Incomplete Markets and Exchange Rates*

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Abstract

We investigate the implications of domestic and international market incompleteness for the co-movement of real exchange rates and relative aggregate consumption, using an analytically tractable two-country, consumption-based asset pricing model with heterogeneous agents. Domestic market incompleteness can recover the observed co-movement as long as periods of exchange rate depreciations coincide with periods of low idiosyncratic risk, making foreign currency returns a poor hedge. Additionally, international market incompleteness allows the model to capture the low observed exchange rate volatility.

Keywords: exchange rates, international business cycles, incomplete markets, heterogeneous agents.

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1. Introduction

We revisit the Backus-Smith condition (Kollmann, 1991; Backus and Smith, 1993) – which describes the cyclicality of exchange rates across countries in terms of co-movement of relative aggregate consumption growth and real exchange rate depreciation – to consider the implications of domestic and international market incompleteness. First, while a large class of representative agent macroeconomic models with international market incompleteness can reconcile the cyclicality of exchange rates in the data, these models fail once we account for no-arbitrage restrictions arising from cross-border trade in multiple assets (Benigno and Küçük, 2012; Lustig and Verdelhan, 2019). Second, the representative agent assumption violates even the domestic Euler equation (Berger, Bocola and Dovis, 2023; Di Tella, Hébert and Kurlat, 2024), so domestic incomplete markets are a reality that exchange rate puzzles must confront.

When international financial markets are complete, a large class of models admits the following relationship:

$$\left(\frac{C_{t+1}}{C^*_{t+1}} / \frac{C_t}{C^*_t}\right)^s = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tag{1}$$

where C_t is Home aggregate consumption, C^*_t is Foreign aggregate consumption, $\mathcal{E}_t$ is the real exchange rate where an increase signifies a depreciation of Home currency, and s is the inverse of the inter-temporal elasticity of substitution. Trade in risk-free Home and Foreign currency denominated bonds implies ex-post redistribution, which must occur entirely through exchange rate movements. For example, following a positive productivity shock in the Home country, the Home currency depreciates leading to higher consumption abroad. Exchange rates thus move to reallocate wealth from Home to Foreign and so are "risky" from the perspective of a Home investor.[1] However, in the data, exchange rates often appreciate when Home consumption rises and constitute the Backus-Smith *puzzle*.

[1]This implication echoes closed economy models with complete markets which imply a correlation of -1 between the representative agent SDF and the market portfolio, see, e.g. Lettau (2002).

When markets are incomplete internationally, the condition above needs to hold only in expectation and so may fail ex-post. Classical contributions in international macroeconomics such as Corsetti, Dedola and Leduc (2008) and Benigno and Thoenissen (2008) show that non-traded risk, which arises from consumption or production complementarities, enables incomplete market models to generate plausible patterns of international risk sharing. We refer to these economic mechanisms as goods market mechanisms (as opposed to financial). However, these models only allow cross-border trade in a single risk-free asset (denominated in either currency). Lustig and Verdelhan (2019) show that, no-arbitrage restrictions from cross-border trade in at least one Home and one Foreign risk-free asset imply prices always co-move negatively with consumption – so incomplete market models cannot resolve the puzzle of excessive risk-sharing *regardless* of goods market frictions and other economy specifics.[2] Since, in practice, the number of assets traded across borders is very high, this result has far-reaching implications for both theory and practice.

Our first contribution is to highlight how goods markets mechanisms can reconcile the Backus-Smith puzzle. We show that any model that can reconcile the comovement of exchange rates and consumption growth (or SDFs) must generate a non-traded component of relative prices which is "safe" from a domestic investor perspective. We then show that moving from frictionless cross-border trade in a single risk-free asset to cross-border trade in just one Home and one Foreign risk-free asset inhibits *any* goods-market mechanisms from reconciling the patterns of risk-sharing observed in the data since the additional no-arbitrage restriction rules out such safe exchange rate movements, regardless of how far the economy is from complete financial markets.

Our second and main contribution is to propose a generalization of the representative agent, two-country model with heterogeneous consumers within each country. Domestic heterogeneity implies the presence of non-traded risk within countries which, if correlated with exchange rate movements, implies such movements cannot be insured

[2] Benigno and Küçük (2012) also find negative comovement in a quantitative international macro model with frictionless cross-border trade in two *nominal* bonds.

by trade just in Home and Foreign risk-free assets.

Our model considers heterogeneous consumers in the Home economy facing idiosyncratic consumption risk, all trading in fully *integrated* international markets. The presence of idiosyncratic risk implies an Euler inequality for aggregate consumption which, in turn, shapes the properties of exchange rates. Even when there is trade in two nominally risk-free assets, the Backus-Smith puzzle can be resolved as long as idiosyncratic consumption risk increases sufficiently with exchange rate depreciation. This condition implies that the Foreign bonds are a poor hedge for consumption risk, consistent with the diversification puzzle in Heathcote and Perri (2013). With trade in a single risk-free asset internationally, we show that the volatility of idiosyncratic risk implies the model can more easily achieve the low risk-sharing benchmark.

We extend the consumption-based asset pricing framework in Cox, Ingersoll and Ross (1985) and Lustig and Verdelhan (2019) to account for uninsurable idiosyncratic consumption risk in the Home economy. We derive the incomplete markets wedge in closed form and present sufficient conditions for low risk sharing to be obtained. We show that while the internationally-complete markets equilibrium can also attain low risk-sharing, it delivers far too volatile exchange rates, so international market incompleteness significantly expands the admissible range of parameters.

Related Literature We extend the work of Benigno and Küçük (2012) and Lustig and Verdelhan (2019), who show that introducing a second internationally traded bond breaks the ability of canonical international macro models to reconcile the Backus-Smith puzzle, to allow for within-country heterogeneous agents. In doing so, we relate to Ramchand (1999), and Kocherlakota and Pistaferri (2007). Ramchand (1999) constructs an income process, generalizing Constantinides and Duffie (1996), which delivers a a modified Cole-Obstfeld balanced trade equilibrium featuring full international risk sharing in a two country open economy heterogeneous consumers environment. Kocherlakota and Pistaferri (2007) study the restrictions imposed by

models with partial insurance domestically but do not speak to the implications of incomplete insurance for the connection of consumption growth with exchange rates. In contrast, we allow for both domestic and internationally incomplete asset markets and study the implications for rationalizing empirical moments of exchange rates.[3]

Our approach relates a nascent literature in international finance. Chernov, Haddad and Itskhoki (2024) investigate how different financial market structures and the mix of locally, globally traded, and unspanned risks contribute to different exchange rate puzzles. Jiang, Krishnamurthy and Lustig (2023) show that Euler equation wedges are necessary to resolve Backus-Smith puzzle.[4] Important precursors have shown that segmentation (Alvarez, Atkeson and Kehoe, 2002; Chien, Lustig and Naknoi, 2020) or intermediation frictions (Gabaix and Maggiori, 2015; Itskhoki and Mukhin, 2021) can resolve various exchange rate puzzles. Our heterogeneous consumers model with integrated markets illustrates that segmentation per se is not required to theoretically reconcile the cyclicality puzzle, and that domestic market incompleteness offers a complementary micro-founded mechanism for understanding exchange rates. In contemporaneous work, Kekre and Lenel (2024*a*) show that permanent discount factor shocks can resolve the comovement and predictability puzzles. Our paper can be seen as providing micro-foundations for such discount factor shocks.

More generally, our model with heterogeneous consumers ties to a large literature investigating whether idiosyncratic risk affects macro-finance aggregates (see, e.g. Mankiw, 1986; Weil, 1992; Guvenen, 2009; Kocherlakota and Pistaferri, 2009; Krueger and Lustig, 2010; Heathcote, Storesletten and Violante, 2014; Werning, 2015; Kaplan,

[3]Relatedly, Leduc (2002) using a calibrated open economy model featuring idiosyncratic and aggregate risk investigated the role of these risks in generating currency premia. Tessari (2021) extends the asset pricing framework of Herskovic, Kelly, Lustig and Van Nieuwerburgh (2016*a*) to an international finance environment and shows that shocks to common idiosyncratic volatility factor are an important pricing factor for carry trade returns.

[4]Verdelhan (2010), Colacito and Croce (2013), Hassan (2013), Karabarbounis (2014), Farhi and Gabaix (2016), and Kekre and Lenel (2024*b*) investigate the role of habits, long-run risk, country-size, home production, rare disasters, and safety shocks respectively in generating observed correlations between cross country asset returns and exchange rate returns. For a treatment of optimal portfolio choice and international risk sharing, see Devereux and Sutherland (2011) and Heathcote and Perri (2013) amongst others.

Moll and Violante, 2018; Auclert, Rognlie and Straub, 2023; Di Tella et al., 2024). In the open economy macro literature, for papers with agent heterogeneity, see Ghironi (2006); Kocherlakota and Pistaferri (2007); Farhi and Werning (2016); Fornaro (2018); De Ferra, Mitman and Romei (2020); Auclert, Rognlie, Souchier and Straub (2021); Guo, Ottonello and Perez (2023); Hong (2023); Acharya and Challe (2024) amongst others.

The rest of the paper is organised as follows. Section 2 provides a preference-free characterization of exchange rate cyclicality and the cosntraints imposed by trade in assets, first with a representative agent and then with heterogeneous agents. Section 3 illustrates how the reduced-form conditions we use can be derived as an equilibrium. Section 4 evaluates the sufficient conditions for heterogeneous agents model to reconcile the Backus Smith puzzle using estimates of idiosyncratic income risk from the literature. Section 5 concludes.

2. Two-country, Representative Agent, Incomplete Markets Model

The starting point of our analysis is the consumption based asset pricing model building on Rubinstein (1974); Lucas (1978); Breeden (1979), and the two-country model in Lucas (1982). Consider a two-country model where M_{t+1} denotes the Home representative household's SDF and M^*_{t+1} denotes the Foreign representative household's SDF. We restrict attention to time-separable utility, e.g. CRRA utility over consumption where $M^{(*)}_{t+1} = \beta \left(\frac{C^{(*)}_{t+1}}{C^{(*)}_t} \right)^{-s}$, as in condition (1).

Home and Foreign households each trade their respective domestic risk-free real bonds with returns R_t and R^*_t respectively frictionlessly (i.e. no borrowing constraints).

No-arbitrage pricing implies:

$$\mathbb{E}_t[M_{t+1}] = 1/R_{t+1}, \tag{2}$$

$$\mathbb{E}_t[M^*_{t+1}] = 1/R^*_{t+1} \tag{3}$$

If the Home (Foreign) households also trade the Foreign (home) bond, and the real exchange rate at time t is denoted with $\mathcal{E}_t$, then we additionally obtain the following two Euler conditions:

$$\mathbb{E}_t\left[M^*_{t+1}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] = 1/R_{t+1}, \tag{4}$$

$$\mathbb{E}_t\left[M_{t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = 1/R^*_{t+1}. \tag{5}$$

We assume SDFs, allocations and prices are jointly log-normal. To close the model without explicitly specifying goods markets, an exchange rate process is needed, which is consistent with equations (2)–(5) above. This problem reduces to finding an exchange rate process that satisfies:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) \tag{6}$$

where $x = \log(X)$. Naturally, the process corresponding to complete markets ($\Delta e_{t+1} = m^*_{t+1} - m_{t+1}$) is one candidate. More generally, as shown in Backus, Foresi and Telmer (2001), the following process also satisfies equation (6):

$$\Delta e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1} \tag{7}$$

where η_{t+1} is an incomplete markets wedge which must satisfy certain conditions imposed by asset trade. Condition (1) restricts the covariance between relative consumption growth and exchange rate depreciations to be positive. In the data, this covariance is negative, hence the Backus-Smith *puzzle*. Turning to SDFs, this implies the cyclicality of exchange rates $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ is counterfactually positive in the model.

Combining (2) and (4), with (7) – which implies that the Home bond is interna-

tionally traded – yields:

$$\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \tag{8}$$

Combining (3) and (5), with (7) – which implies that the Foreign bond is internationally traded – yields:

$$-\mathbb{E}_t[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) - cov_t(m^*_{t+1}, -\eta_{t+1}) \tag{9}$$

These two conditions mirror those studied by Lustig and Verdelhan (2019) and bound the joint dynamics of the incomplete market wedge and the SDFs, carrying strong implications for the macro side of the model. The conditions reflect the return, for a Home and a Foreign investor respectively, from the non-traded component of exchange rates which must compensate for both volatility of returns and the riskiness of the exchange rate.

Replacing the SDFs with the aggregate consumption growth, the incomplete markets wedge is related to:[5]

$$\eta_{t+1} = \log \underbrace{\left(\frac{P_{t+1}}{P_t} \frac{P^*_t}{P^*_{t+1}} \right)}_{\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}} - \log \underbrace{\left(\frac{C_t}{C_{t+1}} \frac{C^*_{t+1}}{C^*_t} \right)^s}_{\frac{M_{t+1}}{M^*_{t+1}}}$$

where P_t is the Home price level, C_t is aggregate consumption, s^{-1} is the inter-temporal elasticity of substitution, and terms with asterisks denote the corresponding Foreign objects. The wedge, η, is often interpreted as the non-traded component of exchange rate movements or the wealth gap, see e.g. Corsetti, Dedola and Leduc (2023*a*).

2.1. Representative Agent and Trade in Risk-free Assets

Having now specified our framework, we illustrate the mechanism through which goods market frictions in incomplete market models can help reconcile the pattern of

[5] For a closed form solution of the incomplete markets wedge in a two-country open economy model, see Pavlova and Rigobon (2007).

international risk-sharing.

Proposition 1 (One Int'l Traded Asset, Representative Agent No-Arbitrage). *When only Foreign bonds are internationally traded such that equations (2), (3) and (5) hold, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ *if and only if*

$$cov_t(m_{t+1}, \eta_{t+1}) \geq var_t(m^*_{t+1} - m_{t+1}) - \log \mathbb{E}_t[e^{\eta_{t+1}}] \tag{10}$$

where,

$$cov_t(m_{t+1}, \eta_{t+1}) = cov_t(m_{t+1}, \Delta e_{t+1}) - cov_t(m_{t+1}, m^*_{t+1}) + var_t(m_{t+1}) \tag{11}$$

Proof. See Appendix A.2. □

The RHS of condition (10) is equal to the volatility of the exchange rate growth under complete markets and is strictly positive. The condition is satisfied if either the non-traded component η_{t+1} leads to relative price fluctuations which are safe from the perspective of a Home investor, as captured by $cov_t(m_{t+1}, \eta_{t+1}) > 0$ or that the volatility of the non-traded component is high. Equation (11) shows that non-traded risk results in relative price fluctuations which are particularly *safe* when the Home SDF is very volatile or when international comovement in SDFs is low relative to the comovement of exchange rates and the Home SDF.[6] In Section 3, we specify a two-country model with multiple goods and investigate the parametrizations consistent with this condition.

The condition (10) provides a general characterization for goods markets mechanisms developed to resolve the Backus-Smith puzzle in models with consumption or production complementarities (Corsetti et al., 2008; Benigno and Thoenissen, 2008) or global value chain fragmentation (Corsetti, D'Aguanno, Dogan, Lloyd and Sajedi, 2023*b*), amongst others. A limitation of Proposition 1, and the models which satisfy it, is that it may exacerbate other exchange rate puzzles– in particular, that of excess volatility of exchange rates. The RHS of equation (10) is equal to the volatility of

[6] Brandt, Cochrane and Santa-Clara (2006) show that SDFs must comove very strongly to explain the relatively low exchange rate volatility in the data.

exchange rates under complete markets, and models with a low volatility will generally fare better in resolving the cyclicality puzzle, which rationalizes why models such as Corsetti et al. (2008) display very low risk premia (Lustig and Verdelhan, 2019, pp 2241).

Cross-border trade in a second risk-free asset prevents the model from reconciling the Backus-Smith puzzle because this constrains the *safety* of relative price movements due to non-traded risk. From Equation (8), when the non-traded component becomes safe for Home investors, lower returns on their Foreign bond holdings ($\mathbb{E}_t[\eta_{t+1}]$) translate to higher returns on Home bonds for Foreign investors ($\mathbb{E}_t[-\eta_{t+1}]$). This is only consistent with no arbitrage if the non-traded component becomes sufficiently risky for Foreign investors which would restore the Backus-Smith correlation.

Corollary 1 (Two Int'l Traded Asset, Representative Agent No-Arbitrage). *As the Home bond also becomes internationally traded without arbitrage,* $cov_t(m_{t+1}, \eta_{t+1}) \rightarrow -\mathbb{E}_t[\eta_{t+1}] + \frac{1}{2} var_t(\eta_{t+1})$ *as in equation* (8)*, then condition* (10) *implies* $var_t(\Delta e_{t+1}) \leq 0$.

Proof. See Appendix A.2. □

As enough assets become traded and markets approach completeness, $\sigma_\eta \rightarrow 0$. However, even if asset markets are far from complete, trade in just two nominally risk-free bonds is sufficient to ensure $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) > 0$ as we detail in Appendix B.2.

2.2. Many (heterogeneous) agents

Assume there is a continuum of consumers i in the Home economy, each facing uninsurable idiosyncratic permanent shocks to their consumption. The financial markets are domestically incomplete with respect to this idiosyncratic risk. For simplicity, we maintain the representative agent assumption in the Foreign economy but we show our results generalize in Appendix YY. We also postpone construction of an equilibrium, with goods market clearing and micro-foundations, and an empirical

evaluation to Section 3. Here, we preview our main result.[7]

Each individual consumer i trades frictionlessly in Home and Foreign risk-free bonds, and hence individual Euler equations hold as in (2)-(5), with M_{t+1} replaced by consumer-specific SDF M^i_{t+1}. With particular assumptions on agents' consumption processes that allow aggregation as in Constantinides and Duffie (1996), we can construct an aggregate Euler equation for an asset with payoff $\tilde{R}_{t+1}$ summarized by:

$$\mathbb{E}_t \left[\underbrace{\beta \left(\frac{C_{t+1}}{C_t} \right)^{-s}}_{M_{t+1}} e^{\tilde{y}_{t+1}} \tilde{R}_{t+1} \right] = 1 \tag{12}$$

where $\tilde{y}_{t+1}$ denotes the log of the variance of the idiosyncratic permanent shocks faced by households, which we assume is jointly log-normally distributed with the other variables. We retain M_{t+1} notation to denote aggregate consumption growth based SDF. For the case of the Home bond $\tilde{R}_{t+1} = R_{t+1}$ and for the Foreign bond $\tilde{R}_{t+1} = \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} R^*_{t+1}$. Combining these aggregate Euler equations, along with (3), (4) yields:

$$var_t(\Delta e_{t+1}) = cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) + cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}) \tag{13}$$

This extended economy admits the exchange rate process described by Equation (7), but with new restrictions imposed by domestic incompleteness on the (international) incomplete markets wedge η_{t+1}, generalizing Lustig and Verdelhan (2019). Cross-border trade in Home and Foreign bonds imply:

$$\begin{aligned} \mathbb{E}_t \left[M_{t+1} e^{-\eta_{t+1}} \right] &= \mathbb{E}_t \left[M^i_{t+1} \right], \\ \mathbb{E}_t \left[M^i_{t+1} \frac{M^*_{t+1}}{M_{t+1}} e^{\eta_{t+1}} \right] &= \mathbb{E}_t \left[M^*_{t+1} \right] \end{aligned}$$

[7] Our approach in this section relates to the wedge accounting methodology of Jiang et al. (2023). However, our micro-foundations in later sections allow us to both test the specific mechanism and define the equilibrium as an outcome of an economic model with standard features of international macroeconomics.

The resulting restrictions are given by:

$$\begin{aligned} \mathbb{E}[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) \quad &- \quad cov_t(m_{t+1}, \eta_{t+1}) \\ &- \quad \log \mathbb{E}[e^{y_{t+1}}] - cov_t(m_{t+1}, y_{t+1}) \end{aligned} \tag{14}$$

$$\begin{aligned} -\mathbb{E}[\eta_{t+1}] = \frac{1}{2} var_t(\eta_{t+1}) \quad &+ \quad cov_t(m^*_{t+1}, \eta_{t+1}) \\ &+ \quad \log \mathbb{E}[e^{y_{t+1}}] + cov_t(m^*_{t+1}, y_{t+1}) + cov_t(\eta_{t+1}, y_{t+1}) \end{aligned} \tag{15}$$

The incomplete markets wedge, or equivalently the non-traded component of exchange rates, must now compensate Home investors for both the aggregate and idiosyncratic volatility, as-well as their risk, i.e. the covariance of valuations and returns.

Proposition 2 (Two Int'l Traded Assets, Many Agents).

The two-country model with two internationally traded bonds and a continuum of heterogeneous Home consumers characterized by Equations (3), (4) and (12) evaluated for assets with payoffs $\tilde{R}_{t+1} = \{R_t, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} R^*_t\}$ *delivers* $cov_t(m^*_{t+1} - \log(\int_i e^{\Delta c^i_t} di)^{-s}, \Delta e_{t+1}) < 0$ *if and only if*

$$1 \geq -\rho_{\tilde{y}_{t+1}, \Delta e_{t+1}} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{y}_{t+1})} \tag{16}$$

where $\rho_{\tilde{y}_{t+1}, \Delta e_{t+1}} \equiv \dfrac{cov_t(\tilde{y}_{t+1}, \Delta e_{t+1})}{\sigma_t(\Delta e_{t+1}) \sigma_t(\tilde{y}_{t+1})}$.

Proof. See Appendix A.2. □

Whereas in a representative agent economy, exchange rate risk becomes traded (is spanned) when households trade in both Home and Foreign real bonds across borders, introducing idiosyncratic consumption risk —which co-moves with the exchange rate— recovers a non-traded component and can deliver negative cyclicality of exchange rates.[8] If variance of the idiosyncratic consumption risk falls with depreciations, that is domestic consumers face lower variance of permanent income shocks at times

[8] Note that we adjust the risk-sharing measure to consider the co-movement of aggregate consumption and exchange rates, in SDF space, consistent with the original definition of the Backus-Smith covariance.

when foreign bonds yield high returns, Foreign bonds are a poor hedge for domestic consumption. This idiosyncratic risk factor allows a *disconnect* of exchange rates with domestic aggregate consumption growth. When the (negative) comovement of idiosyncratic risk factor with exchange rates is sufficiently large, then the model can accommodate a negative Backus-Smith correlation even with trade in two nominally risk-free bonds. The poor hedging property of Foreign bonds arises in this framework due to a particular comovement of the idiosyncratic risk factor with exchange rates, and is consistent with a solution to *diversification puzzle* offered by Heathcote and Perri (2013).

Note that this reconciliation for Backus Smith puzzle in the aggregate, is sensitive to the number of traded assets. In particular, the necessary requirement to obtain low correlation is relaxed as the corollary below details the case of when only the Foreign bond is internationally traded.

Corollary 2 (One Int'l Traded Asset, Many Agents).
When only Foreign bonds are internationally traded and there exist a continuum of heterogeneous Home consumers such that Equations (3)*, and* (12) *evaluated for assets with returns* $\tilde{R}_{t+1} = \{R_t, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} R_t^*\}$ *hold, then* $cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$cov_t(m_{t+1}, \eta_{t+1}) + \log \mathbb{E}_t[e^{\eta_{t+1}}] \geq \quad var(m_{t+1}^* - m_{t+1}) - \log \mathbb{E}_t\left[e^{\tilde{y}_{t+1}}\right] - \; cov_t(\tilde{y}_{t+1}, m_{t+1}^* + \eta_{t+1})$$

Proof. Follows from the Proof of Proposition 1 and and equations (14) and (15). □

The model requires the non-traded component of exchange rates to be sufficiently safe, generalizing Proposition 1 to an environment with idiosyncratic risk. Indeed, in contrast to the case with two internationally traded risk free assets, the condition is relaxed by the variance of the idiosyncratic consumption risk faced by the agents – without relying on the covariance term (i.e. focusing on the case where idiosyncratic and aggregate risk are uncorrelated).

2.3. Risky Assets

In practice, few assets traded across borders are risk-free in real terms, e.g. long-maturity bonds and equity. We next investigate the implications of trading in multiple risky assets, when (nominally) risk-free bonds are not available.

Proposition 3 (Many Assets, Heterogeneous Consumers No-Arbitrage)
When Home and Foreign currency risky assets with returns $\tilde{r}^{(*)}_{t+1}$ *are internationally traded, then* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if:*

$$-\rho_{\tilde{y}_{t+1},\Delta e_{t+1}} + \frac{cov_t(\eta_{t+1} + \tilde{y}_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1})}{\sigma_t(\tilde{y}_{t+1})\sigma_t(\Delta e_{t+1})} \geq \frac{\sigma_t(\Delta e_{t+1})}{\sigma_t(\tilde{y}_{t+1})} \tag{17}$$

Proof. See Appendix A.2. □

When only risky assets are traded, exchange rates need not be spanned, allowing further scope for the cyclicality of exchange rates to be negative. In particular, the negative cyclicality is recovered if the incomplete markets wedge internationally or idiosyncratic risk domestically co-vary positively with the return from a risky foreign asset. Notably, the new (second) term in the Proposition is non-zero only if international financial markets are also incomplete.

If, however, we additionally allow for trade in two nominally risk-free assets the new term in Proposition 3 ($cov_t(\eta_{t+1} + \tilde{y}_{t+1}, \tilde{r}^{k(*)}_{t+1} - \tilde{r}^k_{t+1})$) converges to 0, as stated in the corollary below. This implies that Proposition 2 is the necessary and sufficient condition even when there is trade in many assets. Nonetheless, $\rho_{\tilde{y}_{t+1},\Delta e_{t+1}}$ and $\sigma_t(\Delta e_{t+1})$ are themselves changing as the number of traded assets changes.

Corollary 3 (Many Assets, Many Agents) *Condition* (16) *is the necessary and sufficient condition for* $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$*, even when there is additional trade in arbitrary risky assets.*

2.4. A Closed Form Incomplete Market Wedge

We next consider a two-country model with incomplete international markets where agents have power utility over an exogenous consumption process as in Lustig and Verdelhan (2019), which is closely related to a two-country version of Cox, Ingersoll and Ross (1985). Our framework builds on two further critical ingredients. First, we extend this framework to allow for countercyclical idiosyncratic risk as in Constantinides and Duffie (1996) and Herskovic, Kelly, Lustig and Van Nieuwerburgh (2016*b*) amongst others. Second, we allow for domestic and foreign consumption to be driven by a common factor (see e.g. Brandt, Cochrane and Santa-Clara, 2006).

Suppose that a common factor z_t drives aggregate consumption globally:[9]

$$z_{t+1} = \rho z_t + u_{t+1}$$

Aggregate consumption in the Home country is given by $\Delta c_{t+1} = \sqrt{z_t} u_{t+1}$ while consumption abroad is given by $\Delta c^*_{t+1} = \xi^* \sqrt{z_t} u_{t+1}$, such that $\xi^* > 1$ corresponds to an environment where the shock u_t has a relatively large effect on the Foreign aggregate consumption. The Home and Foreign SDFs defined on aggregate consumption are given by:

$$m_{t+1} = \log \beta - \gamma \Delta c_{t+1} = -\gamma \sqrt{z_t} u_{t+1}, \tag{18}$$

$$m^*_{t+1} = \log \beta - \gamma \xi^* \Delta c_{t+1} = \log \beta - \gamma^* \sqrt{z_t} u_{t+1} \tag{19}$$

where we define γ^* as the product of Foreign risk aversion and ξ^*. The volatility of aggregate consumption is given by z_t and $var_t(m^{(*)}_{t+1}) = \gamma^{(*)2} z_t$. The model above corresponds to the canonical representative agent Lucas model. In the presence of uninsurable idiosyncratic risk, the relevant pricing kernel is instead characterised by (12) and additionally depends on the log of the variance of the idiosyncratic permanent

[9] We focus on the simplest model which can rationalize the results. Common factors are necessary to match comovement of aggregates but also interest rates across countries. Allowing for additional country-specific factors reduces the correlation between SDFs to below 1. However, where we focus on conditional moments, introducing additional linearly independent factors does affect the results.

shocks $\tilde{y}_{t+1}$, for which we specify a linear process:

$$\tilde{y}_{t+1} = \tilde{\alpha}_y + \tilde{\phi}\sqrt{z_t}u_{t+1}, \tag{20}$$

where $\tilde{\phi} = \phi\frac{\gamma(\gamma+1)}{2}$ captures the cyclicality of idiosyncratic risk. Existing research focuses on $\phi < 0$,i.e. countercyclical risk.

Proposition 4 (Equilibrium Incomplete Markets Wedge)

In the model with heterogeneous consumers, satisfying (3), (4), (12) evaluated for $\tilde{R}_{t+1} = \{R_t, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}R_t^*\}$ *(7), the incomplete markets wedge is given by:*[10]

$$\eta_{t+1} = -\frac{1}{2}\left[(\gamma - \gamma^* - \phi)^2 - \kappa\right] + \left[-\phi - \sqrt{(\phi - \gamma - \gamma^*)^2 - \lambda}\right]\sqrt{z_t}u_{t+1} - \left[\sqrt{\lambda - \kappa}\right]\sqrt{z_t}\epsilon_{t+1}, \tag{21}$$

where $\lambda \geq \kappa$ is a parameter governing cross border spanning and

$$\kappa = (\gamma - \gamma^* - \phi)^2 - (\gamma - \gamma^* - \phi)\sqrt{(\gamma - \gamma^* - \phi)^2 - \lambda} \geq 0 \tag{22}$$

where κz_t is the exchange rate volatility.

Proof. See Appendix A.2. □

The benchmark of internationally complete markets is not as obvious when there is uninsurable risk. It turns out this special case obtains when $\eta_{t+1}^{CM} = -\tilde{y}_{t+1} = -\phi\sqrt{z_t}u_{t+1}$, i.e. when exchange rates move to reflect differences in the as-if representative agent SDFs, such that $\lambda^{CM} = \kappa^{CM} = (\gamma - \gamma^* - \phi)^2$. Armed with a closed form expression for the international incomplete markets wedge (21), we can revisit the condition which determines the cyclicality of exchange rates.

[10] Square roots refer to the positive root only. While the negative root is a potential solution, conditional on $\phi < 0$, it violates $\lambda > \kappa$ which we require for a real solution.

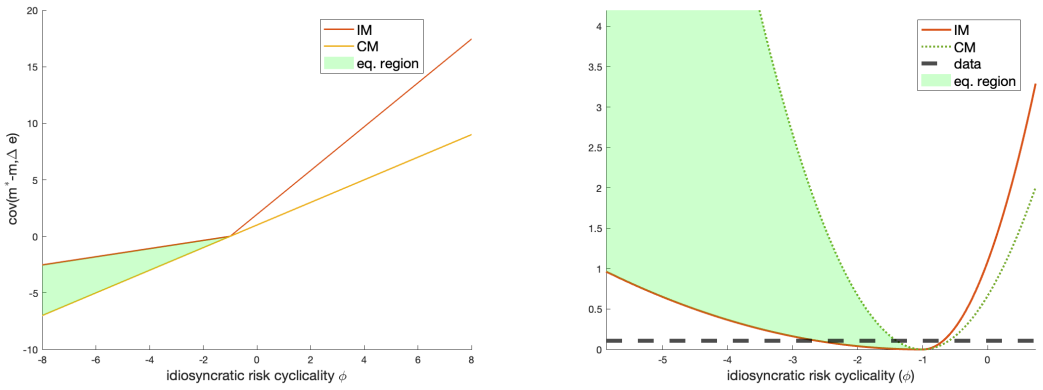
Proposition 5 (Exchange rate cyclicality in Cox et al. (1985) model) *The model with heterogeneous consumers, satisfying* (3),(4), (12) *evaluated for* $\tilde{R}_{t+1} = \{R_t, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} R_t^*\}$, *exchange rate process* (7) *and and incomplete markets wedge* (21), *delivers* $cov_t(m_{t+1}^* - m_{t+1}, \Delta e_{t+1}) < 0$ *if and only if* $var(r_{t+1}) < var(r_{t+1}^*)$.

Proof. See Appendix A.2. □

More specifically, a negative aggregate Backus-Smith correlation obtains when $\phi < \gamma - \gamma^* < 0$, but the advantage of expressing the result in terms of interest rates is that it continues to be true when we allow idiosyncratic risk abroad as well ($\phi^* > 0$).

Figure 1

(a) *Backus-Smith Covariance* **(b)** *Exchange Rate Volatility*



Shaded regions capture the combinations of ϕ and moments of exchange rates admissible under incomplete markets for $\gamma = 1, \xi^* = 2$. We use moments of the z_t process assumed in Lustig and Verdelhan (2019). Solid yellow line reflects complete markets equilibrium. Only the negative root is plotted because the positive root violates $\lambda \geq \kappa$ in the region of interest.

Figure 1 illustrates the results. The shaded area indicates the admissible equilibria under incomplete markets which satisfy (16), where the required restrictions $var(\Delta e_{t+1}) > 0$ and $\lambda > \kappa$ are additionally satisfied. In particular, the green area traces values of λ since as $\lambda \to \kappa$, markets becomes complete.[11] The left panel illustrates that varying ϕ has a direct impact on the cyclicality of exchange rates, *regardless* of the degree of international market incompleteness. The right panel, in

[11] If the negative root associated with (21) is chosen, then low aggregate risk sharing can also be attained for $\phi > \gamma - \gamma^* > 0$, but this does not appear to be the relevant parameter region.

turn, shows both ϕ and the level of market completeness matter for exchange rate volatility.

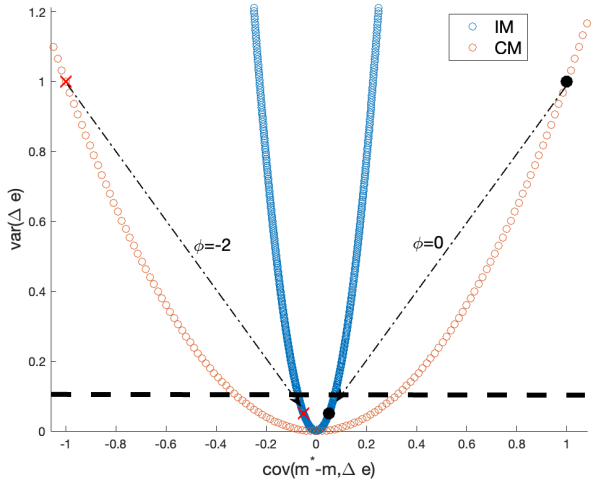
The Role for Market Incompleteness Brandt et al. (2006) claim that if aggregate risk-sharing is low, the observed exchange rates are too smooth. Put differently, in the region where exchange rates are pro-cyclical (in line with the data), canonical models tend to struggle to reconcile the low exchange rate volatility (relative to e.g. stock prices) observed in the data. In the proposition below, we show that introducing idiosyncratic risk can recover pro-cyclical exchange rates consistent with low risk-sharing, but only at the cost of further increasing exchange rate volatility. Plausible levels of exchange rate volatility are around 0.1 and are marked on the right panel of Figure 1 therefore the region of ϕ admissible under complete markets is very small. In contrast, this region can be much larger when international financial markets are incomplete markets, as illustrated in the Proposition below.

Proposition 6 (Idiosyncratic Risk and Exchange Rate Volatility)
If income risk is counter-cyclical ($\phi < 0$)*, exchange rate volatility* (22) *is always higher under international complete markets* $\kappa^{CM} > \kappa^{IM}$.

Proof. Consider (22) for $\phi < 0$ and allow $\lambda > \kappa$ and denote this as κ^{IM}. Imposing complete markets $\lambda^{CM} = (\gamma - \gamma^* - \phi)^2$ and defining this as κ^{CM} confirms the result. □

Taken together, our results imply that while varying the cyclicality of idiosyncratic risk (which requires domestic market incompleteness) allows us to correct the cyclicality of exchange rates, international market incompleteness is critical to correct the volatility of exchange rates. Enriching models in both directions allows more flexibility to match the data – a low exchange rate volatility and a cyclicality that is close to zero but negative in SDF space. Figure 2 show the tight link between volatility and cyclicality and illustrates the limit where there is acyclical idiosyncratic risk (which

Figure 2: *Uninsurable Idiosyncratic Risk vs. Incomplete International Spanning*



Locus of admissible variance and cyclicality of exchange rates in equilibrium. Circles denote different value of idiosyncratic risk cyclicality ϕ. Arrows denote equilibria as international spanning varies holding ϕ.

coincides with the representative agent limit), studied in Chernov et al. (2024). When idiosyncratic risk is not priced, incomplete international spanning can at best deliver a zero covariance and only as exchange rate volatility approaches zero. Instead, as ϕ becomes negative and large, if we allow international markets to be sufficiently incomplete, the model is able to match the data.

3. Equilibrium in a Two-Country Two-Good Economy

Finally, we specify a parsimonious two-country, two-good model with incomplete domestic markets where the aggregate Euler equation (12) described above can be supported as an equilibrium outcome. The model extends the closed economy framework of Constantinides and Duffie (1996) to an international setting. There is a continuum of heterogeneous consumers i in the Home economy trading in *fully integrated* asset markets.

Each agent i derives per-period utility from consumption:

$$u(C_t^i) = \beta(C_{t-1})\frac{C_t^{i\ 1-s}}{1-s} \tag{23}$$

where $\beta(C_{t-1}) = \omega C_{t-1}^{-u}$, with $\omega \in (0,1)$, is the discount factor used as a stationarity-inducing device following Bodenstein (2011), C_t is the aggregate consumption bundle at Home, and the individual i's consumption bundle is given by:[12]

$$C_t^i = \left[\alpha^{\frac{1}{\phi}} C_{H,t}^{i\ \frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_{F,t}^{i\ \frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{24}$$

where ϕ is the trade elasticity and α is the measure of home-bias. The corresponding price index is given by:

$$P_t = \left[\alpha P_{H,t}^{\frac{\phi-1}{\phi}} + (1-\alpha) P_{F,t}^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \tag{25}$$

C^* and P^* are defined symmetrically. The real exchange rate is given by $\mathcal{E} = P^*/P$. Individual i's budget constraint at Home is given by:

$$C_t^i - I_t^i \leq R_t B_{t-1}^i - B_t + \mathcal{E}_t R_t^* B_t^{i*} - \mathcal{E}_t B_{t-1}^{i*} - \int_{j\in J} X_{t-1}^{i,j} + \int_{j\in J} \tilde{R}^j X_t^{i,j} \tag{26}$$

where I_t^i is the income drawn by agent i and we define $\frac{P_{H,t}}{P_t} I_{H,t}^i + \frac{P_{F,t}}{P_t} I_{F,t}^i = I_t^i$, $B_t^{i(*)}$ is the position in Home (Foreign) risk-free bonds and $R_t^{(*)}$ is the corresponding returns and $X^{i,j}$ denote risky assets indexed by j held by agent i. Foreign agents face an analogous maximization, but we assume there is no heterogeneity in the Foreign country for expositional purpose. Finally, goods market clearing requires:

$$\int_i C_{H,t}^i di + C_{H,t}^* = \int_i I_{H,t}^i di + I_{H,t}^*; \quad \int_i C_{F,t}^i di + C_{F,t}^* = \int_i I_{F,t}^i di + I_{F,t}^* \tag{27}$$

where the aggregate endowment processes are given by $I_{H,t}^{(*)} = \rho I_{H,t-1}^{(*)} + (1-\rho) I_H^{(*)} + \epsilon_{H,t}^{(*)}$, $I_{F,t}^{(*)} = \rho I_{F,t-1}^{(*)} + (1-\rho) I_F^{(*)} + \epsilon_{F,t}^{(*)}$ and $\epsilon_{H,t}^{(*)}$ and $\epsilon_{F,t}^{(*)}$ are iid $N(0, \sigma_\epsilon)$ endowment shocks.

No-arbitrage pricing requires that the Euler equations for household i investing in

[12] The endogenous discount factor is not important for our results but is a device to induce a stationary equilibrium in this model.

an arbitrary security with return $\tilde{R}_{t+1}$:

$$\mathbb{E}_t\left[\beta\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-s}\tilde{R}_{t+1}\right]=1 \tag{28}$$

We allow individual consumption draws to be related to the aggregate consumption draw and follows a heteroskedastic log-normal process:

$$\Delta c^i_{t+1}=\log\left(\frac{\delta^i_{t+1}}{\delta^i_t}\frac{C_{t+1}}{C_t}\right)\sim\mathcal{N}(\mu_{c^i_t},\sigma^2_{i,t}) \tag{29}$$

where $\frac{C_{t+1}}{C_t}\equiv\exp\left(\Delta c_{t+1}\right)$ denotes aggregate consumption growth and is drawn from a lognormal distribution given by $\Delta c^{(*)}_{t+1}\sim\mathcal{N}\left(\mu_{C^{(*)}_t},\sigma^2_{C^{(*)}_t}\right)$, where $\mu_{C^{(*)}_t}$ and $\sigma^2_{C^{(*)}_t}$ are conditional mean and variance of this process, and the asterisks denote the corresponding aggregate consumption process for the Foreign economy. The individual consumption exposures satisfy the law of large numbers and are themselves log-normally distributed, i.e. $\delta^i_{t+1}/\delta^i_t$ satisfies $\int_i\delta^i_t\,di=1,\quad\log\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)\sim\mathcal{N}(\mu_{\delta_t},\sigma^2_{\delta_t}),\ \forall t$.[13,14]

Substituting in (29), the individual Euler equation can be written as:

$$\mathbb{E}_t\left[\beta\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)^{-s}\left(\frac{C_{t+1}}{C_t}\right)^{-s}\tilde{R}_{t+1}\right]=1 \tag{30}$$

To arrive at an aggregate Euler equation, we adopt the following process for δ^i_t:

$$\frac{\delta^i_{t+1}}{\delta^i_t}=\exp(\xi^i_{t+1}\sqrt{y}_{t+1}-y_{t+1}/2) \tag{31}$$

where ξ^i_{t+1} are the uninsurable idiosyncratic shocks which are distributed as standard normal for all i and t, independently distributed from the aggregate state y_{t+1}.[15] This process satisfies (3) implying $\log(\delta^i_{t+1}/\delta^i_t)\sim\mathcal{N}(-\frac{y_{t+1}}{2},y_{t+1})$.

The individual Euler equations can be aggregated to condition (12) used in Section

[13]We repeatedly use the result that if $X\sim\mathcal{N}(\mu_X,\sigma_X)$, $e^X\sim\log\mathcal{N}(\mu_X,\sigma_X)$ and if $Y\sim\mathcal{N}(\mu_Y,\sigma_Y)$, then $Z=XY\sim\mathcal{N}(\mu_X+\mu_Y,\sigma_X+\sigma_Y+2\sigma_{XY})$. From this it follows that $e^Z\sim\log\mathcal{N}(\mu_X+\mu_Y,\sigma_X+\sigma_Y+2\sigma_{XY})$.

[14]Together, Equations (29) and (3) thus imply $\mu_{c^i_t}=\mu^\delta_t+\mu_{c^i_t}$ and $\sigma^2_{i,t}=\sigma^2_{\delta,t}+\sigma^2_{C,t}+2\sigma_{\delta_t,C_t}$, where $\sigma^2_{\delta,t}\equiv var_t\left(\log\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)\right)$, and $\sigma_{\delta_t,C_t}\equiv cov_t\left(\log\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right),\log\left(\frac{C_{t+1}}{C_t}\right)\right)$.

[15]The law of large numbers follows from properties of the normal distribution for z^i. Treating y_{t+1} as a constant, and using the moment generating function $M_z(h)=e^{hz}$ for $h\in\mathbb{R}$, $\mathbb{E}[e^{z^i\sqrt{y}-y/2}]=M_z(x)e^{-y/2}=e^0=1$.

(2.2), using the consumption share process (31), as in Constantinides and Duffie (1996). Two key results follow. First, the aggregate Euler equality fails, i.e. Equation (12) features a risk premium over aggregate consumption growth. Second, as a result of the specification of δ^i, the Euler for each individual household is independent of i, indicating all households agree on a common valuation. Assuming individual endowments are given by $I_t^i = \delta_t^i C_t + D_t$, where D_t denotes the aggregate dividend in the economy, conditional on a process for aggregate consumption C_t, standard arguments from Constantinides and Duffie (1996) imply $C_t^i = I_t^i$ is an equilibrium in this economy for all i, t.

Following Constantinides and Duffie (1996), we have assumed all agents are unconstrained. Berger et al. (2023) building on Werning (2015) show that incomplete markets model with a general class of borrowing constraints and income processes can be aggregated to a similar Equation where the growth in consumption shares of the unconstrained agents represents the discount factor wedge, and they measure this wedge directly from the consumption expenditures data. In Section 4, we build on Storesletten, Telmer and Yaron (2004), Bayer, Luetticke, Pham-Dao and Tjaden (2019), and estimate the variance of permanent income process in the data.

3.1. International Equilibrium

We construct an equilibrium where each agent $i \in [0, 1]$ chooses not to trade in financial assets with agents in their own or the foreign country.[16] We first establish that if the Home country is in financial autarky, the individual SDF m_{t+1}^i is independent of i, so starting from an equilibrium where every household is the same, the no-trade allocation constitutes an equilibrium within each country. Then, taking any foreign endowment process as given, we select a process for aggregate domestic endowments such that agents would optimally choose not to trade across borders even when assets

[16] In general, heterogeneous agent models with endogenous portfolio choice are few in the literature. We favor our approach for its analytical tractability in characterizing the exchange rates.

are available. More generally, the endowment processes we consider can be interpreted as the wealth of consumers after all gains from trade have been exhausted.

The adjusted risk-sharing or Backus-Smith condition is then given by:

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s} e^{\frac{s(s+1)}{2}y_{t+1}}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-s}\right] \tag{32}$$

Using (32), replacing C_t with $Y_t = \int_i Y_t^i$ and C_t^* with $Y_t^* = \int_{i^*} Y_t^{i\,*}$ implies equalization of the autarky rates across countries resulting in no trade (Svensson, 1988).

3.2. Goods Market Clearing and Risk Sharing

The equilibrium we construct does not always lie on the static Pareto frontier (goods markets do not clear), so we support such an equilibrium with static goods market wedges.[17] While this may sound undesirable, such wedges are non-zero in the data and play a significant role in impeding risk sharing (Fitzgerald, 2012; Bodenstein, Cuba-Borda, Gornemann and Presno, 2024). Moreover, as illustrated in Section 2, and consistent with Lustig and Verdelhan (2019), such goods markets frictions alone are unable to explain the Backus Smith puzzle with trade in many assets.

Given a level of aggregate consumption C_t, households optimally choose their consumption bundle $\{c^i_{H,t}, c^i_{F,t}\}$, characterized by:[18]

$$\begin{aligned} C^*(C_t) = \max_{\{c^i_{H,t}, c^i_{F,t}\}} \Bigg\{ & g^*(c^{i\,*}_H, c^{i\,*}_F) \quad \text{s.t.} \quad \int_i c^i_H di + \int_{i^*} c^{i\,*}_H di^* = Y_H, \\ & \int_i c^i_F(1+\tau_t)di + \int_{i^*} c^{i*}_F di^* = Y_F \text{ and } g(c^i_H, c^i_F) \geq C^i_t \Bigg\}, \end{aligned} \tag{33}$$

where we define $g(c_H, c_F)$ by (24). The first order condition relates relative goods

[17] This is true of any consumption based asset pricing model. A contribution of our paper is that we specify the wedges needed to support equilibrium and discuss their impact on international risk-sharing.

[18] See Costinot, Lorenzoni and Werning (2014) for a friction-less representative agent framework, and Lloyd and Marin (2024) for a treatment with static wedges. Our generalization to many agents follows from the first order homogeneity of $g(\cdot)$.

consumption across the two countries:

$$\frac{c^i_{H,t}}{c^i_{F,t}} = \left(\frac{\alpha}{1-\alpha}\right)^2 (1+\tau_t)^{\phi} \frac{c^{i*}_{H,t}}{c^{i*}_{F,t}}, \tag{34}$$

The proposition below provides the sufficient condition for goods market clearing in our model.

Proposition 7 (Goods Market Clearing)

Given processes for $\{Y_t, Y_{H,t}, Y_{F,t}, \tau_t\}$ *and the Pareto Frontier* $\{c_H(C_t), c_F(C_t)\}$*, goods markets are efficient and clear if and only if:*

$$\left[Y_t^{\frac{\phi-1}{\phi}} - \alpha^{\frac{1}{\phi}} c_H(Y_t)^{\frac{\phi-1}{\phi}}\right]^{\frac{\phi}{\phi-1}} \left(\hat{b} Y_{H,t} + (1-\hat{b}) c_H(Y_t)\right) = c_H(Y_t) Y_{F,t} (1-\alpha)^{\frac{1}{\phi-1}} \tag{35}$$

where $\hat{b} = (\frac{\alpha}{1-\alpha})^2 (1+\tau_t)^{\phi}$ *and* $\tau_t > -1$.

Proof. Consider the static problem (33). Aggregating (34) over all agents i, combining with market clearing (27) and the (first order homogeneous) aggregator (24) delivers the result. □

To see that goods markets do not need to clear emphin a no-trade equilibrium if wedges are zero, consider (32). Taking $\hat{Y}^*_t = \int_{i^*} Y^{i\,*}_t$ as given, the process $\hat{Y}_t$ ensures a no-trade equilibrium. However, we cannot simultaneously choose $Y_{H,t}$, $Y_{F,t}$ and $\hat{Y}_t$ to satisfy (35), implying the need for wedges.

While this construction may appear undesirable, there are three key issues to flag. First, such static wedges are a salient feature on the data (see Fitzgerald, 2012; Bodenstein et al., 2024). Second, static wedges themselves, even if time-varying, can never lead to low risk-sharing when there is trade in at least two risk-free assets. Third, as the corollary below illustrates, a zero wedge is consistent with equilibrium in the limit of full home bias (note that a very high home bias value of $\alpha > 0.9$ is often estimated in the literature for the U.S., e.g. Colacito and Croce (2013)).

Corollary 2 (Full Home Bias)

As $\alpha \to 1$, static wedges are not needed for markets to clear, i.e. $\tau_t = 0 \; \forall t$ is consistent with (35).

Proof. Taking the limit of (35) implies both LHS and RHS converge to 0, meaning the condition is trivially satisfied.

3.3. Representative Agent Limit

To speak to classical contributions in the literature, consider the representative agent limit ($I^i = I$ for all i), such that $C_t^i = C_t, B_t^i = B_t, B_t^{i*} = B_t^*, X_t^{i,j} = X_t^j$. Lemma 1 describes the autarky limit for prices and allocations, attained at the limit of zero liquidity (Corsetti et al., 2008) and full home-bias limit (Itskhoki and Mukhin, 2021, 2023).

Lemma 1 (Autarky Limit).

In the autarky limit $\alpha \to 1$, $B, B^* \to 0$, the model is summarized by the following equations:

$$\begin{aligned} m_{t+1} &= -s g_{y_{H,t+1}}, \\ m_{t+1}^* &= -s g_{y_{F,t+1}}, \\ \Delta e_{t+1} &= \frac{1}{1-2(1-\phi)} (g_{y_{H,t+1}} - g_{y_{F,t+1}}), \\ \eta_{t+1} &= \left[\frac{1}{1-2(1-\phi)} - s \right] (g_{y_{H,t+1}} - g_{y_{F,t+1}}) \end{aligned}$$

where $g_{y_{t+1}} = y_{t+1} - y_t$. Then, $m_{t+1}, m_{t+1}^*, \eta_{t+1}$ and Δe_{t+1} are jointly normally distributed.

Proof. See Appendix B.3 □

Our approximation technique relies on taking the autarky limit for real quantities. There are two reasons why this limit is attractive. First, we prove the joint log normality of SDFs and the exchange rate at the full home-bias limit as $\alpha \to 1$, whereas this is not the case for $\alpha < 1$. Second, the allocations and prices are invariant to the

number of assets traded.[19] Then, whereas we calculate the covariance in the financial autarky case by direct computation, we back out the implied covariance at the autarky limit, denoted $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$, between SDFs and depreciations when there is trade in assets by imposing $cov_t(m^{(*)}_{t+1}, \eta_{t+1})$ consistent with (8)-(9).

Proposition 3 (Representative agent, Autarky Limit).
The two-country model at the autarky limit can deliver $cov_t^{\to FA}(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) \leq 0$ *conditional on shocks to* $y_{H,t}$ *if* $\frac{-s}{1-2(1-\phi)} \leq 0$ *when at most one asset is traded internationally. With two risk-free assets (and potentially also risky assets) are internationally traded, the model requires* $\phi \to \infty$ *which implies* $var_t(\Delta e_{t+1}) \to 0$.

Proof. See Appendix A.2. □

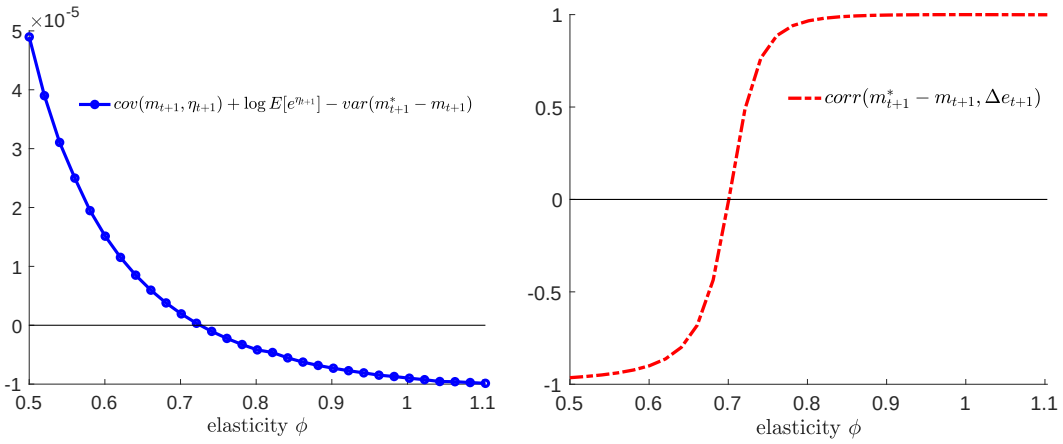
When the trade elasticity is sufficiently low, the non-traded component of exchange rates is risky ($cov(m_{t+1}, \eta_{t+1})) < 0$ and following an increase in Home productivity, demand for Home goods rises so much, that prices must adjust to constrain Foreign consumption of the Home good for markets to clear, as in Corsetti et al. (2008). Under autarky the ability of the model to match risk sharing depends only on ϕ given that $s > 0$. When there is trade in a risk-free asset, condition (10) additionally depends on the inter-temporal elasticity of substitution but this can never have an effect on whether the inequality is satisfied. With trade in two nominally risk-free bonds, a zero covariance between relative SDFs and the exchange rate arises in the limit where $var_t(\Delta e_{t+1})$ approaches zero, but a negative covariance can never be achieved.

Figure 3a illustrates Proposition 1 away from the autarky limit, i.e. $\alpha < 1$. Specifically, condition (10) is satisfied for values over which the plotted line is positive. Figure 3b shows that the sign on the simulated Backus-Smith correlation tracks condition (10). The correlation is negative for low values of ϕ, positive for higher

[19] While the invariance and the log-normality properties also hold at the complete markets allocation or Cole and Obstfeld (1991) limit, these cannot by construction break the Backus-Smith condition.

Figure 3

(a) *Prop. 1 Statistic* **(b)** *Backus-Smith Correlation*



Calibration: $\omega = 0.96, u = 0.005, s = 1, \phi \in [0.5, 1.1], \alpha = 0.6, \alpha^* = 1 - \alpha, \rho = 0.96, Y_H = 1$. Unconditional moments calculated from second-order simulation with one million draws.

values and intersects 0 at approximately the same value for ϕ as Panel (a), even though, away from the autarky limit, the SDFs and exchange rates in the equilibrium model are not exactly log-normal.

4. How much heterogeneity?

In the heterogeneous consumers model (Constantinides and Duffie, 1996), the wedge in the aggregate Euler equation simplifies to the conditional volatility of the permanent idiosyncratic risk.

A direct measurement for the necessary and sufficient conditions laid out in Proposition 4 is that this wedge co-moves negatively with the depreciation of the domestic currency. We consider a proxy of this wedge from the U.S. data proposed in the literature to measure the empirical co-movement. We use the structural estimate of variance of permanent income risk by Bayer et al. (2019, BLPT henceforth) who estimate a quarterly series, using quasi-maximum likelihood methods with Survey of Income and Program Participants (SIPP) panel data spanning 1979Q1 to 2013Q1,

extending Storesletten et al. (2004)'s method of moments estimator for income risk to quasi-maximum likelihood methods.[20] Under a no-trade equilibrium where the households consume their labor income, this estimate proxies for the variance of the consumption risk.

The income volatility series is counter-cyclical. The correlations of the income volatility with de-trended U.S. output and consumption growth are -0.22 and -0.24 respectively.[21]

We take the bilateral real exchange rate of U.S. dollar against seven advanced economies: Canada, France, Germany, Italy, Japan, Spain, and the U.K. We use the end of quarter nominal exchange rates and price indices to construct the quarterly series. An increase in the bilateral exchange rate index denotes a depreciation of the U.S. dollar.

Proposition 4 requires measuring conditional standard deviations of the real exchange rate growth and the variance of income shock process, and conditional correlation of real exchange rate growth with the income shock process. We measure unconditional moments for these variables over 1979Q1-2013Q1. The standard deviations of quarterly real exchange rate averaged across the seven bilateral pairs is 0.11. The standard deviation of the volatility of income shocks in the U.S. is 0.44. Substituting these moments into the right hand side of the inequality (16) in Proposition 2 and using a $s = 10$ Best, Cloyne, Ilzetzki and Kleven (2020), the threshold correlation required to satisfy Proposition 2 is -0.025.

Table 1 reports the correlation of U.S. income shock volatility with bilateral exchange rate against the Backus Smith correlation over 1979Q1-2013Q1.[22] The correlation of income risk with bilateral exchange rate averaged across country pairs

[20] Bayer, Born and Luetticke (2024) use these income risk estimates to evaluate the joint dynamics of income inequality and business cycles in the U.S.

[21] Regression coefficient $\hat{\phi}$ from $y_{t+1} = \alpha + \phi(\text{ detrended output}_t) + \epsilon_t$ is -2.8, and when estimated on consumption growth is -8.

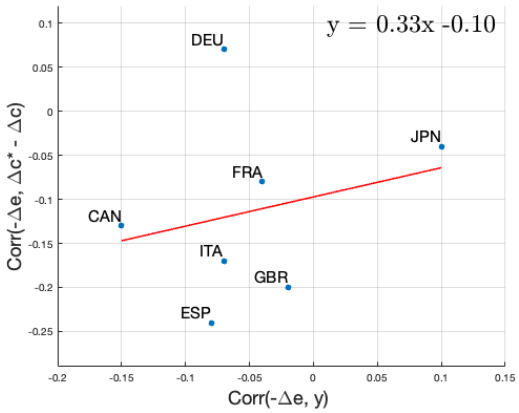
[22] We also check the correlations over 1979Q1-2009Q3. Engel (2016) suggests stopping the measurement of exchange rate moments in 2009Q3. We find similar results over this alternate sample.

Table 1: *Correlation Coefficients for various countries (1979Q1–2013Q1)*

	CAN	DEU	ESP	FRA	GBR	ITA	JPN	Avg
Corr$(-\Delta e,\, y)$	-0.15	-0.07	-0.08	-0.04	-0.02	-0.07	0.10	-0.05
Corr$(-\Delta e,\, \Delta c^* - \Delta c)$	-0.13	0.07	-0.24	-0.08	-0.20	-0.17	-0.04	-0.11

Note: Δe is USD depreciation relative to Foreign currency. Sample: 1979Q1–2013Q1 based on availability of variance of income risk data from BLPT.

Figure 4: *Backus-Smith Correlation against the US Income Shock Volatility Correlation with Exchange Rate Growth*



Note: The figure plots the correlations reported in Table 1.

is around -0.05, a substantially higher number than required by the inequality (16) in Proposition 2.

There is considerable variation in the comovement of U.S. income shock volatility with bilateral exchange rate. Scatter plot in Figure 4 plots the two rows in Table 1. Backus Smith correlation is on the y-axis, and the correlation of income shock volatility in the U.S. with the bilateral real exchange rate growth is on the x-axis. The scatter plot provides a suggestive evidence for the predictions of the heterogeneous consumers model. Namely, more negative is the correlation of income risk with USD depreciation, more negative is the correlation of relative consumption growth with USD depreciation.

While we find evidence supporting the ability of heterogeneous consumers model to

reconcile the Backus-Smith puzzle, our exercise is limited in at least three ways. First, we do not offer a quantitative comparison of the domestic market incompleteness mechanism with a segmentation/intermediation based explanations such as Itskhoki and Mukhin (2021). We view our mechanism as complementary to theirs. Second, we do not have data on the idiosyncratic income shock volatility of other countries. Canada is likely to be similar to the U.S., but countries such as Japan may differ substantially. Third, there is likely a role for recursive preferences Colacito and Croce (2013) or non-separabilities in driving the exchange rate. We leave these explorations to future work.

5. Conclusion

A classical strand of the literature in international macroeconomics has focused on formulating goods-market mechanisms which generate a negative relationship between consumption growth and depreciation– the opposite sign to that implied by the Backus-Smith condition– as long as financial markets are incomplete. We show that any model which achieves this resolution must rely on a non-traded component to relative prices which is "safe" from a domestic investor perspective. However, Lustig and Verdelhan (2019) determine that any two-country model with a representative agent and frictionless trade in Home and Foreign currency denominated risk-free bonds recovers the exchange rate cyclicality implied by complete markets.

We propose a generalization of the model beyond the representative agent no-arbitrage benchmark to a model of heterogeneous consumers trading in integrated markets but facing uninsurable idiosyncratic risk. Introducing domestic heterogeneity generates a non-traded component to exchange rate movements, resulting in low risk-sharing even with cross-border trade in both Home and Foreign bonds. Using estimates of idiosyncratic income risk from the data, domestic market incompleteness is plausibly large to reconcile patterns of cross-country risk sharing.

We consider a consumption based asset pricing model with heteroskedastic aggregate consumption and heterogeneous consumers facing uninsurable idiosyncratic risk, extending Lucas (1982) and Cox et al. (1985) and derive the incomplete markets wedge in closed form. We show that the cyclicality of within-country idiosyncratic risk is critical to reconcile the Backus-Smith correlation. At the same time, internationally incomplete markets are necessary to prevent unrealistically high exchange rate volatility from a model that can reconcile volatile asset prices domestically.

Our paper highlights the need to extend a workhorse international macroeconomic model to allow not only for heterogeneous agents and specifically account for portfolio choice with both idiosyncratic and aggregate risk. This is a challenging endeavour that goes beyond most of the existing literature, see e.g. Auclert, Rognlie, Straub and Ťapák (2024), which either abstracts from heterogeneity (Benigno and Küçük, 2012; Heathcote and Perri, 2013) or delegates the portfolio choice problem to a mutual fund (Auclert et al., 2021).

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A. Appendix

A.1. Additional Derivations for Section 2.

To find the admissible set of processes, consider the log expansions of the above conditions, assuming joint log normality:

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) = -r_{t+1}, \quad (36)$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) = -r^*_{t+1} \quad (37)$$

$$\mathbb{E}_t[m^*_{t+1}] + \frac{1}{2}var_t(m^*_{t+1}) - \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m^*_{t+1}, -\Delta e_{t+1}) = -r_{t+1}, \quad (38)$$

$$\mathbb{E}_t[m_{t+1}] + \frac{1}{2}var_t(m_{t+1}) + \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1}, \Delta e_{t+1}) = -r^*_{t+1}, \quad (39)$$

where lower case levels denote logs, e.g. $\log(M_{t+1}) = m_{t+1}$ and $\Delta e_{t+1} = e_{t+1} - e_t$. Using (36) and (39), and (37) and (38) respectively, yields:

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = -cov_t(m_{t+1}, \Delta e_{t+1}) - \frac{1}{2}var_t(\Delta e_{t+1}), \quad (40)$$

$$\mathbb{E}_t[\Delta e_{t+1}] + r^*_{t+1} - r_{t+1} = cov_t(m^*_{t+1}, -\Delta e_{t+1}) + \frac{1}{2}var_t(\Delta e_{t+1}) \quad (41)$$

A.2. Proofs to Propositions

Proof to Proposition 1 The Backus-Smith condition is related to the covariance $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1})$ which can be rewritten as:

$$cov_t(m^*_{t+1} - m_{t+1}, m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \quad (42)$$

$$= var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1}, \eta_{t+1}) - cov_t(m_{t+1}, \eta_{t+1}) \quad (43)$$

Imposing (9) (international trade in the Foreign asset), but not (8) (international trade in the Home asset) as is done in Lustig and Verdelhan (2019), assuming $\mathbb{E}_t[\eta_{t+1}] = 0$, and rearranging yields the result. □

Proof to Corollary 1 The volatility of the exchange rate is given by:

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) + var_t(\eta_{t+1}) + 2cov_t(m^*_{t+1}, \eta_{t+1}) - 2cov_t(m_{t+1}, \eta_{t+1})$$

Imposing (8) and (9):

$$var_t(\Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) - var_t(\eta_{t+1})$$

Taking the limit $cov_t(m_{t+1}, \eta_{t+1}) \to$ (10) would imply $var_t(\Delta e_{t+1}) < 0$ which cannot be an equilibrium. □

Proof to Proposition 2. Taking a log expansion of (12) for the case of the Home households investing in the Home bond ($\tilde{R}_t = R_t$) can be written as:

$$-r_t = \mathbb{E}_t[m_{t+1}] + \frac{1}{2}[var_t(m_{t+1}) + var_t(\tilde{y}_{t+1}) + 2cov_t(m_{t+1}, \tilde{y}_{t+1})] \tag{44}$$

Turning to the Euler for a Home household i investing in Foreign bonds ($\tilde{R}_t = R^*_t \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}$):

$$\begin{aligned} -r^*_t = \mathbb{E}_t[m_{t+1}] + \frac{1}{2}[var_t(m_{t+1}) + var_t(\tilde{y}_{t+1}) + 2cov_t(m_{t+1}, \tilde{y}_{t+1})] + \\ \mathbb{E}_t[\Delta e_{t+1}] + \frac{1}{2}var_t(\Delta e_{t+1}) + cov_t(m_{t+1} + \tilde{y}_{t+1}, \Delta e_{t+1}) \end{aligned} \tag{45}$$

Combining with (3)-(4), and taking log expansions:

$$var_t(\eta_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) - var_t(\Delta e_{t+1}) - 2cov(\tilde{y}_{t+1}, \Delta e_{t+1}) \tag{46}$$

Then, using $e_{t+1} = m^*_{t+1} - m_{t+1} + \eta_{t+1}$:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(m^*_{t+1} - m_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) \tag{47}$$

In turn, combining (14)-(15) yields

$$0 = var(\eta_{t+1}) + cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1}) + cov_t(\Delta e_{t+1}, \tilde{y}_{t+1})$$

Combining by substituting $cov_t(m^*_{t+1} - m_{t+1}, \eta_{t+1})$ in (47) yields:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var(m^*_{t+1} - m_{t+1}) - var_t(\eta_{t+1}) - cov_t(\Delta e_{t+1}, \tilde{y}_{t+1})$$

Using (46), this implies:

$$cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) = var_t(\Delta e_{t+1}) + cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}).$$

Then, $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ requires $-cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}) > var_t(\Delta e_{t+1})$. Finally, the Cauchy-Schwarz identity implies $|cov_t(\tilde{y}_{t+1}, \Delta e_{t+1})| \leq \sigma_t(\tilde{y}_{t+1})\sigma_t(\Delta e_{t+1})$. Dividing through by $\sigma_t(\tilde{y}_{t+1})\sigma_t(\Delta e_{t+1})$ yields the result.

Note also that we construct the correct measure of the Backus-Smith correlation, in SDF space, when there are heterogeneous agents. If log quantities are jointly normally distributed it follows that the log of the Backus-Smith covariance in this economy is equal to the covariance of the logs:[23]

$$\log\left(cov_t\left(\int_i e^{\Delta c^i_{t+1}} di, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right) = cov_t\left(\log \underbrace{\int_i e^{\Delta c^i_{t+1}} di}_{C_{t+1}}, \Delta e_{t+1}\right) \implies$$

$$\log\left(cov_t\left(\int_i M^i_{t+1} di, \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)\right) = cov_t\left(m_{t+1}, \Delta e_{t+1}\right) \tag{48}$$

□

Proof to Proposition 3 Combining (68)-(71) in Section B.1 yields (78). Using (78), $cov_t(m^*_{t+1} - m_{t+1}, \Delta e_{t+1}) < 0$ if and only if $var_t(\Delta e_{t+1}) + cov_t(\tilde{y}, \Delta e_{t+1}) - cov_t(\eta_{t+1} - \tilde{y}_{t+1}, \tilde{r}^*_{t+1} - \tilde{r}_{t+1}) < 0$. Setting $\tilde{r}^{(*)}_{t+1} = r^{(*)}_t$ and comparing again with (68)-(71) yields additional restrictions imposed by trade in risk-free assets. □

Proof to Proposition 4 The individual Euler equation can be expressed as:

$$\mathbb{E}_t\left[\beta\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-s}\right] = \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\left(\frac{\delta^i_{t+1}}{\delta^i_t}\right)^{-s}\tilde{R}_{t+1}\right] = 1, \tag{49}$$

for any (risky) asset with return $\tilde{R}_{t+1}$, so that e.g. foreign nominally risk-free bonds are captured by $(\mathcal{E}_{t+1}/\mathcal{E}_t)R_{t+1}$. Using

$$\frac{\delta^i_{t+1}}{\delta^i_t} = e^{(z^i_{t+1}\sqrt{y}_{t+1} - y_{t+1}/2)} \tag{50}$$

[23] In general $cov_t(X_{t+1}, Y_{t+1}) = \mathbb{E}_t[X_{t+1}]\mathbb{E}_t[Y_{t+1}]\left(e^{cov_t(U,V)} - 1\right)$ when $U, V \sim \mathcal{N}$.

we show:

$$\mathbb{E}_t\left[\beta\left(\frac{C^i_{t+1}}{C^i_t}\right)^{-s}\right]=\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}e^{\frac{y_{t+1}(s+s^2)}{2}}\tilde{R}_{t+1}\right]=1, \tag{51}$$

where y_t is the cross-sectional volatility of consumption. We assume a process for the incomplete markets wedge:

$$\eta_{t+1}=\beta_0 z_t+\beta_1\sqrt{z_t}u_{t+1}+\beta_2\sqrt{z_t}\epsilon_{t+1}+.... \tag{52}$$

where u are spanned innovations and ϵ are unspanned. We proceed to determine the coefficients of the IM wedge consistent with no arbitrage. Using (46) and denoting $var_t(\Delta e_{t+1})=\kappa z_t+\kappa^{\nu}...$:

$$\underbrace{(\beta_1^2+\beta_2^2)z_t}_{var_t(\eta_{t+1})}=\underbrace{[(\gamma-\gamma^*)^2-\kappa]z_t}_{var(m^*_{t+1}-m_{t+1})-var(\Delta e_{t+1})}-2\underbrace{\phi(\gamma-\gamma^*)z_t}_{cov_t(\tilde{y}_{t+1},m^*_{t+1}-m_{t+1})}-2\underbrace{\phi\beta_1 z_t}_{cov_t(y_{t+1},\eta_{t+1})} \tag{53}$$

which yields:

$$\beta_1=-\phi\pm\sqrt{(\gamma-\gamma^*-\phi)^2-\lambda},\quad \beta_2=\sqrt{\lambda-\kappa} \tag{54}$$

where we require:

$$(\gamma-\gamma^*-\phi)^2\geq\lambda\geq\kappa \tag{55}$$

Using (14)-(15) imply:

$$\beta_0=\frac{1}{2}\left\{(\gamma-\gamma^*)^2-\kappa\right\}-\phi(\gamma-\gamma^*)-\phi\beta_1+\gamma\beta_1-\alpha_y-\frac{1}{2}\phi^2+\gamma\phi \tag{56}$$

and

$$-\beta_0=\frac{1}{2}\left\{(\gamma-\gamma^*)^2-\kappa\right\}-\phi(\gamma-\gamma^*)-\phi\beta_1-\gamma^*\beta_1+\alpha_y+\frac{1}{2}\phi^2-\phi\gamma^*+\phi\beta_1 \tag{57}$$

respectively. Gathering from (57):

$$-\beta_0=\frac{1}{2}(\gamma-\gamma^*)^2-\frac{1}{2}\kappa-\phi\gamma+\frac{1}{2}\phi^2-\gamma^*\beta_1 \tag{58}$$

Subbing β_0 back into (56):

$$\kappa=(\gamma-\gamma^*)^2-\phi(\gamma-\gamma^*)+(\gamma-\gamma^*-\phi)\beta_1\geq 0 \tag{59}$$

This can be rewritten as:

$$\kappa = (\gamma - \gamma^* - \phi)^2 \pm (\gamma - \gamma^* - \phi)\sqrt{(\gamma - \gamma^* - \phi)^2 - \lambda}$$

and completes the characterization of the η process. □

Proof to Proposition 5 Building on Proposition 2, the Backus-Smith covariance is negative if and only if:

$$-cov_t(\tilde{y}_{t+1}, \Delta e_{t+1}) \geq var_t(\Delta e_{t+1}) \quad \longleftrightarrow \tag{60}$$

$$\underbrace{-cov_t(\tilde{y}_{t+1}, m^*_{t+1})}_{\gamma^*\phi z_t} + \underbrace{cov_t(\tilde{y}_{t+1}, m_{t+1})}_{-\gamma\phi z_t} - \underbrace{cov_t(\tilde{y}_{t+1}, \eta_{t+1})}_{\phi\beta_1 z_t} \geq \underbrace{var_t(\Delta e_{t+1})}_{\kappa z_t}$$

$$\leftrightarrow -\phi(\gamma - \gamma^*) - \phi\beta_1 \geq \kappa$$

which implies:

$$-\phi(\gamma - \gamma^*) - \phi\left(-\phi \pm \sqrt{(\gamma - \gamma^* - \phi)^2 - \lambda}\right) \geq \tag{61}$$

$$(\gamma - \gamma^* - \phi)^2 \pm (\gamma - \gamma^* - \phi)\sqrt{(\gamma - \gamma^* - \phi)^2 - \lambda} \geq 0 \tag{62}$$

Simplifying yields:

$$-(\gamma - \gamma^*)[(\gamma - \gamma^*) - \beta_1] \geq 0 \tag{63}$$

Consider the case where $\gamma - \gamma^* < 0$, such that z_t is a common factor and disproportionately drives the Foreign investor's SDF. Then the condition requires $\beta_1 \leq (\gamma - \gamma^*)$, requiring $\phi < \gamma - \gamma^*$ to be satisfied.

To express the condition in terms of observables, we consider:

$$r_{t+1} = \mathbb{E}_t[m_{t+1} + \tilde{y}_{t+1}] + \frac{1}{2}var_t(m_{t+1} + \tilde{y}_{t+1}), \tag{64}$$

$$= \log\beta + \tilde{\alpha}_y + \frac{1}{2}(\phi - \gamma)^2 z_t \tag{65}$$

It then follows that:

$$var_t(r_{t+}) = \frac{1}{4}(\phi - \gamma)^4 var_t(z_t) \tag{66}$$

and analogously:

$$var_t(r^*_{t+}) = \frac{1}{4}(\phi^* - \gamma^*)^4 var_t(z_t) \tag{67}$$

where $\phi^* \to 0$. Then,

$$var_t(r^*_{t+1}) > var_t(r_{t+1}) \leftrightarrow$$
$$\phi < \gamma - \gamma^*$$

as long as $\phi > \gamma$ and $\phi^* > \gamma^*$. □

B. ONLINE APPENDIX

B.1. Trade in Risky Assets

If instead of allowing for trade in both risk-free assets, we allow for trade in Home and Foreign risky assets, then equations (2)–(5) are replaced by:

$$\mathbb{E}_t[M_{t+1} e^{\tilde{y}_{t+1}} \tilde{R}_{t+1}] = 1, \tag{68}$$

$$\mathbb{E}_t[M_{t+1} e^{\tilde{y}_{t+1}} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t} \tilde{R}^*_{t+1}] = 1, \tag{69}$$

$$\mathbb{E}_t[M^*_{t+1} \tilde{R}^*_{t+1}] = 1, \tag{70}$$

$$\mathbb{E}_t[M^*_{t+1} \left(\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right)^{-1} \tilde{R}_{t+1}] = 1, \tag{71}$$

where $\tilde{R}$ and $\tilde{R}^*$ are returns on risky Home and Foreign assets respectively.

Adding cross-border trade in the Home risk free bond will reimpose equation (**??**) and cross-border trade in the Foreign risk free bond will reimpose equation (**??**). Consider the environment in Proposition 1, where only Foreign risk-free bonds are internationally traded.

Suppose Home and Foreign households trade in Home and Foreign currency denominated risky assets $\tilde{R}_{t+1}$ such that (68)- (71) hold. Assuming joint log normality,

the above Euler equations imply:

$$\mathbb{E}_t[m_{t+1}+\tilde{y}_{t+1}]+\frac{1}{2}var_t(m_{t+1}+\tilde{y}_{t+1})+\mathbb{E}_t[\tilde{r}_{t+1}]$$
$$+\frac{1}{2}var_t(\tilde{r}_{t+1})+cov_t(m_{t+1}+\tilde{y}_{t+1},\tilde{r}_{t+1})=0, \quad (72)$$

$$\mathbb{E}_t[m_{t+1}+\tilde{y}_{t+1}]+\frac{1}{2}var_t(m_{t+1}+\tilde{y}_{t+1})+\mathbb{E}_t[\tilde{r}^*_{t+1}]+$$
$$\frac{1}{2}var_t(\tilde{r}^*_{t+1})+\mathbb{E}_t[\Delta e_{t+1}]+\frac{1}{2}var_t(\Delta e_{t+1})+\cdots$$
$$cov_t(m_{t+1}+\tilde{y}_{t+1},\tilde{r}^*_{t+1})+cov_t(m_{t+1}+\tilde{y}_{t+1},\Delta e_{t+1})+cov_t(\Delta e_{t+1},\tilde{r}^*_{t+1})=0, \quad (73)$$

$$\mathbb{E}_t[m^*_{t+1}]+\frac{1}{2}var_t(m^*_{t+1})+\mathbb{E}_t[\tilde{r}^*_{t+1}]+\frac{1}{2}var_t(\tilde{r}^*_{t+1})+cov_t(m^*_{t+1},\tilde{r}^*_{t+1})=0, \quad (74)$$

$$\mathbb{E}_t[m^*_{t+1}]+\frac{1}{2}var_t(m^*_{t+1})-\mathbb{E}_t[\Delta e_{t+1}]+\frac{1}{2}var_t(\Delta e_{t+1})+\mathbb{E}_t[\tilde{r}_{t+1}]+\frac{1}{2}var_t(\tilde{r}_{t+1})+\cdots$$
$$cov_t(m^*_{t+1},\tilde{r}_{t+1})+cov_t(m^*_{t+1},-\Delta e_{t+1})+cov_t(-\Delta e_{t+1},\tilde{r}_{t+1})=0 \quad (75)$$

Combining (72) and (73):

$$\mathbb{E}_t[\Delta e_{t+1}]+\frac{1}{2}var_t(\Delta e_{t+1})+\mathbb{E}_t[\tilde{r}^*_{t+1}]+\frac{1}{2}var_t(\tilde{r}^*_{t+1})-\mathbb{E}_t[\tilde{r}_{t+1}]-\frac{1}{2}var_t(\tilde{r}_{t+1})+\cdots$$
$$cov_t(m_{t+1}+\tilde{y}_{t+1},\Delta e_{t+1})+cov_t(\Delta e_{t+1},\tilde{r}^*_{t+1})+cov_t(m_{t+1}+\tilde{y}_{t+1},\tilde{r}^*_{t+1}-\tilde{r}_{t+1})=0 \quad (76)$$

Combining (74) and (75):

$$-\mathbb{E}_t[\Delta e_{t+1}]+\frac{1}{2}var_t(\Delta e_{t+1})+\mathbb{E}_t[\tilde{r}_{t+1}]+\frac{1}{2}var_t(\tilde{r}_{t+1})-\mathbb{E}_t[\tilde{r}^*_{t+1}]-\frac{1}{2}var_t(\tilde{r}^*_{t+1})+\cdots$$
$$cov_t(m^*_{t+1},-\Delta e_{t+1})+cov_t(-\Delta e_{t+1},\tilde{r}_{t+1})-cov_t(m^*_{t+1},\tilde{r}^*_{t+1}-\tilde{r}_{t+1})=0 \quad (77)$$

Together, the above conditions yield:

$$var_t(\Delta e_{t+1})=cov_t(m^*_{t+1}-m_{t+1}-\tilde{y}_{t+1},\Delta e_{t+1})-cov_t(\eta_{t+1}-\tilde{y}_{t+1},\tilde{r}^*_{t+1}-\tilde{r}_{t+1}) \quad (78)$$

B.2. Degree of Market Completeness and Risk-Sharing

To illustrate that two nominally risk-free asssets suffice to span exchange rates, regardless of market completeness, we turn to a framework in the tradition of Lucas (1978). Agents have time-separable, CRRA preferences over an exogenous consumption

stream. In particular:

$$\Delta c_{t+1} = \sum_{k=1}^{N} g_{y_{k,t+1}}, \tag{79}$$

$$m_{t+1} = -s\Delta c_{t+1}, \tag{80}$$

where $g_{y_{k,t+1}} = y_{k,t+1} - y_{k,t} \sim i.i.d\ \mathcal{N}(\mu_{y_k}, \sigma_{y_k})$ denotes the growth rate of k-th productive unit that comprises the consumption good. Corresponding variables for the Foreign economy are denoted with an asterisk. Start with the case of $N = 1$ productive units, discussed in (Lustig and Verdelhan, 2019, Sec III.A). Frictionless international trade in Home and Foreign risk-free bonds (8)- (9) and additional trade in a Home and a Foreign risky asset (a claim on $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$ respectively) will imply that the incomplete markets wedge η_{t+1} is orthogonal to $g_{y_{k,t+1}}, g^*_{y_{k,t+1}}$, and it then follows that the only equilibrium is $\eta_{t+1} = 0$– i.e. markets are complete. When $N > 1$, additional risky claims need to traded to complete the market. However, for any N, frictionless international trade in just the Home and the Foreign real bonds ensures risk-sharing consistent with complete markets (Corollary 1). Relatedly, Chernov et al. (2024) show that incompleteness alone cannot match exchange rate puzzles, unless there is some lack of integration in financial markets (i.e. only Foreign bonds traded as in Corollary 2).

B.3. An equilibrium model

In this appendix we build on the equilibrium two-country, two-good, endowment model in Section **??**. We derive the autarky limit and express SDFs and prices as functions of exogenous variables. Starting with the static conditions:

$$C = \left[\alpha^{\frac{1}{\phi}} C_H^{\frac{\phi-1}{\phi}} + (1-\alpha)^{\frac{1}{\phi}} C_F^{\frac{1}{\phi}} \right]^{\frac{\phi}{\phi-1}} \tag{81}$$

Relative demand for goods requires:

$$\frac{C_F}{C_H} = \tfrac{1-\alpha}{\alpha} ToT^{-\phi}, \tag{82}$$

$$\frac{C_F^*}{C_H^*} = \tfrac{1-\alpha^*}{\alpha^*} ToT^{-\phi}, \tag{83}$$

where τ denotes the terms of trade.Market clearing requires:

$$C_H + C_H^* = Y_H \tag{84}$$

$$C_F + C_F^* = Y_F \tag{85}$$

The real exchange rate is given by $\mathcal{E} = \frac{P^*}{P}$ and the terms of trade by $ToT = \dfrac{P_F}{P_H}$. The law of one price holds for each good but not the aggregate basket unless $\alpha = \alpha^*$.

Under financial autarky, $PC = P_H Y_H$ and $P^* C^* = P_F Y_F$. Combining this with relative demand yields:

$$C_H + ToT\ C_F = Y_H, \tag{86}$$

$$C_F = \frac{1-\alpha}{\alpha}\left(\frac{1}{ToT}\right)^{\phi} C_H, \quad C_H = Y_H\left[\frac{\alpha}{\alpha + ToT^{1-\phi}(1-\alpha)}\right] \tag{87}$$

Abroad,

$$C_H^* ToT^{-1} + C_F^* = Y_F, \tag{88}$$

$$C_F^* = \frac{1-\alpha^*}{\alpha^*}\left(\frac{1}{ToT}\right)^{\phi} C_H^*, \quad C_H^* = Y_F\left[\frac{\alpha^*}{\alpha^* ToT^{-1} + ToT^{-\phi}(1-\alpha^*)}\right] \tag{89}$$

Balanced trade, and the law of one price, requires $\tau_t C_F = C_{H^*}$ in every period. Using relative demand:

$$ToT_t = \frac{\alpha^* \dfrac{P_t^*}{p_{H,t}}^{\phi} C_t^*}{(1-\alpha)\dfrac{P_t}{p_{F,t}}^{\phi} C_t}$$

Using autarky again:

$$ToT_t = \frac{\alpha^* \dfrac{P_t^*}{p_{H,t}}^{\phi} \dfrac{p_{F,t}^*}{P_t^*} Y_{F,t}^*}{(1-\alpha)\dfrac{P_t}{p_{F,t}}^{\phi} \dfrac{p_{H,t}}{P_t} Y_{H,t}}$$

Imposing $\alpha^* = (1-\alpha)$:

$$ToT_t = \underbrace{\frac{P_t^*}{P_t}^{\phi-1}}_{\mathcal{E}_t^{\phi-1}} \frac{Y_{F,t}^*}{Y_{H,t}} \underbrace{\frac{p_{F,t}}{p_{H,t}}^{1+\phi}}_{ToT_t^{1+\phi}}$$

So:

$$ToT_t^{-\phi} = \mathcal{E}_t^{\phi-1} \frac{Y_{F,t}^*}{Y_{H,t}} \tag{90}$$

A first order approximation and $e = (2\alpha - 1)\tau$,

$$\tau = \frac{y_H - y_F}{1 - 2\alpha(1-\phi)}, \tag{91}$$

$$\Delta e = (2\alpha - 1) * \Delta\tau \tag{92}$$

We now show that at the limit of $\alpha \to 1$, the model coincides with a two-country, two-good (no trade), CAPM. Home and Foreign consumption is given by:

$$c_t = c_{H,t} = y_{H,t} \tag{93}$$

$$c_t^* = c_{F_t}^* = y_{F,t} \tag{94}$$

Assuming $g_{y_{H,t}}$, $g_{y_{F,t}} \sim \mathcal{N}(\mu, \sigma_y^2)$, then $m_{t+1}^{(*)} \sim \mathcal{N}(-s\mu, s^2\sigma_{y_i}^2)$, $i \in \{H, F\}$. We can then construct:

$$\eta_{t+1} = (g_{y_{H,t+1}} - g_{y_{F,t+1}}) \frac{1-s}{1 - 2\alpha(1-\phi)} \tag{95}$$

using (7).

Returning to the financial side of the model, the Home agents' inter-temporal allocation satisfies (2), (5). The international risk sharing condition in the model is given by:

$$\mathbb{E}_t[M_{t+1} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}] = \mathbb{E}_t\left[M_{t+1}^*\right] \leftrightarrow$$

$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s} \frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right] = \mathbb{E}_t\left[\beta\left(\frac{C_{t+1}^*}{C_t^*}\right)^{-s}\right] \tag{96}$$

Critically, if the Foreign risk-free bond was also traded then the second risk-sharing

condition below would also need to be satisfied:

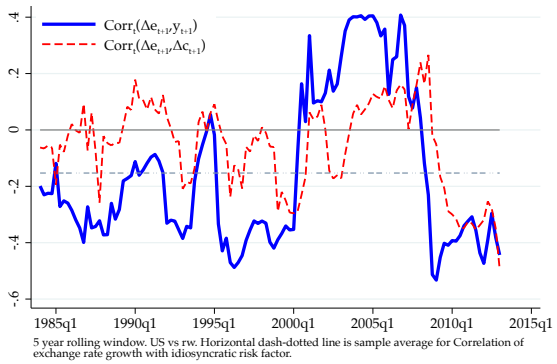
$$\mathbb{E}_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-s}\right]=\mathbb{E}_t\left[\beta\left(\frac{C^*_{t+1}}{C^*_t}\right)^{-s}\frac{\mathcal{E}_t}{\mathcal{E}_{t+1}}\right] \tag{97}$$

Notice that (96) and (97) are the same if approximated to first order, but in general will imply significantly different results.

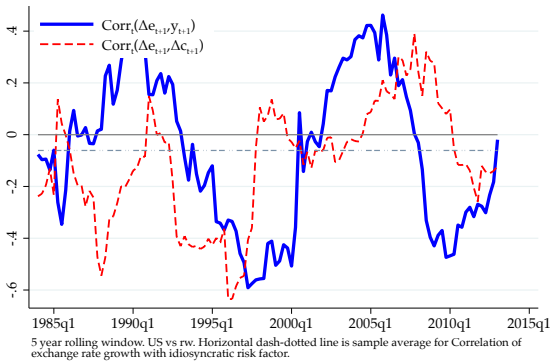
Figure 5: *Correlations by Country*

Figure 6

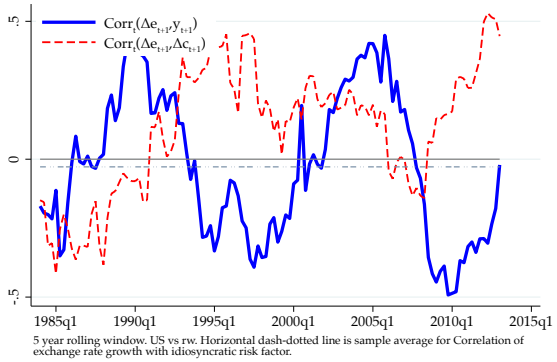
(a) *Canada*



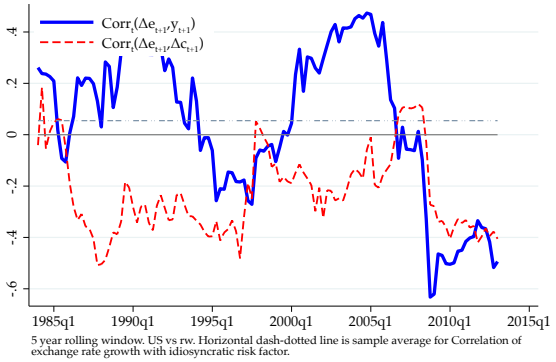
(b) *Italy*



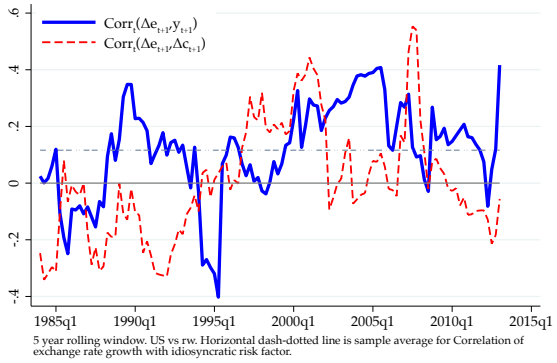
(c) *Germany*



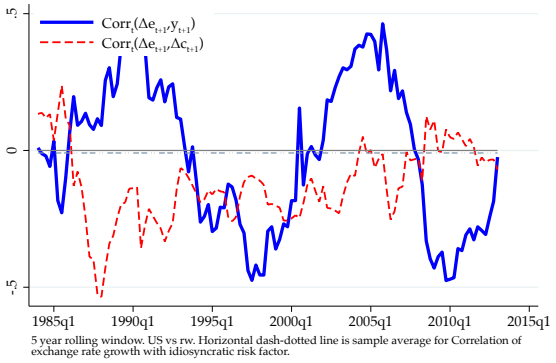
(d) *UK*



(e) *Japan*



(f) *France*



Notes: The dash-dotted line plots the sample average of the 5-year moving average correlation between real exchange rate growth (depreciation of USD against the currency) and the variance of the US idiosyncratic risk factor.