A Theory of International Asset Returns: Country Size and Equity Rebalancing *

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Abstract

This paper provides a theoretical framework to understand the returns of sovereign bonds issued in different currencies in both normal and crisis times. Using a continuous-time two-country Lucas tree model with equity constraint, we show that the country-size effect and the equity-rebalancing effect are the key determinants of sovereign bond returns. The country size effect spills over home production risk to the smaller country through trade and equity rebalancing; equity constraint limits equity rebalancing and contributes to endogenous uncovered interest parity deviations in crisis times. In a period of crisis, the larger country's sovereign bond becomes even safer when the country-size effect collaborates with the equity-rebalancing effect, as is the case with the United States. Our model mechanisms are supported by the empirical evidence for a set of advanced and emerging market economies.

JEL Classification: F31, F41, G15.

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1 Introduction

Sovereign bonds issued in different currencies pay different returns that cannot be compensated by their exchange rate movements. Such persistent difference in bond returns is also known as the failure of the uncovered interest rate parity (UIP), a long-standing puzzle in international finance since the pioneering work of Fama (1984). The failure of UIP occurs both in the time series and in the cross section of countries. Numerous papers in the literature have tried to provide explanations for the failure of UIP in the time series (Backus, Foresi, and Telmer (2001); Alvarez, Atkeson, and Kehoe (2002), Lustig, Roussanov, and Verdelhan (2014); Corsetti and Marin (2020)) and in the cross-section (Lustig and Verdelhan (2007); Lustig, Roussanov, and Verdelhan (2011); Hassan (2013); Hassan and Mano (2019)).

This paper re-visits the failure of UIP in time series and provides a theoretical framework to understand the determinants of UIP premium in both crisis and normal times. A key innovation in the paper is that we leverage insights on portfolio rebalancing from the equity markets to generate rich dynamics in the sovereign bonds and the foreign exchange market. In our model, equity home bias that increases during the crisis regime tightens equity constraint and limits equity rebalancing, leading to return differentials in the sovereign bonds markets and generating endogenous deviations from UIP in the foreign exchange markets.

Our theory shows that the relative size of the country (the country size effect) and the equity rebalancing channel (the equity rebalancing effect) jointly determine sovereign bond returns. Using a continuous-time two-country Lucas tree model with equity constraints, we characterize the model mechanism in closed-form and reconcile the failure of UIP both in normal and in crisis times. Our model mechanisms are supported by empirical evidence and can qualitatively explain the pattern of UIP premium through the interaction of country-size and equity-rebalancing effect in crisis times (e.g., a spike in the global risk sentiment) for a set of advanced and emerging market economies.

In normal times, the two countries in our model – a smaller home country and a larger foreign country – can perfectly share consumption risks through freely adjusting their equity and bond holdings. The country-size effect makes the larger country's bond a global safe asset in normal times as the larger country constitutes most of total world consumption risks through the interna-

tional trade and financial market. Therefore, investors are willing to pay a safety premium for the larger country bond by receiving lower expected returns. This rationalizes the observed UIP premium during normal times. Throughout the paper, we treat the U.S. as the larger foreign country and the G10-currency countries as the smaller home country.

In a period of crisis, both the home and foreign countries are constrained in their equity holdings and have to take on more home risks in consumption than they would ideally prefer. However, the two model mechanisms – namely the country-size and equity-rebalancing effect – work differently for the smaller home and the larger foreign country in crisis. For investors in the smaller home country, the equity-rebalancing effect competes with the country-size effect. If the country-size effect dominates, home bond becomes safer for home investors in crisis and UIP reversal occurs; if the equity-rebalancing effect dominates, the larger foreign country's bond (USD) remains safer for home investors and the flight-to-safety occurs. For investors in the foreign larger country, the equity-rebalancing effect collaborates with the country-size effect and the safety of the larger country's bond is strengthened for home investors in the period of crisis.

In our model, the equity constraint is the key and only departure from an otherwise standard two-country Lucas tree model as in Cochrane, Longstaff, and Santa-Clara (2007). Each country has to hold a positive share of its domestic equity and is capped on its share of foreign equity holdings. The equity constraint delivers the "equity home bias" feature well-documented in the literature (Hau and Rey (2008); Coeurdacier and Rey (2013)) and falls into the class of balance sheet constraints founded by empirical evidence (Du, Tepper, and Verdelhan (2018b); Du, Hébert, and Huber (2019)).[1] In the crisis episodes in our model, we assume that equity constraints are tightened for the home country and equity home bias increases. This assumption is supported by our novel evidence on the cyclicality of the equity rebalancing cycle, and in particular, that equity home bias positively correlates with the global risk sentiments and the crisis periods.

Our model predictions also reconcile with the empirical facts on deviations from the covered interest rate parity (CIP) and convenient yields. The failure of CIP implies a breakdown of the no-arbitrage condition, contrasts the friction-less market assumption, and points to models with financial frictions. There is market segmentation during crisis time in our model with equity con-

[1]For example, Du, Tepper, and Verdelhan (2018b) uses banking regulation to test the balance sheet constraints and shows that the balance sheet constraints have impact on asset prices. Du, Hébert, and Huber (2019) provides direct evidence that the risk of balance sheet constraints becoming tighter is priced.

straints. Such market segmentation limits arbitrage across markets and law of one price is violated in crisis time, which generates the deviations from CIP.

Related Literature. This paper builds on and contributes to the study of currency risk premia, safe assets determination, continuous-time financing and portfolio rebalancings.

Our paper speaks to the gigantic literature on currency risk premium and determinants of uncovered interest parity (UIP) condition. Since the pioneering work of Fama (1984), an older strand of literature focuses mainly on the deviations of UIP in time series, such as Bekaert (1996), Bansal (1997), Backus, Foresi, and Telmer (2001) and Alvarez et al. (2002). Starting with Lustig and Verdelhan (2007), many papers explore UIP deviation in the cross-section of currencies, including Lustig, Roussanov, and Verdelhan (2011), Hassan (2013), and Hassan and Mano (2019)). A few recent papers provide new empirical facts on the cyclicality of UIP and currency premium (Corsetti and Marin (2020); Kalemli-Ozcan and Varela (2021); Jiang, Krishnamurthy, and Lustig (2023)). What's absent in these papers is a unified theory to explain UIP deviations in both normal and crisis times and a model where exchange rates, bonds, and equities are jointly endogenously determined. Our paper provides a parsimonious model that serves such purpose.

Secondly, our two-country Lucas-tree model builds on the classic continuous-time asset pricing framework. Cochrane, Longstaff, and Santa-Clara (2007), Pavlova and Rigobon (2007) and Martin (2011) solve for asset prices in a frictionless two-tree model. A recent two-tree model by Sauzet (2023) studies international portfolio problem in a imperfectly integrated international market with frictions but does not look at the return differentials of bonds nor UIP deviations. The main innovation in our model is to introduce equity constraint that binds during crisis in an otherwise standard two-tree model. The binding equity constraint limits equity rebalancing and generates rich dynamics in the return differentials of sovereign bonds and the exchange rate movements.

In addition, our paper contribute to the empirical literature on the portfolio rebalancing of equities and equity flows, including Hau and Rey (2004), Coeurdacier and Rey (2013), Wynter (2019), Atkeson, Heathcote, and Perri (2022), and Avdjiev, Hardy, Kalemli-Özcan, and Servén (2022). Our paper provides novel empirical evidence on the cyclicality of equity home bias for a set of advanced and emerging market economies.

2 Motivating Empirical Facts

Our research question on the determinants of UIP premium in time series is motivated by the empirical facts on the strong and significant correlation between realized UIP premium and crisis times. The novel fact that equity home bias increases during crises also provides empirical support for the model mechanisms in our paper. Below we document these empirical facts in detail and describe the dataset we use.

2.1 Three Stylized Empirical Facts

We highlight three stylized facts linking UIP premiums to the crisis episodes, as well as its relation to the country-size and the equity rebalancing effect in the crisis episodes. We focus on the a set of advanced economies (G10 currencies) in the main texts and replicate similar empirical facts for a set of emerging market economies in the appendix of the paper[2].

Stylized Fact I*: Realized UIP premium – defined as the excess return in local currencies against the USD – positively correlates with crisis episodes as measured by the VIX index.*

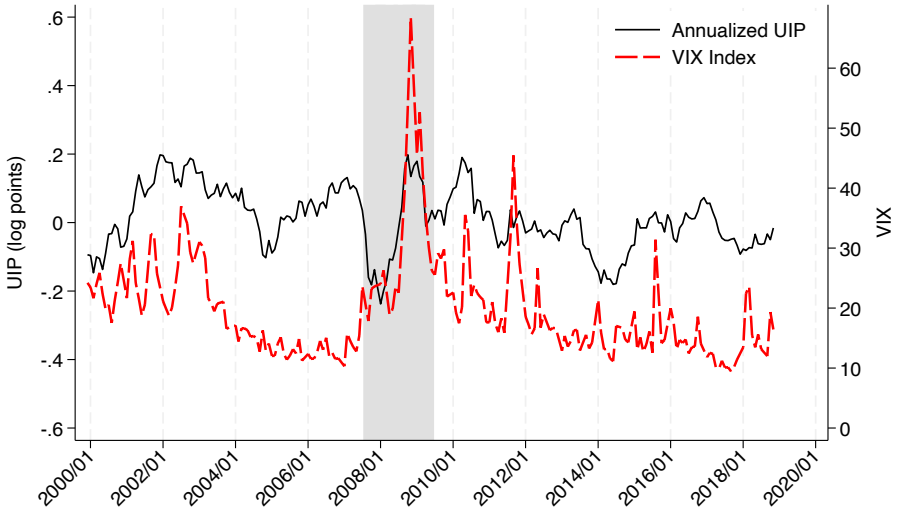
To fix ideas, let us define UIP premium as the excess return of home currency asset against the U.S. Dollar (foreign currency). The home is G10 currencies other than the USD. The expected UIP premium in log points is therefore:

$$\mathbb{E}_t[\lambda_{t+h}] \equiv (i_t - i_t^{\text{US}}) \; - \; (\mathbb{E}_t \; s_{t+h} - s_t)$$

where i_t and i_t^{US} are local and U.S. annualized one-year government bond yields; h is the 12-month horizon. Exchange rate s is in units of local currency per USD; an increase in s would imply local currency depreciation against the USD. When $\mathbb{E}_t[\lambda_{t+h}]$ = 0, the UIP condition holds and there's no excess return from the currency carry trade. If $\mathbb{E}_t\ [\lambda_{t+h}] > 0$, there's positive excess returns for the currency trade that longs home currencies and shorts the USD; vice versa for $\mathbb{E}_t[\lambda_{t+h}] < 0$. In the data, we use realized spot exchange rates, $s_{t+h} - s_t$, to measure ex-post realized UIP premium.

[2]Figures for emerging market economies TBA in the appendix.

Figure 1: Realized UIP Premium Correlates with VIX Index



Note: This panel presents the annualized realized average UIP premium as excess return in local currencies against the USD for countries in the G10 currency group and the VIX index. The UIP premiums are in log points and calculated for one-year government bond yields and shown in the left scale. The VIX index are in red and given by the right sacle. The grey area corresponds to months of the Great Financial Crisis in 2008.

There is a clear positive correlation between the realized local-currency UIP premium and the global risk sentiments measured by the VIX index (Rey (2015); Di Giovanni, Kalemli-Özcan, Ulu, and Baskaya (2022)), as shown in Figure 1. When global risk sentiments are high, local currency UIP premium increases and global investors demand excess premium on the carry trade the longs home currencies (G10) shorts the USD. We also report in Table B.5 the OLS regression of realized UIP premium on VIX index with various controls and the coefficients remain strongly significant. This is consistent with the observed "flight to safety" to the USD during the global financial crisis episodes documented in the literature (Maggiori (2017); Jiang, Krishnamurthy, and Lustig (2023); Kekre and Lenel (2024)).[3]

One caveat in interpreting Figure 1 is whether the results will hold if we use *ex-ante* expected exchange rates for the UIP premium, as the risk appetite shown in VIX would have already been priced in the realized UIP premium. Although we currently lack expected exchange rates data to confirm this, we refer to the key findings of Kalemli-Ozcan and Varela (2021) that using expected UIP premium from the survey data produce no different results compared with the realized UIP premium data. In addition, we provide in Table B.5 in the appendix different regression specifica-

[3]We also plot the realized UIP premium at the currency level, as reported in Table B.1 and B.2 in the Appendix.

tions with lags of VIX as independent variables, which also shown to be significantly correlated with the realized UIP premium.

Stylized Fact II*: The changes in relative country size cannot explain away the variation in the realized UIP premium in time series.*

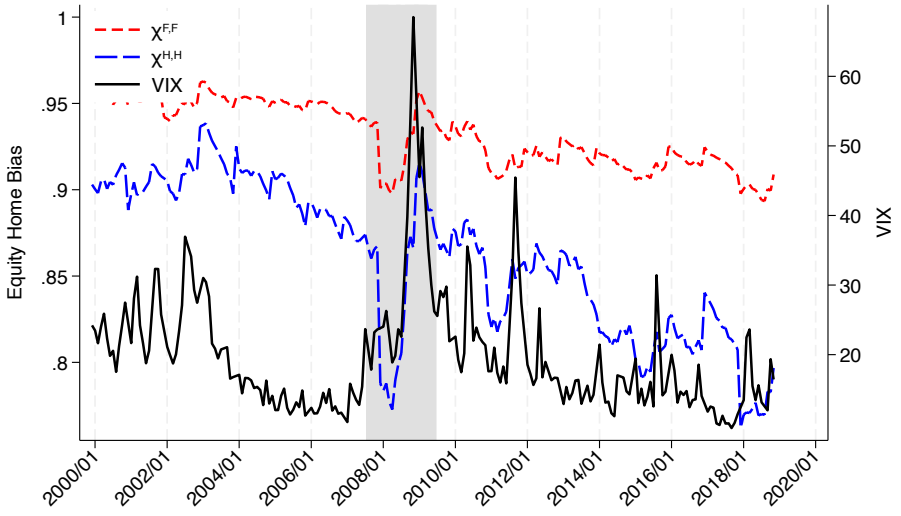
A simple OLS regression of realized UIP premium on relative country size of G10 countries compared to the US produce zero and insignificant correlation, as reported in Table B.5 column (3) in the appendix. The intuition is straightforward: variations in country size (as measured by nominal GDP) are too small in the time series to account for the variations in the UIP premium (Table B.3). Indeed, variations in country size explain variations in UIP premium in the cross-section of countries rather than in the time series, as shown in Table B.4 in appendix. This is consistent with the findings of Hassan (2013), who find negative and statistically significant correlation between the UIP premium and the share country size (measured either by GDP or non-tradables) for th OECD economies.

Stylized Fact III*: The equity rebalancing cycle negatively correlates with crisis episodes as measured by the VIX index and positively correlates with the realized UIP premium. In particular, equity home bias increases during crisis episodes when both VIX index and local-currency UIP premium are high.*

We provide four measures on cross-border equities holdings[4] to measure the equity rebalancing cycle and its relation with crisis episodes. All measures treat G10 currencies other than the USD as home and the USD as foreign. We define $\chi_{H,H}$ as the share of home investors' holdings of home equities over the total supply of home equities; $\chi_{F,H}$ is foreign investors' holdings of home equities over the total supply of home equities; $\chi_{H,F}$ is home investors' holdings of foreign equities over the total supply of foreign equities; Finally, $\chi_{F,F}$ is foreign investors' holdings of foreign equities over the total supply of foreign equities. By construction, $\chi_{H,H}$ and $\chi_{F,H}$ add up to 1 as they share the total supply of home equities as the same denominator; $\chi_{H,F}$ and $\chi_{F,F}$ also add up to 1 and both use the total supply of foreign (USD) equities as the denominator.

[4]Note that the share of equity holdings needs to be distinguished from the equity *portfolio shares* (denoted by θ) that will be introduced later in the model section of the paper. For example, we use $\theta_{H,H}$ to define the share of home equities held by home investors over the total wealth of home investors.

Figure 2: Equity Home Bias Correlates with VIX Index



Note: This panel of figures present the average share of equity holdings or the G10 countries (left scale) and the VIX index (right scale). Home (H) is G10 country, and foreign (F) is the United States. Equity share χ_{HH} is defined as home investors' holdings of home equities over the total supply of home equities.; χ_{FF} is foreign investors' holdings of foreign equities over the total supply of foreign equities (in other words, it measures the equity home bias of U.S.) The shaded line indicates the 2008 financial crisis.

Figure 2 plots the average share of home investors' holdings of home equities ($\chi_{H,H}$) and the share of foreign investors' holdings of foreign equities ($\chi_{F,F}$) together with the VIX index. There's a clear positive correlation between the global risk appetite as measured by the VIX index and the equity rebalancing cycle. Specifically, as global risk sentiment rises, both $\chi_{H,H}$ and $\chi_{F,F}$ increase, suggesting that equity home bias increases for both the G10 countries and the US during the crisis episodes. Together with Stylized Fact I, we document the strong and positive correlation between the equity rebalancing cycles and realized UIP premium. In particular, the crisis episodes are correlated with high global risk sentiments, high realized local-currency UIP premium, and an increase in equity home bias, consistent with the empirical facts in Atkeson, Heathcote, and Perri (2022).[5] We also establish this claim formally through an econometric regression in Table B.6 in the appendix.

[5] Atkeson, Heathcote, and Perri (2022) study the equity holdings of the U.S. and focus the 2008 financial crisis, while we look at a broader set of countries and the cyclicality of the equity rebalancing cycle.

2.2 Data Description

We combine a few public databases to provide empirical evidence on the pattern of UIP premium and its relation with country size and the equity portfolio rebalancings at the financial crisis. The data on exchange rates and government bond yields used to construct UIP premium are from the Bank for International Settlements (BIS). The data on nominal GDP are retrieved from the International Financial Statistics (IFS) provided by the International Monetary Fund (IMF).

External portfolio positions between the US and non-US (home country in our analysis) countries are publicly observable from the US Treasury International Capital (TIC) survey at monthly frequency. The TIC liabilities survey reports the stock dollar value of a non-US country's aggregate holdings of US equity at the aggregate asset class level, and conversely the TIC claims survey reports the stock dollar value of US holdings of the non-US country's equity. The TIC dataset has a wide coverage of countries but it only reports cross-border demand and not demand for domestic equities held by domestic investors. We therefore follow Koijen and Yogo (2020) and use the total market capitalization of all domestic listed firms reported by the World Bank as the total supply of domestic equities. We then use the total foreign demand aggregated from the CPIS data to subtract from the total supply of domestic equities to back out the holdings of domestic equities.

Our sample includes both the group of G10 currencies and the group for emerging market currencies. The G10 currencies besdies the U.S. Dollar (USD) are Euro (EUR), Pound Sterling (GBP), Japanese Yen (JPY), Australian Dollar (AUD), New Zealand Dollar (NZD), Canadian Dollar (CAD), Swiss Franc (CHF), Norwegian Krone (NOK) and the Swedish Krona (SEK). We collapse all G10 currencies other than the US and treat the collective as a single investor country in our analysis. We also replicate the same empirical facts for a set of emerging market currencies in the appendix.[6]

[6][TBA in appendix] We have 12 emerging market currencies in our sample chosen for their availability in the exchange rates, government bond yields and equity holdings data. These 12 currencies are Brazillian Real (BRL), Chilean Peso (CLP), Colombian Peso (COP), Hungarian Forint (HUF), Indonesian Rupiah (IDR), Indian Rupee (INR), Mexican Peso (MXN), Malaysian Ringgit (MYR), Philippine Peso (PHP), Russian Ruble (RUB), Thai Baht (THB) and South African Rand (ZAR).

3 Model Set-up

Time is continuous and infinite horizon, $t \in [0, +\infty)$. There are two countries in the world, home country (denoted by H) and foreign country (denoted by F). For ease of illustration, we will call home country the UK (one of G10-currency countries) and foreign country the US.

Technology Each country is endowed with a tree producing domestic good. The two trees evolve as follows,

$$\frac{dY_{H,t}}{Y_{H,t}} = \mu_{H,t}\, dt + \sigma_{H,t}\, dZ_{H,t}$$
$$\frac{dY_{F,t}}{Y_{F,t}} = \mu_{F,t}\, dt + \sigma_{F,t}\, dZ_{F,t}$$

where $\{\mu_{H,t}, \mu_{F,t}, \sigma_{H,t}, \sigma_{F,t}\}$ are exogenous parameters (or processes). For simplicity, we assume throughout the paper that $\mu_{H,t} = \mu_{F,t} = \mu$ and $\sigma_{H,t} = \sigma_{F,t} = \sigma$.

Preferences In order to highlight our mechanism, we assume homogeneous preference, logarithmic utility, and no consumption home bias for the representative agents of the two countries. The final consumption is a CES aggregate of the two goods produced by the two countries. The expected utility of the representative agent in country i, takes the form

$$\mathbb{E}\int_0^\infty e^{-\rho t} \log C_{H,t}\, dt$$

where

$$C_{H,t} = \left[a^{\frac{1}{\eta}} C_{HH,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} C_{HF,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

a is the share parameter [7]. η is the elasticity of substitution between the two goods, assumed to be greater than 1 and smaller than infinity.

[7] Unlike Pavlova and Rigobon (2008) where a represents the country size, here in our model a is not a key parameter of interest.

Numeraire Define 1 unit of the CES basket of total output $\overline{Y}_t$ as numeraire throughout the paper,

$$\overline{Y}_t \equiv \left[a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

Denote the process of total output $\overline{Y}_t$ as

$$\frac{d\overline{Y}_t}{\overline{Y}_t} = \overline{\mu}_t dt + \overline{\sigma}_t dZ_t$$

where $dZ_t = [dZ_{H,t} \;\; dZ_{F,t}]^T$.

International trade market and exchange rate The international trade market (of home and foreign goods) is frictionless. Denote p_t^H as the price of home good and p_t^F as the price of foreign good. The real exchange rate is given by the relative price of home and foreign good[8],

$$e_t \equiv \frac{p_t^H}{p_t^F} \tag{1}$$

and e_t is also the terms of trade in this model. And denote the endogenous process of real exchange rate, e_t, as

$$\frac{de_t}{e_t} = \mu_t^e dt + (\sigma_t^e)^T dZ_t$$

Equity Each country can issue domestic equity shares in unit supply. The equities are risky claims to domestic trees. Denote S_t^H and S_t^F as the total value of domestic equity and foreign equity respectively. Define $\chi_t^{H,H}$ as the the share of home stock market (apple tree) held by home investor, $\chi_t^{H,F}$ as the share of foreign stock market (orange tree) held by home investor. And similarly define $\chi_t^{F,H}$ as the share of home stock market (apple tree) held by foreign investor and $\chi_t^{F,F}$ as the share of foreign stock market (orange tree) held by foreign investor.

Equity constraint Importantly, equity constraint for home country:

$$0 \leq \chi_t^{H,F} \leq \overline{\chi}^{H,F} \tag{2}$$

[8] An increase of e_t corresponds to an appreciation of home currency relative to foreign currency.

This equation is saying that home investor can not hold more than $\overline{\chi}^{H,F}$ share of foreign equity, nor short-sell foreign equity[9].

And similarly we have equity constraint for foreign country:

$$0 \leq \chi_t^{F,H} \leq \overline{\chi}^{F,H} \tag{3}$$

That is, foreign investor can not hold more than $\overline{\chi}^{F,H}$ share of home equity, nor short-sell home equity.

Sovereign Bond Each country can issue (sovereign) bond in zero net supply. The bonds, denoted as B_t^H and B_t^F, are instantaneously *risk-free in domestic goods* but *not* risk-free in terms of numeraire[10]. The price of home (foreign) bond B_t^H (B_t^F) in terms of numeraire is the same as the price of home (foreign) good p_t^H (p_t^F). Denote $B_t^{H,H}$ as the home bond held by home investors[11] and $B_t^{F,H}$ as the home bond held by foreign investors. And denote $B_t^{H,F}$ as the foreign bond held by home investors and $B_t^{F,F}$ the foreign bond held by foreign investors.

Asset returns We introduce notations for asset returns which are *endogenous* processes. Recall that B_t^H is instantaneously risk-less bond in home good and denote the return process of home bond (in terms of numeraire) as:

$$dr_t^{B^H} = \frac{d(p_t^H B_t^H)}{p_{H,t} B_t^H} = (\mu_{p^H,t} + r_t^H)\, dt + \sigma_{p^H,t}\, dZ_t$$

where $\mu_{p^H,t}$ and $\sigma_{p^H,t}$ are given by the endogenous process

$$\frac{dp_t^H}{p_t^H} = \mu_{p^H,t} t\, dt + \sigma_{p^H,t}\, dZ_t$$

Similarly denote the return process of foreign bond (in terms of numeraire) as:

$$dr_t^{B^F} = \frac{d(p_t^F B_t^F)}{p_{F,t} B_t^F} = (\mu_{p^F,t} + r_t^F)\, dt + \sigma_{p^F,t}\, dZ_t$$

[9]Here one can replace the lower bound 0 to a negative number, say $\underline{\chi}^F$. The key thing is that $\chi_t^{H,F}$ (the share of foreign equity held by home investor) is lower bounded.

[10]Their returns are subject to exchange rate risks through price changes

[11]$B_t^{H,H} > 0$ means lending and $B_t^{H,H} < 0$ means borrowing

Recall that S_t^H is the total value of home equity and define q_t^H as the per unit price of home equity in terms of numeraire $\overline{Y}_t$, that is, $S_t^H = q_t^H \overline{Y}_t$. And postulate the endogenous process of q_t^H as follows

$$\frac{dq_t^H}{q_t^H} = \mu_{q^H,t}dt + \sigma_{q^H,t}dZ \tag{4}$$

The return of home equity in terms of numeraire is given by

$$dr_t^{S^H} = \underbrace{\frac{p_t^H Y_{H,t}}{q_t^H \overline{Y}_t}dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^H \overline{Y}_t)}{q_t^H \overline{Y}_t}}_{\text{capital gain}}$$

and similarly

$$dr_t^{S^F} = \underbrace{\frac{p_t^F Y_{F,t}}{q_t^F \overline{Y}_t}dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^F \overline{Y}_t)}{q_t^F \overline{Y}_t}}_{\text{capital gain}}$$

Forward market Since there is no friction on the bond markets, there naturally exists a FX forward market for home bond and foreign bond. Home investor can enter a FX forward contract (long in home currency and short in foreign currency) with zero cost today which will deliver an instantaneous return $dr_t^{B^H} - dr_t^{B^F}$.[12].

Wealth and portfolio shares We introduce notations for wealth and portfolio shares, which will be determined in equilibrium. Denote the aggregate wealth of home country as W_t^H and the aggregate wealth of foreign country as W_t^F.

Denote $\theta_t^{H,S^H} = \frac{\chi_t^{H,H} S_t^H}{W_t^H}$ as the portfolio share of home equity for home country, $\theta_t^{H,S^F} = \frac{\chi_t^{H,F} S_t^F}{W_t^H}$ as the portfolio share of foreign equity for home country. And similarly, denote $\theta_t^{F,S^H} = \frac{\chi_t^{F,H} S_t^H}{W_t^F}$ as the portfolio share of home equity for foreign country and $\theta_t^{F,S^F} = \frac{\chi_t^{F,F} S_t^F}{W_t^F}$ as the portfolio share of foreign equity for foreign country.

We can similarly define portfolio shares of bonds in home and foreign country. Denote $\theta_t^{H,B^H} = \frac{p_t^H B_t^{H,H}}{W_t^H}$ as the portfolio share of home bond for home country, $\theta_t^{H,B^F} = \frac{p_t^H B_t^{H,F}}{W_t^H}$ as the portfolio share of foreign bond for home country, And similarly, denote $\theta_t^{F,B^H} = \frac{p_t^H B_t^{F,H}}{W_t^F}$ as the portfolio share of home bond for foreign country and $\theta_t^{F,B^F} = \frac{p_t^F B_t^{F,F}}{W_t^F}$ as the portfolio share of foreign bond for foreign country.

[12]In equilibrium, investors will exactly do so as discussed in appendix

Country Size Define relative size of home country as follows.

$$s_t = \frac{a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}}}{a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}} = a^{\frac{1}{\eta}} \left(\frac{Y_{H,t}}{\overline{Y_t}} \right)^{\frac{\eta-1}{\eta}} \tag{5}$$

Optimization problems The optimization problem for home country is as follows:

$$\begin{aligned} \max_{\{C_{HH,t}, C_{HF,t}, \chi_t^{H,H}, \chi_t^{H,f}, \theta_t^{H,B^H}, \theta_t^{H,B^F}\}_{t=0}^{\infty}} \quad & \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log \left(\left[a^{\frac{1}{\eta}} C_{HH,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} C_{HF,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \right) dt \right] \\ \text{s.t.} \quad \frac{dW_t^H}{W_t^H} &= \frac{\chi_t^{H,H} S_t^H}{W_t^H} dr_t^{S^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} dr_t^{S^F} + \theta_t^{H,B^H} dr_t^{B^H} + \theta_t^{H,B^F} dr_t^{B^F} \\ & \quad - \frac{p_t^H C_{HH,t} + p_t^F C_{FF,t}}{W_t^H} dt \\ 1 &= \frac{\chi_t^{H,H} S_t^H}{W_t^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} + \theta_t^{H,B^H} + \theta_t^{H,B^F} \\ 0 &\leq \chi_t^{H,F} \leq \overline{\chi}^{H,F} \end{aligned} \tag{6}$$

The optimization problem for foreign country is similar and discussed in appendix.

Market clearing conditions Home equity market clears,

$$\chi_t^{H,H} + \chi_t^{F,H} = 1 \tag{7}$$

Foreign equity market clears,

$$\chi_t^{H,F} + \chi_t^{F,F} = 1 \tag{8}$$

Home bond market clears,

$$B_t^{H,H} + B_t^{F,H} = 0 \tag{9}$$

And foreign bond market clears,

$$B_t^{H,F} + B_t^{F,F} = 0 \tag{10}$$

Total consumption of home (foreign) good equals total production of home (foreign) good,

$$C_{HH,t} + C_{FH,t} = Y_{H,t} \tag{11}$$

$$C_{HF,t} + C_{FF,t} = Y_{F,t} \tag{12}$$

4 Complete market model

Before solving the model with equity constraints, it is useful to solve for the complete market case which works as a clear illustration of the country size spillover effect.

Postulate two stochastic discount factor processes for the two countries, $\xi_{H,t} = e^{-\rho t}\frac{1}{C_{H,t}}$ and $\xi_{F,t} = e^{-\rho t}\frac{1}{C_{F,t}}$, as

$$\frac{d\xi_{H,t}}{\xi_{H,t}} = -r^f_{H,t}\,dt - m^T_{H,t}\,dZ_t$$

$$\frac{d\xi_{F,t}}{\xi_{F,t}} = -r^f_{F,t}\,dt - m^T_{F,t}\,dZ_t$$

respectively. $m_{H,t}$ is the vector of risk prices in home country and also the consumption risk[13] in the logarithmic utility case.

In the complete market case, there exists a unique stochastic discount factor ξ_t such that $\frac{d\xi_{H,t}}{\xi_{H,t}} = \frac{d\xi_{F,t}}{\xi_{F,t}} = \frac{d\xi_t}{\xi_t}$.

4.1 Sovereign bond safety and country size spillover effect

Proposition 1 (Sovereign bond safety)**.** *Expected return difference between home bond and foreign bond is given by*

$$\frac{\mathbb{E}_t\left[dr_t^{B^H} - dr_t^{B^F}\right]}{dt} = m^T_{H,t}\sigma^e_t = m^T_{F,t}\sigma^e_t \tag{13}$$

where $dr_t^{B^H}$ is the return process for home bond, $dr_t^{B^F}$ is the return process for foreign bond, σ^e_t is the exchange rate risk.

If home country's consumption risk is positively correlated with exchange rate risk (domestic consumption is low when domestic currency depreciates), then home bond earns a positive risk

[13] Consumption risk of a country is defined as the volatility vector of consumption process of that country, $\frac{dC_{H,t}}{C_{H,t}}$.

premium.

If home country's consumption risk is negatively correlated with its exchange rate (domestic consumption is high when domestic currency depreciates), then home bond earns a negative safety premium.

Proof. see appendix □

The intuition is as follows: A bond is considered safe if it has high value when consumption is low, because the bond insures investors against bad times. In our example, US treasury pays lower expected return than UK government bond if GBP depreciates against USD when consumption is low. Because in this case, US treasury is a good hedge for consumption risk and is viewed as safe, while UK government bond does not hedge consumption risk and is viewed as risky. Lustig and Verdelhan (2007) provides empirical evidence for proposition 1.

In our model, uncertainty comes from production fluctuations of the trees. When UK production declines due to a negative shock, the supply of UK good declines, and the relative price of UK good should go up, implying a higher expected return of UK bond. However, this is not the whole story for bond safety. Because the final consumption is an aggregate of both countries' goods, another competing force emerges: the demand for US good increases because of consumption smoothing motive. The final consumption shifts more towards US good than before due to a supply drop of UK good. This positive demand shock for US good will put upward pressure on the expected return of US bond. The next proposition shows that the relative magnitude of the supply force and demand force is determined by the relative country size.

Proposition 2 (Country size spillover effect)**.** *Solving for* (13)*, we have*

$$\frac{\mathbb{E}_t[dr_t^{B^H} - dr_t^{B^F}]}{dt} = \overline{\sigma}_t^T \sigma_t^e = \begin{bmatrix} s_t\sigma_H & (1-s_t)\sigma_F \end{bmatrix} \begin{bmatrix} -\frac{1}{\eta}\sigma_H \\ \frac{1}{\eta}\sigma_F \end{bmatrix} \tag{14}$$

$$= \frac{1}{\eta}(-s_t\sigma_H^2 + (1-s_t)\sigma_F^2) \tag{15}$$

and the safety threshold

$$s^C = \frac{\sigma_F^2}{\sigma_H^2 + \sigma_F^2} \tag{16}$$

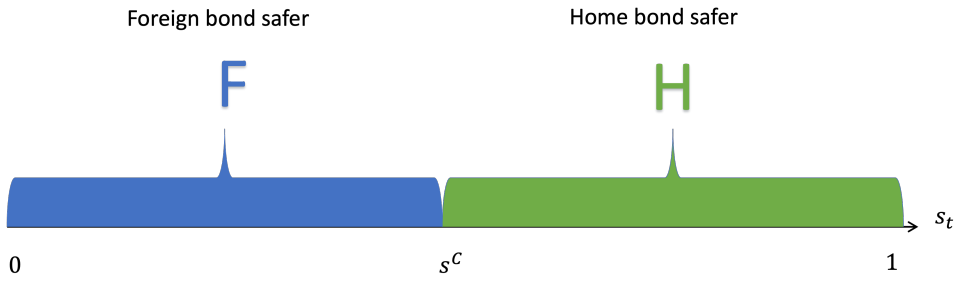
If $s_t < s^C$, home country is a relatively small country and home bond is riskier than foreign bond.

If $s_t > s^C$, home country is a relatively large country, country bond is safer than foreign bond.

Proof. see appendix □

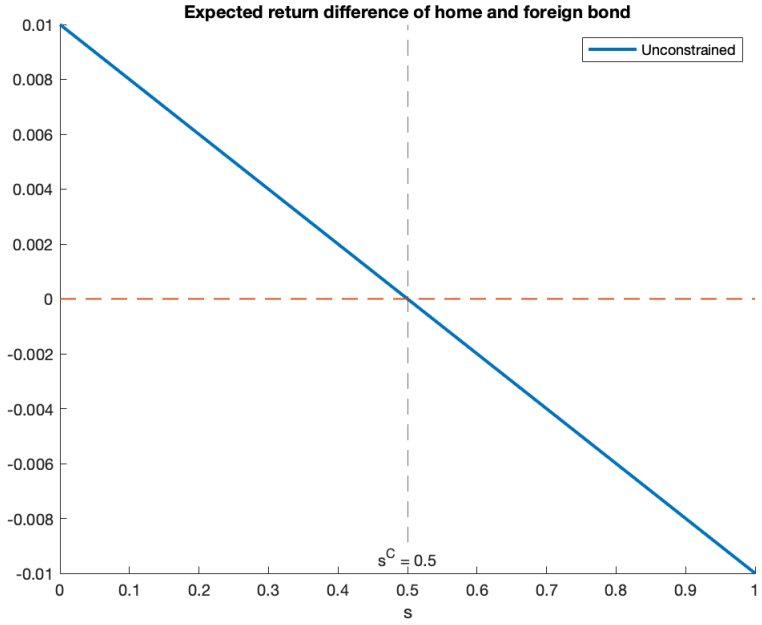
Country size spillover effect states that larger country's bond is safer. US treasury is safer than UK government bond because the size of US economy is a larger than the size of UK economy. Since US contributes a larger share to world consumption, the world consumption risk also consists largely US risk. US treasury becomes a safe asset and pays lower expected return because it is a better hedge for world consumption risk. This is often referred to as the exorbitant privilege of the US Dollar. Going back to the supply force and demand force discussed earlier: the larger the country's share in the world consumption, the larger the magnitude of supply or demand change of its good. When the small country becomes smaller, the large country's production becomes more dominant in world consumption, strengthening the hedging benefit of the large country's bond. On the other hand, when the small country grows larger, world consumption depends less on the large country's production, reducing the hedging benefit of the large country's bond. Country size spillover effect is stronger when there is more asymmetry in s_t and $1 - s_t$. Expected return differences of sovereign bonds are sizable and persistent across countries as discussed in Hassan (2013), which also provides empirical evidence for country size spillover effect.

Figure 3: safety region of home and foreign bond



Note: This figure shows the safety region of home and foreign bond as the relative size of home country s_t changes. s^C is the safety threshold where UIP changes sign.

Figure 4: Expected return difference of home and foreign bond, complete market



Note: The y-axis of this figure shows the expected return difference of home and foreign bond. The x-axis is the relative size of home country s. The dotted vertical line s^C represents the endogenous threshold for UIP changing sign. Parameter values used: growth volatility $\sigma_H^2 = \sigma_F^2 = \sigma^2 = 0.04$, and trade elasticity $\eta = 4$.

4.2 Persistence and asymmetry

In the simplest complete market case, on top of country size spillover effect, there are also persistence effect and asymmetry effect on prices of risks through changes in country size s_t.

The volatility [14] of country size s_t and the prices of risks during normal times are given by

$$\sigma_{s_t} = \frac{\eta - 1}{\eta}(1 - s_t) \begin{bmatrix} \sigma_H \\ -\sigma_F \end{bmatrix} \tag{17}$$

and

$$m_{H,t} = m_{F,t} = \overline{\sigma}_t = \begin{bmatrix} \underbrace{s_t \sigma_H}_{\text{price of home country risk}} \\ \underbrace{(1 - s_t)\sigma_F}_{\text{price of foreign country risk}} \end{bmatrix} \tag{18}$$

[14] Throughout this paper, we denote the volatility of the process $\frac{dX_t}{X_t}$ as volatility of X_t.

Persistence A temporary negative shock to home country's production immediately reduces home country's relative country size, decreases the price of home country risk. In addition, a smaller s_t also affects the magnitude of future shocks on country size s_t as well as prices of risks $\overline{\sigma}_t$. Unlike classic works in macro-finance literatrue (Bernanke et al. (1999) etc), where the persistence of a temporary shock is due to changes in current and future investment, the persistence here in our benchmark model without investment is purely from changes in country size s_t.

Asymmetry A negative shock to home country production affects both price of home country risk and price of foreign country risk through country size spillover effect. This shock affect prices of risks asymmetrically through changes in country size s_t. As in equation (18), the decline of home risk price is mitigated by the smaller size of home country, s_t, while increase of foreign country risk price is amplified by the larger size of foreign country, $1 - s_t$. The same shock thus affects prices of the two countries' bonds and equities asymmetrically.

5 Model with equity constraints

Adding another key ingredient to the model, the equity constraints, we proceed in two steps.

First step, we explore what happens with only one equity constraint for foreign country's holding of home equity, $0 \leq \chi_t^{F,H} \leq \overline{\chi}^{F,H}$. There is a maximum limit on home equity share held by foreign investors and no short-selling of home country's equity is allowed. As shown in the following proposition 3, the two countries can still perfectly share consumption risk and have the same prices of risks as in the complete market case.

Second step, we explore the full model with equity constraints for both countries. There exists an endogenous crisis regime in the model which results in asymmetry and instability of the system.

To highlight the mechanism and simplify some algebra for illustration purpose, we assume symmetric parameters for the two trees $\mu_1 = \mu_2 = \mu$ and $\sigma_H = \sigma_F = \sigma$ hereafter. And taking advantage of symmetry, we focus on analysing home country (Home). The symmetric assumption makes sense in the example of UK and US, as the two countries have similar growth rates and volatilities. And we will focus on empirically relevant case where home country (UK) is a small country relative to foreign country (US).

5.1 Normal regime

Proposition 3. *With only one equity constraint for foreign country's holding of home equity,* $0 \leq \chi_t^{F,H} \leq \overline{\chi}^{F,H}$*, the two countries can perfectly share consumption risk and replicate the complete market case result in the sense that proposition 1 and proposition 2 still hold true.*

Proof. see appendix □

An intuitive way to look at proposition 3 is to count the risks and assets. There are two sources of risks, from the two trees. Even with one equity holding constraint, there are still another three assets that can be freely traded which can span all the possible states of the world. So investors in the two countries can still replicate first best risk-sharing through portfolio re-balancing. Similar to the complete market case, there is indeterminacy in the model with respect to portfolio holdings but not asset returns.

Proposition 4. *With the only equity constraint for foreign country's holding of home equity,* $0 \leq \chi_t^{F,H} \leq \overline{\chi}^{F,H}$*, a special specification for discount rate* $\rho = (\frac{\eta-1}{\eta}\sigma)^2$*, initial condition* s_0*, home country's equity shares and bond holdings are given by*

$$\begin{aligned}
\chi_t^{H,H} &\in [1-\overline{\chi}^{F,H}, 1] \\
\chi_t^{H,F} &= \frac{n_0 - \chi_t^{H,H}\rho q_t^H}{1-\rho q_t^H} \\
\theta_t^{H,B^H} &= \frac{\rho (q_t^H)'(s_t) s_t (1-s_t)(\chi_t^{H,H} - \chi_t^{H,F})(\eta-1)}{n_0} \\
\theta_t^{H,B^F} &= -\theta_t^{H,B^H}
\end{aligned}$$

where

$$q_t^H(s) = \frac{1}{2\rho}\Big(1 + \frac{1-s}{s}\ln(1-s) - \frac{s}{1-s}\ln(s)\Big)$$

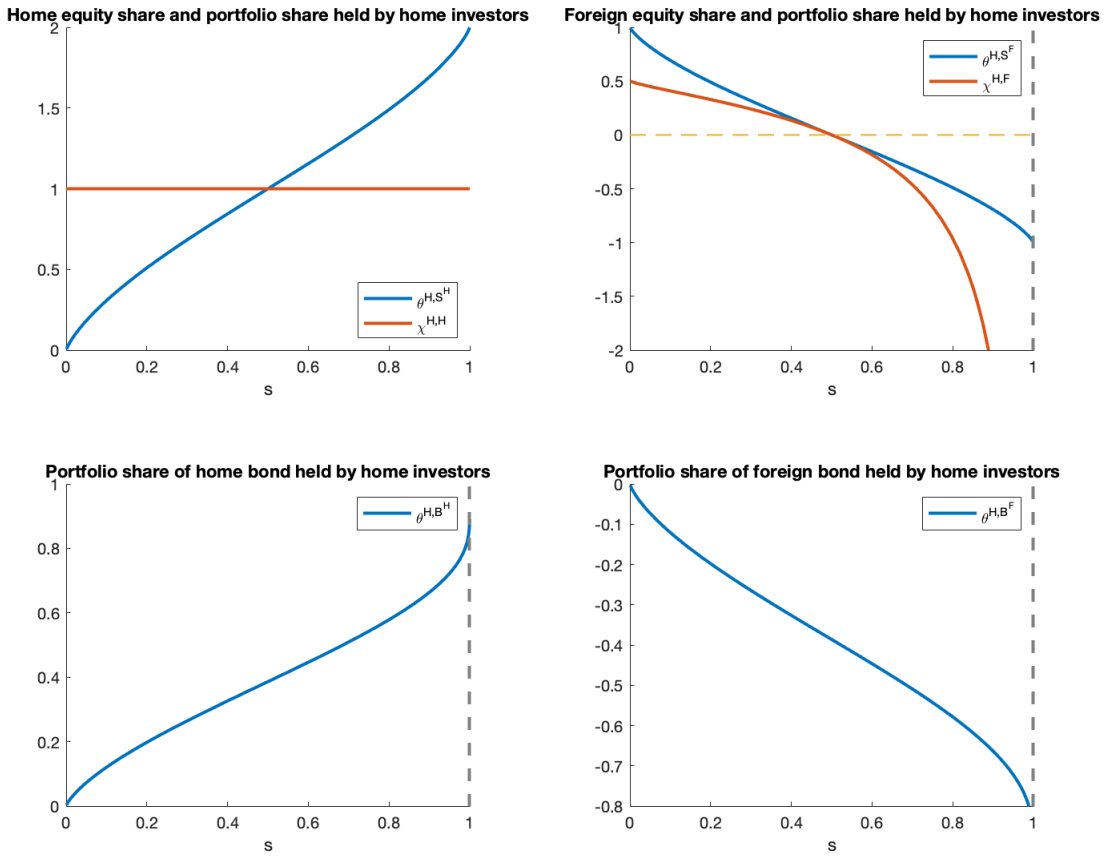
is the per unit price of home equity and taking derivatives with respect to s_t*, we have*

$$(q_t^H)'(s_t) = -\frac{1}{2\rho}\frac{1}{s(1-s)}\Big(1 + \frac{1-s}{s}\ln(1-s) + \frac{s}{1-s}\ln(s)\Big)$$

And $n_0 = q^H(s_0)$ *is the initial wealth share of home country.*

Proof. see appendix □

Figure 5: Equity shares and portfolio shares of home investors



Note: This panel of figures present the equity shares and portfolio shares of home investors solved from the model. For all panels, the x-axis is the relative size of home country s. For the top left panel, the equity share $\chi^{H,H}$ is defined as home investors' holdings of home equities over the total supply of home equities; and the portfolio share of home equity θ^{H,S^H} is defined as home investors' holdings of home equities over their total wealth. For top right panel, $\chi^{H,F}$ is home investors' holdings of foreign equities over the total supply of foreign equities; and the portfolio share of foreign equity θ^{H,S^F} is defined as home investors' holdings of foreign equities over their total wealth. For the bottom left panel, θ^{H,B^H} is defined as home investors' holdings of home bond over their total wealth And for the bottom right panel, θ^{H,B^F} is defined as home investors' holdings of foreign bond over their total wealth. Parameter vlues used: volatility of GDP growth $\sigma^2 = 0.04$, trade elasticity $\eta = 2$, and equity share $\chi^{H,H} = 1$.

From proposition 4 and figure 5, we see that the net borrowing in bonds between the two countries is zero. The two countries smooth their consumption by holding equities and use bonds to help achieve perfect risk-sharing. Both countries go short in foreign bond and long in domestic bond. Because domestic bond is a better hedge for domestic risk, the two countries can offload the

extra domestic risk from domestic equity holding requirement by lending in domestic bond and borrowing in foreign bond.

Comparing figure 4 and the right-bottom panel of figure 5, we see that countries borrow more in foreign bond when their country size grows larger and domestic bond becomes safer, fixing home country's holding of domestic equity share. Because when a country grows larger, its domestic equity price increases, leading to a heavier portfolio weight on domestic equity and thus more domestic risk exposure, which requires more hedging. This is consistent with the empirical fact documented by Du, Pflueger, and Schreger (2020). Until now, the model with only country size spillover effect can explain the UIP violation in normal times and find empirical support from earlier work. However, the model does not have space for crisis yet and is thus silent about what happens in crisis.

5.2 Crisis regime

Moving on to second step, with equity holding constraints for both countries, an endogenous crisis regime emerges and the system moves into the crisis regime when one country falls too small.

Proposition 5 (Crisis regime)**.** *The system moves in to crisis regime if* $s_t \in [0, s^U] \cup [1 - s^U, 1]$*, where* s^U *is the crisis boundary and solves*

$$q_1(s^U) = \frac{n_0 - \overline{\chi}^{H,F}}{\rho(1 - \overline{\chi}^{H,F})} \tag{19}$$

If $s_t \in [0, s^U]$*, we have*

$$\chi_t^{H,H} = 1, \quad \chi_t^{H,F} = \overline{\chi}^{H,F}$$

If $s_t \in [1 - s^U, 1]$*, we have*

$$\chi_t^{H,H} = 1 - \overline{\chi}^{F,H}, \quad \chi_t^{H,F} = 0$$

Proof. see appendix. □

The crisis boundary s^U is the left margin where both countries' equity holding constraints bind and $1 - s^U$ is the right margin where the equity holding constraints bind in the opposite direction.

When $s_t < s^U$, the two countries can perfectly share exchange rate risk through freely adjusting their equity holdings and trading on the FX market. The gains and losses from FX market will be delivered by capital flows induced by equity trading, until both equity constraints bind. we refer to the constrained region as *crisis regime*. In crisis regime, risk-sharing is limited, and asset returns vary discontinuously from in normal regime due to the constraint on equity rebalancing.

5.2.1 Safety spectrum

In crisis regime, equity reblancing is constrained and risk-sharing is limited. This market segmentation drives a wedge between normal time SDF and crisis time SDF, thus a wedge of risk prices between normal times and crisis time. We refer to the effect of the constraints on equity reblancing as the *equity rebalancing effect*[15].

Proposition 6 (equity rebalancing effect)**.** *In crisis region, there exists a wedge between the normal time SDF and crisis time SDF, due to lack of equity rebalancing. For home country investors, denote this wedge as* σ_{n_t}.

If $0 < s_t < s^U$,

$$\sigma_{n_t} = \frac{(1-\overline{\chi}^{H,F})s_t}{(1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}} \frac{\eta-1}{\eta}(1-s_t) \begin{bmatrix} \sigma_H \\ -\sigma_F \end{bmatrix} \tag{20}$$

If $1 - s^U < s_t < 1$,

$$\sigma_{n_t} = \frac{\eta-1}{\eta}(1-s_t) \begin{bmatrix} \sigma_H \\ -\sigma_F \end{bmatrix} \tag{21}$$

symmetrically for foreign investors.

Proof. see appendix □

With two equity constraints and the crisis region, the model exhibits a safety spectrum for each country with four regions identified by three key safety thresholds.

Proposition 7 (Safety spectrum)**.** *With equity holding constraints for both countries,* $0 \leq \chi_t^{F,H} \leq \overline{\chi}^{F,H}$ *and* $0 \leq \chi_t^{H,F} \leq \overline{\chi}^{H,F}$*, and reasonable parameter restrictions on* $(\overline{\chi}^{F,H}, \overline{\chi}^{H,F}, \eta)$*, there are*

[15] To be precise, this is "the lack of equity rebalacing" effect

three key thresholds, normal time safety threshold s^C, crisis boundary s^U, and crisis time safety threshold s^A,

$$s^C = \frac{1}{2} \tag{22}$$

$$q^H(s^U) = \frac{n_0 - \overline{\chi}^{H,F}}{\rho(1 - \overline{\chi}^{H,F})} \tag{23}$$

$$\frac{2(1 - \overline{\chi}^{H,F})(s^A)^2 + (2\overline{\chi}^{H,F}\eta + (1 - \overline{\chi}^{H,F})(\eta - 2))s^A - \eta\overline{\chi}^{H,F}}{(1 - \overline{\chi}^{H,F})s^A + \overline{\chi}^{H,F}} = 0 \tag{24}$$

such that

$$0 < s^A < s^U < s^C \tag{25}$$

When $s^U < s_t < 1 - s^U$, the system stays in normal regime, country size determines bond safety as in proposition 2:

If $s^U < s_t < s^C$, home country is a relatively small country, home country's bond is risky.

If $s^C < s_t < 1 - s^U$, home country is a relatively large country, home country's bond is safe.

When $0 < s_t < s^U$ or $1 - s^U < s_t < 1$, the system moves into crisis regime. Country size spillover effect and the equity rebalancing effect will jointly determine sovereign bond safety.

When $0 < s_t < s^U$, the system moves into crisis regime where country size spillover effect competes with equity rebalancing effect:

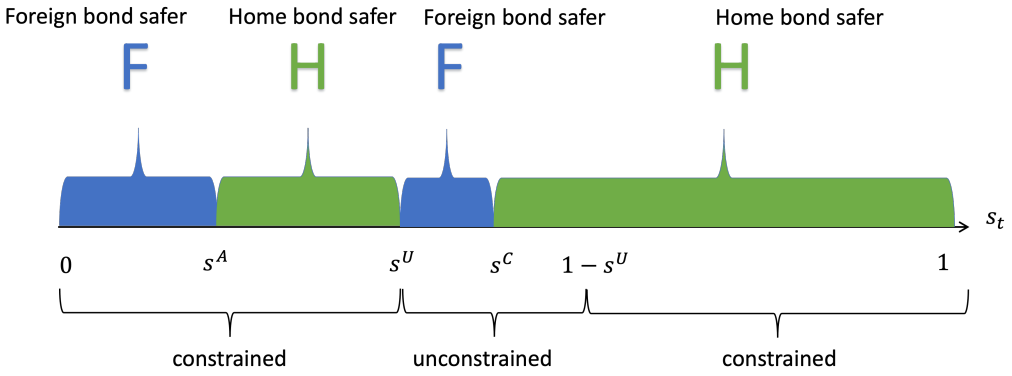
If $s^A < s_t < s^U$, equity rebalancing effect dominates, home country's bond is safe for domestic investors.

If $0 < s_t < s^A$, country size spillover effect dominates, home country's bond is risky for domestic investors.

When $1 - s^U < s_t < 1$, the system moves into crisis regime where country size spillover effect joins forces with equity rebalancing effect: home country's bond is safe for domestic investors.

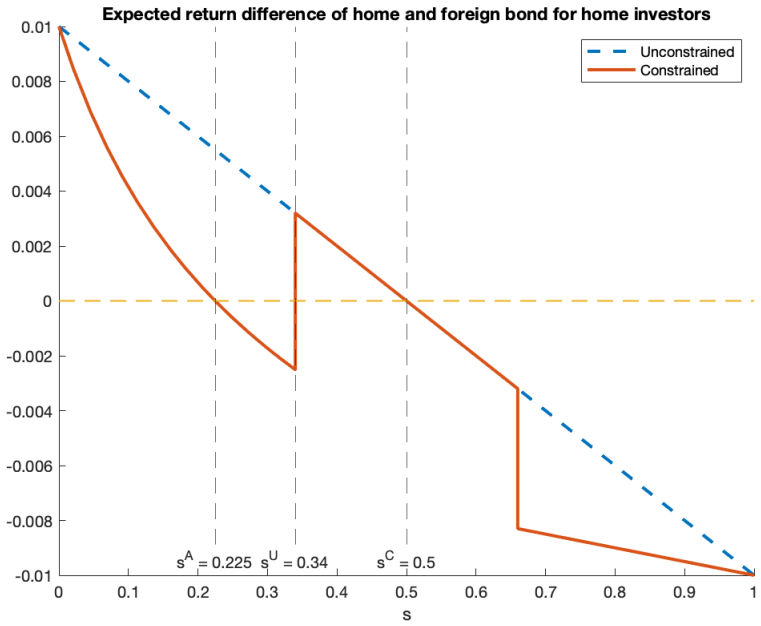
Proof. see appendix □

Figure 6: safety region of home and foreign bond for home investors



Note: This figure shows the safety region of home and foreign bond as the relative size of home country s_t changes, from the perspective of home investors. s^U is the boundary of crisis regime (constrained) as home country falls in relative country size, symmetrically for $1 - s^U$. s^C and s^A are the safety thresholds where UIP changes sign in normal regime (unconstrained) and crisis regime (constrained).

Figure 7: Expected return difference of home and foreign bond for home investors



Note: The y-axis of this figure shows the expected return difference of home and foreign bond from the perspective of home investors. The x-axis is the relative size of home country s. The dotted blue line (unconstrained) corresponds to the complete market model. The solid orange line (constrained) corresponds to the full model with two equity constraints. The three dotted vertical lines (from left to right) s^A, s^U, s^C represents the endogenous threshold for UIP changing sign in crisis regime, the boundary of crisis regime, and the threshold for UIP changing sign in normal regime, respectively. Parameter values used: $\eta = 4$, $\overline{\chi}^{F,H} = \overline{\chi}^{H,F} = 0.2$, $\sigma^2 = 0.04$.

As shown in figure 6, cut by the three thresholds, there are four regions along the safety spectrum. Blue regions represent where foreign bond is safer for domestic investors, and green regions

represent where domestic bond is safer. Figure 7 shows the expected return difference between domestic bond and foreign bond for home country investors.

In normal regime where $s^U < s_t < 1 - s^U$, there is only the familiar country size spillover effect: the larger country's bond is safer. If home country's size continues falling below s^U, the risk-sharing is limited by the equity holding constraints.

In crisis regime, investors in both countries are forced to hold more domestic risk and less foreign risk compared to the perfect risk-sharing scenario in normal regime. Because of limited risk-sharing, domestic bond becomes safer for domestic investors in crisis regime than in normal regime, as it is a better hedge for domestic risk.

equity rebalancing effect improves safety of the domestic bond for domestic investors in crisis regime while country size spillover effect improves safety of the larger country's bond. So in crisis regime, country size spillover effect competes with equity rebalancing effect for the smaller country's investors but collaborates for the larger country's investors.

If the smaller country's size falls in between s^A and s^U, equity rebalancing effect dominates country size spillover effect. The smaller country's domestic bond is safe for domestic investors. If the smaller country falls below s^A, country size spillover effect dominates equity rebalancing effect. The smaller country's domestic bond is risky for domestic investors.

Whereas for the larger country when $1 - s_t > s^C$, its domestic bond is always safe for domestic investors. The safety of the larger country's domestic bond is discontinuously strengthened when $1 - s_t > s^U$ due to equity rebalancing effect.

5.2.2 Domestic amplification

equity rebalancing amplifies the effect of domestic shock. This domestic amplification exists both in the "time series" (compared to normal regime) and in the "cross secrion" (compared to foreign country).

In the crisis regime $[0, s^U]$, the prices of risks for home country investors is given by

$$m_{H,t} = \frac{(1-\overline{\chi}^{H,F})s_t}{(1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}}\sigma_{s_t} + \overline{\sigma}_t = \begin{bmatrix} \underbrace{\left[\frac{(1-\overline{\chi}^{H,F})}{(1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}}\frac{\eta-1}{\eta}(1-s_t)+1\right]s_t\sigma_H}_{\geq s_t\sigma_H, \text{ normal time price of home country risk}} \\ \underbrace{\left[1-\frac{(1-\overline{\chi}^{H,F})s_t}{(1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}}\frac{\eta-1}{\eta}\right](1-s_t)\sigma_F}_{\leq (1-s_t)\sigma_F, \text{ normal time price of foreign country risk}} \end{bmatrix} \tag{26}$$

and the prices of risks for foreign country investors is given by

$$m_{F,t} = \sigma_{1-s_t} + \overline{\sigma}_t = \begin{bmatrix} \underbrace{\frac{1}{\eta}s_t\sigma_H}_{\leq s_t\sigma_H, \text{ normal time price of home country risk}} \\ \underbrace{\left[\frac{\eta-1}{\eta}s_t + (1-s_t)\right]\sigma_F}_{\geq (1-s_t)\sigma_F, \text{ normal time price of foreign country risk}} \end{bmatrix} \tag{27}$$

For home country investors, compared to in normal times, the effect of a shock on domestic risk price is amplified by the factor

$$\frac{(1-\overline{\chi}^{H,F})}{(1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}}\frac{\eta-1}{\eta}(1-s_t)+1 > 1$$

and the effect of a shock on foreign risk price is mitigated by the factor

$$1-\frac{(1-\overline{\chi}^{H,F})s_t}{(1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}}\frac{\eta-1}{\eta} < 1$$

Similarly for foreign country investors, the effect of a shock on domestic risk price is amplified by the factor

$$\frac{\eta-1}{\eta}\frac{s_t}{1-s_t}+1 > 1$$

and the effect of a shock on foreign risk price is mitigated by the factor

$$\frac{1}{\eta} < 1$$

In crisis regime, domestic risk price response to a shock is amplified and foreign risk price response

to a shock is mitigated compared to in normal times due to lack of equity rebalancing. Domestic amplification improves the hedging benefit of domestic bond in crisis compared to in normal times.

In another dimension, comparing the prices of risks between home country investors and foreign country investors, we have

$$\left[\frac{(1-\overline{\chi}^{H,F})}{(1-\overline{\chi}^{H,F})s_t+\overline{\chi}^{H,F}}\frac{\eta-1}{\eta}(1-s_t)+1\right]s_t\sigma_H > s_t\sigma_H > \frac{1}{\eta}s_t\sigma_H$$

where the first term is domestic risk price for home country investors in crisis regime $\left[0, s^U\right]$, the second term is home country risk price in normal regime $\left[s^U, s^C\right]$, and the third term is foreign risk price for foreign country investors in crisis regime $\left[0, s^U\right]$. And similarly

$$\left[1-\frac{(1-\overline{\chi}^{H,F})s_t}{(1-\overline{\chi}^{H,F})s_t+\overline{\chi}^{H,F}}\frac{\eta-1}{\eta}\right](1-s_t)\sigma_F < (1-s_t)\sigma_F < \left[\frac{\eta-1}{\eta}s_t+(1-s_t)\right]\sigma_F$$

where the first term is foreign risk price for home country investors in crisis regime $\left[0, s^U\right]$, the second term is foreign country risk price in normal regime $\left[s^U, s^C\right]$, and the third term is domestic risk price for foreign country investors in crisis regime $\left[0, s^U\right]$. In crisis regime, domestic investors hold more domestic risk than foreign investors and thus require a higher risk premium than foreign investors. Domestic amplification drives up domestic risk price and pushes down foreign risk price in crisis for investors in both countries.

5.2.3 Domestic and global safety

In crisis regime, because of domestic amplification, domestic safety status of bonds may or may not coincide with global safety status of bonds. As shown in the following proposition 8, the smaller country's bond is domestically safe in mild crisis when country sizes are mildly asymmetric and the larger country's bond is globally safe in deep crisis when country sizes are sufficiently asymmetric.

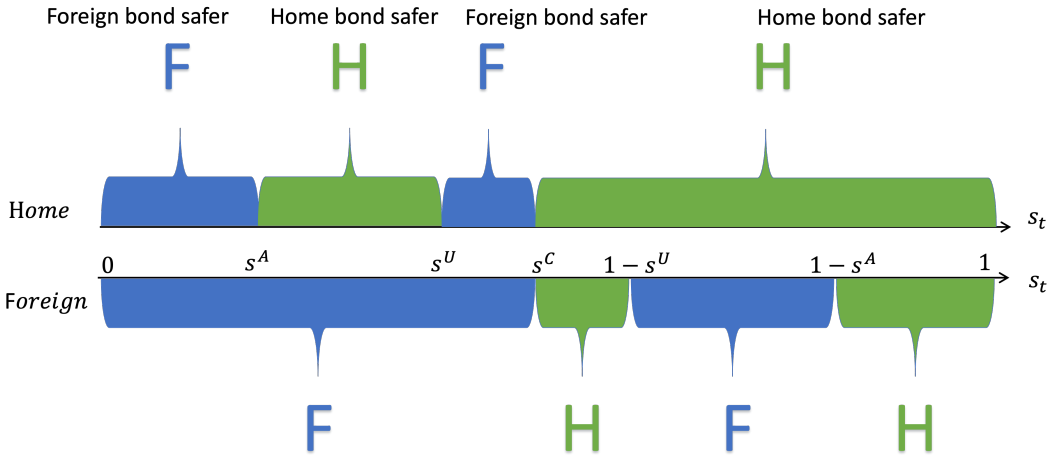
Proposition 8 (Domestic and global safety)**.** *Assume that home country is the smaller country,* $s_t < s^C$.

The smaller country's bond is domestically safe if $s_t \in [s^A, s^U]$.

The larger country's bond is globally safe if and only if $s_t \in [0, s^A] \cup [s^U, s^C]$.

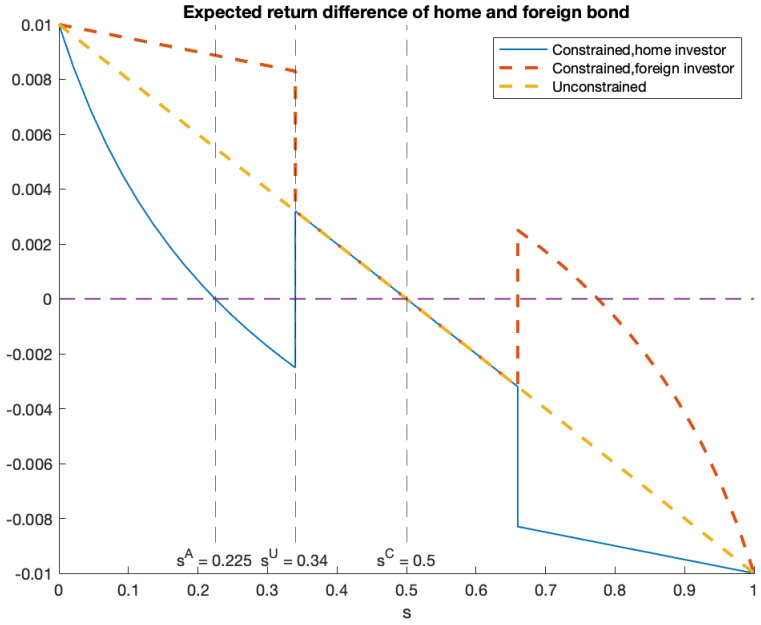
Proof. see appendix. □

Figure 8: Global safety region of home and foreign bond



Note: This figure shows the safety region of home and foreign bond as the relative size of home country s_t changes, from the perspective of both home (the upper axis) and foreign investors (the lower axis). s^U is the boundary of crisis regime (constrained) as home country falls in relative country size, symmetrically for $1 - s^U$. s^C and s^A (symmetrically for $1 - s^A$) are the safety thresholds where UIP changes sign in normal regime (unconstrained) and crisis regime (constrained).

Figure 9: Expected return difference of home and foreign bond for home and foreign investors



Note: The y-axis of this figure shows the expected return difference of home and foreign bond. The x-axis is the relative size of home country s. The solid blue line (constrained, home investor) and the dotted orange line (constrained, foreign investors) shows the expected return difference of home and foreign bond in the full model (with two equity constraints) from the perspective of home investors and foreign investors, respectively. The dotted yellow line (unconstrained) shows the expected return difference of home and foreign bond in complete market model. And the three dotted vertical lines (from left to right) s^A, s^U, s^C represents the endogenous threshold for UIP changing sign in crisis regime, the boundary of crisis regime, and the threshold for UIP changing sign in normal regime, respectively. Parameter values used: $\eta = 4$, $\overline{\chi}^{F,H} = \overline{\chi}^{H,F} = 0.2$, $\sigma^2 = 0.04$.

In crisis regime, binding equity constraints drives a wedge between SDFs of home and foreign investors. Different pricing kernels result in different returns for the same asset. Investors in the two countries disagree on expected return difference between bonds in crisis, as shown in figure 9. Technically, the heterogeneity in SDFs comes from heterogeneity in constraints. The two countries face asymmetric complementary margin requirements on their equity holdings which results in asymmetric Lagrangian multipliers associated with the binding constraints.

For the smaller country, when falling into crisis, country size spillover effect competes with equity rebalancing effect. In mild crisis, equity rebalancing effect dominates country size spillover effect. The smaller country's bond becomes domestically safe. In deep crisis, country size spillover effect dominates equity rebalancing effect. The larger country's bond becomes safe for investors in the smaller country. For investors in the larger country, domestic bond is always safe and domestic bond safety status gets strengthened upon entering crisis regime, see the jumps in figure 9. So the

larger country's bond is globally safe in normal times and in deep crisis, as shown in figure 8.

5.2.4 Market segmentation

Proposition 9 (Market segmentation)**.** *In crisis regime, bond holdings of the two countries are given by*

$$\theta_t^{H,B^H} = \theta_t^{H,B^F} = 0, \quad \theta_t^{F,B^H} = \theta_t^{F,B^F} = 0$$

Proof. see appendix □

Forced by the constraints, investors in both countries hold more domestic risk than desired and would like to offload domestic risk to foreign investors. However, no such security is available because domestic amplification is resulted from dispersion in consumption prices (in terms of numeraire) of the two countries created by binding equity constraints and applies to any real asset. Real bond returns for investors bear the extra domestic risk coming from consumption price and no bond is held or traded between the two countries in crisis even though there is no friction in bond markets. Liquidity drains between the two countries in every asset market[16] and financial dichotoour emerges in crisis regime.

5.2.5 Non-linearity and systemic risk

Non-linearity The non-linearity in asset returns shows up for investors in the smaller country who hold both domestic equity and foreign equity. The non-linearity factor for home country is given by

$$\frac{(1-\overline{\chi}^{H,F})s_t}{(1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}} \tag{28}$$

Taking derivative with respect to s_t, we have

$$\frac{\overline{\chi}^{H,F}(1-\overline{\chi}^{H,F})}{\left[(1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}\right]^2}$$

where we see the non-linearity effect gets stronger when s_t is smaller, consistent with figure 7.

[16]There is still trade happening between the two countries and potential trade in assets within domestic investors.

Systemic risk At crisis boundaries s^U and $1 - s^U$, there are endogenous jumps between normal regime and crisis regime for both countries, which is the systemic risk in the model. The discontinuous change in expected return difference between sovereign bonds for home country investors at s^U (in absolute value), denoted as Δ_{11} is given by

$$\Delta_{11} = \frac{(1-\overline{\chi}^{H,F})s^U}{(1-\overline{\chi}^{H,F})s^U + \overline{\chi}^{H,F}} \frac{\eta - 1}{\eta^2}(1-s^U)(\sigma_H^2 + \sigma_F^2) \tag{29}$$

and similarly denote Δ_{21} as the absolute change in expected return difference between sovereign bonds for foreign country investors at s^U, where

$$\Delta_{21} = \frac{\eta - 1}{\eta^2} s^U (\sigma_H^2 + \sigma_F^2) \tag{30}$$

Since $s^U < \frac{1-\overline{\chi}^{H,F}}{2(1-\overline{\chi}^{H,F})}$, we have

$$\Delta_{11} > \Delta_{21}$$

that is, the smaller country suffers greater systemic risk instability when it falls into crisis.

6 UIP reversal, flight-to-safety, CIP deviations and convenience yields

The model with crisis regime can explain several puzzles strongly associated with crisis periods: UIP reversal, flight to safety, CIP deviations and convenience yields.

First, in crisis regime, both countries price risks differently from normal regime. Binding equity constraints distort asset returns and asset safety in crisis, which explains the UIP reversal and flight to safety in crisis. Second, in crisis regime, the two countries disagree on prices of risks with each other, which explains the CIP deviations and convenience yields: in crisis, UK investors and US investors perceive different returns for exactly the same bond, the US Treasury. The gap between the actual return of US Treasury and the perceived return of US Treasury by foreign investors is also known as the CIP deviations or the convenience yield of the US Treasury, which is a signature of the 2008 Great Financial Crisis (GFC).

6.1 UIP reversal

UIP reversal is the opposite direction of UIP violation in normal times. As documented in Corsetti and Marin (2020), in normal times, US Treasury is a safer asset than UK government bond for UK investors and pays lower expected return as its safety premium. While when crisis hit, UK government bond pays lower expected return and becomes safer than US Treasury for UK investors.

Mapping into the model, in normal times, UK country size s_t is above the crisis boundary s^U but below the normal time safety threshold s^C. US Treasury is safer than UK government bond because of country size spillover effect. If UK economy or US economy suffers a rare loss, s_t falls into $[s^A, s^U]$ or rises to above s^C, UK government bond reverses to become a domestic safe bond and pays lower expected return for domestic investors compared to the US Treasury. In addition, if s_t rises to $[s^C, 1 - s^U]$ or $[1 - s^A, 1]$, UK government bond becomes a global safe bond. As shown in figure 10, UIP reversal between UK and US government bond happens when UK country size s_t falls into the left blue area $[0, s^A]$ from the green area $[s^A, s^U]$.

The model can also speak to emerging economies. For emerging economies whose initial country size is too small and is below s^A, if s_t rises into $[s^A, s^U]$ because the emerging economy grows larger due to its higher growth rate or the larger country, say US, falls into crisis due to rare losses, the emerging economy's domestic bond reverses from being the riskier bond to become the safer bond than US Treasury for domestic investors. As shown in figure 10, UIP reversal happens for emerging countries when their domestic country size s_t rises into the green area $[s^A, s^U]$ from the left blue area $[0, s^A]$.

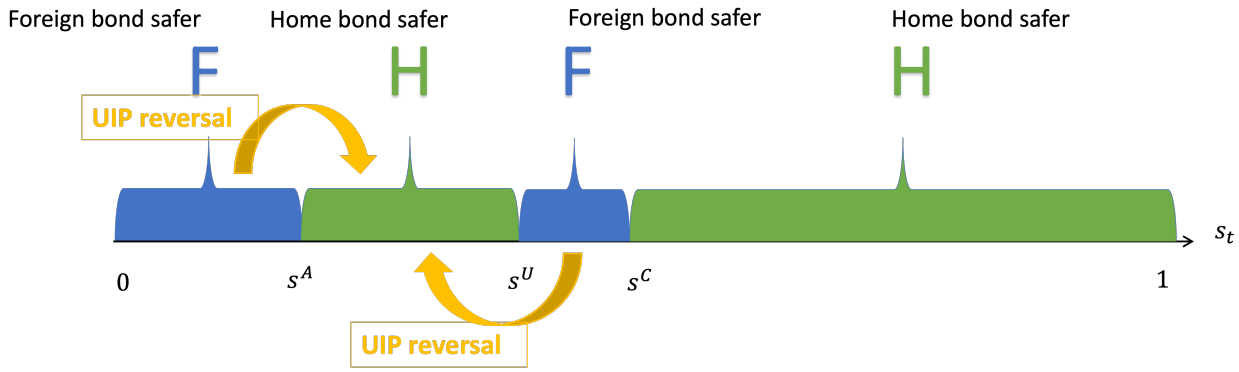


Figure 10: UIP reversal

Note: This figure shows how UIP reversal happens as the relative size of home country s_t changes. s^A, s^U, s^C represents the endogenous threshold for UIP changing sign in crisis regime, the boundary of crisis regime, and the threshold for UIP changing sign in normal regime, respectively.

6.2 Flight to safety

In a flight-to-safety phenomena, investors demand safe assets and push down the return of safe assets in crisis. US Treasury and German bond are good examples for being the global safe asset in crisis and pay lower expected returns than in normal times.

Mapping into the model, if UK country size s_t falls below s^U, the safety of US Treasury for US domestic investors is strengthened by equity rebalancing effect and US Treasury yield drops. If UK country size s_t falls below s^A, US Treasury is the safe asset for both UK investors and US investors, the global safe asset. The smaller UK country size s_t falls, the safer the US Treasury. Because country size spillover effect, which improves US Treasury safety, gets stronger with falling s_t. When UK country size s_t falls into crisis regime $[0, s^U]$, US investors find US Treasury even safer than in normal times and UK investors find US Treasury the safer bond than UK government bond if s_t falls below s^A.

In the case of the European debt crisis, German bond is the safe asset and pays historically low return. In the model, when periphery countries falls deep in crisis and their country size s_t drops below s^A, German bond is the global safe asset and investors are willing to accept an extra low return as the safety premium.

In a deep crisis, country size spillover effect dominates equity rebalancing effect and the larger country's bond is the global safe asset. From the point of view of investors in the smaller country, a flight-to-safety to the larger country's bond happens when s_t falls into the left blue area as shown

in figure 11.

For emerging countries whose initial country size s_t is in the green area $[s^A, s^U]$, if US economy suffers from rare losses and s_t rises into blue area $[s^U, s^C]$, US Treasury becomes a global safe asset. As shown in figure 11, a flight-to-safety to the larger country's bond also happens when s_t rises into the blue area $[s^U, s^C]$ from the green area $[s^A, s^U]$.

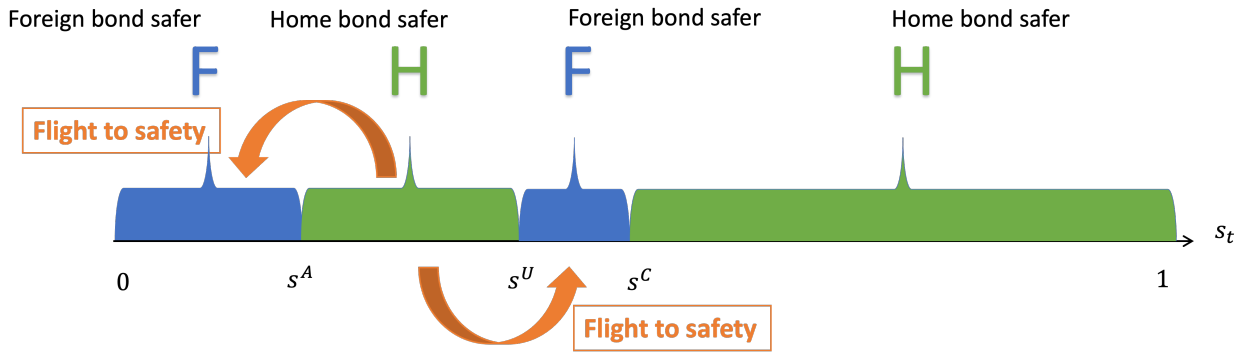


Figure 11: Flight to safety

Note: This figure shows how flight-to-safety happens as the relative size of home country s_t changes. s^A, s^U, s^C represents the endogenous threshold for UIP changing sign in crisis regime, the boundary of crisis regime, and the threshold for UIP changing sign in normal regime, respectively.

6.3 CIP deviations and convenience yields

CIP deviations is a failure of the Law of One Price: assets with the same underlying dividend flow pay different returns. Government bond convenience yield is the return difference between risk-free rate and government bond yield. The relative convenience yield between sovereign bonds are often related to CIP deviations. Du, Im, and Schreger (2018a) studies the US Treasury premium which is defined as the relative convenience yield between US Treasury and other countries' government bonds by measuring CIP deviations between government bond yields. In our model, the larger country's bond enjoys a positive convenience yield relative to the smaller country's bond in crisis regime.

In crisis region, if home (G-10) investors want to borrow US dollar, they can not directly borrow from US dollar cash market with rate dr_t^{F,B^F} because of market segmentation. However, they can borrow domestic currency at rate dr^{H,B^H} and simultaneously enter a forward contract $-dr^{H,B^H} + dr^{H,B^F}$ to sell domestic currency for US dollar in the future. The implied US dollar

rate from FX swap market (or the synthetic dollar rate) is thus dr^{H,B^F}.

Proposition 10. *The CIP condition is violated in crisis regime. The direct US dollar rate from cash market is lower than the synthetic dollar rate implied from FX swap market, that is*

$$\frac{\mathbb{E}_t\left[dr_t^{F,B^F} - dr^{H,B_t^F}\right]}{dt} = r_{F,t}^f - r_{H,t}^f - \frac{1}{1-n_t}\sigma_{n_t}\sigma_{e_t} < 0 \tag{31}$$

Proof. see appendix □

7 Financial development and trade elasticity

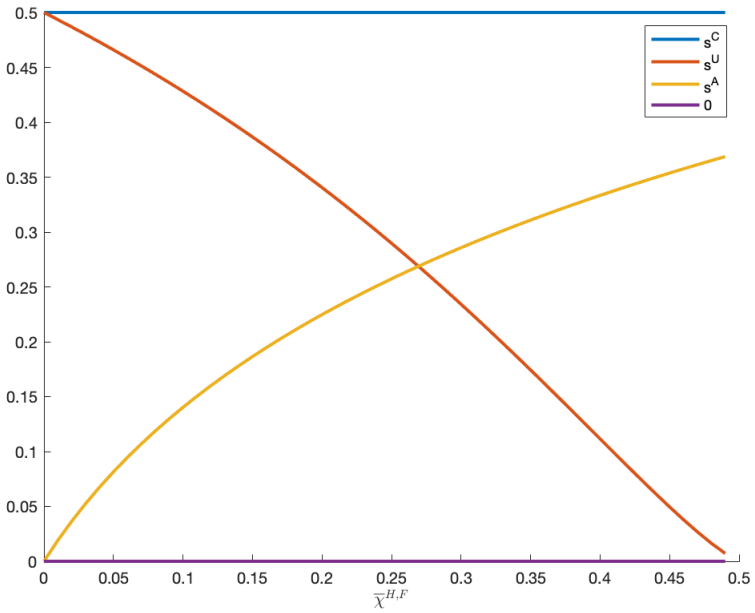
With a stable country size s_t, bond safety can also change with a shift of the safety spectrum due to changes in financial fiction parameters $1-\overline{\chi}^{F,H}$, $\overline{\chi}^{H,F}$, and trade elasticity η.

7.1 Financial development

As discussed in section 5.2, it is the foreign country's financial development that matters for bond safety when domestic country size shrinks and falls into crisis regime. In crisis regime $[0, s^U]$, a tightening of the larger country's equity holding constraint (i.e., a smaller $\overline{\chi}^{H,F}$), has impact on two safety thresholds, the crisis boundary s^U and crisis time safety threshold s^A, the non-linear domestic amplification for home country investors, and the systemic risk instability. Tighter equity constraint makes it harder for consumption smoothing and risk-sharing when there is asymmetry in country sizes, which shifts s^U to the right and expands the crisis regime. Chances of entering and the time spent in the crisis regime are increased. Meanwhile, a tighter constraint strengthens equity rebalancing effect and shifts crisis time safety threshold s^A to the left. The safety region of domestic bond in crisis regime is expanded due to increased hedging benefit of domestic bond.

On the other hand, financial development (i.e., a larger $\overline{\chi}^{H,F}$) reduces the crisis regime coverage, as well as the safety region of domestic bond in crisis regime until the left threshold s^A exceeds the right threshold s^U. With a sufficient loose constraint, equity rebalancing effect is too weak to reverse country size spillover effect and foreign bond is still the safer bond in crisis regime for domestic investors. See equation (19), (24), (20) and figure 12.

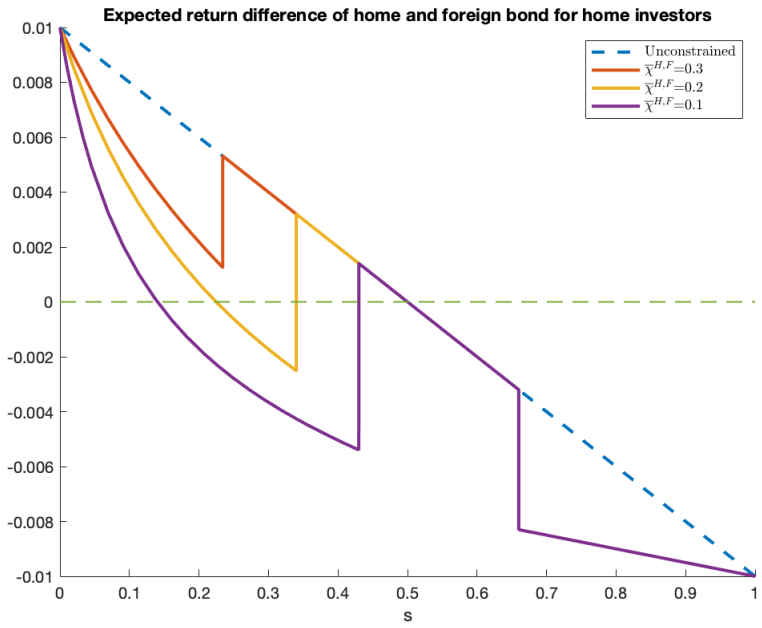
Figure 12: Changes of safety thresholds with respect to equity limit $\overline{\chi}^{H,F}$



Note: This figure shows how the safety thresholds s^C (UIP changing sign in normal regime, the blue line), s^U (boundary of crisis region, the orange line), and s^A (UIP changing sign in crisis regime, the yelow line) change with respect to equity limit $\overline{\chi}^{H,F}$. The y-axis of this figure is the relative size of home country s. The x-axis is the tightness of equity constraint $\overline{\chi}^{H,F}$ (the upper limit of foreign equity share held by home investors). The equity constraint is tighter with smaller $\overline{\chi}^{H,F}$. Parameter values used: $\eta = 4$, $\overline{\chi}^{F,H} = 0.2$, $\sigma^2 = 0.04$.

Financial development also matters for the non-linear domestic amplification effect and systemic risk instability, see figure 13. The larger country's financial development reduces systemic risk instability for both countries when the smaller country falls too small, see equation (29) and (30). And the domestic non-linear effect weakens with foreign financial development, see equation (28).

Figure 13: Changes of expected return difference of home and foreign bond with respect to equity limit $\overline{\chi}^{H,F}$

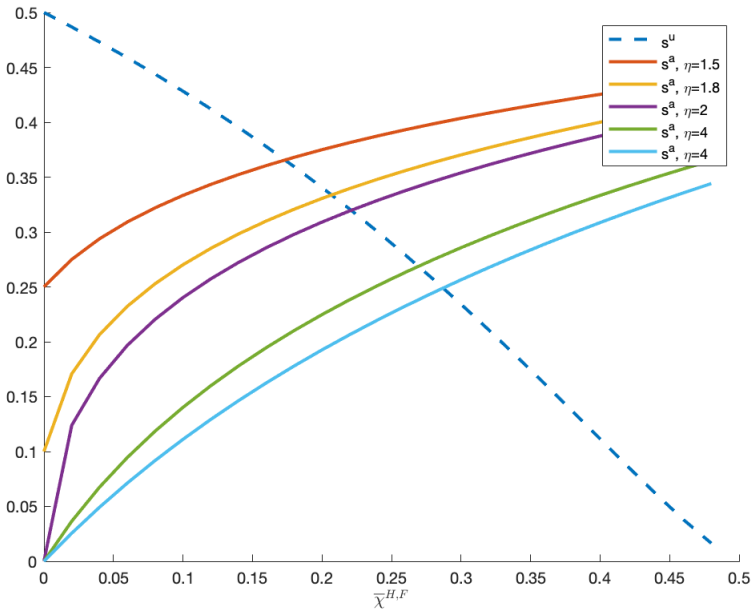


Note: The y-axis of this figure shows the expected return difference of home and foreign bond from the perspective of home investors. The x-axis is the relative size of home country s. The dotted blue line (unconstrained) corresponds to the complete market model. The solid lines corresponds to the full model with different tightness of equity constraint $\overline{\chi}^{H,F}$ (the upper limit of foreign equity share held by home investors). The equity constraint is tighter with smaller $\overline{\chi}^{H,F}$. Parameter values used: $\eta = 4$, $\overline{\chi}^{F,H} = 0.2$, $\sigma^2 = 0.04$.

7.2 trade elasticity

A larger trade elasticity η, which means domestic good and foreign good are more substitutable, does not affect the crisis boundary s^U but shifts the crisis time safety threshold s^A to the left, because substitutability between two goods produced by the two countries weakens country size spillover effect and strengthens equity rebalancing effect. See equation (15), (20), and figure 14. Notice that there is discontinuity at $\overline{\chi}^{H,F} = 0$ when equation (24) degenerates.

Figure 14: Changes of safety threshold with respect to trade elasticity η



Note: The y-axis of this figure shows the relative size of home country s. The x-axis is the tightness of equity constraint $\overline{\chi}^{H,F}$ (the upper limit of foreign equity share held by home investors). ... to be completed

8 Conclusion

This paper provides a theory of sovereign bond safety which is jointly determined by country size and equity rebalancing. Country size spillover effect improves the safety of the larger country's bond, which explains normal time UIP violation. Equity rebalancing, the equity holding constraints in the model, creates endogenous systemic risk instability between normal regime and crisis regime where domestic risk is amplified. The interaction between country size and equity rebalancing in crisis regime explains UIP reversal, flight to safety, sovereign bond CIP deviations and convenience yields at the same time.

References

Fernando Alvarez, Andrew Atkeson, and Patrick J Kehoe. Money, interest rates, and exchange rates with endogenously segmented markets. *Journal of political Economy*, 110(1):73–112, 2002.

Andrew Atkeson, Jonathan Heathcote, and Fabrizio Perri. The end of privilege: A reexamination of the net foreign asset position of the united states. Technical report, National Bureau of Economic Research, 2022.

Stefan Avdjiev, Bryan Hardy, Şebnem Kalemli-Özcan, and Luis Servén. Gross capital flows by banks, corporates, and sovereigns. *Journal of the European Economic Association*, 20(5):2098–2135, 2022.

David K Backus, Silverio Foresi, and Chris I Telmer. Affine term structure models and the forward premium anomaly. *The Journal of Finance*, 56(1):279–304, 2001.

Ravi Bansal. An exploration of the forward premium puzzle in currency markets. *The Review of Financial Studies*, 10(2):369–403, 1997.

Geert Bekaert. The time variation of risk and return in foreign exchange markets: A general equilibrium perspective. *The Review of Financial Studies*, 9(2):427–470, 1996.

Ben S Bernanke, Mark Gertler, and Simon Gilchrist. The financial accelerator in a quantitative business cycle framework. *Handbook of macroeconomics*, 1:1341–1393, 1999.

John H. Cochrane, Francis A. Longstaff, and Pedro Santa-Clara. Two Trees. *The Review of Financial Studies*, 21(1):347–385, 11 2007. ISSN 0893-9454.

Nicolas Coeurdacier and Helene Rey. Home bias in open economy financial macroeconomics. *Journal of Economic Literature*, 51(1):63–115, 2013.

Giancarlo Corsetti and Emile Alexandre Marin. A century of arbitrage and disaster risk pricing in the foreign exchange market. 2020.

Julian Di Giovanni, Şebnem Kalemli-Özcan, Mehmet Fatih Ulu, and Yusuf Soner Baskaya. International spillovers and local credit cycles. *The Review of Economic Studies*, 89(2):733–773, 2022.

Wenxin Du, Joanne Im, and Jesse Schreger. The us treasury premium. *Journal of International Economics*, 112:167–181, 2018a.

Wenxin Du, Alexander Tepper, and Adrien Verdelhan. Deviations from covered interest rate parity. *The Journal of Finance*, 73(3):915–957, 2018b.

Wenxin Du, Benjamin M Hébert, and Amy Wang Huber. Are intermediary constraints priced? Technical report, National Bureau of Economic Research, 2019.

Wenxin Du, Carolin E Pflueger, and Jesse Schreger. Sovereign debt portfolios, bond risks, and the credibility of monetary policy. *The Journal of Finance*, 75(6):3097–3138, 2020.

Eugene F Fama. Forward and spot exchange rates. *Journal of monetary economics*, 14(3):319–338, 1984.

Tarek Hassan. Country size, currency unions, and international asset returns. *The Journal of Finance*, 68(6):2269–2308, 2013.

Tarek A Hassan and Rui C Mano. Forward and spot exchange rates in a multi-currency world. *The Quarterly Journal of Economics*, 134(1):397–450, 2019.

Harald Hau and Hélène Rey. Can portfolio rebalancing explain the dynamics of equity returns, equity flows, and exchange rates? *American Economic Review*, 94(2):126–133, 2004.

Harald Hau and Helene Rey. Home bias at the fund level. *American Economic Review*, 98(2): 333–38, 2008.

Zhengyang Jiang, Arvind Krishnamurthy, and Hanno Lustig. Dollar safety and the global financial cycle. *Review of Economic Studies*, page rdad108, 2023.

Sebnem Kalemli-Ozcan and Liliana Varela. Five facts about the uip premium. Technical report, National Bureau of Economic Research, 2021.

Rohan Kekre and Moritz Lenel. The flight to safety and international risk sharing. *American Economic Review*, 114(6):1650–1691, 2024.

Ralph SJ Koijen and Motohiro Yogo. Exchange rates and asset prices in a global demand system. Technical report, National Bureau of Economic Research, 2020.

Hanno Lustig and Adrien Verdelhan. The cross section of foreign currency risk premia and consumption growth risk. *American Economic Review*, 97(1):89–117, 2007.

Hanno Lustig, Nikolai Roussanov, and Adrien Verdelhan. Common risk factors in currency markets. *The Review of Financial Studies*, 24(11):3731–3777, 2011.

Hanno Lustig, Nikolai Roussanov, and Adrien Verdelhan. Countercyclical currency risk premia. *Journal of Financial Economics*, 111(3):527–553, 2014.

Matteo Maggiori. Financial intermediation, international risk sharing, and reserve currencies. *American Economic Review*, 107(10):3038–71, 2017.

Ian Martin. The forward premium puzzle in a two-country world. Working Paper 17564, National Bureau of Economic Research, November 2011.

Anna Pavlova and Roberto Rigobon. Asset Prices and Exchange Rates. *The Review of Financial Studies*, 20(4):1139–1180, 01 2007. ISSN 0893-9454.

Anna Pavlova and Roberto Rigobon. The role of portfolio constraints in the international propagation of shocks. *The Review of Economic Studies*, 75(4):1215–1256, 2008.

Helene Rey. Dilemma not trilemma: the global financial cycle and monetary policy independence. Technical report, National Bureau of Economic Research, 2015.

Maxime Sauzet. Asset prices, global portfolios, and the international financial system. *Global Portfolios, and the International Financial System (February 2, 2023)*, 2023.

Matthew M Wynter. Why did the equity home bias fall during the financial panic of 2008? *The World Economy*, 42(5):1343–1372, 2019.

Appendix

A Additional Proofs and Derivations

Trade market: Notice that international trade is a static problem for both countries. Since there is no friction in the international trade market and homogeneous preferences, we have that

$$\frac{C_{HH,t}}{C_{HF,t}} = \frac{C_{FH,t}}{C_{FF,t}} = \frac{Y_{H,t}}{Y_{F,t}} \tag{32}$$

Using market clearing condition for home good and foreign good,

$$C_{HH,t} + C_{FH,t} = Y_{H,t}$$
$$C_{HF,t} + C_{FF,t} = Y_{F,t}$$

we have

$$C_{H,t} + C_{F,t} = \overline{Y}_t$$

where

$$C_{H,t} = \left[a^{\frac{1}{\eta}} C_{HH,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} C_{HF,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

$$C_{F,t} = \left[a^{\frac{1}{\eta}} C_{FH,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} C_{FF,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

$$\overline{Y}_t = \left[a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

As a result, the prices of the two goods produced by the two trees are given by

$$p_t^H = \left(\frac{Y_{H,t}}{a\overline{Y}_t} \right)^{-\frac{1}{\eta}} \quad \text{and} \quad p_t^F = \left(\frac{Y_{F,t}}{(1-a)\overline{Y}_t} \right)^{-\frac{1}{\eta}} \tag{33}$$

We have that

$$C_{H,t} = p_t^H C_{HH,t} + p_t^F C_{HF,t}$$
$$C_{F,t} = p_t^H C_{FH,t} + p_t^F C_{FF,t}$$

Recall that country size if defined as follows

$$s_t = \frac{a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}}}{a^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}} = a^{\frac{1}{\eta}} \left(\frac{Y_{H,t}}{\overline{Y}_t} \right)^{\frac{\eta-1}{\eta}}$$

as home country's share of the world total output, i.e. the *country size* of home country, which will turn out to be an important state variable. We have that

$$p_t^H Y_{H,t} = s_t \overline{Y}_t \quad \text{and} \quad p_t^F Y_{F,t} = (1-s_t)\overline{Y}_t$$

The aggregate wealth of home country is

$$W_t^H = \chi_t^{H,H} S_t^H + \chi_t^{H,F} S_t^F + p_t^H B_t^{H,H} + p_t^F B_t^{H,F}$$

The aggregate wealth of foreign country is

$$W_t^F = \chi_t^{F,H} S_t^H + \chi_t^{F,F} S_t^F + p_t^H B_t^{F,H} + p_t^F B_t^{F,F}$$

The optimization problem for home country is as follows:

$$\begin{aligned}
\max_{\{C_{HH,t}, C_{HF,t}, \chi_t^{H,H}, \chi_t^{H,f}, \theta_t^{H,B^H}, \theta_t^{H,B^F}\}_{t=0}^{\infty}} \quad & \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log\left(\left[a^{\frac{1}{\eta}} C_{HH,t}^{\frac{\eta-1}{\eta}} + (1-a)^{\frac{1}{\eta}} C_{HF,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}\right) dt\right] \\
\text{s.t.} \quad & \frac{dW_t^H}{W_t^H} = \frac{\chi_t^{H,H} S_t^H}{W_t^H} dr_t^{S^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} dr_t^{S^F} + \theta_t^{H,B^H} dr_t^{B^H} + \theta_t^{H,B^F} dr_t^{B^F} \\
& \qquad - \frac{p_t^H C_{HH,t} + p_t^F C_{FF,t}}{W_t^H} dt \\
& 1 = \frac{\chi_t^{H,H} S_t^H}{W_t^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} + \theta_t^{H,B^H} + \theta_t^{H,B^F} \\
& 0 \le \chi_t^{H,F} \le \overline{\chi}^{H,F}
\end{aligned} \tag{34}$$

Define home country's wealth share as

$$n_t = \frac{W_t^H}{W_t^H + W_t^F} \tag{35}$$

Lemma 1. *Home country's wealth share $n_t = \frac{W_t^H}{W_t^H + W_t^F}$ is a function of country size s_t. And we always have that*

$$p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = 0 \tag{36}$$

Proof. From optimization problems, we have that

$$C_{H,t} = \rho W_t^H \quad \text{and} \quad C_{F,t} = \rho W_t^H$$

In the aggregate, we have that total consumption equals total output

$$\begin{aligned}
C_{HH,t} + C_{FH,t} &= Y_{H,t} \\
C_{HF,t} + C_{FF,t} &= Y_{F,t}
\end{aligned}$$

And using the result from trade market optimization, we have

$$C_{H,t} + C_{F,t} = \overline{Y_t}$$

That is,

$$\rho W_t^H + \rho W_t^F = \overline{Y_t}$$

The total wealth in the world is given by

$$W_t^H + W_t^F = \frac{\overline{Y_t}}{\rho}$$

There are two state variables, wealth share n_t and country size s_t. In equilibrium, all the prices and quantities must be functions of state variables n_t and s_t. We can rewrite home country's wealth as

$$p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = [n_t - (\chi_t^{H,H} q_t^H + \chi_t^{H,F} q_t^F)]\overline{Y_t}$$

where $q_t^H = \frac{S_t^H}{\overline{Y_t}}$ and $q_t^F = \frac{S_t^F}{\overline{Y_t}}$ are per unit price of home and foreign equity, respectively. Recall that

$$p_t^H Y_{H,t} = s_t \overline{Y_t} \quad \text{and} \quad p_t^F Y_{F,t} = (1 - s_t)\overline{Y_t}$$

we have

$$p_t^H = a^{\frac{1}{\eta-1}} s_t^{-\frac{1}{\eta-1}} \quad \text{and} \quad p_t^F = (1-a)^{\frac{1}{\eta-1}} (1 - s_t)^{-\frac{1}{\eta-1}}$$

And in equilibrium $\theta_t^{H,B^H} = \frac{p_t^H B_t^{H,H}}{W_t^H}$, $\theta_t^{H,B^F} = \frac{p_t^F B_t^{H,F}}{W_t^H}$, q_t^H, q_t^F, $\chi_t^{H,H}$ and $\chi_t^{H,F}$ must be functions of state variables n_t and s_t. We have that

$$n_t = \frac{\chi_t^{H,H} q_t^H + \chi_t^{H,F} q_t^F}{1 - (\theta_t^{H,B^H} + \theta_t^{H,B^F})} \equiv f(s_t, n_t) \tag{37}$$

Equation (37) is an implicit function and we can solve for n_t as a function of s_t.

Now we have that in equilibrium $\theta_t^{H,B^H} = \frac{p_t^H B_t^{H,H}}{W_t^H}$, $\theta_t^{H,B^F} = \frac{p_t^F B_t^{H,F}}{W_t^H}$, q_t^H, q_t^F, $\chi_t^{H,H}$ and $\chi_t^{H,F}$ must be functions of the *only* state variable s_t.

Recall the dynamic budget constraint of home country,

$$\begin{aligned}\frac{dW_t^H}{W_t^H} = &\frac{\chi_t^{H,H} S_t^H}{W_t^H} dr_t^{S^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} dr_t^{S^F} + \theta_t^{H,B^H} dr_t^{B^H} + \theta_t^{H,B^F} dr_t^{B^F} \\ &- \frac{p_t^H C_{HH,t} + p_t^F C_{FF,t}}{W_t^H} dt\end{aligned}$$

And the asset return processes,

$$dr_t^{S^H} = \underbrace{\frac{p_t^H Y_{H,t}}{q_t^H \overline{Y}_t} dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^H \overline{Y}_t)}{q_t^H \overline{Y}_t}}_{\text{capital gain}} = \mu_t^{S^H}\, dt + (\frac{(q^H)'(s_t) s_t}{q_t^H} \sigma_{s_t} + \overline{\sigma}_t) dZ_t$$

and similarly

$$dr_t^{S^F} = \underbrace{\frac{p_t^F Y_{F,t}}{q_t^F \overline{Y}_t} dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^F \overline{Y}_t)}{q_t^F \overline{Y}_t}}_{\text{capital gain}} = \mu_t^{S^F}\, dt + (\frac{(q^F)'(s_t) s_t}{q_t^F} \sigma_{s_t} + \overline{\sigma}_t) dZ_t$$

$$dr_t^{B^H} = \frac{d(p_t^H B_t^H)}{p_{H,t} B_t^H} = (\mu_{p^H,t} + r_t^H)\, dt + \sigma_{p^H,t}\, dZ_t$$

$$dr_t^{B^F} = \frac{d(p_t^F B_t^F)}{p_{F,t} B_t^F} = (\mu_{p^F,t} + r_t^F)\, dt + \sigma_{p^F,t}\, dZ_t$$

And we have

$$\sigma_{p_t^H} = -\frac{1}{\eta - 1}\sigma_{s_t}$$

$$\sigma_{p_t^F} = \frac{s_t}{(\eta - 1)(1 - s_t)}\sigma_{s_t}$$

Since $\rho W_t^H = n_t \overline{Y_t}$, we have

$$\frac{dW_t^H}{W_t^H} = \mu_t^{W^H} dt + (\frac{n'(s_t)s_t}{n_t}\sigma_{s_t} + \overline{\sigma}_t)\, dZ_t$$

Note that

$$\overline{\sigma}_t = [s_t \sigma^H, (1 - s_t)\sigma^F]$$

$$\sigma_{s_t} = \frac{\eta - 1}{\eta}(1 - s_t)[\sigma^H, -\sigma^F]$$

are *linearly independent* for non-degenerate s_t. Matching terms for $\overline{\sigma}_t$, we must have that

$$\frac{\chi_t^{H,H} S_t^H}{W_t^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} = 1 \tag{38}$$

That is

$$p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = 0 \tag{39}$$

□

Corollary 2. *The total capital flow of home country induced by equity trade is given by*

$$\begin{aligned} dQ_t^H = {} & S_t^H d\chi_t^{H,H} - \chi_t^{H,H}(p_t^H Y_{H,t})dt + S_t^F d\chi_t^{H,F} - \chi_t^{H,F}(p_t^F Y_{F,t})dt \\ & + d\chi_t^{H,H} dS^H + d\chi_t^{H,F} dS_t^F \end{aligned} \tag{40}$$

And such capital flow must be financed and absorbed by trading in bonds and consumption goods

$$dQ_t^H = (\theta_t^{B^{H,H}} dr_t^{B^H} + \theta_t^{B^{H,F}} dr_t^{B^F}) W_t^H - (p_t^H C_{HH,t} + p_t^F C_{HF,t})dt \tag{41}$$

Similarly the total capital flow of foreign country induced by equity trade is given by

$$\begin{aligned} dQ_t^F = {} & S_t^H d\chi_t^{F,H} - \chi_t^{F,H}(p_t^H Y_{H,t})dt + S_t^F d\chi_t^{F,F} - \chi_t^{F,F}(p_t^F Y_{F,t})dt \\ & + d\chi_t^{F,H} dS^H + d\chi_t^{F,F} dS_t^F \end{aligned} \tag{42}$$

and such capital flow must be financed and absorbed by trading in bonds and consumption goods

$$dQ_t^F = (\theta_t^{B^{F,H}} dr_t^{B^H} + \theta_t^{B^{F,F}} dr_t^{B^F}) W_t^F - (p_t^H C_{FH,t} + p_t^F C_{FF,t})dt \tag{43}$$

Proof. This follows from Lemma 1. Since we have

$$W_t^H = \chi_t^{H,H} S_t^H + \chi_t^{H,F} S_t^F$$

Taking total differentiation on both sides, we have that

$$dW_t^H = \chi_t^{H,H} S_t^H \, dr_t^{S^H} + \chi_t^{H,F} S_t^F \, dr_t^{S^F} + dQ_t^H \tag{44}$$

Combining with dynamic budget constraint

$$\begin{aligned}\frac{dW_t^H}{W_t^H} = &\frac{\chi_t^{H,H} S_t^H}{W_t^H} \, dr_t^{S^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} \, dr_t^{S^F} + \theta_t^{H,B^H} dr_t^{B^H} + \theta_t^{H,B^F} dr_t^{B^F} \\ &- \frac{p_t^H C_{HH,t} + p_t^F C_{FF,t}}{W_t^H} \, dt\end{aligned}$$

We have that

$$dQ_t^H = (\theta_t^{B^{H,H}} dr_t^{B^H} + \theta_t^{B^{H,F}} dr_t^{B^F}) W_t^H - (p_t^H C_{HH,t} + p_t^F C_{HF,t}) dt$$

Similar proof for foreign country. □

Lemma 3. *In crisis region $s_t \in [0, s^u]$, we have that*

$$\chi_t^{H,H} = 1, \quad \chi_t^{H,F} = \overline{\chi}^{H,F}$$

$$p_t^H B_t^{H,H} = p_t^F B_t^{H,F} = 0$$

Proof. At the crisis region boundary $s_t = s^u$, we have $\chi_t^{H,H} = 1$ and $\chi_t^{H,F} = \overline{\chi}^{H,F}$,

$$dQ_t^H = S_t^H d\chi_t^{H,H} - (p_t^H Y_{H,t}) dt + S_t^F d\chi_t^{H,F} - \overline{\chi}^{H,F} (p_t^F Y_{F,t}) dt + d\chi_t^{H,H} dS^H + d\chi_t^{H,F} dS_t^F$$

For *any* realization of $ds_t < 0$ at $s_t = s^u$, it must be that $d\chi_t^{H,H} \leq 0$ and $d\chi_t^{H,F} \leq 0$. To satisfy this, we must have $d\chi_t^{H,H}$ and $d\chi_t^{H,F}$ are deterministic for any realization of $ds_t < 0$ (thus, any $s < s^u$) at $s_t = s^u$.

Collecting terms for σ_{s_t} and $\overline{\sigma}_t$, we have

$$dQ_t^H - \mathbb{E}[dQ_t^H] = [(S_t^H d\chi_t^{H,H} + S_t^F d\chi_t^{H,F}) \overline{\sigma}_t + (S_t^H d\chi_t^{H,H} \frac{(q^H)'(s_t) s_t}{q_t^H} + S_t^F d\chi_t^{H,F} \frac{(q^F)'(s_t) s_t}{q_t^F}) \sigma_{s_t})] \, dZ_t$$

On the other side, we have

$$dQ_t^H = (\theta_t^{B^{H,H}} dr_t^{B^H} + \theta_t^{B^{H,F}} dr_t^{B^F}) W_t^H - (p_t^H C_{HH,t} + p_t^F C_{HF,t}) dt$$

which only consists of σ_{s_t} risk. As a result of matching terms for $\overline{\sigma}_t$, we have

$$S_t^H d\chi_t^{H,H} + S_t^F d\chi_t^{H,F} = 0$$

That is, $d\chi_t^{H,H} = d\chi_t^{H,F} = 0$. It follows that matching terms for σ_{s_t} on both sides should also be

0, and we have

$$-\theta_t^{B^{H,H}} + \frac{s_t}{1-s_t}\theta_t^{B^{H,F}} = 0$$

Combining with

$$\theta_t^{B^{H,H}} + \theta_t^{B^{H,F}} = 0$$

We have that in crisis region,

$$\theta_t^{B^{H,H}} = \theta_t^{B^{H,F}} = 0$$

□

While in crisis when $s_t \in [0, s^U]$, $P_t^H = (1-\overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}$ and $P_t^F = (1-\overline{\chi}^{H,F})(1-s_t)$.

Because the two countries have different consumption prices which are non-degenerate stochastic processes, the same (real) bond corresponds to different return processes in the two countries. That is, real bond returns bear consumption price risks and real bonds can not help overcome consumption price deviations. As a result, any real bond will not be traded in the constrained equilibrium [17]. The equity holding constraints creates financial friction that can not be overcome by sovereign bonds.

With one equity constraint Recall the wealth of home country and its evolution

$$W_t^H = \chi_t^{H,H} S_t^H + \chi_t^{H,F} S_t^F + p_t^H B_t^{H,H} + p_t^F B_t^{F,F} \tag{45}$$

$$\frac{dW_t^H}{W_t^H} = \frac{\rho\chi_t^{H,H} q_t^H}{n_t} dr_t^{S^H} + \frac{\rho\chi_t^{H,F} q_t^F}{n_t} dr_t^{S^F} + \frac{p_t^H B_{1t}}{W_t^H} dr_t^{B_1} + \frac{p_t^F B_{2t}}{W_t^H} dr_t^{B_2} - \rho dt \tag{46}$$

Denote

$$\frac{dW_t^H}{W_t^H} = \mu_{W_t^H}\, dt + \sigma_{W_t^H}\, dZ_t \tag{47}$$

$$\frac{dW_t^F}{W_t^F} = \mu_{W_t^F}\, dt + \sigma_{W_t^F}\, dZ_t \tag{48}$$

$$\frac{dn_t}{n_t} = \mu_{n_t} dt + \sigma_{n_t} dZ_t \tag{49}$$

There are two risks in this world: the aggregate consumption risk, $\overline{\sigma}_t$, and the distribution risk, σ_{s_t}. Since there are four financial assets, there is some redundancy. With only one equity constraint, the two countries can still perfectly share consumption risk. So we have

$$\sigma_{n_t} = \sigma_{1-n_t} = 0 \tag{50}$$

and

$$\sigma_{W_t^H} = \sigma_{W_t^F} = \overline{\sigma}_t \tag{51}$$

To find the portfolio weights on each asset, we have

$$\overline{\sigma}_t = \frac{\rho\chi_t^{H,H} q_t^H}{n_t}(\sigma_{q_t^H} + \overline{\sigma}_t) + \frac{\rho\chi_t^{H,F} q_t^F}{n_t}(\sigma_{q_t^F} + \overline{\sigma}_t) + \frac{p_t^H B_t^{H,H}}{W_t^H}\sigma_{p_t^H} + \frac{p_t^F B_t^{H,F}}{W_t^H}\sigma_{p_t^F} \tag{52}$$

[17] The symmetric setting in discount rate and preferences matters. (conjecture)

Now we need to find $\sigma_{q_t^H}$. In the complete market case, we have

$$q_t^H = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau} s_\tau d\tau\Big] \tag{53}$$

Using Ito's lemma we have

$$\sigma_{q_t^H} = \frac{(q^H)'(s_t)s_t}{q_t^H}\sigma_{s_t} \tag{54}$$

and since

$$q_t^F = \frac{1}{\rho} - q_t^H \tag{55}$$

we have

$$\sigma_{q_t^F} = -\frac{q_t^H}{q_t^F}\sigma_{q_t^H} = -\frac{(q^H)'(s_t)s_t}{q_t^F}\sigma_{s_t} \tag{56}$$

And we have

$$\sigma_{p_t^H} = -\frac{1}{\eta-1}\sigma_{s_t} \tag{57}$$

$$\sigma_{p_t^F} = \frac{s_t}{(\eta-1)(1-s_t)}\sigma_{s_t} \tag{58}$$

Note that

$$\overline{\sigma}_t^T = [s_t\sigma_1, (1-s_t)\sigma_2] \tag{59}$$

$$\sigma_{s_t}^T = \frac{\eta-1}{\eta}(1-s_t)[\sigma_1, -\sigma_2] \tag{60}$$

are *linearly independent*. Now coming back to the risk of home country's wealth (52) and matching $\overline{\sigma}_t$ term, we have

$$\frac{\rho\chi_t^{H,H}q_t^H}{n_t} + \frac{\rho\chi_t^{H,F}q_t^F}{n_t} = 1 \tag{61}$$

That is

$$W_t^H = \chi_t^{H,F}S_t^H + \chi_t^{H,F}S_t^F \tag{62}$$

and thus

$$p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = 0 \tag{63}$$

To determine portfolio weights on bonds, we match σ_{s_t} terms in home country's wealth

$$\frac{\rho\chi_t^{H,H}q_t^H}{n_t}\frac{(q^H)'(\frac{s_t}{n_t})s_t}{q_t^H} + \frac{\rho\chi_t^{H,F}q_t^F}{n_t}(-\frac{(q^H)'(s_t)s_t}{q_t^F}) + \frac{p_t^H B_t^{H,H}}{W_t^H}(-\frac{1}{\eta-1} - \frac{s_t}{(\eta-1)(1-s_t)}) = 0 \tag{64}$$

Simplified to

$$\frac{\rho(q^H)'(s_t)s_t(\chi_t^{H,H} - \chi_t^{H,F})}{n_t} - \frac{p_t^H B_t^{H,H}}{W_t^H}\frac{1}{(1-s_t)(\eta-1)} = 0 \tag{65}$$

So we have

$$\frac{p_t^H B_t^{H,H}}{W_t^H} = \frac{\rho\chi_t^{H,H}(q^H)'(s_t)s_t(1-s_t)(\chi_t^{H,H} - \chi_t^{H,F})(\eta-1)}{n_t} > 0 \tag{66}$$

and

$$\frac{p_t^F B_t^{H,F}}{W_t^H} = -\frac{p_t^H B_t^{H,H}}{W_t^H} < 0 \tag{67}$$

We can also find the drift of home country wealth share n_t by looking at the drift term of the wealth. Using market clearing condition $B_t^{H,H} = -B_t^{F,H}$ and $B_t^{H,F} = -B_t^{F,F}$,

$$\mu_{W_t^F} = -\frac{n_t}{1-n_t}\mu_{n_t} + \overline{\mu}_t \tag{68}$$

$$= \frac{\mathbb{E}_t[dr_t^{S^F}]}{dt} - \frac{n_t}{1-n_t}\frac{p_t^H B_t^{H,H}}{W_t^H} m_t \sigma_{e_t}^T - \rho \tag{69}$$

and we have that

$$\mu_{q_t^H} = \rho - \frac{s_t}{q_t^H} + \mu_{n_t} - \sigma_{n_t}^2 + \sigma_{q_t^H}\sigma_{n_t} \tag{70}$$

and $\sigma_{n_t} = 0$,

$$\mu_{q_t^H} = \rho - \frac{s_t}{q_t^H} + \mu_{n_t} \tag{71}$$

and thus

$$\frac{\mathbb{E}_t[dr_t^{S^F}]}{dt} = \frac{1-s_t}{q_t^F} + (-\frac{q_t^H}{q_t^F}\mu_{q_t^H} + \overline{\mu}_t + \sigma_{q_t^F}\overline{\sigma}_t^T) \tag{72}$$

Substituting into (68), and solving for μ_{n_t}, we have

$$\mu_{n_t} = \frac{\frac{q_t^H}{q_t^F}\sigma_{q_t^H}\overline{\sigma}_t^T + \frac{n_t\theta_t^{H,B^H}}{1-n_t}\sigma_{e_t}\overline{\sigma}_t^T}{\frac{n_t}{1-n_t} - \frac{q_t^H}{q_t^F}} = 0 \tag{73}$$

as

$$\frac{q_t^H}{q_t^F}\sigma_{q_t^H}\overline{\sigma}_t^T = -\frac{(q^H)'(\frac{s_t}{n_t})s_t(1-s_t)(\eta-1)}{q_t^F}\sigma_{e_t}\overline{\sigma}_t^T = \frac{n_t\theta_t^{H,B^H}}{1-n_t}\sigma_{e_t}\overline{\sigma}_t^T \tag{74}$$

With two equity holding constraints The second step is to explore what will happen with two equity holding constraints. Now the two countries can not always perfectly share consumption risk and the wealth shares are not always constant.

Full model special case: with two equity holding constraints $0 \leq \chi_t^{H,F} \leq \overline{\chi}^{H,F}$, $0 \leq \chi_t^{F,H} \leq \overline{\chi}^{F,H}$, and symmetric parameters $\sigma_1 = \sigma_2 = \sigma$ there are three safety thresholds s^c, s^u and s^a, $n_0 = \frac{1}{2}$:

$$s^c = \frac{1}{2} \tag{75}$$

$$q_1(s^u) = \frac{1-2\overline{\chi}^{H,F}}{2\rho(1-\overline{\chi}^{H,F})} \tag{76}$$

and

$$0 < s^a < s^u \tag{77}$$

with parameter restrictions on $(\overline{\chi}^{H,F}, \eta)$. The equity shares are

$$\chi_{1t} = 1 \tag{78}$$

and

$$\chi_t^{H,F} = \begin{cases} \overline{\chi}^{H,F} & if \quad s_t < s^U(\overline{\chi}^{H,F}) \\ \frac{1-2\rho q_t^H}{2(1-\rho q_t^H)} & if \quad s^U(\overline{\chi}^{H,F}) < s_t < \frac{1}{2} \\ 0 & if \quad s_t > \frac{1}{2} \end{cases} \tag{79}$$

bond holdings are (in equilibrium)

$$\theta_t^{H,B^H} = \theta_t^{H,B^F} = \theta_t^{F,B^H} = \theta_t^{F,B^F} = 0 \tag{80}$$

where

$$\frac{q_t^{H,H}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{s_\tau}{n_\tau}d\tau\Big] \tag{81}$$

Proof. The equity holding constraint binds when $s_t < s^u$. There are no cross-border bond trading. The implied bond returns (within domestic country) are given as follows

$$\frac{\mathbb{E}_t[dr_t^{B^{H,H}}]}{dt} - r_t^{H,f} = m_{1t}^T\sigma_{p_t^H} = (\sigma_{n_t} + \overline{\sigma}_t)\sigma_{p_t^H} \tag{82}$$

$$\frac{\mathbb{E}_t[dr_t^{B^{H,F}}]}{dt} - r_t^{H,f} = m_{1t}^T\sigma_{p_t^F} = (\sigma_{n_t} + \overline{\sigma}_t)\sigma_{p_t^F} \tag{83}$$

$$\frac{\mathbb{E}_t[dr_t^{B^{F,H}}]}{dt} - r_t^{F,f} = m_{2t}^T\sigma_{p_t^H} = (-\frac{n_t}{1-n_t}\sigma_{n_t} + \overline{\sigma}_t)\sigma_{p_t^H} \tag{84}$$

$$\frac{\mathbb{E}_t[dr_t^{B^{F,F}}]}{dt} - r_t^{F,f} = m_{2t}^T\sigma_{p_t^F} = (-\frac{n_t}{1-n_t}\sigma_{n_t} + \overline{\sigma}_t)\sigma_{p_t^F} \tag{85}$$

where $-r_t^{H,f}$ and $-m_{1t}$ are the drift and volatility of home country's SDF and similarly $-r_t^{F,f}$ and $-m_{2t}$ are for foreign country.

$$r_t^{H,f} = \rho + \mu_{n_t} + \overline{\mu}_t + \sigma_{n_t}\overline{\sigma}_t - (\sigma_{n_t} + \overline{\sigma}_t)^2 \tag{86}$$

$$r_t^{F,f} = \rho - \frac{n_t}{1-n_t}\mu_{n_t} + \overline{\mu}_t - \frac{n_t}{1-n_t}\sigma_{n_t}\overline{\sigma}_t - (-\frac{n_t}{1-n_t}\sigma_{n_t} + \overline{\sigma}_t)^2 \tag{87}$$

$$m_{1t} = \sigma_{n_t} + \overline{\sigma}_t \tag{88}$$

$$m_{2t} = \sigma_{1-n_t} + \overline{\sigma}_t \tag{89}$$

When $s_t < s(\overline{\chi}^{H,F})$, we can write out Country 1 and Country 2's wealth as

$$W_t^H = S_t^{H,F} + \overline{\chi}^{H,F}S_t^{H,F} \tag{90}$$

$$W_t^F = (1 - \overline{\chi}^{H,F})S_t^{F,F} \tag{91}$$

And from the optimization of logarithmic utility, we have

$$\rho W_t^H = C_{H,t} = n_t \overline{Y}_t \tag{92}$$

$$\rho W_t^F = C_{F,t} = (1 - n_t)\overline{Y}_t \tag{93}$$

Looking at the volatility of W_t^H,

$$\sigma_{W_t^H} = \sigma_{n_t} + \overline{\sigma}_t \tag{94}$$

$$= \frac{\rho q_t^{H,H}}{n_t}(\sigma_{q_t^{H,H}} + \overline{\sigma}_t) + \frac{\rho \overline{\chi}^{H,F} q_t^{H,F}}{n_t}(\sigma_{q^{H,F}} + \overline{\sigma}_t) \tag{95}$$

and the volatility of W_t^F,

$$\sigma_{W_t^F} = -\frac{n_t}{1 - n_t}\sigma_{n_t} + \overline{\sigma}_t \tag{96}$$

$$= \frac{\rho(1 - \overline{\chi}^{H,F}) q_t^{F,F}}{1 - n_t}(\sigma_{q_t^{F,F}} + \overline{\sigma}_t) \tag{97}$$

For both countries, portfolio weights add up to 1

$$\frac{\rho q_t^{H,H}}{n_t} + \frac{\rho \overline{\chi}^{H,F} q_t^{H,F}}{n_t} = 1 \tag{98}$$

$$\frac{\rho(1 - \overline{\chi}^{H,F}) q_t^{F,F}}{1 - n_t} = 1 \tag{99}$$

so we have

$$\frac{\rho q_t^{H}}{n_t} + \frac{\rho \overline{\chi}^{H,F} q_t^{H,F}}{n_t} + \frac{\rho(1 - \overline{\chi}^{H,F}) q_t^{F,F}}{1 - n_t} = 2 \tag{100}$$

where

$$\frac{q_t^{H,H}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{s_\tau}{n_\tau}d\tau\Big] \tag{101}$$

$$\frac{q_t^{H,F}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{1 - s_\tau}{n_\tau}d\tau\Big] \tag{102}$$

$$\frac{q_t^{F,F}}{1 - n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{1 - s_\tau}{1 - n_\tau}d\tau\Big] \tag{103}$$

That is,

$$\mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}(\frac{s_\tau + \overline{\chi}^{H,F}(1 - s_\tau)}{n_\tau} + \frac{(1 - \overline{\chi}^{H,F})(1 - s_\tau)}{1 - n_\tau})d\tau\Big] = \frac{2}{\rho} \tag{104}$$

Using Feynman-Kac formula, we have

$$\frac{s_t + \overline{\chi}^{H,F}(1 - s_t)}{n_t} + \frac{(1 - \overline{\chi}^{H,F})(1 - s_t)}{1 - n_t} = 2 \tag{105}$$

So n_t is a function of the only state variable s_t (the other solution $n_t = \frac{1}{2}$ is not achievable with equity constraint binding).

$$n_t = (1 - \overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F} \tag{106}$$

$$1 - n_t = (1 - \overline{\chi}^{H,F})(1 - s_t) \tag{107}$$

$$\theta_t^{H,B^H} = \theta_t^{H,B^F} = 0 \tag{108}$$

And also we have

$$\frac{C_{H,t}}{C_{F,t}} = \frac{n_t}{1 - n_t} \tag{109}$$

Now we can solve for prices of risks,

$$m_{1t} = \sigma_{n_t} + \overline{\sigma}_t = \frac{(1 - \overline{\chi}^{H,F})s_t}{(1 - \overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}}\sigma_{s_t} + \overline{\sigma}_t \tag{110}$$

and solve for s^a using

$$\frac{\mathbb{E}_t[dr_t^{B^{H,H}} - dr_t^{B^{H,F}}]}{dt} = m_{1t}(\sigma_{p_t^H} - \sigma_{p_t^F}) < 0 \tag{111}$$

we have

$$\frac{2(1 - \overline{\chi}^{H,F})s^2 + (2\overline{\chi}^{H,F}\eta + (1 - \overline{\chi}^{H,F})(\eta - 2))s - \eta\overline{\chi}^{H,F}}{(1 - \overline{\chi}^{H,F})s_t + \overline{\chi}^{H,F}} > 0 \tag{112}$$

For $0 < s^a < s^u$, we need the right range for parameter pair $(\overline{\chi}^{H,F}, \eta)$. For example, if $\overline{\chi}^{H,F} = 0$, we have $s^u = \frac{1}{2}$ and $s^a = \frac{2-\eta}{2}$. And if $\eta = \infty$, we have $s^a = \frac{\overline{\chi}^{H,F}}{1+\overline{\chi}^{H,F}}$.

With a special parameter case, $\rho = (\frac{\eta-1}{\eta}\sigma)^2$, we can solve everything we need analytically. The price of equity 1 in unconstrained case is

$$q_t^H(s) = \frac{1}{2\rho}\Big(1 + \frac{1-s}{s}\ln(1-s) - \frac{s}{1-s}\ln(s)\Big) \tag{113}$$

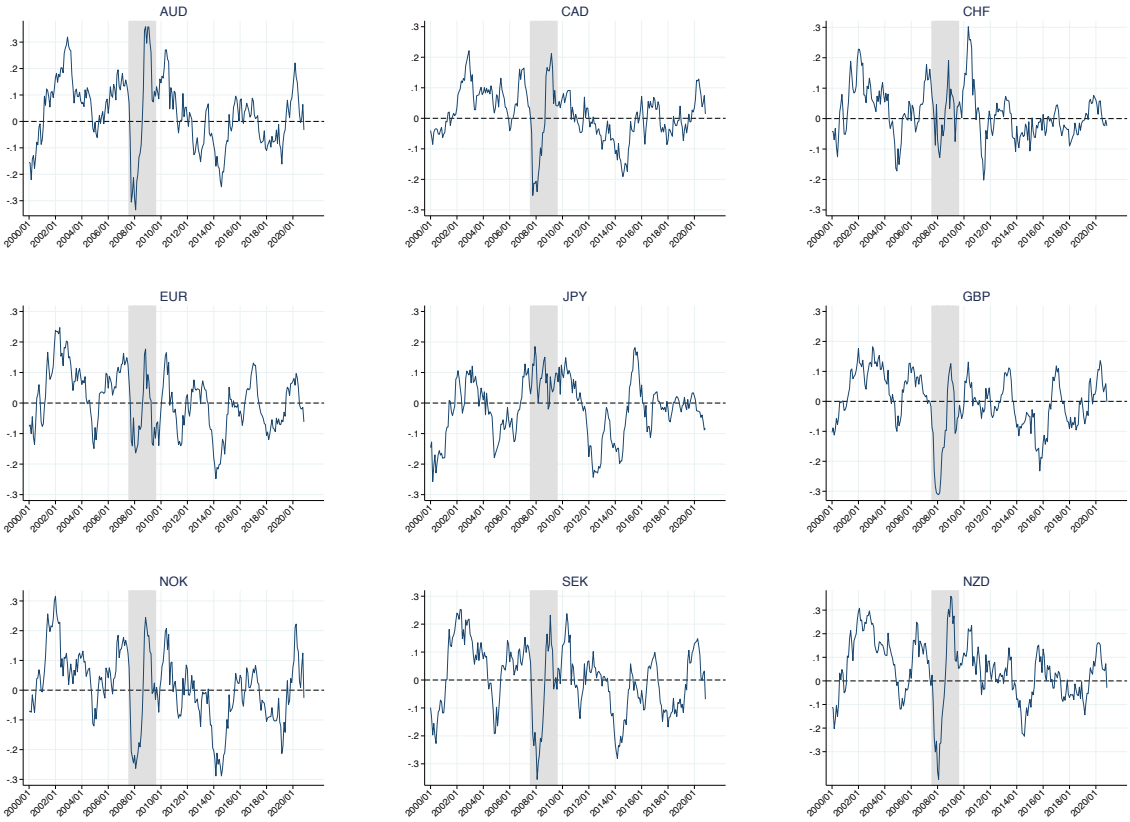
$$q_{1t}'(s) = -\frac{1}{2\rho}\frac{1}{s(1-s)}\Big(1 + \frac{1-s}{s}\ln(1-s) + \frac{s}{1-s}\ln(s)\Big) \tag{114}$$

$$q_{1t}''(s) = -\frac{1}{2\rho}\frac{1}{s^2(1-s)^2}\Big((2s-1) - \frac{(1-s)^2}{s}\ln(1-s) + \frac{s^2}{1-s}\ln(s)\Big) \tag{115}$$

□

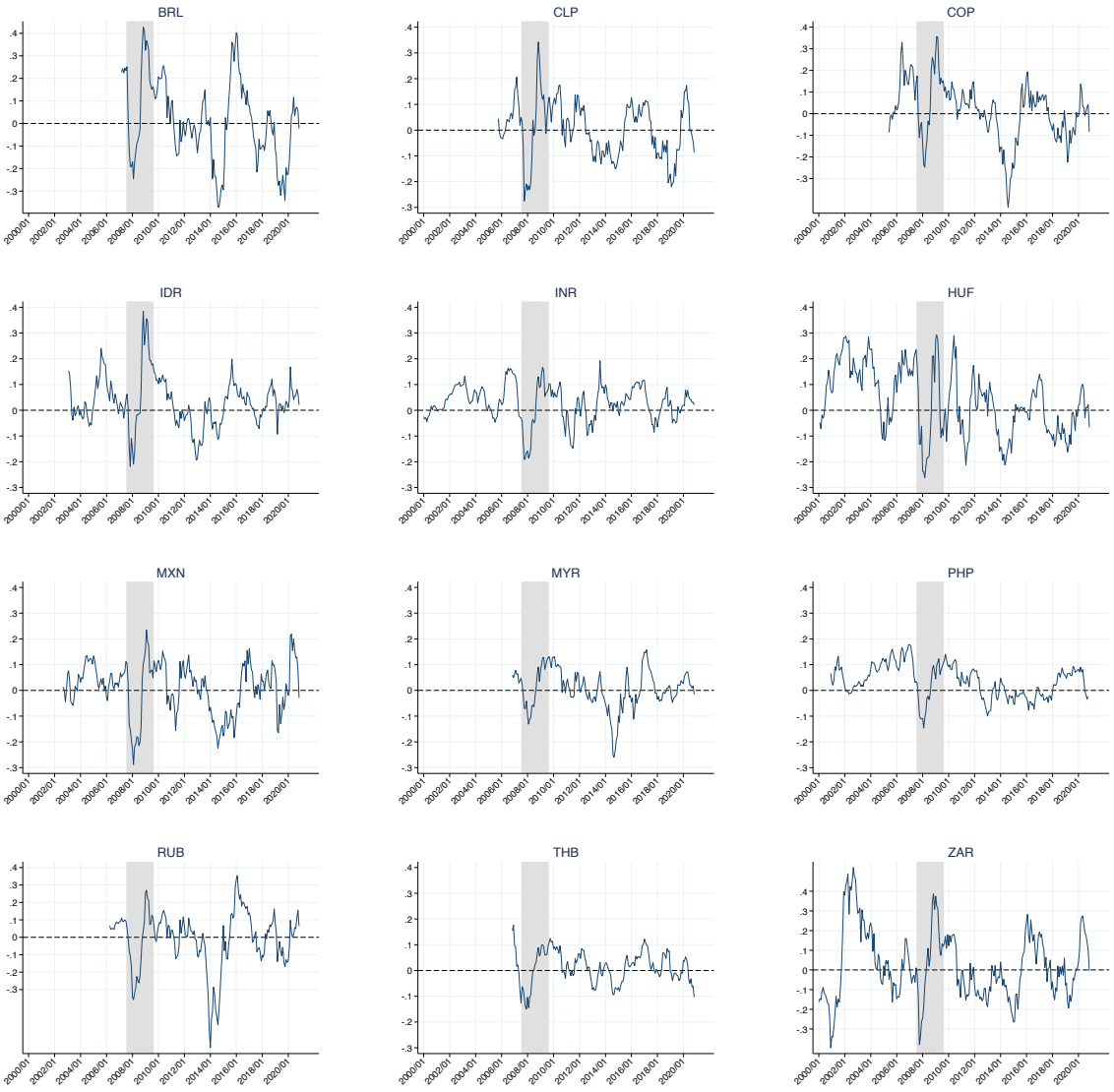
B Additional Graphs and Tables

Table B.1: Annualized UIP Premium for G10 Currencies



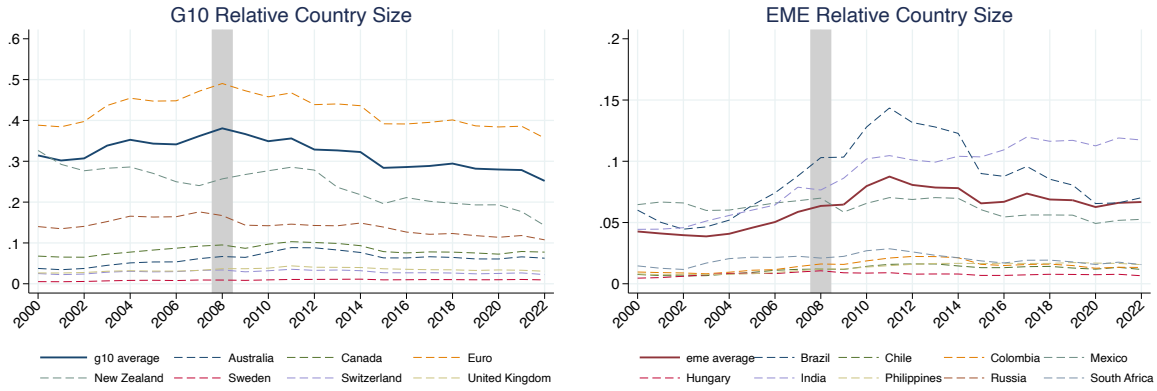
Note: This panel of figures present the pattern of UIP premiums for G-10 currencies over the 2000 - 2021 sample. The UIP premiums are defined as the annualized excess returns of local-currency one-year government bond yields against the synthetic USD yields and are in log points. The grey area corresponds to the months of the Great Financial Crisis in 2008.

Table B.2: Annualized UIP Premium for EME currencies



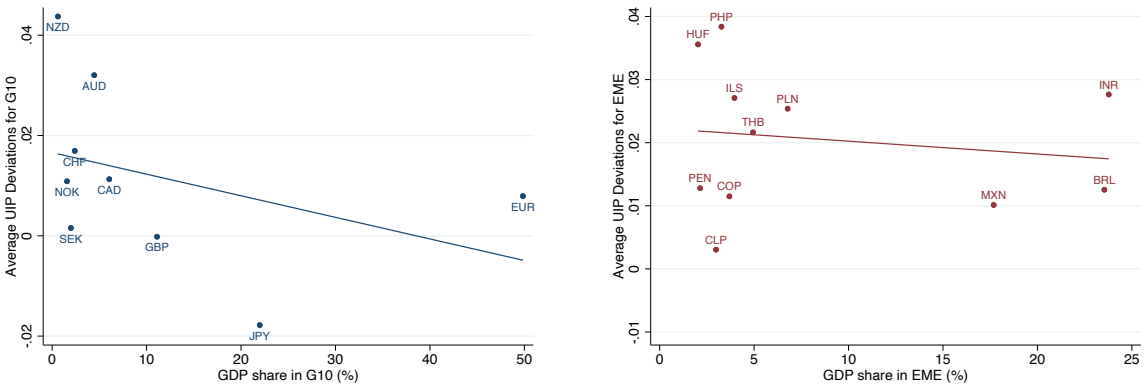
Note: This panel of figures present the pattern of UIP premiums for emerging market economies over the 2000 - 2021 sample. The UIP premiums are defined as the annualized excess returns of local-currency one-year government bond yields against the synthetic USD yields and are in log points. The grey area corresponds to the months of the Great Financial Crisis in 2008.

Table B.3: Relative G10 (or EME) Country Size



Note: This panel of figures present the ratio of nominal GDP of a country over the sum of GDP of the country and of the U.S. The figure for G10 currency countries are on the left and emerging market economies are on the right. Each line represents the country-specific GDP ratio. The bolded line is the weighted average ratio for the G10 group (left) and the EME group (right).

Table B.4: Average UIP Premium and Country Size Share within the G10 (or EME) Group



Note: This panel of figures present the scatter plot of average UIP premium in our sample against the average country size share (measured by nominal GDP) for each currency in their respective G10 currency group (left) or EME group (right). UIP deviations are in log points and are annualized and averaged across time for the sample period of 2000 to 2021. Each dot represent a currency labeled by its currency name.

Table B.5: Annualized UIP Premium on VIX index and Equity Home Bias Measures

	(1)	(2)	(3)	(4)	(5)
VIX_t	0.00340***			0.00200***	0.00245***
	(0.000698)			(0.000656)	(0.000660)
VIX_{t-1}		0.00335***			
		(0.000679)			
Share of GDP over US			-0.0346		
			(0.0616)		
$\chi_{HH,t}$				0.969***	
				(0.128)	
$\chi_{FF,t}$					2.047***
					(0.304)
Constant	-0.0568***	-0.0555***	0.0483	-0.863***	-1.945***
	(0.0148)	(0.0144)	(0.0656)	(0.107)	(0.281)
Observations	213	213	251	203	203
R^2	0.1012	0.1034	0.0013	0.3025	0.2675
Adjusted R^2	0.097	0.099	-0.003	0.296	0.260

Standard errors in parentheses

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: This table gives the regression coefficients of annualized average UIP premium on various independent variables. Depedent variables in all columns are the annualized average UIP premium, defined as the as excess return in local currencies against the USD for countries in the G10 currency group and is in log points and calculated for one-year government bond yields. We treat home (H) is G10 country, and foreign (F) is the U.S. The equity share $\chi_{HH,t}$ defined as home investors' holdings of home equities over the total supply of home equities; The equity share $\chi_{FF,t}$ defined as foreign investors' holdings of foreign equities over the total supply of foreign equities. The share of GDP over US the total share of nominal GDP of the G10 countries over the US GDP.

Table B.6: Equity Home Bias Measures on VIX index

	(1)	(2)	(3)	(4)	(5)	(6)
Dependent Var.	$\chi_{HH,t}$	$\chi_{HH,t}$	$\chi_{HH,t}$	$\chi_{FF,t}$	$\chi_{FF,t}$	$\chi_{FF,t}$
VIX_t	0.00143***		0.000650**	0.000459***		0.000152
	(0.000348)		(0.000324)	(0.000150)		(0.000143)
VIX_{t-1}		0.00147***			0.000474***	
		(0.000349)			(0.000150)	
UIP Premium			0.231***			0.0904***
			(0.0304)			(0.0134)
Constant	0.832***	0.831***	0.845***	0.923***	0.922***	0.928***
	(0.00746)	(0.00748)	(0.00681)	(0.00321)	(0.00322)	(0.00300)
Observations	203	202	203	203	202	203
R^2	0.0778	0.0820	0.2844	0.0445	0.0474	0.2213
Adjusted R^2	0.073	0.077	0.277	0.040	0.043	0.214

Standard errors in parentheses

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Note: This table gives the regression coefficients of equity home bias measures on VIX index with other controls. Dependent variable in columns (1)-(3) is $\chi_{HH,t}$ defined as home investors' holdings of home equities over the total supply of home equities; dependent variable in columns (4)-(6) is $\chi_{FF,t}$, defined as foreign investors' holdings of foreign equities over the total supply of foreign equities. The annualized average UIP premium is defined as the excess return in local currencies against the USD for countries in the G10 currency group and is in log points and calculated for one-year government bond yields. We treat home (H) is G10 country, and foreign (F) is the U.S.