Aggregation Bias in International Business Cycles*

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Abstract

We study a large class of international business cycle models where countries are connected in a global input-output network and quantify how an aggregated one-sector model can result in "aggregation bias" in cross-country shock spillover strength. We show analytically the parameterization under which international spillover may differ in a multi-sector networked model than that in a one-sector counterpart. Quantitatively, aggregation bias in a fully dynamic general equilibrium model with global input-output linkages across sectors and countries tends to generate weaker shock transmission across countries. In our baseline parameterization, aggregation bias results in about 25% lower international spillover in the period between 1990 and 2009 for 20 countries in our sample.

JEL classification: E32, F31.

Keywords: International business cycles, trade linkages, volatility, comovement, input-output, aggregation bias, one-sector, multi-sector model, global value chain.

*The views expressed in this paper are those of the authors and do not necessarily represent the view of the Federal Reserve Bank of San Francisco, or the Federal Reserve Bank system.

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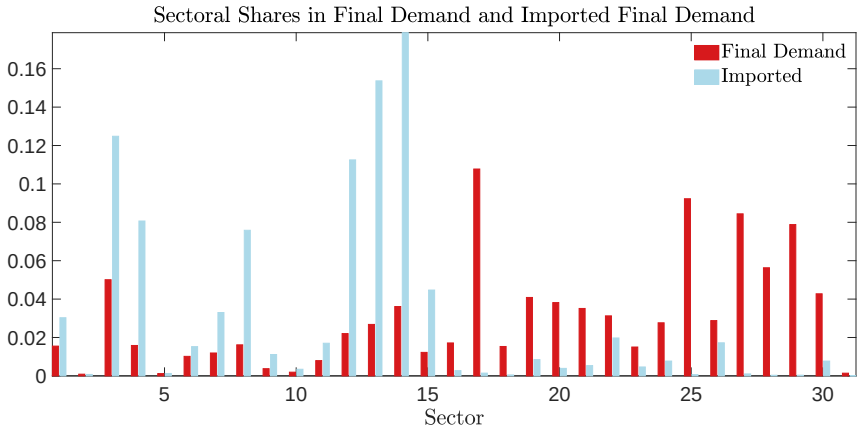
1 Introduction

Understanding the international propagation of business cycles across countries is an important topic in international macroeconomics. A large literature starting with Backus, Kehoe, and Kydland (1994), Heathcote and Perri (2002) and subsequent work including a recent influential paper by Itskhoki and Mukhin (2021) often analyze international business cycles in a context of a one-sector model where each country produces one differentiated good. These models are known to have difficulty matching salient features of the data such as the correlations of output and employment across countries and the volatilities of net exports, and exchange rates. In recent years, with the rise of global value chain, and more available data on input-output tables as well as time series of disaggregated sectoral output, an increasing number of research adopts a multi-sector and multi-country framework. For example, Huo, Levchenko, and Pandalai-Nayar (2023) decompose international comovements due to different shocks in a global economy; our work, Miyamoto and Nguyen (2024), use a multi-country two-sector model to study how trade linkages affect business cycles over time; Bonadio et al. (2021) study the role of global supply chain, and di Giovanni et al. (2023) analyze inflation drivers during COVID–19 on the global economy using a multi-sector model. Since world economies consist of many consumers, firms, sectors, and countries, it is arguable possible to study a further disaggregated model. The question is then whether the aggregate implications on international business cycles, such as shock spillover, are substantially different in a granular model of the world economy such as a multi-sector networked model where countries and sectors are interconnected from those in an aggregated one-sector model counterpart where each country produces one differentiated good and trade with one another. In other words, how important is it to model a granular world economy with heterogeneity for international business cycles?

There are reasons to believe that heterogeneity matters for the transmission of shocks across countries. Take sectors, for example. Sectors are different in many dimensions. Some sectors are much more exposed to international trade than others. As plotted in Figure 1, in a median country in a sample of 25 OECD countries averaged between 1995 and 2011, the first fifteen sectors that include manufacturing, machinery and equipment account for most of the imported final demand while other sectors such as services contribute little to the total imported final demand. In contrast, the first fifteen sectors are small fractions of total final demand—the last few sectors that include services make up most of the final demand. In addition, shocks driving sectoral fluctuations can

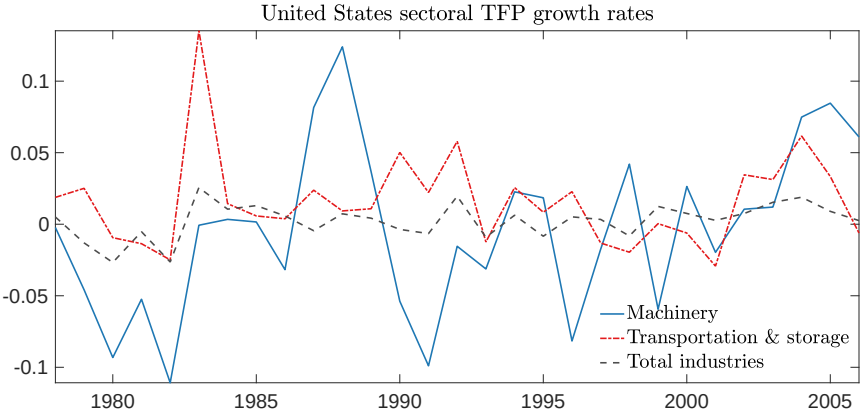
also be markedly different. As plotted in Figure 2, according to the EUKLEM's calculation of US TFP, TFP in the Machinery sector is more volatile than that in the Construction sector, and both are more volatile than the aggregate TFP.

Figure 1: Sectoral Shares in Final Demand



Notes: We compute sectoral shares in final demand and sectoral shares in imported final demand from the world IO table between 1995 and 2011 for 23 countries, then take median across countries for each sector. The red bars are sectoral shares in final demand, and the grey bars are sectoral shares in final imported.

Figure 2: Sectoral and aggregate productivity



Notes: Productivity growth rates for the United States are computed from the EUKLEMS database.

This paper studies a large class of international business cycle models where countries are connected in a global input-output network and investigate how aggregating multi-sector networked model into a one-sector model may bias international spillover strength across countries. We focus on the transmission of shocks across countries in the multi-sector production networked model and compare with that in a one-sector model counterpart where each country produces one differentiated good using a production function and trades with other countries. In both models, countries

trade both intermediate and final goods, and the production function for each firm requires endogenous labor input, intermediate goods and capital services with exogenous productivity processes. Households can borrow and save intertemporally. The model also features variable capital utilization and investment adjustment costs, which have been shown to help better match business cycle moments. The main differences between the two models are the existence of more disaggregated sector in the multi-sector model. In particular, each country's production side consists of S sectors, each of which produces a differentiated good using a production function. These sectors can have different exposure to international trade through the input-output linkages and different productivity shock processes. We use this framework as a laboratory to characterize aggregation bias in a simple version of the model, where we show that aggregation bias arises in most cases due to the different importance of the sectors in the networked economy as well as the heterogeneous nature of sectoral productivity shocks. We then quantify the aggregation bias in the data for 20 countries over time between 1970 and 2009 and find that a one-sector model tends to underpredict shock transmission across countries. In the period between 1970 and 2009, shock spillover is 25% lower in a one-sector model than in the multi-sector model counterpart, on average, over this period.

To build intuition, we first derive analytically the differences in output responses to a foreign shock, or the aggregation bias, in a simplified version of the general framework. In particular, we assume that the home country is a small open economy that takes the expenditure and prices of the rest of the world (ROW) as given. Then, ROW is a closed economy, as changes in prices and quantities in the small home country have negligible effects on ROW. We show that to predict aggregate outcomes of the closed economy, such as the price index, employment and real GDP, we only need the labor supply elasticity and the average sectoral productivity change weighted by sectoral-output-to-aggregate-GDP ratios (Domar weights). This resonates with the results in Hulten (1978). We then take the changes in ROW's expenditure and prices as given and assume financial autarky to obtain a closed-form solution for the equilibrium in the home country. In the log-linearized solution, changes in equilibrium prices and quantities in the home country are linear combinations of foreign expenditure and prices, which are, in turn, linear combinations of foreign sectoral productivity shocks. When labor is the only input in the production function with no intermediate inputs and no capital services, we find that the change in real GDP in the home country depends on an aggregate import price and export-to-GDP ratio of the home country in addition to the foreign expenditure and price index. A one-sector model that matches aggregate outcomes such as the foreign expenditure and price index and the home export-to-GDP ratio will

generate a different output responses in the home country to foreign productivity shocks because it mistakes the foreign price index for home's aggregate import price. In a simple case where each country has two sectors: tradeble and non-tradable sectors, productivity shocks in the tradable sector in the foreign country will lower the home country's import prices more than productivity shocks in the nontradable sector, and thus have a larger impact on the home country's price index and amplify the responses of employment and real GDP in the home country. The corresponding one-sector model misses such a distinction and generates different predictions of the responses of employnent and value added, in general. Furthermore, we show that in the context of the simplified model, this aggregation bias cam disappear when foreign sectors have the same export-to-GDP ratios or when the sectoral shocks are identical. In a more general case with input-output linkages, the aggregation bias also depends on sectoral heterogeneity in intermediate input shares and value-added shares.

Equipped with the intuition from the simplified model, we quantify the importance of the aggregation bias using our full model with the data for 20 advanced countries, each vis-a-vis a ROW. For each pair of countries, the steady states input-output linkages in this multi-sector model are calibrated to match with the average of the World Input-Output Table (WIOD) between 1995 and 2011, which include 31 sectors, so as to match the linkages within and across countries. The calibration for the one-sector model counterpart is an aggregate of the multi-sector WIOD. In addition to heterogeneous input-output sectoral characteristics, we also assume that sectors are also driven by different productivity shock volatilities. Focusing on the role of sectoral heterogeneity in the input-output linkages, we find that the responses of value added in the home country are larger in a multi-sector model than that in a one-sector model when the productivity shocks happen in some sectors such as those in manufacturing. In other words, if shocks are skewed toward sectors that trade more extensively, assuming a one-sector model will underpredict the effects of foreign shock spillover to the domestic economy. However, consistent with the simple model analyses, if shocks happen in all sectors at the same time and change all sectors' productivity by the same amount, cross-country transmission in a one-sector model is similar to that in a multi-sector model. Furthermore, as sectoral shock composition changes over time, the aggregation bias can vary over time without any changes in input-output linkages. Indeed, after backing out productivity shocks by observing sectoral value added for these countries in the same period, our analysis suggests that the bias caused by using a one-sector model is more pronounced during the 2008-2009 Great Recession. On average, a one-sector model generates 25% lower shock spillover than a multi-sector model.

Overall, our results highlight how taking a granular approach with sectoral heterogeneity seriously can theoretically and quantitatively can change the answers to business cycle comovement across countries. Under plausible parameterizations, one-sector models may underpredict the importance of foreign shocks in driving domestic business cycles.

Related Literature Our paper contributes to three stands of literature. First, our analyses help to better understand different frameworks to study international business cycles. As noted above, international business cycle literature include many different frameworks: aggregated one-sector model such as Backus, Kehoe, and Kydland (1994), Heathcote and Perri (2002) and Itskhoki and Mukhin (2021), two-sector model of tradeable and nontradable goods or durable goods such as Corsetti, Dedola, and Leduc (2008), Engel and Wang (2011), Johnson (2013), and Miyamoto and Nguyen (2024), as well as multi-sector model such as Huo, Levchenko, and Pandalai-Nayar (2023). Our paper using a general model that encompasses prominent models in this literature featuring fundamentally different sectors helps to shed light on distinguishing these models based on aggregate implications, as well as the quantitative importance of modeling more granular aspects of the world economy.

Second, our paper is related to understanding the macroeconomic impact of microeconomic shocks. For example Hulten (1978), Gabaix (2011), Carvalho and Tahbaz-Salehi (2019), Baqaee and Farhi (2019), and Baqaee and Rubbo (2022) concentrate on closed economy implications, our paper provides new insights on the transmission of shocks across countries in a networked economy.

Third, we also relate to the growing literature studying the role of sectoral heterogeneity in business cycles. For example, Pasten, Schoenle, and Weber (2023) highlight the role of sectoral heterogeneity in nominal rigidity in amplifying sectoral shocks in a closed economy setting. Pasten, Schoenle, and Weber (2020), Cox et al. (2023), Bouakez, Rachedi, and Santoro (2023a), and Bouakez, Rachedi, and Santoro (2023b) study the propagation of monetary policy shocks and government spending shocks in a multi-sector closed economy. Unlike these papers, we analyze productivity shocks in an open-economy setting and highlight how a one-sector model may have missed the aggregate implications in the multi-sector model. In this sense, we are related to Devereux, Gente, and Yu (2022) who study fiscal shock spillover in a production network model.

Our paper proceeds as follows. In Section 2, we describe our full model in details. We build intuition in Section 3. Section 4 presents the quantitative exercises for the aggregation bias. We concluded in Section 5.

2 The Model

This section describes our model and solution concepts. Our model encompasses several models used in the literature such as Backus, Kehoe, and Kydland (1994), Heathcote and Perri (2002) and Huo, Levchenko, and Pandalai-Nayar (2023) among others. In particular, the model is a general dynamic framework featuring multi-country multi-sector setup with a production network, where countries trade with one another in both intermediate and final goods in each sector. This feature is designed to capture the input-output linkages within and across countries in the data. The world consists of I countries and each country has S sectors. A one-sector model counterpart is when $S = 1$. We denote the set of countries and industries with $\mathcal{I}$ and $\mathcal{S}$, and use i, j to denote individual countries and k, s to denote individual sectors. Each country i has $n(i)$ symmetric representative households. To better match key features of the international business cycle facts, we augment the model with variable capacity utilization and investment adjustment costs. These features have been widely used in the macroeconomic literature as they are minimal departures from standard models, and are supported in the data.

2.1 Households

A representative household in country i maximizes its lifetime expected utility:

$$\max E_0 \sum_{t=0}^{\infty} \beta_t (i) U \left(C_t (i), H_t (i) \right), \tag{1}$$

where $C_t (i)$ denotes consumption in country i at time t, and $H_t (i)$ is hours worked in country i. We do not specify a functional form for the utility function U, following Bilbiie (2009) and Furlanetto and Seneca (2014). We directly impose the values for parameters for the Frisch elasticity of labor supply, the intertemporal elasticity of substitution, and the wealth effects on labor supply in the linearized model. We assume incomplete financial markets, where households can trade non state-contingent bonds, so we use Uzawa-type preferences to induce stationarity. Then, $\beta_t (i)$ evolves as follows:

$$\beta_{t+1} (i) = \beta \left(\frac{C_t (i)}{C_{ss} (i)} \right)^{-\phi_\beta} \beta_t (i), \tag{2}$$

where $\phi_B \geq 0$ is a parameter, $C_{ss} (i)$ is consumption at the steady state, and β is the steady-state discount factor.

The period-by-period budget constraint is given by:

$$C_t(i) + \sum_s p_t^Z(i,s) Z_t(i,s) + \frac{1}{R_t(I)} B_{t+1}(i) ex_t(i) \tag{3}$$
$$= \sum_s W_t(i,s) H_t(i,s) + \sum_s R_t^K(i,s) u_t(i,s) K_t(i,s) + \Pi_t(i) + B_t(i) ex_t(i)$$

where $p_t^Z(i,s)$ is the price of investment goods for sector s, $Z_t(i,s)$ is the investment in sector s, $R_t(I)$ is the real interest rate in country I (the last country), $B_{t+1}(i)$ is the non-contingent bond holding of country i in the unit of consumption in country I, $ex_t(i)$ is the real exchange rate between country i and I, $W_t(i,s)$ is the real wage in sector s, $R_t^K(i,s)$ is the rental rate of capital in sector s, $u_t(i,s)$ is the variable capital utilization rate, $K_t(i,s)$ is capital stock in sector s, and $\Pi_t(i)$ is the sum of profits from firms in country i.

The household allocates hours across sectors, and total hours worked is an aggregator of hours supplied to each sector s with a constant elasticity of substitution as follows:

$$H_t(i) = \left(\sum_s \omega_H(i,s)^{-\frac{1}{\gamma_H}} H_t(i,s)^{\frac{1+\gamma_H}{\gamma_H}} \right)^{\frac{\gamma_H}{1+\gamma_H}}, \tag{4}$$

where γ_H is the elasticity of substitution, $H_t(i,s)$ is hours worked in sector s, and $\omega_H(i,s)$ is the parameter related to the shares of hours worked in each sector at the steady state.

The evolution of capital is standard as follows:

$$K_{t+1}(i,s) = (1 - \delta(u_t(i,s))) K_t(i,s) + Z_t(i,s) \left(1 - S_I \left(\frac{Z_t(i,s)}{Z_{t-1}(i,s)} \right) \right), \tag{5}$$

where $\delta(u_t(i,s))$ is the depreciation rate that depends on the rate of capital utilization rate, and $S_I\left(\frac{Z_t(i,s)}{Z_{t-1}(i,s)}\right)$ is investment adjustment cost. We assume the following functional form for the capital utilization rate:

$$\delta(u_t(i,s)) = \delta_0(i,s) + \delta_a(i,s)(u_t(i,s) - 1) + \frac{\delta_b(i,s)}{2}(u_t(i,s) - 1)^2, \tag{6}$$

where $\delta_0(i,s)$ is the steady-state depreciation rate in sector s. The model without utilization corresponds to $\frac{\delta''(1)}{\delta'(1)}$ approaching infinity, as it is extremely costly to change the utilization. If $\frac{\delta''(1)}{\delta'(1)}$ goes to zero, the marginal cost of changing utilization is constant. The investment adjustment cost

is standard given by

$$S_I\left(\frac{Z_t(i,s)}{Z_{t-1}(i,s)}\right)=\frac{\phi^I(i,s)}{2}\left(\frac{Z_t(i,s)}{Z_{t-1}(i,s)}-1\right)^2, \tag{7}$$

where $\phi^I \geq 0$ is the parameter that governs the investment adjustment cost.

2.2 Firms

There are S sectors in each country. Each firm in each sector produces a differentiated good using capital, labor and a composite intermediate good, which is an aggregate of intermediate goods produced by all countries and sectors. Final consumption and investment are as an aggregate of the sectoral consumption and investment goods. These sectoral final composite goods are an aggregate of goods of the same sector with origins from both domestic and foreign countries. Specifically, firms producing the final composite consumption goods in country i uses sectoral goods $f_C(i,s)$ with the elasticity of substitution γ_{FC} across sectors as follows:

$$C_t(i)=\left[\sum_{s=1}^{S}\left(\omega_{FC}(i,s)\right)^{\frac{1}{\gamma_{FC}}}\left(f_{C,t}(i,s)\right)^{\frac{\gamma_{FC}-1}{\gamma_{FC}}}\right]^{\frac{\gamma_{FC}}{\gamma_{FC}-1}}, \tag{8}$$

where $\omega_{FC}(i,s)$ is the share of sectoral s goods in final consumption. The final sectoral firms produce sectoral goods $f_{C,t}(i,s)$ by combining goods of the same sector with origins from both domestic and foreign countries, $f_{C,t}(i,(j,s))$, given by:

$$f_{C,t}(i,s)=\left[\sum_{j=1}^{I}\left(\omega_{fC}(i,(j,s))\right)^{\frac{1}{\gamma_{fC}}}\left(f_{C,t}(i,(j,s))\right)^{\frac{\gamma_{fC}-1}{\gamma_{fC}}}\right]^{\frac{\gamma_{fC}}{\gamma_{fC}-1}}, \tag{9}$$

where γ_{fC} is the elasticity of substitution of goods across countries, and $\omega_{fC}(i,(j,s))$ is the share of $f_{C,t}(i,(j,s))$ in final sectoral goods for consumption. The production of the sectoral final investment goods is analogous to that of the sectoral final consumption goods, as follows:

$$Z_t(i,s)=\left[\sum_{k=1}^{S}\left(\omega_Z((i,s),k)\right)^{\frac{1}{\gamma_Z}}\left(z_t((i,s),k)\right)^{\frac{\gamma_Z-1}{\gamma_Z}}\right]^{\frac{\gamma_Z}{\gamma_Z-1}}, \tag{10}$$

and

$$z_t((i,s),k)=\left[\sum_{j=1}^{I}\left(\omega_z((i,s),(j,k))\right)^{\frac{1}{\gamma_z}}\left(z_t((i,s),(j,k))\right)^{\frac{\gamma_z-1}{\gamma_z}}\right]^{\frac{\gamma_z}{\gamma_z-1}}, \tag{11}$$

where $z_t((i,s),(j,k))$ is goods for investment bought by country i sector s from country j sector k, and $z_t((i,s),k)$ is the sectoral investment goods after aggregating $z_t((i,s),(j,k))$ over seller country k.

Each firm in country i in sector s produces goods using the following production function:

$$Q_t(i,s) = \left[\omega_Q(i,s)^{\frac{1}{\gamma_Q}} \left[A_t(i,s)\widetilde{K}_t(i,s)^{\alpha_K} H(i,s)^{\alpha_H}\right]^{\frac{\gamma_Q-1}{\gamma_Q}} + (1-\omega_Q(i,s))^{\frac{1}{\gamma_Q}} M_t(i,s)^{\frac{\gamma_Q-1}{\gamma_Q}}\right]^{\frac{\gamma_Q}{\gamma_Q-1}}, \tag{12}$$

where Q_t is gross output in country i sector s, $A_t(i,s)$ is productivity, $\widetilde{K}_t(i,s)$ is the service from capital defined as $\widetilde{K}_t(i,s) \equiv u_t(i,s) K_t(i,s)$, $M_t(i,s)$ is intermediate goods used in sector s, γ_Q is the elasticity of substitution between intermediate goods and the value added, α_K is the share parameter for capital service, α_H is the share parameter for hours worked, and $\omega_Q(i,s)$ are the parameters related to the shares of intermediate goods. Similar to consumption and investment, intermediate input is produced in two steps. First, intermediate goods of both domestic and foreign origins are combined to produce sector k's intermediate goods to be used in sector s country i:

$$m_t((i,s),k) = \left[\sum_{j=1}^{I} (\omega_m((i,s),(j,k)))^{\frac{1}{\gamma_m}} (m_t((i,s),(j,k)))^{\frac{\gamma_m-1}{\gamma_m}}\right]^{\frac{\gamma_m}{\gamma_m-1}}, \tag{13}$$

where $m_t((i,s),(j,k))$ is intermediate good bought by country i sector s from country j sector k with the share $\omega_m((i,s),(j,k))$., and $m_t((i,s),k)$ is intermediate goods after aggregating $m_t((i,s),(j,k))$ over sectors of sellers. Second, these sector-specific intermediate goods are aggregated across sectors to be used in each sector s in country i as follows:

$$M_t(i,s) = \left[\sum_{k=1}^{S} (\omega_M((i,s),k))^{\frac{1}{\gamma_M}} (m_t((i,s),k))^{\frac{\gamma_M-1}{\gamma_M}}\right]^{\frac{\gamma_M}{\gamma_M-1}}, \tag{14}$$

where $\omega_M((i,s),k)$ is the share of the sector-specific intermediate goods.

The profit maximization problem for the firm is defined as

$$\Pi_t(i) \equiv \sum_s \left[p_t^Q(i,s) Q_t(i,s) - W_t(i,s) H_t(i,s) - R_t^K(i,s)\widetilde{K}_t(i,s) - p_t^M(i,s) M_t(i,s)\right]$$

where $p_t^Q(i,s)$ is the relative price of $Q_t(i,s)$, and $p_t^M(i,s)$ is the relative price of intermediate goods $M_t(i,s)$.

2.3 Resource Constraints

The market clearing condition for each sector s in each country i is given by:

$$n(i)\,Q(i,s) = \sum_j n(j)\,f_{C,t}(j,(i,s)) + \sum_j \sum_k n(j)\,z_t((j,k),(i,s)) + \sum_j \sum_k n(j)\,m_t((j,k),(i,s)). \tag{15}$$

The model also requires the asset market clearing condition and $\sum_{s=1}^{S} K_t(i,s) = u_t(i)K_t(i)$.

2.4 Shock Processes and Variable Definition

We close the model with the description of the productivity processes. The productivity in each sector in each country follows an AR(1) process with persistence ρ_A given by

$$lnA_t(i,s) = \rho_A lnA_{t-1}(i,s) + e_t^A(i,s), \tag{16}$$

where $e_t^A(i,s)$ has a standard deviation $\sigma(i,s)$ and mean 0.

We define the real value added for country i, $V_t(i)$ to be equal to the sum of all sector s gross output less the intermediate input usage evaluated at steady-state prices as follows.

$$V_t(i) \equiv \sum_s Q_t(i,s) - \sum_s M_t(i,s). \tag{17}$$

The sectoral value added in each sector s of country i, $V_t(i,s)$, is defined to be sector s gross output less the intermediate input usage evaluated at steady-state prices,

$$V_t(i,s) \equiv Q_t(i,s) - M_t(i,s). \tag{18}$$

3 Aggregation Bias in a Simplified Model

To build intuition on how aggregation bias may arise from a one-sector model, we first analyze a simple case of the general framework to derive analytical expressions for the response of value added output in the home country to shocks in the foreign country. We define the aggregation bias as the difference between the response in the home country in a one-sector model compared to that in the multi-sector model. We then discuss the conditions under which shock spillover in a one-sector model is different from that in a multi-sector model.

More specifically, we solve a simple version of the full model. We assume country 1 is a small open economy (SOE) that trades with the rest of the world (Country 2), so we can approximate the equilibrium variables in Country 2 as a closed economy. Labor is the only input in the production function, there is no capital services or intermediate inputs. Households spend all their income in each period and have no savings. This immediately implies that trade is balanced every period. To simplify the exposition, we assume $\gamma_{FC} = \gamma_{fC} \equiv \gamma_f$ so that the nested CES preferences for consumption is reduced to one layer. In addition, we assume that hours worked in each sector are perfect substitutes, i.e. $\gamma_H = \infty$, and $H_t(i) = \sum_s H_t(i,s)$. Therefore, wages are equalized across sectors. For simplicity, we assume the per-period utility function takes the Greenwood, Hercowitz, and Huffman (1988b) (GHH) preferences:

$$U(C_t(i), H_t(i)) = \frac{\left(C_t(i) - \psi H_t(i)^{1+\frac{1}{\nu}}\right)^{1-\sigma}}{1-\sigma},$$

where ν is the labor supply elasticity. Under the assumption of no savings, employment has a constant elasticity with respect to real wage: $H_t(i) = \left(\frac{W_t(i)/P_t(i)}{\psi\left(1+\frac{1}{\nu}\right)}\right)^{\nu}$, where $P_t(i)$ is the consumers' price index in country i, period t.

We study the cross-country transmission of productivity shocks by characterizing the equilibrium value added in Country 1 in response to a temporary sectoral productivity shock in Country 2, denoted by $\widehat{A}(2,s)$ for $s \in \mathcal{S}$. In all of the expressions below, we omit the time subscript t.

We first show that the equilibrium in Country 2–the closed economy–in a one-sector model can be constructed to be the same as in a multi-sector model, as follows:

Lemma 1 *In a closed economy, we can find an observationally equivalent one-sector economy in terms of changes in real value added, real wage, and hours worked up to a first-order approximation. In particular, this requires the hypothetical one-sector economy to have an aggregate productivity shock $\widehat{A}(2)^{one-sec}$ is the weighted sum of the sectoral productivity shocks in the multi-sector model, where the weight is the sectoral output share:*

$$\widehat{A}(2)^{one-sec} = \sum_s \left(\frac{p^Q_{ss}(2,s) Q_{ss}(2,s)}{\sum_k p^Q_{ss}(2,k) Q_{ss}(2,k)} \widehat{A}(2,s) \right), \tag{19}$$

where $p^Q_{ss}(2,s)$ and $Q_{ss}(2,s)$ are the steady-state price and gross output in sector s, country 2.

Proof. See Appendix B. ■

Equation (19) is closely related to the results in Hulten (1978). To see the connection, we can rewrite it as

$$\frac{\sum_s p_{ss}^Q(2,s)Q_{ss}(2,s)}{V(2)} \times \widehat{A}(2)^{one-sec} = \sum_s \frac{p_{ss}^Q(2,s)Q_{ss}(2,s)}{V(2)} \widehat{A}(2,s), \tag{20}$$

where $V(2)$, as in the general model, denotes value added of Country 2. The right-hand side of equation (20) is the aggregate productivity change due to sectoral shocks according to Hulten (1978) where the ratios $\frac{p_{ss}^Q(2,s)Q_{ss}(2,s)}{V_2}$ are the Domar weights (Domar, 1961). The left-hand side is the corresponding version in a one-sector economy case. Setting the hypothetical aggregate productivity shock following equation (19) or (20) ensures that the aggregate productivity changes are the same in the one-sector and multi-sector models. We can further show that the price index, $\widehat{P}(2)$, is proportional to $\widehat{A}(2)^{one-sec}$. With GHH preferences and log linearization, the change in equilibrium employment is linear in the change of the real wage, i.e. $\widehat{W}(2) - \widehat{P}(2)$. Therefore, the equilibrium change in real wage and employment are all proportional to $\widehat{A}(2)^{one-sec}$. Following Hulten (1978), the change in real value added, or GDP, can be decomposed into the change in aggregate productivity and employment and is also proportional to $\widehat{A}(2)^{one-sec}$.

Taking Country 2's equilibrium changes in prices and quantities as given, Country 1's equilibrium is solved as in the following proposition:

Proposition 1 *Country 1's real value-added can be expressed as*

$$\widehat{V}_t(1) = \frac{\nu \widetilde{a}_{21}^C}{\Omega} \left[-\gamma_f \widehat{p}_t^{CM}(1) + \gamma_f \widehat{p}_t(2) + \widehat{C}_t(2) \right], \tag{21}$$

where $\Omega \equiv 1 + \nu \widetilde{a}_{21}^C - (1-\gamma_f)\left(2 - \widetilde{a}_{21}^C\right)$, $\widehat{p}_t(2)$ *is Country 2's consumer price index, and* $\widehat{p}_t^{CM}(1)$ *is the import price index defined as*

$$\widehat{p}_t^{CM}(1) \equiv \sum_s \frac{p_{ss}^Q(2,s) f_{ss}(1,(2,s))}{\sum_k p_{ss}^Q(2,k) f_{ss}(1,(2,s))} \widehat{p}_t^Q(2,s).$$

Proof. See Appendix B. ■

In the expression above, the change of real value added in Country 1 is the same as the change of hours worked since we focus on its responses to Country 2's sectoral productivity shocks and assume that Country 1's own sectoral productivity is unchanged.

In the proof of Lemma 1, we express the changes in foreign expenditure and the foreign consumer

price index as linear combinations of sectoral productivity shocks $\widehat{A}_t(2,s)$ and both are proportional to $\widehat{A}_2^{one-sec}$. However, this proportionality does not hold for the change in the aggregate import price, $\widehat{P}_t CM(1)$. The lack of a sufficient statistic for $\widehat{P}_t CM(1)$ is the only source of aggregation bias in the simple model without intermediate inputs. More formally, we state this aggregation bias in the following corollary:

Corollary 1 *Denote the actual change in Country 1's real value added as $\widehat{V}_t(1)$ in a multi-sector model and the predicted change in a one-sector model as $\widehat{V}_t(1)^{one-sec}$. The one-sector model is calibrated to match (1) the aggregate variables in Country 2 as in Lemma 1, (2) the steady-state export-to-GDP ratio in Country 1, and (3) the same labor supply elasticity v and elasticity of substitution γ_f as in the multi-sector model. The one-sector model will have a bias in the predicted change in value added as*

$$\widehat{V}_t(1)^{one-sec} - \widehat{V}_t(1) = \frac{\nu \widetilde{a}_{21}^C \gamma_f}{\Omega} \left(\widehat{p}_t^{CM}(1) - \widehat{p}_t(2) \right).$$

The term $\widehat{p}_t^{CM}(1) - \widehat{p}_t(2)$ can be expressed as

$$\widehat{p}_t^{CM}(1) - \widehat{p}_t(2) = \sum_s \left[\omega_C(2,(2,s)) - \tilde{\omega}_C(1,(2,s)) \right] \widehat{A}(2,s), \tag{22}$$

where $\tilde{\omega}_C(1,(2,s))$ is sectoral import share defined as

$$\tilde{\omega}_C(1,(2,s)) \equiv \frac{p_{ss}^Q(2,s) f_{ss}(1,(2,s))}{\sum_k p_{ss}^Q(2,k) f_{ss}(1,(2,k))}.$$

The bias disappears if any of the following conditions holds:

1. *Shocks are symmetric across sectors: $\widehat{A}(2,s) = \widehat{A}(2,1), \forall s$*
2. *Sectoral import shares in Country 1 are the same as sectoral consumption shares in Country 2, i.e., $\omega_C(2,(2,s)) = \tilde{\omega}_C(1,(2,s)), \forall s$.*

Proof. The above corollary is a direct result of Proposition 1. To obtain the predicted change in real value added in the one-sector model $\widehat{V}_t(1)^{one-sec}$, we simply set the change in import price $\widehat{p}_t^{CM}(1)^{one-sec}$ to the change in Country 2's price index, $\widehat{p}_t(2)$, because there is only one tradable sector. When either condition (1) or (2) holds, $\widehat{p}_t CM(1) - \widehat{p}_t(2)$ is zero based on equation (22) so the aggregation bias $\widehat{V}_t(1)^{one-sec} - \widehat{V}_t(1)$ is zero. ■

Intuitively, when all sectors in Country 2 receive the same shock, it is straightforward that the multi-sector model behaves like a one-sector model and that abstracting from sectoral heterogeneity does not lead to any bias in aggregate predictions. The bias also disappears when the sectoral import shares are the same as sectoral consumption shares. This holds, for example, when sectors in Country 2 have the same openness, i.e.,

$$\frac{f_{ss}(1,(2,s))}{f_{ss}(1,(2,s))+f_{ss}(2,(2,s))}=\frac{f_{ss}(1,(2,s))}{Q_{ss}(2,s)}=\frac{f_{ss}(1,(2,1))}{Q_{ss}(2,1)},\forall s\in\mathcal{S}.$$

To see this, suppose there are two sectors. Sector 1 is tradable while sector 2 is nontradable. We must have $\omega_C(2,(2,s))<\tilde{\omega}_C(1,(2,s))=1$. It is immediate that, for non-negative shocks $\widehat{A}(2,s)$, the one-sector model under-predicts transmission if the shock is larger in the tradable sector and over-predicts transmission if the shock is larger in the non-tradable sector.

Do these two conditions hold in the data? As plotted in 1, we find that import and consumption shares are different across sectors and countries. In addition, 2 suggests that changes in sectoral productivity are unlikely similar across sectors. Therefore, aggregation bias is likely to exist in realistic calibration. The question is how large this bias is, so we turn to the quantitative analysis next.

4 Quantitative Analysis

As the above analysis suggests that aggregation bias in terms of shock transmission across countries can depend on both sectoral linkages and sectoral productivity shocks, this section seeks to quantify aggregation bias based on these channels in our full model. We highlight that using a one-sector model tends to underpredict cross-country shock spillover, and that aggregation bias varies over time.

4.1 Data and Calibration

We first describe the data and parameters to calibrate the full model.

Data Our model is calibrated to two countries, each of the 20 countries in our sample vis-a-vis a composite ROW between 1970 and 2008. The 20 countries are Australia, Austria, Belgium, Canada, Germany, Denmark, Spain, Finland, France, Greece, Ireland, Italy, Japan, Korea, Luxembourg, the Netherlands, Portugal, Sweden, the United Kingdom and the United States. Each country has 31

sectors. The names of the sectors corresponding to the sector numbers are in Appendix Table A-1. The sectoral output data come from EUKLEMS and the world IO tables are from the World Input-Output Database between 1995 and 2011.

Calibration The model is calibrated for annual data. Table **??** summarizes the calibrated values for common parameters across countries in our baseline. The discount factor is set to be 0.96, so the implied steady state annual interest rate is about 4%. We set the labor share parameter, α_H, to be 2/3, which is standard in the literature. The depreciation rate at the steady state is 10% a year. We calibrate three parameters for household preferences, which are sufficient for the linearized model. The inverse of the intertemporal elasticity of substitution parameter, σ, is set to be 2, which is along the line of Backus, Kehoe, and Kydland (1995), and others in the international business cycle literature. The inverse Frisch labor supply parameter, ψ, is set to be 1, following Bilbiie (2009) and others. We set the parameter measuring the wealth elasticity of labor supply, κ, to be 0.1, which implies low wealth effects, similar to Miyamoto and Nguyen (2017).[1] We set a small value of the Uzawa preference, ϕ_β, as standard international business cycle model with incomplete markets (0.0001).

In the literature, the range for the elasticities of substitution across sectors and countries varies widely. The elasticity of substitution across countries in the international business cycle literature is typically small, around 1; for example, Backus, Kehoe, and Kydland (1995) set this parameter to 1.5, and Johnson (2013) imposes 1 in his baseline model and analyzes lower elasticities case in his robustness check. In contrast, the estimates in the trade literature are much larger, between 6 and 15. In the baseline, as Johnson (2013), we set the elasticities of substitution across both final and intermediate goods in different sectors and countries to 1 and discuss our quantitative results with respect to this parameter later. With this parameterization, the composite intermediate input is Cobb-Douglas in inputs from different countries. We also set the parameter for variable utilization, δ_b, to be 0.05, implying a highly elastic utilization. The steady-state investment adjustment cost parameter is imposed to be 0.25.

As our full model does not assume small open economy, we calibrate the size of the countries according to the average of the World IO tables between 1995 and 2011. In addition, we set the following steady-state shares in the model based on the average 1995-2011 World IO tables: the shares of intermediate inputs in gross output in each sector in each country ω_q, the shares of

[1] The Greenwood, Hercowitz, and Huffman (1988a) preferences imply a wealth elasticity parameter of 0, and the separable utility function implies this parameter to be 2.

foreign goods in sectoral final consumption, investment and intermediate goods for all sectors and countries $\{\omega_{fC}, \omega_Z, \omega_M\}$, the shares of sectoral composite in final consumption, investment and intermediate goods for all countries $\{\omega_{CF}, \omega_z, \omega_m\}$. These steady-state shares are calibrated to completely replicate the 2 country 31-sector world IO table.

Table 1: Calibrated Parameters

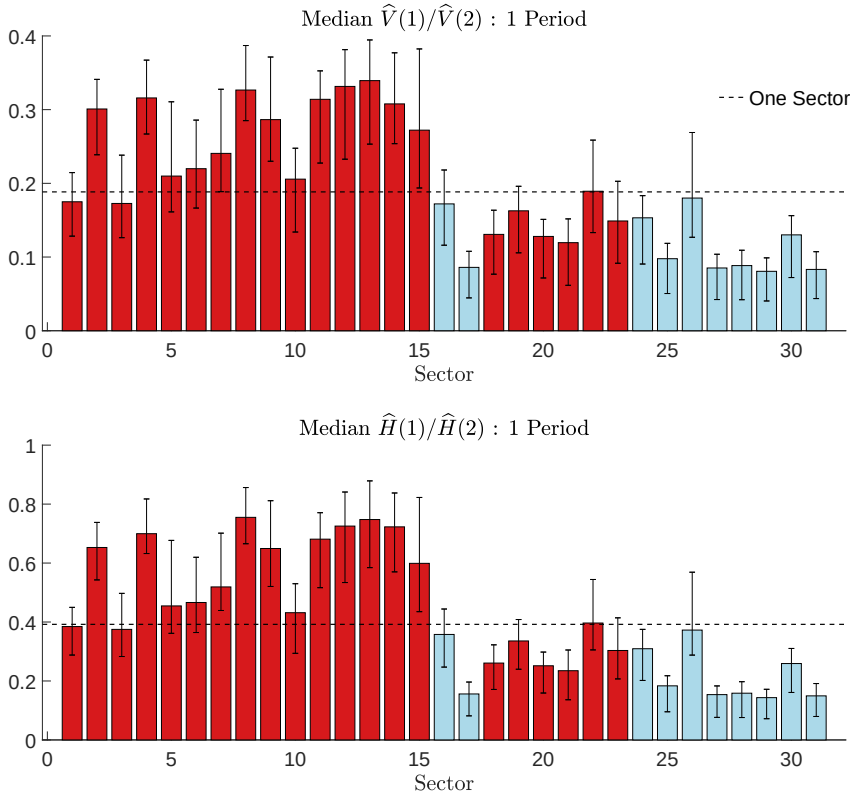
Parameter		Value
β	discount factor	0.96
σ	relative risk aversion	2
κ	wealth effect on labor supply	0.1
ψ	inverse of Frisch elasticity	1
s	investment adjustment cost	0.25
α_H	labor share	2/3
δ	depreciation	0.1
δ_b	capacity utilization	0.15
ϕ_β	Uzawa preference	0.0001
γ_F	elasticity across sectors for final good	1
γ_f	elasticity across countries for final good	1
γ_Z	elasticity across countries for investment goods	1
γ_z	elasticity across sectors for investment goods	1
γ_Q	elasticity between value added and intermediate	1
γ_M	elasticity across sectors for intermediate	1
γ_m	elasticity across countries for intermediate	1
ρ_A	persistence of TFP	0.34

4.2 Aggregation bias and sectoral heterogeneity in IO structure

This section quantifies how much sectoral heterogeneity in input-output linkages can generate aggregation bias in shock spillover. As in the simple model analysis, we consider $\widehat{A}_t(2,s)$, which is a productivity shock in sector s in the foreign country, i.e. country 2. We then compute the relative responses of country 1's value added and hours to the responses of country 2's value added and hours in both the multi-sector model and the one-sector model counterparts. In other words, we compute $\frac{\widehat{V}_t(1)}{\widehat{V}_t(2)}$ and $\frac{\widehat{H}_t(1)}{\widehat{H}_t(2)}$ to $\widehat{A}_t(2,s)$ for each $s \in \mathbb{S}$. This relative responses are the changes in value added output in country 1 for a 1% change in value added output in country 2. Notice that we do not compute $\widehat{V}_t(1)$ and $\widehat{V}_t(2)$ separately like in the simple model. The reason is that the relative impulse responses normalize the transmission of shocks from country 2 to country 1, and sectoral shock sizes do not matter, as the relative responses measure how much value added output in the home country changes for a given 1% change in foreign country, which can be driven by different sectoral shocks.

Figure 3 plots the relative changes in country 1 to country 2 in response to each sectoral productivity shock in country 2 in a median country in our sample of 20 pairs of countries (the

Figure 3: Transmission of sectoral shocks from country 2 to country 1 in a multi-sector model.



Notes: The red (and light blue) bars are the median relative responses across 20 country pairs in sectors with relatively high (and low) trade volume. The lines indicate 25-75 percentile. The horizontal dashed lines are the shock spillover of productivity shock in country 2 to country 1.

red and light blue bars) and its 25-75 percentile bands (the thin lines). Given the heterogeneous input-output linkages of each sector, international spillover of sectoral shocks are also different: sectors with higher trade exposures (the red bars) tend to have larger transmission across countries than those with lower trade exposures. Furthermore, the one-sector model can generate smaller spillover across countries if the actual shock originates from sectors such as Manufacturingm, but a larger spillover if the shock comes from Services sector. These results highlight the differences in international spillover of productivity shocks between a multi-sector and a one-sector models as shocks originate from different sectors which play different roles in the input-output structure. In addition, the one-sector spillover strength is roughly the same as average of all sectoral shocks' spillover.

4.3 Aggregation bias and sectoral shock heterogeneity

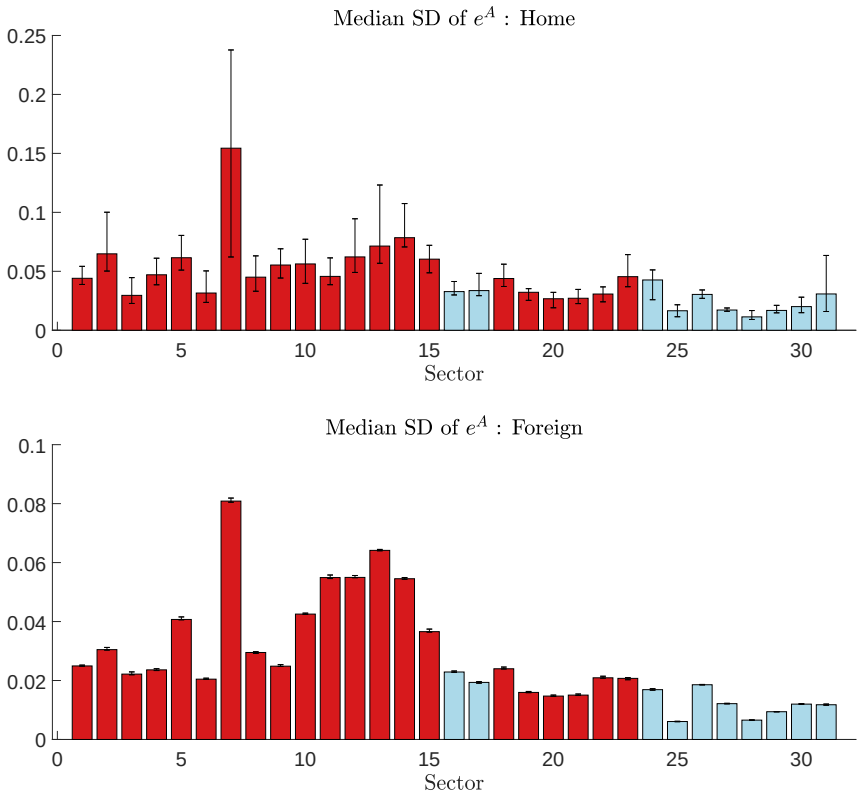
In addition to playing different roles in the input-output structure, sectors can be driven by different productivity shocks, and the sectoral shock composition can be different at each point in time. This implies that aggregation bias can be time varying.

To quantify the aggregation bias in a one-sector model in terms of shock spillover across countries, we first recover all sectoral productivity shocks between 1970 and 2009 using Kalman smoother. To that end, we match sectoral value added in the model for each country with the data between 1970 and 2009 and back out productivity shocks that replicate the data. Figure 4 plot median standard deviation of home and foreign sectoral productivity shocks for all 31 sectors. As expected, shock sizes are heterogeneous across sectors. The sector with the largest shock size is sector 7 (Coke, Refined Petroleum and Nuclear Fuel). Services sectors such as sectors 25 (Real estate activities) have lower shock standard deviations, consistent with the observation that these sectors' value added is less volatile than manufacturing sectors.

What do these heterogeneous shocks imply about the aggregation bias in a one-sector model in terms of shock spillover? We plot in Figure 5 the impulse responses of both home and foreign value added, $\hat{V}_1$ and $\hat{V}_2$, respectively, to the recovered foreign shocks between 1970 and 2009 in a multi-sector model and a one-sector model counterpart. To facilitate comparison between international transmission implied in a multi-sector model and that in a one-sector model, we normalize foreign value added response to be the same between the two models. In almost all years between 1970 and 2009, shock spillover is larger in the multi-sector model than in the one-sector model. For example, in 1973-1974, foreign shock spillovers to home country are only half as large in the one-sector model as in the multi-sector model. Similarly, the one-sector model predicts a transmission strength of only 75% of that in the multi-sector model in the 2008-2009 Global Financial Crisis. In other words, the one-sector model tends to underpredict the transmission of foreign shocks to domestic economy.

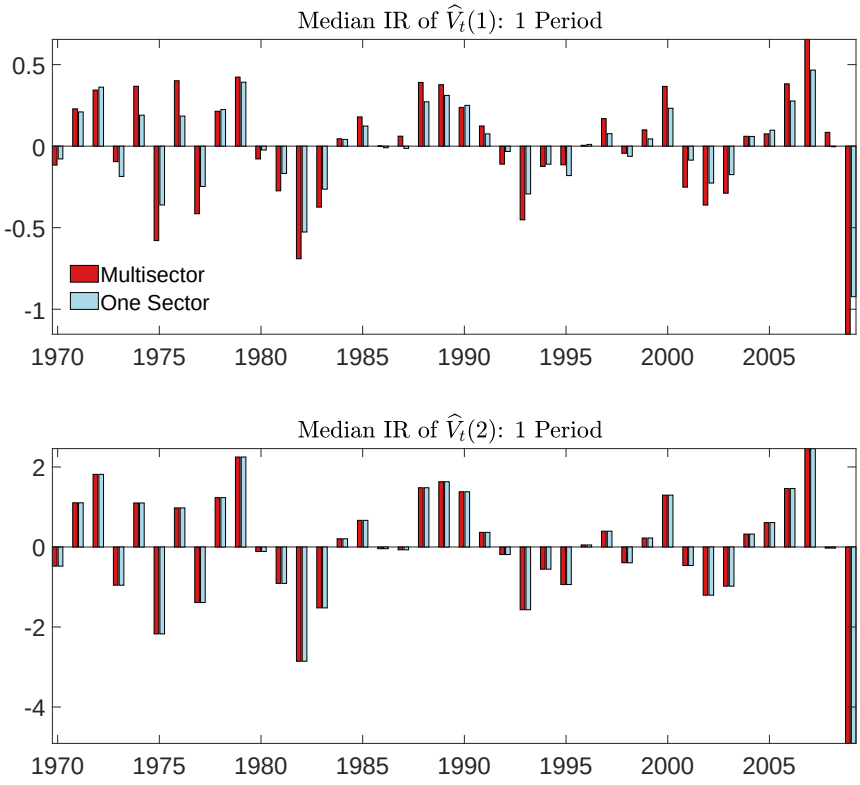
Finally, we plot in Figure 6 the responses of home country to foreign shocks in all countries between 1970 and 2009 in the multi-sector model against that in the one-sector model. The responses are computed in the one-year horizon. The blue dots show the cases when the one-sector model overpredicts transmission and the red dots are when the one sector model underpredicts transmission. Overall, consistent with the median analysis, we find that the one-sector model predicts 25.4% less transmission of foreign shocks to domestic economy over time. This result implies

Figure 4: Recovered shocks in a multi-sector model.



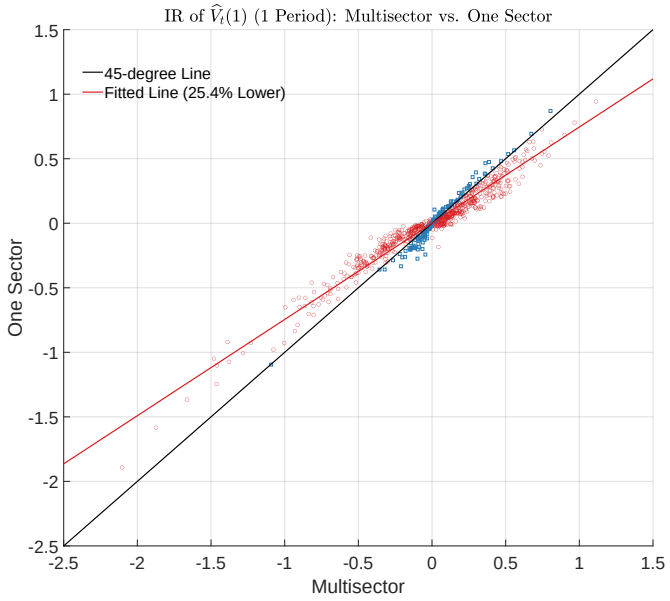
Notes: This figure plots the recovered standard deviations of home (upper panel) and foreign (lower panel) sectoral shocks in a median country. The thin bars plot the 25-75 percentile.

Figure 5: Median aggregation bias across time.



Notes: This figure plots the impulse responses of value added in home (upper panel) and foreign (lower panel) at a one-year horizon in a median country. The red bars are the responses in a multi-sector model and the light blue bars are those in a one-sector model counterpart.

Figure 6: Aggregation bias across countries over time.



Notes: Each point presents the domestic response to a foreign shock in the one-sector against the multi-sector models at each point in time for each pair of countries.

that the multi-sector model should, on average, generate larger comovement across countries than in the one-sector model.

5 Conclusion

We investigate the differences in terms of foreign shock transmission to domestic economies in a multi-sector model and an aggregated one-sector model counterparts. Our analytical solution shows the condition in which international shock spillover predicted in a one-sector model is the same with that in a multi-sector model. Quantitatively, we find that the one-sector model tends to underpredict shock spillover from foreign countries, suggesting that the multi-sector model may help generate larger comovement across countries conditional on productivity shocks compared to the one-sector model. While we consider a model that has been a basis of many international business cycle models, future work can incorporate other frictions to understand how these frictions may amplify or reduce the aggregation bias in a one-sector model.

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Online Appendix

A Tables and Figures

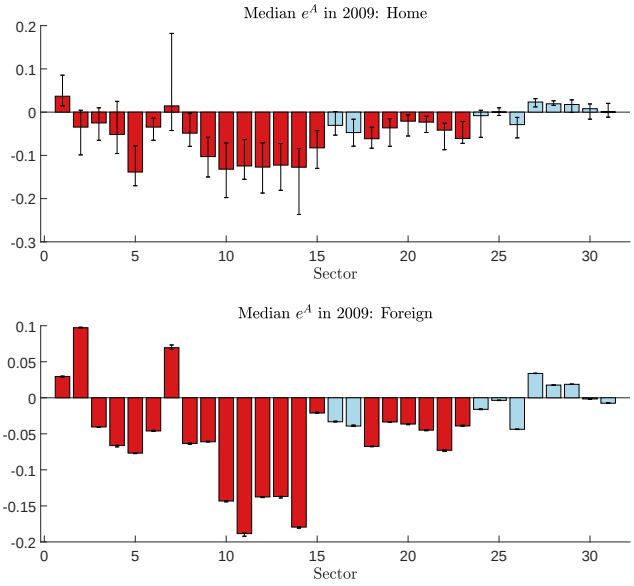
Table A-1: Available Sectors

Sector code	Number	Name
AtB	Sector 1	Agriculture, Hunting, Forestry and Fishing
C	Sector 2	Mining and Quarrying
D15t16	Sector 3	Food, Beverages and Tobacco
D17t19	Sector 4	Textiles, Textile, Leather and Footwear
D20	Sector 5	Wood and of wood and cork
D21t22	Sector 6	Pulp, Paper, Paper, Printing and Publishing
D23	Sector 7	Coke, Refined Petroleum and Nuclear Fuel
D24	Sector 8	Chemicals and Chemical Products
D25	Sector 9	Rubber and Plastics
D26	Sector 10	Other Non-Metallic Mineral
D27t28	Sector 11	Basic Metals and Fabricated Metal
D29	Sector 12	Machinery, Nec
D30t33	Sector 13	Electrical and Optical Equipment
D34t35	Sector 14	Transport Equipment
D36t37	Sector 15	Manufacturing, Nec; Recycling
E	Sector 16	Electricity, Gas and Water Supply
F	Sector 17	Construction
50	Sector 18	Sale, maintenance and repair of motor vehicle and retail sale of fuel
51	Sector 19	Wholesale trade and commission trade, except motor vehicles
52	Sector 20	Retail trade, except of motor vehicles
H	Sector 21	Hotels and Restaurants
60t63	Sector 22	Transport and Storage
64	Sector 23	Post and Telecommunications
J	Sector 24	Financial Intermediation
70	Sector 25	Real estate activities
71t74	Sector 26	Renting of M&EQ and other business activities
L	Sector 27	Public admin and defence
M	Sector 28	Education
N	Sector 29	Health and Social work
O	Sector 30	Other Community, social and personal services
P	Sector 31	Private households with employed persons

B Proofs

In this section, we provide proofs for our theoretical results in Section 3. We derive the change in equilibrium in the more general case with domestic and foreign intermediate inputs and obtain Proposition 1 as a special case. Before any derivation, we first define vectors and matrices that are useful for solving the equilibrium in the general case.

Figure A-1: Sectoral shocks during the Global Financial Crisis 2009.



Notes: The recoverd shocks in each sector in a median country between 2008-2009.

Table B-2: Define other vectors and matrices

Notation	Description	Dimensions	Value
$\mathbf{p}_{Q,i}$	output prices	$S \times 1$	$\left[P_Q(i,s)\right]_s$
$\mathbf{q}_i$	output	$S \times 1$	$\left[Q(i,s)\right]_s$
$\boldsymbol{\lambda_i}$	output share	$S \times 1$	$P_Q(i,s)Q(i,s)/\left(\mathbf{p}_i\right)^T \mathbf{q}_i$
$\mathbf{f}_i$	final consumption	$S \times 1$	$\left[f(i,s)\right]_s$
$\mathbf{a}_i$	productivity	$S \times 1$	$\left[A(i,s)\right]_s$
$\boldsymbol{\omega}_{Q,i} \circ \widehat{\mathbf{a}}_i \equiv \widehat{\mathbf{z}}_i$	change in effective productivity	$S \times 1$	$\left[\omega_Q(i,s)\widehat{A}(i,s)\right]_s$
$\boldsymbol{\omega}_{Q,i}$	value added share in production	$S \times 1$	$\left[\omega_Q(i,s)\right]_s$
$\text{diag}(\mathbf{1} - \boldsymbol{\omega}_{Q,i})\boldsymbol{\Theta}^m_{ij} \equiv \mathbf{A}_{ij}$	input share in gross output	$S \times S$	$(1-\omega_Q(i,s))\theta_m((i,s),(j,k))$
$\boldsymbol{\Psi}^m_{ij}$	share of output used by downstream	$S \times S$	
$\boldsymbol{\Psi}^f_{ij}$	share of output consumed	$S \times 1$	
$\tilde{\boldsymbol{\Theta}}^m_{ii}$	Leontief inverse	$S \times S$	$\left(\mathbf{I} - \text{diag}(\mathbf{1} - \boldsymbol{\omega}_{Q,i})\boldsymbol{\Theta}^m_{ii}\right)^{-1}$
$\tilde{\boldsymbol{\Psi}}^m_{ii}$	Downstream usage inverse	$S \times S$	$\left(\mathbf{I} - \boldsymbol{\Psi}^m_{ii}\right)^{-1}$
$\boldsymbol{\omega}_{C,ij}$	consumption shares	$S \times 1$	$\omega_C(j,(i,s)) \equiv \frac{\theta_f(j,(i,s))}{\sum_s \theta_f(j,(i,s))}$
$\boldsymbol{\Gamma}^w_i$	sector's share of employment	$1 \times S$	$\frac{H(i,s)}{H_i}$

Proof of Lemma 1. To solve the equilibrium in Country 2 as a closed economy, we first apply Walrus Law and normalize the change in wages, $\widehat{W}_2$, to zero. As illustrated in Table B-2, we denote the effective productivity shocks in Country 2 as $\widehat{\mathbf{z}}_2 \equiv \boldsymbol{\omega}_{Q,2} \circ \widehat{\mathbf{a}}_2$, where $\circ$ indicates element-wise product.

Perfect competition and the production function of each sector implies the changes in sectoral prices must follow

$$\widehat{\mathbf{p}}_{Q,2} = -\widehat{\mathbf{z}}_2 + \text{diag}(\mathbf{e}_S - \boldsymbol{\omega}_{Q,2})\boldsymbol{\Theta}^m_{22}\widehat{\mathbf{p}}_{Q,2},$$

where $\mathbf{e}_S$ is a $S \times 1$ vector of ones and the operator diag transforms the vector into a diagonal matrix. The first term on the right-hand side is the direct effect of the productivity shocks, while the second term captures the direct effect of the input prices.

Based on this equation, we can solve the changes in prices as a linear transformation of changes in productivities, $\widehat{\mathbf{p}}_{Q,2} = -(\mathbf{I} - \text{diag}(\mathbf{1} - \boldsymbol{\omega}_{Q,2})\boldsymbol{\Theta}^m_{22})^{-1}\widehat{\mathbf{z}}_2 = -\tilde{\boldsymbol{\Theta}}^m_{22}\widehat{\mathbf{z}}_2$. The change in the aggregate price index is the average of sectoral price changes, weighted by consumption shares $\boldsymbol{\omega}_{C,22}$:

$$\widehat{P}_2 = \boldsymbol{\omega}^T_{C,22}\widehat{\mathbf{p}}_{Q,2} = -\boldsymbol{\omega}^T_{C,22}\tilde{\boldsymbol{\Theta}}^m_{22}\widehat{\mathbf{z}}_2.$$

Due to GHH preferences, the change in employment must satisfy

$$\widehat{H}_2 = \nu(\widehat{W}_2 - \widehat{P}_2) = \nu\boldsymbol{\omega}^T_{C,22}\tilde{\boldsymbol{\Theta}}^m_{22}\widehat{\mathbf{z}}_2.$$

Since Hulten (1978), we know that the change in real value added can be written as

$$\widehat{V}_2 = \sum_s \underbrace{\frac{p_Q(2,s)Q(2,s)}{V_2}}_{\text{Domar weights}} \omega_Q(2,s)\widehat{A}(2,s) + \widehat{H}_2.$$

Note that the Domar weights equals $\boldsymbol{\omega}^T_{C,22}\tilde{\boldsymbol{\Theta}}^m_{22}$. To see this, consider the goods market clearing conditions

$$\begin{aligned}
&(\mathbf{p}_{Q,2}\mathbf{q}_2)^T = (\mathbf{p}_{Q,2}\mathbf{f}_2)^T + (\mathbf{p}_{Q,2}\mathbf{q}_2)^T \,\text{diag}(\mathbf{1} - \boldsymbol{\omega}_{Q,2})\boldsymbol{\Theta}^m_{22} \\
\Leftrightarrow &(\mathbf{p}_{Q,2}\mathbf{q}_2)^T = (\mathbf{p}_{Q,2}\mathbf{f}_2)^T \times \tilde{\boldsymbol{\Theta}}^m_{22} \\
\Leftrightarrow &\frac{1}{V_2}(\mathbf{p}_{Q,2}\mathbf{q}_2)^T = \boldsymbol{\omega}^T_{C,22}\tilde{\boldsymbol{\Theta}}^m_{22}
\end{aligned}$$

Therefore, both changes in employment and value added are proportional to the Domar weights weighted productivity change $\boldsymbol{\omega}^T_{C,22}\tilde{\boldsymbol{\Theta}}^m_{22}\widehat{\mathbf{z}}_2$. We need a one-sector economy in which

$$\boldsymbol{\omega}^T_{C,22}\tilde{\boldsymbol{\Theta}}^m_{22}\widehat{\mathbf{z}}_2 = \frac{\sum_s p_Q(2,s)Q(2,s)}{V_2}\widehat{z}_2^{one-sec},$$

where $\frac{\sum_s p_Q(2,s)Q(2,s)}{V_2}$ is the output-to-value-added ratio (also the Domar weight in a one-sector economy). When labor is the only input, we have $\boldsymbol{\omega}_{Q,2} = \mathbf{e}_s$ thus $\widehat{\mathbf{z}}_2 = \widehat{\mathbf{a}}_2$. Therefore, we have obtained equation (20). ■

We now solve the equilibrium in Country 1 in the more general case with intermediate inputs. We state the equilibrium change in nominal wage (minus the aggregate import price) in the following proposition.

Proposition B-1 *The equilibrium change in nominal wage (minus the aggregate import price) in Country 1 can be expressed as:*

$$\begin{aligned}
&\widehat{W}_1 - \widehat{P}^c_{1,M} = \Big\{(1-\mathbf{\Gamma}^w_1\tilde{\mathbf{\Psi}}^m_{11}\mathbf{\Psi}^f_{11})(1+v\tilde{a}^c_{21}) - (1-\gamma_f)\mathbf{\Gamma}^w_1\tilde{\mathbf{\Psi}}^m_{11}[\mathbf{\Psi}^f_{11}\circ(\tilde{\mathbf{\Theta}}^m_{11}\boldsymbol{\omega}_{Q,1} - \tilde{a}^c_{1L}\mathbf{e}_S) + \mathbf{\Psi}^f_{12}\circ(\tilde{\mathbf{\Theta}}^m_{11}\boldsymbol{\omega}_{Q,1})] \\
&-(1-\gamma_m)\mathbf{\Gamma}^w_1\left[\mathbf{e}_S + \tilde{\mathbf{\Psi}}^m_{11}\left((\tilde{\mathbf{\Theta}}^m_{11}\boldsymbol{\omega}_{Q,1})\circ((\mathbf{\Psi}^m_{11}+\mathbf{\Psi}^m_{12})\mathbf{e}_S) - \tilde{\mathbf{\Theta}}^m_{11}\boldsymbol{\omega}_{Q,1}\right)\right]\Big\}^{-1} \\
&\times\Big\{-(1-\mathbf{\Gamma}^w_1\tilde{\mathbf{\Psi}}^m_{11}\mathbf{\Psi}^f_{11})\widehat{P}^c_{1,M} + (1-\gamma_f)\mathbf{\Gamma}^w_1\tilde{\mathbf{\Psi}}^m_{11}\left[(\mathbf{\Psi}^f_{11}+\mathbf{\Psi}^f_{12})\circ(\tilde{\mathbf{\Theta}}^m_{11}\mathbf{A}_{12}\Delta\widehat{\mathbf{p}}_{Q,2}) + \mathbf{\Psi}^f_{12}(\widehat{P}^c_{1,M}-\widehat{P}_2)\right] \\
&+(1-\gamma_m)\mathbf{\Gamma}^w_1\tilde{\mathbf{\Psi}}^m_{11}\left[(\tilde{\mathbf{\Theta}}^m_{11}\mathbf{A}_{12}\Delta\widehat{\mathbf{p}}_{Q,2})\circ((\mathbf{\Psi}^m_{11}+\mathbf{\Psi}^m_{12})\mathbf{e}_S) - \left(\tilde{\mathbf{\Theta}}^m_{11}\mathbf{A}_{12}+\mathbf{\Psi}^m_{12}\right)\Delta\widehat{\mathbf{p}}_{Q,2}\right] + \mathbf{\Gamma}^w_1\tilde{\mathbf{\Psi}}^m_{11}\left(\mathbf{\Psi}^f_{12}\widehat{P_2C_2} + \mathbf{\Psi}^m_{12}\widehat{\mathbf{p}_{Q,2}\mathbf{q}_2}\right)\Big\},
\end{aligned}$$

where we have defined the following terms

$$\begin{aligned}
\Delta\widehat{\mathbf{p}}_{Q,2} &\equiv \widehat{\mathbf{p}}_{Q,2} - \widehat{P}^c_{1,M}\mathbf{e}_S \\
\tilde{a}^c_{1L} &\equiv \boldsymbol{\omega}^T_{C,11}\tilde{\mathbf{\Theta}}^m_{11}\boldsymbol{\omega}_{Q,1} \\
\tilde{a}^c_{21} &\equiv 1 - \tilde{a}^c_{1L} \\
\widehat{P}^c_{1,M} &\equiv (\tilde{a}^c_{21})^{-1}(\boldsymbol{\omega}^T_{C,11}\tilde{\mathbf{\Theta}}^m_{11}\mathbf{A}_{12} + \boldsymbol{\omega}^T_{C,21})\widehat{\mathbf{p}}_{Q,2}
\end{aligned}$$

Proof. To solve the equilibrium in Country 1, we express all other prices and quantities as functions of the endogenous variable $\widehat{W}_1$, and finally use the labor market clearing condition to solve $\widehat{W}_1$.

Perfect competition and the production functions imply that the changes in sectoral output prices must satisfy

$$\widehat{\mathbf{p}}_{Q,1} = \boldsymbol{\omega}_{Q,1}\widehat{W}_1 + \text{diag}(\mathbf{1}-\boldsymbol{\omega}_{Q,1})\mathbf{\Theta}^m_{12}\widehat{\mathbf{p}}_{Q,2} + \text{diag}(\mathbf{1}-\boldsymbol{\omega}_{Q,1})\mathbf{\Theta}^m_{11}\widehat{\mathbf{p}}_{Q,1}.$$

From this we can solve the change in prices as $\widehat{\mathbf{p}}_{Q,1} = \tilde{\mathbf{\Theta}}^m_{11}(\boldsymbol{\omega}_{Q,1}\widehat{W}_1 + \mathbf{A}_{12}\widehat{\mathbf{p}}_{Q,2})$.

The goods market clearing conditions are

$$\widehat{\mathbf{p}_{Q,1}\mathbf{q}_1} = \mathbf{F}_1 + \mathbf{\Psi}^m_{11}\widehat{\mathbf{p}_{Q,1}\mathbf{q}_1},$$

where the first term collects demand from consumers in both countries and intermediate input usage by sectors in Country 2. The second term reflects intermediate input usage by sectors in Country 1 taking the input shares as fixed. From the above equation, we can solve $\widehat{\mathbf{p}_{Q,1}\mathbf{q}_1} = \tilde{\mathbf{\Psi}}^m_{11}\mathbf{F}_1$. The

s-th element of the vector $\mathbf{F}_1$ is

$$\begin{aligned}
F_{1s} = {} & \psi^f_{1s,1}\left((1-\gamma_f)(\widehat{P}_Q(1,s)-\widehat{P}_1)+\widehat{P_1C_1}\right) \\
& + \sum_{k\in\mathcal{S}} \psi^m_{1s,1k}\left[(1-\gamma_Q)(\widehat{P}_M(1,k)-\widehat{P}_Q(1,k))+(1-\gamma_m)(\widehat{P}_Q(1,s)-\widehat{P}_M(1,k))\right] \\
& + \psi^f_{1s,2}\left((1-\gamma_f)(\widehat{P}_Q(1,s)-\widehat{P}_2)+\widehat{P_2C_2}\right) \\
& + \sum_{k\in\mathcal{S}} \psi^m_{1s,2k}\left[(1-\gamma_Q)(\widehat{P}_M(2,k)-\widehat{P}_Q(2,k))+(1-\gamma_m)(\widehat{P}_Q(1,s)-\widehat{P}_M(2,k))+\widehat{P}_Q(2,k)+\widehat{Q}(2,k)\right].
\end{aligned}$$

We can then write $\mathbf{F}_1$ in matrix format

$$\begin{aligned}
\mathbf{F}_1 = {} & (1-\gamma_f)\mathbf{\Psi}^f_{11}\circ(\widehat{\mathbf{p}}_{Q,1}-\widehat{P}_1\mathbf{e}_S)+\mathbf{\Psi}^f_{11}\widehat{P_1C_1}+(1-\gamma_m)[\widehat{\mathbf{p}}_{Q,1}\circ(\mathbf{\Psi}^m_{11}\mathbf{e}_S)-\mathbf{\Psi}^m_{11}\widehat{\mathbf{p}}_{Q,1}] \\
& + (1-\gamma_f)\mathbf{\Psi}^f_{12}\circ(\widehat{\mathbf{p}}_{Q,1}-\widehat{P}_2\mathbf{e}_S)+\mathbf{\Psi}^f_{12}\widehat{P_2C_2}+(1-\gamma_m)[\widehat{\mathbf{p}}_{Q,1}\circ(\mathbf{\Psi}^m_{12}\mathbf{e}_S)-\mathbf{\Psi}^m_{12}\widehat{\mathbf{p}}_{Q,2}]+\mathbf{\Psi}^m_{12}\widehat{\mathbf{p}_{Q,2}\mathbf{q}}_2,
\end{aligned}$$

where the first and third terms are the changes in demand due to domestic and foreign consumers' substitution between goods from different sectors, and the second and fourth terms reflect changes in the share of the goods as intermediate inputs in each sector. The fifth term is caused by a change in foreign demand for intermediate inputs, taking the input shares as fixed.

According to the GHH preference, labor supply measured in wage bills can be expressed as

$$\widehat{W}_1+\widehat{H}_1=\widehat{W}_1+\nu(\widehat{W}_1-\widehat{P}_1),$$

where $\widehat{P}_1=\boldsymbol{\omega}^T_{C,11}\widehat{\mathbf{p}}_{Q,1}+\boldsymbol{\omega}^T_{C,21}\widehat{\mathbf{p}}_{Q,2}$ is the aggregate price index in Country 1. Substituting in the solution of $\widehat{\mathbf{p}}_{Q,1}$ obtained earlier, we have

$$\widehat{P}_1=\underbrace{\boldsymbol{\omega}^T_{C,11}\tilde{\mathbf{\Theta}}^m_{11}\boldsymbol{\omega}_{Q,1}}_{\tilde{a}^c_{1L}}\widehat{W}_1+\underbrace{(\boldsymbol{\omega}^T_{C,11}\tilde{\mathbf{\Theta}}^m_{11}\mathbf{A}_{12}+\boldsymbol{\omega}^T_{C,21})}_{\text{sum}\equiv\tilde{a}^c_{21}}\widehat{\mathbf{p}}_{Q,2}=\tilde{a}^c_{1L}\widehat{W}_1+\tilde{a}^c_{21}\widehat{P}^c_{1,M},$$

where we have used the definitions of $\tilde{a}^c_{1L}$, $\tilde{a}^c_{21}$ and $\widehat{P}^c_{1,M}$ and the fact that $\boldsymbol{\omega}^T_{C,11}\tilde{\mathbf{\Theta}}^m_{11}\boldsymbol{\omega}_{Q,1}+\left(\boldsymbol{\omega}^T_{C,11}\tilde{\mathbf{\Theta}}^m_{11}\mathbf{A}_{12}+\boldsymbol{\omega}^T_{C,21}\right)\mathbf{e}_S=1$.

Meanwhile, labor demand measured in wage bills can be written as

$$\sum_s\frac{H(1,s)}{H_1}\left(\left(\frac{\widehat{W_1H(1,s)}}{P_Q(1,s)Q(1,s)}\right)+\widehat{P}_Q(1,s)+\widehat{Q}(1,s)\right)=(1-\gamma_Q)\mathbf{\Gamma}^w_1(\widehat{W}_1\mathbf{e}_S-\widehat{\mathbf{p}}_{Q,1})+\mathbf{\Gamma}^w_1\widehat{\mathbf{p}_{Q,1}\mathbf{q}}_1.$$

Equating labor demand and supply and collecting terms, we obtain the solution to $\widehat{W}_1-\widehat{P}^c_{1,M}$

stated in the proposition. ■

Lemma B-2 *The coefficient $\tilde{a}^c_{1L}$ can be written as*

$$\tilde{a}^c_{1L} = \boldsymbol{\omega}_{Q,1} = \boldsymbol{\Gamma}^w_1 \tilde{\boldsymbol{\Psi}}^m_{11} \boldsymbol{\Psi}^f_{11}.$$

When imports from Country 2 are only used for final consumption, i.e., $\boldsymbol{\Theta}^m_{12} = 0$, the coefficient $\tilde{a}^c_{21}$ is the export-to-GDP ratio.

Proof. We need to prove the following relationship

$$\frac{\left(\boldsymbol{\lambda}_1 \circ \boldsymbol{\Psi}^f_{11}\right)^T}{\boldsymbol{\lambda}_1^T \boldsymbol{\omega}_{Q,1}} \left(\mathbf{I} - \operatorname{diag}(\mathbf{1} - \boldsymbol{\omega}_{Q,1})\boldsymbol{\Theta}^m_{11}\right)^{-1} \boldsymbol{\omega}_{Q,1} = \frac{\left(\boldsymbol{\lambda}_1 \circ \boldsymbol{\omega}_{Q,1}\right)^T}{\boldsymbol{\lambda}_1^T \boldsymbol{\omega}_{Q,1}} \left(\mathbf{I} - \boldsymbol{\Psi}^m_{11}\right)^{-1} \boldsymbol{\Psi}^f_{11},$$

where $\boldsymbol{\lambda}_1$ is a vector of sectoral output shares in Country 1.

Note that the denominators are GDP-to-output ratios. We can actually prove a stronger result

$$\left(\boldsymbol{\lambda}_1 \circ \boldsymbol{\Psi}^f_{11}\right)^T \left(\operatorname{diag}(\mathbf{1} - \boldsymbol{\omega}_{Q,1})\boldsymbol{\Theta}^m_{11}\right)^n \boldsymbol{\omega}_{Q,1} = \left(\boldsymbol{\lambda}_1 \circ \boldsymbol{\omega}_{Q,1}\right)^T \left(\boldsymbol{\Psi}^m_{11}\right)^n \boldsymbol{\Psi}^f_{11},\ n = 0, 1, 2, \ldots$$

This expression holds trivially when $n = 0$. When $n > 0$, notice that there is a tight relationship between $\operatorname{diag}(\mathbf{1} - \boldsymbol{\omega}_{Q,1})\boldsymbol{\Theta}^m_{11}$ and $\boldsymbol{\Psi}^m_{11}$ as follows

$$\boldsymbol{\Psi}^m_{11} = \operatorname{diag}(\boldsymbol{\lambda}_1)^{-1} \left(\operatorname{diag}(\mathbf{1} - \boldsymbol{\omega}_{Q,1})\boldsymbol{\Theta}^m_{11}\right)^T \operatorname{diag}(\boldsymbol{\lambda}_1).$$

Substituting this into the right-hand side and apply the fact that $\boldsymbol{\lambda} \circ \mathbf{x} = \operatorname{diag}(\boldsymbol{\lambda}_1)\mathbf{x}$, as well as the fact that the two sides are scalars and to show $LHS = RHS$ is the same as to show $LHS^T = RHS$, we prove the equality.

When $\boldsymbol{\Theta}^m_{12} = \mathbf{A}_{12} = 0$, it is straightforward that $\tilde{a}^c_{21} = \boldsymbol{\omega}^T_{C,21}\mathbf{e}_S$, which equals the import-to-GDP (and export-to-GDP) share. ■

Proof of Proposition B-1. When labor is the only input and no domestic or foreign intermediate inputs are used, we have $\boldsymbol{\Psi}^m = 0, \boldsymbol{\Theta}^m = 0, \tilde{\boldsymbol{\Theta}}^m_{ii} = \mathbf{I}, \tilde{\boldsymbol{\Psi}}^m_{ii} = \mathbf{I}, \boldsymbol{\omega}_{Q,1} = \mathbf{e}_S$. Applying the formula in Proposition B-1, we have

$$\begin{aligned}
\widehat{W}_1 - \widehat{P}^c_{1,M} = &\left\{(1 - \boldsymbol{\Gamma}^w_1 \boldsymbol{\Psi}^f_{11})(1 + v\tilde{a}^c_{21}) - (1 - \gamma_f)\boldsymbol{\Gamma}^w_1 [\boldsymbol{\Psi}^f_{11} \circ (\boldsymbol{\omega}_{Q,1} - \tilde{a}^c_{1L}\mathbf{e}_S) + \boldsymbol{\Psi}^f_{12} \circ \boldsymbol{\omega}_{Q,1}]\right\}^{-1} \\
&\times \left\{-(1 - \boldsymbol{\Gamma}^w_1 \boldsymbol{\Psi}^f_{11})\widehat{P}^c_{1,M} + (1 - \gamma_f)\boldsymbol{\Gamma}^w_1 \left[\boldsymbol{\Psi}^f_{12}(\widehat{P}^c_{1,M} - \widehat{P}_2)\right] + \boldsymbol{\Gamma}^w_1 \boldsymbol{\Psi}^f_{12} \widehat{P_2 C_2}\right\}
\end{aligned}$$

In addition, we know that $\mathbf{\Theta}_{12}^m = \mathbf{A}_{12} = 0$. From Lemma B-2, we know that $\tilde{a}_{1L}^c = \mathbf{\Gamma}_1^w \mathbf{\Psi}_{11}^f, \tilde{a}_{21}^c = \mathbf{\Gamma}_1^w \mathbf{\Psi}_{12}^f$. $\tilde{a}_{21}^c$ is also the export-to-GDP ratio. We can then simplify the above formula and obtain the expression in Proposition 1. ■