-(1-\beta)g+\alpha(1-\eta)\right]\tilde{F}_j(k) = A_j K_0^{1-\beta}k^\beta + \lambda\sum_{x=1}^J \pi_x\left[\tilde{F}_x(k)-\tilde{F}_j(k)\right]. \tag{102}$$

Multiply by π_j both sides and sum across j's to obtain:

$$\sum_{j=1}^J \pi_j\tilde{F}_j(k) = \frac{\bar{A}K_0^{1-\beta}k^\beta}{r-(1-\beta)g+\alpha(1-\eta)}.$$

Substitute this expression back into (102) to obtain:

$$\tilde{F}_j(k) = \left\{ \frac{[r-(1-\beta)g+\alpha(1-\eta)]\,A_j + \lambda\bar{A}}{[r-(1-\beta)g+\alpha(1-\eta)+\lambda]\,[r-(1-\beta)g+\alpha(1-\eta)]} \right\} K_0^{1-\beta} k^{\beta}.$$

Using the definition for $\hat{A}_j$ in (51), i.e.,

$$\hat{A}_j \equiv \frac{[r-(1-\beta)g+\alpha(1-\eta)]\,A_j + \lambda\bar{A}}{r-(1-\beta)g+\alpha(1-\eta)+\lambda},$$

the solution can be rewritten as

$$\tilde{F}_j(k) = \frac{\hat{A}_j K_0^{1-\beta} k^{\beta}}{r-(1-\beta)g+\alpha(1-\eta)}. \tag{103}$$

The first term on the right side of (50) is equal to $F_{j,t}(k) = e^{(1-\beta)gt}\tilde{F}_j(k)$ where $\tilde{F}_j(k)$ is given by (103). The second and third terms correspond to

$$\mathbb{E}\left\{ e^{-r(T-t)} \max_{k'\geq 0} [\upsilon_{j',T}(k') - (k'-k)] \right\} = \mathbb{E}\left[e^{-r(T-t)}\right] k + \mathbb{E}\left\{ e^{-r(T-t)} \max_{k'\geq 0} [\upsilon_{j',T}(k') - k'] \right\}.$$

The first term on the right side is the discounted value of the capital that can be liquidated at price one at time T. Using that $T-t$ is exponentially distributed with mean $1/\alpha(1-\eta)$,

$$\mathbb{E}\left[e^{-r(T-t)}\right] = \int_0^{+\infty} \alpha(1-\eta) e^{-\alpha(1-\eta)s} e^{-rs} ds = \frac{\alpha(1-\eta)}{r+\alpha(1-\eta)}.$$

Hence, $\mathbb{E}\left[e^{-r(T-t)}\right] k = \alpha(1-\eta)k/\left[r+\alpha(1-\eta)\right]$. The second term on the right side of the equation above,

$$\Lambda_{j,t} \equiv \mathbb{E}\left\{ e^{-r(T-t)} \max_{k'\geq 0} [\upsilon_{j',T}(k') - k'] \right\},$$

corresponds to the firm re-optimizing its value by choosing k'. It is independent of k. ■

Proof of Lemma 4. The marginal Tobin's q of a firm with productivity type j and capital k is $\upsilon'_{j,t}(k)$. From (50), it is equal to

$$\upsilon'_{j,t}(k) = \frac{\beta\hat{A}_j}{r-(1-\beta)g+\alpha(1-\eta)} \left(\frac{K_t}{k}\right)^{1-\beta} + \frac{\alpha(1-\eta)}{\alpha(1-\eta)+r}.$$

If the firm accessed the market at $t-\tau$ with productivity type i, its capital is $k_{i,t-\tau}$ where $k_{i,t}/K_t$ is given by (53). Using that K_t grows at rate g,

$$\begin{aligned}
\left(\frac{K_t}{k_{i,t-\tau}}\right)^{1-\beta} &= \left(\frac{e^{g\tau}K_{t-\tau}}{k_{i,t-\tau}}\right)^{1-\beta} \\
&= e^{(1-\beta)g\tau} \left(\frac{K_{t-\tau}}{k_{i,t-\tau}}\right)^{1-\beta} \\
&= \frac{e^{(1-\beta)g\tau} r\,[r+\alpha(1-\eta)-(1-\beta)g]}{\beta\,[r+\alpha(1-\eta)]\,\hat{A}_i},
\end{aligned}$$

where I used (53) to obtain the last equality. Substitute this expression for $(K_t/k_{i,t-\tau})^{1-\beta}$ into $\upsilon'_{j,t}(k)$ above to obtain:

$$q_{ij}(\tau) \equiv \upsilon'_{j,t}(k_{i,t-\tau}) = \frac{e^{(1-\beta)g\tau} r \hat{A}_j}{[r+\alpha(1-\eta)]\,\hat{A}_i} + \frac{\alpha(1-\eta)}{\alpha(1-\eta)+r}.$$

It can be rearranged to obtain (54). ■

Proof of Lemma 5. Along a balanced growth path, $k_{i,t-\tau} = e^{-g\tau}k_{i,t}$. Hence, substituting $\gamma(i,j,\tau)$ by its expression given by (55) into (56),

$$K_{j,t} = \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t} \int_{\mathbb{R}^+} \alpha e^{-(\alpha+g)\tau} \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right] d\tau.$$

We can rewrite the integral on the right side as:

$$\frac{\alpha}{\alpha+g+\lambda}(\mathbb{I}_{\{i=j\}} - \pi_j) + \frac{\alpha}{\alpha+g}\pi_j.$$

Hence,

$$\begin{aligned} K_{j,t} &= \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t} \left[\frac{\alpha}{\alpha+g+\lambda}(\mathbb{I}_{\{i=j\}} - \pi_j) + \frac{\alpha}{\alpha+g}\pi_j\right] \\ &= \frac{\alpha}{\alpha+g+\lambda}\pi_j k_{j,t} + \frac{\alpha\lambda}{(\alpha+g)(\alpha+g+\lambda)}\pi_j \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t}. \end{aligned}$$

From market clearing,

$$\alpha \sum_{i\in\{1,\ldots,J\}} \pi_i k_{i,t} = (\alpha+g)K_t.$$

Hence,

$$K_{j,t} = \frac{\alpha}{\alpha+g+\lambda}\pi_j k_{j,t} + \frac{\lambda\pi_j}{\alpha+g+\lambda}K_t.$$

It can be rewritten as:

$$\frac{K_{j,t}}{K_t} = \pi_j \left[\frac{\alpha\,(k_j/K) + \lambda}{g+\alpha+\lambda}\right].$$

Substitute (k_j/K) by its expression given by (53) to obtain (57). ■

Proof of Lemma 6. Using the closed-form solution for $\upsilon_{j,t}(k_{j,t})$ given by (50), $\phi_t(i,j,\tau)$ given by (59) can be reexpressed as

$$\phi_t(i,j,\tau) = \eta \left[\frac{\hat{A}_j K_t^{1-\beta}\left(k_{j,t}^{\beta} - k_{i,t-\tau}^{\beta}\right)}{r-(1-\beta)g+\alpha(1-\eta)} - \frac{r(k_{j,t}-k_{i,t-\tau})}{\alpha(1-\eta)+r}\right]. \tag{104}$$

Using that $k_{i,t-\tau} = e^{-g\tau}k_{i,t}$ and the notations $\tilde{k}_j \equiv k_{j,t}/K_t$ and $\tilde{\phi}(i,j,\tau) \equiv \phi_t(i,j,\tau)/K_t$, $\tilde{\phi}(i,j,\tau)$ is given by

$$\tilde{\phi}(i,j,\tau) = \eta \left[\frac{\hat{A}_j \left(\tilde{k}_j^{\beta} - e^{-\beta g\tau} \tilde{k}_i^{\beta} \right)}{r - (1-\beta)g + \alpha(1-\eta)} - \frac{r(\tilde{k}_j - e^{-g\tau}\tilde{k}_i)}{\alpha(1-\eta) + r} \right], \tag{105}$$

and $\tilde{k}_j$ is given by (53). Using that $\dot{w} = gw$, w_t given by (58) solves

$$w_t = \frac{\alpha \Phi_t}{r-g}, \tag{106}$$

where a broker's expected profits from a meeting with a firm picked at random in the population of firms are

$$\Phi_t \equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,...,J\}^2} \phi_t(i,j,\tau)\gamma(i,j,\tau)d\tau,$$

where $(i,j,\tau) \in \{1,...,J\}^2 \times \mathbb{R}^+$ is a list composed of the current productivity of the firm, j, the time that has elapsed since its last access to the market, τ, and its productivity at that time, i, and $\gamma(i,j,\tau)$ is the density measure of firms,

$$\gamma(i,j,\tau) = \alpha e^{-\alpha\tau}\pi_i \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j \right].$$

In order to compute Φ_t, we decompose it as follows:

$$\Phi_t = \eta \left[\frac{Q_1 - Q_2}{r - (1-\beta)g + \alpha(1-\eta)} - \frac{r(Q_3 - Q_4)}{\alpha(1-\eta) + r} \right] K_t, \tag{107}$$

where:

1. Q_1 is given by:

$$\begin{aligned} Q_1 &\equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,...,J\}^2} \hat{A}_j \tilde{k}_j^{\beta} \gamma(i,j,\tau)d\tau \\ &= \int_0^{+\infty} \sum_{(i,j)} \hat{A}_j \tilde{k}_j^{\beta} \alpha e^{-\alpha\tau}\pi_i \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j \right] d\tau \\ &= \int_0^{+\infty} \sum_{j} \pi_j \hat{A}_j \tilde{k}_j^{\beta} \alpha e^{-\alpha\tau} d\tau \\ &= \sum_j \pi_j \hat{A}_j \tilde{k}_j^{\beta}. \end{aligned}$$

We move from the second line to the third line by taking the sum over $i \in \{1,...,J\}$ and use the fact that

$$\sum_i \pi_i \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j \right] = \pi_j e^{-\lambda\tau} + (1-e^{-\lambda\tau})\pi_j = \pi_j.$$

We move from the third line to the fourth by changing the order of the sum and integral and noticing that $\int_0^{+\infty} \alpha e^{-\alpha\tau} d\tau = 1$.

2. Q_2 is given by:

$$\begin{aligned} Q_2 &\equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,...,J\}^2} e^{-\beta g\tau} \hat{A}_j \tilde{k}_i^{\beta} \gamma(i,j,\tau) d\tau \\ &= \int_0^{+\infty} \sum_{(i,j)} e^{-\beta g\tau} \hat{A}_j \tilde{k}_i^{\beta} \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau} \mathbb{I}_{\{i=j\}} + (1 - e^{-\lambda\tau}) \pi_j \right] d\tau \\ &= \int_0^{+\infty} \sum_{i} \alpha e^{-(\alpha+\beta g)\tau} \left[e^{-\lambda\tau} \hat{A}_i + (1 - e^{-\lambda\tau}) \bar{A} \right] \pi_i \tilde{k}_i^{\beta} d\tau \\ &= \frac{\alpha}{\alpha + \lambda + \beta g} \sum_i \pi_i \left[\hat{A}_i + \frac{\lambda}{\alpha + \beta g} \bar{A} \right] \tilde{k}_i^{\beta}. \end{aligned}$$

We move from the second line to the third line by taking the sum over $j \in \{1, ..., J\}$ and use the fact that:

$$\sum_j \hat{A}_j \left[e^{-\lambda\tau} \mathbb{I}_{\{i=j\}} + (1 - e^{-\lambda\tau}) \pi_j \right] = e^{-\lambda\tau} \hat{A}_i + (1 - e^{-\lambda\tau}) \sum_j \hat{A}_j \pi_j.$$

Moreover, $\sum_j \hat{A}_j \pi_j = \bar{A}$. We move from the third line to the fourth line by using that

$$\begin{aligned} \int_0^{+\infty} \alpha e^{-(\alpha+\beta g)\tau} e^{-\lambda\tau} \hat{A}_i d\tau &= \frac{\alpha}{\alpha + \lambda + \beta g} \hat{A}_i \\ \int_0^{+\infty} \alpha e^{-(\alpha+\beta g)\tau} (1 - e^{-\lambda\tau}) \bar{A} d\tau &= \frac{\alpha\lambda}{(\alpha + \beta g)(\alpha + \lambda + \beta g)} \bar{A}. \end{aligned}$$

3. Q_3 is given by:

$$\begin{aligned} Q_3 &\equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,...,J\}^2} \tilde{k}_j \gamma(i,j,\tau) d\tau \\ &= \int_0^{+\infty} \sum_{(i,j)} \tilde{k}_j \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau} \mathbb{I}_{\{i=j\}} + (1 - e^{-\lambda\tau}) \pi_j \right] d\tau \\ &= \int_0^{+\infty} \sum_{j} \alpha e^{-\alpha\tau} \pi_j \tilde{k}_j d\tau \\ &= \sum_j \pi_j \tilde{k}_j \end{aligned}$$

4. Q_4 is given by:

$$\begin{aligned} Q_4 &\equiv \int_0^{+\infty} \sum_{(i,j)\in\{1,...,J\}^2} e^{-g\tau}\tilde{k}_i\gamma(i,j,\tau)d\tau \\ &= \int_0^{+\infty} \sum_{(i,j)} e^{-g\tau}\tilde{k}_i\alpha e^{-\alpha\tau}\pi_i\left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right]d\tau \\ &= \int_0^{+\infty} \sum_{i} e^{-g\tau}\alpha e^{-\alpha\tau}\pi_i\tilde{k}_i d\tau \\ &= \frac{\alpha}{\alpha+g}\sum_i \pi_i\tilde{k}_i \end{aligned}$$

From the expressions above,

$$\begin{aligned} Q_1 - Q_2 &= \frac{1}{\alpha+\lambda+\beta g}\sum_j\left[(\lambda+\beta g)\,\hat{A}_j - \frac{\alpha\lambda}{\alpha+\beta g}\bar{A}\right]\pi_j\tilde{k}_j^{\beta} \\ Q_3 - Q_4 &= \frac{g}{\alpha+g}\sum_j \pi_j\tilde{k}_j = \frac{g}{\alpha}, \end{aligned}$$

where we used that, by market clearing, $\sum_j \pi_j\tilde{k}_j = 1 + g/\alpha$. We substitute $Q_1 - Q_2$ and $Q_3 - Q_4$ by their expressions into (107) to obtain (61).

■

Proof of Proposition 7. The equilibrium condition for the real interest rate, (65), can be written as $r = RHS(r)$ where

$$RHS(r) \equiv \frac{\theta\beta\left[\alpha(1-\eta)+r\right]}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)}\left\{\frac{\alpha\theta}{\alpha\theta+r-\rho}\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta},$$

with

$$\hat{A}_j(r) \equiv \frac{\left\{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)\right\}A_j + \theta\lambda\bar{A}}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)+\theta\lambda}.$$

In order to establish that a solution exists and it is such that $r > \min\{\rho, g\}$, I will distinguish three cases.

Case #1: $\theta \geq 1$. The term $\theta - (1-\beta)$ is positive. Thus, the denominators of $RHS(r)$ and $\hat{A}_j(r)$ are positive for all $r \geq \rho$. For all $r \geq \rho$, $RHS(r)$ is a continuous function. As $r \to +\infty$,

$$\begin{aligned} \frac{\theta\beta\left[\alpha(1-\eta)+r\right]}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)} &\to \frac{\theta\beta}{\theta-(1-\beta)} \in (0,+\infty) \\ \frac{\alpha\theta}{\alpha\theta+r-\rho}\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}} &\to 0, \end{aligned}$$

where I used that

$$\lim_{r\to+\infty} \hat{A}_j(r) = A_j.$$

Hence, $\lim_{r\to+\infty} RHS(r) = 0$. A sufficient condition for the existence of a solution to $r = RHS(r)$ such that $r > \rho$ is $RHS(\rho) > \rho$, i.e.,

$$\beta \left\{ \sum_{j=1}^{J} \pi_j \left[\hat{A}_j(\rho) \right]^{\frac{1}{1-\beta}} \right\}^{1-\beta} > \rho.$$

By replacing $\hat{A}_j(\rho)$ by its expression given by (66), i.e.,

$$\hat{A}_j(\rho) \equiv \frac{[\rho + \alpha(1-\eta)] A_j + \lambda \bar{A}}{\rho + \alpha(1-\eta) + \lambda},$$

one obtains (67). The condition $r > g = (r-\rho)/\theta$ is automatically satisfied when $\theta \geq 1$.

Case #2: $1 - \beta \leq \theta < 1$. Since $\theta - (1-\beta) \geq 0$, $RHS(r)$ is continuous in r for all $r \geq \rho$. As in Case #1, a sufficient condition for $r > \rho$ is $RHS(\rho) > \rho$, i.e., (67) holds. The condition, $r > g$, can be rewritten as $r < \bar{r}_\theta \equiv \rho / (1-\theta)$. A sufficient condition for the existence of an equilibrium with $r < \bar{r}_\theta$ is $RHS(\bar{r}_\theta) < \bar{r}_\theta$, i.e., (68) holds.

Case #3: $\theta < 1 - \beta$. Now $\theta - (1-\beta) < 0$. Hence, the denominator of $RHS(r)$ and $\hat{A}_j(r)$ approaches 0 as r approaches $r_0 \equiv [(1-\beta)\rho + \theta\alpha(1-\eta)] / [1-\beta-\theta] > 0$. For all $r < r_0$, $RHS(r)$ is continuous. Moreover, it can be checked that $r_0 > \bar{r}_\theta$, i.e.,

$$\begin{aligned} \frac{(1-\beta)\rho + \theta\alpha(1-\eta)}{1-\beta-\theta} > \frac{\rho}{1-\theta} &\iff (1-\beta)\rho(1-\theta) + \theta\alpha(1-\eta)(1-\theta) > \rho(1-\beta-\theta) \\ &\iff \beta\rho + \alpha(1-\eta)(1-\theta) > 0, \end{aligned}$$

where I used that $1 - \theta > \beta > 0$. The conditions $RHS(\rho) > \rho$ and $RHS(\bar{r}_\theta) < \bar{r}_\theta$, i.e., (67) and (68), guarantee that there exists a $r \in (\rho, \bar{r}_\theta)$.

Next, I establish that any $r > \rho$ solution to (65) and such that $r > g$ is bounded above by r^* given by (47). Consider the first term on the right side of (65),

$$\Psi(\beta) \equiv \frac{\theta\beta[\alpha(1-\eta) + r]}{r[\theta - (1-\beta)] + (1-\beta)\rho + \theta\alpha(1-\eta)}.$$

The derivative with respect to β is

$$\Psi'(\beta) \equiv \frac{\theta^2[\alpha(1-\eta) + r][r - g + \alpha(1-\eta)]}{\{r[\theta - (1-\beta)] + (1-\beta)\rho + \theta\alpha(1-\eta)\}^2} > 0 \text{ for all } r > g.$$

Moreover,

$$\lim_{\beta\to 1} \Psi(\beta) = 1.$$

Hence, for all $r > \rho$ and such that $r > g$, and for all $\beta \leq 1$, $\Psi(\beta) \leq 1$. Thus, from (65),

$$r \leq \left\{ \sum_{j=1}^{J} \pi_j \left[\hat{A}_j(r) \right]^{\frac{1}{1-\beta}} \right\}^{1-\beta}.$$

From (66), the transformation from $\{A_j\}$ to $\left\{ \hat{A}_j(r) \right\}$ is a mean-preserving decrease in spread. Since $x^{\frac{1}{1-\beta}}$ is a convex function for all $\beta < 1$,

$$\sum_{j=1}^{J} \pi_j \left[\hat{A}_j(r) \right]^{\frac{1}{1-\beta}} \leq \sum_{j=1}^{J} \pi_j A_j^{\frac{1}{1-\beta}}.$$

It follows that $r \leq r^*$. ■

Proof of Lemma 7. Part 1. Rewrite (65) as

$$r = \frac{\beta \left[\alpha(1-\eta) + r \right] \hat{A}_J}{r - (1-\beta)g + \alpha(1-\eta)} \left[\left(\frac{\alpha}{\alpha + g} \right) \sum_{j=1}^{J} \pi_j \left(\frac{\hat{A}_j}{\hat{A}_J} \right)^{\frac{1}{1-\beta}} \right]^{1-\beta}.$$

For β close to 1, the right side is continuous in β. For all $j < J$, $\hat{A}_j / \hat{A}_J < 1$ and

$$\lim_{\beta \to 1} \left(\frac{\hat{A}_j}{\hat{A}_J} \right)^{\frac{1}{1-\beta}} = \lim_{\beta \to 1} e^{\frac{1}{1-\beta} \ln\left(\frac{\hat{A}_j}{\hat{A}_J} \right)} = 0.$$

Hence,

$$\lim_{\beta \to 1} \left[\left(\frac{\alpha}{\alpha + g} \right) \sum_{j=1}^{J} \pi_j \left(\frac{\hat{A}_j}{\hat{A}_J} \right)^{\frac{1}{1-\beta}} \right]^{1-\beta} = \lim_{\beta \to 1} \left[\frac{\alpha \pi_J}{\alpha + g} \right]^{1-\beta} = 0,$$

provided $\pi_J < 1$ and $g \geq 0$ (which is satisfied under the conditions of Proposition 7). It follows that

$$\lim_{\beta \to 1} r = \lim_{\beta \to 1} \frac{\beta \left[\alpha(1-\eta) + r \right] \hat{A}_J}{r - (1-\beta)g + \alpha(1-\eta)} = \lim_{\beta \to 1} \hat{A}_J.$$

From (51),

$$\lim_{\beta \to 1} \hat{A}_j = \frac{\left[r + \alpha(1-\eta) \right] A_j + \lambda \bar{A}}{r + \alpha(1-\eta) + \lambda}.$$

From (57),

$$\lim_{\beta \to 1} \frac{K_j}{K} = \lim_{\beta \to 1} \frac{\pi_j}{g + \alpha + \lambda} \left[\alpha \left\{ \frac{\beta \left[\alpha(1-\eta) + r \right] \hat{A}_j}{r \left[r - (1-\beta)g + \alpha(1-\eta) \right]} \right\}^{\frac{1}{1-\beta}} + \lambda \right].$$

$$\lim_{\beta \to 1} \frac{K_j}{K} = \lim_{\beta \to 1} \frac{\pi_j}{g + \alpha + \lambda} \left[\alpha \left(\frac{\hat{A}_j}{\hat{A}_J} \right)^{\frac{1}{1-\beta}} + \lambda \right].$$

For all $j < J$,

$$\lim_{\beta\to 1}\left\{\frac{\beta\left[\alpha(1-\eta)+r\right]\hat{A}_j}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]}\right\}^{\frac{1}{1-\beta}} = \lim_{\beta\to 1}\left(\frac{\hat{A}_j}{A_J}\right)^{\frac{1}{1-\beta}} = 0.$$

Hence, $\lim_{\beta\to 1}(K_j/K) = \lambda\pi_j/(g+\alpha+\lambda)$. If $j = J$,

$$\lim_{\beta\to 1}\left(\frac{K_J}{K}\right) = 1 - \sum_{j=1}^{J-1}\lim_{\beta\to 1}\left(\frac{K_J}{K}\right) = 1 - \frac{\lambda(1-\pi_J)}{g+\alpha+\lambda}.$$

Part 2. If $\eta < 1$, for given $r \geq \rho$, as $\alpha \to +\infty$,

$$\begin{aligned}\frac{\theta\beta\left[\alpha(1-\eta)+r\right]}{r\left[\theta-(1-\beta)\right]+(1-\beta)\rho+\theta\alpha(1-\eta)} &\to \beta \\ \left\{\frac{\alpha\theta}{\alpha\theta+r-\rho}\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta} &\to \left\{\sum_{j=1}^{J}\pi_j\left[\hat{A}_j(r)\right]^{\frac{1}{1-\beta}}\right\}^{1-\beta}.\end{aligned}$$

Hence, the right side of (65) tends to the right side of (71). From (57) the share of capital allocated to firms of type j tends to

$$\lim_{\alpha\to+\infty}\left(\frac{K_j}{K}\right) = \lim_{\alpha\to+\infty}\left[\frac{\alpha\pi_j}{g+\alpha+\lambda}\left\{\frac{\beta\left[\alpha(1-\eta)+r\right]}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]}\right\}^{\frac{1}{1-\beta}}\hat{A}_j^{\frac{1}{1-\beta}} + \frac{\pi_j\lambda}{g+\alpha+\lambda}\right].$$

Using that $\lim_{\alpha\to+\infty}\left[\alpha\pi_j/(g+\alpha+\lambda)\right] = \pi_j$, $\lim_{\alpha\to+\infty}\left[\pi_j\lambda/(g+\alpha+\lambda)\right] = 0$, $\lim_{\alpha\to+\infty}\hat{A}_j = A_j$,

$$\lim_{\alpha\to+\infty}\frac{\beta\left[\alpha(1-\eta)+r\right]}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]} = \lim_{\alpha\to+\infty}\frac{\beta}{r} = \frac{1}{\left(\sum_{j=1}^{J}\pi_j A_j^{\frac{1}{1-\beta}}\right)^{1-\beta}},$$

it follows that

$$\lim_{\alpha\to+\infty}\left(\frac{K_j}{K}\right) = \frac{\pi_j A_j^{\frac{1}{1-\beta}}}{\sum_{j=1}^{J}\pi_j A_j^{\frac{1}{1-\beta}}}.$$

Part 3. Suppose $A_j = \bar{A}$ for all $j \in \{1,...,J\}$. From (66), $\hat{A}_j = \bar{A}$ and, from (65),

$$r = \frac{\alpha(1-\eta)+r}{\alpha(1-\eta)+r-(1-\beta)g}\left(\frac{\alpha}{\alpha+g}\right)^{1-\beta}\beta\bar{A}$$

where $g = (r-\rho)/\theta$. ■

Proof of Proposition 8. Suppose $\eta = 1$. From (73), the real interest rate solves

$$r = \left(\frac{\alpha\theta}{\alpha\theta+r-\rho}\right)^{1-\beta}\beta\bar{A} + (1-\beta)\frac{(r-\rho)}{\theta}.$$

It can be reexpressed as:

$$r = \frac{\theta \left(\frac{\alpha\theta}{\alpha\theta + r - \rho}\right)^{1-\beta} \beta \bar{A} - (1-\beta)\rho}{\theta - 1 + \beta}.$$

Assuming $\theta - 1 + \beta > 0$, the right side is decreasing in r for all $r \geq \rho$. For all $\alpha > 0$,

$$RHS|_{r=\rho} = \frac{\theta\beta\bar{A} - (1-\beta)\rho}{\theta - 1 + \beta}.$$

It is greater than ρ if $\beta\bar{A} > \rho$. For r sufficiently large, $RHS = 0$. Thus, if $\beta\bar{A} > \rho$, there is a unique solution to the equation above and it is such that $r > \rho$. Moreover, since the right side is increase in α, the solution r is also increasing in α. As $\alpha \to 0$, $r \to \rho$. As $\alpha \to +\infty$,

$$r \to r_\infty \equiv \frac{\theta\beta\bar{A} - (1-\beta)\rho}{\theta - 1 + \beta}.$$

Moreover, $r_\infty > \beta\bar{A}$ if and only if $\beta < 1$ and $\beta\bar{A} > \rho$. Thus, if $\beta\bar{A} > \rho$, there exists a $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$, $r > \beta\bar{A}$. ■

Proof of Proposition 9. Part 1: I establish first that

$$\frac{\pi_1 \hat{A}_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} > \frac{\pi_j A_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j A_j^{\frac{1}{1-\beta}}}.$$

By the definition of $\hat{A}_1$, (51), $\hat{A}_1 > A_1$. Using that the lottery $\{(\hat{A}_j, \pi_j)\}$ corresponds to a mean-preserving reduction in spread of the lottery $\{(A_j, \pi_j)\}$, $\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}} < \sum_{j=1}^J \pi_j A_j^{\frac{1}{1-\beta}}$. This proves the inequality. Next, from the fact that $\hat{A}_1 < \hat{A}_2 < .... < \hat{A}_J$, $\hat{A}_1^{\frac{1}{1-\beta}} < \sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}$ and hence

$$1 - \frac{\hat{A}_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} > 0.$$

The inequality

$$\frac{K_1}{K} > \frac{\pi_1 A_1^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j A_j^{\frac{1}{1-\beta}}}$$

follows directly from the observations above and (75).

Part 2: I establish that

$$\frac{\pi_j \hat{A}_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} < \frac{\pi_j A_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j A_j^{\frac{1}{1-\beta}}}.$$

The left side can be rewritten as

$$\frac{\pi_j \hat{A}_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} = \frac{\pi_j}{\sum_{j=1}^J \pi_j (\hat{A}_j/\hat{A}_J)^{\frac{1}{1-\beta}}}.$$

From (51),

$$\frac{\hat{A}_j}{\hat{A}_J} = \frac{[r-(1-\beta)g+\alpha(1-\eta)]\,A_j + \lambda\bar{A}}{[r-(1-\beta)g+\alpha(1-\eta)]\,A_J + \lambda\bar{A}}.$$

It can be checked that this ratio is decreasing in $\alpha(1-\eta)$. As $\alpha(1-\eta) \to +\infty$, $\hat{A}_j/\hat{A}_J \to A_j/A_J$. Hence,

$$\sum_{j=1}^J \pi_j \left(\frac{\hat{A}_j}{\hat{A}_J}\right)^{\frac{1}{1-\beta}} > \sum_{j=1}^J \pi_j \left(\frac{A_j}{A_J}\right)^{\frac{1}{1-\beta}}.$$

This establishes the inequality above. For any α,

$$\frac{\hat{A}_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j \hat{A}_j^{\frac{1}{1-\beta}}} > 1.$$

The inequality

$$\frac{K_J}{K} < \frac{\pi_J A_J^{\frac{1}{1-\beta}}}{\sum_{j=1}^J \pi_j A_j^{\frac{1}{1-\beta}}}$$

follows from the observations above and (75). ■

Proof of Lemma 8.

Replace $\gamma(i,j,\tau)$ by its expression given by (55) into (78) to obtain:

$$\begin{aligned} y_K &= \int_0^{+\infty} \sum_{(i,j)} e^{-\beta g\tau} A_j \tilde{k}_i^\beta \alpha e^{-\alpha\tau} \pi_i \left[e^{-\lambda\tau}\mathbb{I}_{\{i=j\}} + (1-e^{-\lambda\tau})\pi_j\right] d\tau \\ &= \int_0^{+\infty} \sum_i \alpha e^{-(\alpha+\beta g)\tau} \left[e^{-\lambda\tau} A_i + (1-e^{-\lambda\tau})\bar{A}\right] \pi_i \tilde{k}_i^\beta d\tau \\ &= \sum_i \left[\frac{\alpha}{\alpha+\beta g+\lambda}\left(A_i - \bar{A}\right) + \frac{\alpha}{\alpha+\beta g}\bar{A}\right] \pi_i \tilde{k}_i^\beta. \end{aligned}$$

The second equality sums first over all j's and uses that $\sum_j \mathbb{I}_{\{i=j\}} A_j = A_i$ and $\sum_j \pi_j A_j = \bar{A}$. The third equality interchanges the sum and the integral and it integrates over τ. It uses that $\int_0^{+\infty} \alpha e^{-(\alpha+\beta g+\lambda)\tau} d\tau = \alpha/(\alpha+\beta g+\lambda)$ and $\int_0^{+\infty} \alpha e^{-(\alpha+\beta g)\tau} d\tau = \alpha/(\alpha+\beta g)$. Substitute $\tilde{k}_i$ by its expression given by (53), i.e.,

$$\tilde{k}_i^\beta = \left\{\frac{\beta\,[r+\alpha(1-\eta)]}{r\,[r+\alpha(1-\eta)-(1-\beta)g]}\right\}^{\frac{\beta}{1-\beta}} \hat{A}_j^{\frac{\beta}{1-\beta}},$$

one obtains

$$y_K = \left\{ \frac{\beta\left[\alpha(1-\eta)+r\right]}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]} \right\}^{\frac{\beta}{1-\beta}} \sum_i \pi_i \left[\frac{\alpha}{\alpha+\beta g+\lambda}\left(A_i-\bar{A}\right) + \frac{\alpha}{\alpha+\beta g}\bar{A} \right] \hat{A}_i^{\frac{\beta}{1-\beta}}.$$

From the definition of the real interest rate given by (64):

$$\left\{ \frac{\beta\left[\alpha(1-\eta)+r\right]}{r\left[r-(1-\beta)g+\alpha(1-\eta)\right]} \right\}^{\frac{\beta}{1-\beta}} = (\hat{r}^*)^{\frac{-\beta}{1-\beta}} \left(\frac{\alpha+g}{\alpha} \right)^{\beta} \quad \text{where } \hat{r}^* \equiv \left(\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}} \right)^{1-\beta}.$$

Susbtitute this expression into the equation for y_K above to obtain (79). ■

Proof of Proposition 10. A decrease in α or an increase in λ generates a mean-preserving decrease in the spread of the distribution of $\hat{A}_j$. Since $x^{\frac{1}{1-\beta}}$ is a convex function, it reduces $\sum_{j=1}^{J} \pi_j \hat{A}_j^{\frac{1}{1-\beta}}$. Hence, from (80), it reduces y_K. Thus, MIS_Y given by (81) increases. ■

Proof of Proposition 11. Existence and uniqueness. Since $\rho \approx 0$, the growth rate is positive whenever $r > 0$. According to (84), b is a decreasing function of $r > 0$. It is represented by the curve labelled FE in Figure 10. As r goes to 0, b tends to $b_0 > 0$ solution to

$$\frac{b_0}{\alpha(b_0)} = \frac{\eta\left(A_H - A_L\right)\lambda\pi_L}{\left[r+\lambda+\alpha(b_0)(1-\eta)\right]\left[\lambda+\alpha(b_0)\right]\kappa}$$

As r goes to $+\infty$, b approaches 0. According to (85), r is an increasing function of b. It is represented by the curve labelled R in Figure 10. As b goes to $+\infty$, r tends to A_H. As b goes to 0, r approaches r_0 solution to

$$r_0 = \frac{r_0 A_H + \lambda\bar{A}}{r_0+\lambda}.$$

If $\bar{A} > 0$, $r_0 > 0$. Hence, there is a unique pair, $(b, r) \in \mathbb{R}^+ \times (0, A_H)$, solution to (84) and (85).

Part 1: Effects of the entry cost. As κ decreases, the FE curve shifts upward and both R and b increase.

Part 2: Effects of the bargaining power. Suppose that $A_L < 0$ and $\bar{A} = 0$. From (85), r is determined by

$$r = \frac{\left[r+\alpha(b)(1-\eta)\right]A_H}{r+\lambda+\alpha(b)(1-\eta)}.$$

From (84), as η goes to 0, b goes to 0 for any $r > 0$. From (85), $b \to 0$ implies the positive solution is such that $r \to 0$. Consider the other polar case, $\eta \to 1$. Suppose $\alpha(b)(1-\eta)$ converges to a positive quantity. From (85), $r > 0$. From (84), $b < +\infty$, which contradicts $\lim_{\eta \to 1} \alpha(b)(1-\eta) > 0$. Hence, $\alpha(b)(1-\eta) \to 0$ and $r \to 0$. ■

SUPPLEMENTARY APPENDICES

SA1. Distribution of capital across firms

We characterize how capital is distributed across firms. We start with firms with the highest productivity level, H. There is a measure π_H of such firms. They hold a positive capital stock if their productivity type was also at the highest at the time of their last participation in the capital market. Suppose they participated in the market at time $t-\tau$ with $\tau > 0$. How much capital did they leave the market with? The quantity of capital in the market available to be allocated to firms over a time interval of length dt is

$$\text{capital supply at time } t-\tau = \overbrace{\alpha dt K_{t-\tau}}^{\text{second-hand capital}} + \overbrace{\dot{K}_{t-\tau}dt}^{\text{new capital}}.$$

There is a measure αdt of firms who participated the capital market at time $t-\tau$. By the law of large number, their average capital stock is equal to the aggregate capital stock, $K_{t-\tau}$. In addition, there is a production of new capital, $\dot{K}_{t-\tau}dt$, that is instantly traded in the capital market. The measure of investors of the highest productivity type is $\alpha dt \pi_H$. Assuming a symmetric equilibrium where firms of type H leave the market with the same capital, their capital holding is

$$k_{H,t-\tau} = \frac{\alpha K_{t-\tau} + \dot{K}_{t-\tau}}{\alpha \pi_H}. \tag{108}$$

We focus on balanced-growth paths where aggregate variables grow at the same rate g. Hence, $K_{t-\tau} = e^{-g\tau}K_t$ and $\dot{K}_{t-\tau} = gK_{t-\tau}$, which gives

$$k_{H,t-\tau} = \frac{(\alpha+g)e^{-g\tau}K_t}{\alpha \pi_H}. \tag{109}$$

The density measure of firms with productivity type H and capital $k_{H,t-\tau}$ is

$$\gamma_H(\tau) = \alpha e^{-\alpha\tau}\pi_H \left[e^{-\lambda\tau} + (1-e^{-\lambda\tau})\pi_H\right] \quad \text{for all } \tau > 0. \tag{110}$$

This expression is explained as follows. The density measure of the firms who contacted the market at time $t-\tau$ is $\alpha e^{-\alpha\tau}$. Firms that own $k_{H,t-\tau} > 0$ units of capital must have been of type H the last time they participated in the market since otherwise they would have liquidated their capital. There is a fraction π_H of such firms. These firms are still of type H at time t if: either they did not receive a productivity shock during the interval of length τ, with probability, $e^{-\lambda\tau}$; of they received at least one productivity shock, with probability $1-e^{-\lambda\tau}$, and the realization of the last one was H with probability π_H.

Consider next firms with productivity L. Those firms have a positive capital stock equal to $k_{h,t-\tau}$ if they contacted the market at time $t-\tau$ and had a productivity type equal to H. By a similar reasoning as above, the density measure of such firms is

$$\gamma_L(\tau) = \alpha e^{-\alpha\tau}\pi_H(1-e^{-\lambda\tau})\pi_L. \tag{111}$$

The difference relative to (110) is that a firm who is of type L and owns a positive capital stock at time t must have received at least one productivity shock between $t-\tau$ and t, with probability $1-e^{-\lambda\tau}$, and the realization of the last shock is L. The next Lemma makes use of the density functions, (110) and (111), to compute the cumulative distribution, Υ.

Lemma 9 ***(Distribution of capital across firms.)*** *The measure of high-productivity firms who owns no more than k units of capital at time t is*

$$\Upsilon_{H,t}(k)=\pi_H\left\{1-\left(\frac{\alpha+\lambda\pi_H}{\alpha+\lambda}\right)+\frac{\pi_L\alpha}{\alpha+\lambda}\left[\frac{\alpha\pi_H k}{(\alpha+g)K_t}\right]^{\frac{\alpha+\lambda}{g}}+\pi_H\left[\frac{\alpha\pi_H k}{(\alpha+g)K_t}\right]^{\frac{\alpha}{g}}\right\}, \quad (112)$$

for all $k\leq(\alpha+g)K_t/\alpha\pi_H$.

The measure of low-productivity firms who owns no more than k units of capital at time t is:

$$\Upsilon_{L,t}(k)=\pi_L\left\{1-\pi_H\left\{\frac{\alpha}{\alpha+\lambda}\left[\frac{\alpha\pi_H k}{(\alpha+g)K_t}\right]^{\frac{\lambda+\alpha}{g}}-\left[\frac{\alpha\pi_H k}{(\alpha+g)K_t}\right]^{\frac{\alpha}{g}}+\frac{\lambda}{\alpha+\lambda}\right\}\right\}, \quad (113)$$

for all $k\leq(\alpha+g)K_t/\alpha\pi_H$.

Proof. The cumulative measure, $\Upsilon_{H,t}(k)$, is given by:

$$\Upsilon_{H,t}(k)=\pi_H-\int_0^{\ln\left[\frac{(\alpha+g)K_t}{\alpha\pi_H k}\right]^{\frac{1}{g}}}\gamma_H(\tau)d\tau \quad \forall k\in(0,k_{H,t}]\,. \quad (114)$$

The measure of all firms of type H is π_H. Assuming the growth rate is positive, $g>0$, the maximum capital held by a firm of type H at time t is $k_{H,t}=(\alpha+g)K_t/(\alpha\pi_H)$. So, for all $k\geq k_{H,t}$, $\Upsilon_{H,t}(k)=\pi_H$. If $k<k_{H,t}$, we substract the measure of measure who owns a capital stock between k and $k_{H,t}$. These firms last accessed the market after time $t-\tau$ where τ is the solution to $k_{H,t-\tau}=k$, i.e., $\tau=\ln\left[(\alpha+g)K_t/\alpha\pi_H k\right]^{1/g}$. Substitute $\gamma_H(\tau)$ from its expression given by (110) into (114) and rearrange to obtain (112).

Similarly, in order to compute $\Upsilon_{L,t}(k)$, we integrate $\gamma_L(\tau)$ as follows:

$$\Upsilon_{L,t}(k)=\pi_L-\int_0^{\ln\left[\frac{(\alpha+g)K_t}{\alpha\pi_H k}\right]^{\frac{1}{g}}}\gamma_L(\tau)d\tau \quad \forall k\in(0,k_{H,t}]\,.$$

This gives (113). ■

From (112) there is a mass of firms of type H with $k=0$ which is equal to

$$\Upsilon_{H,t}(0^+)=\pi_H-\pi_H\left(\frac{\alpha+\lambda\pi_H}{\alpha+\lambda}\right)=\pi_L\pi_H\left(\frac{\lambda}{\alpha+\lambda}\right).$$

It is explained as follows. A measure π_L of firms had a low productivity at their last contact with the market. Among those firms, a fraction $\lambda/(\alpha+\lambda)$ experienced at least one productivity shock between $t-\tau$ and t and the realization of the last shock was H with probability π_H. Similarly, the measure of type-L firms with zero capital is

$$\Upsilon_{L,t}(0^+) = \pi_L \left(\frac{\alpha + \lambda \pi_L}{\alpha + \lambda} \right).$$

SA2. Generalized idiosyncratic productivity shocks

In this Appendix, I generalize the stochastic process driving firms' idiosyncratic productivity. I assume that there are $J \geq 2$ distinct realizations, indexed by $j \in \{1, ..., J\}$, for a firm's idiosyncratic productivity. Productivities are ranked as follows: $A_1 < A_2 < ... < A_J$. Suppose that A_j follows a continous-time Markov chain with generator matrix $\mathbf{\Lambda} = [\lambda_{ji}]$ where λ_{ji} is the rate at which productivity transitions from j to $i \neq j$ and $\lambda_{jj} = -\sum_{i \neq j} \lambda_{ji}$. The Tobin's q of a firm with idiosyncratic productivity j solves

$$rq_j = A_j + \sum_{i \in \{1,...,J\}} \lambda_{ji} q_i + \alpha(1-\eta)\left(1 - q_j\right). \tag{115}$$

The first term on the right side is the flow output generated by one unit of capital. The second term represents the capital gains or losses as productivity changes. In order to see this, note that

$$\begin{aligned} \sum_{i \in \{1,...,J\}} \lambda_{ji} q_i &= \sum_{i \in \{1,...,J\} \setminus \{j\}} \lambda_{ji} q_i + \lambda_{jj} q_j \\ &= \sum_{i \in \{1,...,J\} \setminus \{j\}} \lambda_{ji} q_i - \sum_{i \in \{1,...,J\} \setminus \{j\}} \lambda_{ji} q_j \\ &= \sum_{i \in \{1,...,J\} \setminus \{j\}} \lambda_{ji} (q_i - q_j), \end{aligned}$$

where, in order to obtain the second equality, I used the definition of λ_{jj}. The last term on the right side of (115) corresponds to the gain from liquidating the capital when the firm has access to the market. Here, I used the observation that $q_j \leq 1$ for all j so that the capital market can clear. The equations (115) can be rewritten in a matrix form as

$$\left[r + \alpha(1-\eta)\right] \mathbf{q} = \mathbf{A} + \mathbf{\Lambda q} + \alpha(1-\eta)\mathbf{1}. \tag{116}$$

Solving for $\mathbf{q}$,

$$\mathbf{q} = \left\{\left[r + \alpha(1-\eta)\right] \mathbf{I} - \mathbf{\Lambda}\right\}^{-1} \left[\mathbf{A} + \alpha(1-\eta)\mathbf{1}\right]. \tag{117}$$

For market clearing, the firms with the highest q_j must be indifferent between acquiring additional capital or not. Hence,

$$\max_{j \in \{1,...,J\}} \{q_j\} = 1. \tag{118}$$

An equilibrium is a $r > \rho$ and $\mathbf{q} \in \mathbb{R}^J$ solution to (117) and (118).

In the main text, I specialized the analysis to $J = 2$. If shocks are i.i.d. so that $\lambda_{ji} = \lambda \pi_i$, with $\lambda > 0$ and $\pi_i \in [0, 1]$, for all i and j, then the restriction to $J = 2$ is with no loss in generality. Otherwise, if shocks are not i.i.d., some insights from $J = 2$ do not carry over to

the case where $J \geq 3$. In particular, it is not necessarily the case that $\arg\max_{j\in\{1,\ldots,J\}}\{q_j\} = J$ even though $\arg\max_{j\in\{1,\ldots,J\}}\{A_j\} = J$. If $\arg\max_{j\in\{1,\ldots,J\}}\{q_j\} \neq J$, then among firms who access the capital market, capital is not reallocated toward the type-J firms with the highest instantaneous productivity.

As an example, suppose $J = 3$, $\lambda_{31} > 0$, $\lambda_{12} > 0$, $\lambda_{23} > 0$, and all the other $\lambda_{ij} = 0$. Thus, type-1 firms become type-2 firms who become type-3 firms who then return to type 1. Moreover, $\eta = 1$. I construct an equilibrium where $\max\{q_j\} = q_2 = 1$. From the HJB equations (115),

$$\begin{aligned} r &= A_2 + \frac{\lambda_{23}}{r+\lambda_{31}}\left[A_3 + \lambda_{31}\left(\frac{A_1+\lambda_{12}}{r+\lambda_{12}}\right)\right] - \lambda_{23} \\ q_1 &= \frac{A_1+\lambda_{12}}{r+\lambda_{12}} \\ q_3 &= \frac{1}{r+\lambda_{31}}\left[A_3 + \lambda_{31}\left(\frac{A_1+\lambda_{12}}{r+\lambda_{12}}\right)\right]. \end{aligned}$$

In order to have an equilibrium, one must check that $q_1 < 1$ and $q_3 < 1$. Suppose λ_{23} is small so that $r \approx A_2$. Then, $q_1 \approx (A_1+\lambda_{12})/(A_2+\lambda_{12}) < 1$. Moreover, provided λ_{31} is large enough, q_3 approaches q_1 and hence $q_3 < 1$. So, if firms transition rapidly from state 3 to state 1 and if state 2 is sufficiently persistent, then the capital has a higher market value when operated by type-2 firms. Moreover, if α is high, most of the capital is owned by type-2 firms in equilibrium even though the first best allocation requires the capital to be held by type-3 firms.

SA3. Rent seeking in the capital market

Suppose that brokers can exert some costly effort to increase their bargaining power, η. The cost of this effort is $\psi(\eta)K$, where $\psi(\eta) = \bar{\psi}\eta$ with $\bar{\psi} > 0$. It is scaled by the aggregate capital stock to obtain a balanced growth path. In practice, we can interpret the investment in bargaining strength as follows: brokers improve their pricing strategy and marketing techniques and acquire information about firms to better price-discriminate.

The problem of the intermediary in (14) is generalized as follows:

$$rw = \max_{\eta \in [0,1]} \left\{ -\psi(\eta)K + \alpha\eta(1-q_L) \int k d\Upsilon(L,k) \right\} + \dot{w}. \tag{119}$$

On the right side of (119), the intermediary chooses its bargaining power, η, at some cost, $\psi(\eta)K$. The expected profits of the intermediary are linear in its bargaining power. Using that $\int k d\Upsilon(L,k) = K_L$, the first-order condition is

$$\eta \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \quad \text{if } \bar{\psi} \begin{cases} < \\ = \\ > \end{cases} \alpha(1-q_L)\hat{k}_L. \tag{120}$$

When deciding to invest in its bargaining power, the intermediary compares the cost $\bar{\psi}$ to the expected return $\alpha(1-q_L)\hat{k}_L$. It chooses to invest in a higher bargaining power when the share of capital that is misallocated increases, when trading frictions get smaller, and when the Tobin's q of low-productivity firms decreases.

We focus on symmetric Nash equilibria where all brokers choose the same η. Substitute $\hat{k}_L$ and q_L by their expressions to rewrite the first-order condition as:

$$\eta \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \quad \text{if } \bar{\psi} \begin{cases} < \\ = \\ > \end{cases} \frac{\theta\alpha\lambda\pi_L \left(A_H - A_L\right)}{\left[r - \rho + \theta\left(\alpha + \lambda\right)\right]\left[r + \lambda + \alpha(1-\eta)\right]}, \tag{121}$$

where r is a decreasing function of η given by (27). The term on the right is increasing in η. Hence, there is at most one average value of η that makes individual brokers indifferent between investing in their barganing strength or not. It follows that the best-response function given by (121) is a weakly increasing step function.

Proposition 12 ***(Endogenous rent seeking.)*** *Suppose* $\theta \geq 1$ *and*

$$\rho < \frac{\rho A_H + \lambda \bar{A}}{\rho + \lambda}. \tag{122}$$

There exist $0 < \bar{\psi}_0 < \bar{\psi}_1 < +\infty$ *such that the following is true:*

1. For all $\bar{\psi} \geq \bar{\psi}_0$, there exists an equilibrium with $\eta = 0$ and $r = r_0 > \rho$ solution to

$$r_0 = \frac{(r_0 + \alpha) A_H + \lambda \bar{A}}{r_0 + \lambda + \alpha}. \tag{123}$$

2. For all $\bar{\psi} \leq \bar{\psi}_1$, there exists an equilibrium with $\eta = 1$ and $r = r_1 > \rho$ solution to

$$r_1 = \frac{r_1 A_H + \lambda \bar{A}}{r_1 + \lambda}. \tag{124}$$

3. For all $\bar{\psi} \in \left(\bar{\psi}_0, \bar{\psi}_1\right)$, there is an interior equilibrium, $(r, \eta) \in \mathbb{R}^+ \times (0, 1)$, solution to

$$\bar{\psi} = \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r - \rho + \theta (\alpha + \lambda)] [r + \lambda + \alpha(1 - \eta)]} \tag{125}$$

$$r = \frac{[r + \alpha(1 - \eta)] A_H + \lambda \bar{A}}{r + \lambda + \alpha(1 - \eta)}. \tag{126}$$

4. As $\alpha \to +\infty$, the length of the interval, $\left(\bar{\psi}_0, \bar{\psi}_1\right)$, is maximum with

$$\lim_{\alpha \to +\infty} \bar{\psi}_0(\alpha) = 0 \quad \textit{and} \quad \lim_{\alpha \to +\infty} \bar{\psi}_1(\alpha) = \frac{\lambda \pi_L (A_H - A_L)}{r_1 + \lambda}.$$

Proof. Part 1. From (121), an equilibrium without rent seeking, $\eta = 0$, if and only if

$$\bar{\psi} \geq \bar{\psi}_0 \equiv \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r_0 - \rho + \theta (\alpha + \lambda)] (r_0 + \lambda + \alpha)},$$

where $r_0(\alpha)$ is the solution to (123). Note that $\bar{\psi}_0$ is a continuous function of α that is equal to 0 when $\alpha = 0$ and $\alpha = +\infty$ and is positive for all $\alpha \in (0, +\infty)$.

Part 2. From (121), an equilibrium with $\eta = 1$ exists if and only if

$$\bar{\psi} \leq \bar{\psi}_1 \equiv \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r_1 - \rho + \theta(\alpha + \lambda)] (r_1 + \lambda)},$$

where r_1 solves (124). From (122), $r_1 > \rho$. Note that r_1 is independent of α. Moreover, $\bar{\psi}_1$ is an increasing function of α that is equal to 0 when $\alpha = 0$ and $\lambda \pi_L (A_H - A_L) / (r_1 + \lambda)$ when $\alpha = +\infty$.

Part 3. Using that $r_0 > r_1$,

$$\bar{\psi}_0 \equiv \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r_0 - \rho + \theta (\alpha + \lambda)] (r_0 + \lambda + \alpha)} < \bar{\psi}_1 \equiv \frac{\theta \alpha \lambda \pi_L (A_H - A_L)}{[r_1 - \rho + \theta(\alpha + \lambda)] (r_1 + \lambda)}.$$

It can be checked that for all $\bar{\psi} \in \left(\bar{\psi}_0, \bar{\psi}_1\right)$, there is an interior equilibrium, $(r, \eta) \in \mathbb{R}^+ \times (0, 1)$, solution to (125)-(126). Equation (125) gives a positive relation between r and η while (126) gives a negative relation between r and η. The solution is interior if r solution to (125) when

$\eta = 1$ is larger than r_1, i.e., $\bar{\psi} < \bar{\psi}_1$, and if r solution to (125) when $\eta = 0$ is less than r_0, i.e., $\bar{\psi} > \bar{\psi}_0$.

Part 4. As $\alpha \to +\infty$,

$$\bar{\psi}_0 \equiv \frac{\theta\lambda\pi_L\left(A_H - A_L\right)}{\left[r_0 - \rho + \theta\left(\alpha + \lambda\right)\right]\left[1 + \left(r_0 + \lambda\right)/\alpha\right]} \to 0,$$

and hence $\lim_{\alpha\to+\infty} \bar{\psi}_0(\alpha) = 0$. From Part 2, $\bar{\psi}_1$ is an increasing function of α that approaches $\lambda\pi_L\left(A_H - A_L\right)/\left(r_1 + \lambda\right)$ as $\alpha \to +\infty$. ■

The first part of Proposition 12 shows an equilibrium without rent seeking by brokers, $\eta = 0$, exists if the marginal cost of rent seeking is larger than some threshold, $\bar{\psi}_0$. This threshold is a nonmonotone function of α which approaches 0 as either $\alpha \to 0$ or $\alpha \to +\infty$. It is maximum for some intermediate value of α. So, when $\bar{\psi}$ is not too large, an equilibrium where brokers have no market power is more likely when the market is either very illiquid or very liquid.

The second part of Proposition 12 shows an equilibrium with maximum rent seeking, $\eta = 1$, exists if the marginal cost of rent seeking is lower than some threshold, $\bar{\psi}_1$. This threshold is an increasing function of α that is equal to 0 when $\alpha = 0$ and that is positive as $\alpha \to +\infty$. So, an equilibrium with monopolist brokers is more likely in liquid markets.

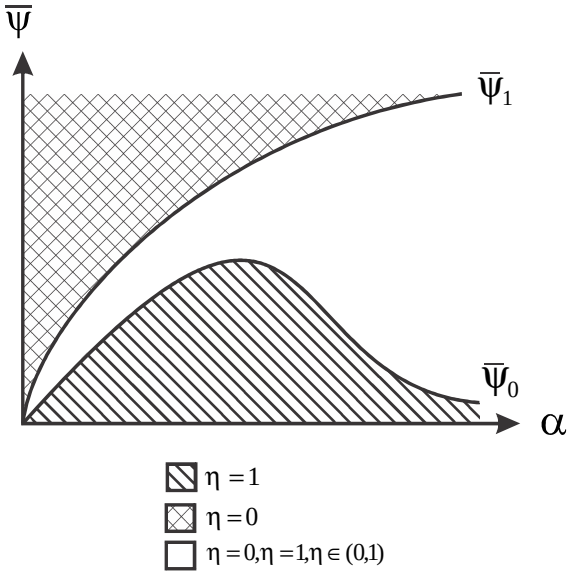


Figure 13: Typology of balanced-growth equilibria with endogenous rent seeking

The threshold $\bar{\psi}_0$ is lower than $\bar{\psi}_1$. Hence, when $\bar{\psi}$ is between $\bar{\psi}_0$ and $\bar{\psi}_1$, there are multiple equilibria. There is a high-growth equilibrium where brokers do not invest in their bargaining power and there is a low-growth equilibrium where brokers have maximum bargaining power. The share of capital that is misallocated is the same in both equilibria.

However, the competitive equilibrium has a higher real interest rate and a higher growth rate. There is also an intermediate equilibrium where $\eta \in (0,1)$. From (125)-(126), the real interest rate solves

$$\bar{\psi} = \frac{\theta\alpha\left(A_H - r\right)}{r - \rho + \theta\left(\alpha + \lambda\right)}.$$

It increases in α and decreases in $\bar{\psi}$. The typology of equilibria is represented in Figure 13 in the parameter space $(\alpha, \bar{\psi})$.

We now consider equilibria at the frictionless limit where $\alpha \to +\infty$. The threshold $\bar{\psi}_0$ goes to 0 so that an equilibrium with $\eta = 0$ always exists at the frictionless limit. The threshold below which $\eta = 1$ is such that

$$\lim_{\alpha \to +\infty} \bar{\psi}_1 \equiv \frac{\lambda\pi_L\left(A_H - A_L\right)}{r_1 + \lambda} > 0.$$

So there can also exist equilibria with monopolist brokers when trading frictions vanish. We now turn to interior equilibria and denote $\hat{\alpha}_\infty = \lim_{\alpha \to +\infty} \alpha(1-\eta)$. From (125) and (126),

$$\begin{aligned} \bar{\psi} &= \frac{\lambda\pi_L\left(A_H - A_L\right)}{A_H - \bar{\psi} + \lambda + \hat{\alpha}_\infty}. \\ r &= A_H - \bar{\psi}. \end{aligned}$$

So, as the limit, $\eta \to 1$ but the effective meeting rate of firms, $\alpha(1-\eta)$, remains bounded. The real interest rate is below the one of a frictionless economy, A_H.

SA4. Growth, money, and capital reallocation

In the main text, I introduced a notion of illiquidity in the market for capital goods by assuming that it takes time to trade and reallocate capital goods across firms. An alternative notion of liquidity is as an attribute to an asset that can serve as means of payment, such as money. So far, we have assumed that firms faced no liquidity constraint when financing their acquisition of new capital. They can borrow against their future income and promise to repay their debts. We now relax this assumption and assume that in order to purchase capital, the firm must hold liquid assets, e.g., money or government bonds. It will allow us to revisit a classical question in macroeconomics since Tobin (1965), Sidrauski (1967), Brock (1974), and Stockman (1981), the relationship between money (or liquidity) and economic growth.[25]

Modified environment

I make the following assumptions in order to generate a role for liquidity. Stocks, private debt, or claims on physical capital are not accepted as means of payment in the capital market. The only accepted means of payments are government-issued liabilities, such as money or bonds. Firms can rent liquid assets from financial institutions at the price s in a competitive market opened continuously through time. The renting price of liquidity, $s \geq 0$, is the interest-rate spread between illiquid financial assets (i.e., stocks held by firms) and liquid assets, which represents the cost incurred by financial institutions to hold liquid assets in order to rent them. For instance, if the liquid asset takes the form of money and the inflation rate is $\pi > -r$, then $s = r + \pi$. If the liquid asset takes the form of real government bonds that pays a real interest rate r^g, then $s = r - r^g$. Firms can adjust their holdings of liquid assets in-between meetings with brokers, but liquidity cannot be adjusted instantly at the time of the meetings. Hence, liquidity must be held prior to the meetings.

I generalize the description of firms' idiosyncratic productivity shocks as follows. As before, the most productive firms have a productivity equal to A_H. Each firm receives an opportunity to draw a new productivity at Poisson arrival rate λ from the distribution $G(A) = \pi_H G_H(A) + \pi_L G_L(A)$. The distribution $G_H(A)$ is a step function that assigns mass one to A_H. The distribution $G_L(A)$ is continuous with support $(0, A_H)$. These assumptions imply that the distribution of firms across productivities has a mass point at A_H and is continuous otherwise. We represent the distribution of values for A in Figure 14.

[25] Ireland (1994) revits the relationship between money and growth in the context of an AK model where the payment system is endogenous.

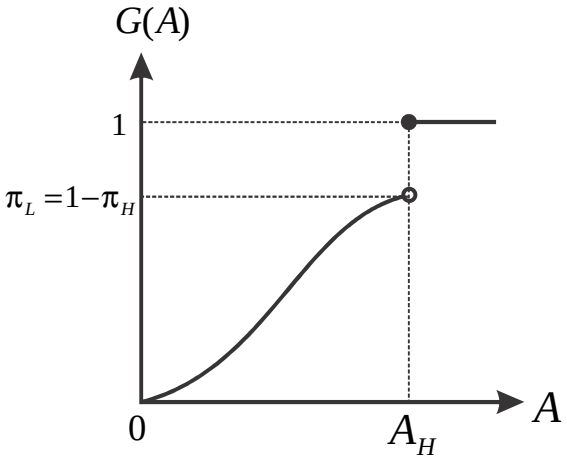


Figure 14: Distribution of productivity shocks.

The bargaining problem

In order to simplify the bargaining problem between the firm and the intermediary, we make the following assumption. While the acquisition of new capital, $k' - k$, must be financed with liquid assets, the payment of the intermediation fees to the brokers can be financed with debt, i.e., the firm can commit to pay ϕ in the future at an interest rate equal to r.[26]

Let us start with the bargaining problem between a firm with capital, k, real money balances, m, and productivity, A, and a broker. The terms of trade solve:

$$(k', \phi) = \arg \max_{k' \in [0, k+m]} \left[\upsilon(k', A) - (k' - k) - \phi - v(k, A) \right]^{1-\eta} [\phi]^{\eta} . \tag{127}$$

The novelty is that the acquisition of new capital, $k' - k$, must be financed with liquid assets, $k' - k \leq m$. The value function of the firm does not depend on m which is a control variable that can be adjusted instantly at any point in time. By assumption, the payment of ϕ is not subject to a liquidity constraint and can be financed with debt. The outcome of the negotiation is

$$k' \in \arg \max_{k' \in [0, k+m]} \left[\upsilon(k', A) - (k' - k) - v(k, A) \right] \tag{128}$$

$$\phi = \eta \max_{k' \in [0, k+m]} \left[\upsilon(k', A) - (k' - k) - v(k, A) \right] . \tag{129}$$

From (128) the choice of capital is the one the firm would make if it had direct access to the capital market. As before, from (129), the intermediation fee represents a fraction η of the gains from trade. It is as if the firm evolves in an economy where brokers have no bargaining power but the contact rate with the market is $\alpha(1 - \eta)$.

[26] See Lagos and Zhang (2020) for a similar assumption.

Firm valuations and Tobin's q

The lifetime expected profits of firms are determined by the following HJB equation:

$$\begin{aligned} r\upsilon(k,A) &= \max_{m\geq 0}\Bigg\{-sm + Ak + \lambda\int_{[0,A_H]}\left[\upsilon(k,x)-\upsilon(k,A)\right]dG(x) \\ &\quad + \alpha(1-\eta)\max_{k'\in[0,k+m]}\left[\upsilon(k',A)-(k'-k)-\upsilon(k,A)\right]\Bigg\}, \end{aligned} \tag{130}$$

where m denote the choice of liquid assets. The first term on the right side is the cost of renting liquid assets, sm. The acquisition of new capital, $k'-k$, is bounded above by the firm's holdings of liquid asset, i.e., $k'-k\leq m$. The second term corresponds to the arrival of productivity shocks drawn from the distribution G. The last term describes the arrival of a meeting with a broker.

I guess that $\upsilon(k,A)=q_Ak$. The value functions in (130) can be rewritten as:

$$rq_Ak = Ak + \lambda\int_{[0,A_H]}(q_x-q_A)\,kdG(x) + \Omega(k,A), \tag{131}$$

where the last term represents the problem of the firm,

$$\Omega(k,A)\equiv\max_{0\leq k'\leq k+m}\left\{-sm+\alpha(1-\eta)\left(q_A-1\right)\left(k'-k\right)\right\}. \tag{132}$$

The firm chooses its holdings of liquid assets and its future capital in order to maximize the value of its investment net of the renting cost of liquidity. If $q_A>1$, then $k'=k+m$ and there is a positive and bounded solution for m if $-s+\alpha(1-\eta)\left(q_A-1\right)=0$. Assuming q_A is increasing in A, in any monetary equilibrium this equality must hold for the highest productivity, $A=A_H$. If we denote q_H the value of q_A evaluated at $A=A_H$,

$$q_H = 1+\frac{s}{\alpha(1-\eta)}. \tag{133}$$

The Tobin's q of the most productive firms is greater than one in order to compensate for the holding cost of liquid assets that is necessary to finance the acquisition of capital. Note the magnitude of the effect of s on q_H, $\partial q_H/\partial s=\left[\alpha(1-\eta)\right]^{-1}$ depends on the search and bargaining frictions in the market. The effect is stronger if the market is more illiquid, α is low, and if brokers' bargaining power is high, η is close to one. Assuming (133) holds,

$$\Omega(k,A)=\alpha(1-\eta)\left(1-q_A\right)k\mathbb{I}_{\{q_A\leq 1\}}.$$

So, $\Omega(k,A)$ is proportional to k and from (131) the conjecture $\upsilon(k,A)=q_Ak$ is verified.

We divide both sides of (131) to obtain:

$$rq_A = A+\lambda\pi_H\left(q_H-q_A\right)+\lambda\pi_L\int\left(q_x-q_A\right)dG_L(x)+\alpha(1-\eta)\left(1-q_A\right)\mathbb{I}_{\{q_A\leq 1\}}. \tag{134}$$

The last term on the right side of (134) captures the gain when firms liquidate their capital. The following Lemma gives a closed-form solution for q_A.

Lemma 10 ***(Tobin's q and liquidity.)*** *The Tobin's q of a firm with productivity A is given by*

$$q_A = 1 + \frac{r+\lambda}{r+\lambda+\alpha(1-\eta)\mathbb{I}_{\{A<B\}}}\left[\frac{s}{\alpha(1-\eta)} - \frac{A_H - A}{r+\lambda}\right], \tag{135}$$

where B, the productivity level such that $q_B = 1$, solves

$$\frac{s}{\alpha(1-\eta)} = \frac{A_H - B}{r+\lambda}. \tag{136}$$

Proof of Lemma 10. Let B the value of productivity such that $q_B = 1$. From (134),

$$\frac{\partial q_A}{\partial A} = \frac{1}{r+\lambda+\alpha(1-\eta)\mathbb{I}_{\{A\leq B\}}}.$$

So, the value function is piecewise linear with a kink at $A = B$. By integrating $\partial q_A/\partial A$ over the intervals (A, A_H) and (A, B), we obtain:

$$q_A = \begin{cases} q_H - \frac{A_H - A}{r+\lambda} \\ q_B - \frac{B-A}{r+\lambda+\alpha(1-\eta)} \end{cases} \quad \forall A \in \begin{cases} (B, A_H) \\ (0, B). \end{cases} \tag{137}$$

Using that $q_B = 1$ and $q_H = 1 + s/\alpha(1-\eta)$, it follows from (137) that B is determined by (136). From (137) and (136), the expression for the Tobin's q can be written more compactly as (135). ■

For all $A < B$, firms liquidate their capital whenever they access a broker whereas for all $A > B$, they hoard their capital. For given r, the productivity threshold above which firms hoard their capital, B, increases with s. From (135), the passthrough from the liquidity spread, s, to the Tobin's q increases with the productivity level. Note that r is endogenous and will also depend on s.

Equilibrium real interest rate

We close the description of the equilibrium with the determination of the real interest rate. In the case of $A = A_H$, the HJB equation (134) simplifies to

$$rq_H = A_H + \lambda\pi_L \int (q_x - q_H)\, dG_L(x). \tag{138}$$

By plugging q_H given by (133) into (138), the real interest rate is determined by

$$r\left(1 + \frac{s}{\alpha(1-\eta)}\right) = A_H - \lambda\pi_L \int_0^B \left(\frac{A_H - A + s}{r+\lambda+\alpha(1-\eta)}\right) dG_L(A) \tag{139}$$
$$-\lambda\pi_L \int_B^{A_H} \frac{A_H - A}{r+\lambda} dG_L(A).$$

An equilibrium is a pair, (r, B), solution to (136)-(139). When liquidity is costless, $s = 0$, $B = A_H$ and r solves

$$r = A_H - \frac{\lambda \pi_L}{r + \lambda + \alpha(1-\eta)} \int_0^{A_H} G_L(A) dA.$$

It corresponds to (27) when G_L is a step function.

Proposition 13 ***(The cost of liquidity and economic growth.)*** *As s increases, the rate of return of capital and the growth rate of the economy decreases.*

In order to illustrate this result, we return to the case of two productivity levels by assuming that G_L is a Dirac measure assigning mass one to $A_L \in (0, A_H)$. We obtain the following Corollary.

Corollary 4 *Suppose $G_L(A) = \mathbb{I}_{\{A \geq A_L\}}$. There is a unique balanced-growth-path equilibrium and it is such that as s increases, r and g decrease while q_L and q_H increase. Moreover, there is a pair, $(s_0, r_0) \in \mathbb{R}^2_+ > 0$, solution to*

$$r_0 (r_0 + \lambda) = (r_0 + \lambda \pi_L) A_L + \lambda \pi_H A_H \tag{140}$$

$$s_0 = \frac{\alpha(1-\eta)(A_H - A_L)}{r_0 + \lambda}, \tag{141}$$

such that the following is true:

1. *For all $s < s_0$, the equilibrium features $q_L < 1$, i.e., low-productivity firms liquidate their capital when given the opportunity. The share of the aggregate capital stock held by low-productivity firms is*
$$\hat{k}_L = \frac{\lambda \pi_L}{g + \alpha + \lambda}.$$
2. *For all $s > s_0$, the equilibrium features $q_L > 1$, i.e., low-productivity firms hold onto their capital. The share of the aggregate capital stock held by low-productivity firms is*
$$\hat{k}_L = \frac{\lambda \pi_L}{g + \lambda}.$$

Proof of Corollary 4. From (139), the real interest rate, r, is the solution to:

$$r \left[1 + \frac{s}{\alpha(1-\eta)} \right] = A_H - \frac{\lambda \pi_L (A_H - A_L + s)}{r + \lambda + \alpha(1-\eta)} \qquad \text{if } r < \hat{r} \tag{142}$$

$$= A_H - \frac{\lambda \pi_L (A_H - A_L)}{r + \lambda} \qquad \text{if } r > \hat{r}, \tag{143}$$

where, from (136),

$$\hat{r} \equiv \frac{\alpha(1-\eta)\left(A_H - A_L\right)}{s} - \lambda. \tag{144}$$

First, we establish that there is a unique s, denoted s_0, such that $r = \hat{r}$. We denote $r_0 = \hat{r}(s_0)$. The pair, (s_0, r_0), solves (143) and (144), i.e.,

$$\begin{aligned} r_0\left[1 + \frac{s_0}{\alpha(1-\eta)}\right] &= A_H - \frac{\lambda\pi_L(A_H - A_L)}{r_0 + \lambda} \\ r_0 &= \frac{\alpha(1-\eta)\left(A_H - A_L\right)}{s_0} - \lambda. \end{aligned}$$

They can be rearranged to correspond to (140)-(141).

Next, we establish that there is a unique r solution to (142)-(143). Denote r_1 the solution to (142) and r_2 the solution to (143). Both solutions are unique and belong to the interval $(0, A_H)$. Moreover, both solutions are decreasing functions of s. For all $r < \hat{r}$,

$$\frac{\lambda\pi_L\left(A_H - A_L + s\right)}{r + \lambda + \alpha(1-\eta)} < \frac{\lambda\pi_L(A_H - A_L)}{r + \lambda}.$$

It follows that if $r_1 < \hat{r}$, then $r_2 < \hat{r}$ and hence r_2 cannot be a solution to (142)-(143). Conversely, if $r_2 > \hat{r}$, then $r_1 > \hat{r}$ and hence r_1 cannot be the solution (142)-(143).

If $s < s_0$, the dynamics of the capital held by high-productivity firms is given by (17). The Tobin's q for low-productivity firms solves

$$rq_L = A_L + \lambda\pi_H\left(q_H - q_L\right) + \alpha(1-\eta)\left(1 - q_L\right).$$

Solving for q_L and substituting q_H by its expression:

$$q_L = \frac{A_L + \lambda\pi_H q_H + \alpha(1-\eta)}{r + \lambda\pi_H + \alpha(1-\eta)}.$$

As s increases, q_H increases and r decreases. Hence, q_L increases.

If $s > s_0$, the capital stocks in high- and low-productivity firms evolve according to:

$$\dot{K}_H = \lambda\pi_H K_L - \lambda\pi_L K_H + \dot{K} \tag{145}$$

$$\dot{K}_L = \lambda\pi_L K_H - \lambda\pi_H K_L. \tag{146}$$

According to (145), the stock of capital held by high productivity firms increases with the flow flow of low-productivity firms that receive a high productivity shock, it decreases with the flow of high-productivity firms that receive a low productivity shock, and it increases with the flow of new capital goods entering the market. Using that $\dot{K}_H/K_H = \dot{K}_L/K_L = \dot{K}/K$, it follows that the capital share are equal to:

$$\begin{aligned} \hat{k}_H &\equiv \frac{K_H}{K} = \frac{\lambda\pi_H + g}{\lambda + g} \\ \hat{k}_L &\equiv \frac{K_L}{K} = \frac{\lambda\pi_L}{g + \lambda}. \end{aligned}$$

The Tobin's q for low-productivity firms solves

$$rq_L = A_L + \lambda\pi_H \left(q_H - q_L\right).$$

Solving for q_L and substituting q_H by its expression:

$$q_L = \frac{A_L + \lambda\pi_H q_H}{r + \lambda\pi_H}.$$

As s increases, q_H increases and r decreases. Hence, q_L increases. ■

If the cost of holding liquidity, s, is smaller than some threshold, s_0, then q_L is less than one so that low-productivity firms liquidate their capital when they access the capital market. As s increases, the rate of return of capital decreases and hence the growth rate of the economy decreases. Moreover, the Tobin's qs, q_H and q_L, increase. The distribution of capital across productivity types is the same as the one in the economy without liquidity constraints. The cost of liquidity only affects the misallocation of capital, $\hat{k}_L$, through g, i.e., a larger share of the capital is misallocated as s increases because the rate at which new capital is formed slows down.

If s is larger than s_0, then the equilibrium is such that $q_L > 1$, in which case low-productivity firms choose not to liquidate their capital. The fact that low-productivity firms hoard their capital raises the misallocation of capital relative to an economy with a low cost of liquidity.

From (141) the threshold s_0 decreases with η. So, as brokers' bargaining power increases, the economy is more likely to be in the economy where low-productivity firms hoard capital instead of selling it. As a result, in the presence of liquidity constraints, η affects both the rate at which capital is accumulated and its distribution across firms.

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