Addiction and Illegal Markets*

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Abstract

This paper studies dynamic contracts in illegal addictive markets where individuals' tastes for addictive goods develop through prolonged consumption and contract enforcement is limited. Our theoretical analysis uncovers the optimality of a 'free-first-dose' strategy where sellers intensify buyers' addiction by offering consumption credit to newcomers. More specifically, we show that buyers default a certain portion of the debts for early period consumption but are never imposed any penalty on the equilibrium path. This implies that illegal markets might favor non-violent interactions over violent ones, defying the stereotypical association of illegality with violence. Furthermore, we show that buyers' consumption of addictive goods is socially excessive and that equilibrium odds in illegal gambling markets exhibit a puzzling phenomenon known as the long-shot bias. We discuss these insights in the context of illegal sports wagering, narcotics, and religious sects.

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1 Introduction

Individuals' tastes for addictive goods develop through prolonged consumption, while economic transactions are shaped by their dynamic incentives and expectations about future interactions when contract enforcement is limited. These dual dynamics are crucial in understanding optimal trading mechanisms in addictive markets, where sellers manage dynamic incentives of buyers who become addicted by consumption over time, and formal enforcement is absent because the activity itself is illegal or requires severe infringement of individual rights. Our study aims to clarify this complex relationship, a topic that remains overlooked in existing literature on dynamic contracts.

While our model can capture market dynamics across a variety of illicit markets, we chose illegal gambling markets as our focal industry. This selection is primarily underscored by the remarkable scale of the global gambling market, which significantly eclipses the market values of other major industries, including tobacco and pharmaceuticals. It is estimated that the illicit segment of the gambling market alone may surpass an astounding $1,700 billion globally (United Nations Office on Drugs and Crime (2021)). Furthermore, illegal gambling serves as an exemplary model for studying other illegal markets, as it embodies many of the features found in other illegal markets, as discussed below.

In the model, a bookmaker (seller) illegally holds horse race betting for a finite number of periods, and gamblers (buyers) become addicted to gambling.[1] In each period, the bookmaker chooses either 'cash-bet' or 'book-bet' as a format for horse race betting. Under the cash-bet format, gamblers are required to immediately pay the amounts they bet whereas under the book-bet format, gamblers defer payments. Since gambling is illegal, there are no formal institutions to enforce debt repayments from gamblers, and the bookmaker is exposed to default risks under the book-bet, in contrast to the cash-bet.[2] Since gamblers always default all their debts in the last period, the bookmaker has no choice but to rely on the cash-bet in that period. On the contrary, the bookmaker may choose the book-bet in the other periods, together with a punishment instrument to reduce the default risk. The cornerstone of our model is the potential for gambler addiction, a factor the bookmaker is aware of. In this context, we characterize the market equilibrium, detailing the bookmaker's optimal punishment strategy in the event of default, and the gamblers' betting trajectory.

This model captures several key features of illegal markets. First, transactions are not protected. Second, sellers, such as those in illegal gambling or illicit drug markets, extend consumption credit to buyers. Third, the model details how sellers may enforce non-formal punitive measures to counteract default risk. These measures can range from

[1] Our main model focuses on two periods to simplify the argument. We extend the basic model to allow for an infinite time horizon in Section 7.

[2] We omit consideration of liquidity risk, which could arise from a widespread default, potentially hindering the bookmaker's ability to pay winning gamblers.

exclusion from future transactions to the exertion of psychological or physical violence, tactics seldom seen or severely restricted within legal markets.[3]

Our first theoretical finding reveals that sellers utilize a 'free-first-dose' approach. Much like drug dealers who may offer a free sample to cultivate dependency in their clientele, we find that sellers strategically allows the buyers to initially borrow and consume beyond their means. This tactic is designed to cultivate an addiction, thereby ensuring increased future revenues.

This result unveils that in equilibrium, the seller anticipates and accepts a certain degree of default from each buyer. More specifically, we show that the seller can always replicate the equilibrium outcome attained under any punishment policy by the following *bang-bang* punishment policy (see Proposition 2 below): buyers are allowed to default up to a certain upper bound, which we call *maximum forgiveness*, and are never punished unless they default more than it in equilibrium. Furthermore, the seller can be weakly better off by this bang-bang punishment policy upon any other punishment policy. Such "allowed default" plays a strategic role: it cultivates buyers' addiction by inducing large consumption amounts in the initial period, making them willing to consume aggressively later on.

The second main finding probes further into this dynamic, assessing the influence of violence on market equilibrium. Interestingly, while violence might enforce more prompt debt repayments, it could simultaneously weaken the initial 'free-first-dose' incentive, thereby stifling the development of addiction. This paradoxical outcome proposes that illegal markets might favor non-violent interactions over violent ones, defying the stereotypical association of illegality with violence.

Third, we investigate welfare implications about illegal market equilibrium and show that buyers may consume more addictive goods than the social optimum, that is, their consumption becomes "socially excessive". This is because sellers try to extract buyers' surplus by making them addicted to consuming goods in the initial period and then increasing their willingness-to-pay for a later period consumption.

Fourth, we show that equilibrium odds in illegal gambling markets exhibit the feature of so called *long-shot bias*, which has been well-known as an empirical puzzle in the existing literature, such that buyers (gamblers) place larger bets on long-shots than favorites (Chiappori et al. (2019), Ottaviani and Sørensen (2008)). That is, gamblers bet more on the horses which are less likely to win the race. In contrast to the previous studies about the long-shot bias, we explain such a phenomenon in the view point of the seller's strategy to manage gamblers' default incentives in illegal gambling markets. Moreover, the long-shot bias emerges as an equilibrium feature despite buyers' rational expectations, and odds not manipulated by the bookmakers. The key intuition behind this result is that betting on long-shots becomes relatively less costly as gamblers can partially default on their debts. In turn, the distribution of odds mechanically becomes more homogeneous, which creates the long-shot bias.

[3]In Section 7, the model is formally extended to more general illegal market settings.

Finally, we extend the model is two main directions. Firstly, we extend the model to account for endogenous commission fees. This seems particularly relevant in our context, since one could argue that negative fees - or bet subsidies - are substitutes to book-bet strategies. We show that this is not the case, as the bookmaker still relies on book-bet strategies when commission fees are endogenous. Rather than being substitute, bet subsidies and book-bet seems rather complement. Secondly, we extend the model and relax the assumption that the bookmaker can commit to the punishment policy announced at the beginning of the game. We also assume an infinite time horizon, with a long-lived bookmaker and short-lived gamblers. When non-repayment from the book-maker is punished by gamblers, the bookmaker faces a dynamic enforcement constraint. We find that this constraint is more easily satisfied when addiction is strong, as foregoing future bets becomes particularly costly for the bookmaker. This result again supports the prediction that addiction eases transactions in the absence of formal enforcement.

The dynamics within illegal markets of addictive goods, as highlighted in our model, bear a striking resemblance to the concept of relational contracts. In these markets, the relational contracts formed between participants, such as bookmakers and gamblers or drug dealers and users, are not only built on mutual trust and reputation but are also deeply influenced by the addictive nature of the transactions. For instance, when bookmakers extend credit to gamblers, or when drug dealers employ a 'free-first-dose' strategy, they are initiating a relational contract that exploits the addictive behavior of the other party. This addiction becomes a critical component of the agreement, ensuring repeat engagement and a form of enforced loyalty. Thus, addiction serves not only as a driving force for consistent market participation but also as an informal enforcement mechanism that underpins these relational contracts, sustaining the market's equilibrium in the absence of formal legal structures.

The predictions of our model find particular support in the functioning of different markets. First, the model's theoretical predictions find empirical support in the functioning of the illegal gambling market. Despite the challenges associated with data collection in such opaque markets, Mehmood and Chen (2022) collect and provide some stylized facts from the illegal horse race betting market in Pakistan. The authors calculate that approximately 55% of wagers within this market are placed on credit. Our model is able to square this puzzling manifestation as the 'free-first-dose' strategy in action. According to our model, this approach is not merely about extending credit; it is a calculated bid to cultivate gambling addiction, aiming to ensure a lucrative and enduring customer base. Mehmood and Chen (2022) collected data on debt repayment too, and found that on average, 35% of gamblers do not repay their debt in full. The authors give suggestive evidence that despite the significant rate of default, violence is rare in this market. Only 0.5% of gamblers express fear of violent repercussions for unpaid debts. This empirical evidence underscores the bookmakers' strategic choice to foster addiction through leniency and trust rather than through coercion and fear, a choice that aligns with our model's predictions.

Second, our model's implications also find support in the functioning of illegal drug markets, where the "free-first dose" strategy is a well-documented method to foster dependency among new users. Galenianos and Gavazza (2017) estimate a model using data on the crack cocaine market in the United States. The authors rely on the STRIDE dataset, which contains records of the acquisition of illegal drugs by undercover agents and DEA informants. Hence, the data may contain information about purchases from "new" consumers. Consistent with the prediction of our model, Galenianos and Gavazza (2017) find that drug dealers do not necessarily provide low-quality products to new users by diluting (or "cutting") the products. That way, drug dealers may invest in relationships, and cultivate stronger addictions. Finally, the nuanced role of violence in illegal drug markets has been studied, among others, by Curtis and Wendel (2000) in their analysis of the heroin market of New York City. Violence can either be a tool for enforcement, or something deliberately avoided to maintain customer loyalty. This mirrors our model's predictions about the complex interplay between coercion and addiction sustainability.

Finally, the dynamics explored in our model can also be observed in the interaction of religion and addiction, particularly within the context of religious sects. Initial engagements often come with tangible benefits and a strong sense of community, serving as the "free-first dose" that attracts individuals and fosters attachment (Dawson (1998)). As commitment deepens, a form of "debt repayment" emerges, where members feel obliged to contribute more significantly, financially or otherwise, driven by emotional or spiritual indebtedness.

Related Literature. This paper contributes to the literature in several ways. First, we contribute to the large and multifaceted literature on addiction. In a seminal article, Becker and Murphy (1988) developed a theory of addiction were decision-makers have rational expectations about the consequences of their choices. Since then, several works investigated the rational nature of addiction and explored addictions in various environments.[4] Several papers investigated the interaction between addiction and market structure (Becker, Grossman and Murphy (1994), Chaloupka (1991), Fethke and Jagannathan (1996), Driskill and McCafferty (2001)). We complement these studies most specifically, as we investigate addiction in illegal markets. This aspect seems particularly crucial, as addictive goods frequently fall into the realm of illegality.

Second, this paper also contributes to the literature on relational contracts. This literature addressed issues related to how transactions are self-enforced without formally written contracts (for example, Levin (2003); Malcomson (2012)). However, to our knowledge, there are few theoretical attempts to investigate how individuals' preferences over addictions are interlinked with optimal trading arrangements in the lack of formal enforcement institutions. Because illegal markets of addictive goods are characterized by both addiction preferences and informal contracts, it is an important research venture to un-

[4]See, among others, Hoch and Loewenstein (1991), Orphanides and Zervos (1995), Chaloupka and Warner (2000), Gruber and Koszegi (2001), Courtemanche, Heutel and McAlvanah (2014), Grossman and Chaloupka (2017), Schilbach (2019), and Allcott, Gentzkow and Song (2022).

derstand what informal trading arrangements emerge when individuals consume addicted goods and engage in informal contract agreements that may be sustained via relational contracts. More specifically, both addiction preferences and relational contracts dynamically evolve altogether: individuals accumulate the tastes for addictions by consuming addicted goods over time while relational contracts are sustained by individuals' future concerns. Therefore, it is important to address how these two-way dynamic incentives are intertwined to characterize optimal trading mechanisms in illegal markets without formal enforcement institutions.

Third, this paper contributes to the theoretical literature on illicit drug markets. Galenianos, Pacula and Persico (2012) and Galenianos and Gavazza (2017) propose search theoretic models where sellers of illicit drugs are randomly matched buyers and choose qualities of drugs that are not observable to buyers. The main focuses of these papers are on the moral hazard problem of sellers, and, more specifically, the issue of whether sellers choose low quality for first-time buyers or offer high qualities for loyal buyers. We complement their work by focussing more specifically on buyers' addiction preferences, which dynamically interact with their default incentives, and sellers' optimal selling strategy.

Finally, this paper contributes to the literature on decision-making that leverages gambling data. Most notably, gambling data have been used to recover risk preference (Jullien and Salanié (2000), Feess, Müller and Schumacher (2016), Chiappori et al. (2019)), and behavioral biases affecting risky choices (Snowberg and Wolfers (2010), Losak, Weinbach and Paul (2023)). We complement these studies by focusing on addiction in illegal gambling markets.

2 Model

2.1 Gambling Market for Horse Race Betting

We present the model of an illegal market where a seller sells addictive goods to buyers who form the tastes for addictions over time and their transactions are illegal. Specifically, to fix the idea, we consider the gambling market for horse race betting. We will discuss how our model can be applied to other illegal markets than horse race betting in Section 7.3 later.

There are two periods, denoted by $t = 1, 2$, and a bookmaker runs a horse race betting in each period. There are n gamblers and m horses where $n \geq 2$ and $m \geq 2$. We denote by I and J the sets of gamblers and horses. We use the feminine pronoun for the bookmaker and masculine pronoun for each gambler in what follows.

We first drop time index $t = 1, 2$ and describe how the gambling market for horse race betting works. We consider the so called *parimutuel betting* that is common in many wagering markets including not only horse race betting but also other sports betting. In the parimutuel betting gamblers decide how much bet to place before they know the odds of outcomes. In particular we focus on the "win bet" that gamblers bet on which horse to come first, given expectations about the odds of horses. Suppose that gambler i places

bet of $a_{i,j} \geq 0$ on horse $j \in J$, where amounts bet are measured in terms of the numèraire good. We denote by $\mathbf{a}_i = (a_{i,1}, ..., a_{i,m})$ a profile of amounts bet by gambler i on all horses. Given a collection of amounts bet by all gamblers $\{\mathbf{a}_i\}_{i \in I}$, the bookmaker receives the commission fee ("house take") as a constant fraction $f \in (0,1)$ of the aggregate amount bet $\sum_{i \in I, j \in J} a_{i,j}$, that is, the bookmaker receives the commission fee equal to $f \sum_{i \in I, j \in J} a_{i,j}$. We will maintain throughout the basic model the assumption that the commission fee $f \in (0,1)$ is exogenously given.[5] We will discuss the extension of the model to allow endogenous commission fees in Section 7.1.

In the parimutuel betting the bookmaker does not face a market risk; the winning bets are paid with the losing bets, net of the fee f. The odds of horse j, denoted by h_j, is defined as follows:

$$\begin{aligned} h_j &\equiv \frac{(1-f) \times \text{Amount bet on all horses}}{\text{Amount bet on horse } j} \\ &= \frac{(1-f) \sum_{i \in I, j \in J} a_{i,j}}{\sum_{i \in I} a_{i,j}}, \end{aligned} \tag{1}$$

provided the denominator is positive.[6] [7] That is, the odds of horse j is determined by reflecting how much gamblers bet on horse j relative to the aggregate amounts bet on all horses. Specifically, the odds of horse j is higher as gamblers bet less for it. We define $h_j \equiv 0$ when nobody bets on horse j.

We denote by $p_j \in (0,1)$ the probability that horse j wins the race. The returns of horse racing are determined as follows: when horse j wins the race, gambler i who places bet of $a_{i,j} > 0$ on horse j will receive from the bookmaker the payout equal to the odds of horse j times the amount he bet on horse j, that is, $h_j a_{i,j}$. When gambler i bets $A_i \equiv \sum_{j \in J} a_{i,j}$ in total, his net return equals to $h_j a_{i,j} - A_i$, which is positive (when he wins) or negative (when he loses).[8]

2.2 Addiction Preference

We now turn to the two-period model ($t = 1, 2$). We denote by $a_{i,j,t} \geq 0$ the amount bet by gambler i on horse j in period $t = 1, 2$. We then define by $A_{i,t} \equiv \sum_{j \in J} a_{i,j,t}$ the

[5] The commission/participation fee of the bookmaker is about 5% in the case of the illegal horse racing we observed in Pakistan.

[6] The odds h_j defined here represents the gross return of betting one dollar on winning horse j. Alternatively, we can call its net return $h_j - 1$ odds.

[7] Levitt (2004) shows that bookmakers might also take some market risks when setting odds. Accounting for this would slightly change equation (1), although the logic and the results of this paper would remain unchanged.

[8] There may be the case that horse j wins the race but nobody bets on that horse. Then the practice used in that case may be that each gambler i is refunded $(1-f)A_i$ which equals to the amount bet A_i minus the commission fee of the bookmaker fA_i. However, since we will focus on the equilibrium in which odds are positive for any horse as we will explain more details later, such case never arises.

amount bet by gambler i in total in period $t = 1, 2$. To model gamblers' preferences, we follow the "gambling-in-utility" approach (Conlisk (1993)) as follows. Gamblers have the preferences over both gambling activities and private consumption of the numèraire good. More specifically, when gambler i bets $A_{i,1}$ in total and consumes the private good of $c_{i,1}$ in period 1, he obtains the payoff of $v(A_{i,1}) + c_{i,1}$ in period 1. Here, $v(A_{i,1})$ is the gambler i's direct payoff obtained from enjoying the gambling by betting $A_{i,1}$ in total in period 1. We assume that v is continuously differentiable, strictly concave, and $v(0) = 0$. Additionally, gambler i obtains the payoff of $u(A_{i,2}, S_i) + c_{i,2}$ in period 2 when he bets $A_{i,2}$ in total and consumes the private good of $c_{i,2}$ in period 2. Here the payoff obtained from gambling activities in period 2 is given by $u(A_{i,2}, S_i)$ that depends on how much to bet $A_{i,2}$ in total in period 2 as well as how much the gambler is addicted by the first-period gambling, which we capture by the *addiction stock* S_i:

$$S_i = \alpha + \beta A_{i,1} \tag{2}$$

where $\alpha \geq 0$ and $\beta > 0$. Due to $\beta > 0$, the addiction stock S_i positively depends on how much gambler i bet in period 1. Furthermore, $\alpha \geq 0$ represents the fixed benefit from participating in the gambling in period 1: $\alpha > 0$ when $\sum_{j \in J} a_{i,j,1} > 0$ and $\alpha = 0$ otherwise. Therefore, gamblers develop more addiction preferences in period 2 by accumulating the addiction stock $S_i = \alpha + \beta A_i$ which increases as they engage in more gambling activities in period 1 (they place larger bet A_i in period 1).

We assume that the second-period utility u is continuously differentiable and strictly concave with respect to $A_{i,2}$ with $u(0, S) = 0$ as well as it is increasing in the addiction stock S_i. In what follows we denote by u_A and u_S partial derivatives of u with respect to $A_{i,2}$ and S_i: $u_A \equiv \partial u / \partial A$ and $u_S \equiv \partial u / \partial S$. We assume the boundary conditions on u_A such that $u_A(0, S) > 1$ and $u_A(\infty, S) < f$ for all $S \geq 0$, which will ensure the interior solution.

In summary gambler i obtains total payoffs over two periods as follows

$$v\left(A_{i,1}\right) + c_{i,1} + \left\{u\left(A_{i,2}, S_i\right) + c_{i,2}\right\} \tag{3}$$

where we assume no time discounting between the two periods for simplicity.

2.3 Default and Punishment

The gambling market for horse race betting is illegal such that there are no formal institutions to enforce payments from gamblers to the bookmaker.

There are two forms of betting: cash-bet format and book-bet format. Under the cash-bet, gamblers decide how much to bet and then immediately pay the bookmaker the chosen amount. Under the book-bet, gamblers also decide on their bet amount, but payment is deferred. They are only required to settle their debts after a certain period of time. The amount they owe is the initial bet minus any winnings and potential additional fees. Essentially, the book-bet functions as a loan from the bookmaker. In Pakistan's

illegal horse gambling market, book-bets are commonly utilized, with the expectation that gamblers repay their debts within a week. However, since the gambling market is illegal and not formally organized, gamblers may renege on debt paybacks to the bookmaker. Indeed, when gamblers use a book-bet strategy and lose, they may refuse to pay the bookmaker. The bookmaker cannot however legally force gamblers to pay back their debts, because the gambling is illegal.

The bookmaker chooses which cash-bet or book-bet to use in period 1, although she must rely only on the cash-bet in period 2 because period 2 is the last period, so gamblers will never make paybacks in period 2 when the gambling is held as the book-bet in period 2. We will show later that, even though the book-bet causes default risks to the bookmaker, she may prefer the book-bet to the cash-bet.

In order to deter the default by gamblers under the book-bet the bookmaker can resort to a punishment policy announced in period 1. More specifically, we consider two instruments to punish the gamblers who default on debt repayments, both of which are relevant in illegal markets.

First, the bookmaker can use *the exclusion policy* q that specifies the probability $q \in [0,1]$ to exclude each defaulting gambler from the participation in the second-period gambling (see Mehmood and Chen (2022) for an empirical assessment on this sort of punishment.[9]) Suppose that a gambler owes a debt $z \geq 0$ to repay to the bookmaker but defaults an amount $D \in [0,z]$ in the end of period 1; he pays back only $z - D$ to the bookmaker in the end of period 1. Then the gambler is excluded from the second-period gambling with the probability $q(D,z,A) \in [0,1]$ that is contingent on how much to bet A in total and how much to owe debt z and default $D \in [0,z]$ in period 1.

Second, the bookmaker can also impose an utility-based penalty $\xi(D,z,A) \in [0,H]$ on the gambler who places bet of $A \geq 0$ in total, and owes debt of $z \geq 0$ but defaults $D \in [0,z]$ in period 1.[10] Here $H \geq 0$ is the exogenous upper bound for available utility-based penalties. For example, as often discussed about illegal markets, the buyers who defaulted on payments may face the violence threatened by sellers and associated crime groups. The availability of violence as a threat to enforce debt repayments depends on legal and police institutions of a society, which we capture by the exogenous upper bound for possible penalties $H \geq 0$. Additionally, gamblers may lose social reputation and suffer from being labelled bad social images when they default promised payments (see Mehmood and Chen (2022) for related penalties imposed on defaulting gamblers in the illegal horse race betting in Pakistan.) We can allow the case that the utility-based penalty is never available, that is, $H = 0$.

In the beginning of period 1 the bookmaker offers and commits to a *punishment policy* $\{q, \xi\}$.

Two remarks are in order.

[9] Mehmood and Chen (2022) empirically investigate different punishment forms to exclude defaulting gamblers in the illegal horse race betting in Pakistan.

[10] See Dubey, Geanakoplos and Shubik (2005) for a related approach to default and punishment in market equilibrium framework.

First, to simplify, we suppose throughout the main analysis that the default incentive is relevant only for gamblers under the book-bet while it is not for the bookmaker. That is, the bookmaker cannot renege on payouts to gamblers once the market odds are determined. We discuss the mechanism by which the bookmaker can endogenously commit to enforce payouts as well as punishment policies in Section 7.2 and Appendix. In particular we present the dynamic model in which the bookmaker who is a long-lived player interacts with gamblers who are short-lived players over infinite periods. Then we show that there exists an equilibrium in which the bookmaker self-enforces agreed upon payouts and punishment policies. In this respect we can view the current two-period setting as a short-cut of such a dynamic equilibrium.[11] As a result, the bookmaker can make promised payouts to gamblers according to the market odds $\{h_j\}_{j\in J}$.

Second, because of some exogenous reasons such as institutional and historical constraints, the bookmaker cannot use other mechanisms than the punishment policy $\{q, \xi\}$ we mentioned above. In other words there are limits on available instruments which the bookmaker can use. For example, the commission fee $f \in (0, 1)$ is exogenous so that the bookmaker cannot control it, and it cannot be negative. Also the bookmaker cannot charge fixed fees to extract surplus from gamblers at the outset of period 1.[12] The main purpose of the paper is not to consider general mechanism design problems but rather focus on the realistic situation in which the bookmaker can use only limited set of instruments. This is the reasonable scenario when informal rules and institutions governing illegal gambling markets have been already established historically so that the current bookmakers cannot drastically change them. We will discuss how our main insights still remain valid even when the bookmaker chooses commission fees later. We will also show that commission fees are endogenously constrained by the commitment problem in illegal gambling markets, thereby justifying our assumption that the bookmaker has only limited instruments (see Section 7 below).

Additionally, the punishment policy $\{q, \xi\}$ relies only on the information about how much to bet A in total and how much to default $D \in [0, z]$ but not on other detailed information such as the identities of gamblers.

[11] Besides the dynamic mechanism we will discuss later, there are other justifications for the bookmaker to commit to payouts and punishment policies. For instance, the bookmaker may hire agents who are paid constant wages and work on her behalf. These agents are simply required to execute the payout and punishment policy according to ex ante specified rule. Then, since they are paid constant wages regardless of horse racing outcomes, they have no incentives to deviate from the ex ante specified rule. In fact, gambling may be managed by several betting stations but not the bookmaker herself. This might play a role for the bookmaker to commit to payouts and punishment policy.

[12] If the bookmaker can charge both commission fees and fixed fees without any restriction, she would set commission fees equal to zero in both periods together with a certain fixed fee charged in the beginning of period 1 such that all the surplus of gamblers is extracted. By an analogy to non-linear pricing by a monopoly, such two-part tariff can achieve the efficient allocation and give the bookmaker all the first-best surplus. However, such two-part tariff scheme encounters the difficulty to enforce in illegal markets where the bookmaker may renege on payouts to gamblers. As we will show in the extended model in Section 7, it becomes more difficult for the bookmaker to commit to payouts when the commission fee f is smaller. Therefore, the two-part tariff with zero commission fee may not be self-enforcing.

2.4 Gambling Equilibrium

We now define equilibrium in the gambling market for horse race betting in each period, $t = 1, 2$, given a gambling format (cash-bet or book-bet). The gambling market opens for horse race betting after the bookmaker chooses its format, the cash-bet or the book-bet, in each period.[13] In the parimutuel betting gamblers do not know market odds when they decide how much to bet on horses. One reasonable equilibrium concept used in such a market is *rational expectations equilibrium* in which (i) gamblers optimally decide how much to bet on horses $\mathbf{a}_i = (a_{i,1}, ..., a_{i,m})$ by forming expectations about market odds $(h_1, ..., h_m)$, and (ii) market odds $(h_1, ..., h_m)$ are determined according to condition (1) in order to be consistent with the amounts $(\mathbf{a}_1, ..., \mathbf{a}_n)$ bet by gamblers in the market. Therefore, the gamblers' expectations about the odds are self-fulfilled in equilibrium. In this definition each gambler takes his expectations about market odds $(h_1, ..., h_m)$ as given, and hence believes that he cannot influence the determination of market odds. This is the reasonable case when the number of gamblers n is so large that each of them has no market powers, as considered in the standard notion of competitive equilibrium.

We call a profile of amounts bet by gamblers and the odds $\{\mathbf{a}_1, ..., \mathbf{a}_n, h_1, ..., h_m\}$ that satisfy these conditions (i) and (ii) *gambling equilibrium*. In our two-period model we will look for a gambling equilibrium in each period $t = 1, 2$, given the bookmaker's strategy. In particular, to avoid complication, we will focus on the gambling equilibrium in which every horse attracts some positive amounts bet, so the odds of all horses are positive, $h_j > 0$ for all $j \in J$.[14] [15]

2.5 Timing of the Game

We consider the following timing of the games under the cash-bet and the book-bet used in period 1 (see Figure 1 for the cash-bet and Figure 2 for the book-bet).

When the bookmaker uses the cash-bet in both periods, the following game is repeated twice:

1. Gamblers decide how much to bet $\mathbf{a}_i = (a_{i,1}, ..., a_{i,m})$, by forming expectations about the odds $\{h_j\}_{j \in J}$ and taking them as exogenously given. Then gamblers immediately pay the bookmaker the amounts they bet.

2. Given amounts $\{\mathbf{a}_i\}_{i \in I}$ bet by gamblers, the equilibrium odds $\{h_j\}_{j \in J}$ are determined by condition (1).

[13] As we noted, only the cash-bet is used in period 2 because it is the last period so that gamblers never repay debts under the book-bet.

[14] In such equilibrium there are no horses which win the race but nobody bet on: no gamblers are refunded $(1 - f)A_i$.

[15] This restriction can be justified when each gambler places a small but positive bet $\varepsilon > 0$ on each horse by a "mistake". Alternatively, there are some irrational gamblers who always bet a positive but small amount $\varepsilon > 0$ on each horse. Then our equilibrium might be viewed as the limit as such irrational/mistaken bet ε goes to zero.

3. The wining horse is determined, and the bookmaker makes payouts to gamblers.

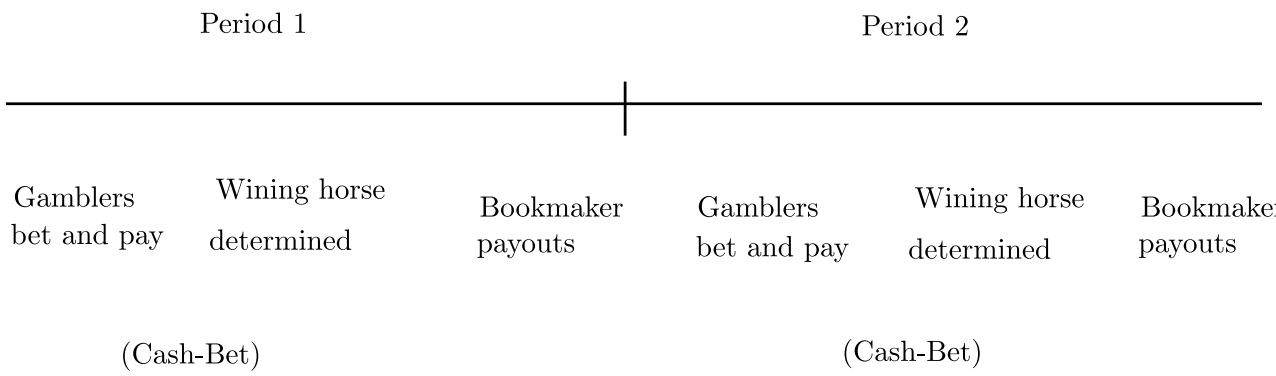


Figure 1: Timing of the game under cash-bet

Next we consider that the bookmaker uses the book-bet in period 1 while using the cash-bet in period 2. As we mentioned, gamblers may default on paybacks under the book-bet because they defer payments to the end of period 1 and no formal contracts exist to enforce their payments. When gambler i bets $\mathbf{a}_i$ and horse j wins the race, he receives the payout $h_j a_{i,j}$, which equals to horse j's odds h_j multiplied by his bet $a_{i,j}$ on horse j, from the bookmaker and pays the amount bet A_i in total, thereby obtaining the net return of $h_j a_{i,j} - A_i$ which is positive or negative. Then gambler i owes the debt equal to $z_{i,j} \equiv \max\{A_i - h_j a_{i,j}, 0\}$ and can default any amount $D_{i,j} \in [0, z_{i,j}]$, in which case he will pay back only $z_{i,j} - D_{i,j}$ to the bookmaker.

The game under the book-bet proceeds as follows.

In period 1:

1. The bookmaker offers a punishment policy $\{q, \xi\}$.
2. Gamblers decide how much to bet $\mathbf{a}_i = (a_{i,1}, ..., a_{i,m})$, by forming expectations about the odds $\{h_j\}_{j \in J}$ and taking them as exogenously given.
3. Given amounts $\{\mathbf{a}_i\}_{i \in I}$ bet by gamblers, the equilibrium odds $\{h_j\}_{j \in J}$ are determined by condition (1).
4. The wining horse is determined.
5. Gamblers decide how much to default on paybacks. When gambler i defaults an amount of $D_{i,j} \in [0, z_{i,j}]$, he will be excluded from the second-period gambling with probability $q(D_{i,j}, z_{i,j}, A_i) \in [0, 1]$ as well as he will be imposed an utility-based penalty $\xi(D_{i,j}, z_{i,j}, A_i) \in [0, H]$.

In period 2:

1. Those who were not excluded in the end of period 1 decide how much to bet under the cash-bet, by forming expectations about the odds.

2. The odds are determined by condition (1), the winning horse is determined, and the bookmaker makes payouts to gamblers.

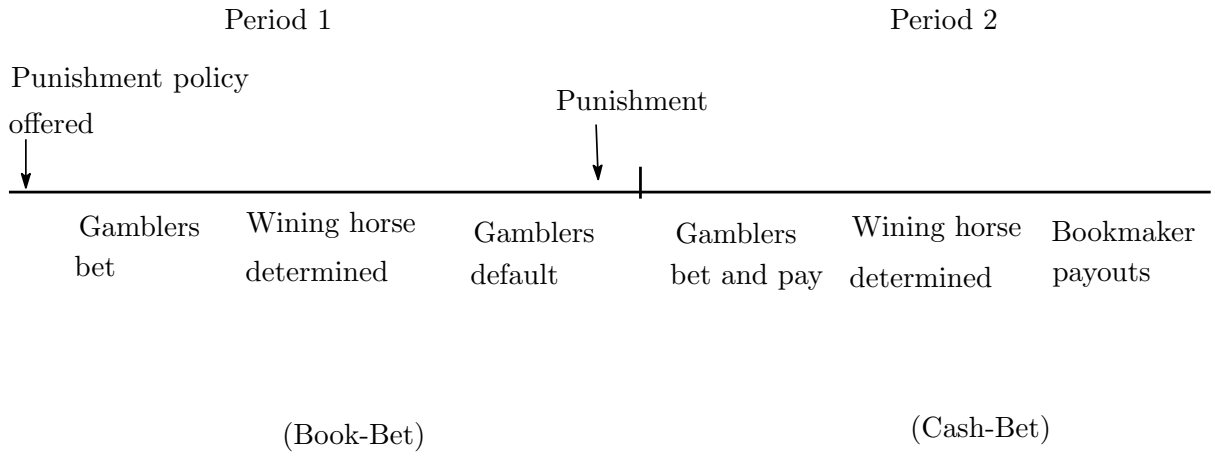


Figure 2: Timing of the game under book-bet

We proceed to provide equilibrium characterization as follows. First, we provide the equilibrium outcome in period 2, in which the bookmaker holds the cash-bet. Given this, we then show that the optimal punishment policy for the bookmaker under the book-bet in period 1 becomes the bang-bang form and has the feature of "free-first-dose" strategy such that gamblers are allowed to default but are never punished in equilibrium.

3 Gambling Equilibrium in Period 2

We begin with gambling equilibrium in period 2, given the addiction stock of gamblers $(S_1, ..., S_n)$ that has been already determined by the amounts they bet in period 1, $(A_{1,1}, ..., A_{n,1})$, where $S_i = \alpha + \beta A_{i,1}$.

Suppose that there are $I_2 \subseteq I$ gamblers who are allowed to participate in the gambling in period 2, according to a exclusion probability q specified in the punishment policy. Although it may be the case that $I_2 \neq I$ in general, we will simplify the following argument by setting $I_2 = I$; all gamblers participate in period 2. As we mentioned, gambling is always held as the cash-bet in period 2 regardless of which the cash-bet or the book-bet was used in period 1. Under the cash-bet gamblers cannot default. To save notation, we drop time index $t = 2$ from subscripts of all the variables in this section.

Gambler i's payoff in period 2 is given by

$$u(A_i, S_i) + c_{i,j} \tag{4}$$

where $c_{i,j}$ is his private consumption of the numèraire good when horse j wins the race, and it equals to the net return $R_{i,j}$ he earns, given by $c_{i,j} = R_{i,j} \equiv h_j a_{i,j} - A_i$.[16] Then, gambler i places bet of $\mathbf{a}_i$ so as to maximize his expected payoff

$$u\left(A_i, S_i\right) + \sum_{j \in J} p_j \left\{h_j a_{i,j} - A_i\right\} \tag{5}$$

given expectations about the odds $\{h_j\}_{j \in J}$.

We denote by $\hat{a}_{i,j} > 0$ the equilibrium amount bet by gambler i on horse j, and by $\hat{A}_i \equiv \sum_{j \in J} \hat{a}_{i,j}$ the aggregate amount bet by gambler i. Let also denote by $\hat{A} \equiv \sum_{i \in I} \hat{A}_i$ the aggregate amount bet by all gamblers and by $\hat{a}_j \equiv \sum_{i \in I} \hat{a}_{i,j}$ the aggregate amount bet by all gamblers on horse $j \in J$.

Then, according to (1), equilibrium odds $\{\hat{h}_j\}_{j \in J}$ are determined as follows[17]

$$\hat{h}_j = \frac{(1-f)\hat{A}}{\hat{a}_j}, \quad \text{for } j \in J. \tag{6}$$

We then obtain the following proposition (we relegate all proofs to the Appendix.)

Proposition 1. *Given a collection of addiction stocks of gamblers $(S_1, ..., S_n)$, gambling equilibrium in period 2 is characterized as follows:*

- *Gambler i who has addiction stock S_i bets the amount $\hat{A}_i = \hat{A}(S_i)$ in total satisfying*

$$u_A(\hat{A}_i, S_i) - f = 0. \tag{7}$$

- *Gambler i who has addiction stock S_i obtains the equilibrium payoff given as follows:*

$$\hat{U}(S_i) \equiv u(\hat{A}(S_i), S_i) - f\hat{A}(S_i). \tag{8}$$

Proposition 1 states that gambler i's equilibrium aggregate bet $\hat{A}(S_i)$ is determined by the first-order optimality condition (7), which depends on his addiction stock S_i. When the gambler's marginal utility of betting u_A increases as he is addicted more (that is, $u_{AS} \equiv \partial u^2 / \partial A \partial S$), he will bet more aggressively as he is addicted more; $\hat{A}$ is increasing in S_i. This result implies that the bookmaker has the incentive to let gamblers bet more in period 1, and hence make them addicted more, inducing larger amounts bet in period 2. Also, as we can see from (8), gambler's equilibrium payoff of $\hat{U}(S_i)$ increases with his addiction stock S_i; $\hat{U}'(S_i) = u_S(\hat{A}(S_i), S_i) > 0$. Note also that $\hat{U}(S) > 0$ holds due to $u_A(0, S) > f$

[16] We here allow $c_{i,j}$ to be negative for simplicity. We can however ensure non-negative consumption $c_{i,j} \geq 0$ by assuming that gamblers are exogenously endowed a large income $w > 0$ in each period.

[17] Note that we are considering the equilibrium in which $\hat{h}_j > 0$ for all $j \in J$, so $\hat{a}_j > 0$ for all $j \in J$.

By Proposition 1, the bookmaker obtains the payoff of $f\sum_{i\in I}\hat{A}(S_i)$ in period 2, which equals to the commission fee times the aggregate amount bet by participating gamblers.[18]

4 "Free-First-Dose" Strategy

4.1 Bang-Bang Punishment Policy

We move to period 1 and characterize the optimal punishment policy $\{q,\xi\}$ chosen by the bookmaker in period 1. Then we show that, when the book-bet is used in period 1, the bookmaker optimally adopts the 'free-first-dose' strategy in period 1 by reducing initial constraints weighing on gamblers to foster addictions. In order to save notations, we drop time index $t=1$ from the variables chosen in period 1. The equilibrium outcome in the second-period game is summarized by the aggregate amount bet $\hat{A}(S_i)$ and equilibrium payoff $\hat{U}(S_i)$ for each gambler $i\in I$, as shown by (7) and (8) in Proposition 1.

Under the book-bet, gambler i does not immediately pay the amount bet $A_i=\sum_j a_{i,j}$ to the bookmaker, but it is deferred in the end of period 1. When horse j wins the race, gambler i has the debt of $z_{i,j}\equiv\max\{0,A_i-h_ja_{i,j}\}$. Given his debt $z_{i,j}$, gambler i can default any amount of $D_{i,j}$ such that

$$0\leq D_{i,j}\leq z_{i,j}. \tag{9}$$

When gambler i places bet A in total and then defaults $D\in[0,z]$ in period 1, he will then lose the second-period equilibrium payoff $\hat{U}(S)$ with probability $q(D,z,A)\in[0,1]$ and be imposed an utility-based penalty $\xi(D,z,A)$ in the end of period 1. In total gambler i incurs the expected loss of

$$\psi(D,z,A)\equiv q(D,z,A)\hat{U}(S)+\xi(D,z,A) \tag{10}$$

by defaulting an amount $D\in[0,z]$ in period 1. Note that

$$0\leq\psi(D,z,A)\leq\hat{U}(S)+H,\quad\text{for any } A\geq 0 \text{ and any } D\in[0,z].$$

In what follows we will consider only $\psi(D,z,A)$ instead of treating q and ξ separately because what matters for each gambler is the total expected loss of default. We will thus call ψ a *punishment policy* in the following.

Given a punishment policy ψ offered by the bookmaker, we consider the resulting equilibrium in period 1 as follows: gamblers decide how much bet to place by forming expectations about the odds $\{h_j\}_{j\in J}$. Then the horse race outcome is realized and the equilibrium odds $\{h_j\}_{j\in J}$ are determined such that the gamblers' expectations are self-fulfilled according to (1). Following this, gamblers decide how much to default, and only

[18] When $I_2\neq I$, the bookmaker's payoff in period 2 is modified to $f\sum_{i\in I_2}\hat{A}(S_i)$.

those who are not excluded will participate in the gambling market in period 2; they then obtain the second-period equilibrium payoff $\hat{U}(S_i)$ as shown by (8) in Proposition 1.

When horse j wins the race and gambler i defaults $D_{i,j} \in [0, z_{i,j}]$ in the end of period 1, he will obtain the following *continuation* payoff:

$$h_j a_{i,j} - A_i + D_{i,j} - \psi(D_{i,j}, z_{i,j}, A_i) + \hat{U}(S_i). \tag{11}$$

Gambler i decides how much to default $D_{i,j} \in [0, z_{i,j}]$ for each winning horse $j \in J$, given the punishment policy ψ and realized odds $\{h_j\}_{j \in J}$. The *ex ante* payoff of gambler i in period 1 is then given by

$$v(A_i) + \sum_{j \in J} p_j \{h_j a_{i,j} - A_i + D_{i,j} - \psi(D_{i,j}, z_{i,j}, A_i))\} + \hat{U}(S_i). \tag{12}$$

Now we define the "bang-bang" punishment policy $\tilde{\psi}$: each gambler is imposed the lowest penalty $\underline{\psi}(z, A)$ unless he defaults more than a cutoff value $\tilde{D}(z, A)$ while he is imposed the maximum penalty $\hat{U}(S) + H$ otherwise. That is, it is defined as follows:

$$\tilde{\psi}(D, z, A) \equiv \begin{cases} \underline{\psi}(z, A) & \text{if } D \leq \tilde{D}(z, A) \\ \\ \hat{U}(S) + H & \text{otherwise} \end{cases} \tag{13}$$

We call the above cut-off value $\tilde{D}(z, A)$ the *maximum forgiveness*.

Suppose that a punishment policy ψ implements the gambling equilibrium outcome $\{\mathbf{a}^*, \mathbf{h}^*\}$ under the book-bet in period 1 such that gamblers bet $\mathbf{a}^* \equiv \{\mathbf{a}_i^*\}_{i \in I}$ and the associated odds $\mathbf{h}^* \equiv \{h_j^*\}_{j \in J}$ are determined in order to satisfy condition (1) in period 1: $h_j^* = (1-f) \sum_i A_i^* / \sum_i a_{i,j}^*$ for each $j \in J$. Following this, gambler i places the equilibrium bet $\hat{A}(S_i^*)$ in total in period 2 where $S_i^* \equiv \alpha + \beta A_i^*$ is the addiction stock of gambler i in the equilibrium (Proposition 1). Let denote by $\hat{\mathbf{A}} \equiv (\hat{A}_1(S_1^*), ..., \hat{A}_n(S_n^*))$ the profile of these equilibrium bets in period 2. The entire equilibrium outcome in the two-period game is therefore given by the collection $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$. Let denote by $z_{i,j}^* \equiv \max\{0, A_i^* - h_j^* a_{i,j}^*\}$ the equilibrium debt gambler i owes when horse j wins the race.

Then we show that the bookmaker can replace any punishment policy ψ by the bang-bang punishment policy $\tilde{\psi}$ defined as (13) such that she can implement the same equilibrium outcome $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$ as that attained under the original policy ψ, and can be weakly better off by the new policy $\tilde{\psi}$.

Proposition 2. *Suppose that the bookmaker adopts the book-bet in period 1 and that a punishment policy ψ implements an equilibrium outcome $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$. Then there always exists a bang-bang punishment policy $\tilde{\psi}$, defined as (13), such that the bookmaker can implement the same equilibrium outcome $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$, and can be weakly better off from the*

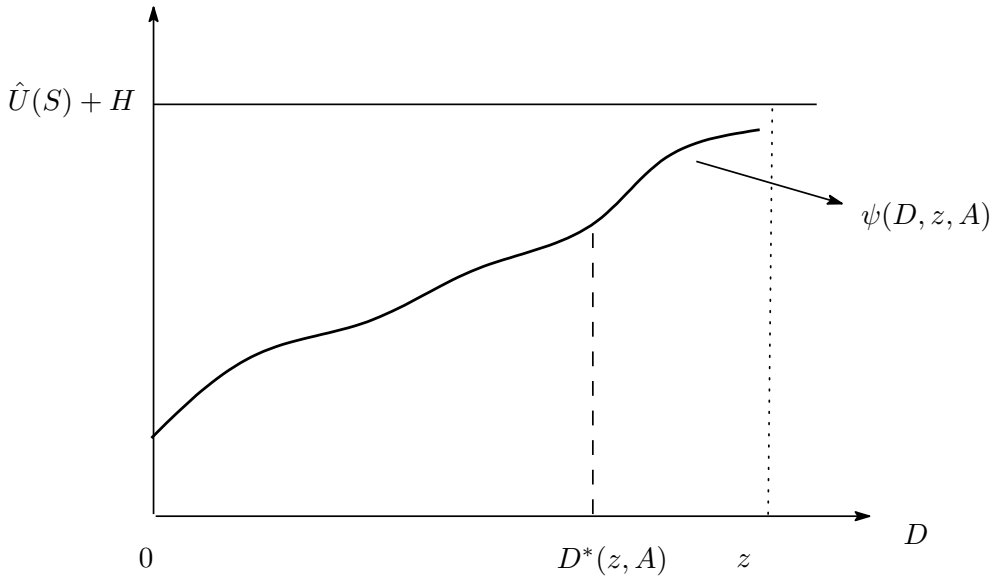


Figure 3: Punishment policy

original punishment policy ψ.

Proposition 2 states that we can confine our attention only to the bang-bang punishment policy, defined as (13), without loss of generality.

To see the intuition behind Proposition 2, consider any punishment policy ψ (as given in Figure 3) and suppose that there exists an equilibrium $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$ implemented by ψ. Thus in any subgame given any $A_i \geq 0$ and any $z_{i,j} \geq 0$, gambler i optimally chooses the default level $D_{i,j} \in [0, z_{i,j}]$, denoted by $D^*(z_{i,j}, A_i) \in [0, z_{i,j}]$.[19] Gambler i is then imposed a penalty $\psi(D^*(z_{i,j}, A_i), z_{i,j}, A_i)$, and chooses the optimal default $D^*(z_{i,j}, A_i)$ so as to maximize his continuation payoff:

$$\max_{D} \; D - \psi(D, z_{i,j}, A_i)$$

subject to $0 \leq D \leq z_{i,j}$.

Then we define the bang-bang punishment policy $\tilde{\psi}$ as follows: First, we define the lowest possible penalty as

$$\underline{\psi}(z, A) \equiv \inf_{0 \leq D \leq z} \psi(D, z, A).$$

Next we set the maximum forgiveness equal to

$$\tilde{D}(z_{i,j}, A_i) \equiv D^*(z_{i,j}, A_i) - \{\psi(D, z, A) - \underline{\psi}(z, A)\} \tag{14}$$

[19] Because we are supposing the existence of the equilibrium $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$, the optimal default choice must exist in any subgame, ensuring that $D^*(z_{i,j}, A_i)$ exists for any $A_i \geq 0$ and any $z_{i,j} \geq 0$.

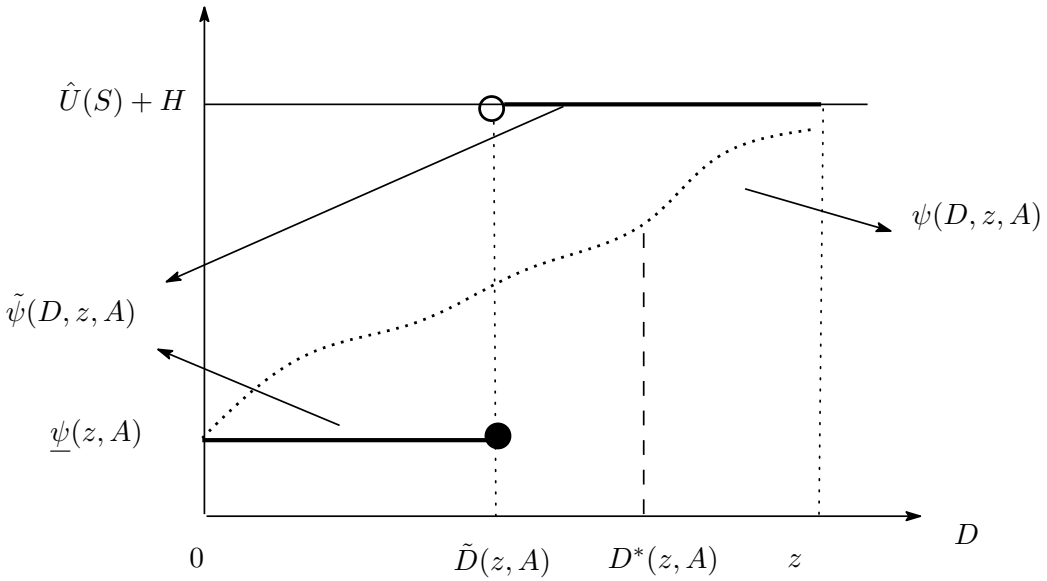


Figure 4: Bang-bang punishment policy

where we can show that $\tilde{D}(z,A) \geq 0$ (Lemma A1 in the Appendix) and $\tilde{D}(z,A) \leq z$ for all $A \geq 0$ and all $z \geq 0$. Then gambler i is punished according to the bang-bang punishment policy $\tilde{\psi}$ defined as (13) for the maximum forgiveness $\tilde{D}(z_{i,j}, A_i)$ defined as (14) (see Figure 4).

This newly defined policy can induce gamblers to choose the same amounts bet as those in the equilibrium under the original punishment policy ψ and induce them to default the amount equal to $\tilde{D}(z_{i,j}, A_i)$ for each wining horse $j \in J$. Note first that under the newly defined punishment policy, gambler i never defaults less than the maximum forgiveness, $D < \tilde{D}(z_{i,j}, A_i)$. This is because gambler i is never punished as long as $D < \tilde{D}(z_{i,j}, A_i)$, but then he can slightly increase the amount to default without being imposed any punishment. Second, we show that gambler i never defaults more than the maximum forgiveness, $D > \tilde{D}(z_{i,j}, A_i)$. If this is the case, gambler i obtains the ex post payoff equal to $D - \{\hat{U}(S) + H\} + \hat{U}(S) = D - H$ when horse j wins the race in period 1 because he will be most harshly punished. However, due to $\hat{U}(S_i) \geq 0$ and the definition of $\tilde{D}(z_{i,j}, A_i)$, we can verify that

$$\begin{aligned} D - H &\leq D - \psi(D, z_{i,j}, A_i) + \hat{U}(S_i) \\ &\leq \max_{D \in [0, z_{i,j}]} D - \psi(D, z_{i,j}, A_i) + \hat{U}(S_i) \\ &= D^*(z_{i,j}, A_i) - \psi(D^*(z_{i,j}, A_i), z_{i,j}, A_i) + \hat{U}(S_i) \\ &= \tilde{D}(z_{i,j}, A_i) - \underline{\psi}(z_{i,j}, A_i) + \hat{U}(S_i) \end{aligned}$$

so that gambler i cannot be better off by deviating from the default of $\tilde{D}(z_{i,j}, A_i)$, in which case he is least harshly punished and obtains the ex post payoff of $\tilde{D}(z_{i,j}, A_i) - \underline{\psi}(z_{i,j}, A_i) +$

$\hat{U}(S_i)$. Therefore, gambler i's optimal choice is to default $D_{i,j} = \tilde{D}(z_{i,j}, A_i)$ when horse j wins the race.

Given the above result, gambler i places bets $\mathbf{a}_i$ so as to maximize the following expected payoff in period 1:

$$v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + \tilde{D}(z_{i,j}, A_i) - \underline{\psi}(z_{i,j}, A_i) + \hat{U}(S_i)\}. \tag{15}$$

Due to the definition of $\tilde{D}(\cdot, \cdot)$ (see (14)), the above payoff (15) is equivalent to

$$v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + D^*(z_{i,j}, A_i) - \psi(D_j^*(z_{i,j}, A_i), z_{i,j}, A_i)) + \hat{U}(S_i)\},$$

which is same as what gambler i would obtain under the original punishment policy ψ and the same odds $\mathbf{h}^*$ as those in the original equilibrium. Therefore, in period 1 gamblers bet the same amounts $\mathbf{a}^*$ as those in the original equilibrium with the same equilibrium odds $\mathbf{h}^*$. This also leads to the same equilibrium amounts bet $\hat{\mathbf{A}}$ as those in the original equilibrium in period 2. Therefore, without loss of generality we can confine our attention to the bang-bang punishment policy $\tilde{\psi}$ defined as (13).

Additionally, we associate the lowest possible probability to exclude defaulting gamblers, defined as $\underline{q}(z_{i,j}^*, A_i^*) \equiv \inf_{0 \leq D \leq} q(D, z_{i,j}^*, A_i^*)$, with the lowest equilibrium penalty $\underline{\psi}(z_{i,j}^*, A_i^*)$. This can benefit the bookmaker because more gamblers can participate in the second-period gambling than those under the original punishment policy ψ.

The bookmaker hence obtains the following payoff by the bang-bang punishment $\tilde{\psi}$ modified in the above way:

$$\Pi(\tilde{\psi}) \equiv \sum_{i \in I} \sum_{j \in J} p_j \{A_i^* - h_j^* a_{i,j}^* - \tilde{D}(z_{i,j}^*, A_i^*) + (1 - \underline{q}(z_{i,j}^*, A_i^*)) f \hat{A}(S_i^*)\}.$$

The bookmaker's payoff under the original punishment policy ψ is given by

$$\Pi(\psi) \equiv \sum_{i \in I} \sum_{j \in J} p_j \{A_i^* - h_j^* a_{i,j}^* - D(z_{i,j}^*, A_i^*) + (1 - q(D(z_{i,j}^*, A_i^*), A_i^*)) f \hat{A}(S_i^*)\}.$$

We then obtain $\Pi(\tilde{\psi}) \geq \Pi(\psi)$ due to $\tilde{D}(z_{i,j}^*, A_i^*) \leq D^*(z_{i,j}^*, A_i^*)$ and $\underline{q}(z_{i,j}^*, A_i^*) \leq q(D^*(z_{i,j}^*), z_{i,j}^*, A_i^*)$, thereby showing that the bookmaker can be weakly better off by the bang-bang punishment policy $\tilde{\psi}$.

The bang-bang punishment policy $\tilde{\psi}$ can bring two gains to the bookmaker. First, gamblers reduce the amount to default from $D^*(z_{i,j}, A_i)$ to $\tilde{D}(z_{i,j}, A_i)$. This can benefit the bookmaker. Second, more gamblers can participate in the gambling in period 2 so that the market size in period 2 does not shrink under the bang-bang punishment policy. These two positive effects make the bookmaker never worse off by the bang-bang punishment policy.

By using Proposition 2, we show that the bookmaker can maximize her payoff by allowing gamblers to default without any punishment.

Proposition 3. *Suppose that the book-bet is used in period 1 and a bang-bang punishment policy* $\tilde{\psi}$ *maximizes the bookmaker's payoff* $\Pi(\tilde{\psi})$ *with its associated equilibrium* $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$. *Then we obtain the following:*

(i) *Each gambler* i *defaults the amount equal to the maximum forgiveness* $\tilde{D}(z^*_{i,j}, A^*_i) \geq 0$ *and is never punished, that is,* $\underline{\psi}(z^*_{i,j}, A^*_i) = 0$, *whenever the default* $\tilde{D}(z^*_{i,j}, A^*_i)$ *is strictly positive on the equilibrium path.*

(ii) *Without loss of generality we can set the lowest expected penalty among gamblers, defined as* $\min_{i\in I} \sum_{j\in J} p_j \underline{\psi}(z^*_{i,j}, A^*_i)$, *to be to zero.*

The key prediction of Proposition 3 (i) is that, whenever gamblers default strictly *positive* amounts in equilibrium, they are never punished at all: equilibrium punishment becomes zero $\underline{\psi}(z^*_{i,j}, A^*_i) = 0$ on the path of play. This implies that defaulting gamblers are never excluded from participating in the second-period gambling and are imposed no utility-based penalties such as violence even for defaulting on paybacks. Therefore, the bookmaker optimally adopts the "free-first-dose" strategy such that she induces gamblers to place large bets initially by allowing them to default without punishment and then fosters addictions, resulting in large bets in the later period. Such "allowed default" plays a strategic role to compensate individuals for being addicted to gambling over time. This tactic is often adopted by sellers of addicted goods such that individuals get goods for free initially and increase the willingness to pay for the goods later after they are addicted.

Proposition 3 (i) can help explain seemingly paradoxical evidence about illegal markets that sellers do not often penalize buyers even when they default on payments. For example, Mehmood and Chen (2022) collected the date about illegal horse race betting in Pakistan, and showed that on average 35% of gamblers in their sample do not repay their debts in full. Mehmood and Chen (2022) also found that only 0.5% of surveyed gamblers reported any apprehension of encountering violence in case of non-repayment. These facts look surprising at a first glance: a significant fraction of gamblers defaults on debt repayments but few of them are penalized. Proposition 3 provides a theoretical insight to explain these paradoxical observations. Furthermore, despite the high frequency of gamblers' defaults, race clubs (bookmakers) allow gamblers to place bets by credit rather than cash. Mehmood and Chen (2022) reported that 55% of wagers in their sample are placed on credit, as opposed to cash-in-advance bets. We will provide a rationale for this phenomenon later (see Proposition 5 below).

Proposition 3 (ii) shows that we can confine our attention only to the bang-bang punishment policy $\tilde{\psi}$ such that some gambler, say $i \in I$, is imposed no penalties on the equilibrium path regardless of whether he defaults a strictly positive amount. This is because $\min_{i\in I} \sum_{j\in J} p_j \underline{\psi}(z^*_{i,j}, A^*_i) = 0$ and $\underline{\psi}(z^*_{i,j}, A^*_i) \geq 0$ together imply that $\underline{\psi}(z^*_{i,j}, A^*_i) = 0$ is satisfied for any $j \in J$ for such gambler i.

4.2 Optimal Punishment Policy under the Book-bet

We now turn to derive the optimal punishment policy ψ chosen by the bookmaker, provided that she uses the book-bet in period 1. Thanks to Proposition 2, without loss of generality we can focus only on the bang-bang punishment policy $\tilde{\psi}$ defined as (13), when the bookmaker uses the book-bet in period 1. In particular, since all gamblers are identical in terms of their payoff functions, we pay our attentions to the symmetric equilibrium in which all gamblers place bets and default according to the same strategy: $a^*_{i,j,1} = a^*_j$ and $A^*_{i,1} = A^*$ for all $i \in I$ in period 1 and $A_{i,2} = \hat{A}(S^*)$ for all $i \in I$ in period 2 where $S_i = S^* \equiv \alpha + \beta A^*$ for all $i \in I$. We denote by $\mathbf{a}^* = (a^*_1, ..., a^*_m)$ an equilibrium profile of amounts bet by each gambler in period 1, where a^*_j is the amount bet by each gambler on horse j, and $A^* = \sum_{j \in J} a^*_j$ is the amount bet by each gambler in total. Thus we obtain the equilibrium odds as $h^*_j = (1-f)A^*/a^*_j$ for $j \in J$.

We now consider the bang-bang punishment policy $\tilde{\psi}$ that maximizes the bookmaker's payoff $\Pi(\tilde{\psi})$, which we can simplify to

$$\Pi(\tilde{\psi}) \equiv n \left\{ A^* - \sum_{j \in J} p_j \{ h^*_j a^*_j + \tilde{D}(z^*_j, A^*) \} \right\} + nf\hat{A}(S^*) \tag{16}$$

in the symmetric equilibrium where

$$z^*_j \equiv \max\{A^* - h^*_j a^*_j, 0\} = fA^*$$

is the equilibrium debt that each gambler should pay back to the bookmaker conditional on horse j wining the race. The first term of (16) is the bookmaker's payoffs in period 1: she receives the aggregate amount bet A^* from each gambler and gives him the expected payout equal to $\sum_j p_j h^*_j a^*_j$ and loses the defaulted amount $\tilde{D}(z^*_j, A^*)$ conditional on horse $j \in J$ wining the race. The second term of (16) is the bookmaker's payoffs from holding the cash-bet in period 2, where each gambler bets $A_2 = \hat{A}(S^*)$ in total in period 2 according to Proposition 1. We call a punishment policy $\tilde{\psi}$ *optimal* when it induces the symmetric equilibrium outcome $\{\mathbf{a}^*, \mathbf{h}^*, \hat{A}(S)\}$ maximizing the bookmaker's payoff $\Pi(\tilde{\psi})$.

We say that an aggregate amount bet A^* by each gambler is *implementable* when there exists an equilibrium in which each gambler chooses A^* in total in period 1, defaults the amount equal to the maximum forgiveness $\tilde{D}(z^*_j, A^*)$, and is imposed the lowest penalty $\underline{\psi}(z^*, A^*)$, in association with the corresponding equilibrium odds $\mathbf{h}^* = \{h^*_j\}_{j \in J}$ that satisfies (1) and is consistent with equilibrium amount bet $\mathbf{a}^*$ in period 1. Thanks to Proposition 3 (ii) we can set the lowest penalty $\underline{\psi}(z^*, A^*)$ to be zero without loss of generality in the symmetric equilibrium maximizing the bookmaker's payoff $\Pi(\tilde{\psi})$.

For an aggregate amount bet A^* to be implementable, it must satisfy a certain set of constraints by taking into account the default incentive of gamblers.

First, the equilibrium payoff of each gambler is given by

$$\begin{aligned} V^* &\equiv v(A^*) - fA^* + \tilde{D}(z^*, A^*) - \underline{\psi}(z^*, A^*) + \hat{U}(S^*) \\ &= v(A^*) - fA^* + \tilde{D}(z^*, A^*) + \hat{U}(S^*) \end{aligned}$$

where the lowest penalty $\underline{\psi}(z^*, A^*)$ is set to be zero as mentioned above. For each gambler not to deviate from the equilibrium bet A^*, the following *incentive compatibility* (IC) constraint must be satisfied:

$$V^* \geq \max_{\mathbf{a}_i, \{D_{i,j}\}_{j \in J}} v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_i)\} + \hat{U}(S_i). \quad (17)$$

That is, the gambler's equilibrium payoff V^* must be at least as large as the deviation payoff obtained by placing different bets $\mathbf{a}_i \neq \mathbf{a}^*$ from the equilibrium one and making optimal defaults subsequently, yielding the payoff equal to the right hand side of (17).

Second, determining the amount a_j^* bet by each gambler on horse $j \in J$ is equivalent to determining its odds h_j^* given A^*, due to the definition of equilibrium odds $h_j^* = (1 - f)A^*/a_j^*$. Then since $A^* = \sum_j a_j^*$ holds, we have the following identity:

$$\sum_{j \in J} (1/h_j^*) = 1/(1 - f). \quad (18)$$

Third, each gambler cannot default more than his debt $z \geq 0$, that is, the maximum forgiveness $\tilde{D}$ must satisfy

$$0 \leq \tilde{D}(z, A) \leq z \quad (19)$$

for all $z \geq 0$ and all $A \geq 0$.

Finally, available penalties are constrained by the upper bound:

$$0 \leq \tilde{\psi}(D, z, A) \leq \hat{U}(S) + H \quad (20)$$

for all $D \in [0, z]$, all $z \geq 0$ and all $A \geq 0$.

We consider the *best* equilibrium that maximizes the bookmaker's payoffs over two periods

$$fA^* - \tilde{D}(z^*, A^*) + f\hat{A}(S^*)$$

subject to all the constraints obtained above, that is, it is determined by the solution to the following problem:

Problem B:

$$\max_{\mathbf{h}^*, A^*, \tilde{\psi}, \tilde{D}} \; fA^* - \tilde{D}(z^*, A^*) + f\hat{A}(S^*)$$

subject to (17)-(20).

One remark is here that the odds $\mathbf{h}^*$ in period 1 are determined in the gambling market equilibrium as defined in Section 2.4. Therefore the bookmaker cannot directly choose the market odds themselves. By the best equilibrium, we focus on the gambling market equilibrium such that the odds $\mathbf{h}^*$ are consistent with equilibrium condition (1) and maximize the bookmaker's payoffs.

We denote by Π^b the maximum value of Problem B, which gives the bookmaker the largest payoff attained under the book-bet.

5 Equilibrium Characterization under the Book-Bet

In this section we characterize the best equilibrium for the bookmaker under the book-bet, which solves Problem B given above. The fundamental difficulty for this purpose is that the IC constraints given by (17) are so complicated that we need to handle many possible deviations of gamblers. First, we address this issue and show the condition under which we can confine our attention only to a single IC constraint without loss of generality, thereby reducing the number of possible deviations by gamblers. Second, given this result, we show that the best equilibrium for the bookmaker under the book-bet exhibits the following interesting features: (i) equilibrium odds have so called *long-shot bias* such that gamblers place more bets on the long-shots which are less likely to win the race, and (ii) equilibrium amounts bet are "socially excessive" in the sense that these are larger than the social optimal bets that maximize the social welfare of the economy.

5.1 Equilibrium Odds: Long-Shot Bias

First, we identify the lower bound payoff of each gambler in any symmetric equilibrium under the book-bet. Suppose that gambler i places bet of $\mathbf{a}_i$ in period 1 and then defaults all the debt $z_{i,j} \equiv \max\{A_i - h_j^* a_{i,j}, 0\}$, that is, $D_{i,j} = z_{i,j}$ for all $j \in J$, followed by being punished most harshly $\tilde{\psi}(D_{i,j}, z_{i,j}, A_i) = \hat{U}(S_i) + H$ no matter what horse race outcome is realized in period 1. This deviation gives gambler i the payoff of

$$\begin{aligned}\overline{V} &\equiv \max_{\mathbf{a}_i}\ v(A_i) + \sum_{j\in J} p_j \max\{h_j^* a_{i,j} - A_i, 0\} - \{\hat{U}(S_i) + H\} + \hat{U}(S_i) \\ &= \max_{\mathbf{a}_i}\ v(A_i) + \sum_{j\in J} p_j \max\{h_j^* a_{i,j} - A_i, 0\} - H.\end{aligned}$$

This is a lower bound for gambler i's payoff. Furthermore, in order to achieve the payoff of $\overline{V}$, gambler i's optimal deviation is to bet all the amount of A_i to the horse that maximizes the expected net return $p_j \max\{h_j^* - 1, 0\}$, provided that he defaults all debts when such horse j wins the race but $h_j^* < 1$ or loses the race, in which either case he pays nothing.

Therefore, gambler i's deviation payoff $\overline{V}$ can be re-written as follows

$$\overline{V} = \max_{A} v(A) + \max_{j \in J} \max\{p_j(h_j^* - 1), 0\}A - H. \tag{21}$$

Then, for $\mathbf{a}^*$ to be implementable, the following IC must be satisfied:

$$V^* \equiv v(A^*) - fA^* + \tilde{D}(z^*, A^*) + \hat{U}(S^*) \geq \overline{V} \tag{IC*}$$

Note that incentive constraint (17) implies IC* because each gambler can always default all the debt whatever horse race outcome is in period 1, followed by being most harshly punished $\tilde{\psi}(D_{i,j}, z_{i,j}, A_i) = \hat{U}(S_i) + H$. Therefore, IC* is a necessary condition for (17) to hold. We can also show its converse under Assumption 1 given below and greatly simplify the characterization of optimal punishment policy.

Without loss of generality, suppose that $p_1 \leq \cdots \leq p_m$. We make the assumption that horses are not so heterogeneous with respect to the probabilities to win the race.

Assumption 1. $(1 - p_1)(p_m - p_1) \leq p_1$.

When the winning probabilities of horses are close to each other, $p_1 - p_m$ can be small so that Assumption 1 is satisfied. Then we show the following result.

Proposition 4. *Suppose that Assumption 1 is satisfied. Then the best equilibrium for the bookmaker under the book-bet is characterized as follows: (i) the amount bet A^* per gambler in total in period 1 is given by a solution to the following problem:*

Problem B*:

$$\max_{A^* \geq 0, \tilde{D}^*} \Phi(A^*) \equiv fA^* - \tilde{D}^* + f\hat{A}(S^*)$$

subject to IC* *and* $0 \leq \tilde{D}^* \leq fA^*$, *where* $\tilde{D}^* \equiv \tilde{D}(z^*, A^*)$.

(ii) Furthermore, equilibrium odds h_j^* are given by $h_j^* = k^*/p_j + 1$ for each $j \in J$, where $k^* \in (p_1, \infty)$ is a unique solution to

$$\sum_{j \in J} \frac{p_j}{p_j + k^*} = \frac{1}{1 - f}. \tag{22}$$

We provide an intuitive explanation for Proposition 3 below.

Suppose that the bookmaker wants to implement a profile of amounts bet $\mathbf{a}^* = (a_j^*)_{j \in J}$ with its total amount $A^* \equiv \sum_j a_j^*$ from each gambler in period 1 in a symmetric equilibrium

with market odds $\mathbf{h}^* = (h_j^*)_{j\in J}$. We show that under Assumption 1 IC* constraint is sufficient to detect all possible deviations as follows.

First, gambler i places a deviation bet $a_{i,j} \neq a_j^*$ with its total amount $A_i \equiv \sum_{j\in J} a_{i,j} \neq A^*$ in period 1. Due to IC*, the bookmaker can easily detect this deviation by imposing the most severe penalty $\underline{\psi}(z, A) = \hat{U}(S) + H$ for any deviation bet $A \neq A^*$. Next, gambler i follows the equilibrium amount bet A^* in total but changes its allocation among horses by placing the equilibrium bets $a_{i,j} = a_j^*$ for some subset of horses $\tilde{J} \subseteq J$ while placing different bets $a_{i,j} \neq a_j^*$ for others $j \notin \tilde{J}$. Therefore, gambler i allocates the amount $A^* - \sum_{j\in\tilde{J}} a_j^*$ among horses belonging to $J \setminus \tilde{J}$ and defaults the debts $z_{i,j}$ for those horses. However, under Assumption 1 horses are not so heterogeneous with respect to their wining probabilities such that p_j and p_ℓ do not differ much for any $j \neq \ell$. Therefore the equilibrium expected gross returns of betting one dollar $p_j h_j^* = k^* + p_j$ do not differ much among different horses as well. Given this, gamblers can gain little from allocating the amount $A^* - \sum_{j\in\tilde{J}} a_j^*$ among the horses in $J \setminus \tilde{J}$ which do not differ much in their expected gross returns. Instead of doing this, the gambler can be better off by allocating all the amount A^* to a single horse and default all the debts for all other horses, yielding at most the deviation payoff $\overline{V}$ to him. However, this is not profitable due to IC* again.

Next we see how equilibrium odds are determined in order to minimize the gambler's deviation payoff $\overline{V}$. By setting equilibrium odds as $h_j^* = k^*/p_j + 1$ defined in Proposition 3, we can minimize the deviation payoff of each gambler $\overline{V}$. To see this, note that, when a gambler bets one dollar on horse j, he will receive the payout $h_j^* - 1$ whenever it is positive and horse j wins the race while he can default paybacks in all other cases. Therefore, the expected net return of betting one dollar on horse j is given by $\max\{p_j(h_j^* - 1), 0\}$. To minimize the deviation payoff $\overline{V}$, market odds $\{h_j^*\}_{j\in J}$ should be chosen to minimize the largest expected return, $\max_{j\in J} p_j(h_j^* - 1)$, implying that $p_j(h_j^* - 1) = k$ for all $j \in J$ for some constant k. To show this, suppose that $p_\ell(h_\ell^* - 1) = \max_i p_i(h_i^* - 1) > p_j(h_j^* - 1)$ for some $\ell \neq j$. Then we can slightly reduce $h_\ell^* > 0$ while increasing $h_j^* > 0$ slightly, making the feasibility condition $\sum_j (1/h_j^*) = 1/(1-f)$ unchanged. This reduces the gambler's deviation payoff $\overline{V}$. Therefore, the expected returns taking account into the default incentive should be equalized across horses, implying that $p_j(h_j^* - 1) = k$ for all $j \in J$. Then, by using (18) and (22), we can determine the value of such k, given by k^* in Proposition 3, and can re-write the gambler's deviation payoff $\overline{V}$ as follows

$$\overline{V} = \max_{A\geq 0} v(A) + \max\{k^*, 0\}A - H.$$

The equilibrium odds determined in the above way $h_j^* = k^*/p_j + 1$ exhibit the feature of so called *long-shot bias*: gamblers place more bets on the horses which are less likely to win the race. This bias has been observed in many gambling markets and has been known as an empirical puzzle in the literature because such bias is not consistent with the standard theory of expected utility maximization (Chiappori et al. (2019),Ottaviani and

Sørensen (2008)). To understand why the long-shot bias exists in our equilibrium, note that equilibrium odds $h_j^* = k^*/p_j + 1$ imply that

$$p_1 h_1^* \leq p_1 h_2^* \leq \cdots \leq p_m h_m^* \tag{23}$$

because p_j is increasing in j. This means that the expected gross returns $p_j h_j^*$ to bet one dollar are higher for horses having higher wining probabilities. Define the fraction of amounts bet on horse j as

$$r_j \equiv \sum_{i \in I} a_{i,j} / \sum_{i \in I} \sum_{j \in J} a_{i,j}$$

measuring how much fraction of bets horse j attracts from gamblers. In the equilibrium we have $r_j = r_j^* \equiv a_j^*/A^*$ so that $h_j^* = (1-f)/r_j^*$ holds by the definition of the odds. Then, since $\sum_j r_j = 1$ and $\sum_j p_j = 1$ hold by the definition, we verify from (23) that there must exist a unique j^* such that

$$p_1/r_1^* \leq \cdots \leq p_{j^*}/r_{j^*}^* \leq 1 \leq \cdots \leq p_m/r_m^*.$$

Thus, for $j < j^*$ we have $p_j < r_j^*$ while, for $j > j^*$ we have $p_j > r_j^*$. Therefore, gamblers place more bets (with larger r_j) on the horses (with lower j) which are less likely to win the race (with lower p_j), implying that gamblers overbet on long-shots while they underbet on favorites. This result is in contrast to the situation in which gamblers place bets to maximize expected payoffs *without* default incentives. In that case the expected returns of betting one dollar must be equal across horses in equilibrium:

$$p_j h_j^* - 1 = p_\ell h_\ell^* - 1, \quad j \neq \ell$$

which yields $p_j = r_j$ for all $j \in J$, so long-shot bias never emerges. However, as shown above, if gamblers can default on paybacks under the book-bet, the expected return of betting one dollar *taking into account the default incentive* must be equal across horses:

$$p_j(h_j^* - 1) = p_\ell(h_\ell^* - 1), \quad j \neq \ell$$

in order to minimize the gambler's deviation payoff $\overline{V}$. Such default incentive of gamblers is the key to obtain the long-shot bias in our equilibrium.

The existing studies have tried to explain the long-shot bias in the view point of gamblers' risk-loving or behavioral preferences (Chiappori et al. (2019)). In contrast we provide a new rationale for the long-shot bias in the perspective of strategic defaults without these preferences of gamblers. Our result shows that the long-shot bias emerges as an equilibrium feature of illegal gambling markets where individuals may renege on debts due to the lack of formal contracts and bookmakers optimally responds to such strategic default by punishment policies. In our theory market odds with the long-shot bias can minimize the default incentive of gamblers. Proposition 3 has the important implication that the long-shot bias might be more prevalent in illegal gambling markets than legal gambling

markets. This theoretical finding is worth addressing further to match actual data about illegal horse race and sports betting, which will be left for future research.

5.2 Socially Excessive Addiction

Next we investigate welfare implications about the gambling market equilibrium obtained in Proposition 3. To this end, we define the social welfare of the economy as the sum of all players' payoffs over two periods. Since transfers made between the bookmaker and gamblers are cancelled out in total, the social welfare is equivalent to the sum of gamblers' gross payoffs over two periods as follows:

$$W(A_1, A_2) \equiv n\{v(A_1) + u(A_2, S)\}$$

where $S = \alpha + \beta A_1$, and $A_t \geq 0$ is the total amount bet per gambler in period $t = 1, 2$. We then define by (A_1^{fb}, A_2^{fb}) *the first-best* amounts bet to maximize the social welfare $W(A_1, A_2)$.

Now we consider the optimal punishment policy ψ with the equilibrium odds $\mathbf{h}^*$, and the associated equilibrium amounts bet $(A^b, \hat{A}(S^b))$ in periods 1 and 2 where A^b is the optimal solution to Problem B* given in Proposition 5 and $S^b \equiv \alpha + \beta A^b$ is the addition stock corresponding to A^b.

Regarding the addictive preference of gamblers, we make the following assumption.

Assumption 2. $\hat{U}(S) > A$ *for any* $A \geq 0$.

Assumption 2 says that gambler's addiction preference, captured by the parameter values $\alpha > 0$ and $\beta > 0$, is so strong that his second-period payoff $\hat{U}(S)$ is sufficiently large to cover any amount bet $A \geq 0$ in the first period. To see this, assume that the marginal utility of gambling u_A is increasing in the addiction stock S and its effect is so large that $u_{SA}^2 \geq u_{SS}u_{AA}$. Then, by the definition of $\hat{U}(S)$, we have $\hat{U}'(S) = u_S(\hat{A}(S), S) > 0$, and thus $\partial\hat{U}/\partial A = \beta\hat{U}'(S) > 0$ as well as $\partial\hat{U}^2/\partial^2 A = \beta^2\{u_{SA}\hat{A}'(S) + u_{SS}\} \geq 0$ because $\hat{A}'(S) = -u_{AS}/u_{AA}$. When α and β are so large that $\beta u_S(\hat{A}(\alpha), \alpha) > 1$ and $\hat{U}(\alpha) \geq 0$ are satisfied, we obtain $\hat{U}(S) > A$ for all $A \geq 0$.

We then show the following result.

Proposition 5. *Suppose that Assumption 1-2 are satisfied, and that the commission fee* $f \in (0, 1)$ *is sufficiently small. Then the equilibrium amounts bet* A^b *and* $\hat{A}(S^b)$*, which solve Problem B* of Proposition 5, are socially excessive in the sense that these are larger than the first-best amounts bet* A_1^{fb} *and* A_2^{fb} *in both periods, that is,* $A^b > A_1^{fb}$ *and* $\hat{A}(S^b) > A_2^{fb}$.

The intuition behind this result is as follows.

We define the *constrained* social welfare as the sum of all gamblers' gross payoffs over two periods, $v(A) + u(\hat{A}(S), S)$, where the amount bet in period 2 is constrained by the

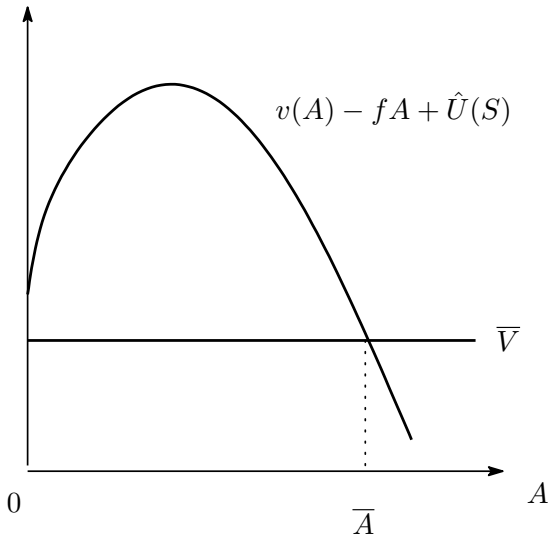


Figure 5: Equilibrium Default

gambling market equilibrium condition such that $\hat{A}(S)$ satisfies the equilibrium condition (17): $u_A(\hat{A}(S), S) = f$. We also define the *constrained* efficient amounts bet $(A^{**}, \hat{A}(S^{**}))$ that maximize the constrained social welfare $v(A) + u(\hat{A}(S), S)$ over $A \geq 0$. Then, instead of comparing the equilibrium amounts bet $(A^b, \hat{A}(S^b))$ that solve Problem B* with the first-best amounts bet (A_1^{fb}, A_2^{fb}), we will make a comparison between the equilibrium amounts bet $(A^b, \hat{A}(S^b))$ and the constrained efficient amounts bet $(A^{**}, \hat{A}(S^{**}))$. This alternative comparison makes sense when the commission fee f is sufficiently small because then the constrained efficient amounts bet can be sufficiently close to the first-best amounts bet.

Now we consider which A^b or A^{**} is larger. The key for this exercise is that the bookmaker needs to compensate gamblers when she wants them to place larger bets than some cutoff value, denoted by $\overline{A}$, such that gambler's equilibrium payoff $v(A) - fA + \hat{U}(S)$ is lower than his deviation payoff $\overline{V}$ for $A > \overline{A}$, as depicted in Figure 5.[20] Such compensation is made by allowing gamblers to default up to a certain amount without any penalty. Then there are two cases depending on whether or not the bookmaker's optimal choice is to allow gamblers to default positive amounts without penalties, in which case IC* is binding with a positive maximum forgiveness $\tilde{D}^* \equiv \tilde{D}(z^*, A^*) > 0$.

First, suppose that a positive maximum forgiveness $\tilde{D}^* > 0$ is required to satisfy IC* as equality

$$\tilde{D}^* = \overline{V} - \{v(A) - fA + \hat{U}(S)\} \geq 0$$

[20] To simply the argument, we assume that such $\overline{A}$ is unique.

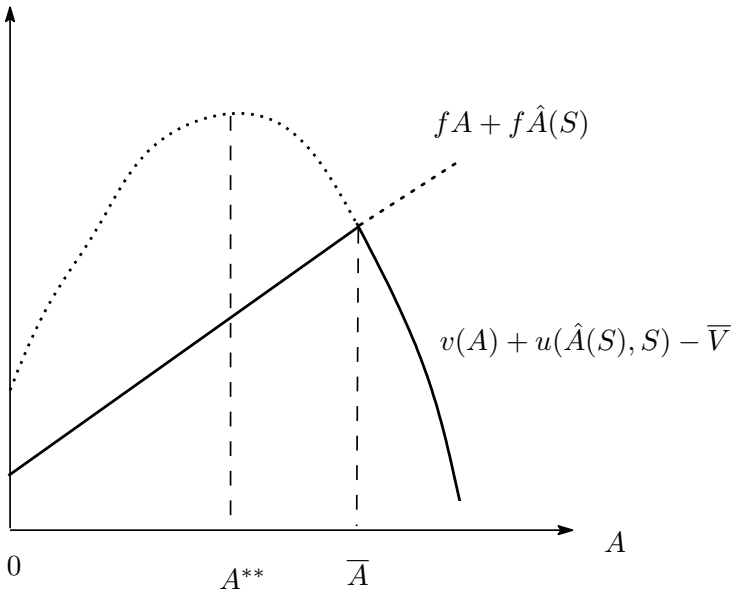


Figure 6: Equilibrium and Social Welfare

at the optimal solution to Problem B*.[21] Then the bookmaker's objective $fA - \tilde{D}^* + f\hat{A}(S)$ is aligned with the constrained social welfare $v(A) + u(\hat{A}(S), S)$:

$$fA - \tilde{D}^* + f\hat{A}(S) = v(A) + u(\hat{A}(S), S) - \overline{V}$$

up to the constant $\overline{V}$. Therefore, the bookmaker's choice of A^b coincides with the one maximizing the constrained social welfare: $A^b = A^{**}$. This is the case when the cutoff value $\overline{A}$ is smaller than the constrained efficient amount bet A^{**}, as shown in Figure 6 where the bookmaker's payoff is depicted as a function of the first period amount bet A per gambler.[22]

Next suppose that the optimal maximum forgiveness $\tilde{D}^*$ is zero at the solution to Problem B*. Therefore the bookmaker's payoff per gambler is given by $fA + f\hat{A}(S)$, which is increasing in $A \geq 0$. Thus the amount bet A in period 1 should be large as much as possible until IC* is binding:

$$\overline{V} = v(A) - fA + \hat{U}(S)$$

which yields the optimal choice of $A^b = \overline{A}$. Thus the equilibrium amount bet A^b in period 1 becomes larger than the constrained efficient amount bet A^{**}, as depicted in Figure 7

[21] To ensure the constraint $\tilde{D}^* \leq fA^{**}$, we need $v(A) + \hat{U}(S) \geq \overline{V}$, which is however satisfied at $A = A^{**}$ when f is sufficiently small.

[22] The bookmaker's payoff equals to $fA + f\hat{A}(S)$ when $A \leq \overline{A}$ so that the optimal default is zero $\tilde{D}^* = 0$, and it equals to $v(A) + u(\hat{A}(S), S) - \overline{V}$ when $A > \hat{A}$ so that the optimal default is positive, $\tilde{D}^* > 0$, respectively.

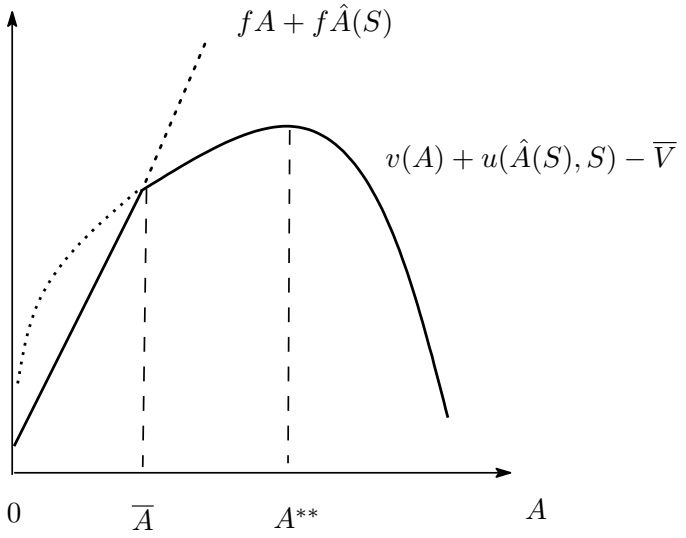


Figure 7: Socially Excessive Bet

where the bookmaker's payoff is maximized at the cutoff value $\overline{A}$ at which the optimal default is zero, $\tilde{D}^* = 0$, in Problem B*. In that case gamblers are addicted to gambling too much in the view point of constrained efficiency because they accumulate larger addiction stock $S^b = \alpha + \beta A^b$ than the constrained efficient level $S^{**} = \alpha + \beta A^{**}$.

Intuitively, when gamblers' addiction preference is so strong, IC* is slack at the constrained efficient level A^{**} when they do not default any amount, that is, $\tilde{D}^* = 0$. In other words gamblers are willing to place larger bet than the efficient level A^{**} even when they are not compensated by any positive default. In such a situation the optimal policy chosen by the bookmaker should set the maximum forgiveness to the lowest possible level, zero, $\tilde{D}^* = 0$, and increase the amount bet A in period 1 as much as possible until the gambler's equilibrium payoff $v(A) - fA + \hat{U}(S)$ is reduced down to the deviation payoff $\overline{V}$. In that way the bookmaker extracts the surplus from gamblers. This makes the equilibrium amount bet A^b in period 1 larger than the constrained efficient one A^{**}. Since the first-best amounts bet (A_1^{fb}, A_2^{fb}) are close to the constrained efficient one $(A^{**}, \hat{A}(S^{**}))$ when the commission fee $f \in (0, 1)$ is sufficiently small, by continuity the equilibrium amounts bet $(A^b, \hat{A}(S^b))$ can be larger than the first-best amounts bet (A_1^{fb}, A_2^{fb}).

6 Optimal Gambling Format: Cash-bet or Book-bet?

We next address the issue about which book-bet or cash-bet becomes optimal for the bookmaker to use in period 1. We then show that gamblers' addiction preferences play the crucial role to determine the optimal gambling format in period 1. More specifically, we show that the bookmaker prefers the book-bet to the cash-bet when gamblers are strongly addicted to gambling. On the contrary, when gamblers are not addicted to gambling at

all, the cash-bet may be optimal for the bookmaker. These results can help understand why sellers often provide credits to newcomers for purchasing addictive goods despite the default risk in illegal markets.

6.1 Equilibrium under the Cash-Bet in Period 1

Suppose that the bookmaker chooses the cash-bet in period 1 as well as in period 2. Under the cash-bet gamblers cannot default in period 1 and can then surely participate in the gambling in period 2. As a result, gambler i places bet $\mathbf{a}_i$ in period 1 to maximize the following expected payoff

$$v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i\} + \hat{U}(S_i) \tag{24}$$

given expectations about equilibrium odds $\{h_j^*\}_{j \in J}$.

In the symmetric equilibrium the market odds are determined by $h_j^* = (1-f)/p_j$ for each $j \in J$.[23]. Given this, each gambler places the bet of total amount A in period 1 to maximize the expected payoff given by (24), which we can re-write by $v(A) - fA + \hat{U}(S)$. We denote by $\tilde{A}$ the equilibrium amount bet by each gambler in total in period 1, that is, $\tilde{A}$ maximizes $v(A) - fA + \hat{U}(S)$. Also we denote by $\hat{A}(\tilde{S})$ the associated amount bet by each gambler in total in period 2 where $\tilde{S} \equiv \alpha + \beta \tilde{A}$.

Then the bookmaker obtains the following payoffs over two periods

$$\Pi_c \equiv n\{f\tilde{A} + f\hat{A}(\tilde{S})\}$$

by using the cash-bet in period 1.

6.2 Book-Bet or Cash-Bet?

We now compare the bookmaker's payoff Π^c under the cash-bet with that under the book-bet Π^b which attains the maximum value in Problem B given in Section 4. We do not impose Assumption 1 in this section.

The bookmaker is exposed to default risks when she uses the book-bet in period 1 in contrast to the cash-bet. However, according to Mehmood and Chen (2022), in the illegal horse race betting in Pakistan more than 50% of gamblers are allowed to place bets by credit rather than cash-in-advance bets. If we focus only on default costs, this observation is puzzling. Why does the bookmaker extend credit to gamblers even when they default? In the following we show the condition under which the bookmaker can gain from using the book-bet in period 1 by strategically inducing gamblers' defaults and making them addicted to gambling, resulting in large amounts bet in period 2.

[23] Again we here focus on the equilibrium in which the odds are positive for any horse.

Which book-bet or cash-bet the bookmaker prefers in period 1 depends on how strongly gamblers are addicted to gambling in the first period. We measure the degree of gamblers' addictions by the parameter values α and β that determine the addiction stock $S_i = \alpha + \beta A_i$ in period 2. When α and β are larger, gamblers are addicted more by engagements in the first period gambling. We then show that, when gamblers' addictions are so strong that α and β are sufficiently large to satisfy Assumption 2, the bookmaker prefers the book-bet to the cash-bet in period 1: the bookmaker obtains a higher payoff Π_b under the book-bet than the payoff Π_c under the cash-bet. On the contrary, in the absence of addiction preferences in the sense that $\alpha = \beta = 0$, the bookmaker may prefer the cash-bet to the book-bet under a certain condition. Therefore, gamblers' addictions cause a significant effect on the bookmaker's choice of the gambling format.

When gamblers are never addicted ($\alpha = \beta = 0$, so $S_i = 0$), we define by $u(A_{i,2}) \equiv u(A_{i,2}, 0)$ the second-period utility of gambler i who places bet of $A_{i,2}$ in total. Since the differences in payoff functions between the two periods do not matter when there is no addiction, we assume that the payoff functions of gambling are identical in both periods: $v(A) \equiv u(A)$ for all $A \geq 0$. We define by $\tilde{A}^*$ the amount bet to maximize $u(A) - fA$ over $A \geq 0$. Note that each gambler obtains the payoff of $\hat{U} \equiv \max_{A \geq 0} u(A) - fA$ in period 2 when $S = 0$. Assume also that $\overline{u} \equiv \max_{A \geq 0} u(A) < +\infty$.

We then show the following result.

Proposition 6. *(i) Suppose that gamblers are strongly addicted ($\alpha > 0$ and $\beta > 0$) such that Assumption 2 is satisfied. Then the book-bet becomes optimal for the bookmaker in period 1 ($\Pi_b \geq \Pi_c$). (ii) Suppose that gamblers are not addicted at all ($\alpha = \beta = 0$). Then, if $u(\tilde{A}^*) + H < 2f\tilde{A}^*$, it becomes optimal for the bookmaker to hold the cash-bet in period 1 ($\Pi_c > \Pi_b$).*

When gamblers are so strongly addicted that Assumption 2 is satisfied, the bookmaker can gain from allowing them to default without punishment initially and making them addicted in the later period. This induces gamblers to place large bets in the second period, thereby benefiting the bookmaker more than the default cost in the first period. This gives the first part of Proposition 5. This result helps understand why the bookmaker adopts the book-bet even when she faces default risks, in consistent with the evidence about illegal horse race betting in Pakistan (Mehmood and Chen (2022)): 55% of wagers at the race club are placed on credits instead of cash-in-advance bets, suggesting that credit-based betting becomes a significant component of the race club's economic system.

Regarding the second part of Proposition 5, the bookmaker faces the trade-off between the book-bet and the cash-bet as follows: first, under the book-bet gamblers can always default all debt repayments in period 1 and then quit the gambling in period 2. This "take-money-run-away" strategy can give each gambler the payoff of $\overline{u} \equiv \max_A v(A)$. Therefore, the bookmaker must leave at least this rent $\overline{u}$ to each gambler when she uses the book-bet

in period 1.[24] This rent never arises under the cash-bet. Second, if gamblers have the addiction preference, there is the benefit for the bookmaker to use the book-bet relative to the cash-bet such that gamblers are addicted to gambling when they are allowed to default in period 1 and then subsequently bet aggressively in period 2. However, such gains disappear when gamblers are not addicted at all. Then only the benefit of using the book-bet is that the allowed default can be used as a tool to compensate gamblers for inducing large amounts bet. However, such benefit is outweighed by the first two negative effects when gamblers place a relatively large bet $\tilde{A}^*$ under the cash-bet as assumed in Proposition 5 ($u(\tilde{A}^*) + H < 2f\tilde{A}^*$). Then, the bookmaker prefers the cash-bet to the book-bet.

7 Extensions

7.1 Endogenous Commission Fees

One might wonder if the bookmaker can more directly subsidize gamblers to place large bets in period 1 without allowing them to default. More specifically, the bookmaker may cultivate gamblers' addiction preferences by offering a negative commission fee in period 1.

To address this issue, we extend the basic model by allowing the bookmaker to choose a commission fee $f_t \in (-\infty, \overline{f}]$ in period $t = 1, 2$ where $\overline{f} < 1$ ensuring that market odds become positive: $h_{j,t} = (1 - f_t)A_t/a_{j,t} \geq 0$ for any $j \in J$ and $t = 1, 2$. Here f_t can be negative so that the bookmaker can directly subsidize gamblers to place bets. We suppose that the bookmaker offers and commits to commission fees (f_1, f_2) in the beginning of period 1.

We first consider that the bookmaker adopts the cash-bet in period 1 and chooses the commission fees (f_1, f_2) to maximize her payoffs over two periods, given by $n\{f_1A_1 + f_2A_2\}$ subject to the incentive constraint such that each gambler optimally places the bet of $A_t \geq 0$ in total in period $t = 1, 2$ to maximize his payoffs as follows:

$$(A_1, A_2) \in \arg \max_{A_1'' \geq 0, A_2'' \geq 0} v(A_1'') - f_1A_1'' + u(A_2'', S'') - f_2A_2''$$

where $S'' = \alpha + \beta A_1''$.

We denote by (f_1^c, f_2^c) the optimal commission fees chosen by the bookmaker under the cash-bet. We also denote by (A_1^c, A_2^c) the corresponding amounts bet in two periods, which satisfy the above incentive constraint of gamblers. Here the second-period amount bet A_2^c satisfies the optimality condition as $u_A(A_2^c, S^c) = f_2^c$, where $S^c \equiv \alpha + \beta A_1^c$. We then define by

$$\hat{U}(S; f_2^c) \equiv \max_{A \geq 0} u(A, S) - f_2^c A$$

[24] This is never the case when Assumption 2 is satisfied because gamblers always avoid default and participate in the gambling in period 2 whatever amounts they bet in period 1.

the second-period equilibrium payoff of each gambler, given an addiction stock S and the optimal commission fee f_2^c in period 2.

Next we consider the book-bet. We make the similar assumption to Assumption 2: gamblers' addiction preference is sufficiently strong such that they obtain large addiction gains from the first-period gambling, given the commission fee f_2^c optimally chosen under the cash-bet.

Assumption 3. $\hat{U}(S; f_2^c) > A$ *for all* $A \geq 0$.

We then show that under Assumption 3 the bookmaker still prefers the book-bet to the cash-bet even when commission fees (f_1, f_2) are endogenous. The idea is similar to Proposition 5. Suppose that in the beginning of period 1 the bookmaker offers the commission fees (f_1^c, f_2^c), which are the optimal choice under the cash-bet, and the bang-bang punishment policy $(\tilde{\psi}, \tilde{D})$ such that the maximum forgiveness is set to be zero, $\tilde{D}(z, A) = 0$ for all $A \geq 0$ and all $z \geq 0$. Then, under Assumption 3 gamblers default no positive amounts for any amount bet $A \geq 0$ in period 1: if gambler i defaults $D_{i,j} > 0$, then he is most harshly punished and loses the second-period payoff $\hat{U}(S_i; f_2^c)$. Then his payoff is at most $h_j^* a_{i,j} - A_i + D_{i,j} - H \leq -H$ but, if he does not default, he obtains a larger payoff $h_j^* a_{i,j} - A_i + \hat{U}(S_i; f_2^c) > 0$ under Assumption 3. Therefore, gambler i never defaults and faces the same payoff as that under the cash-bet with the optimal commission fees (f_1^c, f_2^c):

$$v(A_i) - f_1^c A_i + \hat{U}(S_i; f_2^c).$$

Then gamblers place the same bets (A_1^c, A_2^c) as those chosen under the cash-bet with the commission fees (f_1^c, f_2^c). The bookmaker thus obtains at least the same payoff $n\{f_1^c A_1^c + f_2^c A_2^c\}$ as that attained under the cash-bet, and may be better off further by choosing the commission fees and punishment policy optimally.

The main message is here that, when gamblers' addictive preferences are so strong that Assumption 3 holds, the bookmaker can be better off by the book-bet upon the cash-bet even if negative commission fee (subsidy) is allowed and is endogenously chosen. In other words subsidizing gamblers via negative commission fees is not sufficient but adding the book-bet with allowed default yields more gains to the bookmaker. The bookmaker can cultivate more addictive preferences of gamblers by allowing them to default in an early period rather than giving them only subsidies via negative commission fees.

7.2 Endogenous Commitment by the Bookmaker

We have so far assumed that the bookmaker can commit to the punishment policy announced in the beginning of period 1 as well as promised payouts in both period 1 and period 2. As we discussed, this assumption is made to simplify the analysis by paying our main focus to default incentives of only gamblers but not the bookmaker.

In this section we provide the extension of the basic model such that the bookmaker endogenously self-enforces promised payouts and punishment policies. Then we view the two-period model presented so far as a short-cut of the dynamic equilibrium in which the bookmaker's payouts and punishment policy are self-enforcing. Since the full analysis of the dynamic model is complicated, we relegate its details to the Appendix and discuss only its basic idea in this section.

We consider the following dynamic extension of the two-period model. Time is discrete and is extended over infinity $t = 1, 2, ...$ The bookmaker is a long-lived player who lives forever over time while gamblers are short-lived players who live only for one period. For example, the bookmaker is an organization which runs the horse race betting over time, so it is the going-concern over infinite periods.

To maintain the consistency with the two-period model, we suppose that each period t is divided into two sub-periods, which we call "sub-period 1" and "sub-period 2", as follows. Sub-period 1 corresponds to the game played as "period 1" in the basic model while sub-period 2 corresponds to the game played as "period 2" in the basic model respectively. More specifically, in the beginning of period t, new gamblers are born and enter the economy. Then these gamblers and the bookmaker play the "two-period game" in sub-period 1 and sub-period 2 within period t in the same way as they do in the basic model. In the end of period t these gamblers leave the economy and are replaced by new gamblers.

The information structure of this dynamic game is as follows. For each gambler i born in period t, there is a corresponding gambler born in next period $t + 1$, who can observe the events happened between the bookmaker and gambler i in period t. We call such gambler *successor* of gambler i, denoted by $\phi(i)$. This information structure can be used to construct the punishment equilibrium in which the bookmaker will lose her payoffs from future gambling if she deviates against gambler i in the current period, for example, by reneging on payouts to him: when such deviation happens, gambler i and all his successors will punish the bookmaker by not participating in the gambling. This punishment is possible because the information about the deviation made by the bookmaker against i in period t will be transmitted over all i's successors directly or indirectly in the future.

We maintain Assumptions 1 and 2 and discuss the equilibrium in the above dynamic game in which the bookmaker implements an aggregate amount bet A from each gambler in sub-period 1 and the corresponding amount bet $\hat{A}(S)$ from each gambler in sub-period 2 within each period. In such an equilibrium the bookmaker obtains the same payoff of $\Phi(A) \equiv fA - \tilde{D}^* + f\hat{A}(S)$ as that defined in Proposition 3. In contrast to the basic model, we need to take into account the following constraint, which we call *dynamic enforcement* (DE) constraint:

$$\frac{\delta\Phi(A)}{1-\delta} \geq (1-f)\hat{A}(S) \tag{DE}$$

where $S = \alpha + \beta A$ is the addiction stock per gambler and $\delta \in [0, 1)$ is the discount factor of the bookmaker across two successive periods.[25]

[25] We assume no time discounting between two sub-periods within each period.

To understand DE constraint, we consider the equilibrium in which each gambler places the bet of A in total under the book-bet in sub-period 1 and $\hat{A}(S)$ in total under the cash-bet in sub-period 2 within each period. Also, each gambler places the equilibrium bet of $\hat{a}_j$ on horse j under the cash-bet in sub-period 2, resulting in the equilibrium odds $\hat{h}_j \equiv (1-f)/p_j$ in sub-period 2. Then the bookmaker is required to give the payout of $\hat{h}_j \hat{a}_j = (1-f)\hat{A}(S)$ to each gambler i conditional on horse j wining the race in sub-period 2. Since the payout is not formally enforced, the bookmaker must be given the incentive not to renege on the equilibrium payout of $\hat{h}_j \hat{a}_j = (1-f)\hat{A}(S)$. If the bookmaker deviates from this payout against some gambler today, she can save that amount today but she will be punished by the gambler's successors who will quit her in the future, thereby losing the future discounted value of payoffs $\delta\Phi(A)/(1-\delta)$. DE constraint ensures that the bookmaker has no incentives to make such deviation against any gambler in each period.[26]

In the Appendix we show that there exists an equilibrium in the above dynamic game such that (i) the bookmaker follows promised payouts and punishment policy $\{\tilde{\psi}, \tilde{D}\}$ as those considered in the basic model, and (ii) she implements the aggregate amount bet A from each gambler in order to maximize her payoff $\Phi(A)$ subject to DE constraint given above.

We obtain two important implications about the above dynamic equilibrium as follows.

First, when the commission fee f is smaller, DE constraint becomes tighter so that the bookmaker finds it more difficult to commit to payouts. Thus the self-enforcing condition on the side of the bookmaker severely constrains the use of non-positive commission fees. This result provides a reason about why transfer schemes such as two-part tariffs which combine zero commission fees with fixed fees do not work effectively when the bookmaker's commitment to payouts is limited. Therefore, we may justify the assumption made in the basic model that the commission fee is bounded away from zero, $f > 0$, on the ground of limited commitment capacity by the bookmaker.

Second, addiction preferences help self-enforce relational contract agreements between the bookmaker and gamblers over time. The parameter of the model capturing addiction preferences is given by $\beta > 0$ that determines how likely/quickly gamblers are addicted to gambling over time. Here, the increase in β has two effects: first, as β increases, gamblers accumulate larger addiction stock $S_i = \alpha + \beta A_i$, so they place larger bets $\hat{A}(S_i)$ under the cash-bet in sub-period 2 within each period. This increases the payouts that the bookmaker must commit to give gamblers, raising the right hand side of DE constraint and hence making her commitment more difficult. Second, as β increases, the bookmaker can

[26] There are also two other possible deviations by the bookmaker: she may renege on the net payout $h_j a_{i,j} - A_i$ under the book-bet in sub-period 1 and may not exclude those who defaulted in sub-period 1. However, in the Appendix we will show that these deviations do not constrain the bookmaker's payoffs. The reasons for this are as follows. First, gamblers owe debts but do not receive positive net payouts under the book-bet in equilibrium, so there are no rooms for the bookmaker to renege on net payouts under the book-bet. Second, if the bookmaker does not implement the announced punishment policy against some gamblers in sub-period 1, she will be immediately punished by these gamblers and their successors who will quit the bookmaker.

exploit larger profits from future gamblers who will accumulate larger addiction stock and then bet more. This raises the left hand side of DE constraint, making the bookmaker's commitment easier.

We can show that the latter effect dominates the former one when the bookmaker is sufficiently patient, so the bookmaker can more easily self-enforce promised payouts when gamblers' addictions become stronger. To see this, we differentiate both sides of DE constraint with respect to the addiction preference parameter β to obtain

$$\frac{\delta}{1-\delta}\frac{\partial \Phi(A)}{\partial \beta} - (1-f)\frac{\partial \hat{A}(S)}{\partial \beta} \tag{25}$$

for a given amount bet A per gambler. Assuming that $u_{AS} > 0$, we obtain $\partial\Phi(A)/\partial\beta > 0$.[27] Then the first term of (25) can dominate the second one when the discount factor $\delta \in [0, 1)$ is close to 1, in which case larger addiction preference β can weaken DE constraint to implement an amount bet A from each gambler. This result suggests that individuals' addictions can *complement* relational contract agreements, implying that informal markets where goods/services are traded without formal contracts can work more effectively even without well-functioning institutions when individuals are addicted more by consuming goods/services. This implication is useful to understand how and why illegal markets such as illicit drugs and illegal wagering markets work efficiently even without formal enforcement.

7.3 Applications to Other Illegal Markets

In this section we discuss how our results can include a broad range of applications to understand how illegal markets work beyond the specific setting of illegal horse race betting. More specifically, we show that our model can be applied to any illegal markets, such as illegal drugs, where sellers sell addictive goods to buyers who form addiction preferences over time, and their transactions are illegal and are hence not formally enforceable.

For this purpose, we make the following interpretation of the basic model of horse race betting: first, the bookmaker and gamblers are interpreted as a "seller" and "buyers" respectively, where the seller sells an addictive good to buyers. Second, the amount bet $A_{i,t}$ by gambler i in period $t = 1, 2$ is interpreted as the consumption level of the addictive good by buyer i in period $t = 1, 2$. Buyers decide how much to consume the addictive good in each period, and are addicted to consuming the good in period 2, which we capture by the addiction stock $S_i = \alpha + \beta A_{i,1}$ again. Third, the seller has two formats to sell the addictive good: one is the cash-in-advance purchase, meaning that buyers must immediately pay the seller when they purchase the good. The other is the credit-based purchase, meaning that buyers can defer payments. The former corresponds to the cash-bet format while the latter

[27] Note that $\Phi(A) = v(A) + u(\hat{A}(S), S) - \overline{V}$ when $\tilde{D}^* > 0$ holds at the optimum of Problem B* and $\Phi(A) = fA + f\hat{A}(S)$ when $\tilde{D}^* = 0$ holds at the optimum respectively. In either case Φ is increasing in β because $\hat{A}$ and $u(A, S)$ are increasing in β given $u_{AS} > 0$.

corresponds to the book-bet format respectively as we considered in the model of horse race betting. Fourth, we interpret the commission fee of the bookmaker $f > 0$ as the price of the addictive good, which buyers pay the seller for consuming one unit of the addictive good. Here the price f is exogenous as made in the basic model or it can be chosen by the seller as considered in the extended model in Section 7.1.

Given the above interpretation of the model, individual i chooses how much to consume the good $A_i \geq 0$ in period 2 to maximize his second-period payoff $u(A_i, S_i) - fA_i$. This gives him the second-period payoff $\hat{U}(S_i)$, which is same as (8) obtained in Proposition 1. In period 1 individual i decides how much to consume $A_i \geq 0$ and how much to default $D_i \in [0, fA_i]$ to maximize his payoffs over two periods

$$v(A_i) - fA_i + D_i - \psi(D_i, fA_i) + \hat{U}(S_i)$$

where $\psi(D_i, fA_i)$ is the penalty imposed when he is supposed to pay fA_i but defaults $D_i \in [0, fA_i]$. This is essentially same as the payoff defined in the case of illegal horse race betting. Then, by using a similar logic to Proposition 2, we can show that the optimal punishment policy for the seller can be the bang-bang form. Furthermore, since each individual can always default all the payment $D = fA$, followed by being most harshly punished, he can obtain at least

$$\tilde{V} \equiv \max_A v(A) - H.$$

We can then replace $\overline{V}$ by $\tilde{V}$ in Problem B* to obtain the optimal punishment policy for the seller. Thus we can show a similar characterization result to Proposition 3 except for the equilibrium odds which are specific to the horse race betting but not to other illegal goods markets. Thus most of the propositions we have so far obtained still remain valid in a broad range of illegal markets where sellers sell addictive goods to buyers who form addiction preferences over time, and transactions of the goods are illegal and are hence not formally enforceable.

8 Case studies

In this section, we discuss several case studies. We first discuss illegal horse betting, a case that closely aligns with the formal model. We then turn to illegal drugs and religious sects.

8.1 Illegal Horse Betting in Pakistan

Gathering data from markets prone to addiction and weak contract enforcement poses significant challenges due to the clandestine nature of these transactions, which are deliberately concealed to evade legal penalties. The first case study we present focuses on the underground horse race betting market in Pakistan, where Mehmood and Chen (2022) collected data, which we discuss below in light of the main predictions of the model.

The horse races take place every Sunday from noon to 6 pm, with races scheduled every 30 minutes. Gambling takes place at betting stations inside the premises of the race club. The entry at the club requires a ticket of PKR 500 (USD 2.25), with anyone who has a ticket allowed entry into the club and, by default, the ability to bet at any of the 12 betting stations that issue identical odds. Every station charges a constant 5% participation fee, and gamblers are allowed to bet on credit. Below, we discuss our three main theoretical results in light of available data.

Free-first dose effect. In their examination of the horse race betting market, Mehmood and Chen (2022) leave an intriguing puzzle unaddressed: the policy allowing patrons to place credit bets of up to PKR 5000, or roughly USD 20, upon entry to the race club. The economic rationale for such an institutional arrangement is not immediately apparent, raising questions about its profitability. Approximately 55% of wagers at the race club are placed on credit, as opposed to cash-in-advance bets, suggesting that credit betting constitutes a significant component of the race club's economic ecosystem. Our model offers a simple explanation. Offering bets on credit cultivates gamblers' addiction, ensuring a steady stream of future revenues.

Debt repayment. Mehmood and Chen (2022) find that on average, 35% of gamblers do not repay their debt in full. Our model explains this surprising feature as well. Indeed, we find (Proposition 2) that bookmakers allow a certain degree of default on the equilibrium path. That way, gamblers can place larger bets, so their addiction becomes stronger.

Violence. Mehmood and Chen (2022) collected data about perceived violence from betters. Their findings reveal that such occurrences are exceedingly rare. To be specific, a mere 0.5% of surveyed gamblers reported any apprehension of encountering violence in case of non-payment situations. This empirical observation is in line with anecdotal evidence, as exemplified by recent ethnographic research. For instance, during interviews, Paa'h Sadiq, a prominent bookie and key informant, expressed astonishment at the mere suggestion of violence in his line of work. He aptly countered, "Do I look like Amresh Puri [famous Indian actor, notably known for villainous roles]? You guys see too many gangster films. Gambling debts are debts of honor. If I resort to violence, I lose honor and the [very] right to collect debts." (Mahar, 2022, p.5) These testimonies underscore the exceptional rarity of violence associated with unpaid gambling debts within this betting market. These evidence are also in line with the predictions of the model. We find that while violence might enforce more prompt debt repayments, it could simultaneously weaken the ability of book-makers to cultivate addictions.

8.2 Illegal Drugs

Illegal drug markets are among the most elusive yet economically significant sectors, marked by their adaptability and resilience. While empirical data on these markets are scarce due to the illicit and secretive nature of transactions, qualitative evidence provides insights into their operational dynamics and the strategies employed by participants.

Free-first dose effect. The strategy of offering the first dose for free or at a significantly reduced price to potential new users is a well-known method in drug markets, aimed at fostering dependency. Another illustration of the free-first dose effect is through drug dealers' incentive to dilute ("cut") the products they sell. This dilution is unobservable to buyers until after they consume, and thus creates a moral hazard issue (Galenianos and Gavazza (2017), Galenianos, Pacula and Persico (2012)). Refraining from cutting on drugs can be seen as a particular example of the free-first dose effect. Indeed, on the one hand, offering high-quality products to new consumers is costly for drug dealers. On the other hand, offering high-quality products cultivates stronger addictions. Galenianos and Gavazza (2017) estimate a model using data on the crack cocaine market in the United States. Their estimation reveals that although they are short-lived, relationships between buyers and sellers are valuable to sellers, as regular buyers consume more frequently and account for the vast majority of crack cocaine purchases. Galenianos and Gavazza (2017) rely on the STRIDE dataset, which contains records of acquisitions of illegal drugs by undercover agents and DEA informants. Hence, the data may contain more specifically information about purchases from "new" consumers. Consistent with our model, Galenianos and Gavazza (2017) find that drug dealers do not necessarily cut on quality in the STIDE dataset, suggesting that creating relations and cultivating addictions might be valuable.

Debt repayment. Credit plays an important part in drug distribution, acting as a pivotal mechanism for sustaining and expanding consumer bases in environments where immediate payment may not be an option. This reliance on credit not only facilitates transactions but also embeds a level of trust and dependency between dealers and users (Jacobs and Wright (2006)). This aspect of illegal drug markets is consistent with our model, where offering addictive goods on credit strengthens addiction and future transactions.

Violence. The role of violence in drug markets is multifaceted, serving both as a tool for enforcement and as a potential deterrent to the stability of these markets. The strategic use of violence, or the deliberate avoidance thereof, is a critical consideration for drug dealers who must balance the immediate benefits of enforcing payment and loyalty against the long-term consequences of scaring away customers or attracting law enforcement attention. In their analysis of the heroin market of New York city, Curtis and Wendel (2000) illustrate how violence is strategically used or avoided to maintain market stability and customer loyalty. The authors suggest that while violence can be an effective means of debt collection, it can also undermine the very foundation of trust and repeat business upon which these markets rely.

8.3 Religious Sects

Religious sects, particularly those with more exclusive or intense commitment requirements, can sometimes exhibit dynamics similar to the patterns of addiction and enforcement seen in the contexts of gambling and drug markets. The initial engagement with these groups is often marked by a welcoming atmosphere and various forms of support, which can be

seen as analogous to the "free-first dose" effect. Over time, the deepening of commitment can introduce elements of indebtedness to maintain consumption and addiction.

Free-first dose effect. Many religious sects initially offer new members a sense of belonging, community, and sometimes tangible benefits such as meals, social support, or even housing. This welcoming approach serves to attract individuals seeking community or spiritual fulfillment, providing an initial "dose" of the benefits of membership with little to no upfront cost. This phase can be critical for building attachment to the group and its beliefs (Dawson (1998)).

Debt repayment. As members become more integrated into the sect, they may feel an increasing obligation to contribute financially, dedicate time, or engage in proselytizing activities. This sense of obligation can be akin to "debt repayment," where the perceived debt is not just financial but also emotional or spiritual. Members might believe that their salvation, enlightenment, or the well-being of their community depends on their contributions. In some cases, the failure to meet these expectations can lead to feelings of guilt or indebtedness, further binding members to the group as they strive to "repay" their perceived debt.

Violence. While physical violence is rare within the vast majority of religious sects, subtler forms of coercion, such as psychological pressures or social ostracism, may be employed by more controlling groups to ensure adherence to their norms. The complexity and diversity of these coercive practices, especially when balanced against the need for voluntary and genuine commitment, present challenges that our model might not fully capture.

In the economic literature on religion, Iannaccone (1992) argued that costly practices within religious cults allow for the screening of free-riders and increase the overall benefits for group members. Our model complements Iannaccone (1992), suggesting that addiction might be a channel through which costly practices in religious groups are self-enforced. Additionally, in a world of increasing religious pluralism, cultivating an addiction to a specific cult might also be a way to decrease group members' outside options. Hence, our work also connects to the literature on religious competition, suggesting an explanation for the persistence of multiple sects.[28]

9 Conclusion

In this study, we introduce a dynamic model to analyze addiction dynamics in markets lacking formal enforcement mechanisms. Our model delineates how sellers employ a 'free-first-dose' strategy to foster addiction, thereby establishing a reliable consumer base. Key to our findings is the strategic tolerance of non-repayment by sellers, which, paradoxically, deepens addiction and customer loyalty. Contrary to prevalent views, our results suggest that early-stage violence and coercion may detrimentally affect addiction cultivation, offering new insights into the operation of illegal markets.

[28] On the literature on religious markets, see, among others, McBride (2008) and McBride (2010).

Understanding of the economic and psychological forces at play suggests novel policies for effective regulation of non-legal markets. Targeting the "free-first-dose' strategy could significantly disrupt the cycle of addiction, curtailing the pathway to long-term dependency. Surprisingly, protecting sellers against defaults may alter market dynamics.

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10 Appendix A: Proofs

10.1 Proof of Proposition 1

In period 2 gambler i places bet $\mathbf{a}_i$ to maximize his expected payoff

$$u(A_i, S_i) + \sum_{j \in J} p_j \{h_j a_{i,j} - A_i\},$$

given expectations about market odds $\{h_j\}_{j \in J}$. Then the equilibrium bet $\hat{a}_{i,j} > 0$ placed by gambler i on horse j is determined by the following first-order optimality condition:

$$u_A\left(\hat{A}_i, S_i\right) + p_j h_j - 1 = 0, \quad j \in J. \tag{A1}$$

Since (A1) holds for any $j \in J$ in the equilibrium, we obtain

$$p_j h_j = p_\ell h_\ell, \quad \text{for any } j, \ell \in J.$$

By using the definitions of $\hat{a}_j \equiv \sum_{i \in I} \hat{a}_{i,j}$ and $\hat{A} \equiv \sum_{i \in I} \hat{A}_i$, we have

$$p_j(1-f)\hat{A}/\hat{a}_j = p_\ell(1-f)\hat{A}/\hat{a}_\ell,$$

so that

$$\hat{a}_j/\hat{a}_\ell = p_j/p_\ell$$

for any $j, \ell \in J$. Therefore, gamblers bet more on the horses which are more likely to win the race. Then we verify that

$$\begin{aligned} \hat{A} &= \hat{a}_j + \sum_{\ell \neq j} \hat{a}_\ell \\ &= \hat{a}_j \left(1 + \sum_{\ell \neq j} (\hat{a}_\ell/\hat{a}_j) \right) \\ &= \hat{a}_j \left(1 + \sum_{\ell \neq j} (p_\ell/p_j) \right) \end{aligned}$$

so that gamblers place the bet equal to

$$\hat{a}_j = \frac{\hat{A}}{1 + \sum_{\ell \neq j} (p_\ell/p_j)}$$

on horse $j \in J$ in total in equilibrium. Note that the amount bet by each gambler on each horse $\hat{a}_{i,j}$ is indeterminate. Only the aggregate amount $\hat{a}_j$ bet by all gamblers on horse $j \in J$ is determined.

Since $h_j = (1-f)\hat{A}/\hat{a}_j$ holds for $j \in J$ in equilibrium, we substitute $\hat{a}_j$ into h_j in order to obtain the following:

$$p_j h_j = (1-f), \quad \text{for } j \in J.$$

Therefore, the equilibrium amount bet $\hat{A}_i$, given by (A1) above, is determined by the following condition:

$$u_A(\hat{A}_i, S_i) - f = 0, \quad \text{for } i \in I$$

yielding the desired result (7) in the main text. This gives the equilibrium amount $\hat{A}_i = \hat{A}(S_i)$ bet by gambler i in total, which depends on his addiction stock S_i.

As a result, we can write the equilibrium second-payoff of gambler i who has the addiction stock S_i as follows

$$\begin{aligned}\hat{U}(S_i) &\equiv u(\hat{A}(S_i), S_i) + \sum_{j\in J} p_j h_j \hat{a}_{i,j} - \hat{A}(S_i) \\ &= u(\hat{A}(S_i), S_i) + (1-f)\sum_{j\in J} \hat{a}_{i,j} - \hat{A}(S_i) \\ &= u(\hat{A}(S_i), S_i) + (1-f)\hat{A}(S_i) - \hat{A}(S_i) \\ &= u(\hat{A}(S_i), S_i) - f\hat{A}(S_i)\end{aligned}$$

yielding the desired result (8) in the main text. Q.E.D.

10.2 Proof of Proposition 2

Consider any punishment policy ψ implementing an equilibrium $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$ in the two-period game. That is, following the punishment policy ψ offered by the bookmaker, there exists the resulting equilibrium in which gambler $i \in I$ places equilibrium bet $\mathbf{a}_i^* \equiv (a_{i,1}^*, ..., a_{i,m}^*)$ in period 1. The associated equilibrium odds in period 1 are given by $\mathbf{h}^* \equiv \{h_j^*\}_{j\in J}$ such that $h_j^* = (1-f)\sum_{i\in I} A_i^* / \sum_{i\in I} a_{i,j}^*$.[29] Let $A_i^* \equiv \sum_{j\in J} a_{i,j}^*$ denote the aggregate amount bet by gambler i in the equilibrium.

Since we are here supposing that there exists an equilibrium $\{\mathbf{a}^*, \mathbf{h}^*, \hat{\mathbf{A}}\}$ implemented by a punishment policy ψ, in any subgame given any $A_i \geq 0$ and any $z_{i,j} \geq 0$, gambler i must optimally choose the default level $D_{i,j} \in [0, z_{i,j}]$, denoted by $D^*(z_{i,j}, A_i) \in [0, z_{i,j}]$. That is, $D^*(z, A) \in [0, z]$ solves

$$\max_{0\leq D\leq z} \; D - \psi(D, z, A)$$

given $z \geq 0$ and $S = \alpha + \beta A$.

Note that the functional form of $D^*(\cdot,\cdot)$ is the same for all gamblers because all of them face the same functional forms of $\psi(\cdot,\cdot)$ and $\hat{U}(\cdot)$. However, the realized amount of default $D^*(z_{i,j}, A_i)$ can differ across gamblers because they may face the different race outcomes $z_{i,j}$ and place different bets $\mathbf{a}_i$.

Define

$$\underline{\psi}(z, A) \equiv \inf_{D\in[0,z]} \psi(D, z, A)$$

for $z \geq 0$ and $A \geq 0$.

Then we show the following claim.

Lemma A1. *$D^*(z, A) - \psi(D^*(z, A), z, A) \geq -\underline{\psi}(z, A)$ holds for all $z \geq 0$ and all $A \geq 0$.*

[29] Note that we focus on the equilibrium in which every horse has positive odds, that is, $\sum_{i\in I} a_{i,j}^* > 0$.

Proof. Due to the optimal choice of $D^*(z,A)$, it must be that

$$D^*(z,A) - \psi(D^*(z,A),z,A) \geq D - \psi(D,z,A)$$

for all $D \in [0,z]$. Also, due to the definition of $\underline{\psi}(z,A)$, for any $\varepsilon > 0$, some $D'' \in [0,z]$ exists such that

$$\underline{\psi}(z,A) + \varepsilon > \psi(D'',z,A).$$

Therefore, we obtain

$$\begin{aligned} D^*(z,A) - \psi(D^*(z,A),z,A) &\geq D'' - \psi(D'',z,A) \\ &> D'' - \underline{\psi}(z,A) - \varepsilon \\ &\geq -\underline{\psi}(z,A) - \varepsilon \end{aligned}$$

for all small $\varepsilon > 0$, so that

$$D^*(z,A) - \psi(D^*(z,A),z,A) > -\underline{\psi}(z,A) - \varepsilon$$

for all small $\varepsilon > 0$. By letting $\varepsilon \to 0$, we obtain the desired result. Q.E.D.

Gambler i places bet $\mathbf{a}_i$ to maximize the following expected payoff in period 1:

$$v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + D^*(z_{i,j}, A_i) - \psi(D^*(z_{i,j}, A_i), z_{i,j}, A_i)\} + \hat{U}(S_i) \qquad \text{(A2)}$$

given the equilibrium odds $\mathbf{h}^* = \{h_j^*\}_{j \in J}$. These gamblers' choices lead to the equilibrium amounts bet $\mathbf{a}^*$ in period 1 and the associated amounts bet $\hat{\mathbf{A}}$ in period 2 respectively.

Define

$$\tilde{D}(z,A) \equiv D^*(z,A) - \{\psi(D^*(z,A),z,A) - \underline{\psi}(z,A)\}$$

where $\tilde{D}(z,A) \geq 0$ due to Lemma A1, and $\tilde{D}(z,A) \leq z$ because $D^*(z,A) \leq z$. Then we define the following new punishment policy:

$$\tilde{\psi}(D,A) \equiv \begin{cases} \underline{\psi}(z,A) & \text{if } D \leq \tilde{D}(z,A) \\ \\ \hat{U}(S) + H & \text{otherwise} \end{cases}$$

Note that the new policy $\tilde{\psi}$ has the bang-bang form.

We show that the new policy defined above can induce gamblers to place the same bets as those under the original policy ψ while reducing defaulted amounts.

First, gamblers never choose $D < \tilde{D}(z,A)$: otherwise, they can slightly increase D while keeping $D \leq \tilde{D}(z,A)$ so that they can still face the same penalty $\underline{\psi}(z,A)$. This deviation can strictly improve their payoffs. Therefore, we consider only the case that $D \geq \tilde{D}(z,A)$.

Suppose that gambler i chooses $\mathbf{a}_i = (a_{i,1}, ..., a_{i,m})$ and D_{ij} where $D_{ij} \geq \tilde{D}(z_{i,j}, A_i)$ for each $j \in J$. Let

$$J_i^* \equiv \{j \in J \mid D_{ij} = D^*(z_{i,j}, A_i)\}.$$

Then, for any profile of amounts bet $\mathbf{a}_i$, we can show that

$$\begin{aligned}
& v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_i)\} + \hat{U}(S_i) \\
= \; & v(A_i) + \sum_{j \in J_i^*} p_j \{h_j^* a_{i,j} - A_i + \tilde{D}(z_{i,j}, A_i) - \underline{\psi}(z_{i,j}, A_i)\} \\
& + \sum_{j \notin J_i^*} p_j \{h_j^* a_{i,j} - A_i + D_{i,j} - H - \hat{U}(S_i)\} + \hat{U}(S_i) \\
\leq \; & v(A_i) + \sum_{j \in J_i^*} p_j \{h_j^* a_{i,j} - A_i + D^*(z_{i,j}, A_i) - \psi(D^*(z_{i,j}, A_i), z_{i,j}, A_i)\} \\
& + \sum_{j \notin J_i^*} p_j \{h_j^* a_{i,j} - A_i + D_{i,j} - \psi(D_{i,j}, z_{i,j}, A_i)\} + \hat{U}(S_i) \\
\leq \; & v(A_i) + \sum_{j \in J_i^*} p_j \{h_j^* a_{i,j} - A_i + D^*(z_{i,j}, A_i) - \psi(D^*(z_{i,j}, A_i), z_{i,j}, A_i)\} \\
& + \sum_{j \notin J_i^*} p_j \{h_j^* a_{i,j} - A_i + D^*(z_{i,j}, A_i) - \psi(D^*(z_{i,j}, A_i), z_{i,j}, A_i)\} + \hat{U}(S_i) \\
= \; & v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + D^*(z_{i,j}, A_i) - \psi(D^*(z_{i,j}, A_i), z_{i,j}, A_i)\} + \hat{U}(S_i) \\
\leq \; & \max_{\mathbf{a}_i} \; v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + D^*(z_{i,j}, A_i) - \psi(D^*(z_{i,j}, A_i), z_{i,j}, A_i)\} + \hat{U}(S_i) \\
= \; & v(A_i^*) + \sum_{j \in J} p_j \{h_j^* a_{i,j}^* - A_i^* + D^*(z_{i,j}^*, A_i^*) - \psi(D^*(z_{i,j}^*, A_i), z_{i,j}^*, A_i^*)\} + \hat{U}(S_i^*) \\
= \; & v(A_i^*) + \sum_{j \in J} p_j \{h_j^* a_{i,j}^* - A_i^* + \tilde{D}(z_{i,j}^*, A_i^*) - \underline{\psi}(z_{i,j}^*, A_i^*)\} + \hat{U}(S_i^*)
\end{aligned}$$

where the first inequality follows from $\psi(D, z, A) \leq \hat{U}(S) + H$, the second inequality from the optimal choice of $D^*(z, A)$ to maximize $D - \psi(D, z, A)$ over $0 \leq D \leq z$, and the final equality from the definition of $\tilde{D}(z, A)$ respectively.

Therefore, gambler i never deviates from the amount bet $\mathbf{a}_i^*$ which he chooses in the original equilibrium with the equilibrium odds $\mathbf{h}^*$. Thus the bang-bang punishment policy $\{\tilde{\psi}, \tilde{D}\}$ implements the same amounts bet as those in the equilibrium under the original punishment policy $\{\psi, D\}$.

We next decompose the lowest equilibrium penalty $\underline{\psi}(z_{i,j}^*, A_i^*)$ under the above bang-bang punishment policy $\tilde{\psi}$ into q and ξ as follows: define $\underline{q}(z, A) \equiv \inf_{0 \leq D \leq z} q(D, z, A)$

and

$$\underline{\xi}(z,A) \equiv \underline{\psi}(z,A) - \underline{q}(z,A)\hat{U}(S).$$

Since $\underline{\psi}(z,A) = \inf_{0\le D\le z}\{q(D,z,A)\hat{U}(S)+\xi(D,z,A)\} \ge \inf_{0\le D\le z} q(D,z,A)\hat{U}(S) = \underline{q}(z,A)\hat{U}(S)$, we have $\underline{\xi}(z,A) \ge 0$. Also, since $\underline{\psi}(z,A) \le \inf_{0\le D\le z} q(D,z,A)\hat{U}(S) + H = \underline{q}(z,A)\hat{U}(S) + H$, we have $\underline{\xi}(z,A) \le H$. Thus $\underline{\xi}(z,A) \in [0,H]$ holds. Then we set $\underline{\psi}(z_{i,j^*}, A_i^*) = \underline{q}(z_{i,j}^*, A_i^*)\hat{U}(S_i^*) + \underline{\xi}(z_{i,j}^*, A_i^*)$ for $i \in I$ and $j \in J$.

Given this modification, the corresponding payoff of the bookmaker who uses the newly defined policy $\tilde{\psi}$ is then given as follows

$$\sum_{i\in I}\sum_{j\in J} p_j\{A_i^* - h_j^* a_{i,j}^* - \tilde{D}(z_{i,j}^*, A_i^*) + (1 - \underline{q}(z_{i,j}^*, A_i^*))f\hat{A}(S_i^*)\}.$$

This is not less than her payoff under the original punishment policy ψ:

$$\sum_{i\in I}\sum_{j\in J} p_j\{A_i^* - h_j^* a_{i,j}^* - D^*(z_{i,j}^*, A_i^*) + (1 - q(D^*(z_{i,j}^*, A_i^*), z_{i,j}^*, A_i^*))f\hat{A}(S_i^*)\}$$

because $\underline{q}(z,A) \le q(D,z,A) \in [0,1]$ and $\tilde{D}(z,A) \le D^*(z,A)$. Q.E.D.

10.3 Proof of Proposition 3

(i) Suppose contrary to the claim that $\tilde{D}(z_{i,j}^*, A_i^*) > 0$ implies that $\underline{\psi}(z_{i,j}^*, A_i^*) > 0$ for some $i \in I$ and $j \in J$ under the punishment policy $\tilde{\psi}$ that maximizes the bookmaker's payoff. Let $\tilde{I}$ and $\tilde{J}$ denote the set of such $i \in I$ and $j \in J$. Then define the new punishment policy $\tilde{\psi}''$ as follows: define $\tilde{D}''(z_{i,j}^*, A_i^*) \equiv \tilde{D}(z_{i,j}^*, A_i^*) - \varepsilon$ and $\underline{\psi}''(z_{i,j}^*, A_i^*) \equiv \underline{\psi}(z_{i,j}^*, A_i^*) - \varepsilon$ for any $(i,j) \in \tilde{I} \times \tilde{J}$, for a small $\varepsilon > 0$, while $\tilde{D}''(z,A) = \tilde{D}(z,A)$ and $\underline{\psi}''(z,A) = \underline{\psi}(z,A)$ for all other values of z and A. Then the punishment policy $\tilde{\psi}''$ newly defined in this way induces gamblers to place the same equilibrium bets as them under the original policy ψ because

$$\begin{aligned}
& v(A_i^*) + \sum_{j\in J} p_j\{h_j^* a_{i,j}^* - A_i^* + \tilde{D}''(z_{i,j}^*, A_i^*) - \underline{\psi}''(z_{i,j}^*, A_i^*)\} + \hat{U}(S_i^*) \\
=\ & v(A_i^*) + \sum_{j\in J} p_j\{h_j^* a_{i,j}^* - A_i^* + \tilde{D}(z_{i,j}^*, A_i^*) - \underline{\psi}(z_{i,j}^*, A_i^*)\} + \hat{U}(S_i^*) \\
\ge\ & v(A_i) + \sum_{j} p_j\{h_j^* a_{i,j} - A_i + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_i)\} + \hat{U}(S_i) \\
=\ & v(A_i) + \sum_{j\in J} p_j\{h_j^* a_{i,j} - A_i + D_{i,j} - \tilde{\psi}''(D_{i,j}, z_{i,j}, A_i)\} + \hat{U}(S_i)
\end{aligned}$$

for any $\mathbf{a}_i$ and any $D_{i,j} \in [0, z_{i,j}]$.

We next define the exclusion probability q'' and the utility-based penalty ξ'' under $\tilde{\psi}''$. Define $q^*_{i,j} \equiv q(\tilde{D}(z^*_{i,j}, A^*_i), z^*_{i,j}, A^*_i)$ and $\xi^*_{i,j} \equiv \xi(\tilde{D}(z^*_{i,j}, A^*_i), z^*_{i,j}, A^*_i)$ as the exclusion probability and the utility-based penalty used in the equilibrium under ψ. Then we define $\underline{q}''(z^*_{i,j}, A^*_i)$ and $\underline{\xi}''(z^*_{i,j}, A^*_i)$ for $(i,j) \in \tilde{I} \times \tilde{J}$ as follows: if $q^*_{i,j} > 0$, then define $\underline{q}''(z^*_{i,j}, A^*_i) \equiv q^*_{i,j} - \rho_{i,j}$ for a small $\rho_{i,j} > 0$ such that $\rho_{i,j}\hat{U}(S^*_i) = \varepsilon$, and $\underline{\xi}''(z^*_{i,j}, A^*_i) = \xi^*_{i,j}$. If $q^*_{i,j} = 0$, then set $\underline{q}''(z^*_{i,j}, A^*_i) = 0$ while $\underline{\xi}''(z^*_{i,j}, A^*_i) \equiv \xi^*_{i,j} - \varepsilon$. Also define $q''(D,z,A) = q(D,z,A)$ and $\xi''(D,z,A) = \xi(D,z,A)$ for all other values of (D,z,A).

By the new punishment policy $\tilde{\psi}''$ defined above, the bookmaker's payoff is changed to

$$\sum_i \sum_j p_j \{A^*_i - h^*_j a^*_{i,j} - \tilde{D}''(z^*_{i,j}, A^*_i) + (1 - \underline{q}''(z^*_{i,j}, A^*_i))\hat{A}(S^*_i)\}$$

which can be greater than the payoff under the original punishment policy $\tilde{\psi}$.

(ii) Suppose contrary to the claim that $\min_{i \in I} \sum_{j \in J} p_j \underline{\psi}(z^*_{i,j}, A^*_i) > 0$. Then we have $\sum_j p_j \underline{\psi}(z^*_{i,j}, A^*_i) > 0$ for all $i \in I$. We define by $j(i) \in J$ the horse for which $\underline{\psi}(z^*_{i,j(i)}, A_i) > 0$ for each $i \in I$.

For a small $\rho > 0$, define $\varepsilon_{i,j(i)} \equiv \rho / p_{j(i)}$ for each $i \in I$. Then define the new punishment policy $\tilde{\psi}''$ as follows: set $\underline{\psi}''(z^*_{i,j(i)}, A^*_i) \equiv \underline{\psi}(z^*_{i,j(i)}, A^*_i) - \rho$ and $\tilde{\psi}''(D,z,A) = \tilde{\psi}(D,z,A)$ for all other values of (D,z,A) than $(z^*_{i,j(i)}, A^*_i)_{i \in I}$ while keeping the maximum forgiveness $\tilde{D}(z,A)$ unchanged. Then we show that the newly defined punishment policy $\tilde{\psi}''$ can still implement the same equilibrium outcome while making the bookmaker weakly better off.

First, if gambler i chooses $A_i \neq A^*_i$, he obtains the payoff of

$$v(A_i) + \sum_j p_j \{h^*_j a_{i,j} - A_i + D_{i,j} - \tilde{\psi}''(D_{i,j}, z_{i,j}, A_i)\} + \hat{U}(S_i)$$

which is equal to

$$V''_i \equiv v(A_i) + \sum_j p_j \{h^*_j a_{i,j} - A_i + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_i)\} + \hat{U}(S_i)$$

because $\tilde{\psi}''(D,z,A) = \tilde{\psi}(D,z,A)$ for all $(z,A) \neq (z^*_{i,j(i)}, A^*_i)$. However, if gambler i follows the equilibrium bet $\mathbf{a}^*_i$, he obtains the payoff of

$$\begin{aligned}
& v(A^*_i) + \sum_j p_j \{h^*_j a^*_{i,j} - A^*_i + \tilde{D}(z^*_{i,j}, A^*_i) - \underline{\psi}''(z^*_{i,j}, A^*_i)\} + \hat{U}(S^*_i) \\
&= v(A^*_i) + \sum_j p_j \{h^*_j a^*_{i,j} - A^*_i + \tilde{D}(z^*_{i,j}, A^*_i) - \underline{\psi}(z^*_{i,j}, A^*_i)\} + \hat{U}(S^*_i) + \rho \\
&\equiv V^*_i + \rho \\
&> V^*_i.
\end{aligned}$$

Since the original punishment policy $\tilde{\psi}$ implements $\mathbf{a}_i^*$, it must be that $V_i^* \geq V_i''$, implying that gambler i never deviates from $\mathbf{a}_i^*$.

Next suppose that gambler i chooses $A_i = A_s^*$ for some $s \neq i$ but $\mathbf{a}_i \neq \mathbf{a}_i^*$. Define the set of horses J'' such that $a_{i,j} = a_{s,j}^*$ holds. Then we have $z_{i,j} = A_s^* - h_j^* a_{i,j} = z_{s,j}^*$ for $j \in J''$, so that, if gambler i defaults $D_{i,j} \leq \tilde{D}(z_{s,j}^*, A_s^*)$, $\underline{\psi}''(z_{s,j}^*, A_s^*)$ is applied as the penalty. By J^* define the subset of J'' as follows: $J^* \subset J''$ such that $D_{i,j} \leq \tilde{D}(z_{s,j}^*, A_s^*)$. Then, if $j(s) \in J^*$, $\underline{\psi}''(z_{s,j(s)}^*, A_s^*) = \underline{\psi}(z_{s,j(s)}^*, A_s^*) - \rho/p_{j(s)}$ is applied. By using these facts, we verify that gambler i obtains at most the following deviation payoff:

$$
\begin{aligned}
& v(A_s^*) + \sum_{j \in J^*} p_j \{h_j^* a_{s,j}^* - A_s^* + \tilde{D}(z_{s,j}^*, A_s^*) - \underline{\psi}''(z_{s,j}^*, A_s^*)\} \\
& + \sum_{j \in J'' \setminus J^*} p_j \{h_j^* a_{s,j}^* - A_s^* + D_{i,j} - \tilde{\psi}''(D_{i,j}, z_{s,j}^*, A_s^*)\} \\
& + \sum_{j \notin J''} p_j \{h_j^* a_{i,j} - A_s^* + D_{i,j} - \tilde{\psi}''(D_{i,j}, z_{i,j}, A_s^*)\} + \hat{U}(S_s^*).
\end{aligned}
$$

Since $\tilde{\psi}''$ coincides with $\tilde{\psi}$ except only for $(z_{s,j(s)}^*, A_s^*)$, we can rewrite the above payoff by

$$
\begin{aligned}
& v(A_s^*) + \sum_{j \in J^*} p_j \{h_j^* a_{s,j}^* - A_s^* + \tilde{D}(z_{s,j}^*, A_s^*) - \underline{\psi}(z_{s,j}^*, A_s^*)\} \\
& + \sum_{j \in J'' \setminus J^*} p_j \{h_j^* a_{s,j}^* - A_s^* + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{s,j}^*, A_s^*)\} \\
& + \sum_{j \notin J''} p_j \{h_j^* a_{i,j} - A_s^* + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_s^*)\} + \hat{U}(S_s^*) + \mathbf{1}_{j(s)} \rho \\
& = v(A_s^*) + \sum_{j \in J} p_j \{h_j a_{i,j} - A_s^* + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_s^*)\} + \hat{U}(S_s^*) + \mathbf{1}_{j(s)} \rho
\end{aligned}
$$

where $\mathbf{1}_{j(s)}$ takes 1 only when $j(s) \in J^*$ and zero otherwise. Since $\tilde{\psi}$ implements $\mathbf{a}_i^*$, it must be that

$$
V_i^* \geq v(A_s^*) + \sum_{j \in J} p_j \{h_j a_{i,j} - A_s^* + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_s^*)\} + \hat{U}(S_s^*)
$$

for any $\mathbf{a}_i \neq \mathbf{a}_i^*$ with $A_i = A_s^*$ for any $s \in I$. Thus, the above deviation payoff cannot be larger than $V_i^* + \mathbf{1}_{j(s)}\rho$, which is less than the equilibrium payoff $V_i^* + \rho$ obtained under the new punishment policy $\tilde{\psi}''$. Therefore gambler i never makes the above deviation.

Finally, we set the exclusion probability q'' under $\tilde{\psi}''$ as follows: if $q_{i,j(i)}^* \equiv q(\tilde{D}(z_{i,j(s)}^*, A_i^*), z_{i,j(i)}^*, A_i^*) > 0$, then define $\underline{q}''(z_{i,j(i)}^*, A_i^*) \equiv q_{i,j(i)}^* - \varepsilon_{i,j(i)} \geq 0$ where $\varepsilon_{i,j(i)} \hat{U}(S_i^*) = \rho$. Otherwise define $\underline{q}''(z_{i,j(i)}^*, A_i^*) = 0$. Also define $q''(D, z, A) = q(D, z, A)$ for all $(z, A) \notin \{(z_{i,j(i)}^*, A_i^*)\}_{i \in I}$. Given these definitions, we set the utility-based penalty ξ'' as follows: if $q_{i,j(i)}^* > 0$, then

we set $\xi''(z^*_{i,j(i)}, A^*_i) = \xi(z^*_{i,j(i)}, A^*_i)$ while otherwise $\xi''(z^*_{i,j(i)}, A^*_i) \equiv \xi(z^*_{i,j(i)}, A^*_i) - \rho$ (because $\underline{\psi}(z^*_{i,j(i)}, A^*_i) > 0$). Also define $\xi''(D, z, A) = \xi(D, z, A)$ for all $(z, A) \notin \{(z^*_{i,j(i)}, A^*_i)\}_{i \in I}$. Then the bookmaker is never worse off by the newly defined punishment policy $\tilde{\psi}''$. Q.E.D.

10.4 Proof of Proposition 4

We set the candidate for the optimal solution to Problem B as follows:

$$\underline{\psi}(z^*, A^*) = 0,$$

$$\tilde{D}(z^*, A^*) \equiv \max\{0, \{\overline{V} - \{v(A^*) - fA^* + \hat{U}(S^*)\}\},$$
$$\tilde{D}(z, A) = 0, \quad \underline{\psi}(D, z, A) = \hat{U}(S) + H \ \text{ if } z \neq z^* \text{ or } A \neq A^*,$$

and

$$h^*_j = 1 + k^*/p_j, \quad j \in J$$

where $k^* \in (-p_1, \infty)$ is the unique solution to

$$\sum_{j \in J} \left(\frac{p_j}{p_j + k^*} \right) = 1/(1 - f). \tag{A3}$$

We now show that, by using the candidate for the optimal policy defined above, IC* becomes sufficient for (17) in the main text to hold under Assumption 1.

We first show that the odds defined above minimize the deviation payoff of gamblers $\overline{V}$. To this end, fix arbitrary odds $\{h^*_j\}_{j \in J}$. Then we show that some $\ell \in J$ exists such that $a_{i,\ell} = A_i$ and $a_{i,j} = 0$ for any $j \neq \ell$, and we can then re-write $\overline{V}$ as follows

$$\overline{V} = \max_{\mathbf{a}_i} v(A_i) + \max\{0, p_\ell(h^*_\ell - 1)A_i\} - H$$

where $\ell \in J$ maximizes $p_j(h^*_j - 1)$ over J. If this is not the case, some ℓ and j (with $\ell \neq j$) exist such that gambler i places bets of $a_{i,\ell} > 0$ and $a_{i,j} > 0$ on ℓ and j as well as $p_\ell(h^*_\ell a_{i,\ell} - A_i) \geq 0$ and $p_j(h^*_j a_{i,j} - A_i) \geq 0$ to achieve $\overline{V}$. Then the gambler can be however weakly better off by betting all A_i to either ℓ or j depending on whether or not to have $p_\ell h^*_\ell \geq p_j h^*_j$: when $p_\ell h^*_\ell \geq p_j h^*_j$, the gambler should choose $a_{i,\ell} = A_i$ and $a_{i,j} = 0$ to obtain a weakly higher payoff $p_\ell h^*_\ell A_i - p_\ell A_i \geq p_\ell(h^*_\ell a_{i,\ell} - A_i) + p_j(h^*_j a_{i,j} - A_i)$. Therefore, without loss of generality we can suppose that the gambler puts all A_i to horse ℓ that maximizes $p_j(h^*_j - 1)$ over $j \in J$.

We now consider the odds $\{h^*_j\}_{j \in J}$ to minimize the gambler's deviation payoff $\overline{V}$. Suppose that some j exists such that

$$\max_{j \in J} p_j(h^*_j - 1) > p_j(h^*_j - 1).$$

Then we can slightly reduce $h^*_\ell > 0$ for any ℓ for which $p_\ell(h^*_\ell - 1) = \max_{j\in J} p_j(h^*_j - 1)$ and slightly increase $h^*_j > 0$ for any $j \neq \ell$ while keeping $\sum_{j\in J}(1/h^*_j) = 1/(1-f)$ unchanged. This can strictly reduce $\overline{V}$ or make it unchanged. Therefore, without loss of generality we set

$$p_j(h^*_j - 1) = k, \quad \text{for any } j \in J$$

for some constant k. Then $h^*_j = 1 + k/p_j$ for $j \in J$ so that $1/(1-f) = \sum_{j\in J}(1/h^*_j) = \sum_{j\in J}(p_j/(p_j+k))$, which uniquely determines $k^* \in (-p_1, \infty)$.

We now show that IC* is not only necessary but also sufficient for (17) to hold under $\tilde{\psi}$ defined above. First, suppose that $A_i \neq A^*$. Then gambler i faces $\tilde{\psi}(D, z_{i,j}, A_i) = \hat{U}(S_i)+H$ for any $D \in [0, z_{i,j}]$ and any $j \in J$, so he optimally defaults all the debt; $D_{i,j} = z_{i,j}$. Thus gambler i obtains at most

$$\begin{aligned} &\max_{\mathbf{a}} v(A) + \max_{j\in J} p_j \max\{h^*_j - 1, 0\}A - H \\ &= \max_{A} \; v(A) + \max\{0, k^*\}A - H \\ &= \overline{V} \end{aligned}$$

which is however not larger than the equilibrium payoff V^* due to IC*. Second, suppose that $A_i = A^*$ but $z_{i,j} \neq z^*$ for some $j \in J$. Let $\tilde{J} \subseteq J$ bet the set of horses such that $z_{i,j} = z^*$ and hence $a_{i,j} = a^*_j$. Then gambler i faces $\tilde{\psi}(D_{i,j}, z_{i,j}, A^*) = \hat{U}(S^*) + H$ for any $D_{i,j} \in [0, z_{i,j}]$ and any $j \notin \tilde{J}$, and obtains at most

$$\max_{\mathbf{a}_i} v(A^*) + \sum_{j\in\tilde{J}} p_j\{h^*_j a^*_j - A^* + \tilde{D}^*(z^*, A^*)\} + \sum_{j\notin\tilde{J}} p_j\{\max\{h^*_j a_{i,j} - A^*, 0\} - \hat{U}(S^*) - H\} + \hat{U}(S^*)$$

subject to $\sum_{j\notin\tilde{J}} a_{i,j} = A^* - \sum_{j\in\tilde{J}} a^*_j$. Letting $\tilde{P} \equiv \sum_{j\in\tilde{J}} p_j$ and noting that $h^*_j a^*_j = (1-f)A^*$, this payoff is bounded above by

$$v(A^*) + \tilde{P}\{-fA^* + \tilde{D}^*\} + \max\left\{0, \max_{j\notin\tilde{J}} p_j h^*_j \left(A^* - \sum_{j\in\tilde{J}} a^*_j\right) - p_j A^*\right\} - (1-\tilde{P})(\hat{U}(S^*)+H) + \hat{U}(S^*)$$

which equals to

$$\begin{aligned} &v(A^*) + \tilde{P}\{-fA^* + \tilde{D}^*\} \\ &+ \max\left\{0, \max_{j\notin\tilde{J}} \left\{p_j h^*_j \left(1 - (1-f)\sum_{j\in\tilde{J}} \left(\frac{p_j}{p_j + k^*}\right)\right) - p_j\right\}\right\} A^* - (1-\tilde{P})(\hat{U}(S^*) + H) + \hat{U}(S^*) \end{aligned}$$

because $a_j^* = (1-f)A^*/h_j^*$ and $h_j^* = 1 + k^*/p_j$. Due to IC*, this payoff is not larger than the equilibrium payoff V^* if

$$\frac{1}{1-\tilde{P}} \max\left\{0, \left\{ p_j h_j^* \left(1 - (1-f) \sum_{j \in \tilde{J}} \left(\frac{p_j}{p_j + k^*} \right) \right) - p_j \right\} \right\} \leq \max\{k^*, 0\} \tag{A4}$$

for all $j \notin \tilde{J}$. When the maximum of the left hand side of (A4) is zero for all $j \notin \tilde{J}$, this inequality is trivially satisfied. Therefore, suppose that the above maximum is not zero for some $j \notin \tilde{J}$, implying that $0 < p_j h_j^* (A^* - \sum_{j \in \tilde{J}} a_j^*) - p_j A^*$. Then we have

$$\begin{aligned} 0 &< p_j h_j^* \left(A^* - \sum_{j \in \tilde{J}} a_j^* \right) - p_j A^* \\ &\leq p_j (h_j^* - 1) A^* \\ &= k^* \end{aligned}$$

so that $k^* > 0$. Therefore, by using $p_j h_j^* = k^* + p_j$, we can rewrite (A4) by

$$(p_\ell + k^*)(1-f) \sum_{j \in \tilde{J}} (p_j/\tilde{P}) \left(\frac{1}{p_j + k^*} \right) \geq k^* \tag{A5}$$

for any $\ell \notin \tilde{J}$ such that $p_\ell h_\ell^* (A^* - \sum_{j \in \tilde{J}} a_j^*) - p_\ell A^* > 0$. Since $p_j \leq p_m$ for all $j \neq m$, we obtain that

$$\sum_{j \in \tilde{J}} (p_j/\tilde{P}) \frac{1}{p_j + k^*} \geq \frac{1}{p_m + k^*}.$$

Since $p_1 \leq p_\ell$, the left hand side of (A5) is bounded below from

$$(1-f) \frac{p_1 + k^*}{p_m + k^*}.$$

Then the desired inequality (A5) is satisfied if

$$(1-f) \frac{p_1 + k^*}{p_m + k^*} \geq k^*. \tag{A6}$$

Also, since $\sum_j (p_j/p_j + k^*) = 1/(1-f)$ and $p_1 \leq p_j$ for all $j \neq 1$, we have $k^* \leq \hat{k} \equiv 1 - f - p_1$. We now define the following function

$$g(k) \equiv (1-f) \frac{p_1 + k}{p_m + k} - k$$

for $k \geq 0$, where $g(0) > 0$ and $g(\infty) = -\infty$ as well as $g'' < 0$. Then, inequality (A6) holds if $g(\hat{k}) \geq 0$ so that

$$(1-f)\frac{p_1 + \hat{k}}{p_m + \hat{k}} \geq \hat{k}.$$

This can be written by $(1-f)^2 \geq (1-f-p_1)(p_m - p_1 + 1 - f)$ so that $p_1(p_m - p_1) \geq (1-f)(p_m - 2p_1)$. When $p_m \leq 2p_1$, this inequality is satisfied. When $p_m \geq 2p_1$, this inequality holds if $p_1(p_m - p_1) \geq p_m - 2p_1$ (when $f = 0$), which is satisfied due to Assumption 1: $p_1 \geq (1-p_1)(p_m - p_1)$.

Therefore, IC* is sufficient for A^* to be implementable under Assumption 1. Then we can re-write Problem B by Problem B* given in the main text. This completes the proof of the first half of Proposition 3. Furthermore, as we have shown above, the equilibrium odds which solve Problem B are given by $h_j^* = k^*/p_j + 1$, completing the proof of the second half of Proposition 3. Q.E.D.

10.5 Proof of Proposition 5

Define $S^{fb} \equiv \alpha + \beta A_1^{fb}$ and $S^{**} \equiv \alpha + \beta A^{**}$. Then, under Assumption 2 we can verify that, when f is sufficiently small,

$$v(A^{fb}) - fA^{fb} + \hat{U}(S^{fb}) > \overline{V} \tag{A7}$$

and

$$v(A^{**}) - fA^{**} + \hat{U}(S^{**}) > \overline{V}. \tag{A8}$$

To see this, note that for the value of k^* satisfying (A3), we have $k^* \leq 1 - f - p_1 \leq 1 - f$ as shown in the proof of Proposition 3. Then under Assumption 2 we obtain $-fA + \hat{U}(S) > (1-f)A \geq \max\{k^*, 0\}A$ for all $A \geq 0$, and hence

$$\begin{aligned} \max_{A \geq 0} v(A) - fA + \hat{U}(S) &> \max_{A \geq 0} v(A) + \max\{k^*, 0\}A - H \\ &= \overline{V}. \end{aligned}$$

When $f \in (0,1)$ is sufficiently small, the payoff in the left hand side is close to

$$v(A^{**}) - fA^{**} + \hat{U}(S^{**}) \simeq v(A^{**}) + u(\hat{A}(S^{**}), S^{**}) = \max_A v(A) + u(\hat{A}(S), S)$$

thereby resulting in (A8). Also, when f is small, A^{**} and A_1^{fb} are close to each other as well, leading to (A7). We fix such small f in what follows.

Claim 1: $\overline{V} \leq v(A^b) - fA^b + \hat{U}(S^b)$ so that $\tilde{D}^* = 0$ holds at the optimum of Problem B*.

Proof. Suppose that

$$\overline{V} > v(A^b) - fA^b + \hat{U}(S^b)$$

so that $\tilde{D}^* = \overline{V} - \{v(A^b) - fA^b + \hat{U}(S^b)\}$. Then the bookmaker's payoff is given by

$$fA^b - \tilde{D}^* + f\hat{A}(S^b) = v(A^b) + u(\hat{A}(S^b), S^b) - \overline{V}.$$

If $A^b \neq A^{**}$, then the bookmaker can increase her payoff by changing A^b slightly without violating $\overline{V} > v(A^b) - fA^b + \hat{U}(S^b)$. Therefore we have $A^b = A^{**}$ but then

$$v(A^{**}) - fA^{**} + \hat{U}(S^{**}) < \overline{V}$$

contradicting to (A8). Therefore, it must be that

$$\overline{V} \leq v(A^b) - A^b + u(\hat{A}(S^b), S^b)$$

so that $\tilde{D}^* = 0$ holds at the optimum. Q.E.D.

Claim 2: $A^b = A_{\max} \equiv \max\{A \mid \overline{V} \leq v(A) - fA + \hat{U}(S)\}$.

Proof. Due to Claim 1, at the optimum of Problem B* the bookmaker's payoff is given by $fA^b + f\hat{A}(S^b)$ where $\overline{V} \leq v(A^b) - fA^b + \hat{U}(S^b)$. Since $fA + \hat{A}(S)$ is increasing in A, it must be that the bookmaker chooses the largest A among those such that $\overline{V} \leq v(A) - fA + \hat{U}(S)$. Q.E.D.

Claim 3: $A^b > A_1^{fb}$ and $\hat{A}(S^b) > A_2^{fb}$.

Proof. Suppose that $A_1^{fb} \geq A^b = A_{\max}$. First, if $A_1^{fb} = A_{\max}$, then

$$v(A_1^{fb}) - fA_1^{fb} + \hat{U}(S^{fb}) \geq \overline{V}.$$

However, we verify from (A7) that a slight increase from A_1^{fb} to A'' results in $v(A'') - fA'' + \hat{U}(S'') \geq \overline{V}$ so that we obtain $A_{\max} \geq A'' > A_1^{fb}$, contradicting to the supposition that $A_1^{fb} = A_{\max}$. Thus suppose that $A_1^{fb} > A_{\max}$. Then we have

$$v(A_1^{fb}) - fA_1^{fb} + \hat{U}(S^{fb}) < \overline{V} \tag{A9}$$

because otherwise inequality (A9) is reversed so that our supposition of $A_1^{fb} > A_{\max}$ contradicts to the definition of $A_{\max}$. However, then (A9) contradicts to (A7), and hence we obtain $A^b = A_{\max} > A_1^{fb}$.

Given the result that $A^b > A^{fb}$, we obtain $\hat{A}(S^b) \simeq A^*(S^b) > A^*(S^{fb}) = A_2^{fb}$ when $f > 0$ is sufficiently small where $A^*(S)$ is defined as A_2 maximizing $u(A_2, S)$, and thus $A_2^{fb} = A^*(S^{fb})$. Q.E.D.

10.6 Proof of Proposition 6

(i) We show that it becomes optimal for the bookmaker to hold the book-bet rather than the cash-bet in period 1, under Assumption 2. Holding the cash-bet in period 1 is equivalent to holding the book-bet in period 1 with the maximum forgiveness equal to $\tilde{D}(z, A) = 0$ for all $A \geq 0$ and all $z \geq 0$ as well as the following penalty: $\tilde{\psi}(D, z, A) = 0$ when $D = 0$ and $\tilde{\psi}(D, z, A) = \hat{U}(S) + H$ when $D > 0$ for any $z \geq 0$ and any $A \geq 0$.

To see this, note that under such book-bet gambler i never defaults: if gambler i defaults $D_{i,j} > 0$ when horse j wins the race, he will obtain the ex post payoff of $h_j a_{i,j} - A_i + D_{i,j}$ whereas, if he does not default, he will obtain $h_j a_{i,j} - A_i + \hat{U}(S_i)$. Since $0 < D_{i,j} \leq z_{i,j} \equiv \max\{A_i - h_j a_{i,j} 0\}$ and $-A_i + \hat{U}(S_i) > 0$ for all $A_i \geq 0$ under Assumption 2, the former payoff is smaller than the latter, $h_j a_{i,j} - A_i + D_{i,j} - H \leq 0 < h_j a_{i,j} - A_i + \hat{U}(S_i)$. Therefore, gambler i never defaults ex post. Given this result, gambler i bets $\mathbf{a}_i$ to maximize his expected payoff

$$v(A_i) + \sum_{j \in J} p_j \{h_j a_{i,j} - A_i\} + \hat{U}(S_i).$$

This is equivalent to the case that the bookmaker holds the cash-bet in period 1 as well as in period 2. Then the bookmaker can always choose the punishment policy under the book-bet that implements the same payoff as attained under the cash-bet. The bookmaker may be better off further by optimally choosing the punishment policy $\{\tilde{\psi}, \tilde{D}\}$. Therefore under Assumption 2 the bookmaker weakly prefers the book-bet to the cash-bet.

(ii) Define $\tilde{A}^*$ as the amount bet A maximizing $u(A) - fA = v(A) - fA$. Suppose that gamblers are not addicted to gambling at all: $\alpha = \beta = 0$. Then, we show that the bookmaker prefers the cash-bet to the book-bet in period 1, provided that $u(\tilde{A}^*) + H < 2f\tilde{A}^*$.

Recall that $\overline{u} \equiv \max_{A \geq 0} u(A) < +\infty$. Suppose that $\mathbf{a}^* \equiv (a_j^*)_{j \in J}$ and $A^* \equiv \sum_{j \in J} a_j^*$ are implemented in an equilibrium under a bang-bang punishment policy $\tilde{\psi}$. Note that the gambler's equilibrium payoff in period 2 is given by $\hat{U} = \max_A u(A) - fA$ when $S \equiv 0$. Then gambler i's equilibrium payoff is given by

$$\begin{aligned} & u(A^*) + \sum_{j \in J} p_j \{h_j^* a_j^* - A^* + \tilde{D}(z_j^*, A^*) - \underline{\psi}(z_j^*, A^*) + \hat{U}\} \\ = \; & u(A^*) - fA^* + \sum_j p_j \{\tilde{D}(z_j^*, A^*) - \underline{\psi}(z_j^*, A^*)\} + \hat{U} \end{aligned}$$

where the equilibrium odds are given by $h_j^* = (1 - f)A^*/a_j^*$.

Gambler i can always choose A_i to maximize $u(A)$, and default all the amounts required to pay back, $D_{i,j} = \max\{A_i - h_j^* a_{i,j}, 0\}$, followed by being imposed the largest penalty $\tilde{\psi}(D, z, A) = \hat{U} + H$, and then quitting the gambling in period 2. By doing this, gambler i can secure at least the payoff of $\overline{u} - H$. Therefore, for such a deviation to be unprofitable,

it must be that

$$u(A^*) - fA^* + \sum_j p_j\{\tilde{D}(z_j^*, A^*) - \underline{\psi}(z_j^*, A^*)\} + \hat{U} \geq \overline{u} - H$$

so that

$$\sum_j p_j \tilde{D}(z_j^*, A^*) \geq \overline{u} - \{u(A^*) - fA^* + \hat{U} + H\}.$$

Since each gambler bets $A_2 = \tilde{A}^*$ maximizing $u(A) - fA$ in period 2, the expected payoff of the bookmaker from using the book-bet in period 1 is given by

$$\begin{aligned}
\Pi_b &\equiv n\left\{A^* - \sum_j p_j\{h_j^* a_j^* + \tilde{D}(z_j^*, A^*)\}\right\} + nfA_2 \\
&= n\left\{fA^* - \sum_j p_j \tilde{D}(z_j^*, A^*)\right\} + nf\tilde{A}^* \\
&\leq n\{u(A^*) + \hat{U} - \overline{u} + H\} + nf\tilde{A}^*
\end{aligned}$$

where the equality follows from the fact that $\sum_j p_j h_j^* a_j^* = (1-f)A^*$. When the bookmaker uses the cash-bet in period 1 as well as in period 2, she obtains the payoff of $\Pi_c \equiv 2nf\tilde{A}^*$. If $\Pi_b \geq \Pi_c$, it must be then that

$$u(A^*) - \overline{u} + H + \hat{U} \geq f\tilde{A}^*.$$

Since $\overline{u} \geq u(A^*)$, this inequality implies that $\hat{U} + H \geq f\tilde{A}^*$ and hence $u(\tilde{A}^*) + H \geq 2f\tilde{A}^*$ due to the definition of $\hat{U} = u(\tilde{A}^*) - f\tilde{A}^*$, contradicting to the assumption that $2f\tilde{A}^* > u(\tilde{A}^*) + H$. Therefore, $\Pi_c > \Pi_b$ must be satisfied. Q.E.D.

11 Appendix B: Endogenous Commitment by the Bookmaker

In this appendix we extend the two-period model presented in the main text in such a way that the bookmaker endogenously self-enforces promised payouts and punishment policies.

11.1 Dynamic Game with A Sequence of Short-Lived Gamblers

Time is discrete and is extended over infinity $t = 1, 2, ...$ The bookmaker is a long-lived player who lives forever over time while gamblers are short-lived players who live only for one period (in sub-period 1 and sub-period 2 defined below).

To maintain the consistency with the two-period model, we suppose that each period t is divided into two sub-periods, which we call "sub-period 1" and "sub-period 2", as follows.

Sub-period 1 corresponds to the game played as "period 1" in the basic model while sub-period 2 corresponds to the game played as "period 2" in the basic model respectively. More specifically, in sub-period 1 of period t, (i) new gamblers are born and enter the market; (ii) the bookmaker chooses the cash-bet or the book-bet, and announces the punishment policy ψ; (iii) gamblers decide how much to bet; (iv) after the horse race outcome is realized, gamblers decide how much to default; (v) the bookmaker decides whether or not to pay the payouts that are determined by the horse race outcome as well as she decides whether or not to implement the announced punishment policy. In sub-period 2 of period t, (i) gamblers place bets under the cash-bet;[30] (ii) the bookmaker decides whether or not to renege on payouts; (iii) gamblers leave the economy and are replaced by newly born gamblers.

The bookmaker discounts her payoffs across successive two periods at $\delta \in [0, 1)$ while there are no time discounting between sub-period 1 and sub-period 2 within each period.

We introduce the "quitting option" between the bookmaker and each gambler i in the beginning of sub-period 1 and sub-period 2 of each period: the bookmaker decides whether or not to exercise the quitting option against gambler i (exclude him from the gambling) while gambler i decides whether or not to quit the gambling. The bookmaker and gamblers simultaneously make the quitting decision. Then, the relationship between the bookmaker and gambler i ends once at least one of them decides to quit. In that case the bookmaker obtains no revenues from gambler i while gambler i obtains the reservation payoff normalized to zero.

11.2 Information Structure

We consider the following information structure: each gambler i born in period t observes what punishment policy the bookmaker has offered and how much she has paid to him within the entire period of t. In addition, for each gambler i in period t, there exists a corresponding gambler, denoted by $\phi(i)$, born in next period $t+1$ who can observe whether or not the bookmaker and gambler i terminated their relationship in period t as well as how much payouts the bookmaker made to gambler i in sub-period 2 of period t. We then define $\phi(i), \phi(\phi(i)), \phi(\phi(\phi(i))), ...$ recursively.

We call the sequence of gamblers $\phi(i), \phi(\phi(i)), ...$ *successors* of gambler i. The role of gambler$i$'s successors is to punish the bookmaker when she deviated from the promised payouts or announced punishment policy against gambler i in each period: if the bookmaker deviates against gambler i in some period t, gambler i or/and his immediate successor $\phi(i)$ will punish the bookmaker by exercising the quitting option. Here successor $\phi(i)$ can know directly or indirectly the bookmaker's deviation against gambler i in period t: $\phi(i)$ observes whether or not gambler i terminated the relationship with the bookmaker in sub-period 1 and whether or not the bookmaker did not give the promised payouts to gambler i under the cash-bet in sub-period 2 within period t. The next successor $\phi(\phi(i))$ can observe that

[30] The bookmaker has no choices but the cash-bet in sub-period 2: gamblers never repay debts if the book-bet is used in sub-period 2 because they leave the economy in sub-period 2.

the relationship between the bookmaker and gambler $\phi(i)$ was terminated in period $t+1$. Then $\phi(\phi(i))$ will exercise the quitting option in period $t+2$, and so on. In this way all i's successors will punish the bookmaker once she has made the deviation against i in period t.

Let denote by $\phi^{-1}(i)$ the gambler whose immediate successor is i.

11.3 Self-Enforcing Equilibrium

Throughout the following, we will maintain Assumption 1-2 made in the main text, ensuring (i) that the book-bet becomes optimal for the bookmaker in sub-period 1 within each period and (ii) that the bookmaker's per period payoff is reduced to

$$\Phi(A) \equiv fA - \tilde{D}^* + f\hat{A}(S)$$

where $\tilde{D}^*$ satisfies the IC* constraint:

$$v(A^*) - fA^* + \tilde{D}^* + \hat{U}(S^*) \geq \overline{V}. \qquad \text{(IC}^*\text{)}$$

and $0 \leq \tilde{D}^* \leq z^* \equiv fA^*$.

We consider how the bookmaker implements an aggregate amount bet A^* per gambler in sub-period 1 within each period and obtains the corresponding payoff $\Phi(A^*)$ per gambler in each period even *without* formal commitment to promised payouts and punishment policy. To this end, we focus on the bang-bang punishment policy $\{\tilde{\psi}, \tilde{D}\}$ as used in the proof of Proposition 3 as follows.

(*) Punishment policy $\tilde{\psi}$:

$$\underline{\psi}(z^*, A^*) = 0,$$

$$\underline{\psi}(z, A) = \hat{U}(S) + H, \quad \text{for any } (z, A) \neq (z^*, A^*)$$

and the maximum forgiveness $\tilde{D}(z, A) \in [0, z]$ satisfies

$$\tilde{D}(z^*, A^*) = \max\{0, \overline{V} - \{v(A^*) - fA^* + \tilde{+}\hat{U}(S^*)\}\},$$

$$\tilde{D}(z, A) = 0, \quad \text{for any } (z, A) \neq (z^*, A^*)$$

together with the equilibrium odds:

$$h_j^* \equiv k^*/p_j + 1, \quad j \in J.$$

Define $\tilde{D}^* \equiv \tilde{D}(z^*, A^*)$. Note that $q(\tilde{D}^*, z^*, A^*) = \xi(\tilde{D}^*, z^*, A^*) = 0$ and that $q(D, z, A) = 1$ and $\xi(D, z, A) = \hat{U}(S)+H$ for any $(z, A) \neq (z^*, A^*)$ according to the bang-bang punishment policy $\tilde{\psi}$ defined above.

In addition to the IC* constraint, we need to take into account the *dynamic enforcement* (DE) constraint as discussed in the main text:

$$\frac{\delta\Phi(A^*)}{1-\delta} \geq (1-f)\hat{A}(S^*) \tag{DE}$$

where $S = \alpha + \beta A^*$.

Now consider the equilibrium in which in each period the bookmaker implements the aggregate amount bet A^* in sub-period 1 where each gambler places bet $a_j^* \equiv p_j A^*$ on each horse $j \in J$, and implements the aggregate amount bet $\hat{A}(S^*)$ in sub-period 2 from each gambler where $S^* \equiv \alpha + \beta A^*$. Recall that we set $h_j^* \equiv k^*/p_j + 1$ for the equilibrium odds of horse $j \in J$ in sub-period 1. We also define by $\hat{h}_j \equiv (1-f)/p_j$ for each $j \in J$ the equilibrium odds in sub-period 2 and $\hat{a}_j \equiv p_j \hat{A}(S^*)$ the associated equilibrium amount bet by each gambler on horse j under the cash-bet in sub-period 2 within each period. On the equilibrium path the bookmaker obtains $\Phi(A^*)$ per gambler within each period.

There are three possible deviations by the bookmaker: first, the bookmaker may renege on payouts for some gamblers under the book-bet in sub-period 1. Second, the bookmaker may not implement the announced punishment policy against some gamblers in the end of sub-period 1. Third, the bookmaker may renege on payouts for some gamblers under the cash-bet in sub-period 2.

In the first and second deviations we consider the punishment equilibrium as follows: when the bookmaker did not implement the announced punishment policy $\tilde{\psi}$ against gambler i or she did not give the payout of $R_{i,j} \equiv h_j a_{i,j} - A_i$ to gambler i in sub-period 1 of period t (given $R_{i,j} > 0$), she and gambler i exercise the quitting option simultaneously in the beginning of sub-period 2 of period t. In the beginning of period $t+1$ the bookmaker and i's successor $\phi(i)$ will exercise the quitting option simultaneously. Repeating this, all the successors of gambler i, $\phi(i), \phi(\phi(i)), ...$, and the bookmaker will terminate their relationships in the future. Therefore, the bookmaker will obtain no payoffs in any future period.

In the third deviation we consider the punishment equilibrium as follows: when the bookmaker did not give gambler i the promised payout $h_j a_{i,j}$ under the cash-bet in sub-period 2 of period t, all i's successors and the bookmaker will terminate their relationships in all the future periods. As a result, the bookmaker will lose all future payoffs from period $t+1$.

Define the following condition for $S_i = \alpha + \beta A_i$:

$$\frac{\delta}{1-\delta}\Phi(A^*) \geq (1-f)\hat{A}(S_i) \tag{DE-1}$$

By $D^*(z, A) \in [0, z]$ we also define the optimal deafult as follows: $D^*(z^*, A^*) = \tilde{D}^*$ and $D^*(z, A) = z$ for any $(z, A) \neq (z^*, A^*)$.

Then we consider the following strategy profile of the bookmaker and gambler i.

The bookmaker's strategy in period t:

- Sub-period 1:
  - The bookmaker exercises the quitting option against gambler i if she has deviated against gambler $\phi^{-1}(i)$ from the equilibrium payouts and/or punishment policy $\tilde{\psi}$ in the previous period. If the bookmaker has not deviated, she offers the book-bet and the punishment policy $\{\tilde{\psi}, \tilde{D}\}$.
  - Suppose that the bookmaker offered $\{\tilde{\psi}, \tilde{D}\}$ and that DE-1 holds for addiction stock S_i. Then the bookmaker pays $R_{i,j} \equiv \max\{h_j a_{i,j} - A_i, 0\}$ to gambler i when horse j wins the race if

$$\begin{aligned} & z_{i,j} - D^*(z_{i,j}, A_i) + (1 - q(D^*(z_{i,j}, A_i), z_{i,j}, A_i)) f\hat{A}(S_i) + \frac{\delta}{1-\delta}\Phi(A^*) \\ & \geq \ R_{i,j}, \end{aligned}$$

    given the winning odds $h_j > 0$ where $z_{i,j} \equiv \max\{A_i - h_j a_{i,j}, 0\}$.[31] Otherwise, she will not pay gambler i. Suppose that the bookmaker offered $\{\tilde{\psi}, \tilde{D}\}$ but DE-1 fails to hold given S_i. Then the bookmaker will not pay gambler i.
  - The bookmaker excludes the gamblers who defaulted according to the announced punishment policy $\tilde{\psi}$ regardless of how much she has paid to them.
- Sub-period 2:
  - If DE-1 holds, the bookmaker does not exercise the quitting option against gambler i and she quits otherwise.
  - When the bookmaker holds the cash-bet and the horse race outcome is realized, she will pay $h_j a_{i,j}$ to gambler i if

$$\frac{\delta}{1-\delta}\Phi(A^*) \geq h_j a_{i,j} \tag{DE-2}$$

    given the winning odds $h_j > 0$.[32] Otherwise, she will give no payouts to gambler i.

Gambler i's strategy in period t:

[31] We are here focusing on the situation in which market odds $h_j > 0$ are positive for all $j \in J$ again, as we discussed in footnote 13 of the main text: for example, each gambler places a small but positive bet $\varepsilon > 0$ on any horse by a mistake or other irrational reasons.

[32] Again we focus on the equilibrium in which $h_j > 0$ for all $j \in J$.

- Sub-period 1: Suppose that the either one of the following occurs: (i) the bookmaker has deviated against $\phi^{-1}(i)$ in the previous period; (ii) gambler $\phi^{-1}(i)$ has quit in the previous period; (iii) the bookmaker did not offer the book-bet and/or the equilibrium punishment policy $\{\tilde{\psi}, \tilde{D}\}$ in the current period. Then gambler i quits. Otherwise, he will place bet $\{a_j^*\}_{j\in J}$. When gambler i places the bet of $\{a_j^*\}_{j\in J}$ and the bookmaker pays the promised payout $h_j^* a_j^*$ for wining horse j, he will default $\tilde{D}(z_j^*, A^*)$. When gambler i places the bet of $a_{i,j} \neq a_j^*$, he will default all the debt of $z_{i,j}$.
- Sub-period 2: if DE-1 fails or the bookmaker deviates from the promised payout $R_{i,j} \equiv \max\{h_j a_{i,j} - A_i, 0\}$ in sub-period 1, gambler i will quit. Otherwise, he will place bet $\{\hat{a}_j\}_{j\in J}$ such that $A_i = \hat{A}(S_i)$ and $\hat{a}_j \equiv p_j \hat{A}(S_i)$ for $j \in J$.

When all the players follow the prescribed strategies, the bookmaker obtains the payoff of $\Phi(A^*)$ every period while each gambler born in period t bets the aggregate amount of A^* and defaults $\tilde{D}(z^*, A^*)$ in sub-period 1, followed by betting $\hat{A}(S^*)$ in total in sub-period 2. Then the equilibrium payoff of each gambler born in each period is given by

$$V^* \equiv v(A^*) - fA^* + \tilde{D}(z^*, A^*) + \hat{U}(S^*).$$

Given the equilibrium amount bet $\hat{A}(S^*)$ by each gambler in sub-period 2, for the bookmaker to honor the equilibrium payouts $(1-f)\hat{A}(S^*)$ for each gambler, DE-1 must be satisfied at $A_i = \hat{A}(S^*)$, yielding DE constraint. The best equilibrium which the bookmaker can implement is obtained to solve the following problem.

Problem E:

$$\max_{A^*\geq 0, 0\leq \tilde{D}^* \leq z^*} \Phi(A^*)$$

subject to the IC* constraint:

$$v(A^*) - fA^* + \tilde{D}^* + \hat{U}(S^*) \geq \overline{V} \tag{IC*}$$

and the DE constraint:

$$\frac{\delta \Phi(A^*)}{1-\delta} \geq (1-f)\hat{A}(\alpha + \beta A^*). \tag{DE}$$

We then show the following result.

Proposition B1. *Suppose that Assumption 1-2 are satisfied. Then there exists an equilibrium in which the bookmaker obtains the payoff of $\Phi(A^*)$ per gambler attaining the maximum in the Problem E every period.*

Proof. Consider the strategies of the bookmaker and gamblers described above.

Step 1. First, we consider sub-period 2 of period t, provided that DE-1 is satisfied so that the bookmaker and gambler i born in sub-period 1 of period t have not quit.

Suppose then that the bookmaker offered the cash-bet and then horse j won the race in sub-period 2. If the bookmaker pays $h_j a_{i,j}$ to gambler i, she expects to obtain the continuation value $\delta\Phi(A^*)/(1-\delta)$ from subsequent gamblers in future periods. If the bookmaker reneges on the payout of $h_j a_{i,j}$ against gambler i, all the successors of gambler i will quit the gambling, thereby yielding the payoff of zero to the bookmaker in the future. Therefore, it becomes optimal for the bookmaker to pay $h_j a_{i,j}$ to gambler i if (DE-2) is satisfied in sub-period 2 of period t.

Given this, we consider the incentive of gamblers in sub-period 2. Recall the equilibrium odds of $\hat{h}_j = (1-f)/p_j$ for $j \in J$, and take any profile of amounts bet by gambler i, $\mathbf{a}_i$ in sub-period 2 of each period. Then define the set of horses $\tilde{J}_i$ such that DE-2 holds at $h_j = \hat{h}_j \equiv (1-f)/p_j$ for each $j \in J$. When gambler i places bet $\mathbf{a}_i$ in sub-period 2, he will obtain the following expected payoff

$$u(A_i, S_i) + \sum_{j \in \tilde{J}_i} p_j \{\hat{h}_j a_{i,j} - A_i\} + \sum_{j \notin \tilde{J}_i} (-A_i)$$

because the bookmaker will renege on payouts for gambler i when horse $j \notin \tilde{J}_i$ wins the race. Then we verify that

$$\begin{aligned} & u(A_i, S_i) + \sum_{j \in \tilde{J}_i} p_j \{\hat{h}_j a_{i,j} - A_i\} + \sum_{j \notin \tilde{J}_i} (-A_i) \\ \leq\ & u(A_i, S_i) + \sum_{j \in J} p_j \{\hat{h}_j a_{i,j} - A_i\} \\ \leq\ & \hat{U}(S_i). \end{aligned}$$

On the contrary, if gambler i places the equilibrium bet of $\hat{a}_j \equiv p_j \hat{A}(S_i)$ for each $j \in J$, he can ensure that

$$\frac{\delta\Phi(A^*)}{1-\delta} \geq \hat{h}_j \hat{a}_j = (1-f)\hat{A}(S_i)$$

due to DE-1. Therefore, the bookmaker never reneges on payouts, implying that gambler i obtains the equilibrium payoff of $\hat{U}(S_i)$ in sub-period 2. Then gambler i never deviates from betting $\hat{\mathbf{a}}$ where $\hat{a}_j \equiv p_j \hat{A}(S_i)$ for each $j \in J$, given DE-1.

Suppose next that DE-1 fails. Then in the beginning of sub-period 2 the bookmaker and gambler i will terminate the relationship. Furthermore, i's successors will terminate the relationship as well in all the future periods, giving the bookmaker the continuation value of zero.

Step 2. Second, consider the incentive of the bookmaker in the end of sub-period 1,

provided that gambler i defaulted $D_{i,j}$. If the bookmaker follows the equilibrium strategy, she will exclude gambler i who defaulted $D_{i,j}$ according to the exclusion probability q specified in the punishment policy $\tilde{\psi}$. This in turn implies that the bookmaker will obtain the following continuation payoff from the end of sub-period 1:

$$(1 - q(D_{i,j}, z_{i,j}, A_i))f\hat{A}(S_i) + \frac{\delta\Phi(A^*)}{1-\delta} \geq 0 \tag{B1}$$

from gambler i if DE-1 holds and zero otherwise respectively. If the bookmaker deviates not to punish the defaulting gambler i, all i's successors will quit. Therefore, the bookmaker obtains the continuation value of zero, which is not however profitable by condition (B1).

Third, consider the incentive of the bookmaker in the stage in which she makes payouts to gamblers when horse j wins the race in sub-period 1 of period t. Recall that $z_{i,j} \equiv \max\{A_i - h_j a_{i,j}, 0\}$ denotes the net payback from gambler i to the bookmaker when horse j wins the race.

Suppose first that DE-1 holds. Recall that $D^*(z_{i,j}, A_i) \in [0, z_{i,}]$ is the optimal default by gambler i, given the punishment policy $\tilde{\psi}$ specified as (*), when he places the bet of A_i in total and owes the debt of $z_{i,j}$ in sub-period 1. Note that it becomes optimal for gambler i to choose $D^*(z^*, A^*) = \tilde{D}^*$ and $D^*(z, A) = z$ for any $(z, A) \neq (z^*, A^*)$, given the punishment policy $\tilde{\psi}$ specified as (*).

Then, if the bookmaker follows the promised payout of $R_{i,j} \equiv \max\{h_j a_{i,j} - A_i, 0\}$ to gambler i, she will obtain the following continuation value of her payoffs:

$$-R_{i,j} + z_{i,j} - D^*(z_{i,j}, A_i) + (1 - q(\tilde{D}(z_{i,j}, A_i), z_{i,j}, A_i))f\hat{A}(S_i) + \frac{\delta\Phi(A^*)}{1-\delta} \tag{B2}$$

from gambler i and his successors. Here, gambler i will optimally defaults the amount $D^*(z_{i,j}, A_i)$ and subsequently places the bet of $\hat{A}(S_i)$ in sub-period 2, provided that the bookmaker will optimally punish him according to the equilibrium punishment policy $\tilde{\psi}$ as well as she holds the cash-bet and never reneges on the equilibrium payout $\hat{h}_j\hat{a}_j$ in sub-period 2 due to the supposition that DE-1 is satisfied. On the contrary, if the bookmaker deviates from $R_{i,j}$, then she will obtain the continuation value of zero because gambler i will quit in sub-period 2 and all the successors of gambler i will quit as well. Note that, if gambler i will quit in sub-period 2, he always defaults on all the debt of $z_{i,j}$ in sub-period 1 and will never place bet in sub-period 2. Therefore, the bookmaker follows the promised payout $R_{i,j}$ when the above value (B2) is non-negative.

Suppose next that DE-1 fails in some period t. Then, the bookmaker and gambler i will simultaneously quit in the beginning of sub-period 2, followed by the termination of relationships between the bookmaker and i's successors in all the future periods. Therefore, the bookmaker and gambler i obtain the payoffs of zero from sub-period 2 of period t. Given this continuation outcome, gambler i defaults all the debt $z_{i,j}$ in sub-period 1 of period t. Anticipating this, the bookmaker reneges on payouts to gambler i in sub-period 1 of period t.

We now define by J_i^* the set of horses for which DE-1 is satisfied and (B2) becomes non-negative at $h_j^* \equiv k^*/p_j + 1$ for each $j \in J$, given a profile of amounts bet $\mathbf{a}_i$ by gambler i. Note that $h_j^* a_{i,j} > A_i$ must hold for any $j \notin J_i^*$, so $z_{i,j} = 0$ and hence $D_{i,j} = 0$ for any $j \notin J_i^*$.[33] Then, if gambler i places bet $\mathbf{a}_i$ in sub-period 1, he expects to obtain

$$v(A_i) + \sum_{j \in J_i^*} p_j \{h_j^* a_{i,j} - A_i + D^*(z_{i,j}, A_i) - \tilde{\psi}(z_{i,j}, A_i)\} + \sum_{j \notin J_i^*} p_j \{-\tilde{\psi}(0, z_{i,j}, A_i)\} + \hat{U}(S_i)$$

where the third bracket term captures the fact that for $j \notin J_i^*$ the bookmaker reneges on payouts followed by the termination of the relationship between herself and gambler i in sub-period 2 today. Since $h_j^* a_{i,j} > A_i$ for any $j \notin J_i^*$, we verify that for any $\mathbf{a}_i$ and any $D_{i,j} \in [0, z_{i,j}]$,

$$\begin{aligned}
&v(A_i) + \sum_{j \in J_i^*} p_j \{h_j^* a_{i,j} - A_i + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_i)\} + \sum_{j \notin J_i^*} p_j \{-\tilde{\psi}(0, z_{i,j}, A_i)\} + \hat{U}(S_i) \\
&\leq v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_i)\} \\
&\quad + \sum_{j \notin J_i^*} p_j \{h_j^* a_{i,j} - A_i - \tilde{\psi}(0, z_{i,j}, A_i)\} + \hat{U}(S_i) \\
&= v(A_i) + \sum_{j \in J} p_j \{h_j^* a_{i,j} - A_i + D_{i,j} - \tilde{\psi}(D_{i,j}, z_{i,j}, A_i)\} + \hat{U}(S_i) \\
&\leq v(A^*) + \sum_{j \in J} p_j \{h_j^* a_j^* - A^* + \tilde{D}(z^*, A^*) + \hat{U}(S^*)\} \\
&= v(A^*) - f A^* + \tilde{D}^* + \hat{U}(S^*) \\
&= V^*
\end{aligned}$$

where the last inequality follows from the same step as the proof of Proposition 3 (IC* is sufficient to detect gambler's deviation from $\mathbf{a}^*$ under Assumption 1). If gambler i places the bet of $\mathbf{a}^*$, then DE-1 holds and (B2) can be non-negative at $h_j = h_j^*$,[34] implying that gambler i obtains the equilibrium payoff of V^*. Thus, each gambler never deviates from choosing A^* in sub-period 1. Therefore, the bookmaker attains the payoff of $\Phi(A^*)$ in each period where A^* solves Problem E.

[33] When (B2) is negative, $R_{i,j} > 0$ must hold because $z_{i,j} \geq D^*(z_{i,j}, A_i)$, $\Phi(A^*) \geq 0$, and $\hat{A}(S_i) \geq 0$.
[34] This is because $R_{i,j} = h_j^* a_j^* - A^* = -f A^* < 0$.