Are New Keynesian Models Useful When Trend Inflation is Not Low?*

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Abstract

It is well-known that the steady state of the widely-studied New Keynesian (NK) models based on Calvo-pricing that are most suitable for monetary policy analysis does not exist beyond the 4% to 7% trend inflation threshold, rendering them not useful for monetary policy analysis when trend inflation is not low. This 'steady state problem,' where the optimal relative price and price dispersion shoot to infinity and output plummets to zero, is usually resolved by assuming price indexation or by increasing the price-adjustment frequency with trend inflation. These resolutions are, however, unsatisfactory and not supported by evidence. We show that the root of the steady state problem is not the Calvo pricing as commonly assumed, but rather the popular Dixit-Stiglitz demand structure in NK models. We provide sufficient conditions that any demand structure must satisfy for steady-state equilibrium to always exist independently of the level of trend inflation. We then propose a model with storage costs that satisfies the conditions to augment the Dixit-Stiglitz and Kimball-demand aggregators and show that it has relatively superior properties, bringing the corresponding demand functions closer to the micro and macro evidence. The model also predicts that if trend inflation is not sufficiently low, inflation rates will become much harder to tame and require monetary policy to be tightened for longer.

Keywords: New Keynesian models, Calvo pricing, trend inflation, steady state problem, demand functions.
JEL Codes: E31, E32, E52

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1 Introduction

The literature on positive trend inflation has long recognized that the New Keynesian Model with Calvo (1983) price setting does not have a solution if trend inflation exceeds some low single-digit threshold (in the 4% to 7% range).[1] However, empirical evidence shows that much larger long-run levels of inflation do persist in many countries for long periods of time, even in developed economies. Most of the existing literature agrees that this result arises because Calvo pricing implies that a firm might not receive the exogenous signal to re-optimize its price for a long period of time, even though with low probability.

Besides the well-known theoretical elegance in modelling nominal rigidity, the Calvo model also does a decent job of matching empirical micro evidence. In addition, the time-dependent Calvo model is also shown to be equivalent to a large class of state-dependent pricing models.[2] So, on both theoretical and empirical grounds, using Calvo pricing to characterize nominal rigidity remains popular not only in the academic literature on New Keynesian models but also in models used at central banks for monetary policy analysis. Thus, the non-existence of the steady state—the *steady state problem*—remains embedded at the core of this class of models.

In this paper, we show that the steady state problem under trend inflation arises not because of the Calvo pricing assumption but due to another modelling assumption commonly used in macroeconomic models, namely, the Dixit and Stiglitz (1977) constant-elasticity-of-substitution (CES) consumption aggregator. This assumption leads to a tractable constant-elasticity demand function for all goods in the economy and allows for adding monopolistic competition to macroeconomic models. However, it is actually in its implied demand function that the root of the steady state problem lies, as it diverges to infinite relative demand when relative prices approach zero. This property simultaneously creates several problems for the New Keynesian model: First, it creates a threshold (an upper bound) in the level of trend inflation consistent with the existence of a steady state for plausible model

[1] While the threshold is somewhat higher (7-9%) in models with strategic substitutability in pricing decisions such as King and Wolman (1996) and Ascari (2004), when strategic complementarities are present, as recommended in Woodford (2003), Bakhshi et al. (2023) show that the threshold becomes quite low, about 4-5%, especially if output growth is also taken into consideration. See the first part of Section 4 for an analytical discussion on this threshold. Cogley and Sbordone (2008) find that in the US, the time-varying trend in inflation was never above 5% (their Figure 1) between 1960 and 2003, and hence the condition for the existence of the steady state is satisfied for this low trend inflation period.

[2] Klenow and Kryvtsov (2008) show that the Calvo model matches six of the eight stylized facts in the microdata underlying the Consumer Price Index, being even better than some state-dependent models. In line with this result, Costain and Nakov (2011, 2023) build and test a model nesting both Calvo (time-dependent) and Golosov and Lucas (2007) fixed menu costs (state-dependent) models. They find that the parameterization that best fits microdata has low state dependence, implying a Phillips curve closer but not the same as the one implied by the Calvo model. Previously, Bakhshi et al. (2023) showed that the Calvo model approximates the inflation dynamics generated from the Dotsey et al. (1999) state-dependent model. More recently, Auclert et al. (2024) show that in a broad class of menu cost models, the first-order dynamics of aggregate inflation is first-order equivalent to a mixture of two time-dependent models (e.g. the Calvo model), reflecting the extensive and intensive margins of price adjustment.

parameters; second, it causes the marginal costs of non-readjusting firms and price dispersion to rise sharply when trend inflation approaches the upper bound; third, it makes the output level fall rapidly (almost vertically) as trend inflation approaches this limit.

To demonstrate the role of the popular CES demand structure in generating the steady state problem, we start by considering a model with general demand structures along with the Calvo pricing assumption. We prove that the condition for steady-state equilibrium to always exist *independently* of the level of trend inflation is that general demand structures remain finite when relative prices approach zero. The CES demand structure fails to satisfy this condition, which is the source of the steady state problem. This is the main theoretical result of our paper.[3]

We then show that our result can be applied to any preference structure or demand aggregator. In models with the Kimball (1995) aggregator, for example, Kurozumi and Van Zandweghe (2016, 2024) note that positive trend inflation does not cause the steady state problem. We provide the underlying reason why the Kimball aggregator avoids the steady state problem — it does so only when the curvature is sufficiently large. But the empirical evidence does not support large curvature, as shown in Klenow and Willis (2016) and Dossche et al. (2010). Assuming a large curvature leads to two additional issues. First, it truncates the distribution support of the relative prices and, therefore, cannot match the price distribution observed in microdata. Second, output rises with trend inflation above the level observed with flexible prices, which oddly means that the distorted economy performs better than the flexible price economy. In addition, the implied positive correlation between output and trend inflation is not supported by cross-country evidence either. Thus, simply replacing the Dixit-Stiglitz demand structure with the Kimball aggregator does not offer a compelling solution to the steady state problem.

We, therefore, propose a novel approach that considers demand functions for NK models consistent with *any* level of trend inflation. The method is based on including a feature in any demand structure, including commonly used ones like the Dixit and Stiglitz (1977) or the Kimball (1995). Our premise is that agents never face infinite demand, as consuming requires storage and processing costs that increase with the purchased volume—unbounded instantaneous consumption or instantaneous

[3]Our solution strongly departs from the two usual remedies to mechanically resolve the steady state problem. First, by assuming full price indexation to trend inflation. Empirical evidence from macro and microdata, however, suggests that there is very small indexation on individual prices (see e.g., Bils and Klenow (2004), Cogley and Sbordone (2005), Klenow and Kryvtsov (2008), Klenow and Malin (2010), and Levin et al. (2005)). So, this is not a satisfactory resolution. Second, by mechanically increasing the Calvo probability of price adjustment with trend inflation, that is, by introducing some state-dependence with respect to trend inflation. But, as Bakhshi et al. (2023) show, the elasticity of the Calvo probability with respect to trend inflation needs to be implausibly high. Put differently, one has to assume essentially that prices are near-flexible even at single-digit inflation rates, rendering the New Keynesian model not useful for any monetary policy analysis since nominal rigidity is essential to account for the effects of monetary policy on an economy.

reselling is never feasible. With storage costs, the augmented demand function based on Dixit-Stiglitz preferences avoids the steady state problem. Under Kimball aggregation with storage costs, the calibrated elasticities and superelasticities align more closely with micro evidence. Even when applied to the simpler CES aggregation, superelasticities rise to positive levels, inducing the demand function to have a smoothed-out kinked form that does not diverge to infinity. This feature allows it to be used with the Calvo model for all levels of trend inflation. Importantly, this property holds for any level of storage costs, no matter how small. We also find that storage costs strongly attenuate welfare losses and the increase in price dispersion as trend inflation rises, making them more in line with the findings of Nakamura et al. (2018) and Sheremirov (2020). We embed the augmented demand function based on Dixit-Stiglitz with storage costs in a textbook general equilibrium NK model as a proof of concept and study the dynamics at large rates of trend inflation. When the level of trend inflation exceeds about 10%, we find that inflation becomes less responsive to monetary policy, more persistent, and harder to tame, as it lingers much longer and requires a much greater output sacrifice to bring it down. These properties are in line with recent empirical results found by Canova and Forero (2024). The authors estimate a Markov-Switching model for the US with two states (high and low inflation) from 1960 to 2023. They find that, after contractionary monetary policy shocks, inflation rates do not fall as much and become more persistent in high-inflation states when compared to low-inflation states. These results align well with the experience of many countries whose inflation rates that have faced high double-digit inflation rates.

The remainder of the paper is organized as follows: Section 2 reviews related literature on micro and macro evidence. In Section 3, we present the model with a general preference structure and discuss how Calvo (1983) price setting is affected in this general framework. Section 4 presents the main result of our paper, in the form of a theorem that describes the general conditions that demand functions must satisfy so that there always exists a determinate steady-state equilibrium independently of the level of trend inflation. Section 5 discusses demand functions compatible with the theorem, assessing models such as Kimball aggregation (Section 5.1) and presenting another contribution of our paper, i.e., models with storage costs (Section 5.2). Section 6 presents simulations, and Section 7 concludes.

2 Micro and Macro Empirical Evidence

Before presenting the formal model, in which we consider a generic functional form for demand functions, we present some micro and macro evidence. This evidence gauges the features demand

functions should comprise and the macro predictions they should imply when used in NK models.

There is a vast empirical literature, based on micro data, strongly suggesting that actual demand functions are "kinked", in the sense that superelasticity (curvature) is positive. For context, often-used Dixit and Stiglitz (1977) demand functions have constant positive elasticity and zero superelasticity. Recent empirical literature using large scanner data generally finds relatively low, but still positive, values for elasticities (ξ) and superelasticities (η). Micro evidence shown below suggests that price elasticities likely range in $\xi^{micro} \in [1.0, 5.0]$, while superelasticities lie in the narrower interval of $\eta^{micro} \in [1.5, 2.0]$.

For the US, Burya and Mishra (2022) find the representative elasticity and superelasticity to be about $\xi = 4.8$ and $\eta = 1.8$.[4] For Europe, Dossche et al. (2010) find the median elasticity and superelasticity across all products and sectors are rather small, at $\xi = 1.4$ and $\eta = 0.8$,[5] whereas Beck and Lein (2015) find that the median (weighted mean) elasticities and superelasticities are $\xi = 3.6(2.1)$ and $\eta = 1.4(1.5)$.[6]

As for relative prices in the US, Kaplan and Menzio (2015) results suggest that the empirical distribution of relative prices[7] is approximately symmetric, leptokurtic (fat-tailed), and has large dispersion, even when controlling for exactly the same product (same UPC barcode - Universal Product Code) or allowing for strong substitutability. Under Brand Aggregation or Brand and Size Aggregation, in which products have at least the same features and the same size, aligning with what economists usually consider commodity goods, the authors find that the empirical standard deviations of relative prices, relative to the sample average price, are 0.25 (Brand aggregation[8]) and 0.36 (Brand and Size[9]).[10] Moreover, the authors' reported empirical quantile ratios for relative prices, i.e., $r_{(0.90,0.50)} = 1.38$ (Brand) and $r_{(0.90,0.50)} = 1.55$ (Brand and Size), allow us to conclude that the 90% quantile for US prices can be 38% or even 55% larger than the average price. These statistics are im-

[4] See Figures 1 and 3 in Burya and Mishra (2022). Values retrieved for the median Herfindahl-Hirschman Index (HHI) level in their sample (0.14). The authors use weekly firm-level data on prices and quantities (2007 to 2015) from a subsample of the ACNielsen Retail Scanner Database (35,000 US stores).

[5] The authors use bi-weekly scanner data with 15,000 items from Jan 2002 to Apr 2005.

[6] The authors use scanner data from Belgium (2,000 households), Germany (12,000 households), and the Netherlands (4,000 households), provided by AiMark (Advanced International Marketing Knowledge), with about 190,000 products and 2 million individual shopping trips from 2005 through 2008.

[7] The dataset contains 300 million transactions by 50,000 households for 1.4 million goods in 54 US geographic markets.

[8] In their Brand aggregation, products share the same features and the same size, but may have different brands and different Universal Product Code (UPC).

[9] In their Brand and Size aggregation, products are only required to share the same features, even though they might have different sizes, brands, or UPCs.

[10] Using a different dataset, Klenow and Willis (2016) find the standard deviation of relative prices to be 0.14 (regular prices) and 0.19 (posted prices: regular and sales). They also recognize that the standard deviation is higher if including price changes due to product turnover, seasonal changeovers, and temporary stockouts. The dataset the authors consider is a subset of the CPI Commodities and Services Survey (US Bureau of Labor Statistics), which includes 85,000 items per month. Specifically, they assess 14,000 price items from the three New York, Los Angeles, and Chicago from January 1988 to December 2004.

portant as they shed light on the distribution support of relative prices that demand functions used in economic models should be consistent with. We highlight that the authors' empirical histograms show that relative prices in both aggregation types have non-negligible masses even when prices are twice as large as the average price.

Turning to macro cross-sectional facts, we also present evidence on the effects of trend inflation that models should comply with. In short, we present evidence on how trend inflation affects consumption level and price dispersion. International evidence from different countries suggests that trend inflation is negatively correlated (not causal) with per capita consumption levels (e.g., Bleaney (1999)). Even though standard trend-inflation NK models with Dixit and Stiglitz (1977) aggregation also predict a fall in consumption as trend inflation rises (see e.g., Ascari (2004), Levin et al. (2007), Yun (2005), Bakhshi et al. (2023), Ascari and Sbordone (2014), Alves (2014, 2018), and Khan et al. (2020)), the predicted fall is implausibly strong.

The second important macro stylized fact is the relationship between the level of trend inflation and price dispersion. While standard trend-inflation NK models with Dixit and Stiglitz (1977) aggregation predict that price dispersion strongly increases with trend inflation, ultimately inducing welfare to plunge at low levels of long-run inflation, recent literature finds that this relationship is actually weak, even though still positive. That is, price dispersion slightly rises for larger levels of trend inflation. For instance, while Nakamura et al. (2018) find that the size of price changes did not increase in response to the Great Inflation of the late 1970s and early 1980s in the United States, Sheremirov (2020) finds that the positive relationship between price dispersion and inflation is only significant for regular prices. Sale prices, which are included in analyses with all prices, actually dampen this effect.

In the next section, we consider a formal model based on generic demand functions and staggered price setting, and show the conditions the former must satisfy for the general equilibrium to exist independently of the level of trend inflation. We aim to find demand functions that simultaneously satisfy those conditions and are consistent with the micro and macro evidence summarized in this section.

3 The Model

We follow Woodford (2003) to describe the standard new-Keynesian model with Calvo (1983) price setting and flexible wages. The economy consists of a representative infinite-lived household that consumes an aggregate bundle and supplies differentiated labor to a continuum of differentiated firms indexed by $z \in (0,1)$. Firms produce and sell goods in a monopolistic competition environment.

We depart from this structure by considering a broader class of demand functions.

3.1 Households

The representative household consumes $c_t(z)$ units of each differentiated good $z \in (0,1)$ at price $p_t(z)$. Consumption over all differentiated goods is aggregated into a bundle C_t. Prices across all firms are aggregated into a consumption price index P_t, which is defined as $P_t C_t \equiv \int_0^1 p_t(z)\, c_t(z)\, dz$.

The household supplies $h_t(z)$ hours of labor to each firm z, at a differentiated nominal wage $W_t(z) = P_t w_t(z)$, where $w_t(z)$ is the real wage. Disutility over hours is $v_t(z) \equiv \chi h_t(z)^{1+\nu} / (1+\nu)$, where ν^{-1} is the Frisch elasticity of labor supply. The household's aggregate disutility function is $v_t \equiv \int_0^1 v_t(z)\, dz$. The aggregate consumption bundle C_t provides utility $u_t \equiv \epsilon_t \left(C_t^{1-\sigma} - 1\right) / (1-\sigma)$, where σ^{-1} is the intertemporal elasticity of substitution and ϵ_t is a preference shock. The household's instantaneous utility is $u_t - v_t$.

Financial markets are complete. We consider a general budget constraint

$$P_t C_t + E_t q_{t+1} S_{t+1} + B_t \leq S_t + I_{t-1} B_{t-1} + P_t \int_0^1 w_t(z)\, h_t(z)\, dz + d_t \tag{1}$$

where E_t is the time-t expectations operator, S_t is the state-contingent value of the portfolio of financial securities held at the beginning of period t, B_t is the stock of government-issued bonds held at the end of period t, d_t denotes nominal dividend income, $I_t = (1+i_t)$ is the gross nominal interest rate at period t, i_t is the riskless one-period nominal interest rate, and q_{t+1} is the stochastic discount factor from $(t+1)$ to t.

The household chooses the sequence of C_t, $h_t(z)$, B_t, and S_{t+1} to maximize its welfare measure $\mathcal{W}t \equiv \max Et \sum_{\tau=t}^{\infty} \beta^{\tau-t} (u_\tau - v_\tau)$, subject to the budget constraint and a standard no-Ponzi condition, where β denotes the subjective discount factor. In equilibrium, the Lagrange multiplier λ_t on the budget constraint and the optimal labor supply function satisfy $\lambda_t = u_t' / P_t$ and $w_t(z) = v_t'(z) / u_t'$, where $u_t' \equiv \partial u_t / \partial C_t$ is the marginal utility of consumption, $v_t'(z) \equiv \partial v_t(z) / \partial h_t(z)$ is the marginal disutility of hours.[11] The optimal consumption plan and the dynamics of the stochastic discount factor, which satisfies $E_t q_{t+1} = 1/I_t$, are described by the following Euler equations

$$1 = \beta E_t \left(\frac{u_{t+1}'}{u_t'} \frac{I_t}{\Pi_{t+1}} \right) \quad ; \; q_t = \beta \frac{u_t'}{u_{t-1}'} \frac{1}{\Pi_t} \tag{2}$$

where $\Pi_t \equiv \frac{P_t}{P_{t-1}} = 1 + \pi_t$ is the gross inflation rate at period t.

[11] As usual, an equilibrium is defined as the equations describing the first-order conditions of the representative household and firms, a transversality condition $\lim_{T \to \infty} E_T q_{t,T} S_T = 0$, where $q_{t,T} \equiv \Pi_{\tau=t+1}^{T} q_\tau$, and the market clearing conditions.

3.1.1 General Demand Functions

Recall that P_t is the average price of the household's expenditure basket. In this regard, let $\wp_t(z) \equiv \frac{p_t(z)}{P_t}$ denote the relative price of firm z. For demand considerations, it is also convenient to define an additional vector of sufficient statistics $P_{s,t}$, describing the state of prices in the economy. It can be generally implicitly defined as a weighted average of individual prices, with state-dependent weights:

$$P_{s,t} \equiv \int_0^1 g\left(\wp_t(z), \wp_{s,t}\right) p_t(z)\, dz \tag{3}$$

where $\wp_{s,t} = \frac{P_{s,t}}{P_t}$ is the relative price of $P_{s,t}$ and $g\left(\wp(z), \wp_s\right)$ are weights, satisfying $g(1,1) = 1$, $g\left(\wp(z), \wp_s\right) \in (0,1)$, and $\int_0^1 g\left(\wp(z), \wp_s\right) dz = 1$. For instance, after considering a particular case of Kimball (1995) consumption aggregation, Dotsey and King (2005), Levin et al. (2007), Harding et al. (2022), and Kurozumi and Van Zandweghe (2024) find a utility-based demand function that depends not only on the aggregate price P_t but also the simple arithmetic average of prices $P_{s,t} = \int_0^1 p_t(z)\, dz$. In this particular case, $g\left(\wp t(z), \wp s, t\right) = 1$ for all $\wp_t(z)$ and $\wp_{s,t}$.

In the spirit of Gagliardone et al. (2023), we consider a general class of relative demand functions $\frac{c_t(z)}{C_t} = f\left(\wp_t(z), \wp_{s,t}\right)$, where $f\left(\wp, \wp_s\right)$ is continuous and differentiable in its domain, satisfying $f\left(\wp, \wp_s\right) \geq 0$, $f(1,1) = 1$ and $f_1\left(\wp, \wp_s\right) \leq 0$, $\forall\left(\wp, \wp_s\right)$ in its domain, where $f_1\left(\wp, \wp_s\right) \equiv \frac{\partial f(\wp, \wp_s)}{\partial \wp}$. A particular case of $P_{s,t}$ is the mean price $P_{m,t} \equiv \int_0^1 p_t(z)\, dz$.

Since the aggregate price satisfies $P_t = \int_0^1 p_t(z) \frac{c_t(z)}{C_t} dz$, we obtain a general formulation for P_t:

$$P_t \equiv \int_0^1 f\left(\wp_t(z), \wp_{s,t}\right) p_t(z)\, dz \tag{4}$$

where we assume that $P_{s,t}$ and P_t grow at the same rate in the steady state. Finally, firm z's price elasticity $\xi_t(z) \equiv -\frac{p_t(z)}{c_t(z)} \frac{\partial c_t(z)}{\partial p_t(z)}$ and the superelasticity of demand $\eta_t(z) \equiv \frac{p_t(z)}{\xi_t(z)} \frac{\partial \xi_t(z)}{\partial p_t(z)}$ are:

$$\xi_t(z) = -\frac{f_1\left(\wp_t(z), \wp_{s,t}\right)}{f\left(\wp_t(z), \wp_{s,t}\right)} \wp_t(z) \quad ; \eta_t(z) = 1 + \xi_t(z) + \frac{f_{11}\left(\wp_t(z), \wp_{s,t}\right)}{f_1\left(\wp_t(z), \wp_{s,t}\right)} \wp_t(z) \tag{5}$$

3.2 Price Setting

Each firm $z \in (0,1)$ produces a differentiated good using the technology $y_t(z) = \mathcal{A}_t h_t(z)^{\varepsilon}$, where $h_t(z)$ is its demand for labor, $\mathcal{A}_t$ is the aggregate technology shock and $\varepsilon \in (0,1)$. The market clearing condition $c_t(z) = y_t(z)$, $\forall z$, implies that the aggregate output across all firms $z \in (0,1)$ satisfies $Y_t = C_t$.

Since firm-specific hours $h_t(z)$ are the only production input, the firm's real payroll cost is $co_t(z) =$

$w_t(z)\, h_t(z)$. Taking wages as given, the firm's real marginal cost $mc_t(z) \equiv \frac{\partial co_t(z)}{\partial y_t(z)}$ is:

$$mc_t(z) = w_t(z)\frac{\partial h_t(z)}{\partial y_t(z)} = \frac{\chi}{\varepsilon}\frac{(Y_t)^{(\sigma+\omega)}}{\epsilon_t(\mathcal{A}_t)^{(1+\omega)}}\left[f\left(\wp_t(z), \wp_{s,t}\right)\right]^{\omega}$$

where $\omega \equiv \frac{(1+\nu)}{\varepsilon} - 1$ is a composite parameter. As for the firm's real payroll cost, it can be written as $co_t(z) = \frac{(Y_t)^{(1+\sigma+\omega)}}{\epsilon_t(\mathcal{A}_t)^{(1+\omega)}}\left[f\left(\wp_t(z), \wp_{s,t}\right)\right]^{(1+\omega)}$.

Under flexible prices, all firms set the same price when maximizing profits $p_t(z)\, y_t(z) - P_t co_t(z)$. Optimal pricing requires $\left(1 - \frac{1}{\xi_t^n(z)}\right)\wp_t^n(z) = mc_t^n(z)$, where superscript '$n$' denotes natural equilibrium and $\xi_t^n(z)$ is the firm price-demand elasticity. Since all optimal prices are the same, $\xi_t^n(z) = \xi^n$ is constant and we have $\wp_t^n(z) = 1$, $\wp_{s,t}^n = 1$, and $f\left(\wp_t^n(z), \wp_{s,t}^n\right) = 1$. Therefore, δ affects the monopolistic static markup $\mu \equiv \frac{\wp_t^n}{mc_t^n}$ under flexible prices:

$$\mu = \frac{1}{\left(1 - \frac{1}{\xi^n}\right)} \tag{6}$$

In addition, under flexible prices, all firms produce the same level in equilibrium $y_t^n = Y_t^n$, where Y_t^n is the natural output:

$$\left(Y_t^n\right)^{(\sigma+\omega)} = \frac{1}{\mu}\frac{\varepsilon}{\chi}\epsilon_t(\mathcal{A}_t)^{(1+\omega)} \quad ; \xi^n = -f_1(1,1) > 0 \tag{7}$$

This result also implies that, in general price settings, the total real and marginal costs are:

$$\begin{aligned} co_t(z) &= \left(\frac{1}{\mu}\frac{\varepsilon}{\chi}\right)^{\frac{(1+\sigma+\omega)}{(\sigma+\omega)}}\left(\epsilon_t(\mathcal{A}_t)^{(1+\omega)}\right)^{\frac{1}{(\sigma+\omega)}}(X_t)^{(1+\sigma+\omega)}\left[f\left(\wp_t(z), \wp_{s,t}\right)\right]^{(1+\omega)} \\ mc_t(z) &= \frac{1}{\mu}(X_t)^{(\sigma+\omega)}\left[f\left(\wp_t(z), \wp_{s,t}\right)\right]^{\omega} \end{aligned} \tag{8}$$

where $X_t \equiv \frac{Y_t}{Y_t^n}$ is the gross output gap.

As under the standard Calvo (1983) pricing, with probability $(1-\alpha)$, the firm optimally readjusts its price to $p_t(z) = p_t^*$. With probability α, the firm sets the price with partial indexation according to $p_t(z) = p_{t-1}(z)\,\Pi_t^{ind}$, where $\Pi_t^{ind} \equiv \Pi_{t-1}^{\gamma}$ is the gross indexation rate, $\Pi_t \equiv \frac{Pt}{P_{t-1}}$ is the gross inflation rate, and $\gamma \in (0,1)$.[12] When optimally readjusting at period t, price $p_t(z) = p_t^*$ maximizes the present value of nominal profit flows $E_t \sum_{j=0}^{\infty} \alpha^j q_{t,t+j}\left[\Pi_{t,t+j}^{ind} p_t(z)\, y_{t+j}(z) - P_{t+j} co_{t+j}(z)\right]$, given the demand function and the price setting structure, where $q_{t,t+j}$ is the cumulated nominal stochastic discount factor from period $(t+j)$ to t, recursively defined as $q_{t,t} = 1$, $q_{t,t+1} = q_{t+1}$, and $q_{t,t+j} \equiv$

[12] We allow for price indexation even though empirical evidence from macro and micro data suggests that there is very small indexation on individual prices. Full indexation is the particular case in which $\gamma = 1$.

$q_{t+1}q_{t+1,t+j}$ for $j \geq 1$.

If firm z has last optimally readjusted its price at period t, its marginal cost and demand functions at $(t+j)$ are $mc_{t,t+j}(z) = \frac{1}{\mu}\left(X_{t+j}\right)^{(\sigma+\omega)}\left[f\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\frac{p_t(z)}{P_t}, \wp_{s,t+j}\right)\right]^{\omega}$ and $\frac{y_{t+j}(z)}{Y_{t+j}} = f\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\frac{p_t(z)}{P_t}, \wp_{s,t+j}\right)$,[13] where $\Pi_{t,t+j}$ and $\Pi^{ind}_{t,t+j}$, for $j \geq 1$, are the cumulated gross inflation and indexation rates from period t to $(t+j)$, recursively defined as $\Pi_{t,t} = \Pi^{ind}_{t,t} = 1$, $\Pi_{t,t+1} = \Pi_{t+1}$, $\Pi^{ind}_{t,t+1} = \Pi^{ind}_{t+1}$, $\Pi_{t,t+j} \equiv \Pi_{t+1}\Pi_{t+1,t+j} = \Pi_{t,t+j-1}\Pi_{t+j}$, and $\Pi^{ind}_{t,t+j} \equiv \Pi^{ind}_{t+1}\Pi^{ind}_{t+1,t+j} = \Pi^{ind}_{t,t+j-1}\Pi^{ind}_{t+j}$. Most importantly, note that $mc_{t,t+j}(z)$ is not the marginal cost $mc_{t+j}(z)$, as the former depends on the state at period t and cumulated rates from t to $(t+1)$.

In this context, all optimally readjusting firms have the same first order condition for $p_t(z) = p^*_t$ in equilibrium. Therefore, it can be conveniently written as in the following system:

$$\begin{aligned}
0 = \quad & E_t \sum_{j=0}^{\infty} \alpha^j q_{t,t+j}\mathcal{G}_{t,t+j}\Pi^{ind}_{t,t+j} f\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\wp^*_t, \wp_{s,t+j}\right) \\
+ \quad & E_t \sum_{j=0}^{\infty} \alpha^j q_{t,t+j}\mathcal{G}_{t,t+j}\Pi^{ind}_{t,t+j}\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\wp^*_t\right) f_1\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\wp^*_t, \wp_{s,t+j}\right) \\
- \quad & E_t \sum_{j=0}^{\infty} \alpha^j q_{t,t+j}\mathcal{G}_{t,t+j}\Pi^{ind}_{t,t+j} f_1\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\wp^*_t, \wp_{s,t+j}\right)\frac{mc^*_{t,t+j}}{(1+\delta)} \\
mc^*_{t,t+j} = \quad & \frac{1}{\mu}\left(X_{t+j}\right)^{(\sigma+\omega)}\left[f\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\wp^*_t, \wp_{s,t+j}\right)\right]^{\omega}
\end{aligned} \tag{9}$$

where $\wp^*_t \equiv \frac{p^*_t}{P_t}$, $\mathcal{G}_t \equiv \frac{Y_t}{Y_{t-1}}$ denotes the gross output growth rate, and $\mathcal{G}_{t,t+j}$ is the cumulated gross growth rate, defined as $\mathcal{G}_{t,t} = 1$, $\mathcal{G}_{t,t+1} = \mathcal{G}_{t+1}$, and $\mathcal{G}_{t,t+j} \equiv \mathcal{G}_{t+1}\mathcal{G}_{t+1,t+j}$ for $j \geq 1$.

Note that infinite sums involving $f\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\wp^*_t, \wp_{s,t+j}\right)$, $f_1\left(\frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\wp^*_t, \wp_{s,t+j}\right)$ and $mc^*_{t,t+j}$ do not generally allow for recursive representations, and so steady state computations must be done numerically after considering a finite sum $j = \{0, ..., J\}$, for a large J. This is true even in commonly used models based on Kimball aggregation. In this regard, in Appendix C we propose a very flexible approach to approximate each sum into components that allows for recursive forms. This approximation is quite useful for deriving log-linearizations.

Lastly, price aggregations (3) and (4) imply

$$\wp_{s,t} = (1-\alpha)\sum_{j=0}^{\infty}\alpha^j g\left(\frac{\Pi^{ind}_{t-j,t}}{\Pi_{t-j,t}}\wp^*_{t-j}, \wp_{s,t}\right)\frac{\Pi^{ind}_{t-j,t}}{\Pi_{t-j,t}}\wp^*_{t-j} \tag{10}$$

$$1 = (1-\alpha)\sum_{j=0}^{\infty}\alpha^j f\left(\frac{\Pi^{ind}_{t-j,t}}{\Pi_{t-j,t}}\wp^*_{t-j}, \wp_{s,t}\right)\frac{\Pi^{ind}_{t-j,t}}{\Pi_{t-j,t}}\wp^*_{t-j} \tag{11}$$

[13] That is, considering cumulative indexation, the relative price is $\frac{\Pi^{ind}_{t,t+j}p_t(z)}{P_{t+j}} = \frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\frac{p_t(z)}{P_t}$.

3.3 Quarterly Benchmark Calibration

We calibrate the model parameters at the quarterly frequency. As in Cooley and Prescott (1995), we set the subject discount factor at $\beta = 0.99$ and the elasticity to hours at the production function at $\varepsilon = (1 - 0.36)$. We set $\alpha = 0.60$ as the degree of price stickiness, which is consistent with micro and macro evidence.[14] Since empirical evidence from macro and micro data suggests that there is non-existent or very small indexation on individual prices, we set $\gamma = 0$.[15] Using central estimates (the modes of the posterior distributions) obtained by Smets and Wouters (2007), we set the reciprocal of the elasticity of intertemporal substitution at $\sigma = 1.39$. As for the reciprocal of the Frisch elasticity, we set it at $\nu = 1$ for a compromise between micro estimates and macro evidence on total hours fluctuation over the business cycle.[16] Finally, based on median estimates from Cogley and Sbordone (2008) and Ascari and Sbordone (2014), we set the monopolistic static markup of $\mu = 1.12$.[17]

4 Steady State Convergence

For notation purposes, barred variables stand for steady state levels. We want to assess which conditions ensure a determinate steady-state equilibrium with no output growth, i.e. $\bar{\mathcal{G}} = 1$, for different levels of trend inflation $\bar{\Pi} = (1 + \bar{\pi})$. We define determinate steady-state equilibria as those in which all infinite summations in the steady-state equation implied by (9) converge.

Except for the general demand function, the model previously described is otherwise a typical example of the standard New Keynesian framework with Calvo staggered price setting: it has monopolistic competition, standard functional forms, only one source of nominal rigidity and shocks to preferences and technology.

Under those circumstances, the generally accepted paradigm in the literature on trend inflation is that there is a low upper limit for trend inflation consistent with a determinate steady-state equi-

[14] Nakamura and Steinsson (2008), using microdata from 1988 to 2005, estimate the median duration between price changes at roughly 4.5 months (including sales) and 10 months (excluding sales). Their findings are similar to those obtained in Bils and Klenow (2004). The median durations τ_m are consistent with $\alpha = 0.63$ and $\alpha = 0.81$ in quarterly fequency, using $\tau_m = -\log(2) / \log(\alpha)$. As for the macro evidence, Cogley and Sbordone (2008), for instance, report $\alpha = 0.588$ as their median estimate, while Smets and Wouters (2007) report $\alpha = 0.65$ as the mode estimate, using the full sample period from 1966:1 to 2004:4.

[15] For instance, this evidence is found in Bils and Klenow (2004), Cogley and Sbordone (2005), Cogley and Sbordone (2008), Klenow and Kryvtsov (2008), Klenow and Malin (2010), Levin et al. (2005) and Smets and Wouters (2007).

[16] In this regard, even though Chetty et al. (2011) finds a smaller value for ν^{-1} (i.e. a larger value for ν) on the micro side, recent evidence suggests that earlier estimates of micro elasticities for ν^{-1} might be downwardly biased, as their inference approaches did not account for important features in households composition between: (i) male and female workers; (ii) age; and (iii) primary and secondary earners. See, for example, Keane and Rogerson (2012), Peterman (2016-01), and Bredemeier et al. (2023)

[17] In a Dixit and Stiglitz (1977) aggregation model, with the elasticity of substitution set at $\theta = 9.5$, the markup is $\mu = \frac{\theta}{(\theta-1)} = 1.12$.

librium (see e.g. Ascari and Sbordone (2014)). Indeed, in the particular case of $\delta = 0$ and consumption aggregation as in Dixit and Stiglitz (1977), we obtain the usual relative demand function $\frac{y_t(z)}{Y_t} = (\wp_t(z))^{-\theta}$ considered in standard New Keynesian frameworks, where $\theta > 1$ is the elasticity of substitution between goods, satisfying $\theta = \frac{\mu}{(\mu-1)}$. In the standard NK model, given a trend inflation level $\bar{\Pi}$, the steady state equilibrium with no output growth only exists if $\bar{\Pi} < \min\left[\left(\frac{1}{\alpha}\right)^{\frac{1}{(\theta-1)(1-\gamma)}}, \left(\frac{1}{\alpha\beta}\right)^{\frac{1}{\theta(1+\omega)(1-\gamma)}}\right]$. Using the benchmark calibration, the annualized upper limit for trend inflation is 7.40%. If we had assumed a calibration more compatible with micro evidence in the labor market,[18] with $\nu = \frac{1}{0.59} = 1.69$, the annualized upper limit would be much smaller, at 5.44%.

We formally show below, this inflation upper bound in standard New Keynesian models arises because the usual Dixit and Stiglitz (1977) demand function $f(\wp_t(z), \wp_{s,t}) = (\wp_t(z))^{-\theta}$ and its derivative $f_1(\wp_t(z), \wp_{s,t}) = -\theta(\wp_t(z))^{-(\theta+1)}$ have a singularity point at $\wp_t(z) \to 0$. And so, if the relative price approaches zero, this relative demand diverges to infinite. This is exactly what happens in between optimal price readjustments in case of $\gamma < 1$. After j periods since last optimal readjustment, with probability α^j, we have $\wp_{t+j}(z) = \frac{\Pi^{ind}_{t,t+j}}{\Pi_{t,t+j}}\wp^*_t$. If inflation is positive on average, $\Pi_{t,t+j}$ grows faster than $\Pi^{ind}_{t,t+j}$ as j gets larger, which in turn causes $\wp_{t+j}(z)$ to approach zero. As a consequence, if $\bar{\Pi}$ is not sufficiently small, relative demand grows faster than the probability-adjusted discounting rate, causing all infinite summation terms to diverge.

As discussed below, the convergence at any level of trend inflation requires a weak restriction in the relative demand function. Intuitively, if the relative demand function is never infinite (in absolute value), including at zero relative prices $\wp_t(z) \to 0$, then the infinite sums in the price setting system converge in the steady state for virtually all levels of trend inflation. This idea is formalized in the Theorem 1 presented below.

Assumptions: Under the Calvo staggered price setting ($\alpha > 0$) with partial indexation ($\gamma < 1$), as previously described, consider the generic relative demand function $\frac{y}{Y} = f(\wp, \wp_s)$ described in Section 3.1.1, where $\wp_s \equiv \frac{P_s}{P}$, where P_s and P grow at the same rate in any steady state, such that $f(\wp, \wp_s)$ is a non-negative, continuous and differentiable function in $(\wp, \wp_s) \in (\mathbb{R}^*_+ \times \mathbb{R}^*_+)$ and non-increasing in $\wp \in R^*_+$. Let $f_1(\wp, \wp_s) \equiv \frac{\partial f(\wp, \wp_s)}{\partial \wp}$ denote the partial derivative of f with respect to $\wp$.

Theorem 1 *If $f(\wp, \wp_s)$ and $\wp \cdot f_1(\wp, \wp_s)$ are finite and defined at all their domain, including at $\wp \to 0$ and $\wp \to \infty$,[19] there always exists a steady state equilibrium (with no output growth) for any level of trend inflation ($\bar{\Pi} = 1 + \bar{\pi}$), provided that it is not extremely negative, i.e. $\bar{\Pi} > (\alpha)^{\frac{1}{(1-\gamma)}}$. For any other level of*

[18] See Table 1 in Chetty et al. (2011).

[19] Formally, $0 \leq \lim_{\wp \to 0} f(\wp, \wp_s) < \infty$, $0 \leq \lim_{\wp \to \infty} f(\wp, \wp_s) < \infty$, $-\infty < \lim_{\wp \to 0} \wp \cdot f_1(\wp, \wp_s) \leq 0$, and $-\infty < \lim_{\wp \to \infty} \wp \cdot f_1(\wp, \wp_s) \leq 0$.

trend inflation, including all positive values, both the optimal relative price and the output-gap converge to finite steady state levels.

The proof is shown in Appendix A.

Note that the requirement that the threshold level of inflation be greater than a certain negative value, $\bar{\pi} > \left[(\alpha)^{\frac{1}{(1-\gamma)}} - 1\right]$, is easily satisfied. We do not observe extremely negative levels of trend inflation in the data. For instance, if $\alpha = 0.60$, $\beta = 0.99$, and $\gamma = 0$, steady state levels cease to exist if $\bar{\pi} < (\alpha\beta - 1) = -40\%$ in quarterly frequency (-87% annually). Therefore, the feasibility inequality in Theorem 1 does not pose a practical restriction. Of course, the existence of a steady-state equilibrium does not preclude it to be inconsistent with economic reasoning in terms of sign and scale of equilibrium levels of all endogenous variables. For instance, we must still impose usual restrictions on parametrization to ensure positive levels of output.

Hahn (2022) adopts an alternative approach to cope with the fact that the Dixit and Stiglitz (1977) demand function diverges to infinity when relative prices approaches zero. His approach, however, is to keep the standard demand function while allowing firms not to satisfy demand all the times. In a continuous time model with sticky prices, the author introduces an optimal rationing mechanism, by curtailing supply to its optimal level.

By contrast, our approach is to investigate the root of the steady state problem, and propose conditions for demand functions in the standard approach used in the literature of supply meeting demand at any price level in equilibrium.[20]

At this point, once ensuring that a meaningful economic steady state exists, we can also assess what the general demand function $\frac{y_t(z)}{Y_t} = f\left(\wp_t\left(z\right), \wp_{s,t}\right)$ implies for real rigidity in price setting. In the natural equilibrium, i.e. flexible prices ($\alpha = 0$), system (9) simplifies into

$$f\left(\wp_t^n, \wp_{s,t}^n\right) + \left(\wp_t^n\right) f_1\left(\wp_t^n, \wp_{s,t}^n\right) = \left(1 - \frac{1}{\xi^n}\right) f_1\left(\wp_t^n, \wp_{s,t}^n\right) \left[f\left(\wp_t^n, \wp_{s,t}^n\right)\right]^{\omega} \left(X_t\right)^{(\sigma+\omega)}$$

Using the fact that $f\left(1,1\right) = 1$, the last relation is easily log-linearized about the economy steady-state with flexible prices, in which $\overline{\wp}^n = \overline{\wp}_s^n = 1$:

$$\widehat{\wp}_t^n = -\frac{\left[\frac{f_2(1,1)+f_{12}(1,1)}{1+f_1(1,1)} - \frac{f_{12}(1,1)}{f_1(1,1)} - \omega f_2(1,1)\right]}{\left[\frac{2f_1(1,1)+f_{11}(1,1)}{1+f_1(1,1)} - \frac{f_{11}(1,1)}{f_1(1,1)} - \omega f_1(1,1)\right]} \widehat{\wp}_{s,t} + \frac{(\sigma+\omega)}{\left[\frac{2f_1(1,1)+f_{11}(1,1)}{1+f_1(1,1)} - \frac{f_{11}(1,1)}{f_1(1,1)} - \omega f_1(1,1)\right]} \hat{x}_t$$

[20] However, if we were to consider the more general case in which demand is not necessarily met all the time, the proof of a theorem for steady state existence would follow the same lines as the proof of Theorem 1, and so we will not assess this case in this paper. Basically, it is enough to consider instead that firms have production schedules $\bar{\mathfrak{f}}\left(\wp, \wp_s\right) \leq f\left(\wp, \wp_s\right)$ that can be smaller than demand. And so, optimal pricing will be a composite function of $\bar{\mathfrak{f}}\left(\wp, \wp_s\right)$ instead of $f\left(\wp, \wp_s\right)$. In this case, Hahn (2022) approach could be seen as a particular case of using production schedules, as the author consider that production matches the minimum between optimal supply and demand.

where $\widehat{\wp}_t^n$, $\widehat{\wp}_{s,t}$ and $\hat{x}_t$ are log-deviations from steady state levels.

Following Ball and Romer (1990) approach, we compute the content $\psi_{\text{real}} \equiv \frac{1}{\kappa_{\text{real}}}$ of real rigidities in this model, where $\kappa_{\text{real}} \equiv \frac{\partial \widehat{\wp}_t^n}{\partial \hat{x}_t}$ is the pass-through from output gap to prices. If prices are rigid ($\alpha > 0$), κ_{real} is part of the output-gap coefficient in the Phillips curve. Evaluating equations (5) in the steady-state equilibrium with flexible prices, we obtain a simple result to general demand-driven real rigidities[21] as a function of the natural elasticity ξ^n and superelasticity η^n:[22]

$$\kappa_{\text{real}} = \frac{(\sigma+\omega)}{1+\frac{\eta^n}{(\xi^n-1)}+\omega\xi^n} \quad ; \; \psi_{\text{real}} = \frac{1}{\kappa_{\text{real}}} \tag{12}$$

Therefore, the demand structure is a relevant source of real rigidities. As for the role of changes in ξ^n and η^n, notice that $\frac{\partial(\psi_{\text{real}})}{\partial \eta^n} = \frac{1}{(\sigma+\omega)}\frac{1}{(\xi^n-1)}$ and $\frac{\partial(\psi_{\text{real}})}{\partial \xi^n} = -\frac{1}{(\sigma+\omega)}\left(\frac{\eta^n}{(\xi^n-1)^2}-\omega\right)$. We conclude that, no matter the form of the demand function, there must be the case that: (i) increases in natural superelasticity η^n leads to larger (smaller) real rigidity ψ_{real} if natural elasticity ξ^n is larger (smaller) than unity; and (ii) increases in natural elasticity ξ^n leads to larger (smaller) real rigidity ψ_{real} if natural superelasticity η^n is smaller (larger) than $\omega\,(\xi^n-1)^2$.

Given its importance for both steady-state determinacy under trend inflation and as a source of real rigidity, the question is whether there are utility-based demand functions that simultaneously satisfy: (i) micro and macro empirical support for demand functions, relative prices, and macroeconomic relations; and (ii) Theorem 1 conditions for steady-state existence under trend inflation.

5 Demand Functions Consistent With Theorem 1

Now the question is whether there are utility-based demand functions that simultaneously satisfy: (i) Theorem 1 conditions, especially those related to finite demand and slope at zero relative price; and (ii) micro and macro empirical support, as described in Section 2.

Under monopolistic competition models with a continuum of firms, as we consider in this paper, the class of Kimball (1995) demand functions, especially the one proposed by Dotsey and King (2005), has been often used in recent literature.[23] Using Theorem 1, we show that having a sufficiently large

[21] For that, we easily compute $\kappa_{\text{real}} \equiv \frac{\partial \widehat{\wp}_t^n}{\partial \hat{X}_t} = \frac{(\sigma+\omega)}{\frac{2f_1(1,1)+f_{11}(1,1)}{1+f_1(1,1)}-\frac{f_{11}(1,1)}{f_1(1,1)}-\omega f_1(1,1)}$, and apply the definitions in (5).

[22] In Burya and Mishra (2022), the authors derive a similar but simpler result in a model with linear production function, log utility to consumption and no disutility to work, which implies $\omega = 0$. The authors show the pass-through $\kappa_p \equiv \frac{\kappa_{\text{real}}}{(\sigma+\omega)}$ from marginal costs to prices. When $\omega = 0$, their inverse real rigidity metrics is then $\kappa_{pt} = \frac{(\xi^n-1)}{(\xi^n-1)+\eta^n}$.

[23] If we were to extend our modelling approach to also consider oligopoly models with a finite number of firms, instead of only monopolistic competition models with an infinite number of firms in the continuum $z \in (0,1)$, we have a broad set of demand functions satisfying Theorem 1 conditions, as oligopoly demand functions are typically bounded. In this regard, the Atkeson and Burstein (2008) and Wang and Werning (2022) oligopoly models with N firms are strong candidates to be

curvature parameter is the necessary condition for Kimball demand functions to be consistent with all levels of trend inflation. However, under large curvature, Kimball demand functions have three features that are at odds with what micro and macro evidence suggests (see Section 2): (i) superelasticities become much larger than the micro evidence range; (ii) they fail to accommodate a sizable mass of relative prices found in the US empirical distribution; and (iii) if used in NK macroeconomic models with Calvo pricing in lieu of Dixit and Stiglitz (1977) demand functions, Kimball-based NK models predict that the distorted output (due to nominal rigidities) becomes much larger than the flexible-prices output as trend inflation rises. These facts are in line with the recent critiques and findings on Kimball-based NK models found in the literature (see e.g., Dossche et al. (2010), Beck and Lein (2015), Klenow and Willis (2016), and Kurozumi and Van Zandweghe (2016, 2024)).

5.1 Issues with the Kimball Aggregator

Withing the broad class of Kimball (1995) consumption aggregation, Dotsey and King (2005) propose a particular functional form that has been frequently used in the literature (e.g. Levin et al. (2007), Harding et al. (2022) and Kurozumi and Van Zandweghe (2024)). As we better detail in Appendix B, the implied demand function is:

$$\frac{c_t(z)}{C_t} = f(\wp_t(z), \wp_{s,t}) = \begin{cases} \frac{1}{(1+\varphi)}\left(\frac{\wp_t(z)}{\wp_{k,t}}\right)^{\varpi} + \frac{\varphi}{(1+\varphi)} & ;\text{if } \left(\frac{\wp_t(z)}{\wp_{k,t}}\right) \leq (-\varphi)^{\frac{1}{\varpi}} \\ 0 & ;\text{if } \left(\frac{\wp_t(z)}{\wp_{k,t}}\right) > (-\varphi)^{\frac{1}{\varpi}} \end{cases}$$

where $\wp_{k,t} \equiv (1+\varphi) - \varphi\wp_{s,t}$, $\wp_{k,t} \equiv \frac{P_{k,t}}{P_t}$ is an auxiliary composite relative price, $P_{s,t} \equiv \int_0^1 p_t(z)\,dz$ is the average price, and P_t is the aggregate price, implicitly defined by $1 = \int_0^1 \left(\frac{\wp_t(z)}{\wp_{k,t}}\right)^{-\frac{1}{(\mathfrak{m}-1)}} dz$. The composite parameters are $\varpi \equiv \frac{\mu_k(1+\varphi)}{(1-\mu_k)}$ and $\mathfrak{m} \equiv \frac{\mu_k(1+\varphi)}{(1+\mu_k\varphi)}$, where $\mu_k \geq 1$ is the elasticity parameter, which matches the implicit markup rate μ under flexible prices, and $\varphi \leq 0$ sets the aggregation curvature. If $\varphi = 0$, the demand curve simplifies into the standard Dixit and Stiglitz (1977) form.

As for Theorem 1 conditions, note first that $f(\wp, \wp_s)$ and $\wp f_1(\wp, \wp_s)$ are zero when $\left(\frac{\wp}{\wp_k}\right) > (-\varphi)^{\frac{1}{\varpi}}$. Therefore, this function trivially converges to a finite and defined level when $\wp \to \infty$. It remains to verify the conditions for convergence when $\wp \to 0$. Notice that its power ϖ flips into a positive value only when $\varphi < -1$. Only in this case, i.e. when its curvature is sufficiently large, the relative demand $f(\wp, \wp_s)$ and $\wp f_1(\wp, \wp_s)$ converge to finite and defined levels at $\wp \to 0$, i.e. $\lim_{\wp\to 0} f(\wp, \wp_s) = f(0,1) = \frac{\varphi}{(1+\varphi)}$ and $\lim_{\wp\to 0} \wp f_1(\wp, \wp_s) = \frac{\varpi}{(1+\varphi)}\left(\frac{0}{\wp_k}\right)^{\varpi} = 0$.

When $\varphi < 0$, Kimball demand function has positive superelasticities (see Appendix B), which

applied to NK models with trend inflation in future extensions.

makes it qualitatively in line with micro evidence, as described in Section 2. However, quantitatively, Kimball elasticities and superelasticities in estimated/calibrated macroeconomic models tend to be much larger than their empirical microdata counterparts, i.e. $\xi^{micro} \in [1.0\ ,\ 5.0]$ and $\eta^{micro} \in [1.5\ ,\ 2.0]$. To illustrate this property, consider those levels under flexible prices, i.e. $\xi^n = \frac{\mu_k}{(\mu_k - 1)}$ and $\eta^n = (-\varphi)\frac{\mu_k}{(\mu_k - 1)}$. Since $\varphi < -1$ for the existence of the steady state, and μ_k matches the static markup $\mu = 1.12$ under flexible prices, we obtain $\xi^n = 9.3$ and $\eta^n > 9.3$.

In macroeconomic models, extremely negative values for φ have been estimated/calibrated the literature. For the US, the parameters is usually estimated/calibrated in the range $\varphi \in [-16, -2]$.[24] If the static markup is set at the usual low levels of $\mu_k = \mu = 1.12$, even at the smallest curvature in the range, the implied superelasticity $\eta^n = 18.6$ is much larger than what microevidence suggests.

In addition, the large macro curvatures imply that the upper limit for relative prices $(-\varphi)^{\frac{1}{\varphi}}$ is just slightly larger than unity. In this case, there is no demand for prices that are set slightly above the average price, and the distribution of relative prices is strongly asymmetric to the left. For illustration, Klenow and Willis (2016) show that about 15% of goods end up with zero relative demand when the demand function is Kimball-based with large curvature. Relative to actual economies, to have a mass of price setting firms with zero demand is an odd implication. The Kimball aggregator implied distribution of relative prices is, therefore, not in line with empirical evidence when the curvature is large. See a detailed discussion in Appendix B.3.

As for macroeconomic implications, Kurozumi and Van Zandweghe (2016, 2024) use large curvature parameters and find that that, under Calvo price setting, the steady state aggregate output actually increases as trend inflation rises and becomes much larger that the output level achieved under flexible prices, which is independent of trend inflation. Figure 1 illustrates this point, plotting the steady state natural (flexible prices) output $\bar{Y}^n$ and the aggregate output $\bar{Y}$ for different levels of annual trend inflation, under the benchmark calibration and different curvature levels $\varphi \leq 0$.[25]

[24] Some typical values for the US are the following: (i) $\varphi = -12.2$ in Harding et al. (2022); (ii) $\varphi = -2.6$ in Kurozumi and Van Zandweghe (2024); (iii) $\varphi = -8$ in Levin et al. (2007); and (iv) $\varphi = -3.79$ in Smets and Wouters (2007). In addition, obtaining a better marginal likelihood statistics for model comparison, Harding et al. (2022) re-estimate Smets and Wouters (2007) model with a different prior distribution and obtain $\varphi = -16.37$.

[25] See Appendix B.1 for more details.

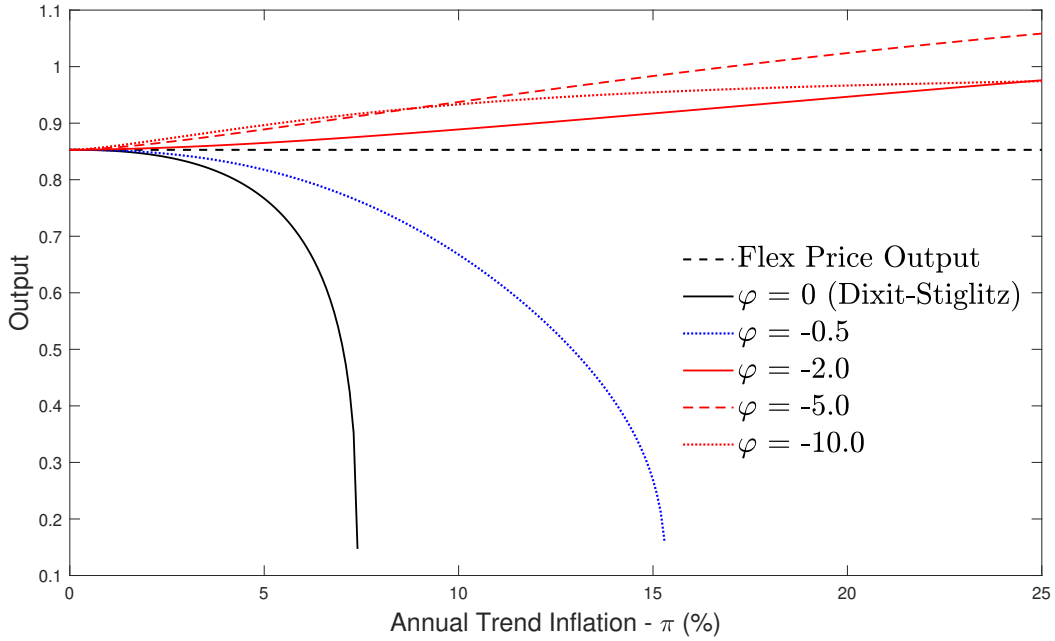


Figure 1: Output and Trend Inflation

Notes: The natural (flexible prices) output $\bar{Y}^n$ does not vary with trend inflation. The steady state problem exists if: $-1 \leq \varphi \leq 0$.

In equilibrium, the natural output $\bar{Y}^n$ (flexible prices) is invariant with respect to trend inflation. Nonetheless, the model predicts that output strongly increases as long run inflation rises, becoming up to 20% larger than the natural output when annual trend inflation is at 25%. That is, Kimball aggregation predicts that the distortion created by trend inflation under sticky prices actually induces a better output level than that obtained in an economy with flexible prices.

In light of those results, we propose in the next section a remedy to attenuate the issues induced by Kimball demand functions.

5.2 Storage and processing costs

Between purchasing a non-durable good and consuming it within a specific period, it is not uncommon for individuals to face extra costs added to its sticker price. Those costs can rise either from direct monetary causes or from efforts, which then can be translated into a monetary price. And more importantly, they might be resilient even if the sticker price is set at zero. For instance, apple trees can be very tall, requiring an effort to pick apples even when they are free. And an orange tree about the same height requires the same effort, even though being a different good. Since there is only so much fruit individuals can carry down the trees, the extra costs should increase with consumed fruit

volumes, as consuming more fruit requires more climbing.

We can indirectly compute the extra price added to the sticker price by quantifying the effort (energy, abilities, etc.) needed to climb the tree in every period we want to consume a fruit. And we highlight that acquiring them has a complementary nature with consuming the fruit, as individuals would not "buy" more effort goods and less fruit unities if effort becomes relatively cheaper than fruits.

Sometimes, the costs can be directly measured in monetary units, for consuming the good might require post-purchase accompanying extra cost from handling, shipping and storing the goods within the period. Again, even if the good's sticker price is set at zero, those extra costly activities still remain. And their cost in many cases depend on good volumes and weights, rather than good types. Those properties again characterize a complementarity rather than a substitutability between consumed goods and the extra cost sources.

Of course, features such as rarity, fragility and perishability also matters. The extra costs might also vary across different individuals and across time. In this paper, for simplicity, we abstract from those possibilities and do not take a definite instance on the most realistic nature of extra costs. In all cases, extra costs prevent individuals to consume infinite amounts of goods, even though that is what they would like to if sticker prices were to approach zero in the absence of extra costs.[26] For short, we use the term "storage costs" to characterize this class of extra-cost models. Here, we consider a simple structure to make the case that this class of models can be used to assess the economy at all levels of trend inflation.

And lastly, the extra costs might be simply wasted (deadweight loss) or might be recovered somehow into the economy. Notice that the first type of extra costs generates more distortions than the second type, as there are no firms or individuals able to accrue the losses individuals bear. For simplicity, we only explore the second type.

As will be clear from the steps shown in the next section, applying storage costs for any general preference framework is straightforward. In this paper, for simplicity, we derive the implied storage costs demand function under the general Kimball (1995) framework and show that, for a given curvature parameter φ, increasing storage costs pushes down the implied elasticities and superelasticities, making them more in line with what microdata suggests. Larger storage costs also push the relative prices threshold further away, giving more room to accommodate empirical distributions. In addition, even when considering the extreme case of $\varphi = 0$ (standard Dixit and Stiglitz (1977) aggregation), the

[26] Since firms are assumed to satisfy any demand level, and all individuals face the same price, there is no incentive for over purchasing goods intended for reselling.

resulting demand function satisfies Theorem 1 conditions for convergence under any level of trend inflation and any level of storage costs, no matter how small it is. We also highlight that, when applied to Dixit and Stiglitz (1977) aggregation, storage costs pushes up the elasticities and superelasticities, towards microdata estimates.

5.2.1 Storage prices

Building upon the general framework described in Section 3.1, we assume that consuming $c_t(z)$ units of good z at sticker price $p_t(z)$ requires paying extra $\delta P_{s,t}$ to an intermediate representative firm for processing, handling and storing, where $\delta \geq 0$ is the storage rate. As storing only depends on volumes, independently of the good type, each unit of storage has the same price $P_{s,t}$. Therefore, the household's total expenditure is $\int_0^1 (p_t(z) + \delta P_{s,t}) c_t(z) dz$.[27]

In this context, let us initially define the aggregate price P_t and consumption C_t as in $(1+\delta) P_t C_t \equiv \int_0^1 (p_t(z) + \delta P_{s,t}) c_t(z) dz$. As we show below, in equilibrium the aggregate price also satisfies $P_t C_t \equiv \int_0^1 p_t(z) c_t(z) dz$, which is in line with the typical definition we mention in Section 3.1.

The representative storage firm's nominal revenue is $R_{s,t} = \int_0^1 (\delta P_{s,t}) c_t(z) dz = \delta P_{s,t} C_{s,t}$, where $C_{s,t} \equiv \int_0^1 c_t(z) dz$ is the average consumption (arithmetic mean). In order to intermediate processing, handling and storing, we assume that the storage firm pays $\delta p_t(z)$ per unit to each firm $z \in (0,1)$, so its nominal cost is $Cost_{s,t} = \int_0^1 (\delta p_t(z)) c_t(z) dz = \delta \int_0^1 p_t(z) c_t(z) dz$. This representative firm sets its price at a zero-profit condition $R_{s,t} = Cost_{s,t}$. Coupling this condition with the aggregate price definition $(1+\delta) P_t C_t \equiv \int_0^1 (p_t(z) + \delta P_{s,t}) c_t(z) dz$ allows us to obtain two important results (see Appendix C):

$$P_t C_t = \int_0^1 p_t(z) c_t(z) dz \quad ; \; P_{s,t} = P_t \frac{C_t}{C_{s,t}} \tag{13}$$

where $C_{s,t} \equiv \int_0^1 c_t(z) dz$. Note that those results do not depend on any particular preference structure or consumption aggregation.

Before proceeding, we need small changes to adapt the general results shown in Section 3. On the household side, after substituting the total expenditure $(1+\delta) P_t C_t$ for $P_t C_t$ in the budget constraint, the optimal labor supply curve becomes $w_t(z) = (1+\delta) v_t'(z) / u_t'$. Firms results also change a little bit, as we show further on in Section 5.2.3. Before presenting it, we assess the consequences of having storage costs under Kimball and Dixit-Stiglitz aggregation. In this context, Section 5.2.2 below derives the resulting demand curve and studies its properties.

[27] Here, $P_{s,t}$ does not have the same meaning as in Kimball's approach, detailed in Section B.

5.2.2 Demand function under storage costs

If the representative household is subject to storage costs and has Kimball-type preferences, as described in Section B, total expenditure minimization gives us the implied demand function and price aggregation:

$$\frac{c_t(z)}{C_t} = \begin{cases} \frac{1}{(1+\varphi)}\left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right)^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}} + \frac{\varphi}{(1+\varphi)} & ;\text{if } \left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right) \leq (-\varphi)^{\frac{1}{\omega}} \\ 0 & ;\text{if } \left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right) > (-\varphi)^{\frac{1}{\omega}} \end{cases} \tag{14}$$

$$\wp_{k,t} = (1+\varphi) - \varphi\wp_{sk,t} \quad ; 1 = \int_0^1 \frac{(\wp_t(z)+\delta\wp_{s,t})}{(1+\delta)\wp_{sk,t}}dz \quad ; 1 = \int_0^1 \left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right)^{-\frac{1}{(\mathfrak{m}-1)}} dz$$

where again $\varphi \leq 0$ is the curvature parameter, μ_k denote the Kimball elasticity parameter, $\mathfrak{m} \equiv \frac{\mu_k(1+\varphi)}{(1+\mu_k\varphi)}$ and $\omega \equiv \frac{\mu_k(1+\varphi)}{(1-\mu_k)} = -\frac{\mathfrak{m}}{(\mathfrak{m}-1)}$. Paralleling the notation used in Section 3.1.1, we define $\wp_t(z) \equiv \frac{p_t(z)}{P_t}$ as the relative price of firm z, and $\wp_{s,t} = \frac{P_{s,t}}{P_t}$ as the a relative price of storage costs. In addition, we also define $\wp_{sk,t} \equiv \frac{P_{sk,t}}{P_t}$ and $\wp_{k,t} \equiv \frac{P_{k,t}}{P_t}$.

In case of non-zero storage costs, i.e. $\delta \neq 0$, note that $f(\wp, \wp_s, \wp_{sk})$ and $\wp f_1(\wp, \wp_s, \wp_{sk})$ are always finite and defined for any relative price $\wp$, even if $\varphi = 0$. Therefore, any small level of storage costs is sufficient to ensure that Theorem 1 conditions are always satisfied. It implies that now we can use any curvature to assess the NK model at all levels of trend inflation, as long as δ is not zero.

When $\varphi = 0$, we have the standard Dixit and Stiglitz (1977) aggregation $C_t = \left(\int_0^1 c_t(z)^{\frac{\theta-1}{\theta}} dz\right)^{\frac{\theta}{\theta-1}}$, where $\theta = \frac{\mu_k}{(\mu_k-1)} > 1$ is the elasticity of substitution between goods. In this case, the demand function with storage costs is $\frac{c_t(z)}{C_t} = (1+\delta)^{\theta}(\wp_t(z)+\delta\wp_{s,t})^{-\theta}$, where $(1+\delta)^{1-\theta} = \int_0^1 (\wp_t(z)+\delta\wp_{s,t})^{1-\theta} dz$.[28] Even though assuming storage costs seems to pose only small changes into the demand function and price aggregation, when compared to those obtained under standard Dixit-Stiglitz aggregation, it allows the demand function $\frac{c_t(z)}{C_t} = (1+\delta)^{\theta}(\wp_t(z)+\delta\wp_{s,t})^{-\theta}$ to be quasi-kinked and more in line with empirical micro-evidence, as presented in Section 2. In addition, this demand function is always positive for all levels of relative prices, and has no relative-price threshold beyond which the demand is zero.

As better detailed further on, Figure 2 depicts the demand function (log-log), price elasticities and price superelasticities for different levels of storage cost rates $\delta \in [0, 0.50]$ and curvature parameters $\varphi \in \{0, -2.0\}$, keeping the static markup fixed at $\mu = 1.12$.

[28] In line with general definition (3), we can also rewrite $P_{s,t}$ as a weighted average $P_{s,t} = \int_0^1 g(\wp_t(z), \wp_{s,t}) p_t(z) dz$ of individual prices $p_t(z)$, where $g(\wp_t(z), \wp_{s,t}) = \frac{(\wp_t(z)+\delta\wp_{s,t})^{-\theta}}{\int_0^1 (\wp_t(z)+\delta\wp_{s,t})^{-\theta} dz}$. The proof is in Appendix C.1.

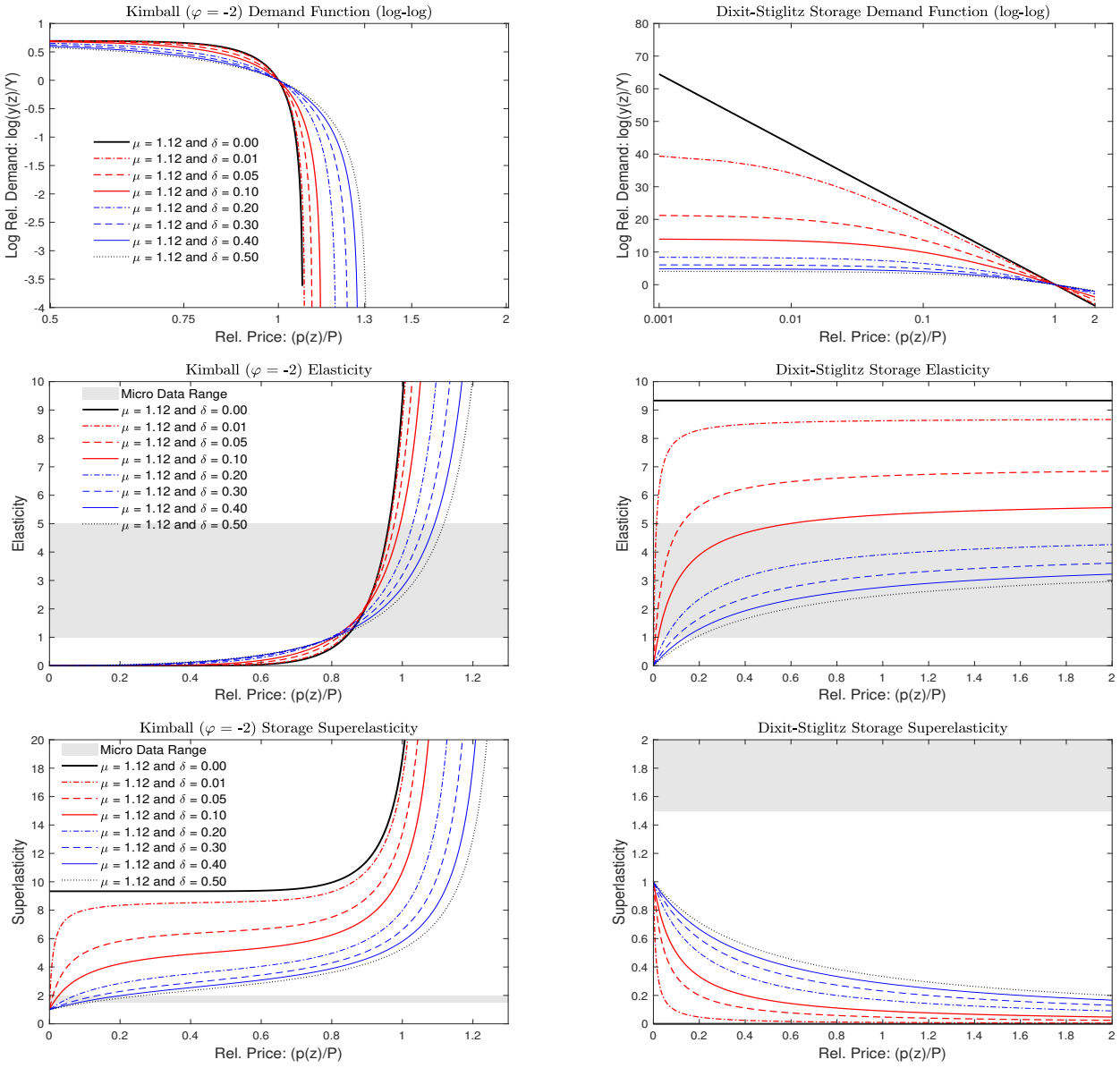


Figure 2: Demand, Elasticities and Superelasticities - Storage Costs

Notes: In top panels, the demand function is plotted using $\log(y(z)/Y)$.and $\log(p(z)/P)$. For simulations, we fix $\wp_s = \wp_k = 1$ and recompute μ_k for each value of δ in order to keep the static markup at $\mu = 1.12$.

For each relative price $\wp_t(z)$, the firm z's price elasticity and superelasticity are: (i) $\xi_t(z) = -\frac{\omega\left(\frac{\wp_t(z)}{(1+\delta)\wp_{k,t}}\right)\left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right)^{(\omega-1)}}{\left[\left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right)^{\omega}+\varphi\right]}$ and $\eta_t(z) = \frac{1}{(\wp_t(z)+\delta\wp_{s,t})}\left[\frac{\varphi\omega\wp_t(z)}{\left[\left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right)^{\omega}+\varphi\right]}+\delta\wp_{s,t}\right]$, if $\left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right) \leq (-\varphi)^{\frac{1}{\omega}}$; or (ii) $\xi_t(z) = 0$ and $\eta_t(z) = 0$, if $\left(\frac{\wp_t(z)+\delta\wp_{s,t}}{(1+\delta)\wp_{k,t}}\right) > (-\varphi)^{\frac{1}{\omega}}$. In Figure 2, we recompute μ_k for each pair (δ, φ), in order to keep the static markup fixed at $\mu = 1.12$. For that, we use $\mu = \frac{\mu_k}{(1+\delta)[1-\delta(\mu_k-1)]}$, obtained from equation (6), applied to the model's elasticity under flexible prices $\xi^n = -\frac{1}{(1+\delta)}\frac{\mu_k}{(1-\mu_k)}$.

With the standard Dixit and Stiglitz (1977) aggregation, i.e. when $\varphi = 0$ and $\theta = \frac{\mu_k}{(\mu_k-1)}$, we obtain $\xi_t(z) = \theta\frac{\wp_t(z)}{(\wp_t(z)+\delta\wp_{s,t})} \geq 0$ and $\eta_t(z) = \delta\frac{\wp_{s,t}}{(\wp_t(z)+\delta\wp_{s,t})} \geq 0$. Note that, if $\delta \neq 0$, the demand elasticity $\xi_t(z)$ and superelasticity $\eta_t(z)$ are price-dependent even if $\varphi = 0$. As Figure 2 shows, accounting for storage costs allows demand functions derived from Dixit and Stiglitz (1977) aggregation to be quasi-kinked.

In the remainder of the paper, we assess the dynamic properties of storage cost models when firms have sticky prices. For that, we consider the simpler Dixit-Stiglitz aggregation with storage costs as a proof of concept to study the implied New-Keynesian model for small and large levels of trend inflation.

5.2.3 Firms

Consider the Dixit-Stiglitz Storage case, as a proof of concept. Using the market clearing condition $y_t(z) = c_t(z), \forall z$, the aggregate and average levels of output satisfy $Y_t = C_t$ and $Y_{s,t} = C_{s,t}$. Therefore, firm z's demand function is $\frac{y_t(z)}{Y_t} = f(\wp_t(z), \wp_{s,t}) = (1+\delta)^{\theta}(\wp_t(z)+\delta\wp_{s,t})^{-\theta}$.

The firm's revenue is now $(1+\delta)p_t(z)c_t(z)$. As we mentioned before, optimal labor supply curve under storage costs is $w_t(z) = (1+\delta)\frac{\chi}{\epsilon_t}h_t(z)^{\nu}(Y_t)^{\sigma}$. Adapting the results shown in Section 3.2, optimal pricing under flexible prices now requires $\frac{(1+\delta)[\theta-(1+\delta)]}{\theta}\wp_t^n = mc_t^n$, where $mc_t^n = \frac{(1+\delta)^{(1+\theta\omega)}[\theta-(1+\delta)]}{\theta}\left(\wp_t^n+\delta\wp_{s,t}^n\right)^{-\theta\omega}$ is the marginal cost under flexible prices. Therefore, we conclude that the natural output evolves according to:

$$(Y_t^n)^{(\sigma+\omega)} = \frac{1}{(1+\delta)\mu}\frac{\varepsilon}{\chi}\epsilon_t(\mathcal{A}_t)^{(1+\omega)} \tag{15}$$

where $\mu \equiv \frac{\wp_t^n}{mc_t^n}$ is the static markup under flexible prices and storage costs:

$$\mu = \frac{\mu_\delta}{(1+\delta)}\ ;\ \mu_\delta \equiv \frac{\theta}{\theta-(1+\delta)} \tag{16}$$

This last result allows us to design a strategy to calibrate θ as a function of markup μ and storage

cost rate δ: $\theta = \frac{\mu(1+\delta)^2}{[\mu(1+\delta)-1]}$. Note that, for a given a steady state markup μ, the elasticity of substitution θ decreases with δ, when it is not unreasonably large. In particular, at the benchmark low markup level $\mu = 1.12$, the elasticity of substitution can be as low as $\theta = 5$ even for a small storage cost rate of $\delta = 0.10$. For larger storage cost rates, θ falls to about $\theta = 3.6$ when $\delta \in [0.5, 1.0]$. Those low levels for the elasticity of substitution θ are consistent with microdata estimates found by Broda and Weinstein (2006). The authors find that consumers have low elasticities of substitution across similar goods in most categories, with the median elasticity being estimated at about $\theta = 3$.

With Calvo price setting under storage costs, we can simplify firm z's optimal pricing equation (9) into the following system:

$$\begin{aligned}
1 = &\quad \mu_\theta \tfrac{N_t^*}{D_t^*} \\
N_t^* = &\quad E_t \sum_{j=0}^{\infty} \alpha^j q_{t,t+j} \mathcal{G}_{t,t+j} \Pi_{t,t+j}^{ind} \left(z_{t,t+j}^*\right)^{-(1+\theta)} \left(\tfrac{mc_{t,t+j}^*}{(1+\delta)} + \delta \wp_{s,t+j}\right) \\
D_t^* = &\quad E_t \sum_{j=0}^{\infty} \alpha^j q_{t,t+j} \mathcal{G}_{t,t+j} \Pi_{t,t+j}^{ind} \left(z_{t,t+j}^*\right)^{-\theta} \\
mc_{t,t+j}^* = &\quad \tfrac{(1+\delta)^{\theta\omega}}{\mu} \left(X_{t+j}\right)^{(\sigma+\omega)} \left(z_{t,t+j}^*\right)^{-\theta\omega} \\
z_{t,t+j}^* = &\quad \tfrac{\Pi_{t,t+j}^{ind}}{\Pi_{t,t+j}} \wp_t^* + \delta \wp_{s,t+j}
\end{aligned} \tag{17}$$

where again $\wp_t^* \equiv \frac{p_t^*}{P_t}$ and $X_t = \frac{Y_t}{Y_t^n}$. Here, $\mu_\theta \equiv \frac{\theta}{(\theta-1)}$ is the primordial markup without storage costs. Note that we also use three additional auxiliary variables, i.e. N_t^*, D_t^* and $z_{t,t+j}^*$.

In this framework, price aggregation $(1+\delta)^{1-\theta} = \int_0^1 \left(\frac{p_t(z)}{P_t} + \delta \frac{P_{s,t}}{P_t}\right)^{1-\theta} dz$, average output and storage pricing evolve according to:

$$\begin{aligned}
&(1+\delta)^{1-\theta} = (1-\alpha) \textstyle\sum_{j=0}^{\infty} \alpha^j \left(z_{t-j,t}^*\right)^{1-\theta} \\
&Y_{s,t} = (1+\delta)^{\theta} Y_t \left(\wp_{\delta,t}\right)^{-\theta} \\
&\left(\wp_{\delta,t}\right)^{-\theta} = (1-\alpha) \textstyle\sum_{j=0}^{\infty} \alpha^j \left(z_{t-j,t}^*\right)^{-\theta} \\
&\wp_{s,t} = \tfrac{Y_t}{Y_{s,t}}
\end{aligned} \tag{18}$$

Now, the infinite sum is on $z_{t-j,t}^* = \left(\frac{\Pi_{t-j,t}^{ind}}{\Pi_{t-j,t}} \wp_{t-j}^* + \delta \wp_{s,t}\right)$ instead of $z_{t,t+j}^* = \left(\frac{\Pi_{t,t+j}^{ind}}{\Pi_{t,t+j}} \wp_t^* + \delta \wp_{s,t+j}\right)$. Most importantly, $z_{t,t+j}^*$ and $z_{t-j,t}^*$ enter systems (17) and (18) raised to non-positive integer powers. This fact prevents the equations to have recursive forms. In order to cope with that, we present a precise approximation in next section, allowing those terms to have recursive forms in log-linearizations.

5.2.4 Aggregates and Welfare

Let $h_t \equiv \int_0^1 h_t(z)\,dz$ denote the aggregate working hours. Given the production function $y_t(z) = \mathcal{A}_t h_t(z)^{\varepsilon}$ and demand function $\frac{y_t(z)}{Y_t} = (1+\delta)^{\theta}\left(\wp_t(z) + \delta\wp_{s,t}\right)^{-\theta}$, we conclude that aggregate hours evolve according to $h_t = (1+\delta)^{\frac{\theta}{\varepsilon}}\left(\frac{Y_t}{\mathcal{A}_t}\right)^{\frac{1}{\varepsilon}}\Lambda_{y,t}$, where $\Lambda_{y,t} \equiv \int_0^1 \left(\wp_t(z) + \delta\wp_{s,t}\right)^{-\frac{\theta}{\varepsilon}} dz$. Therefore, following the vast literature of price dispersion, we can write the aggregate output as $Y_t = \frac{1}{\mathfrak{d}_{y,t}}\mathcal{A}_t(h_t)^{\varepsilon}$, where $\mathfrak{d}_{y,t} \equiv (1+\delta)^{\theta}\left(\Lambda_{y,t}\right)^{\varepsilon}$ is the production-relevant metric of price dispersion. Using Calvo price setting, note that $\Lambda_{y,t} = (1-\alpha)\sum_{j=0}^{\infty}\alpha^j\left(z^*_{t-j,t}\right)^{-\frac{\theta}{\varepsilon}}$.

As for welfare considerations, recall that $\mathcal{W}_t \equiv (u_t - v_t)$ is the relevant instantaneous welfare metric, where $u_t \equiv \epsilon_t\frac{(Y_t)^{(1-\sigma)}-1}{(1-\sigma)}$ is the consumption utility and $v_t \equiv \int_0^1 v_t(z)\,dz$ is the aggregate disutility of working hours, in which $v_t(z) \equiv \frac{\chi}{(1+\nu)}h_t(z)^{(1+\nu)}$. Given the production and demand functions, we can write the aggregate disutility as $v_t = \mathfrak{d}_{v,t}\frac{\chi}{(1+\nu)}(h_t)^{(1+\nu)}$, where $\mathfrak{d}_{v,t} = \frac{\Lambda_t}{\left(\Lambda_{y,t}\right)^{(1+\nu)}}$ is the welfare-relevant metric of price dispersion, $\Lambda_t \equiv \int_0^1 \left(\wp_t(z) + \delta\wp_{s,t}\right)^{-\theta_1} dz$, and $\theta_1 = \theta(1+\omega)$ is a composite parameter. Under Calvo price setting, note that $\Lambda_t = (1-\alpha)\sum_{j=0}^{\infty}\alpha^j\left(z^*_{t-j,t}\right)^{-\theta_1}$.

In the equilibrium with flexible prices ($\alpha = 0$), we obtain $\Lambda^n_{y,t} = (1+\delta)^{-\frac{\theta}{\varepsilon}}$, $\Lambda^n_t = (1+\delta)^{-\theta_1}$, and $\mathfrak{d}^n_{y,t} = \mathfrak{d}^n_{v,t} = 1$. In this equilibrium, the instantaneous welfare evolves according to $\mathcal{W}^n_t \equiv (u^n_t - v^n_t) = \epsilon_t\frac{(Y^n_t)^{(1-\sigma)}-1}{(1-\sigma)} - \frac{\chi}{(1+\nu)}\left(\frac{Y^n_t}{\mathcal{A}_t}\right)^{(1+\omega)}$, where $u^n_t = \epsilon_t\frac{(Y^n_t)^{(1-\sigma)}-1}{(1-\sigma)}$, $v^n_t = \frac{\chi}{(1+\nu)}(h^n_t)^{(1+\nu)}$, and $h^n_t = \left(\frac{Y^n_t}{\mathcal{A}_t}\right)^{\frac{1}{\varepsilon}}$.

Therefore, following Schmitt-Grohe and Uribe (2007), we can compute the consumption-equivalent welfare metric as a distorted output level Y^{eq}_t that would prevail in a equilibrium with flexible prices in order to keep the welfare level as the one obtained with sticky prices ($\mathcal{W}_t$). That is, Y^{eq}_t satisfies:

$$\epsilon_t\frac{\left(Y^{eq}_t\right)^{(1-\sigma)}-1}{(1-\sigma)} - \frac{\chi}{(1+\nu)}\left(\frac{Y^{eq}_t}{\mathcal{A}_t}\right)^{(1+\omega)} = \mathcal{W}_t = \epsilon_t\frac{(Y_t)^{(1-\sigma)}-1}{(1-\sigma)} - \mathfrak{d}_{v,t}\frac{\chi}{(1+\nu)}\left(\mathfrak{d}_{y,t}\frac{Y_t}{\mathcal{A}_t}\right)^{(1+\omega)}$$

In this regard, we defined $X^{eq}_t \equiv \frac{Y^{eq}_t}{Y^n_t}$ as the consumption-equivalent output gap.

5.2.5 Steady State Properties

Using the steady state relations shown in Appendix C.3, Figure 3 shows how relevant steady state levels vary with different levels of trend inflation $\bar{\pi}$ and different storage cost rates δ. For this, we use the benchmark calibration defined in Section 3.3, and use (16) recompute θ for each value of δ to keep the static markup at $\mu = 1.12$.

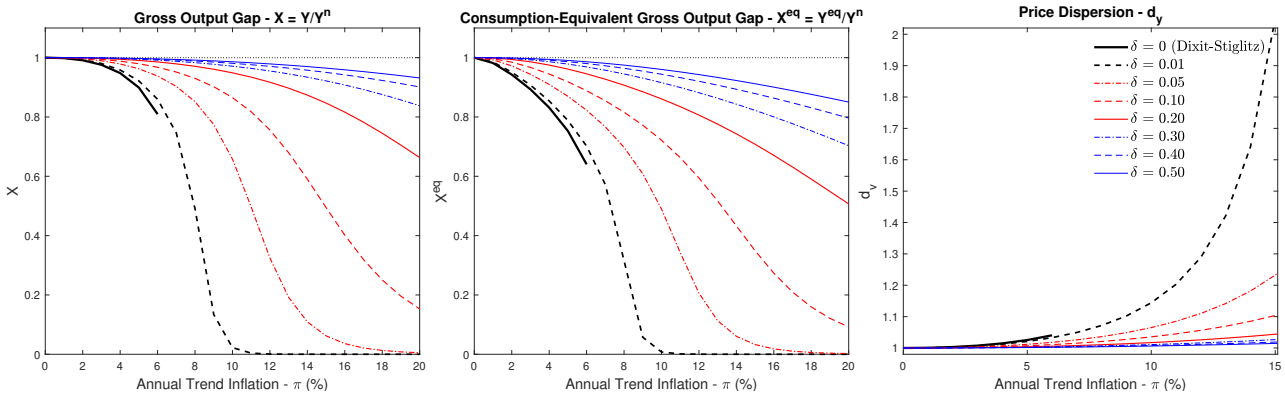


Figure 3: Steady State Levels - Storage Costs

As predicted, steady state levels now exist for all levels of trend inflation as long as $\delta > 0$. In the left panel, note that the gross output gap falls as trend inflation rises. It is interesting to note that using larger values for δ makes output smoothly decline with respect to the natural output, avoiding the sharp fall observed under $\delta = 0$ (Dixit-Stiglitz). If δ is very small, the model is able to present a seamless continuation of what standard NK models (Dixit-Stiglitz) predict for the steady state, but now without the upper limit on trend inflation, which is 7.40% using the benchmark calibration. In the middle panel, the consumption-equivalent output gap behaves similarly to the output gap shown in the left panel. However, due to the effect of disutility of hours, $\bar{X}^{eq}$ is always smaller than $\bar{X}$. Finally, in the right panel, note that the presence of storage costs strongly attenuates the price dispersion caused by trend inflation. We highlight these results as recent micro evidence on price dispersion suggests that it only weakly increases as inflation rises (e.g., Nakamura et al. (2018) and Sheremirov (2020)).

6 Simulations

In this section, we assess impulse responses using the log-linearized model presented at Appendix C.4. Using the benchmark calibration, recall that the annualized upper limit for trend inflation is 7.40% under standard Dixit and Stiglitz (1977) preferences and zero storage costs ($\delta = 0$).[29] However, as we show in Section 5.2.2, the demand function function under storage costs satisfies Theorem 1, and so setting $\delta > 0$ is a sufficient condition for the existence of steady state equilibrium at any level of trend inflation. Therefore, we set[30] $\delta = 0.05$ and consider impulse responses for a range of different levels of annual trend inflation, from 0% to 35%. Setting a low value for δ, as we do, implies that the dynamics are very similar to those obtained under standard NK models when trend inflation are smaller than

[29] See Section 4.

[30] Recall that equation (16) allows us to calibrate θ as a function of markup μ and storage cost rate δ.

the upper limit. However, as $\delta > 0$, it allows us to explore the dynamics at larger long-run inflation rates, past the usual upper limit.

For dynamic simulations, we assume that the central bank has a mandated gross inflation target $\bar{\Pi} \geq 1$ (or $\bar{\pi} \geq 0$) and follows a flexible Taylor-type rule to achieve it. We consider the specification $\left(\frac{I_t}{\bar{I}}\right) = \epsilon_{i,t} \left(\frac{I_{t-1}}{\bar{I}}\right)^{\phi_i} \left[\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi} \left(\frac{X_t}{\bar{X}}\right)^{\phi_x} \left(\frac{X_t}{X_{t-1}}\right)^{\phi_{gx}} \left(\frac{Y_t}{\bar{Y}}\right)^{\phi_y} \left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_{gy}}\right]^{1-\phi_i}$, where $\epsilon_{i,t}$ is the monetary policy shock, $\phi_i \in (0,1)$ is the policy smoothing parameter, and the response parameters ϕ_π, ϕ_x, ϕ_{gx}, ϕ_y and ϕ_{gy} are consistent with stability and determinacy in equilibria with rational expectations.

Reacting to output gap growth or output growth is in line with the findings of Coibion and Gorodnichenko (2011) and Khan et al. (2020) as it generates more stabilizing properties when the trend inflation is not zero. In addition, reacting to output growth is in line with Walsh (2003), Orphanides and Williams (2007), and Coibion and Gorodnichenko (2011), and is empirically relevant.

In order to minimize any inertial dynamics coming from shocks and policy, we consider all shocks as being pure white noise, without any AR component, and set $\phi_i = 0$ in the Taylor rule. This rule is completed by setting $\phi_\pi = 1.80$ and $\phi_{gx} = 1.60$. The remaining policy parameters are set at zero. Since $\phi_i = 0$, the calibrated values for ϕ_π and ϕ_{gx} are set so responses to monetary policy shocks (annualized nominal interest rates) at zero trend inflation are roughly in line with empirical evidence for the US: a less than one-to-one response in annualized inflation, and a more than one-to-one response in aggregate output. Impulse responses to utility $\hat{\epsilon}_t$ and technology $\widehat{\mathcal{A}}_t$ are not too different from those obtained in the trend inflation literature. But we do find that responses to monetary policy shocks $\hat{\epsilon}_{i,t}$ give us important insights as trend inflation gets large.

Figure 4 shows the impulse responses for different levels of trend inflation. The top two rows show the responses under low annual trend inflation, from 0 to 10%. And the bottom two rows assesses responses under high trend inflation, from 10 to 35%. Since we are considering a range of different trend inflation levels, we normalize the shocks amplitude in order to make comparisons easier. That is, in each level of trend inflation, the amplitude of monetary policy shocks $\hat{\epsilon}_{i,t}$ is such that the annualized nominal interest rate $\hat{\imath}_t$ has unit response at period 1, when the one-off shock hit.

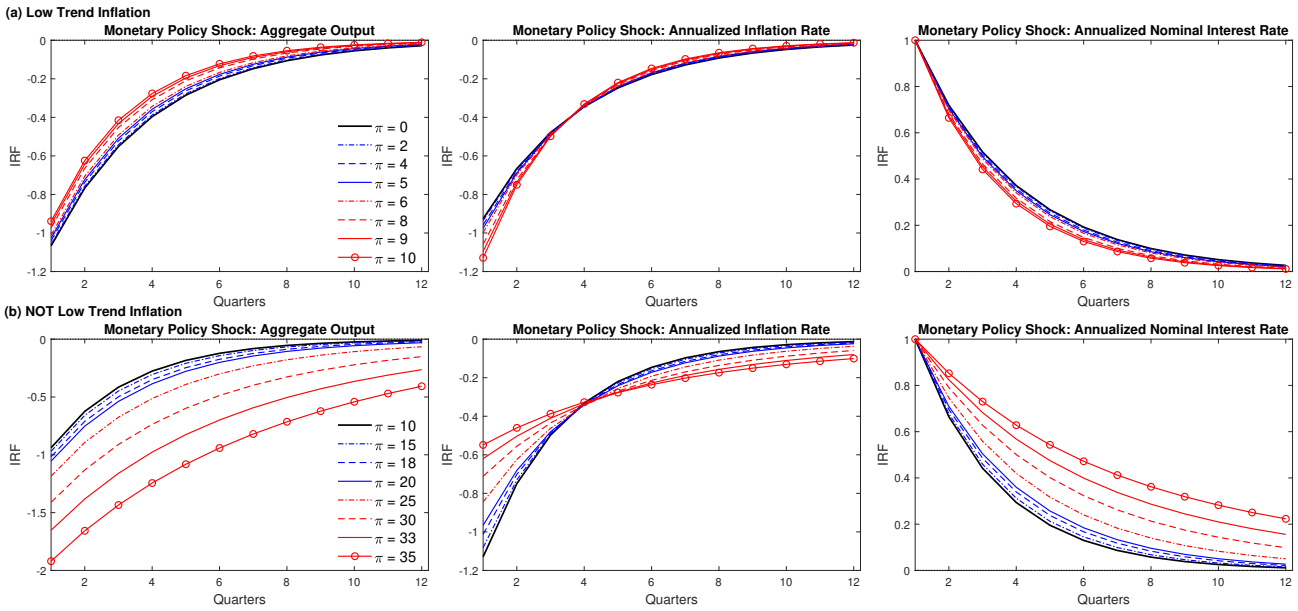


Figure 4: Impulse responses to monetary policy shocks

Notes: Impulse responses to Monetary Policy shock $\hat{\epsilon}_{i,t}$ with $\delta = 0.05$, in the Dixit-Stiglitz model with Storage Costs. In the top row, we consider low levels of trend inflation ($0 \leq \bar{\pi} \leq 10$). In the bottom row, we consider "not low" levels of trend inflation ($10 \leq \bar{\pi} \leq 35$). Shocks are normalized so that nominal interest rate $\hat{\imath}_t$ has unit response at period 1 for all levels of trend inflation.

When trend inflation is low, up to about the 10%, responses behave similarly to what we observe in standard NK models with trend inflation, in the sense that the amplitude of inflation rate responses increase with trend inflation, whereas output has its response amplitudes decreased. However, we see that there is a reversal in this pattern at high levels of trend inflation. From this point on, amplitudes of inflation responses decrease, while that of output increase, as trend inflation gets higher. It means that it is harder for central banks to curb inflation hikes and bring it down, when the average inflation sits above the 10% level. Close inspection of monetary policy shocks at large trend inflation reveals that monetary policy sharply looses its power. As trend inflation rises from the 10% level to 35%, inflation responses to nominal interest rates fast mutes down to half the amplitude it has at the 10% level. Inflation rates also become much more inertial. At the same time, output doubles its fall and requires much longer to recover. In a nutshell, the model predicts that inflation becomes much harder to tame at larger levels of trend inflation, as it lingers for much longer and requires much more output sacrifice in order to bring it down.

We highlight that these properties are in line with recent empirical results found by Canova and Forero (2024). The authors estimate a Markov-Switching model for the US with two states (high and

low inflation) from 1960 to 2023. They find that, after contractionary monetary policy shocks, inflation rates do not fall as much and become more persistent in high-inflation states when compared to low-inflation states.

7 Conclusion

We provide a resolution to a well-known issue: the steady state of the widely-studied New Keynesian (NK) models based on Calvo-pricing does not exist beyond a low single-digit trend inflation threshold, rendering them not useful for monetary policy analysis when trend inflation is not low. This 'steady state problem', where the relative price and price dispersion shoot to infinity and output goes to zero, can be mechanically resolved by assuming price indexation or increasing the price-adjustment frequency with trend inflation. These resolutions are, however, unsatisfactory and not supported by evidence. The main contribution of the paper is to establish that the root of the steady state problem is not Calvo pricing, as commonly assumed, but rather the popular Dixit-Stiglitz demand structure in NK models. We consider a general demand structure with the feature that demand remains finite when relative prices increase and show that the steady state always exists with Calvo pricing for any trend inflation level. Using this framework, we assess the properties of the Kimball-demand aggregator, which avoids the steady state problem but creates new ones. We then present a model with storage costs to augment the Dixit-Stiglitz and Kimball-demand aggregators and show that it resolves the steady state problem. Our findings show that modification of the demand structure can ensure that NK models are useful in evaluating alternative monetary policies for reducing inflation when trend inflation is not low.

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A Proof of Theorem 1

Theorem 1 *Under the Calvo staggered price setting ($\alpha > 0$) with partial indexation ($\gamma < 1$), previously described, consider the generic relative demand function $\frac{y}{Y} = f(\wp, \wp_s)$ described in Section 3.1.1, where $\wp_s \equiv \frac{P_s}{P}$, where P_s and P grow at the same rate in any steady state, such that $f(\wp, \wp_s)$ is a non-negative, continuous and differentiable function in $(\wp, \wp_s) \in (\mathbb{R}_+^* \times \mathbb{R}_+^*)$ and non-increasing in $\wp \in R_+^*$. Let $f_1(\wp, \wp_s) \equiv \frac{\partial f(\wp, \wp_s)}{\partial \wp}$ denote the partial derivative of f with respect to $\wp$. If $f(\wp, \wp_s)$ and $\wp \cdot f_1(\wp, \wp_s)$ are finite and defined at all their domain, including at $\wp \to 0$ and $\wp \to \infty$, i.e. $0 \leq \lim_{\wp \to 0} f(\wp, \wp_s) < \infty$, $0 \leq \lim_{\wp \to \infty} f(\wp, \wp_s) < \infty$, $-\infty < \lim_{\wp \to 0} \wp \cdot f_1(\wp, \wp_s) \leq 0$, and $-\infty < \lim_{\wp \to \infty} \wp \cdot f_1(\wp, \wp_s) \leq 0$, then there always exists a steady state equilibrium (with no output growth) for any level of trend inflation ($\bar{\Pi} = 1 + \bar{\pi}$), provided that $\bar{\Pi} > (\alpha)^{\frac{1}{(1-\gamma)}}$, i.e. trend inflation $\bar{\pi}$ cannot be extremely negative. For any other level of trend inflation, including all positive values, both the optimal relative price and output-gap converge to finite steady state levels.*

Proof. Consider that all shocks are kept at their means, i.e. $\epsilon_t = \bar{\epsilon}$ and $\mathcal{A}_t = \overline{\mathcal{A}}$, at all periods, so that there are no stochastic uncertainties. Also consider that gross trend inflation is kept constant at $\bar{\Pi} = 1 + \bar{\pi}$. Since we assume that P_s and P grow at the same rate in any steady state, it must be the case that $\bar{\wp}_s$ is independent of j.

For simplicity sake, let us define the function $\tilde{f}(\wp, \wp_s) \equiv \wp \cdot f_1(\wp, \wp_s)$, where $f_1(\wp, \wp_s) \equiv \frac{\partial f(\wp, \wp_s)}{\partial \wp}$.

Now, since we assume in Section 3.1.1 that the weight function is bounded, i.e. $g(\wp, \wp_s) \in (0, 1)$, we must have that $g(0, \wp_s)$ and $\lim_{\wp \to \infty} g(\wp, \wp_s)$ exist and are both finite. Given the theorem assumptions, for a fixed value $\bar{\wp}_s$, let $f_0 \equiv \lim_{\wp \to 0} f(\wp, \bar{\wp}_s)$, $f_\infty \equiv \lim_{\wp \to \infty} f(\wp, \bar{\wp}_s)$, $\tilde{f}_0 \equiv \lim_{\wp \to 0} \tilde{f}(\wp, \bar{\wp}_s)$, $\tilde{f}_\infty \equiv \lim_{\wp \to \infty} \tilde{f}(\wp, \bar{\wp}_s)$, $g_0 \equiv \lim_{\wp \to 0} g(\wp, \bar{\wp}_s)$, and $g_\infty \equiv \lim_{\wp \to \infty} g(\wp, \bar{\wp}_s)$ denote the implied finite limits, i.e. $0 \leq f_0 < \infty$, $0 \leq f_\infty < \infty$, $-\infty < \tilde{f}_0 \leq 0$, $-\infty < \tilde{f}_\infty \leq 0$, $0 < g_0 < 1$ and $0 < g_\infty < 1$.

In addition, $\bar{q}_{t,t+j} = \left(\frac{\beta}{\bar{\Pi}}\right)^j$, $\bar{\Pi}_{t,t+j}^{ind} = (\bar{\Pi}^\gamma)^j$, and $\bar{\mathcal{G}}_{t,t+j} = 1$. Therefore, if existent, the pricing steady state relations implied by the system in equations (9), (10) and (11) are:

$$
\begin{aligned}
0 = & \sum_{j=0}^{\infty} \left(\frac{\alpha\beta}{\bar{\Pi}^{(1-\gamma)}}\right)^j f\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right) - \sum_{j=0}^{\infty} \left(\frac{\alpha\beta}{\bar{\Pi}^{(1-\gamma)}}\right)^j \left[-\tilde{f}\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right)\right] \\
+ & \sum_{j=0}^{\infty} (\alpha\beta)^j \left(1 + \tfrac{1}{\xi^n}\right) \frac{(\bar{X})^{(\sigma+\omega)}}{\bar{\wp}^*} \left[-\tilde{f}\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right)\right] \left[f\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right)\right]^\omega \\
\wp_s = & (1-\alpha) \sum_{j=0}^{\infty} \left(\frac{\alpha}{\bar{\Pi}^{(1-\gamma)}}\right)^j (\bar{\wp}^*)\, g\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right) \\
1 = & (1-\alpha) \sum_{j=0}^{\infty} \left(\frac{\alpha}{\bar{\Pi}^{(1-\gamma)}}\right)^j (\bar{\wp}^*)\, f\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right)
\end{aligned}
$$

where

$$
\tilde{f}\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right) = \left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}\right) f_1\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right)
$$

It is not hard to recognize this result as a system involving five different non-negative power series, whose general format is $g_b(\rho | \bar{\wp}^*, \bar{\wp}_s) \equiv \sum_{j=0}^{\infty} \rho^j b_j(\bar{\wp}^*, \bar{\wp}_s)$, where $\rho \geq 0$. Function $b_j(\bar{\wp}^*, \bar{\wp}_s) \geq 0$ depends on j, but is independent of ρ. Of course, each one of those power series has specific parameter ρ and function $b_j(\cdot, \cdot)$, i.e. ρ can be either $\frac{\alpha\beta}{\bar{\Pi}^{(1-\gamma)}}$, $\alpha\beta$ or $\frac{\alpha}{\bar{\Pi}^{(1-\gamma)}}$, and $b_j(\cdot, \cdot)$ depends on different combinations of $f\left(\frac{1}{\bar{\Pi}^{(1-\gamma)j}} \bar{\wp}^*, \bar{\wp}_s\right)$, $\tilde{f}\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right)$ and $g\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right)$. Note that, fixing the values of $\bar{\wp}^*$ and $\wp_s$, $b_j(\bar{\wp}^*, \bar{\wp}_s)$ is a well-defined, finite and non-negative sequence in j.

All we have to do is to show that all those five non-negative power series converge. If $\bar{\Pi}^{(1-\gamma)} = 1$ (zero trend inflation), we have the trivial case in which all power series in the system are actually geometric series. In this case, given the assumption that $f(\wp, \wp_s)$, $f(\wp, \wp_s)$ and $g(\wp, \wp_s)$ are finite and defined at all their domain, convergence is always ensured, for $0 \leq \alpha < 1$ and $0 \leq \beta < 1$ in the model.

When $\bar{\Pi}^{(1-\gamma)} > 1$ (positive trend inflation), or $\bar{\Pi}^{(1-\gamma)} < 1$ (negative trend inflation), a more elaborated test is shown below.

Since the relative demand and weight functions are general, there is nothing precluding $b_j(\bar{\wp}^*, \bar{\wp}_s)$ to be zero at some points. Therefore, we resort to an auxiliary power series $g_\zeta(\rho) \equiv \sum_{j=0}^{\infty} \rho^j \zeta$, defined for an arbitrary fixed and strictly positive value $\zeta > 0$. Obviously, $g_\zeta(\rho)$ converges as long as $|\rho| < 1$, which is the case whenever $\alpha\beta < 1$, $\alpha\beta < \bar{\Pi}^{(1-\gamma)}$ and $\alpha < \bar{\Pi}^{(1-\gamma)}$. Since $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, the restriction is simplified to $\alpha < \bar{\Pi}^{(1-\gamma)}$.

In this context, let us consider an augmented power series defined as $\tilde{g}(\rho|\bar{\wp}^*, \bar{\wp}_s) \equiv g(\rho|\bar{\wp}^*, \bar{\wp}_s) + g_\zeta(\rho) = \sum_{j=0}^{\infty} \rho^j \left[\zeta + b_j(\bar{\wp}^*, \bar{\wp}_s)\right]$. By construction, all terms in this power series are strictly positive. It implies that convergence can be verified using the sufficient Ratio test, which states that the power series $\tilde{g}(\rho) = \sum_{j=0}^{\infty} \rho^j \left[\zeta + b_j(\bar{\wp}^*, \bar{\wp}_s)\right]$ converges if the limiting ratio $T_{ratio} \equiv |\rho| \lim_{j\to\infty} \left| \frac{\left[\zeta + b_{j+1}(\bar{\wp}^*, \bar{\wp}_s)\right]}{\left[\zeta + b_j(\bar{\wp}^*, \bar{\wp}_s)\right]} \right|$ is smaller than unity, i.e. if $T_{ratio} < 1$, and diverges if $T_{ratio} > 1$. At the boundary $T_{ratio} = 1$, the test is mute, as the power series can either converge or diverge, depending on its functional form.

Note that $\lim_{j\to\infty} \tilde{f}\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right) = \tilde{f}_0$ when $\bar{\Pi}^{(1-\gamma)} > 1$, whereas $\lim_{j\to\infty} \tilde{f}\left(\frac{\bar{\wp}^*}{\bar{\Pi}^{(1-\gamma)j}}, \bar{\wp}_s\right) = \tilde{f}_\infty$ when $\bar{\Pi}^{(1-\gamma)} < 1$. Since f_0, f_∞, $\tilde{f}_0$, $\tilde{f}_\infty$, g_0 and g_∞ are all finite and defined, it is easy to verify that the limiting ratio is $T_{ratio} = |\rho|$ for all five augmented power series, regardless of whether $\bar{\Pi}^{(1-\gamma)} > 1$ or $\bar{\Pi}^{(1-\gamma)} < 1$. And so, the ratio test predicts that all of them converge if $|\rho| < 1$. Since the auxiliary power series converges, it implies that all five original power series also converge under the same conditions.

Therefore, we conclude that the system implies a convergent relation and provides an implicit solution for the steady state levels $\bar{\wp}^*$, $\bar{\wp}_s$ and $\bar{X}$, as long as $\alpha < \bar{\Pi}^{(1-\gamma)}$.

Since $\bar{\Pi} = 1 + \bar{\pi}$, the condition is always satisfied for non-negative trend inflation ($\bar{\Pi} \geq 1$). Only for some negative trend inflation, the condition can be violated. That is, convergence is only achieved if trend inflation $\bar{\pi}$ is not extremely negative, i.e. when $\bar{\pi} > (\alpha)^{\frac{1}{(1-\gamma)}} - 1$. ■

B Kimball Aggregator

In Kimball (1995), consumption over all differentiated goods $c_t(z)$ are aggregated into a bundle C_t, according to $1 = \int_0^1 G\left(\frac{c_t(z)}{C_t}\right) dz$, where function $G(\varkappa)$ satisfies $G(1) = 1$, $G'(\varkappa) > 0$, and $G''(\varkappa) < 0$, for all $\varkappa \geq 0$. In this context, Dotsey and King (2005) propose the particular functional form $G\left(\frac{c_t(z)}{C_t}\right) = \frac{\mathfrak{m}}{1+\varphi}\left[(1+\varphi)\frac{c_t(z)}{C_t} - \varphi\right]^{\frac{1}{\mathfrak{m}}} + 1 - \frac{\mathfrak{m}}{1+\varphi}$, where $\mathfrak{m} \equiv \frac{\mu_k(1+\varphi)}{(1+\mu_k\varphi)}$ is a composite parameter, $\mu_k \geq 1$ is the elasticity parameter, which matches the implicit markup rate μ under flexible prices, and $\varphi \leq 0$ sets the aggregation curvature. If $\varphi = 0$, $G(\cdot)$ simplifies into the standard Dixit and Stiglitz (1977) aggregation form. Allowing for smooth-kinked demand function, it has also been used by Levin et al. (2007), Harding et al. (2022) and Kurozumi and Van Zandweghe (2024).

According the notation used in Section 3.1, this model sets $\delta = 0$. The literature typically derives the utility-based demand function by choosing $c_t(z)$ to minimize expenditure $P_t C_t \equiv \int_0^1 p_t(z)\, c_t(z)\, dz$, subject to only one restriction, the Kimball aggregation $1 = \int_0^1 G\left(\frac{c_t(z)}{C_t}\right) dz$. The implied demand func-

tion and implied price aggregation are:

$$\frac{c_t(z)}{C_t} = f\left(\wp_t\left(z\right), \wp_{s,t}\right) = \begin{cases} \frac{1}{(1+\varphi)}\left(\frac{\wp_t(z)}{\wp_{k,t}}\right)^{\varpi} + \frac{\varphi}{(1+\varphi)} & ; \text{if } \left(\frac{\wp_t(z)}{\wp_{k,t}}\right) \leq (-\varphi)^{\frac{1}{\varpi}} \\ 0 & ; \text{if } \left(\frac{\wp_t(z)}{\wp_{k,t}}\right) > (-\varphi)^{\frac{1}{\varpi}} \end{cases} \tag{19}$$

$$\wp_{k,t} \equiv (1+\varphi) - \varphi\wp_{s,t} \quad ; \; P_{s,t} \equiv \int_0^1 p_t\left(z\right) dz \quad ; \; 1 = \int_0^1 \left(\frac{\wp_t(z)}{\wp_{k,t}}\right)^{-\frac{1}{(\mathfrak{m}-1)}} dz$$

where $\varpi \equiv \frac{\mu_k(1+\varphi)}{(1-\mu_k)} = -\frac{\mathfrak{m}}{(\mathfrak{m}-1)}$, $P_{k,t}$ is an auxiliary composite price aggregate, $P_{s,t}$ is the average price. Paralleling the notation used in Section 3.1.1, we define $\wp_t\left(z\right) \equiv \frac{p_t(z)}{P_t}$ as the relative price of firm z, and $\wp_{s,t} = \frac{P_{s,t}}{P_t}$ as the average relative price. We also set $\wp_{k,t} \equiv \frac{P_{k,t}}{P_t}$ as the auxiliary composite relative price. In addition, it is straightforward to verify that the price aggregation $1 = \int_0^1 \left(\frac{\wp_t(z)}{(1+\varphi)-\varphi\wp_{s,t}}\right)^{-\frac{1}{(\mathfrak{m}-1)}} dz$ is equivalent to $P_t = \int_0^1 p_t\left(z\right) f\left(\wp_t\left(z\right), \wp_{s,t}\right) dz$.

Under this type of Kimball's aggregation, the firm z's price $p_t\left(z\right)$ elasticity and superelasticity are: (i) $\xi_t\left(z\right) = -\varpi\left(\frac{\wp_t(z)}{\wp_{k,t}}\right)^{\varpi}\left[\left(\frac{\wp_t(z)}{\wp_{k,t}}\right)^{\varpi} + \varphi\right]^{-1}$ and $\eta_t\left(z\right) = \varpi\varphi\left[\left(\frac{\wp_t(z)}{\wp_{k,t}}\right)^{\varpi} + \varphi\right]^{-1}$, if $\left(\frac{\wp_t(z)}{\wp_{k,t}}\right) \leq (-\varphi)^{\frac{1}{\varpi}}$; or (ii) $\xi_t\left(z\right) = 0$ and $\eta_t\left(z\right) = 0$, if $\left(\frac{\wp_t(z)}{\wp_{k,t}}\right) > (-\varphi)^{\frac{1}{\varpi}}$.

In macroeconomic models, the way to generate empirically observed persistent non-neutrality in aggregate output is to combine real and nominal rigidities. However, empirical evidence suggest that price stickiness is not so large.[31] Therefore, macroeconomists tend to use theoretical models with large real rigidities (see e.g. Ball and Romer (1990), Basu (1995), Blanchard and Gali (2007)). In this regard, Kimball's implied real rigidity can be easily computed using (12), evaluated in the steady state equilibrium with flexible prices:[32]

$$\kappa_{\text{real}}^{Kimball} = \frac{(\sigma+\omega)}{1-\mu\varphi+\frac{\mu}{(\mu-1)}\omega} \quad ; \; \psi_{\text{real}}^{Kimball} = \frac{1}{\kappa_{\text{real}}}$$

Therefore, for a given preferences/production structure represented by σ and ω, a large degree of large real rigidity $\psi_{\text{real}}^{Kimball}$ can be achieved with a convenient balance between a large demand curvature $(\varphi << 0)$ and a appropriate markup $\mu > 1$.

[31] As in e.g. Bils and Klenow (2004) and Nakamura and Steinsson (2008), estimated median duration between price changes ranges from about 4.5 months, when sales are included, to 10 months, when they are excluded.

[32] Considering that μ here is the gross markup rate, the component $(1-\mu\varphi)$ is the same found in Levin et al. (2007) and Harding et al. (2022), as their models implies $\omega = 0$.

B.1 Kimball NK Model

In general equilibrium, based on the generic model shown in Section 3, we have:

$$
\begin{aligned}
1 = \quad & \beta E_t \left(\tfrac{\epsilon_{t+1}}{\epsilon_t} \left(\tfrac{Y_t}{Y_{t+1}} \right)^{\sigma} \tfrac{I_t}{\Pi_{t+1}} \right) \\
q_t = \quad & \beta \tfrac{\epsilon_t}{\epsilon_{t-1}} \left(\tfrac{Y_{t-1}}{Y_t} \right)^{\sigma} \tfrac{1}{\Pi_t} \\
\left(\tfrac{I_t}{\bar{I}} \right) = \quad & \epsilon_{\mathfrak{i},t} \left(\tfrac{I_{t-1}}{\bar{I}} \right)^{\phi_{\mathfrak{i}}} \left[\left(\tfrac{\Pi_t}{\bar{\Pi}} \right)^{\phi_\pi} \left(\tfrac{X_t}{\bar{X}} \right)^{\phi_x} \left(\tfrac{Y_t}{Y_{t-1}} \right)^{\phi_{gy}} \left(\tfrac{Y_t}{\bar{Y}} \right)^{\phi_y} \right]^{(1-\phi_{\mathfrak{i}})} \\
\left(Y_t^n \right)^{(\sigma+\omega)} = \quad & \tfrac{1}{\mu} \tfrac{\varepsilon}{\chi} \epsilon_t \left(\mathcal{A}_t \right)^{(1+\omega)} \\
X_t = \quad \tfrac{Y_t}{Y_t^n} \quad & ; \; q_{t,t+j} = \quad q_{t+1} q_{t+1,t+j} \qquad \text{for } j \geq 1 \text{ and } q_{t,t} = 1 \\
\mathcal{G}_t = \quad \tfrac{Y_t}{Y_{t-1}} \quad & ; \; \Pi_{t,t+j} = \quad \Pi_{t+1} \Pi_{t+1,t+j} \qquad \text{for } j \geq 1 \text{ and } \Pi_{t,t} = 1 \\
\Pi_t^{ind} = \quad \Pi_{t-1}^{\gamma} \quad & ; \; \Pi_{t,t+j}^{ind} = \quad \Pi_{t+1}^{ind} \Pi_{t+1,t+j}^{ind} \qquad \text{for } j \geq 1 \text{ and } \Pi_{t,t}^{ind} = 1 \\
& ; \; \mathcal{G}_{t,t+j} = \quad \mathcal{G}_{t+1} \mathcal{G}_{t+1,t+j} \qquad \text{for } j \geq 1 \text{ and } \mathcal{G}_{t,t} = 1
\end{aligned}
$$

$$
\begin{aligned}
\wp_{s,t} = \quad & (1-\alpha) \, \wp_t^* + \alpha \tfrac{\Pi_t^{ind}}{\Pi_t} \wp_{s,t-1} \\
(1+\varphi) = \quad & \wp_{k,t} + \varphi \wp_{s,t} \\
\left(\wp_{k,t} \right)^{-\frac{1}{(\mathfrak{m}-1)}} = \quad & (1-\alpha) \left(\wp_t^* \right)^{-\frac{1}{(\mathfrak{m}-1)}} + \alpha \left(\tfrac{\Pi_t^{ind}}{\Pi_t} \right)^{-\frac{1}{(\mathfrak{m}-1)}} \left(\wp_{k,t-1} \right)^{-\frac{1}{(\mathfrak{m}-1)}} \\
\wp_t^* = \quad & \varphi \left(\mathfrak{m}-1 \right) \left(\wp_t^* \right)^{\left(1+\frac{\mathfrak{m}}{(\mathfrak{m}-1)} \right)} \tfrac{N_{1,t}}{D_t} + \tfrac{\mathfrak{m}}{\mu} \tfrac{N_{2,t}}{D_t} \\
D_t = \quad & \left(\wp_{k,t} \right)^{\frac{\mathfrak{m}}{(\mathfrak{m}-1)}} + \alpha E_t q_{t+1} \mathcal{G}_{t+1} \Pi_{t+1} \left(\tfrac{\Pi_{t+1}}{\Pi_{t+1}^{ind}} \right)^{\frac{1}{(\mathfrak{m}-1)}} D_{t+1} \\
N_{1,t} = \quad & 1 + \alpha E_t q_{t+1} \mathcal{G}_{t+1} \Pi_{t+1}^{ind} N_{1,t+1} \\
N_{2,t} = \quad & \mu E_t \textstyle\sum_{j=0}^{\infty} q_{t,t+j} \alpha^j \mathcal{G}_{t,t+j} \Pi_{t,t+j} \left(\tfrac{\Pi_{t,t+j}}{\Pi_{t,t+j}^{ind}} \right)^{\frac{\mathfrak{m}}{(\mathfrak{m}-1)}} \left(\wp_{k,t+j} \right)^{\frac{\mathfrak{m}}{(\mathfrak{m}-1)}} mc_{t,t+j}^* \\
mc_{t,t+j}^* = \quad & \tfrac{1}{\mu} \left(X_{t+j} \right)^{(\sigma+\omega)} \left[\tfrac{1}{(1+\varphi)} \left(\tfrac{\Pi_{t,t+j}^{ind}}{\Pi_{t,t+j}} \tfrac{\wp_t^*}{\wp_{k,t+j}} \right)^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}} + \tfrac{\varphi}{(1+\varphi)} \right]^{\omega}
\end{aligned}
$$

Since power ω in the equation for $mc_{t,t+j}^*$ is not a positive integer, we cannot write $N_{2,t}$ in a finite recursive way. Therefore, simulations are to be carried out using the same approximation we use for the storage cost model.

B.2 Steady state

Given an exogenous level of trend inflation $\bar{\Pi}$, the steady state levels can be numerically obtained as follows. First, we compute $\bar{I}$, $\bar{q}$, and $\bar{Y}^n$:

$$
\bar{I} = \tfrac{\bar{\Pi}}{\beta} \quad ; \; \bar{q} = \tfrac{\beta}{\bar{\Pi}} \quad ; \; \left(\bar{Y}^n \right)^{(\sigma+\omega)} = \tfrac{1}{\mu} \tfrac{\varepsilon}{\chi} \bar{\epsilon} \left(\overline{\mathcal{A}} \right)^{(1+\omega)}
$$

Next, we use a numerical code to solve the following non-linear system for relative prices $\bar{\wp}^*$, $\bar{\wp}_s$, and $\bar{\wp}_k$:

$$
\bar{\wp}^* = \tfrac{(1+\varphi)}{\left[\left(\frac{1-\alpha}{1-\bar{\alpha}_{k1}} \right)^{-(\mathfrak{m}-1)} + \varphi \frac{(1-\alpha)}{(1-\bar{\alpha}_{k3})} \right]} \quad ; \; \bar{\wp}_s = \tfrac{(1-\alpha)}{(1-\bar{\alpha}_{k3})} \bar{\wp}^* \quad ; \; \bar{\wp}_k = (1+\varphi) - \varphi \bar{\wp}_s
$$

where

$$\bar{\alpha}_{k1} = \alpha\left(\bar{\Pi}\right)^{\frac{(1-\gamma)}{(\mathfrak{m}-1)}} \quad ; \; \bar{\alpha}_{k2} = \alpha\bar{\Pi}^{\frac{\mathfrak{m}(1-\gamma)}{(\mathfrak{m}-1)}} \quad ; \; \bar{\alpha}_{k3} = \alpha\bar{\Pi}^{-(1-\gamma)}$$

The following step is to find the gross output gap $\bar{X}$:

$$\bar{S}_d \equiv \sum_{j=0}^{\infty}\left(\bar{\alpha}_{k2}\beta\right)^j\left[\frac{1}{(1+\varphi)}\left(\frac{\bar{\alpha}_{k2}}{\alpha}\right)^j\left(\frac{\bar{\wp}^*}{\bar{\wp}_k}\right)^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}+\frac{\varphi}{(1+\varphi)}\right]^{\omega} \quad ; \; \bar{D} = \frac{\left(\bar{\wp}_k\right)^{\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}}{(1-\bar{\alpha}_{k1}\beta)} \quad ; \; \bar{N}_1 = \frac{1}{(1-\bar{\alpha}_{k3}\beta)}$$

$$\bar{N}_2 = \frac{\mu}{\mathfrak{m}}\left(\bar{\wp}^*\right)\left(\bar{D}-\varphi\left(\mathfrak{m}-1\right)\left(\bar{\wp}^*\right)^{\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}\bar{N}_1\right) \quad ; \; \left(\bar{X}\right)^{(\sigma+\omega)} = \frac{\bar{N}_2}{\left(\bar{\wp}_k\right)^{\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}}\frac{1}{\bar{S}_d} \quad ; \; \bar{Y} = \bar{X}\bar{Y}^n$$

If $\frac{\bar{\alpha}_{k2}}{\alpha} \leq 1$ the infinite sum $\bar{S}_d$ converges when $\left(\bar{\alpha}_{k2}\beta\right) < 1$. If $\frac{\bar{\alpha}_{k2}}{\alpha} > 1$, it converges when $\left(\bar{\alpha}_{k2}\beta\right)\left(\frac{\bar{\alpha}_{k2}}{\alpha}\right)^{\omega} < 1$.

The infinite sum $\bar{S}_d$ is generally numerically retrieved by using a finite sum in $j = \{0, 1, ..., J\}$ for a large J. In this paper, we use $J = 10000$. For numerical stability when $\frac{\bar{\alpha}_{k2}}{\alpha} > 1$, $\bar{S}_d$ is better computed using $\bar{S}_d = \sum_{j=0}^{\infty}\left(\bar{\alpha}_{k2}\beta\left(\frac{\bar{\alpha}_{k2}}{\alpha}\right)^{\omega}\right)^j\left[\frac{1}{(1+\varphi)}\left(\frac{\bar{\wp}^*}{\bar{\wp}_k}\right)^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}+\frac{\varphi}{(1+\varphi)\left(\frac{\bar{\alpha}_{k2}}{\alpha}\right)^j}\right]^{\omega}$.

Alternatively, if ω is a positive integer, it is feasible to derive an exact closed form solution for $\bar{S}_d$. For that, we only need to expand the term in brackets and obtain a couple of infinite sums that allow for closed form solutions. For instance, if $\omega = 2$, $\frac{\left(\bar{\alpha}_{k2}\right)^2}{\alpha^2}\beta < 1$ and $\bar{\alpha}_{k2}\beta < 1$, we obtain:

$$\begin{aligned}
S \equiv & \sum_{j=0}^{\infty}\left(\bar{\alpha}_{k2}\beta\right)^j\left[\frac{1}{(1+\varphi)}\left(\frac{\bar{\alpha}_{k2}}{\alpha}\right)^j\left(\frac{\bar{\wp}^*}{\bar{\wp}_k}\right)^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}+\frac{\varphi}{(1+\varphi)}\right]^2 \\
= & \sum_{j=0}^{\infty}\left(\bar{\alpha}_{k2}\beta\right)^j\left[\frac{1}{(1+\varphi)^2}\left(\frac{\bar{\wp}^*}{\bar{\wp}_k}\right)^{-\frac{2\mathfrak{m}}{(\mathfrak{m}-1)}}\left(\left(\frac{\bar{\alpha}_{k2}}{\alpha}\right)^2\right)^j+2\frac{1}{(1+\varphi)}\left(\frac{\bar{\wp}^*}{\bar{\wp}_k}\right)^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}\left(\frac{\bar{\alpha}_{k2}}{\alpha}\right)^j+\frac{\varphi^2}{(1+\varphi)^2}\right] \\
= & \frac{\left(\bar{\wp}^*/\bar{\wp}_k\right)^{-\frac{2\mathfrak{m}}{(\mathfrak{m}-1)}}}{(1+\varphi)^2}\sum_{j=0}^{\infty}\left(\frac{\left(\bar{\alpha}_{k2}\right)^2}{\alpha^2}\beta\right)^j+\frac{2\left(\bar{\wp}^*/\bar{\wp}_k\right)^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}}{(1+\varphi)}\sum_{j=0}^{\infty}\left(\frac{\left(\bar{\alpha}_{k2}\right)^2}{\alpha}\beta\right)^j+\frac{\varphi^2}{(1+\varphi)^2}\sum_{j=0}^{\infty}\left(\bar{\alpha}_{k2}\beta\right)^j \\
= & \frac{1}{(1+\varphi)^2}\frac{\left(\bar{\wp}^*/\bar{\wp}_k\right)^{-\frac{2\mathfrak{m}}{(\mathfrak{m}-1)}}}{\left(1-\frac{\left(\bar{\alpha}_{k2}\right)^2}{\alpha^2}\beta\right)}+\frac{2}{(1+\varphi)}\frac{\left(\bar{\wp}^*/\bar{\wp}_k\right)^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}}{\left(1-\frac{\left(\bar{\alpha}_{k2}\right)^2}{\alpha}\beta\right)}+\frac{\varphi^2}{(1+\varphi)^2}\frac{1}{(1-\bar{\alpha}_{k2}\beta)}
\end{aligned}$$

B.3 Constrained Demand

Given the extra demand kink at $\left(\frac{p_t(z)}{P_{k,t}}\right) = (-\varphi)^{\frac{1}{\omega}}$, this particular case of Kimball's aggregation implies firms will typically not set any price $p_t(z)$ larger than $(-\varphi)^{\frac{1}{\omega}}\left[(1+\varphi)P_t - \varphi P_{s,t}\right]$, as would lead to zero demand. If $\varphi = 0$, in particular, the threshold $(-\varphi)^{\frac{1}{\omega}}$ is infinity. And so the restriction $\frac{c_t(z)}{C_t} \geq 0$ is never biding under Dixit and Stiglitz (1977) aggregation. If $\varphi < 0$, however, we argue that the condition for non-zero demand $\left(\frac{p_t(z)}{P_{k,t}}\right) < (-\varphi)^{\frac{1}{\omega}}$ might not always hold with real data. That is, an empirical test for this type of aggregation is to verify whether empirical values of relative prices $\frac{p_t(z)}{P_{k,t}}$ are smaller than $(-\varphi)^{\frac{1}{\omega}}$.

Here, as it is not the main scope of this paper, we do not propose a sophisticated formal econometric test. Rather, we propose a simple approach. So, the question is whether we can find a way to compute P_t and $P_{s,t}$ using typical moments from price samples, which would provide us with an estimate for $P_{k,t} \equiv \left[(1+\varphi)P_t - \varphi P_{s,t}\right]$. And here lies a slight caveat. While $P_{s,t} \equiv \int_0^1 p_t(z)\,dz$ is the simple average price, which can be easily estimated using sample price means, P_t has no obvious empirical counterpart. Therefore, P_t is not easily empirically retrievable without relying on a general equilibrium model.

In order to tackle this issue, we propose a second-order approximation approach. Consider the typical price aggregation into P_t, abstracting from the relative price threshold. It can be written as $(P_{k,t})^{-\frac{1}{(\mathfrak{m}-1)}} = \int_0^1 (p_t(z))^{-\frac{1}{(\mathfrak{m}-1)}} dz$. Note that a second-order approximation of $(p_t(z))^{-\frac{1}{(\mathfrak{m}-1)}}$, about the average price $P_{s,t} \equiv \int_0^1 p_t(z)\, dz$, is:

$$(p_t(z))^{-\frac{1}{(\mathfrak{m}-1)}} \approx (P_{s,t})^{-\frac{1}{(\mathfrak{m}-1)}} - \frac{(P_{s,t})^{-\frac{\mathfrak{m}}{(\mathfrak{m}-1)}}}{(\mathfrak{m}-1)} (p_t(z) - P_{s,t}) + \frac{1}{2} \frac{\mathfrak{m}(P_{s,t})^{-\frac{2\mathfrak{m}-1}{(\mathfrak{m}-1)}}}{(\mathfrak{m}-1)^2} (p_t(z) - P_{s,t})^2$$

It implies that

$$(P_{k,t})^{-\frac{1}{(\mathfrak{m}-1)}} = \int_0^1 (p_t(z))^{-\frac{1}{(\mathfrak{m}-1)}} dz \approx (P_{s,t})^{-\frac{1}{(\mathfrak{m}-1)}} \left[1 + \frac{1}{2} \frac{\mathfrak{m}}{(\mathfrak{m}-1)^2} \int_0^1 \left[\frac{p_t(z)}{P_{s,t}} - P_{s,t}\right]^2 dz\right]$$

Therefore, we obtain the following relation between $P_{k,t}$ and $P_{s,t}$:

$$\begin{aligned} P_{k,t} &\approx \left[1 + \frac{1}{2} \frac{\mathfrak{m}}{(\mathfrak{m}-1)^2} s_{s,t}^2\right]^{-(\mathfrak{m}-1)} P_{s,t} \\ s_{s,t}^2 &\equiv \int_0^1 \left[\frac{p_t(z)}{P_{s,t}} - 1\right]^2 dz \end{aligned} \tag{20}$$

where $P_{k,t} \equiv [(1+\varphi) P_t - \varphi P_{s,t}]$. Since $P_{s,t} \equiv \int_0^1 p_t(z)\, dz$ and $\int_0^1 \frac{p_t(z)}{P_{s,t}} dz = 1$, $s_{s,t}$ is the cross-section standard deviation of relative prices $\frac{p_t(z)}{P_{s,t}}$, which is a measure of relative price dispersion.

Recalling that $\mathfrak{m} \equiv \frac{\mu_k(1+\varphi)}{(1+\mu_k\varphi)}$, it is not hard to verify that: (i) $\frac{p_t(z)}{P_{s,t}} < \frac{p_t(z)}{P_{k,t}} < \frac{p_t(z)}{P_t}$, if $\varphi \in (-1, 0]$; and (ii) $\frac{p_t(z)}{P_{k,t}} \leq \frac{p_t(z)}{P_{s,t}} < \frac{p_t(z)}{P_t}$, if $\varphi \leq -1$. In both cases, all three relative prices are very close to each other whenever price dispersion is $s_{s,t}$ small.

Since the cross section average relative price is unity, i.e. $\int_0^1 \frac{p_t(z)}{P_{s,t}} dz = 1$, we can reasonable conclude that $\int_0^1 \frac{p_t(z)}{P_{k,t}} dz$ is also close to unity. And there lies a potential empirical issue with this type Kimball's demand function. Recall that its relative price constraint for non-zero demand is $\left(\frac{p_t(z)}{P_{k,t}}\right) \leq (-\varphi)^{\frac{1}{\varpi}}$, for $\varpi \equiv \frac{\mu_k(1+\varphi)}{(1-\mu_k)}$.

In the literature, the upper limit $(-\varphi)^{\frac{1}{\varpi}}$ for relative prices is generally very close to unity when φ is set, or implied, using common values estimated or calibrated for the US. In order to verify this property, consider first that μ_k matches the implicit markup rate μ under flexible prices. In this case, some typical calibrations for the US are the following ones: (i) $\mu = 1.10$, $\varphi = -12.2$ and $(-\varphi)^{\frac{1}{\varpi}} = 1.021$ in Harding et al. (2022); (ii) $\mu = 1.17$, $\varphi = -8$ and $(-\varphi)^{\frac{1}{\varpi}} = 1.043$ in Levin et al. (2007);[33] and (iii) $\mu = 1.61$ (estimated), $\varphi = -3.79$ and $(-\varphi)^{\frac{1}{\varpi}} = 1.198$ in Smets and Wouters (2007).[34] In addition, obtaining a better marginal likelihood statistics for model comparison,[35] Harding et al. (2022) re-estimate Smets and Wouters (2007) model with a different prior distribution. Their new posterior modes imply $\mu = 1.34$ (estimated), $\varphi = -16.37$ and $(-\varphi)^{\frac{1}{\varpi}} = 1.047$.

As for the empirical dispersion of relative prices, we consider the Kaplan and Menzio (2015) results described in Section 2. In particular, we make a conservative choice by considering the authors' Brand Aggregation, in which products have at least the same features and the same size, and so are in line

[33] In Levin et al. (2007), the elasticity of substitution between goods ϵ can be mapped into our notation as $\mu = \frac{\epsilon}{(\epsilon-1)}$. The authors calibrated $\epsilon = 7$, and so $\mu = \frac{7}{6}$.

[34] In Smets and Wouters (2007), the demand's curvature parameter ϵ_p can be mapped into our notation as $\epsilon_p = -\frac{\mu\varphi}{(\mu-1)}$. The authors calibrated $\epsilon_p = 10$ and estimated the gross markup rate at $\mu = 1.61$.

[35] The authors obtain a marginal likelihood gain of 5 log points.

with what economists usually think about commodity goods. Under this aggregation, the authors find that the empirical standard-deviation of relative prices, relatively to the sample average price $P_{a,t}$, is 0.25.[36] Notice that, under this type of Kimball aggregation, $P_s = P_a$. As depicted in Section 2, the authors' findings imply that a 80% confidence interval for empirical relative prices in the US are at least ranging from $\left(\frac{p(z)}{P_s}\right)_{0.10} = 0.68$ to $\left(\frac{p(z)}{P_s}\right)_{0.90} = 1.38$.

Using approximation (20), with standard deviation $s_s = 0.25$, and considering the authors' different calibration options for μ and φ, we are able to compute the implied 80% confidence intervals for $\left(\frac{p(z)}{P_k}\right)$ as follows:

$$\left(\frac{p(z)}{P_k}\right)_{0.10} = \left(\frac{P_s}{P_k}\right)\left(\frac{p(z)}{P_s}\right)_{0.10} \quad ; \quad \left(\frac{p(z)}{P_k}\right)_{0.90} = \left(\frac{P_s}{P_k}\right)\left(\frac{p(z)}{P_s}\right)_{0.90}$$

Therefore, considering different calibration options for μ and φ, Table 1 verifies whether the implied 80% confidence intervals for $\left(\frac{p(z)}{P_k}\right)$ are at least totally included in the feasibility region $\left(\frac{p(z)}{P_k}\right) \leq (-\varphi)^{\frac{1}{\omega}}$. Of course, this back-of-the-envelope analysis is by no means meant to be a formal hypothesis test, but the fact that all 90% quantiles surpass the theoretical Kimball's upper limit $(-\varphi)^{\frac{1}{\omega}}$ strongly suggests that an important fraction of relative prices are larger than the implied Kimball's upper limit for relative prices $(-\varphi)^{\frac{1}{\omega}}$. This conclusion is specially so for cases in which $(-\varphi)^{\frac{1}{\omega}}$ is very close to unity. This result is in line with simulations carried out by Klenow and Willis (2016), who find that about 15% of goods end up with zero relative demand when the demand function is Kimball-based with large curvature.

Table 1: Kimball's Relative Prices - Empirical confidence intervals and Kimball's upper limit

Authors	μ	φ	$\left(\frac{P_s}{P_k}\right)$	$(-\varphi)^{\frac{1}{\omega}}$	$\left(\frac{p(z)}{P_k}\right)_{0.90}$
Harding et al. (2022)^a	1.10	−12.2	0.95	1.02	1.31
Harding et al. (2022)^b	1.34	−16.4	0.93	1.05	1.28
Levin et al. (2007)	1.17	−8.0	0.92	1.04	1.27
Smets and Wouters (2007)	1.61	−3.8	0.88	1.20	1.21

Notes: The empirical relative price 90% quantile is computed using Equation (20) and Kaplan and Menzio (2015) estimates. Kimball's relative price upper bound is $(-\varphi)^{\frac{1}{\omega}}$. Harding et al. (2022) first (a) calibrates $\mu = 1.10$ and $\varphi = -12.2$; and then (b) estimates $\mu = 1.34$ and $\varphi = -16.4$ using Smets and Wouters (2007) model with a different prior distribution. Again, ω is defined as $\omega \equiv \frac{\mu_k(1+\varphi)}{(1-\mu_k)}$, and $\mu_k = \mu$.

C Storage Cost Model

The representative storage firm's nominal revenue is $R_{s,t} = \int_0^1 P_{s,t} c_{s,t}(z)\, dz = \delta P_{s,t} C_{s,t}$, where $C_{s,t} \equiv \int_0^1 c_t(z)\, dz$ is the average consumption (arithmetic mean). As shown further on, the average consumption $C_{s,t}$ will only be the same as the aggregate consumption C_t when all goods z are perfect substitutes. We assume that the storage firm production function is such that it needs to consume $c_{s,t}(z)$ units of goods produced by each firm $z \in (0,1)$, so its nominal cost is $Cost_{s,t} = \int_0^1 p_t(z)\, c_{s,t}(z)\, dz = \delta \int_0^1 p_t(z)\, c_t(z)\, dz$. This representative firm sets its price at a zero-profit condition $R_{s,t} = Cost_{s,t}$. Coupling this condition with the aggregate price definition $(1+\delta) P_t C_t \equiv$

[36] Here, we are abstracting from frequency considerations the authors dealt with when using empirical data.

$\int_0^1 (p_t(z) + \delta P_{s,t}) c_t(z)\, dz$ allows us to obtain two important results:

$$\begin{aligned}
(1+\delta) P_t C_t = &\int_0^1 (p_t(z) + \delta P_{s,t}) c_t(z)\, dz = \int_0^1 p_t(z) c_t(z)\, dz + \delta P_{s,t} \int_0^1 c_t(z)\, dz \\
= &\int_0^1 p_t(z) c_t(z)\, dz + \delta P_{s,t} C_{s,t} = \int_0^1 p_t(z) c_t(z)\, dz + R_{s,t} \\
= &\int_0^1 p_t(z) c_t(z)\, dz + Cost_{s,t} = \int_0^1 p_t(z) c_t(z)\, dz + \delta \int_0^1 p_t(z) c_t(z)\, dz \\
= &(1+\delta) \int_0^1 p_t(z) c_t(z)\, dz \\
\therefore P_t C_t = &\int_0^1 p_t(z) c_t(z)\, dz
\end{aligned}$$

Since $R_{s,t} = Cost_{s,t}$, it implies that $P_{s,t} C_{s,t} = \int_0^1 p_t(z) c_t(z)\, dz = P_t C_t$. Therefore, we obtain two important results:

$$P_t C_t = \int_0^1 p_t(z) c_t(z)\, dz \quad ; \; P_{s,t} = P_t \tfrac{C_t}{C_{s,t}}$$

where $C_{s,t} \equiv \int_0^1 c_t(z)\, dz$.

C.1 Aggregate price as a weighted average of individual prices

Since $C_{s,t} \equiv \int_0^1 c_t(z)\, dz$ and $P_{s,t} = P_t \frac{C_t}{C_{s,t}}$, we obtain:

$$\frac{P_t}{P_{s,t}} = \int_0^1 \frac{c_t(z)}{C_t} dz = \int_0^1 (1+\delta)^\theta \left(\frac{p_t(z)}{P_t} + \delta \frac{P_{s,t}}{P_t} \right)^{-\theta} dz$$

Moreover, since $P_t = \int_0^1 p_t(z) f(\wp_t(z), \wp_{s,t})\, dz$, we obtain:

$$P_t = \int_0^1 p_t(z) (1+\delta)^\theta \left(\frac{p_t(z)}{P_t} + \delta \frac{P_{s,t}}{P_t} \right)^{-\theta} dz$$

Therefore, the storage price can be written as follows:

$$P_{s,t} = \frac{\int_0^1 \left(\frac{p_t(z)}{P_t} + \delta \frac{P_{s,t}}{P_t} \right)^{-\theta} p_t(z)\, dz}{\int_0^1 \left(\frac{p_t(z)}{P_t} + \delta \frac{P_{s,t}}{P_t} \right)^{-\theta} dz} = \int_0^1 g(\wp_t(z), \wp_{s,t}) p_t(z)\, dz$$

where $g(\wp_t(z), \wp_{s,t}) = \frac{(\wp_t(z) + \delta \wp_{s,t})^{-\theta}}{\int_0^1 (\wp_t(z) + \delta \wp_{s,t})^{-\theta} dz}$.

C.2 General Equilibrium

The composite parameters are:

$$\omega \equiv \tfrac{(1+\nu)}{\varepsilon} - 1 \quad ; \; \theta_1 \equiv \theta(1+\omega) \quad ; \; \mu_\theta \equiv \tfrac{\theta}{(\theta-1)} \quad ; \; \mu_\delta \equiv \tfrac{\theta}{\theta-(1+\delta)} \quad ; \; \mu = \tfrac{\mu_\delta}{(1+\delta)}$$

The dynamic equations are:

$$
\begin{aligned}
1 = &\quad \beta E_t \left(\tfrac{\epsilon_{t+1}}{\epsilon_t} \left(\tfrac{Y_t}{Y_{t+1}} \right)^{\sigma} \tfrac{I_t}{\Pi_{t+1}} \right) \\
q_t = &\quad \beta \tfrac{\epsilon_t}{\epsilon_{t-1}} \left(\tfrac{Y_{t-1}}{Y_t} \right)^{\sigma} \tfrac{1}{\Pi_t} \\
\left(\tfrac{I_t}{\bar{I}} \right) = &\quad \epsilon_{\mathfrak{i},t} \left(\tfrac{I_{t-1}}{\bar{I}} \right)^{\phi_{\mathfrak{i}}} \left[\left(\tfrac{\Pi_t}{\bar{\Pi}} \right)^{\phi_\pi} \left(\tfrac{X_t}{\bar{X}} \right)^{\phi_x} \left(\tfrac{Y_t}{Y_{t-1}} \right)^{\phi_{gy}} \left(\tfrac{Y_t}{\bar{Y}} \right)^{\phi_y} \right]^{(1-\phi_{\mathfrak{i}})} \\
\left(Y_t^n \right)^{(\sigma+\omega)} = &\quad \tfrac{1}{(1+\delta)\mu} \tfrac{\varepsilon}{\chi} \epsilon_t \left(\mathcal{A}_t \right)^{(1+\omega)} \\
Y_{s,t} = &\quad (1+\delta)^{\theta} Y_t \left(\wp_{\delta,t} \right)^{-\theta}
\end{aligned}
$$

$$
\begin{aligned}
X_t = &\quad \tfrac{Y_t}{Y_t^n} \quad ; \; q_{t,t+j} = \quad q_{t+1} q_{t+1,t+j} \quad \text{for } j \geq 1 \text{ and } q_{t,t} = 1 \\
\mathcal{G}_t = &\quad \tfrac{Y_t}{Y_{t-1}} \quad ; \; \Pi_{t,t+j} = \quad \Pi_{t+1} \Pi_{t+1,t+j} \quad \text{for } j \geq 1 \text{ and } \Pi_{t,t} = 1 \\
\wp_{s,t} = &\quad \tfrac{Y_t}{Y_{s,t}} \quad ; \; \Pi_{t,t+j}^{ind} = \quad \Pi_{t+1}^{ind} \Pi_{t+1,t+j}^{ind} \quad \text{for } j \geq 1 \text{ and } \Pi_{t,t}^{ind} = 1 \\
\Pi_t^{ind} = &\quad \Pi_{t-1}^{\gamma} \quad ; \; \mathcal{G}_{t,t+j} = \quad \mathcal{G}_{t+1} \mathcal{G}_{t+1,t+j} \quad \text{for } j \geq 1 \text{ and } \mathcal{G}_{t,t} = 1
\end{aligned}
$$

$$
\begin{aligned}
(1+\delta)^{1-\theta} = &\quad (1-\alpha) \textstyle\sum_{j=0}^{\infty} \alpha^j \left(\tfrac{\Pi_{t-j,t}^{ind}}{\Pi_{t-j,t}} \wp_{t-j}^* + \delta \wp_{s,t} \right)^{1-\theta} \\
\left(\wp_{\delta,t} \right)^{-\theta} = &\quad (1-\alpha) \textstyle\sum_{j=0}^{\infty} \alpha^j \left(\tfrac{\Pi_{t-j,t}^{ind}}{\Pi_{t-j,t}} \wp_{t-j}^* + \delta \wp_{s,t} \right)^{-\theta} \\
1 = &\quad \mu_\theta \tfrac{N_t^*}{D_t^*} \\
N_t^* = &\quad E_t \sum_{j=0}^{\infty} \alpha^j q_{t,t+j} \mathcal{G}_{t,t+j} \Pi_{t,t+j}^{ind} \left(z_{t,t+j}^* \right)^{-(1+\theta)} \left(\tfrac{mc_{t,t+j}^*}{(1+\delta)} + \delta \wp_{s,t+j} \right) \\
D_t^* = &\quad E_t \sum_{j=0}^{\infty} \alpha^j q_{t,t+j} \mathcal{G}_{t,t+j} \Pi_{t,t+j}^{ind} \left(z_{t,t+j}^* \right)^{-\theta} \\
mc_{t,t+j}^* = &\quad \tfrac{(1+\delta)^{\theta\omega}}{\mu} \left(X_{t+j} \right)^{(\sigma+\omega)} \left(z_{t,t+j}^* \right)^{-\theta\omega} \\
z_{t,t+j}^* = &\quad \tfrac{\Pi_{t,t+j}^{ind}}{\Pi_{t,t+j}} \wp_t^* + \delta \wp_{s,t+j}
\end{aligned}
$$

As for the remaining aggregates and welfare measures, they are:

$$\mathcal{W}_t = u_t - v_t \qquad ; \; u_t = \epsilon_t \frac{(Y_t)^{(1-\sigma)}-1}{(1-\sigma)}$$

$$h_t = \left(\mathfrak{d}_{y,t}\frac{Y_t}{\mathcal{A}_t}\right)^{\frac{1}{\varepsilon}} \qquad ; \; v_t = \frac{\chi}{(1+\nu)}\mathfrak{d}_{v,t}\left(h_t\right)^{(1+\nu)}$$

$$\mathfrak{d}_{y,t} = (1+\delta)^{\theta}\left(\Lambda_{y,t}\right)^{\varepsilon} \qquad ; \; \mathfrak{d}_{v,t} = \frac{\Lambda_t}{\left(\Lambda_{y,t}\right)^{(1+\nu)}}$$

$$\Lambda_{y,t} = (1-\alpha)\sum_{j=0}^{\infty}\alpha^j\left(z^*_{t-j,t}\right)^{-\frac{\theta}{\varepsilon}} \qquad ; \; \Lambda_t = (1-\alpha)\sum_{j=0}^{\infty}\alpha^j\left(z^*_{t-j,t}\right)^{-\theta_1}$$

$$\mathcal{W}^n_t = u^n_t - v^n_t \qquad ; \; u^n_t = \epsilon_t \frac{(Y^n_t)^{(1-\sigma)}-1}{(1-\sigma)}$$

$$h^n_t = \left(\frac{Y^n_t}{\mathcal{A}_t}\right)^{\frac{1}{\varepsilon}} \qquad ; \; v^n_t = \frac{\chi}{(1+\nu)}\left(h^n_t\right)^{(1+\nu)}$$

$$\epsilon_t \frac{\left(Y^{eq}_t\right)^{(1-\sigma)}-1}{(1-\sigma)} - \frac{\chi}{(1+\nu)}\left(\frac{Y^{eq}_t}{\mathcal{A}_t}\right)^{(1+\omega)} = \mathcal{W}_t \quad ; \; X^{eq}_t \equiv \frac{Y^{eq}_t}{Y^n_t}$$

C.3 Steady State

For any variable χ_t, its steady state level is defined as $\bar{\chi}$. The steady state equilibrium can be numerically obtained as follows. First, we compute $\bar{I}$, $\bar{q}$, and $\bar{Y}^n$:

$$\bar{I} = \frac{\bar{\Pi}}{\beta} \quad ; \; \bar{q} = \frac{\beta}{\bar{\Pi}} \quad ; \; \left(\bar{Y}^n\right)^{(\sigma+\omega)} = \frac{1}{(1+\delta)\mu}\frac{\varepsilon}{\chi}\bar{\epsilon}\left(\overline{\mathcal{A}}\right)^{(1+\omega)}$$

Next, we use a numerical code to solve the following non-linear system for relative prices $\bar{\wp}^*$, $\bar{\wp}_s$, and $\bar{\wp}_\delta$, in which the infinite sums are retrieved by using finite sums in $j = \{0, 1, ..., J\}$ for a large J. In particular, we consider $J = 10000$:

$$\begin{aligned}
&(1+\delta)^{1-\theta} = (1-\alpha)\textstyle\sum_{j=0}^{\infty}\alpha^j\left(\frac{\bar{\wp}^*}{(\bar{\Pi})^{(1-\gamma)j}} + \delta\bar{\wp}_s\right)^{-(\theta-1)}\\
&\left(\bar{\wp}_\delta\right)^{-\theta} = (1-\alpha)\textstyle\sum_{j=0}^{\infty}\alpha^j\left(\frac{\bar{\wp}^*}{(\bar{\Pi})^{(1-\gamma)j}} + \delta\bar{\wp}_s\right)^{-\theta}\\
&\bar{\wp}_s = (1+\delta)^{-\theta}\left(\bar{\wp}_\delta\right)^{\theta}
\end{aligned}$$

After computing the relative prices, we pin down the following composite parameters:

$$\Sigma_{N1} \equiv \sum_{j=0}^{\infty}\left(\bar{\alpha}_1\beta\right)^j\left(\bar{z}^*_j\right)^{-(1+\theta_1)} \quad \Sigma_{N2} \equiv \sum_{j=0}^{\infty}\left(\bar{\alpha}_1\beta\right)^j\left(\bar{z}^*_j\right)^{-(1+\theta)} \quad \Sigma_{D1} \equiv \sum_{j=0}^{\infty}\left(\bar{\alpha}_1\beta\right)^j\left(\bar{z}^*_j\right)^{-\theta}$$

where $\bar{z}^*_j = \left(\frac{\bar{\wp}^*}{(\bar{\Pi})^{(1-\gamma)j}} + \delta\bar{\wp}_s\right)$, $\bar{\alpha}_1 \equiv \frac{\alpha}{\bar{\Pi}^{(1-\gamma)}}$ and $\bar{\alpha}_2 \equiv \frac{\alpha}{\bar{\Pi}^{2(1-\gamma)}}$.

Since the storage cost demand function satisfies the conditions of Theorem 1, we know that the infinite sums converge. Therefore, we can retrieve them numerically, by considering finite sums up to a very large horizon J, i.e. $j \in \{0, 1, ..., J\}$. Again, we use $J = 10000$. For avoiding numerical issues arising from dealing with very large numbers when $\delta > 0$, we proceed as follows. First, for each infinite sum of the form $\Sigma_\varphi \equiv \sum_{j=0}^{\infty}\left(ß\right)^j\left(\bar{z}^*_j\right)^{-\varphi}$, where ß<1, we define its normalized peer $\tilde{\Sigma}_\varphi \equiv$

$\sum_{j=0}^{\infty} (ß)^j \left(\tilde{z}_j^*\right)^{-\varphi}$, where $\tilde{z}_j^* \equiv \frac{\bar{z}_j^*}{\delta \bar{\wp}_s} = \left(1 + \frac{1}{(\bar{\Pi})^{(1-\gamma)j}} \frac{\bar{\wp}^*}{\delta \bar{\wp}_s}\right)$. Therefore, whenever $\delta > 0$, we can accurately approximate $\tilde{\Sigma}_\varphi$ using $\sum_{j=0}^{J} (ß)^j \left(\tilde{z}_j^*\right)^{-\varphi}$. After retrieving $\tilde{\Sigma}_\varphi$, we compute $\Sigma_\varphi = (\delta \bar{\wp}_s)^{-\varphi} \tilde{\Sigma}_\varphi$.

After pinning down the gross output gap $\bar{X} = \left(\frac{\mu_\delta}{\mu_\theta} (1+\delta)^{-\theta\omega} \frac{[\Sigma_{D1} - \mu_\theta(\delta \bar{\wp}_s)\Sigma_{N2}]}{\Sigma_{N1}}\right)^{\frac{1}{(\sigma+\omega)}}$, we compute the aggregate $\bar{Y} = \bar{X}\bar{Y}^n$ and average $\bar{Y}_s = \frac{\bar{Y}}{\bar{\wp}_s}$ output levels. As for the remaining aggregates and welfare measures, they are:

$$\begin{array}{lll}
\bar{\mathcal{W}}^n = \bar{u}^n - \bar{v}^n & \bar{\mathcal{W}} = \bar{u} - \bar{v} & ;\ \bar{\Lambda} = (1-\alpha)\,\Sigma_\Lambda \\
\bar{h}^n = \left(\frac{\bar{Y}^n}{\bar{\mathcal{A}}}\right)^{\frac{1}{\varepsilon}} & \bar{h} = \left(\bar{\mathfrak{d}}_y \frac{\bar{Y}}{\bar{\mathcal{A}}}\right)^{\frac{1}{\varepsilon}} & ;\ \bar{\Lambda}_y = (1-\alpha)\,\Sigma_y \\
\bar{u}^n = \bar{\epsilon}\frac{(\bar{Y}^n)^{(1-\sigma)}-1}{(1-\sigma)} & ;\ \bar{u} = \bar{\epsilon}\frac{(\bar{Y})^{(1-\sigma)}-1}{(1-\sigma)} & ;\ \bar{\mathfrak{d}}_y = (1+\delta)^{\theta}\left(\bar{\Lambda}_y\right)^{\varepsilon} \\
\bar{v}^n = \frac{\chi}{(1+\nu)}\left(\bar{h}^n\right)^{(1+\nu)} & ;\ \bar{v} = \frac{\chi}{(1+\nu)}\bar{\mathfrak{d}}_v\left(\bar{h}\right)^{(1+\nu)} & ;\ \bar{\mathfrak{d}}_v = \frac{\bar{\Lambda}}{\left(\bar{\Lambda}_y\right)^{(1+\nu)}}
\end{array}$$

As for the consumption-equivalent welfare metrics, we use numerical methods to solve the following non-linear equation:

$$\bar{\epsilon}\frac{(\bar{Y}^{eq})^{(1-\sigma)}-1}{(1-\sigma)} - \frac{\chi}{(1+\nu)}\left(\frac{\bar{Y}^{eq}}{\bar{\mathcal{A}}}\right)^{(1+\omega)} = \mathcal{W}_t$$

After that, we compute $\bar{X}^{eq} = \frac{\bar{Y}^{eq}}{\bar{Y}^n}$ as the consumption-equivalent output gap.

In the particular case of $(\bar{\Pi})^{(1-\gamma)} = 1$, it is possible to obtain a closed form solution:

$$\bar{\alpha}_1 = \bar{\alpha}_2 = \alpha \quad ;\ \bar{\wp}^* = \bar{\wp}_s = \bar{X} = 1 \quad ;\ \bar{\wp}_\delta = \bar{z}_j^* = (1+\delta)$$

The remaining steady state levels, when $(\bar{\Pi})^{(1-\gamma)} = 1$, are then easily retrieved using the same relations previously shown.

C.4 Log-Linearized Model

Given an exogenous level of trend inflation $\bar{\Pi}$, we start by defining $\bar{z}_j^* \equiv \left(\frac{\bar{\wp}^*}{(\bar{\Pi})^{(1-\gamma)j}} + \delta \bar{\wp}_s \right)$ and the following composite parameters:

$$\begin{aligned}
&\Sigma_{N1} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1 \beta)^j \left(\bar{z}_j^* \right)^{-(1+\theta_1)} \quad ; \Sigma_{N2} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1 \beta)^j \left(\bar{z}_j^* \right)^{-(1+\theta)} \quad ; \Sigma_{N3} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_2 \beta)^j \left(\bar{z}_j^* \right)^{-(2+\theta_1)} \\
&\Sigma_{N4} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_2 \beta)^j \left(\bar{z}_j^* \right)^{-(2+\theta)} \quad ; \Sigma_{N5} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1 \beta)^j \left(\bar{z}_j^* \right)^{-(2+\theta_1)} \quad ; \Sigma_{N6} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1 \beta)^j \left(\bar{z}_j^* \right)^{-(2+\theta)} \\
&\Sigma_{D1} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1 \beta)^j \left(\bar{z}_j^* \right)^{-\theta} \quad ; \Sigma_{D2} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_2 \beta)^j \left(\bar{z}_j^* \right)^{-(1+\theta)} \quad ; \Sigma_{D3} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1 \beta)^j \left(\bar{z}_j^* \right)^{-(1+\theta)} \\
&\Sigma_{s1} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1)^j \left(\bar{z}_j^* \right)^{-\theta} \quad ; \Sigma_{s2} \equiv \sum_{j=0}^{\infty} \alpha^j \left(\bar{z}_j^* \right)^{-\theta} \quad ; \Sigma_{\delta 1} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1)^j \left(\bar{z}_j^* \right)^{-(1+\theta)} \\
&\Sigma_{\delta 2} \equiv \sum_{j=0}^{\infty} \alpha^j \left(\bar{z}_j^* \right)^{-(1+\theta)} \quad ; \Sigma_{\alpha} \equiv \sum_{j=0}^{\infty} \alpha^j \left(\bar{z}_j^* \right)^{-(\theta-1)} \quad ; \Sigma_{\Lambda} \equiv \sum_{j=0}^{\infty} \alpha^j \left(\bar{z}_j^* \right)^{-\theta_1} \\
&\Sigma_{y} \equiv \sum_{j=0}^{\infty} \alpha^j \left(\bar{z}_j^* \right)^{-\frac{\theta}{\varepsilon}}
\end{aligned} \tag{21}$$

Since the storage cost demand function satisfies the conditions of Theorem 1, we know that the infinite sums converge. When $\delta > 0$, we proceed as in Appendix C.3 and accurately approximate $\tilde{\Sigma}_{\varphi}$ using $\sum_{j=0}^{J} (ß)^j \left(\tilde{z}_j^* \right)^{-\varphi}$, where $J = 10000$. After retrieving $\tilde{\Sigma}_{\varphi}$, we compute $\Sigma_{\varphi} = (\delta \bar{\wp}_s)^{-\varphi} \tilde{\Sigma}_{\varphi}$.

For computing the model loglinearized equilibrium, we also need the augment the set of composite parameters:

$$\begin{aligned}
&\omega \equiv \tfrac{(1+\nu)}{\varepsilon} - 1 \quad ; \bar{\theta}_1 \equiv \theta (1+\omega) \quad ; \mu_{\theta} \equiv \tfrac{\theta}{(\theta-1)} \quad ; \mu_{\delta} \equiv \tfrac{\theta}{\theta-(1+\delta)} \quad ; \mu = \tfrac{\mu_{\delta}}{(1+\delta)} \\
&\kappa_{\delta} \equiv (1+\delta)^{(\theta\omega-1)} (\bar{X})^{(\sigma+\omega)} \quad ; \bar{\theta}_2 \equiv (\delta \bar{\wp}_s) \theta \tfrac{\Sigma_{\delta 2}}{\Sigma_{s2}} \quad ; \bar{\alpha}_1 \equiv \tfrac{\alpha}{\bar{\Pi}^{(1-\gamma)}} \quad ; \bar{\alpha}_2 \equiv \tfrac{\alpha}{\bar{\Pi}^{2(1-\gamma)}} \quad ; \mu_2 \equiv \tfrac{\theta_2}{(\theta_2-1)}
\end{aligned}$$

For $k \in \{0, 1, 2, ...\}$ and for each composite parameter of the form $\Sigma_{\varphi} \equiv \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}_j^* \right)^{-\varphi}$, as defined in (21), we consider the following k-based composite parameters:

$$\Theta_{k,\varphi} \equiv \left(\bar{z}_k^* \right)^{-\varphi} \quad ; \Sigma_{k,\varphi} \equiv \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}_{j+k}^* \right)^{-\varphi}$$

For instance, $\Sigma_{k,N1}$ and $\Theta_{k,N1}$ are then defined as $\Sigma_{k,N1} \equiv \sum_{j=0}^{\infty} (\bar{\alpha}_1 \beta)^j \left(\bar{z}_{j+k}^* \right)^{-(1+\theta_1)}$ and$\Theta_{k,N1} \equiv \left(\bar{z}_k^* \right)^{-(1+\theta_1)}$. Note also that $\Sigma_{0,\varphi} = \Sigma_{\varphi}$.

Finally, the following composite parameters are necessary for deriving the aggregate supply sys-

tem under storage costs and trend inflation:

$$\Omega_1 \equiv \frac{(1-\bar{\theta}_2)}{\theta}\Theta_{0,s1} + (\delta\bar{\wp}_s)\,\Theta_{0,\delta 1} \quad ; \; \Omega_2 \equiv \Theta_{0,s1}\frac{\Sigma_{1,\delta 1}}{\Sigma_{1,s1}} - \Theta_{0,\delta 1} \quad ; \; \Omega_3 \equiv \frac{(1-\bar{\theta}_2)}{\theta}\Sigma_{1,s1} + (\delta\bar{\wp}_s)\,\Sigma_{1,\delta 1}$$
$$\Phi_{\Sigma 1,k,1} \equiv 1 + \Phi_{\Sigma,k,1}$$
$$\Phi_{\Sigma,k,1} \equiv \frac{\mu_\theta}{\mu}\kappa_\delta\Sigma_{k,N1} - \Sigma_{k,D1} + (\delta\bar{\wp}_s)\,\mu_\theta\Sigma_{k,N2}$$
$$\Phi_{\Theta,k,1} \equiv \frac{\mu_\theta}{\mu}\kappa_\delta\Theta_{k,N1} - \Theta_{k,D1} + (\delta\bar{\wp}_s)\,\mu_\theta\Theta_{k,N2}$$
$$\Phi_{\Sigma,k,2} \equiv \frac{\mu_\theta}{\mu}\kappa_\delta\left(1+\bar{\theta}_1\right)\Sigma_{k,N5} - \theta\Sigma_{k,D3} + (\delta\bar{\wp}_s)\,\mu_\theta\,(1+\theta)\,\Sigma_{k,N6}$$
$$\Phi_{\Theta,k,2} \equiv \frac{\mu_\theta}{\mu}\kappa_\delta\left(1+\bar{\theta}_1\right)\Theta_{k,N5} - \theta\Theta_{k,D3} + (\delta\bar{\wp}_s)\,\mu_\theta\,(1+\theta)\,\Theta_{k,N6}$$
$$\Phi_{\Sigma,k,3} \equiv \frac{\mu_\theta}{\mu}\kappa_\delta\left(1+\bar{\theta}_1\right)\Sigma_{k,N3} - \theta\Sigma_{k,D2} + (\delta\bar{\wp}_s)\,\mu_\theta\,(1+\theta)\,\Sigma_{k,N4}$$
$$\Phi_{\Theta,k,3} \equiv \frac{\mu_\theta}{\mu}\kappa_\delta\left(1+\bar{\theta}_1\right)\Theta_{k,N3} - \theta\Theta_{k,D2} + (\delta\bar{\wp}_s)\,\mu_\theta\,(1+\theta)\,\Theta_{k,N4}$$
$$\Phi_{\Theta,k,4} \equiv \Phi_{\Theta,k,2} - \mu_\theta\Theta_{k,N2}$$

In general, for any variable χ_t, its loglinearized version is defined as $\hat{\chi}_t \equiv \log\left(\frac{\chi_t}{\bar{\chi}}\right)$, keeping the same case as in the original variable, e.g. $\hat{Y}_t = \log\left(\frac{Y_t}{\bar{Y}}\right)$. For gross rates, though, we represent its loglinearized version in lower cases, e.g. $\hat{\pi}_t = \log\left(\frac{\Pi_t}{\bar{\Pi}}\right)$. Usual loglinearizations from the general part of the model, i.e. comprising equations independent of pricing structure, leads to the following system:

$$\begin{aligned}
\hat{Y}_t &= E_t\hat{Y}_{t+1} - \tfrac{1}{\sigma}E_t\left[(\hat{\imath}_t - \hat{\pi}_{t+1}) + (\hat{\epsilon}_{t+1} - \hat{\epsilon}_t)\right] \\
\hat{q}_t &= \sigma\left(\hat{Y}_{t-1} - \hat{Y}_t\right) - \hat{\pi}_t + (\hat{\epsilon}_t - \hat{\epsilon}_{t-1}) \\
\hat{\imath}_t &= \phi_{\mathrm{i}}\hat{\imath}_{t-1} + (1-\phi_{\mathrm{i}})\left[\phi_\pi\hat{\pi}_t + \phi_x\hat{x}_t + \phi_{gy}\left(\hat{Y}_t - \hat{Y}_{t-1}\right) + \phi_y\hat{Y}_t\right] + \hat{\epsilon}_{\mathrm{i},t} \\
(\sigma+\omega)\,\hat{Y}^n_t &= \hat{\epsilon}_t + (1+\omega)\,\widehat{\mathcal{A}}_t \\
\hat{\pi}^{ind}_t &= \gamma\hat{\pi}_{t-1} \quad ; \quad \hat{x}_t = \hat{Y}_t - \hat{Y}^n_{tt-1} \quad ; \quad \widehat{\mathfrak{g}}_t = \hat{Y}_t - \hat{Y}_{t-1}
\end{aligned}$$

Note that $E_t\hat{q}_{t+1} = -\hat{\imath}_t$. Lastly, as we show in Appendix C.4.1, the price setting equations presented in systems (17) and (18) imply the following loglinearized equations under storage costs and trend inflation:

Aggregate Supply:

$$\begin{aligned}
\hat{\ell}_t &= -E_t\left(\hat{\pi}_{t+1} - \hat{\pi}^{ind}_{t+1}\right) + (1-\sigma)\,E_t\left(\hat{x}_{t+1} - \hat{x}_t\right) + \hat{\tilde{\xi}}_t \\
\hat{\wp}^*_t &= \bar{\alpha}_2\beta\Psi_{0,\alpha 2}E_t\left(\hat{\pi}_{t+1} - \hat{\pi}^{ind}_{t+1}\right) + \tfrac{1}{(\bar{\wp}^*)}\left[\Psi_{0,x}\hat{x}_t + \bar{\alpha}_1\beta\Psi_{0,\ell}\hat{\ell}_t - (\delta\bar{\wp}_s)\,\Psi_{0,\wp}\hat{\wp}_{s,t}\right] \\
&+ \tfrac{1}{(\bar{\wp}^*)}\bar{\alpha}_1\beta\Psi_{0,\alpha 2}\Psi_{1,x1}E_t\hat{S}_{1,\alpha 1,t+1} - \bar{\alpha}_2\beta\Psi_{0,\alpha 2}\Psi_{1,\alpha 2}E_t\hat{S}_{1,\alpha 2,t+1}
\end{aligned}$$

Storage and Price Stickiness:

$$\begin{aligned}
\hat{\wp}^*_t &= \bar{\alpha}_1\psi_4\left(\hat{\pi}_t - \hat{\pi}^{ind}_t\right) + \bar{\alpha}_1\psi_5\left[\tfrac{(1-\bar{\theta}_2)}{\theta}\hat{S}_{1,s,t-1} + (\delta\bar{\wp}_s)\,\psi_6\hat{S}_{1,\delta,t-1}\right] \\
\hat{\wp}_{s,t} &= -\bar{\alpha}_1\psi_1\left(\hat{\pi}_t - \hat{\pi}^{ind}_t\right) + \bar{\alpha}_1\psi_2\left[\hat{S}_{1,s,t-1} - \psi_3\hat{S}_{1,\delta,t-1}\right] \\
\hat{\wp}_{\delta,t} &= \tfrac{1}{\theta}\hat{\wp}_{s,t} \\
\hat{Y}_{s,t} &= \hat{Y}_t - \hat{\wp}_{s,t}
\end{aligned}$$

$$
\begin{array}{rl}
\textbf{Ancillary} & \textbf{Variables for } k = \{0,1,2,...\} \\
\widehat{\mathcal{S}}_{k,\alpha 1,t} = & \frac{\Psi_{k,x}}{\Psi_{k,x1}}\hat{x}_t + \bar{\alpha}_1\beta\frac{\Psi_{k,\ell}}{\Psi_{k,x1}}\hat{\ell}_t - (\delta\bar{\wp}_s)\frac{\Psi_{k,\wp}}{\Psi_{k,x1}}\hat{\wp}_{s,t} + \bar{\alpha}_1\beta\Psi_{(k+1),\alpha 1}E_t\hat{S}_{(k+1),\alpha 1,t+1} \\
\widehat{\mathcal{S}}_{k,\alpha 2,t} = & -\bar{\alpha}_2\beta E_t\left(\hat{\pi}_{t+1} - \hat{\pi}^{ind}_{t+1}\right) + \bar{\alpha}_2\beta\Psi_{(k+1),\alpha 2}E_t\hat{S}_{(k+1),\alpha 2,t+1} \\
\widehat{\mathcal{S}}_{k,s,t} = & \bar{\alpha}_1\left(\hat{\pi}_t - \hat{\pi}^{ind}_t\right) - \Psi_{k,\wp s}\hat{\wp}^*_t + \bar{\alpha}_1\Psi_{(k+1),s}\hat{S}_{(k+1),s,t-1} \\
\widehat{\mathcal{S}}_{k,\delta,t} = & \bar{\alpha}_1\left(\hat{\pi}_t - \hat{\pi}^{ind}_t\right) - \Psi_{k,\wp\delta}\hat{\wp}^*_t + \bar{\alpha}_1\Psi_{(k+1),\delta}\hat{S}_{(k+1),\delta,t-1}
\end{array}
$$

$$
\begin{array}{rl}
\textbf{Aggregate} & \textbf{Shock:} \\
\hat{\xi}_t = & \frac{(1+\omega)}{(\sigma+\omega)}E_t\left[(\hat{\epsilon}_{t+1} - \hat{\epsilon}_t) + (1-\sigma)\left(\widehat{\mathcal{A}}_{t+1} - \widehat{\mathcal{A}}_t\right)\right]
\end{array}
$$

where

$$
\begin{array}{llll}
\psi_1 \equiv (\bar{\wp}^*)\frac{\Sigma_{1,s1}\Omega_2}{\Sigma_{s2}\Omega_1} & ; \psi_2 \equiv (\bar{\wp}^*)\frac{\Theta_{0,\delta 1}\Sigma_{2,s1}}{\Sigma_{s2}\Omega_1} & ; \psi_3 \equiv \frac{\Theta_{0,s1}\Sigma_{2,\delta 1}}{\Theta_{0,\delta 1}\Sigma_{2,s1}} & \psi_4 \equiv \frac{\Omega_3}{\Omega_1} \\
\psi_5 \equiv \frac{\Sigma_{2,s1}}{\Omega_1} & ; \psi_6 \equiv \frac{\Sigma_{2,\delta 1}}{\Sigma_{2,s1}} & ; \Psi_{k,\wp s} \equiv \frac{\Theta_{k,s1}}{\Sigma_{(k+1),s1}} & ; \Psi_{k,\wp\delta} \equiv \frac{\Theta_{k,\delta 1}}{\Sigma_{(k+1),\delta 1}} \\
\Psi_{k,x} \equiv \frac{\mu_\theta}{\mu}\kappa_\delta(\sigma+\omega)\frac{\Theta_{k,N1}}{\Phi_{\Sigma,k,3}} & ; \Psi_{k,x1} \equiv \frac{\Phi_{\Sigma 1,(k+1),1}}{\Phi_{\Sigma,k,3}} & ; \Psi_{k,\ell} \equiv \frac{\Phi_{\Sigma,(k+1),1}}{\Phi_{\Sigma,k,3}} & ; \Psi_{k,\wp} \equiv \frac{\Phi_{\Theta,k,4}}{\Phi_{\Sigma,k,3}} \\
\Psi_{k,\alpha 1} \equiv \frac{\Phi_{\Sigma 1,(k+1),1}}{\Phi_{\Sigma 1,k,1}} & ; \Psi_{k,\alpha 2} \equiv \frac{\Phi_{\Sigma,(k+1),3}}{\Phi_{\Sigma,k,3}} & ; \Psi_{k,s} \equiv \frac{\Sigma_{(k+1),s1}}{\Sigma_{k,s1}} & ; \Psi_{k,\delta} \equiv \frac{\Sigma_{(k+1),\delta 1}}{\Sigma_{k,\delta 1}}
\end{array}
$$

As in Alves (2014), $\hat{\xi}_t$ is an aggregate shock term that collects the effects of the technology shock $\widehat{\mathcal{A}}_t$ and the utility shock $\hat{\epsilon}_t$. For our simulations, we truncate the infinite recursive system of ancillary variables at $\bar{k} = 40$. With this approximation, we substitute $E_t\widehat{\mathcal{S}}_{40,\alpha 1,t+1}$ for $E_t\widehat{\mathcal{S}}_{41,\alpha 1,t+1}$, $E_t\widehat{\mathcal{S}}_{40,\alpha 2,t+1}$ for $E_t\widehat{\mathcal{S}}_{41,\alpha 2,t+1}$, $\widehat{\mathcal{S}}_{40,s,t-1}$ for $\widehat{\mathcal{S}}_{41,s,t-1}$, and $\widehat{\mathcal{S}}_{40,\delta,t-1}$ for $\widehat{\mathcal{S}}_{41,\delta,t-1}$.

C.4.1 Deriving the Log-Linearized Supply System

Since $\bar{z}^*_j = \left(\frac{\bar{\wp}^*}{\bar{\Pi}^{j(1-\gamma)}} + \delta\bar{\wp}_s\right)$, and given the steady state relations $\frac{(1+\delta)^{-(\theta-1)}}{(1-\alpha)} = \Sigma_\alpha$, $\frac{(\bar{\wp}_\delta)^{-\theta}}{(1-\alpha)} = \Sigma_{s2}$ and $\bar{\wp}_s = \frac{(1+\delta)^{-\theta}}{(1-\alpha)\Sigma_{s2}}$, direct loglinearization of the pricing systems (17) and (18) initially gives the following equations.

$$
\begin{array}{rl}
\hat{Y}_{s,t} = & \hat{Y}_t - \theta\hat{\wp}_{\delta,t} \\
\hat{\wp}_{s,t} = & \theta\hat{\wp}_{\delta,t} \\
0 = & -(\bar{\wp}^*)\hat{S}_{s1,t} + (\delta\bar{\wp}_s)\Sigma_{s2}\hat{\wp}_{s,t} \\
\hat{S}_{s1,t} \equiv & \sum_{j=0}^{\infty}(\bar{\alpha}_1)^j\left(\bar{z}^*_j\right)^{-\theta}\left(\left(\hat{\pi}_{t-j,t} - \hat{\pi}^{ind}_{t-j,t}\right) - \hat{\wp}^*_{t-j}\right) \\
 & \\
\Sigma_{s2}\hat{\wp}_{\delta,t} = & -(\bar{\wp}^*)\hat{S}_{\delta 1,t} + (\delta\bar{\wp}_s)\Sigma_{\delta 2}\hat{\wp}_{s,t} \\
\hat{S}_{\delta 1,t} \equiv & \sum_{j=0}^{\infty}(\bar{\alpha}_1)^j\left(\bar{z}^*_j\right)^{-(1+\theta)}\left(\left(\hat{\pi}_{t-j,t} - \hat{\pi}^{ind}_{t-j,t}\right) - \hat{\wp}^*_{t-j}\right)
\end{array}
$$

$$
\bar{D}\hat{D}^*_t = \mu_\theta\bar{N}\hat{N}^*_t
$$

$$
\begin{aligned}
\bar{N}\hat{N}_t = \;& \tfrac{\kappa_\delta}{\mu}\hat{S}_{N1,t} + (\delta\bar{\wp}_s)\,\hat{S}_{N2,t} - \tfrac{(\bar{\wp}^*)\left(1+\bar{\theta}_1\right)\kappa_\delta}{\mu}\left(\hat{S}_{N3,t} + \Sigma_{N3}\hat{\wp}^*_t\right) \\
- \;& (1+\theta)\,(\delta\bar{\wp}_s)\,(\bar{\wp}^*)\left(\hat{S}_{N4,t} + \Sigma_{N4}\hat{\wp}^*_t\right) - \tfrac{(\delta\bar{\wp}_s)\left(1+\bar{\theta}_1\right)\kappa_\delta}{\mu}\hat{S}_{N5,t} - (1+\theta)\,(\delta\bar{\wp}_s)^2\,\hat{S}_{N6,t} \\
\hat{S}_{N1,t} \equiv \;& E_t\sum_{j=0}^{\infty}(\bar{\alpha}_1\beta)^j\left(\bar{z}^*_j\right)^{-\left(1+\bar{\theta}_1\right)}\left[\left(\hat{q}_{t,t+j} + \widehat{\mathfrak{g}}_{t,t+j} + \hat{\pi}^{ind}_{t,t+j}\right) + (\sigma+\omega)\,\hat{x}_{t+j}\right] \\
\hat{S}_{N2,t} \equiv \;& E_t\sum_{j=0}^{\infty}(\bar{\alpha}_1\beta)^j\left(\bar{z}^*_j\right)^{-(1+\theta)}\left[\left(\hat{q}_{t,t+j} + \widehat{\mathfrak{g}}_{t,t+j} + \hat{\pi}^{ind}_{t,t+j}\right) + \hat{\wp}_{s,t+j}\right] \\
\hat{S}_{N3,t} \equiv \;& -E_t\sum_{j=0}^{\infty}(\bar{\alpha}_2\beta)^j\left(\bar{z}^*_j\right)^{-\left(2+\bar{\theta}_1\right)}\left(\hat{\pi}_{t,t+j} - \hat{\pi}^{ind}_{t,t+j}\right) \\
\hat{S}_{N4,t} \equiv \;& -E_t\sum_{j=0}^{\infty}(\bar{\alpha}_2\beta)^j\left(\bar{z}^*_j\right)^{-(2+\theta)}\left(\hat{\pi}_{t,t+j} - \hat{\pi}^{ind}_{t,t+j}\right) \\
\hat{S}_{N5,t} \equiv \;& E_t\sum_{j=0}^{\infty}(\bar{\alpha}_1\beta)^j\left(\bar{z}^*_j\right)^{-\left(2+\bar{\theta}_1\right)}\hat{\wp}_{s,t+j} \\
\hat{S}_{N6,t} \equiv \;& E_t\sum_{j=0}^{\infty}(\bar{\alpha}_1\beta)^j\left(\bar{z}^*_j\right)^{-(2+\theta)}\hat{\wp}_{s,t+j}
\end{aligned}
$$

$$
\begin{aligned}
\bar{D}\hat{D}^*_t = \;& \hat{S}_{D1,t} - \theta\,(\bar{\wp}^*)\left(\hat{S}_{D2,t} + \Sigma_{D2}\hat{\wp}^*_t\right) - \theta\,(\delta\bar{\wp}_s)\,\hat{S}_{D3,t} \\
\hat{S}_{D1,t} \equiv \;& E_t\sum_{j=0}^{\infty}(\bar{\alpha}_1\beta)^j\left(\bar{z}^*_j\right)^{-\theta}\left(\hat{q}_{t,t+j} + \widehat{\mathfrak{g}}_{t,t+j} + \hat{\pi}^{ind}_{t,t+j}\right) \\
\hat{S}_{D2,t} \equiv \;& -E_t\sum_{j=0}^{\infty}(\bar{\alpha}_2\beta)^j\left(\bar{z}^*_j\right)^{-(1+\theta)}\left(\hat{\pi}_{t,t+j} - \hat{\pi}^{ind}_{t,t+j}\right) \\
\hat{S}_{D3,t} \equiv \;& E_t\sum_{j=0}^{\infty}(\bar{\alpha}_1\beta)^j\left(\bar{z}^*_j\right)^{-(1+\theta)}\hat{\wp}_{s,t+j}
\end{aligned}
$$

Since the discounted sums do not allow for finite recursive representations, we use the following Lemmas to help us obtain simpler expressions.

Lemma 1 *Consider generic forward and backward equations* $\hat{S}^f_t \equiv E_t\sum_{j=0}^{\infty}(\beta)^j\left(\bar{z}^*_j\right)^{-\phi}\left(\hat{\varkappa}^a_{t,t+j} + \hat{\varkappa}^b_{t+j}\right)$ *and* $\hat{S}^l_t \equiv \sum_{j=0}^{\infty}(\beta)^j\left(\bar{z}^*_j\right)^{-\phi}\left(\hat{\varkappa}^a_{t-j,t} + \hat{\varkappa}^b_{t-j}\right)$*, where* $\beta\in(0,1)$ *is a discounting parameter,* $\hat{\varkappa}^a_{\tau_1,\tau_2}$ *is a cumulative variable from* τ_1 *to* τ_2*, while* $\hat{\varkappa}^b_\tau$ *is a spot variable at period* τ*. Since* $\hat{\varkappa}^a_{t,t+j} = \hat{\varkappa}^a_{t+1} + \hat{\varkappa}^a_{t+1,t+j}$*,* $\hat{\varkappa}^a_{t-j,t} = \hat{\varkappa}^a_t + \hat{\varkappa}^a_{t-j,t-1}$*, and* $\hat{\varkappa}^a_{t,t} = 0$*, the infinite sums lead to the following infinite recursive systems, for* $k = \{0,1,2,...,\infty\}$*:*

$$
\begin{aligned}
\hat{S}^f_t = \;& \hat{S}^f_{0,t} \\
\hat{S}^f_{k,t} = \;& \left(\Sigma_{k,\phi} - \Theta_{k,\phi}\right)E_t\hat{\varkappa}^a_{t+1} + \Theta_{k,\phi}\hat{\varkappa}^b_t + \beta E_t\hat{S}^f_{(k+1),t+1} \\
\hat{S}^l_t = \;& \hat{S}^l_{0,t} \\
\hat{S}^l_{k,t} = \;& \left(\Sigma_{k,\phi} - \Theta_{k,\phi}\right)\hat{\varkappa}^a_t + \Theta_{k,\phi}\hat{\varkappa}^b_t + \beta\hat{S}^l_{(k+1),t-1} \\
& \textit{where} \\
\hat{S}^f_{k,t} \equiv \;& E_t\sum_{j=0}^{\infty}(\beta)^j\left(\bar{z}^*_{j+k}\right)^{-\phi}\left(\hat{\varkappa}^a_{t,t+j} + \hat{\varkappa}^b_{t+j}\right) \\
\hat{S}^l_{k,t} \equiv \;& \sum_{j=0}^{\infty}(\beta)^j\left(\bar{z}^*_{j+k}\right)^{-\phi}\left(\hat{\varkappa}^a_{t-j,t} + \hat{\varkappa}^b_{t-j}\right) \\
\Theta_{k,\phi} \equiv \;& \left(\bar{z}^*_k\right)^{-\phi} \quad , \quad \Sigma_{0,\phi} = \Sigma_\phi \equiv \sum_{j=0}^{\infty}(\beta)^j\left(\bar{z}^*_j\right)^{-\phi} \\
\Sigma_{k,\phi} \equiv \;& \sum_{j=0}^{\infty}(\beta)^j\left(\bar{z}^*_{j+k}\right)^{-\phi} = \tfrac{1}{\beta}\left[\Sigma_{(k-1),\phi} - \Theta_{(k-1),\phi}\right]
\end{aligned}
$$

Proof. As for $\Sigma_{k,\phi} \equiv \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi}$, note that:

$$\begin{aligned}
\Sigma_{k,\phi} \equiv & \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} = \sum_{j=-1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} - (ß)^{-1} \left(\bar{z}^*_{k-1}\right)^{-\phi} \\
= & \sum_{j=0}^{\infty} (ß)^{j-1} \left(\bar{z}^*_{j-1+k}\right)^{-\phi} - (ß)^{-1} \left(\bar{z}^*_{k-1}\right)^{-\phi} = \frac{1}{ß}\left[\sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k-1}\right)^{-\phi} - \left(\bar{z}^*_{k-1}\right)^{-\phi}\right]
\end{aligned}$$

For the forward infinite sum, we obtain:

$$\begin{aligned}
\hat{S}^f_{k,t} \equiv & E_t \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t,t+j} + \hat{\varkappa}^b_{t+j}\right) \\
= & E_t \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t,t+j}\right) + E_t \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^b_{t+j}\right) \\
= & E_t \sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t,t+j}\right) + E_t \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^b_{t+j}\right) \\
= & E_t \sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t+1} + \hat{\varkappa}^a_{t+1,t+j}\right) + \left(\bar{z}^*_k\right)^{-\phi} \hat{\varkappa}^b_t + E_t \sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^b_{t+j}\right) \\
= & \left(\sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi}\right) E_t \hat{\varkappa}^a_{t+1} + \left(\bar{z}^*_k\right)^{-\phi} \hat{\varkappa}^b_t + E_t \sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t+1,t+j} + \hat{\varkappa}^b_{t+j}\right) \\
= & \left(\Sigma_{k,\phi} - \left(\bar{z}^*_k\right)^{-\phi}\right) E_t \hat{\varkappa}^a_{t+1} + \left(\bar{z}^*_k\right)^{-\phi} \hat{\varkappa}^b_t + ßE_t \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k+1}\right)^{-\phi} \left(\hat{\varkappa}^a_{t+1,t+1+j} + \hat{\varkappa}^b_{t+1+j}\right) \\
= & \left(\Sigma_{k,\phi} - \Theta_{k,\phi}\right) E_t \hat{\varkappa}^a_{t+1} + \Theta_{k,\phi} \hat{\varkappa}^b_t + ßE_t \hat{S}^f_{(k+1),t+1}
\end{aligned}$$

And lastly, for the backward sum, we obtain:

$$\begin{aligned}
\hat{S}^l_{k,t} \equiv & \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t-j,t} + \hat{\varkappa}^b_{t-j}\right) = \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t-j,t}\right) + \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^b_{t-j}\right) \\
= & \sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t-j,t}\right) + \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^b_{t-j}\right) \\
= & \sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_t + \hat{\varkappa}^a_{t-j,t-1}\right) + \left(\bar{z}^*_k\right)^{-\phi} \hat{\varkappa}^b_t + \sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^b_{t-j}\right) \\
= & \left(\sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi}\right) \hat{\varkappa}^a_t + \left(\bar{z}^*_k\right)^{-\phi} \hat{\varkappa}^b_t + \sum_{j=1}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi} \left(\hat{\varkappa}^a_{t-j,t-1} + \hat{\varkappa}^b_{t-j}\right) \\
= & \left(\Sigma_{k,\phi} - \left(\bar{z}^*_k\right)^{-\phi}\right) \hat{\varkappa}^a_t + \left(\bar{z}^*_k\right)^{-\phi} \hat{\varkappa}^b_t + ß \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k+1}\right)^{-\phi} \left(\hat{\varkappa}^a_{t-1-j,t-1} + \hat{\varkappa}^b_{t-1-j}\right) \\
= & \left(\Sigma_{k,\phi} - \Theta_{k,\phi}\right) \hat{\varkappa}^a_t + \Theta_{k,\phi} \hat{\varkappa}^b_t + ß\hat{S}^l_{(k+1),t-1}
\end{aligned}$$

■

The recursive systems are infinite, for $\hat{S}^l_{k,t}\left(\hat{S}^f_{k,t}\right)$ depends on $\hat{S}^l_{(k+1),t-1}\left(E_t\hat{S}^f_{(k+1),t+1}\right)$, instead of $\hat{S}^l_{k,t-1}\left(E_t\hat{S}^f_{k,t+1}\right)$, for $k = \{0, 1, 2, ..., \infty\}$. However, since coefficients $\left(\Sigma_{k,\phi} - \Theta_{k,\phi}\right)$ and $\Theta_{k,\phi}$ converges asymptotically as k rises, the equations at a conveniently chosen large level $\bar{k}$ can be approximated by finite recursions, using $\hat{S}^l_{k,t-1}\left(E_t\hat{S}^f_{k,t+1}\right)$, instead of $\hat{S}^l_{(k+1),t-1}\left(E_t\hat{S}^f_{(k+1),t+1}\right)$. In this paper, we set $\bar{k} = 40$.

From Lemma 1, note that the dicounting parameter on $E_t\hat{S}^f_{(k+1),t+1}$ and $\hat{S}^l_{(k+1),t-1}$ is always ß. Therefore, Corollary 2 derives similar results for linear combinations of the form $\hat{S}^f_{a,t} \equiv \sum_{n=1}^{N} {}_n\hat{S}^f_{n,t}$ and

$\hat{S}^l_{a,t} \equiv \sum_{n=1}^{N} {}_n\hat{S}^l_{n,t}$, where $\hat{S}^f_{n,t}$ and $\hat{S}^l_{n,t}$ are generic forward and backward ancillary variables.

Corollary 2 *Consider linear combinations of forward and backward ancillary variables of the form* $\hat{S}^f_{k,a,t} \equiv \sum_{n=1}^{N} \mathfrak{w}_{kn}\hat{S}^f_{k,n,t}$ *and* $\hat{S}^l_{k,a,t} \equiv \sum_{n=1}^{N} \mathfrak{w}_{kn}\hat{S}^l_{k,n,t}$*, where* ${}_n$ *is a n-specific real-valued parameter,* $\mathfrak{w}_k$ *is a k-specific real-valued parameter, and* $\hat{S}^f_{k,n,t}$ *and* $\hat{S}^l_{k,n,t}$ *are generic forward and backward ancillary variables, as defined in Lemma 1, with the same discounting parameter* $ß \in (0,1)$*, but with specific powers* ϕ_n*, specific cumulative variables* $\hat{\varkappa}^a_{n,\tau_1,\tau_2}$ *and specific spot variables* $\hat{\varkappa}^b_{n,\tau}$*. That is,* $\hat{S}^f_{k,n,t} \equiv E_t \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi_n} \left(\hat{\varkappa}^a_{n,t,t+j} + \hat{\varkappa}^b_{n,t+j}\right)$ *and* $\hat{S}^l_{k,n,t} \equiv \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi_n} \left(\hat{\varkappa}^a_{n,t-j,t} + \hat{\varkappa}^b_{n,t-j}\right)$*. The following infinite recursive systems describe the dynamics for* $\hat{S}^f_{k,a,t}$ *and* $\hat{S}^l_{k,a,t}$*, for* $k = \{0,1,2,...,\infty\}$*:*

$$
\begin{aligned}
\hat{S}^f_{a,t} &= \hat{S}^f_{0,a,t} \\
\hat{S}^f_{k,a,t} &= \sum_{n=1}^{N} \mathfrak{w}_{kn}\left(\Sigma_{k,n,\phi} - \Theta_{k,n,\phi}\right) E_t\hat{\varkappa}^a_{n,t+1} + \sum_{n=1}^{N} \mathfrak{w}_{kn}\Theta_{k,n,\phi}\hat{\varkappa}^b_{n,t} + \tfrac{\mathfrak{w}_k}{\mathfrak{w}_{(k+1)}} ß E_t \hat{S}^f_{(k+1),a,t+1} \\
\hat{S}^l_{a,t} &= \hat{S}^l_{0,a,t} \\
\hat{S}^l_{k,a,t} &= \sum_{n=1}^{N} \mathfrak{w}_{kn}\left(\Sigma_{k,n,\phi} - \Theta_{k,n,\phi}\right) \hat{\varkappa}^a_{n,t} + \sum_{n=1}^{N} \mathfrak{w}_{kn}\Theta_{k,n,\phi}\hat{\varkappa}^b_{n,t} + \tfrac{\mathfrak{w}_k}{\mathfrak{w}_{(k+1)}} ß \hat{S}^l_{(k+1),a,t-1} \\
&\textit{where} \\
\hat{S}^f_{k,a,t} &\equiv \sum_{n=1}^{N} \mathfrak{w}_{kn}\hat{S}^f_{k,n,t} \\
\hat{S}^l_{k,a,t} &\equiv \sum_{n=1}^{N} \mathfrak{w}_{kn}\hat{S}^l_{k,n,t} \\
\Theta_{k,n,\phi} &\equiv \left(\bar{z}^*_k\right)^{-\phi_n} \quad , \quad \Sigma_{0,n,\phi} = \Sigma_{n,\phi} \equiv \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_j\right)^{-\phi_n} \\
\Sigma_{k,n,\phi} &\equiv \sum_{j=0}^{\infty} (ß)^j \left(\bar{z}^*_{j+k}\right)^{-\phi_n} = \tfrac{1}{ß}\left[\Sigma_{(k-1),n,\phi} - \Theta_{(k-1),n,\phi}\right]
\end{aligned}
$$

Proof. From Lemma 1, note that the discounting parameter on $E_t\hat{S}^f_{(k+1),t+1}$ and $\hat{S}^l_{(k+1),t-1}$ is always ß. Therefore, we can find similar results for $\hat{S}^f_{a,t}$ and $\hat{S}^l_{a,t}$:

$$
\begin{aligned}
\hat{S}^f_{k,a,t} \equiv \sum_{n=1}^{N} \mathfrak{w}_{kn}\hat{S}^f_{k,n,t} &= \sum_{n=1}^{N} \mathfrak{w}_{kn}\left[\left(\Sigma_{k,n,\phi} - \Theta_{k,n,\phi}\right) E_t\hat{\varkappa}^a_{n,t+1} + \Theta_{k,n,\phi}\hat{\varkappa}^b_{n,t} + ß E_t\hat{S}^f_{(k+1),n,t+1}\right] \\
&= \sum_{n=1}^{N} \mathfrak{w}_{kn}\left(\Sigma_{k,n,\phi} - \Theta_{k,n,\phi}\right) E_t\hat{\varkappa}^a_{n,t+1} + \sum_{n=1}^{N} \mathfrak{w}_{kn}\Theta_{k,n,\phi}\hat{\varkappa}^b_{n,t} + \tfrac{\mathfrak{w}_k ß}{\mathfrak{w}_{(k+1)}} E_t \sum_{n=1}^{N} \mathfrak{w}_{(k+1)n}\hat{S}^f_{(k+1),n,t+1} \\
&= \sum_{n=1}^{N} \mathfrak{w}_{kn}\left(\Sigma_{k,n,\phi} - \Theta_{k,n,\phi}\right) E_t\hat{\varkappa}^a_{n,t+1} + \sum_{n=1}^{N} \mathfrak{w}_{kn}\Theta_{k,n,\phi}\hat{\varkappa}^b_{n,t} + \tfrac{\mathfrak{w}_k ß}{\mathfrak{w}_{(k+1)}} E_t\hat{S}^f_{(k+1),a,t+1}
\end{aligned}
$$

$$
\begin{aligned}
\hat{S}^l_{k,a,t} \equiv \sum_{n=1}^{N} \mathfrak{w}_{kn}\hat{S}^l_{k,n,t} &= \sum_{n=1}^{N} \mathfrak{w}_{kn}\left[\left(\Sigma_{k,n,\phi} - \Theta_{k,n,\phi}\right) \hat{\varkappa}^a_{n,t} + \Theta_{k,n,\phi}\hat{\varkappa}^b_{n,t} + ß\hat{S}^l_{(k+1),n,t-1}\right] \\
&= \sum_{n=1}^{N} \mathfrak{w}_{kn}\left(\Sigma_{k,n,\phi} - \Theta_{k,n,\phi}\right) \hat{\varkappa}^a_{n,t} + \sum_{n=1}^{N} \mathfrak{w}_{kn}\Theta_{k,n,\phi}\hat{\varkappa}^b_{n,t} + \tfrac{\mathfrak{w}_k ß}{\mathfrak{w}_{(k+1)}} \sum_{n=1}^{N} \mathfrak{w}_{(k+1)n}\hat{S}^l_{(k+1),n,t-1} \\
&= \sum_{n=1}^{N} \mathfrak{w}_{kn}\left(\Sigma_{k,n,\phi} - \Theta_{k,n,\phi}\right) \hat{\varkappa}^a_{n,t} + \sum_{n=1}^{N} \mathfrak{w}_{kn}\Theta_{k,n,\phi}\hat{\varkappa}^b_{n,t} + \tfrac{\mathfrak{w}_k ß}{\mathfrak{w}_{(k+1)}}\hat{S}^l_{(k+1),a,t-1}
\end{aligned}
$$

■

Therefore, using Lemma 1, and using the fact that $\mu_2 = (\delta\bar{\wp}_s)\frac{\theta}{(\theta_2-1)}\frac{\Sigma_{\delta 2}}{\Sigma_{s2}}$, we simplify the first six equations of our system into:

$$\begin{aligned}
\hat{Y}_{s,t} &= \hat{Y}_t - \hat{\wp}_{s,t} \\
\hat{\wp}_{\delta,t} &= \tfrac{1}{\theta}\hat{\wp}_{s,t} \\
\hat{\wp}_{s,t} &= -\bar{\alpha}_1(\bar{\wp}^*)\frac{\Sigma_{1,s1}\Omega_2}{\Sigma_{s2}\Omega_1}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \bar{\alpha}_1(\bar{\wp}^*)\frac{1}{\Sigma_{s2}\Omega_1}\left[\Theta_{0,\delta 1}\hat{S}_{1,s1,t-1} - \Theta_{0,s1}\hat{S}_{1,\delta 1,t-1}\right] \\
\hat{\wp}_t^* &= \bar{\alpha}_1\frac{\Omega_3}{\Omega_1}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \bar{\alpha}_1\frac{1}{\Omega_1}\left[\frac{(1-\bar{\theta}_2)}{\theta}\hat{S}_{1,s1,t-1} + (\delta\bar{\wp}_s)\hat{S}_{1,\delta 1,t-1}\right] \\
\hat{S}_{k,s1,t} &= \bar{\alpha}_1\Sigma_{(k+1),s1}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \Theta_{k,s1}\hat{\wp}_t^* + \bar{\alpha}_1\hat{S}_{(k+1),s1,t-1} \\
\hat{S}_{k,\delta 1,t} &= \bar{\alpha}_1\Sigma_{(k+1),\delta 1}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \Theta_{k,\delta 1}\hat{\wp}_t^* + \bar{\alpha}_1\hat{S}_{(k+1),\delta 1,t-1}
\end{aligned}$$

for $k = \{0, 1, 2, ..., \infty\}$.

We make a scale transformation in the ancillary variables: $\hat{S}_{k,s,t} \equiv \frac{\hat{S}_{k,s1,t}}{\Sigma_{(k+1),s1}}$ and $\hat{S}_{k,\delta,t} \equiv \frac{\hat{S}_{k,\delta 1,t}}{\Sigma_{(k+1),\delta 1}}$.And so, we obtain:

$$\begin{aligned}
\hat{Y}_{s,t} &= \hat{Y}_t - \hat{\wp}_{s,t} \\
\hat{\wp}_{\delta,t} &= \tfrac{1}{\theta}\hat{\wp}_{s,t} \\
\hat{\wp}_{s,t} &= -\bar{\alpha}_1(\bar{\wp}^*)\frac{\Sigma_{1,s1}\Omega_2}{\Sigma_{s2}\Omega_1}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \bar{\alpha}_1(\bar{\wp}^*)\frac{\Theta_{0,\delta 1}\Sigma_{2,s1}}{\Sigma_{s2}\Omega_1}\left[\hat{S}_{1,s,t-1} - \frac{\Theta_{0,s1}\Sigma_{2,\delta 1}}{\Theta_{0,\delta 1}\Sigma_{2,s1}}\hat{S}_{1,\delta,t-1}\right] \\
\hat{\wp}_t^* &= \bar{\alpha}_1\frac{\Omega_3}{\Omega_1}\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) + \bar{\alpha}_1\frac{\Sigma_{2,s1}}{\Omega_1}\left[\frac{(1-\bar{\theta}_2)}{\theta}\hat{S}_{1,s,t-1} + (\delta\bar{\wp}_s)\frac{\Sigma_{2,\delta 1}}{\Sigma_{2,s1}}\hat{S}_{1,\delta,t-1}\right] \\
\hat{S}_{k,s,t} &= \bar{\alpha}_1\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \frac{\Theta_{k,s1}}{\Sigma_{(k+1),s1}}\hat{\wp}_t^* + \bar{\alpha}_1\frac{\Sigma_{(k+2),s1}}{\Sigma_{(k+1),s1}}\hat{S}_{(k+1),s,t-1} \\
\hat{S}_{k,\delta,t} &= \bar{\alpha}_1\left(\hat{\pi}_t - \hat{\pi}_t^{ind}\right) - \frac{\Theta_{k,\delta 1}}{\Sigma_{(k+1),\delta 1}}\hat{\wp}_t^* + \bar{\alpha}_1\frac{\Sigma_{(k+2),\delta 1}}{\Sigma_{(k+1),\delta 1}}\hat{S}_{(k+1),\delta,t-1}
\end{aligned} \tag{22}$$

As for remaining 12 equations of the optimal pricing system, after using Lemma 1 applied to discounted sums, we aggregate ancillary variables with discounting $\bar{\alpha}_1\beta$ or $\bar{\alpha}_2\beta$ into two synthetic ancillary variables, $\hat{S}_{k,\alpha 1,t}$ and $\hat{S}_{k,\alpha 2,t}$, defined as follows:

$$\begin{aligned}
\hat{S}_{k,\alpha 1,t} &\equiv \frac{\left[\frac{\mu_\theta}{\mu}\kappa_\delta\hat{S}_{k,N1,t} - \hat{S}_{k,D1,t} + \delta\bar{\wp}_s\left(\mu_\theta\hat{S}_{k,N2,t} - \frac{\mu_\theta}{\mu}\left(1+\bar{\theta}_1\right)\kappa_\delta\hat{S}_{k,N5,t} - \delta\bar{\wp}_s\mu_\theta(1+\theta)\hat{S}_{k,N6,t} + \theta\hat{S}_{k,D3,t}\right)\right]}{\Phi_{\Sigma 1,(k+1),1}} \\
\hat{S}_{k,\alpha 2,t} &\equiv \frac{\left[\frac{\mu_\theta}{\mu}\kappa_\delta\left(1+\bar{\theta}_1\right)\hat{S}_{k,N3,t} - \theta\hat{S}_{k,D2,t} + \delta\bar{\wp}_s\mu_\theta(1+\theta)\hat{S}_{k,N4,t}\right]}{\Phi_{\Sigma,(k+1),3}}
\end{aligned}$$

Therefore, using Corollary 2, we obtain the following simplified system for $k = \{0, 1, 2, ..., \infty\}$:

$$\begin{aligned}
\hat{\wp}_t^* &= \bar{\alpha}_2\beta\frac{\Phi_{\Sigma,1,3}}{\Phi_{\Sigma,0,3}}E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^{ind}\right) + \frac{1}{(\bar{\wp}^*)}\left[\frac{\mu_\theta}{\mu}\kappa_\delta(\sigma+\omega)\frac{\Theta_{0,N1}}{\Phi_{\Sigma,0,3}}\hat{x}_t + \bar{\alpha}_1\beta\frac{\Phi_{\Sigma,1,1}}{\Phi_{\Sigma,0,3}}\hat{\ell}_t - (\delta\bar{\wp}_s)\frac{\Phi_{\Theta,0,4}}{\Phi_{\Sigma,0,3}}\hat{\wp}_{s,t}\right] \\
&+ \frac{1}{(\bar{\wp}^*)}\bar{\alpha}_1\beta\frac{\Phi_{\Sigma 1,2,1}}{\Phi_{\Sigma,0,3}}E_t\hat{S}_{1,\alpha 1,t+1} - \bar{\alpha}_2\beta\frac{\Phi_{\Sigma,2,3}}{\Phi_{\Sigma,0,3}}E_t\hat{S}_{1,\alpha 2,t+1} \\
\hat{S}_{k,\alpha 1,t} &= \frac{\mu_\theta\kappa_\delta(\sigma+\omega)}{\mu}\frac{\Theta_{k,N1}}{\Phi_{\Sigma 1,(k+1),1}}\hat{x}_t + \bar{\alpha}_1\beta\frac{\Phi_{\Sigma,(k+1),1}}{\Phi_{\Sigma 1,(k+1),1}}\hat{\ell}_t - (\delta\bar{\wp}_s)\frac{\Phi_{\Theta,k,4}}{\Phi_{\Sigma 1,(k+1),1}}\hat{\wp}_{s,t} + \bar{\alpha}_1\beta\frac{\Phi_{\Sigma 1,(k+2),1}}{\Phi_{\Sigma 1,(k+1),1}}E_t\hat{S}_{(k+1),\alpha 1,t+1} \\
\hat{S}_{k,\alpha 2,t} &= -\bar{\alpha}_2\beta E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^{ind}\right) + \bar{\alpha}_2\beta\frac{\Phi_{\Sigma,(k+2),3}}{\Phi_{\Sigma,(k+1),3}}E_t\hat{S}_{(k+1),\alpha 2,t+1}
\end{aligned}$$

where $\hat{\ell}_t \equiv E_t\left(\hat{q}_{t+1} + \hat{\mathfrak{g}}_{t+1} + \hat{\pi}_{t+1}^{ind}\right)$ and $\hat{\xi}_t$ is the aggregate shock:

$$\begin{aligned}
\hat{\ell}_t &= -E_t\left(\hat{\pi}_{t+1} - \hat{\pi}_{t+1}^{ind}\right) + (1-\sigma)E_t\left(\hat{x}_{t+1} - \hat{x}_t\right) + \hat{\xi}_t \\
\hat{\xi}_t &= \frac{(1+\omega)}{(\sigma+\omega)}E_t\left[(\hat{\epsilon}_{t+1} - \hat{\epsilon}_t) + (1-\sigma)\left(\widehat{\mathcal{A}}_{t+1} - \widehat{\mathcal{A}}_t\right)\right]
\end{aligned}$$