Influencer Marketing and Product Competition*

Lin William Cong† Siguang Li§

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Abstract

With the rise of social media and (live)streaming platforms, firms and brand owners increasingly depend on influencers to attract consumers, who care about both product quality and consumer-influencer interaction. Sellers thus compete in both influencer and product markets. We develop a theory to understand the influencer labor market and its interaction with product competition. We find: (i) more powerful influencers sell better quality products for sellers with weaker direct-sale capacity, facilitating seller competition; (ii) advances in the intermediation technology can lead to non-monotonic changes in influencers' payoff and income inequality; (iii) style pluralism mitigates market concentration by horizontally differentiating the consumer experience but serves as either complements (intermediate dispersion) or substitutes (small or large dispersions) to vertical product differentiation; and (iv) influencers may inefficiently under-invest in consumer outreach and unidirectional exclusivity can improve welfare in less competitive markets. Together, our findings highlight the novelty of the influencer economy and establish its theoretical baselines for future studies.

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Keywords: Creator Economy, Digital Economy, Internet Celebrity, IO, Streaming.

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†Cornell SC Johnson College of Business (Johnson) and NBER. Email: will.cong@cornell.edu
§Society Hub, HKUST, Guangzhou. Email: siguangli@hkust-gz.edu.cn

1 Introduction

The past decade has witnessed the rise of the influencer economy (also known as "Wang Hong economy" and more recently dubbed by the media as the "creator economy") in both developed and emerging economies including the United States, Europe, China, and Latin America, etc. It prominently features large-scale social media marketing (e.g., live-streaming), testimonial endorsements, and precise product placements and distributions by people and organizations who have a purported expertise or influence.[1] This phenomenal growth is driven by the exponential growth of e-commerce and digital platforms and is further accelerated by the recent COVID-19 pandemic (e.g., Sinha, 2021) and the emerging generative AI. In this new digital economy, influencers come in variety and from diverse backgrounds, including content creators, celebrities, and idols, and key opinion leaders (KOL) (Williams, 2016).[2] They manage their own fan bases who are drawn to their talent, charisma, wisdom, appearance, etc., and profit by helping brand owners and service providers promote various products to potential consumers.[3] The thriving influencer marketing and creator economy not only generate sizable revenues but also unleash huge potential in job creation.[4]

[1]The influencer or creator economy generally refers to the independent businesses and side hustles launched by self-employed individuals who make money off of their knowledge, skills, or following. CB Insights (2021) provides an excellent introduction to the industry. Many influencers generate up to seven-figure income. Wei Ya, once the most famous influencer, made a fortune of over 1 billion CNY in live-streamed presales on Single's Day (Nov 11, the black Friday equivalent in China) alone in 2019 (China Business Industry Research Institute, 2020). Famous Instagram influencers like Huda Kattan or Eleonora Pons net up to 6 figures per post. The top writers on Substack can make up to $1 million annually. Youtube paid 30B to creators in 2019-2021 (CB Insights, 2021).

[2]Content creators derive from "YouTube stars" marketed by YouTube in as early as 2011 (Lorenz, 2019). Now, it can be anyone who creates any form of content online, including TikTok videos and Clubhouse audio. For example, on Twitch, daily users can watch live streams of video games played by others through Streamlabs, and tips paid out on Twitch alone are estimated to be $141 million. Unlike live stream creators, internet celebrities on Instagram and the like can post about or live stream special travel or dining experience, or simply routine daily lives. Many rely on physical attributes alone without actively creating content. For example, Instagram enables brand owners to sell products through idols that attract consumers who simply seek to see them. Similarly, KOLs can target a specific demographic in an interactive manner, making product sales more engaging by sharing their own thoughts and ideas.

[3]Influencers touch almost all aspects of life, including entertainment, fashion, food, movies, music, sports, etc., and increasingly utilize (low cost and easy-to-spread) short videos . Although there are many ways to monetize the influence, such as compensation for content creation or interaction with fans, influencers' largest income is still from commercials and e-commerce traffic direction.

[4]The global influencer marketing has grown from a mere U.S. $1.7 billion in 2016, to a size of $16.4 billion in 2012, and was set to reach $21 billion in 2023 (Geyser, 2023). For comparison, the global television advertising spending was valued at $141 billion in 2021, and the global ad spend for print advertising is estimated at $50.38 billion in 2022 (Okoronkwo, 2022). Goldman-Sachs (2023) predicts that the total addressable market of the creator economy is estimated to double in size from the current $250 billion to half a trillion dollars by 2027, comparable to the current total e-commerce market revenue. Meanwhile, according to the National Bureau of Statistics, the total market size of China's influencer economy is expected to grow from $210 billion in 2020 to $1.035 trillion in 2025. On the other hand, regarding job creation, Influencity (2023) estimates more than 10.2 million Instagram creators and 10.1 Tiktok creators in the United States,

Clearly, despite having to share revenues with influencers, sellers increasingly adopt influencer-intermediated sales. Examples include beauty and cosmetic companies such as Estee Lauder, Revolution Beauty and REFY Beauty, which have led the way in their use of influencers with success stories.[5] In contrast, according to Insider, nearly 3,200 stores from more than 20 major retailers closed in the United States in 2023 (Reuter, 2023). Furthermore, given the explosion of product varieties in online retail sales (Baslandze et al., 2023), influencers can still effectively reach niche markets and local demographics, even without increasing product varieties (Han, 2023).[6] Several questions naturally arise: (a) Why can influencer marketing grow so quickly? (b) What is its impact on product competition? (c) How does the influencer labor market evolve? We provide initial answers to these questions by developing a novel game-theoretic model in which sellers depend on influencers to acquire customers and compete in both the product market and influencers' labor market, with potentially endogenous and heterogeneous influence.

Specifically, we model three important groups of agents—sellers (who are also producers), influencers, and consumers—and allow pairwise group interactions through the product market, influencers' labor market, and social media platforms (for influencers to connect with consumers). Sellers or brand owners hire influencers to sell products to consumers, which may interact with direct sales. Consumers are uniformly located on a unit circle in the "type" space with consumption utilities determined by both the true quality of the product and the style, status, identity, etc.—things that draw people toward influencers selling the product on social media like Instagram, or Alibaba (known as Da Ren). Agents interact in sequential stages: (i) sellers make production decisions, (ii) sellers hire and match with influencer(s) in the labor market, and (iii) consumers choose which influencer to follow and consume the products the influencer promotes. We solve the model backward and discuss economic

and 10.2 million and 18.9 million Instagram influencers for Europe and Latin America. PJdaren (2023) has identified 10.1 million influencers in China whose fan base exceeds 10K followers. As of 2022, the number of influencers and creators is estimated to be 200-300 million (Edelman Data & Intelligence Linktree creater report, Geyser, 2023). Furthermore, Griffith (2023) estimates 165 million new creators since the beginning of the pandemic, including 86 million in the United States, 73 million in Brazil, 19 million in Germany, 18 million in Japan, 17 million in South Korea, 16 million in the United Kingdom, 16 million in France, 10 million in Spain, and 6 million in Australia. According to Goldman-Sachs (2023), the current 50 million influencers will continue to grow at a 10-20% compound annual rate in the next five years.

[5]Specifically, Estee Lauder announced the decision to allocate 75% of its marketing budget on social media influencers even in 2019. Meanwhile, Fast Forward and REFY announced in late 2022 that they achieved revenues of eight digits, a success unheard of for a business in its second year of trading, and Revolution Beauty's earnings increased more than 70% in May 2022 compared to the previous year (Peak, 2022).

[6]Here are several useful statistics. For marketers, more than 80% intend to have a delicate budget for influencer marketing, with 23% spending more than 40% of their budget on influencer marketing (Geyser, 2023). For consumers, 36.4% of all respondents prefer influencer posts as the primary way to try new products, with 56% of them purchasing a product after seeing it used by an influencer (IZEA, 2022).

insights and predictions in each stage. We also discuss endogenous influence acquisition.

We consider the setting in which two sellers compete in the labor market of influncers, each hiring one influencer, in addition to competing in the product market. We find that influencer-intermediated sales feature assortative matching under Nash bargaining: sellers of better products work with more powerful influencers due to the complementarity between influence power and product quality. Moreover, influencer-intermediated sales generally enjoy local monopoly power regardless of product quality when influencer styles are sufficiently distinct. For one seller-influencer group to crowd out the rival group, the style difference between influencers needs to be sufficiently small, and both the influencers' power gap and the product quality gap must be sufficiently large. Finally, technological advances in intermediation can affect influencers' payoffs and income inequality, first benefitting them by scaling their customer outreach, but eventually hurting them in the labor market as the crowded intermediation space transforms into more fierce price competition.

We then move one stage back to endogenize sellers' production decisions. Specifically, we investigate how influencer heterogeneity affects horizontal and vertical product differentiations compared to traditional economies. We find that influencers' style difference is a substitute for horizontal product differentiation. Specifically, when the influencers' style difference is small, sellers differentiate products to reduce competition. When the difference is large, sellers hire influencers and have no incentive to differentiate products because divergent influencers give sellers less elastic demand for their products. However, both the product quality gap and the influencer power gap complement horizontal differentiation because the strong seller competes more aggressively when selling superior products or working with a more powerful influencer. This leads to lower profits for the weaker seller, spurring her to differentiate horizontally.

Sellers can also vertically differentiate by improving product quality to win market shares. In general, when influencers' style difference increases, the return from investing in quality also increases. When influencers' style difference is sufficiently large, both groups can break even and choose high product quality, giving a minimal vertical differentiation. When the style difference is sufficiently small, both groups choose low product quality, resulting in minimal vertical differentiation again. Only for intermediate style differences does vertical differentiation occur because the investment profit allows one and only one seller-influencer pair investing in quality to break even.

Next, we introduce direct sales and analyze how they compete and interact with influencer-intermediated sales. We generalize the model so that sellers are positioned in the center of

the circle with a significant transportation cost in direct sales to reflect the fundamental friction that companies traditionally do not participate in the users' social lives. We show that when sellers have comparable capacity in traditional marketing and sales, all main results remain robust in the presence of direct sales. This general setting offers two additional insights though: First, influencer marketing can foster seller entry and competition by allowing sellers with weak traditional sales to bypass the transportation cost barrier. Second, sellers with weak traditional marketing channels have a stronger incentive to hire influencers, leading to a negative assortative matching between direct-sale capacity and influence power (for any given product quality). Both contribute to the rapid growth of influencer marketing.

We also extend the model to allow for endogenous influence style and power. While the assortative matching between sellers and influencers remains under endogenous style selection, socially inefficient under-investment in power can arise due to: (i) influencers do not internalize the externality on consumer welfare when acquiring influencer power; and (ii) the threat of intense price pushback can further discourage influencers from investing in power, and any potential gain in consumers is dominated by lowered product prices.[7] We further consider "unbalanced matching" to better understand the welfare implications of exclusivity contracts and regulatory policies. We find that requiring balanced seller-influencer matching can encourage seller competition, whereas unidirectional exclusivity contracts are welfare-improving for sufficiently differentiated products and uncontested influencer markets.

Overall, our theory explains and predicts several trends in the influencer market. First, influencer marketing favors large brands who can afford to invest more in improving product quality and match with more powerful influencers, but benefits small and less established firms by facilitating the entry of sellers and enabling those with weak direct sales capacity to bypass the transportation cost barrier to access niche markets and accumulate reputation, creating both market concentration and greater entry at the same time. Second, influencers improve the consumer experience because their style pluralism effectively differentiates products horizontally and enhances investments in product improvements. Third, as digital technology and AI continue to improve, the aggregate wage and income inequality for influencers will first increase and then decrease, while the growth of the influencer market always benefits the seller and the total welfare.[8] Finally, regarding policy interventions and designs, mutual exclusivity can foster competition and improve total welfare in a congested influencer market and helps prevent "arms race" among influencers in power acquisition. In

[7] Over-investment may ensue under a monopolist seller. See Remark 3 for a detailed discussion.

[8] See Section OA.3 for a detailed discussion of influencer competition.

the long run, establishing creator-friendly platforms and providing additional support to less established influencers will likely boost the growth of the influencer economy.

Literature. Our study adds foremost to the nascent literature on the influencer or creator economy. Previous studies have focused on the relationship between influencers and digital platforms or multichannel networks (MCNs), especially the rules of revenue sharing (Bhargava, 2022; Jain and Qian, 2021), disclosure by internet influencers (Mitchell, 2021), search technology and advice transparency (Fainmesser and Galeotti, 2021), influencer cartels (Hinnosaar and Hinnosaar, 2021), and firms' optimal affiliation with influencers (Pei and Mayzlin, 2019). Instead, we analyze seller competition and seller-influencer matching, which in turn affect product differentiation and endogenous influence acquisition. Recently, Bian et al. (2023) provide empirical evidence for our model predictions about the influencer labor market evolution and its dependence on technology.

Our study is thus related to the broad literature on marketing and industrial organization (e.g., Salop, 1979). We analyze their interaction in the context of fast-emerging influencer economy. Studies on advertising have focused on the aggregate and cross-sectional levels of advertising and its welfare implications (Becker and Murphy, 1993; Spence and Owen, 1977; Butters, 1978; Dixit and Norman, 1978; Grossman and Shapiro, 1984; Nichols, 1985; Stegeman, 1991; Nelson, 1974; Johnson and Myatt, 2006). Most models assume no media or focus only on the informational effects or nuisance costs on viewers of advertisements (e.g., Johnson, 2013). Moreover, most studies do not endogenize the locations of media stations, and the ones that do (e.g., Gal-Or and Dukes, 2003; Dukes, 2004) typically take sellers' product differentiation as exogenous. We study influencers whose matching with sellers is affected by the consumer base, and analyze endogenous product differentiation and influencers' style choices simultaneously. We consider the level of advertising or its informational role in reduced-form through influence, focusing on the complementarity between the multiple dimensions of consumer utility from following influencers and consuming products.

More recently, Amaldoss and He (2010) study how firms strategically target consumers to avoid intense price competition. Several studies in marketing analyze how firms compete in the effort to hire influencers, including the intensity of the advertisement, the competitive targeting of influencers in a network, and the structure of the network and its influence on prices, firm profits, and consumer surplus (Galeotti and Goyal, 2009; Katona, 2018). In particular, Fainmesser and Galeotti (2021) analyze search quality, advice transparency, and influencer strategy in the market for online influence. We differ by analyzing the interaction

of seller-influencer matching and product market competition. Furthermore, we add to the discussion on exclusivity contracting and the link between unidirectional exclusivity contracts and bargaining (e.g., Gal-Or, 1997; Dukes and Gal-Or, 2003) by contrasting unidirectional with mutual exclusivity contracts in their impact on welfare in the influencer economy.

Finally, our study is related to studies on the endogenous emergence of intermediation. Middlemen can emerge for various reasons, including concentrating delegated monitoring incentives (Diamond, 1984), reducing search frictions (Rubinstein and Wolinsky, 1987; Duffie et al., 2005), absorbing excess inventory demand or supply in the financial market (Glosten and Milgrom, 1985), facilitating information production in asset management (Gârleanu and Pedersen, 2018). Recent articles have also explored the measurement of financial intermediation (Philippon, 2015), efficient intermediation chains (Glode and Opp, 2016; Glode et al., 2019), and the self-selection of intermediaries (Zhong, 2023).[9] Our study complements this literature by focusing on influencer intermediaries on social media platforms and their impact on product market competition.

The remainder of the paper is organized as follows. Section 2 presents the baseline model. Section 3 studies seller competition and influencer hiring. Section 4 explores influencer heterogeneity and product differentiations. Section 5 introduces direct sales and compares them with influencer-intermediated sales. Section 6 investigates endogenous influence and exclusivity contracting. Section 7 discusses the empirical implications and concludes. The appendix and online appendix contain the proofs and extended discussions.

2 Model Setup

Two risk-neutral, profit-maximizing sellers indexed by $k \in \{1, 2\}$ each sells a unit consumption product/service of a common value y_k. We denote the kth seller's utility by U_k. They work with influencers for interactive marketing and outreach (we allow direct sales later). Two representative influencers with "style" $\theta_j \in \mathbf{S}^1$, where $j \in \{1, 2\}$ indexes them and $\mathbf{S}^1 := \{s \in \mathbb{R}^2 : s_1^2 + s_2^2 = 1\}$. Style could refer to identity, fashion taste, and other things that attract people to influencers on Instagram, Tiktok, Alibaba, etc., among the recent proliferation of social networks, digital platforms, and broadcast channels. Let

[9] Judge (2022) shows how overgrown middlemen became the backbone of modern capitalism and the cause of many of its ailments because complexity and opacity leave our systems more fragile. In a way, buying through influencers can be more direct than the multiple layers of suppliers and distributors in the traditional economy. However, influencers are more than just selling products and bring in the social aspect, which is relevant for trust, information distribution, and communication. The stake of influencers' reputation can also provide good incentives for them to provide value to consumers.

$I_j \in \mathbb{R}_+$ denote the jth influencer's power of influence, which captures her persuasiveness and charisma among followers, as well as general technological advances (e.g., digital platforms and generative AI) that facilitate influencer-follower interactions and allow influencers to create products or content with more appeal to follower consumers.

Traditional advertising channels through televisions, newspapers, etc., can be viewed as having a low persuasiveness since TV commercials, for example, entail limited interaction with the audience, and capital-intensive advertising makes it difficult to instill authenticity in product recommendations. In contrast, social interactions can dramatically increase the authenticity of influencers and the consumer experience. Differential power then reflects distinctions among celebrities, macro-influencers, and micro-influencers, and influencer attributes and intermediation technologies jointly determine the effective consumer base.

A continuum of consumers with a measure 2π is uniformly located on $\mathbf{S}^1$. They derive utilities from the baseline quality of products, as well as from having similar styles or affinity with certain influencers (McDonald, 2019). A greater affinity can be interpreted as how effectively an influencer helps a consumer understand, select and enjoy the product purchased, potentially due to trust built over time or the influencer's expertise or influencer-specific product demo.[10] $\forall x_1, x_2 \in \mathbf{S}^1$, we define $\|x_1 - x_2\|$ to be the distance along the short arc.

Consumer i's utility buying from Seller k through Influencer j of type θ_j is:

$$u_i^j(x_i, \theta_j, y_k) = \begin{cases} y_k * (1 - \|x_i - \theta_j\|/I_j) - p_k, & \text{if a unit good is consumed,} \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where p_k is the unit price charged for the consumption good. We suppress the superscript j whenever there is no confusion. Here, brand owners and product sellers depend on influencers to sell goods, with only consumers having $\|x_i - \theta_j\| \leq I_j$ entering the aggregate demand. Importantly, the utility of the consumer depends on the affinity for style. Our specification aims to adequately capture the typical reasons for engaging celebrities or influencers in advertising campaigns: attention-grabbing, persuasion through expertise, and global outreach (Moeran, 2003).[11] The influence power includes direct attention grabbing,

[10] Malmendier and Veldkamp (2022) provide a microfoundation through information resonance, i.e., information resonates with recipients when they identify with the person who communicates it.

[11] Advertising through conventional technology, e.g., through TV/newspaper, are often extremely costly. We are cognizant that such a cost is used to indicate the existence of a price premium that can guarantee contractual performance in competitive equilibrium (Klein and Leffler, 1981). However, in an influencer economy with digital platforms and the proliferation of social-commercial network apps, the cost is relatively low and would not serve such a function for disclosing the presence of a large sunk "selling" costs.

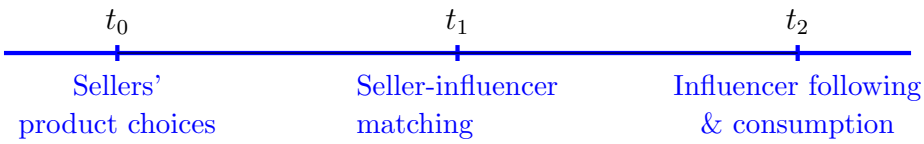


Figure 1: Timeline

either through vacuous "human pseudo-events" in the words of American historian Daniel Boorstin or through skills or entertainment. Expertise and global, cross-cultural outreach can manifest through the combination of location and power.

Timeline. Influencers' type and power are the accumulation of knowledge and skills since childhood and are therefore taken as given in the baseline model. Sellers first decide on the products and subsequently hire influencers. The consumers then choose which influencer to follow and consume the products offered. In Section 3, we take influencers' type and power, as well as the sellers' products as given, in order to focus on the sellers' hiring of influencers and influencers' impact on consumption. In Section 4, we allow sellers to endogenize the products for sale. Our main findings are robust to having product decisions following seller-influencer matching. In practice, firms decide on their business operations before exploring marketing channels, which is what our setup captures.[12]

Matching and bargaining protocols. We use a general (bilateral) Nash bargaining protocol for the negotiation once influencers are hired by sellers.[13] Let γ and $(1-\gamma)$ denote the bargaining power assigned to the seller and the influencer, respectively.[14] Once sellers and influencers are matched, they have exogenous outside options, e.g., from revisiting the influencer market, which we normalize to zero for simplicity. Anticipating such bargaining processes, sellers and influencers endogenously match. Our baseline setup focuses on one-to-one match—often the case in practice, the seller-influencer contracts either feature mutual exclusivity clauses or they are all allowed to have multiple relationships so that the matching is balanced. This negotiation-based approach is realistic and popular for setting the advertising price in the media industry (Dukes and Gal-Or, 2003; Gal-Or, 1997).

[12] In Section 6.1, we endogenize influencers' power (potentially interpreted as skill training over the intermediate term) and type (potentially interpreted as culture, talent, or interest cultivated in the long run).

[13] Standard for modeling, e.g., commercial media and healthcare (Gal-Or, 1999; Dukes and Gal-Or, 2003).

[14] Our Nash bargaining setup is consistent with the convention in the sales agents literature that the seller sets the price, and agents take commission fees. Since all agents have identical bargaining power in our symmetric setting, it is equivalent to maximize sales revenue. Moreover, our modeling of product pricing is also consistent with the practice of how influencers get paid. In 2023, 74% of influencers are paid by percentage of sales and product levels (Geyser, 2023).

The joint matching and bargaining problem is nontrivial. In specifying the protocols, we strive to balance tractability, transparency, convention in the literature, coherence with our non-repeated game set-up, and realism. In fact, many key results are independent of how the surplus is divided between matched sellers and influencers, as long as they care about the group surplus. We discuss unbalanced matching and the welfare implications of contract exclusivity in Section 6.2 where sellers can require exclusive relationship and impose non-compete clauses, as seen in many nascent markets for influencers.

Influencer marketing and intermediation. Influencers are intermediaries who place products into consumers' hands, while adding to consumers' purchase experience. We shall clarify the role of influencers and interpret the additional utility induced in Equation (1) as broadly capturing many realistic scenarios.

First, the specification can represent how the interaction with influencers enhances the experience of the product/service. For example, Alessandra Sales, the vice president of growth at Ipsy, once told McKinsey that transgender creators helped the beauty brand connect with the transgender community. It is also common for followers to derive sustained benefits from purchasing through influencers because they can coordinate followers who made purchases to form a community to discuss additional usage of products or user experience, etc. Such communities are typically difficult for sellers to foster directly.

Second, the additional utility term can also represent some direct financial benefits, such as special discounts or bundled products from sellers that an influencer gets for followers, which is typically hard to obtain by dispersed consumers due to a lack of coordination and collective bargaining.

Third, influencers can convince sellers to modify product features to better fit the style or tastes of their followers, of which sellers are not directly aware unless they conduct costly surveys. In fact, many influencers get involved in the design of products, and some influencers can exclusively reserve new product arrivals for sales to their followers. Anecdotes include the NOCTA product line, Nike's collaboration with the famous rapper Drake, and Dunkin's long-term sponsorship with Charli D'Amelio, one of the biggest stars of TikTok with 144.5 million followers. Dunkin even released a new beverage named "The Charli."

A single-influencer benchmark. Consider a monopolist seller working with a single influencer. When $I < \pi$, not all consumers are within reach of the influencer. Fix the price p and the quality of the product y, only consumers with a non-negative utility are served,

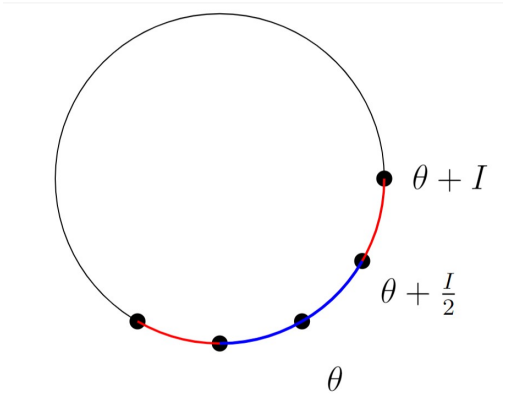


Figure 2: The circular market

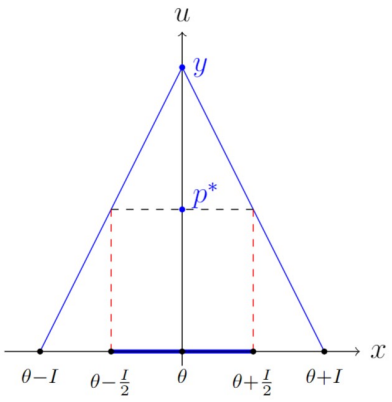


Figure 3: The monopolist pricing

which determines the demand $D(p) = 2(1 - p/y)I$. When $I \geq \pi$, the influencer can reach all consumers. Specifically, if p is sufficiently low such that $p \leq y(1 - \pi/I)$, $D(p) = 2\pi$; otherwise, $D(p) = 2(1 - p/y)I$. We can further analyze the monopolist pricing strategy and the profit as follows. For any given values of (I, y), the optimal pricing strategy p^* and the resulting profit Π for a monopolist seller can be derived (see Appendix A.1):

$$p^* = \begin{cases} \frac{y}{2}, & \text{if } I < 2\pi \\ y\left(1 - \frac{\pi}{I}\right), & \text{if } I \geq 2\pi \end{cases} \quad \textit{and} \quad \Pi = \begin{cases} \frac{yI}{2}, & \text{if } I < 2\pi \\ 2\pi y\left(1 - \frac{\pi}{I}\right), & \text{if } I \geq 2\pi \end{cases} \tag{2}$$

The seller's payoff U_1 and the influencer's wage w_1 are given by $U_1 = \gamma\Pi$ and $w_1 = (1-\gamma)\Pi$.

Depending on I, a monopolist seller can choose to target a subpopulation or the whole demographic of consumers. Figures 2 and 3 illustrate the potential (red) and actual (blue) consumer base and the monopolist pricing strategy in Eq. (2) for an influencer located at θ. The vertical axis corresponds to consumer utility with the optimal price p^* in the red dashed line, and consumers are denoted by the thick blue line.

3 Seller Competition and Influencer Hiring

We now consider seller competition in both the influencers' labor market and the product market, with implications for influencer-intermediated sales. A seller-influencer group is characterized by the 3-tuple (y_m, θ_m, I_m) for $m = 1, 2$. Without loss of generality, we assume that $y_1 \geq y_2$ and $I_1 \geq I_2$. Unless otherwise stated, we consider balanced matching in which each seller is matched with at most one influencer and vice versa. Denote by w_j Influencer j's wage, and by U_k Seller k's payoff. Let $\beta := \|\theta_1 - \theta_2\|$ denote the style dispersion between

the two influencers. Denote by $k(j)$ the identity of the matched seller for Influencer j.

3.1 Assortative Matching Between Sellers and Influencers

Do brands of higher quality hire more powerful influencers? Indeed, given style locations for the influencers, positive assortative matching typically ensues in equilibrium. This echoes practitioners' concern that bigger brands crowd out smaller ones in hiring better influencers. That said, positive assortative matching also provides a rationale for the rise of influencer marketing. By amplifying the outreach of better products, influencer marketing can potentially place better products into consumers' hands and improve the total welfare.

Proposition 1 (Positive Assortative Matching)**.** *$k(j) = j$ for $j = 1, 2$ when one of the following holds: (i) $\beta \geq \frac{1}{2}I_1 + \frac{1}{2}I_2$, that is, the market is uncongested, (ii) one seller dominates the entire market, or (iii) influencers are maximally distanced and both sellers get market shares, that is, $\beta = \pi$ and $I_j \geq \pi$.*

Proposition 1 presents fairly general sufficient conditions for assortative matching to arise. Specifically, (i) shows the emergence of assortative matching when the influence power is relatively small compared to the style dispersion. The seller with a more valuable good can offer to hire a more powerful influencer by proposing a higher wage because the seller-influencer group's total profit is supermodular in influencer power and product quality parameter.[15] Since influencers do not increase the common value of consumption y, the product with a better quality is always priced higher.

Furthermore, in (ii), assortative matching emerges when market competition is intense and a single seller can dominate all other sellers. If only a single seller can survive, it has to be the strong seller because consumers sufficiently loyal to her always keep purchasing from her. Furthermore, if the strong seller can defeat the other seller by hiring the relatively weak influencer, it only makes the strong seller more powerful by hiring the strong influencer in the product market. These two facts jointly lead to assortative matching.

Finally, in (iii), assortative matching also occurs when influencers are maximally distanced (i.e., $\beta = \pi$), which can be an outcome of influencers' endogenous choices of style, as we discuss later. The condition $I_2 \geq \pi$, combined with $I_1 \geq I_2$, means that the market is crowded and influencers compete on both sides. The intuition remains. The strong seller, by hiring a more powerful influencer, enjoys an advantage in competing against the other seller and getting a bigger consumer share and thus a large joint profit.

[15] A twice-differentiable function $f : X \times Y \to \mathbb{R}$ is supermodular iff $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ for all $(x, y) \in X \times Y$.

3.2 Market Dominance in the Influencer Economy

Despite positive assortative matching, influencer marketing can empower smaller brands and endow them with market power in niche markets and subsequently maintain seller competition instead of having one seller dominate. To see this, we first establish a set of sufficient conditions under which a single seller-influencer group grabs the entire market. Compared to traditional markets without influencers, it is more difficult to achieve such market dominance, which now requires relatively large gaps in both product quality and influence power, as well as a small style dispersion between influencers.

Proposition 2 (Market Dominance)**.** *Fix $I_1 \leq 2\pi$. If $\frac{\beta}{I_1} \leq \min\{\frac{1}{2} - \frac{y_2}{y_1}, \frac{1}{2} - \frac{I_2}{I_1}\}$, then Seller 1 hires Influencer 1 and sets $p_1^* = \frac{y_1}{2}$ to completely force Seller 2 out of the market. Payoffs for sellers and influencers are $U_1 = \frac{\gamma y_1 I_1}{2}$, $w_1 = \frac{(1-\gamma) y_1 I_1}{2}$, and $U_2 = w_2 = 0$.*

Unlike direct price competition, it is not enough to have a greater influence power or a much better product to force the rival out in product competition. Suppose that Seller 1 has a superior product. Seller 2 can still compete to gain some market share because Influencer 2, who works with Seller 2, creates a sufficiently large "product differentiation" ($\theta_1 \neq \theta_2$). Here, the sufficient conditions in Proposition 2 involve multi-dimensional heterogeneity and are stringent, as illustrated next.

3.3 Heterogeneity and Market Power

We illustrate how influencers help sellers gain "local" market power through three examples, each involving a distinct form of heterogeneity. We not only show that sufficient conditions in Proposition 2 are easily violated, but also provide insights on how sellers compete in the presence of product quality difference and influencer heterogeneity.

Example 1 (Heterogeneous Product Quality)**.** *Consider $y_1 \geq y_2$, $\beta = 0$, and $I_1 = I_2 = I \leq \pi$. In equilibrium, $k(j) = j$ for $j = 1, 2$. After matching, the two groups choose $(p_1^C, p_2^C) = \left(\frac{2y_1(y_1-y_2)}{4y_1-y_2}, \frac{y_2(y_1-y_2)}{4y_1-y_2}\right)$. Seller 1 targets consumers with $\|x-\theta\| \leq \frac{2Iy_1}{4y_1-y_2}$ and Seller 2 targets those with $\frac{2Iy_1}{4y_1-y_2} < \|x-\theta\| \leq \frac{3Iy_1}{4y_1-y_2}$. Furthermore, the group profits are given by:*

$$\Pi_1^C = \frac{8Iy_1^2(y_1-y_2)}{(4y_1-y_2)^2}, \quad \textit{and} \quad \Pi_2^C = \frac{2Iy_1y_2(y_1-y_2)}{(4y_1-y_2)^2}. \tag{3}$$

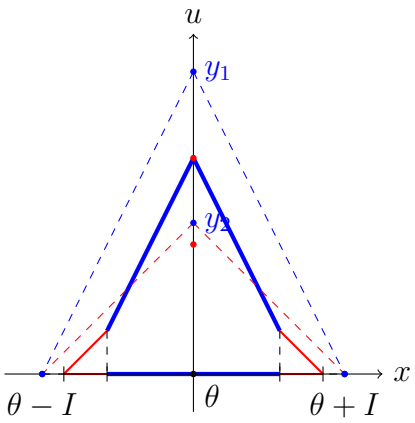


Figure 4: Heterogeneous product quality

Payoffs for sellers and influencers are given by:

$$U_1 = \gamma\Pi_1^C,\ U_2 = \gamma\Pi_2^C,\ w_1 = (1-\gamma)\Pi_1^C,\ \textit{and}\ w_2 = (1-\gamma)\Pi_2^C.$$

Figure 4 illustrates the equilibrium. Specifically, Seller 1, who offers a relatively superior product and works with Influencer 1, targets consumers with high loyalty, and thus a high willingness to pay. We illustrate these clients with the blue line. In contrast, Seller 2, which offers a relatively inferior product, attracts those consumers not targeted by Seller 1, and we illustrate them with the red line. The intuition is that when $y_1 > y_2$, the product offered by Seller 1 is still more attractive to consumers with high loyalty (that is, $\|x - \theta\| \to 0$) even if Seller 2 sets $p_2^* = 0$. Therefore, Seller 1 has incentives to set a positive price $p_1^* > 0$, which implies that Seller 2 can make a profit by attracting consumers not targeted by Seller 1.

Furthermore, when the quality gap shrinks (that is, $y_1 \downarrow y_2$), it converges to the outcome of Bertrand competition with $(p_1^C, p_2^C) = (0, 0)$. In contrast, when $y_1 \gg y_2$, it converges to an equilibrium with $(p_1^C, p_2^C) = (y_1/2, y_2/4)$, which means that the high quality product is priced at its monopoly price, while the low quality product is priced at a monopoly price in the residual market after removing the market share taken by the strong seller. Note that the sufficient conditions in Proposition 2 are violated in Example 1.

To further appreciate the content of Example 1, we compare it with traditional economies in which two sellers with quality indices $y_1 > y_2$ engage in vertical product competition. Bertrand price competition implies that $p_1 = y_1 - y_2$, $p_2 = 0$ and that all consumers buy from Seller 1. Note that a key difference is that Seller 2 has zero market share in this benchmark economy. In contrast, Seller 2 has a strictly positive market share, even if there is no horizontal style dispersion between the two influencers.

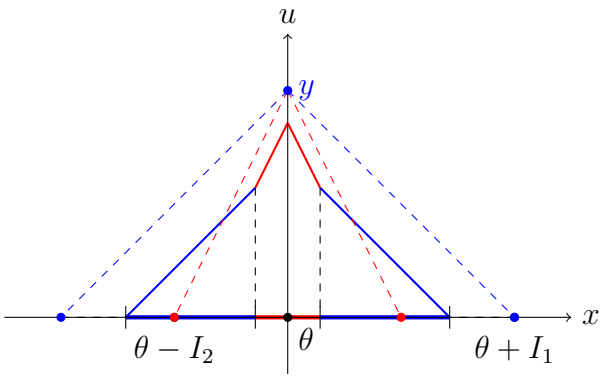


Figure 5: Heterogenous influence power

Without influencers, all consumers are concentrated in a mass, which always leads to the dominance of the superior product. In contrast, with influencers, even without horizontal differentiation, consumers are heterogeneous and distributed in a nontrivial decentralized way. In particular, consumers' willingness to pay for the product differs greatly when located near the influencer's style location, whereas consumers are more homogeneous but less valuable when located away from the center. Hence, the strong seller, by targeting the more valuable consumers, can establish a great advantage in revenue through influencer marketing. However, this revenue advantage also deteriorates when it comes to distant consumers, creating an incentive conflict with attracting the most valuable consumer for the dominant seller. This prevents perfect price competition as in traditional economies. Hence, as long as there exists a difference in product quality, both sellers can survive and secure a share of client, and thus a positive profit.

Example 2 (Heterogeneous Influence Power)**.** *Consider* $I_2 \leq I_1 \leq \pi$, $\beta = 0$ *and* $y_1 = y_2 = y$. *Again,* $k(j) = j$ *for* $j = 1, 2$ *and equilibrium prices are set at* $(p_1^C, p_2^C) = \left(\frac{2y(I_1-I_2)}{4I_1-I_2}, \frac{y(I_1-I_2)}{4I_1-I2}\right)$. *Seller 1 targets consumers with* $\frac{I_1 I_2}{4I_1-I_2} < \|x-\theta\| \leq \frac{I_1(2I_1+I_2)}{4I_1-I_2}$ *and Seller 2 targets those with* $\|x-\theta\| \leq \frac{I_1 I_2}{4I_1-I_2}$. *Group profits are given by:*

$$\Pi_1^C = \frac{4I_1^2(I_1-I_2)y}{(4I_1-I_2)^2}, \quad \textit{and} \quad \Pi_2^C = \frac{I_1 I_2(I_1-I_2)y}{(4I_1-I_2)^2}.$$

and payoffs for sellers and influencers are given by:

$$U_1 = \gamma\Pi_1^C,\ U_2 = \gamma\Pi_2^C,\ w_1 = (1-\gamma)\Pi_1^C,\ \textit{and}\ w_2 = (1-\gamma)\Pi_2^C.$$

Figure 5 illustrates the equilibrium. Specifically, Seller 1, who works with Influencer 1, targets consumers sufficiently distant from the influencer's style, as illustrated with blue

lines, mainly to avoid tough price competition in the product market. Indeed, they can afford this because the strong influence power alleviates the utility loss from consumers. In contrast, Seller 2, who works with Influencer 2, targets consumers with high loyalty, and we depict it with red lines. For the equilibrium, note that $p_1 \geq p_2$. Otherwise, Seller 2 will be forced out of the market. However, $p_1 = 0$ is sub-optimal for Seller 1 because consumers with $I_2 < \|x - \theta\| \leq I_1$ prefer Seller 1. Thus, Seller 1 can be better off by charging a low and positive price, which further implies that Seller 2 can also make a positive profit by attracting consumers close to θ. Furthermore, when $\frac{I_1}{I_2} \to \infty$, prices satisfy $(p_1^C, p_2^C) \to (\frac{y}{2}, \frac{y}{4})$, and when $\frac{I_1}{I_2} \to 1$, $(p_1^C, p_2^C) \to (0, 0)$, which coincides with outcomes in standard Bertrand competitions.

By comparing Example 1 and 2, we can see that the competition mode depends on the way heterogeneity arises. Specifically, when it is driven by the consumption value of the product, the stronger group focuses on attracting consumers with a taste similar to that of the influencer. In contrast, when it is related to how easily the influencer attracts followers, the stronger group focuses on those consumers not reachable by the weaker group and sacrifices the loyal followers in the sense of taste proximity.

Example 3 (Heterogeneous Influencer Style)**.** *Consider $\beta > 0$, $I_1 = I_2 = I$ and $y_1 = y_2 = y$. Define $\beta_0 := \frac{2}{67}(-7 + 5\sqrt{10})I \approx 0.263I$. Again, $k(j) = j$ for $j = 1, 2$. After matching, the kth seller-influencer group's price p_k^C and total profit Π_k^C satisfy:*

(i) when $\beta > I$ holds, $p_k^C = \frac{y}{2}$ and $\Pi_k^C(\beta, y) = \frac{yI}{2}$;

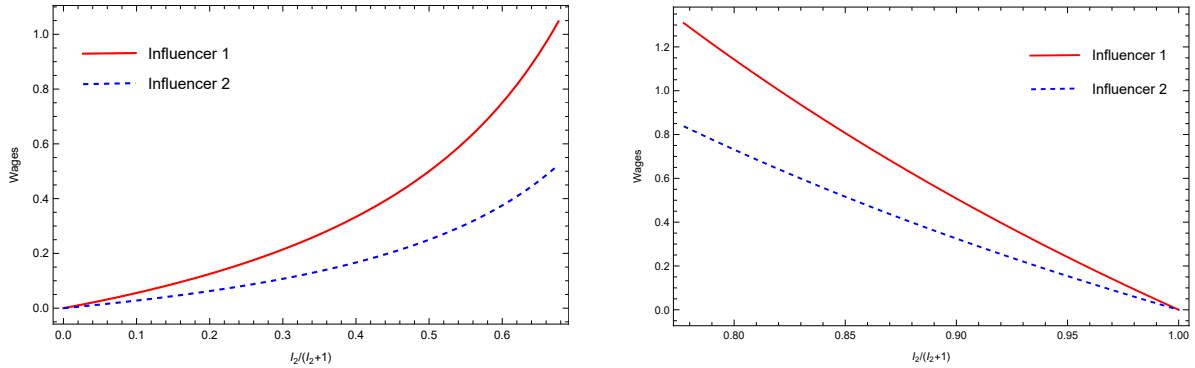
(ii) when $\frac{6}{7}I < \beta \leq I$ holds, $p_k^C = y * \left(1 - \frac{\beta}{2I}\right)$ and $\Pi_k^C(\beta, y) = y * \beta\left(1 - \frac{\beta}{2I}\right)$;[16]

(iii) when $\beta_0 \leq \beta \leq \frac{6}{7}I$ holds, $p_k^C = \frac{y}{5I}(2I + \beta)$ and $\Pi_k^C(\beta, y) = \frac{3y}{50I} * (2I + \beta)^2$.

Furthermore, $U_1 = \gamma\Pi_1^C$, $U_2 = \gamma\Pi_2^C$, $w_1 = (1 - \gamma)\Pi_1^C$, and $w_2 = (1 - \gamma)\Pi_2^C$.

Note that when $\beta < \beta_0$, there exists no pure strategy equilibrium. Once again, the stronger seller cannot dominate the market. To summarize, the three examples above jointly show that it is insufficient to achieve market dominance by introducing one-dimensional heterogeneity in an influencer economy.

[16]Note that there exist multiple equilibria when $\frac{6I}{7} \leq \beta \leq I$ and we focus on symmetric equilibrium such that $p_1^* = p_2^*$ in Example 3. Specifically, all equilibria are characterized as follows: (i) when $\beta \in [6I/7, 13I/14]$, $p_1^* + p_2^* = (2 - \beta/I)y$ with $p_k^* \in [10y/7 - \beta y/I, 4y/7]$ and $\Pi_k^C \in [24yI/49, 2y(10I - 7\beta)(7\beta - 3I)/(49I)]$; and (ii) when $\beta \in (13I/14, I]$, $p_1^* + p_2^* = (2 - \beta/I)y$ with $p_k^* \in [y/2, 3y/I - \beta y/I]$ and $\Pi_k^C \in (4\beta y - 2y\beta^2/I - 3yI/2, yI/2]$.



(a) Uncongested market (b) Congested market

Figure 6: Intermediation Advances and Influencers' Labor Market

3.4 Intermediation Technology and Labor Market Outcomes

How do advances in intermediation technologies, such as digital platforms, AI, as well as exogenous shocks to their adoption (e.g., after the Covid-19 pandemic) affect influencers' labor market? For arbitrarily $\gamma < 1$, $\beta > 0$, and $I_1 > I_2$, we broadly define advances in the intermediation technology as $\Delta I_1 \geq \Delta I_2 > 0$.

Proposition 3 (Technological Advances and Influencer Labor Market)**.** *When* $(I_1 - I_2)y_1 > 4\pi(y_1 - y_2)$ *and* $I_1 < \beta$*, we have:*

(i) The influencers' aggregate payoff is non-monotonic in the intermediation technology (i.e., $\Delta(w_1 + w_2) > 0$ *for small* $\Delta I_1 < \beta - I_1$*, and* $\Delta(w_1 + w_2) < 0$ *for some large* ΔI_2*); and*

(ii) The wage gap between influencers is non-monotonic in the intermediation technology (i.e., $\Delta(w_1 - w_2) > 0$ *for small* $\Delta I_1 < \beta - I_1$*, and* $\Delta(w_1 - w_2) < 0$ *for some large* ΔI_2*).*

Proposition 3 studies the implications on the influencer labor market when new technological advances allow for more powerful intermediation under certain conditions (e.g., asymmetry in influence power, a small gap in product quality, and relatively small initial influence power). We find that advances in the intermediation technology first benefits influencers and then hurts them in the labor market. A similar pattern also emerges for the influencers' wage gap. Intuitively, when the influencer market is not congested, stronger intermediation always transforms into larger revenues. However, when the market becomes crowded, stronger intermediation transforms into more fierce price competition, which harms the capacity to generate revenues. Figures 6a and 6b illustrate this insight. The parameters are given by: $\gamma = 0.5$, $\beta = \pi$, $y_1 = y_2 = 1$ and $I_1 = 2I_2$.[17] Specifically, the solid red line

[17]There exists multiple equilibrium when $\frac{2\pi}{3} < I_2 < \frac{10\pi}{9}$. We only plot the two cases where $I_2 \leq \frac{2\pi}{3}$ and $I_2 \geq \frac{10\pi}{9}$ to avoid equilibrium selection.

is the wage for Influencer 1, and the dashed blue line is the wage for Influencer 2. Figure 6a shows that technological advances in intermediation generate an increase in both total wages and the wage gap when the influencer market is not congested, while Figure 6b shows an opposite effect in a congested influencer market.

4 Influencer Style and Product Differentiations

Having derived how sellers hire influencers and price products, we now endogenize sellers' production. Specifically, we investigate how influencer heterogeneity affects horizontal and vertical product differentiations, compared to traditional economies.

4.1 Horizontal Product Differentiation

We investigate how style dispersion affects the horizontal product differentiation by sellers. Our key observation, perhaps quite intuitively, is that influencers' style difference and horizontal product specialization are substitutes. However, both the product quality gap and the influencer power difference complement horizontal differentiation.

To model horizontal differentiation, we stipulate that Seller k can either choose to hire an influencer with a given style or pay a cost f_E to nurture a new influencer with influence power $\widehat{I}_k$ and select any style $\theta \in \mathbf{S}^1$. This is equivalent to producing a differentiated product of style θ with an effective "transportation cost" $\widehat{I}_k$. To avoid trivial cases and allow room for horizontal differentiation, let $I := \max\{I_1, I_2, \widehat{I}_1, \widehat{I}_2\} \leq \frac{\pi}{2}$. We denote by β^* the optimal style dispersion after horizontal differentiation, and by $\Pi_k(\beta|\widehat{I}_1, \widehat{I}_2, I_1, I_2, y_1, y_2)$ Seller k's profit when the initial style dispersion is β. Then, Seller k engages in horizontal differentiation if

$$f_E < \Pi_k(\beta^*|\widehat{I}_1, \widehat{I}_2, I_1, I_2, y_1, y_2) - \Pi_k(\beta|\widehat{I}_1, \widehat{I}_2, I_1, I_2, y_1, y_2).$$

Otherwise, Seller k hires one influencer and accepts his style location. Intuitively, we expect that $\Pi_k(\beta|I_1, I_2, y_1, y_2)$ is increasing in β, and thus the incentive for horizontal differentiation is negatively associated with ex-ante style disperion β.

To isolate the effects of influencer marketing on product differention from influencers' labor matching, we focus on $I_j = \widehat{I}_j = I \leq \frac{\pi}{2}$ for $j \in \{1, 2\}$.[18] For $l \neq k$, we define:

[18] We relax this in the online Appendix OA.1.

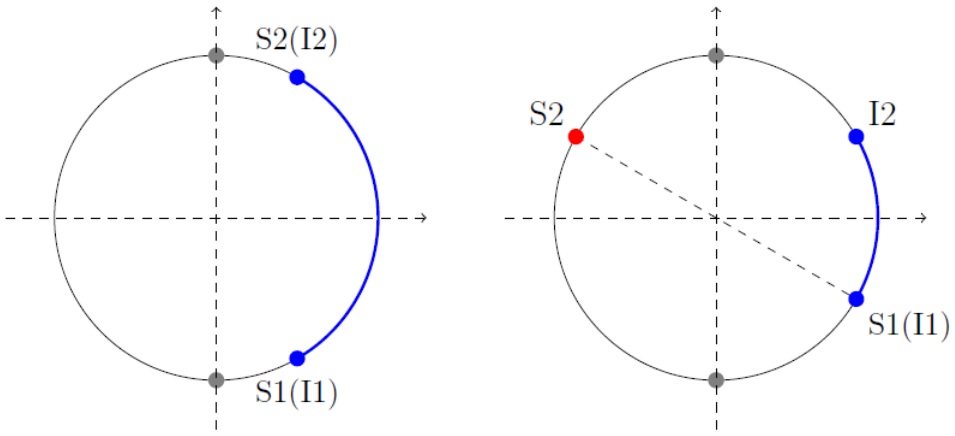


Figure 7: Style dispersion and horizontal differentiation

$$\Pi_k(\beta|y_1,y_2,I)=\frac{y_k(2y_k+y_l)(\beta y_l(3y_k+4y_l)+2I(2y_k^2+3y_ky_l+y_l^2))^2}{I(y_k+y_l)(8y_k^2+19y_ky_l+8y_l^2)^2}.$$

Denote $b=\beta/I$ and $t=y_1/y_2$ with $t\geq 1$. Then, $\Pi_k(\beta|y_1,y_2,I)=H_k(b,t)y_2I$, where

$$H_1(b,t)=\frac{t(1+2t)(2+8t+4t^2+b(4+3t))^2}{(1+t)(8+19t+8t^2)^2},$$
$$H_2(b,t)=\frac{(2+t)(4+(8+3b)t+(2+4b)t^2)^2}{(1+t)(8+19t+8t^2)^2}.$$

We further define: $a=\frac{3}{10}$ and $c=\frac{10(1+t)^2+2t}{12(1+t)^2+t}\in(5/6,6/7]$. We arrive at the next proposition.

Proposition 4. *The influencers' style dispersion is a substitute for horizontal product differentiation. For* $\frac{y_1I}{2}-H_1(c,t)y_1I<f_E<\frac{y_2I}{2}-H_2(a,t)y_2I$, *we have:*

(i) Seller 1 has a stronger incentive to differentiate products horizontally for $aI\leq\beta<\frac{I}{2}$; *otherwise, for* $\frac{I}{2}\leq\beta\leq cI$, *Seller 2 has a stronger incentive to differentiate products.*

(ii) There exists $\widetilde{\beta}\in(aI,cI)$ *such that each seller hires an influencer and accepts his style location when* $\widetilde{\beta}\leq\beta\leq cI$, *and (only) one seller pays* f_E *and selects a location such that* $\beta^*\geq I$ *when* $aI\leq\beta<\widetilde{\beta}$.

(iii) Sellers are more willing to differentiate horizontally when faced with a more powerful opponent seller. Specifically, fix y_2, *the cutoff* $\widetilde{\beta}$ *increases strictly in the ratio* $t=\frac{y_1}{y_2}$ *when Seller 2 chooses horizontal differentiation (that is,* $\frac{\partial\widetilde{\beta}}{\partial t}>0$*). Similarly, fix* y_1, $\widetilde{\beta}$ *increases strictly in* $\tau=\frac{y_2}{y_1}$ *when Seller 1 chooses horizontal differentiation (that is,* $\frac{\partial\widetilde{\beta}}{\partial\tau}>0$*).*

Figure 7 illustrates the equilibrium in Proposition 4. The left sub-figure entails a case with a large style difference. The style locations for Influencers 1 and 2 are marked in blue,

while the two gray nodes illustrate an optimal style separation. The blue arc corresponds to β. In this case, both sellers hire influencers to save the fixed cost in product differentiation and adapt to the influencers' styles. Therefore, the dispersion of style between influencers serves as a substitute for horizontal differentiation. A sufficiently large dispersion, albeit not maximal, removes the incentive to horizontally differentiate. Furthermore, the right subfigure illustrates the equilibrium when influencers' style dispersion is small so that once a seller works with an influencer, the other seller differentiates in style (red dot) to reduce competition, leading to maximum style differentiation between the seller-influencer groups.

Proposition 4 generates two economic insights. First, the stronger seller is more motivated to use horizontal differentiation when the style dispersion is small. Intuitively, the weaker seller has a stronger incentive to use horizontal differentiation, which is indeed the case when the style dispersion is relatively large. However, as the style dispersion decreases, the intense price competition leads to deteriorating profits for both sellers, with a much larger decrease for the strong seller. Second, a seller is more willing to use horizontal differentiation when faced with a more powerful competitor because strong competition decreases market share and profits, necessitating horizontal product differentiation.

Remark 1 (Power Difference Complements Horizontal Differentiation)**.** *It is tempting to generalize the substitution effect in Proposition 4 and argue that influencer heterogeneity always substitutes horizontal differentiation. However, this is wrong when influencer heterogeneity manifests in a difference in power. Influencer power difference, similar to product quality gap, complements horizontal differentiation (that is,* $\frac{\partial \widetilde{\beta}}{\partial(I_1/I_2)} > 0$*), as illustrated in Proposition A.1 in Section OA.1, which implies that a larger gap in influence power strengthens the incentive to differentiate horizontally. The power difference enables the stronger seller to compete more aggressively, lowering both prices and profits for the seller who works with the weaker influencer. This, in turn, incentivizes sellers to differentiate more.*

Remark 2 (Perfectly Symmetric Case)**.** *An undesirable limitation of Proposition 4 is* $c < \frac{6}{7}$*. Ideally and intuitively, we could expect* $c = 1$*, so there exists a single cutoff point* $\widetilde{\beta}$ *such that no seller wants to differentiate products for all* $\beta > \widetilde{\beta}$*, rather than for* $\widetilde{\beta} < \beta < cI$*. When* $\beta \in (cI, I)$*, the profit functions may not be differentiable and there exists a continuum of equilibria, which are characterized by a system of inequalities. However, in the perfectly symmetric case where* $y_1 = y_2$ *and* $I_1 = I_2$*, we can verify this single cutoff property (that is,* $c = 1$*) for consistent equilibrium selection rules including the highest-payoff, the lowest-payoff and the symmetric-payoff equilibrium. For example, we select the symmetric equilibrium for*

both sellers in Proposition A.2.

4.2 Vertical Product Differentiation

To model the relationship between influencers' style dispersion and vertical product differentiation, we allow sellers to incur a cost to improve product quality and potentially gain larger market shares. In general, the endogenous vertical differentiation is non-monotonic in influencers' style dispersion, with an endogenous quality gap only observed for intermediate style dispersions. Specifically, consider two sellers who initially make products of identical quality $y_k = \underline{y}$, but $\beta \geq 0$. We normalize $\underline{y} = 1$ and assume $I_1 = I_2 = 1$ to remove confounding issues in labor matching. Influencers can pay a fixed R&D cost F_V to improve quality to $\bar{y} = y > 1$. We also use "L" and "H" to denote "Low quality" and "High quality". Let $V_1(\beta, y)$ denote the increment profit change from quality investment when the competing group selects low quality,

$$V_1(\beta, y) \equiv \Pi^1_{H,L} - \Pi_{L,L} = \frac{y(1+2y)\big(2+8y+4y^2+\beta(4+3y)\big)^2}{(1+y)\left(8+19y+8y^2\right)^2} - \frac{3}{50}(2+\beta)^2,$$

and by $V_2(\beta, y)$ the investment benefit when the opponent also selects high quality,

$$V_2(\beta, y) \equiv \Pi_{H,H} - \Pi^1_{L,H} = \frac{3y}{50}(2+\beta)^2 - \frac{(2+y)\big(4+8y+2y^2+\beta y(3+4y)\big)^2}{(1+y)\left(8+19y+8y^2\right)^2}.$$

Now, define $\bar{\beta}$ and $\overline{\beta}$ such that $V_1(\underline{\beta}, y) = F_V$ and $V_2(\overline{\beta}, y) = F_V$. Note that both $V_1(\beta, y)$ and $V_2(\beta, y)$ are increasing in style dispersion. Thus, $\underline{\beta}$ is the break-even point for quality investment, given that the opponent selects low quality. Similarly, $\overline{\beta}$ is the break-even point for investment when the opponent selects high quality. Recall that $\beta_0 = \frac{2}{67}(-7+5\sqrt{10})I_2 \approx 0.263$, a number we use in the figure illustrations.

Proposition 5 (Non-monotonic Vertical Differentiation)**.** *Assume that: (i)* $\beta_0 \leq \beta \leq \frac{5}{6}$*; (ii)* $\overline{y}(1-\beta_0) \leq \underline{y}$*; and (iii)* $V_1(\beta_0, y) < F_V < V_2(\frac{5}{6}, y)$*. Then, there exist* $\overline{\beta}$ *and* $\underline{\beta}$ *such that:*

(a) when $\beta \geq \overline{\beta}$*, there exists a unique Nash equilibrium* (H, H)*;*

(b) when $\underline{\beta} \leq \beta < \overline{\beta}$*, there are two asymmetric Nash equilibria:* (H, L) *and* (L, H)*;*

(c) when $\beta_0 \leq \beta < \underline{\beta}$*, there exists a unique Nash equilibrium* (L, L)*.*

Proposition 5 characterizes a non-monotonic relationship between influencer style dispersion and vertical product differentiation. In particular, vertical differentiation arises only for

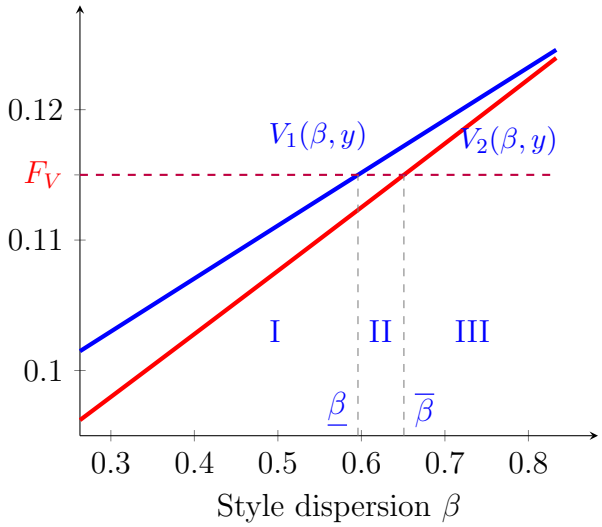


Figure 8: Vertical product differentiation

intermediate values of β. Note that: (i) ensures the existence of pure strategy equilibrium and simplifies the algebra in the proof; (ii) ensures that under asymmetric investment (H, L), both groups can survive in the product market; and (iii) focuses on the most interesting cost range in which non-monotonicity arises.[19]

Figure 8 illustrates the equilibrium configuration in Proposition 5 with $y = \frac{5}{4}$. Specifically, the horizontal axis β corresponds to style dispersion. The two functions, $V_1(\beta, y)$ and $V_2(\beta, y)$, corresponding to the blue and red solid lines in the figure, measure the profit gap between "high quality" and "low quality," given the other group's choice. Fix the investment cost $F_V > 0$ (purple dashed line). There are three regions, "I", "II" and "III", separated by two cutoffs, $\underline{\beta}$ and $\overline{\beta}$. In Region III, $V_2(\beta, y) > F_V$, and thus both groups choose high quality. In Region II, $V_2(\beta, y) < F_V$ and $V_1(\beta, y) > F_V$, which means that only one group chooses high quality. In Region I, both groups choose low quality.

Proposition 5 implies that, when style dispersion (β) increases from low values, we initially see more vertical differentiation (and thus influencers' style dispersion and vertical product differentiation are complements). In contrast, as β further increases when it already takes a large value, we see less vertical differentiation (and thus these two are substitutes). Intuitively, for a small β, intense competition greatly limits the return on investment in high quality. Thus, both groups choose low quality, and the vertical differentiation is minimal. For an intermediate β, the competition is less intense, which improves the investment return provided that the opponent does not invest. Finally, for a large β, the competition is minimal and even when both groups invest, allowing both to at least break even with the investment.

[19]For $F_V \leq V_1(\beta_0, y)$, (H, H) is the unique Nash equilibrium, so is (L, L) for $F_V > V_1(\frac{5}{6}, y)$.

Given that the investment profit is strictly increasing in influencer style dispersion, we no longer observe vertical differentiation in equilibrium.[20]

5 Direct Sales VS. Influencer-Intermediated Sales

To further understand why influencers emerge and assess their economic impact, we introduce direct sales to capture real-life situations in which either sellers sell products directly or a seller uses a conventional advertisement without targeting a particular audience. The corresponding friction is then the relatively high transportation cost for direct sales because consumers do not interact socially with sellers or traditional marketers.[21] In contrast, influencers' advantage lies in their close interaction with followers on digital social platforms.[22]

With direct sales, the ith consumer's utility can be written as:

$$u_i(x_i, y, p) = y * \max\{(1-c), 0\} - p. \tag{4}$$

Here, y and p are the quality index and the price. The transportation cost parameter $c \in (0, 1]$ captures brand reputation, product perception, consumer goodwill, product acceptance rate, and market penetration power. The baseline model can be viewed as a special case with a sufficiently high transporation cost $c \geq 1$ to rule out direct sales. Furthermore, Equation (4) implies that all consumers are equally distant from the seller, and thus all get an identical utility (that is, the seller is located in the center of the circle). We consider two distinct cases, *price discrimination* and *uniform pricing*, which differ in the sellers' ability to charge direct sales prices differently from influencer-intermediated prices.

[20]The non-monotonicity in Proposition 5 relies on the Bertrand competition for small β in the region I. Intuitively, Bertrand competition occurs when quality improvement is limited and/or influencers' influence power is comparable. Thus, the region I might disappear when Bertrand competition vanishes, either because quality improvement is significant or the power gap is large. Instead, we might observe less vertical differentiation as the influencers' style dispersion increases.

[21]Here, the transportation cost can be the traditional marketing cost, the inertia cost of switching to new products, the uncertainty of quality or mismatch for experience goods, the cost of reaching new niche markets, or costly surveys conducted to understand consumers' taste, etc.

[22]Influencers do more than just selling products. They form a social and entertainment network, and the rise of influencers highlights a bigger trend in our economy, that is, the blurring of boundaries of social activities and economics. Influencers are more trustworthy in the eyes of followers because they know the needs of their followers better and thus can serve as natural referrals when consumers are skeptical about products. Meanwhile, influencers are the best choice for sellers to approach new niche markets when the quality of products offered by entrants is unknown to consumers. Influencers can use their charisma and reputation to help consumers reduce the inertia cost of switching to new products.

5.1 Equilibrium with Price Discrimination

We start with the case of price discrimination. Denote by p_k^d and p_k^i the prices charged by Seller k in direct sales and influencer-intermediated sales. First, we consider a single seller and a single influencer and assume that $I_1 \leq 2\pi$. Under the optimal pricing strategy, the monopolist seller charges $p_1^d = (1-c)y_1$ and $p_1^i = (1-c/2)y_1$.[23] Specifically, the seller targets consumers with $\|x-\theta_1\| \leq \frac{c_1 I_1}{2}$ through influencer-intermediated sales and other consumers through direct sales. Now, the effective power, measured by the size of consumers who prefer influencer-intermediated sales to direct sales before paying the prices, decreases to $c_1 I_1$. Then, influencer-intermediated sales can be viewed as monopoly pricing after charging the maximum price in direct sales, and thus only consumers within the effective power range are targeted. This generates a total profit of $\Pi = 2\pi * (1-c_1)y_1 + \frac{1}{2}c_1^2 y_1 I_1$.

Next, we analyze the setup with two sellers and two influencers.

Lemma 1 (Equilibrium with Direct Sales)**.** *Consider $y_1 \geq y_2$, $I_1 \geq I_2$, $\beta \geq \frac{c_1 I_1 + c_2 I_2}{2}$ and $(1-c_1)y_1 = (1-c_2)y_2$. In equilibrium, $k(j) = j$ for $j \in \{1,2\}$. After matching, Seller k sets $p_k^d = 0$ for direct sales and sets $p_k^i = \frac{c_k y_k}{2}$. Influencer j, who works with Seller $k(j)$, targets consumers with $\|x-\theta_j\| \leq \frac{c_{k(j)} I_j}{2}$ for influencer-intermediated sales. Furthermore, the group profits are given by $\Pi_{k(j),j} = \frac{c_{k(j)}^2 y_{k(j)} I_j}{2}$ for $j = 1,2$. Payoffs for sellers and influencers are given by $U_k = \gamma \Pi_k$ and $w_j = (1-\gamma)\Pi_{k(j)}$ for $k = 1,2$ and $j = 1,2$.*

Lemma 1 first verifies the robustness of our main results when sellers have comparable traditional marketing capacity in direct sales. Indeed, when $(1-c_1)y_1 = (1-c_2)y_2$ holds, Equation (4) reduces to:

$$u(x, \theta_j | c_k, y_k, p_k) = y_k(c_k - \|x-\theta_j\|/I_j) - p_k, \tag{5}$$

after we remove the impact of direct sales. The similarity between Equations (5) and (1) implies that our main results on assortative matching, market dominance, and product differentiation, etc., remain robust in the presence of direct sales.

Two new insights emerge from Lemma 1 once we consider heterogeneous marketing capacity in direct sales. First, relative to traditional economies with only direct sales, influencer

[23] To derive the optimal pricing, note that $p_1^i \geq (1-c_1)y_1$. Otherwise, if $p_1^i < (1-c_1)y_1$, the seller prefers direct sales alone and sets $p_1^d = (1-c_1)y_1$. Note also that $p_1^d = (1-c_1)y_1$. If not, suppose that $p_1^d < (1-c_1)y_1$. Then we can increase both p_1^d and p_1^i by a small amount $\delta > 0$ without altering the consumers' purchase decisions, resulting in a profitable deviation. We now optimize over p_1^i the profit $\Pi = (2\pi - 2s_1) * p_1^d + 2s_1 * p_1^i$, where $s_1 = I_1 * (1 - p_1^i/y_1)$ is derived from the indifference condition that $y(1-s_1/I_1) - p_1^i = (1-c_1)y_1 - p_1^d = 0$, we get $p_1^i = (1-c_1/2)y_1$.

marketing can foster seller entry and competition by allowing sellers with weak traditional sales to bypass the transportation cost barrier. Consider $(1-c_1)y_1 > (1-c_2)y_2$ and $y_2 > y_1$. With only direct sales, Seller 1, say the incumbent, by leveraging the low transportation cost advantage, can effectively force Seller 2, the potential entrant with a superior good, out of the product market.[24] In contrast, with influencer marketing, Seller 2 can effectively bypass the transportation cost barrier by employing influencers to gain access to niche markets.

Second, because sellers with weak traditional marketing channels have a stronger incentive to hire influencers, the transportation cost parameter c_k, together with product quality, jointly determine sellers' incentives to hire influencers. Consequently, negative assortative matching occurs, in a spirit similar to that of Proposition 1, between direct sales capacity and influence power when the quality of the product is fixed (that is, $y_1 = y_2 = y$). Proposition 6 below considers a representative case such that $c_1 < c_2$ and $I_1 > I_2$ (that is, Seller 1 has stronger direct sales capacity and Influencer 1 has a larger influence power).

Proposition 6 (Negative Assortative Matching)**.** *Consider that* $y_1 = y_2 = y$, $\beta > \frac{c_2(I_1+I_2)}{2}$ *and* $\frac{\gamma c_2}{2} < (c_2 - c_1)$. *Then:* $k(j) = 2-j$, *that is, Seller 2, who has weaker direct sales capacity, is matched with the more powerful influencer, Influencer 1; Furthermore, in equilibrium,* $p_1^d = (c_2 - c_1)y$, $p_2^d = 0$ *and* $p_1^i > p_2^i = \frac{c_1 y}{2}$.

Similar to Proposition 1, Proposition 6 also helps explain the fast growth of influencer marketing despite the direct sales channel. Negative assortative matching between direct sales capacity and influencer power encourages product competition and efficient product placement, which improves consumer welfare. To see this, note that consumers always receive a baseline surplus equal to $(1 - c_2)y$ from direct sales and a heterogeneous extra surplus rent when buying from influencers with similar styles. Seller 1, which has stronger direct sales capacity, charges a higher price in influencer-intermediated sales compared to Seller 2. This is because Seller 1 earns a positive profit from direct sales (that is, $p_1^d > 0$) and thus faces an additional incentive compatibility condition $\gamma p_1^i > p_1^d$ when placing products by influencers. Therefore, more consumers receive an extra surplus from influencer marketing under negative assortative matching, leading to higher consumer welfare.

Interestingly, a large difference in direct sales capacity not only weakens but can also reverse the positive assortative seller-influencer matching involving product quality and influencer power in our baseline model, as the next example illustrates.

[24] When the condition $(1 - c_1)y_1 \neq (1 - c_2)y_2$ is violated, say, $(1 - c_1)y_1 > (1 - c_2)y_2$, sellers still engage in Bertrand price competition and thus $p_2^d = 0$ and $p_1^d \leq (1 - c_1)y_1 - (1 - c_2)y_2$.

Example 4 (Reversed Seller-Influencer Match)**.** *Fix a small $\varepsilon > 0$. Consider $y_1 = y_2 + \varepsilon$, $2\pi \gg I_1 > I_2$, $\beta > \frac{I_1+I_2}{2}$, $c_2 = 1$, and $\frac{y_2}{2} > (1-c_1)y_1 > \frac{1}{2}y_1 - \varepsilon$. In short, Good 1 slightly outperforms Good 2 in quality. In direct sales, Seller 2 cannot reach consumers who buy only from Seller 1. Thus, $p_2^d = 0$ and $p_1^d \leq (1-c_1)y_1$. Furthermore, Seller 1 will charge a price $p_1^d \to (1-c_1)y_1$ because direct sales are far too important when $2\pi \gg I_1$. This, in turn, implies that Seller 2 will charge a monopolist price $p_2^i \to \frac{y_2}{2}$, and the total profit from working with Influencer 1 approaches $\frac{1}{2}y_2 I_1$. In contrast, Influencer 1, when working with Seller 1, can generate a total profit about $\frac{1}{2}c_1^2 y_1 I_1 \approx \frac{1}{8}y_1 I_1$,[25] which is less than $\frac{1}{2}y_2 I_1$. Therefore, under Nash bargaining, Influencer 1 chooses to work with Seller 2. The negative assortative matching implies that influencer marketing is more useful for those with limited access to traditional marketing channels.*

5.2 Equilibrium Under Uniform Pricing

We next consider uniform pricing between direct sales and influencer-intermediated sales. Again, we start with a single seller and an influencer and assume that $I \leq 2\pi$ and $0 \leq c < 1$. When $p^d \leq (1-c)y$, direct sales occur and the total profit is $\Pi(p^d) = 2\pi p^d$. In contrast, in influencer-intermediated sales, when $p^i \geq (1-c)y$, the demand is $I(1-p^i/y)$, leading to a profit of $\Pi(p^i) = 2I(1-p^i/y)p^i$. Note that $\Pi(p)$ might be discontinuous at $p = (1-c)y$. We can further optimize over p to simplify the total profit as $\Pi^* = \max\left\{\frac{yI}{2}, 2\pi(1-c)y\right\}$. Thus, the seller uses direct sales only when $2\pi(1-c)y \geq \frac{\gamma yI}{2}$. More precisely, the seller chooses direct sales only when: (i) the transportation cost c is small; (ii) the influence power I is small; and (iii) the seller's bargaining power γ is small. Note that the choice of marketing channel is independent of the quality of the product.

Next, we analyze the case with two sellers and two influencers with balanced matching. For simplicity, we assume that $y_k = y$, $c_k = c \in [\frac{1}{2}, 1]$ and $I_j = I \ll \beta$ where $\beta = \|\theta_1 - \theta_2\|$. Denote by q and $(1-q)$ the probability of using influencer-intermediated sales and direct sales, and by p the price charged in direct sales and $f(p)$ the corresponding density function.

Lemma 2. *With uniform pricing, (i) when $\frac{\gamma yI}{2} \geq (2\pi - I)(1-c)y$, both sellers use influencer-intermediated sales and charge the price $p_k = \frac{y}{2}$; (ii) when $(2\pi - I)(1-c)y > \frac{\gamma yI}{2}$, both sellers*

[25] Note that given that $\beta > \frac{I_1+I_2}{2}$, the pricing strategy for Seller 1 in influencer-intermediated sales is independent of that of Seller 2. Thus, when $p_1^d \to (1-c_1)y_1$, we have $p_1^i \to (1-c_1/2)y_1$ with an effective client base approximately given by $c_1 I_1$, just as shown in the case of a monopolist seller.

use a symmetric mixed strategy $(q, f(p))$ *with* $p \in [\underline{P}, \bar{P}]$*, where* $\bar{P} = (1-c)y$*,*

$$q = \frac{\gamma I}{2(1-c)(2\pi - I)}, \quad f(p) = \frac{\gamma y I}{4\pi(1-q)p^2}, \quad \textit{and} \quad \underline{P} = \frac{2\gamma y I}{2q(2\pi - I) + 4\pi(1-q)}.$$

Under uniform pricing, there are two insights. First, an increase in transportation costs or influencers' intermediation power induces sellers to use more influencer marketing. When this occurs, all the main results hold with slight modifications. As that of price discrimination, influencer marketing facilitates seller entry and competition.[26] Moreover, unlike the case of price discrimation, influencer marketing can alleviate competition in direct sales, and helps avoid Bertrand competition. Now, sellers randomize between influencer-intermediated sales and direct sales, even when influencer marketing is not sufficiently desirable. However, this might hurt welfare as fewer consumers are served.

6 Endogenous Influence and Exclusivity Contracting

We now extend the baseline model of influencer marketing (Sections 2 and 3) to consider: (i) influencers may endogenize their influence, either in power or style, and (ii) sellers and influencers may not have one-on-one matching. Both have important welfare implications.

6.1 Endogenous Influence Acquisition

Influencers sometimes incur a cost to boost their influence power, e.g., through livestreaming training or multi-homing on various media platforms. We gain novel insights considering such endogenous influence: First, socially insufficient power acquisition can arise due to externalities and intense price competition under balanced matching. Second, we show that assortative seller-influencer matching holds in the long run when we allow for style selection.

Note that the utility from consumption is bounded above, which means that many influencers might spend effort to acquire power, and too many endogenously become influencers in practice. We show that this concern about the arms race among influencers is not justified in a duopoly product market. Instead, socially insufficient power acquisition can arise due to externalities and intense price competition under balanced matching. For simplicity,

[26] Consider $y_1 = y_2 = y$, $I_1 = I_2 = I$, $1/2 < c_1 < c_2 < 1$ and $(2\pi - I)(1-c_1)y > \frac{1}{2}\gamma y I > (2\pi - I)(1-c_2)y$. First, Seller 2 is always forced out of the product market in traditional economies with only direct sales. Second, the condition $(2\pi - I)(1-c_1)y > \frac{1}{2}\gamma y I > (2\pi - I)(1-c_2)y$ implies that Seller 1 chooses direct sales, while Seller 2 chooses influencer-intermediated sales.

consider two influencers with maximum distance $\beta = \pi$ and initial power I. Either can pay a fixed cost $C_P > 0$ to increase the influence power to $kI > I$.

Proposition 7 (Socially Insufficient Power Acquisition). *Fix $\beta = \pi$. There exists under-investment in influencer power when: (i) $kI \leq \pi/2$ and $\frac{1}{2}(1-\gamma)(k-1)yI < C_P < 2(k-1)yI$; or (ii) $I \geq \frac{3}{2}\pi$ and $0 \leq C_P < \frac{\pi^2 y(k-1)(7k+2)}{18k(k+1)I}$.*

There are two forces driving this result. First, influencers do not internalize the externality on consumer welfare when acquiring influence power. When the market for influencers is not crowded, power acquisition can increase consumer utility, and it can exhibit under-investment when the positive externality on consumer welfare is not internalized. The bargaining power clearly matters, and a large γ reduces the incentive for power acquisition and causes under-investment because condition (i) is more likely to hold. Second, the intense price competition between the two sellers acts as a disincentive for influencers to invest in power when the influencer market is congested because it further intensifies the existing competition in the product market. In particular, any potential gain in consumers is dominated by the lowered product prices, even when power acquisition costs very little.

Remark 3 (Monopolist and Over-investment in Influence). *The endogenous acquisition of influence power also depends on the structure of the product market. In Proposition 7, both forces induce under-investment. In contrast, when there is a monopolist seller, intense price pushback no longer exists. However, a third channel arises: influencers exert a negative externality on other influencers when hired by the same monopoly seller. Depending on the congestion of the influencer market and the bargaining power, these three forces can generate over-investment, efficient investment, and under-investment in power acquisition. More discussion can be found in Section OA.4.*

Remark 4 (Endogenous Style Selection). *Our main results are robust when we allow endogenous style selection. We only prove a special case of an uncongested influencer market in which $\frac{I_1+I_2}{2} \leq \pi$. Let β^* denote the optimal style dispersion for influencers. When selecting the style location is costless, $\beta^* \geq \frac{I_1+I_2}{2}$. Then, by Proposition 1(i), the assortative matching result follows. This implies that influencers, whenever possible, have an incentive to minimize competition and target niche markets.*[27]

[27] A related question is whether it helps to let the seller endogenously decide influencers' style locations since they are more informed about their own products? The answer is no. We illustrate this with a single seller and assume that $I_j = I < \pi$. Denote by β_S^* the seller's preferred style dispersion. Then: (i) if her bargaining power is large (i.e., $\gamma > \frac{1}{2}$), the seller prefers maximal style dispersion (i.e., $\beta_S^* \geq I$); (ii) if $\gamma = \frac{1}{2}$,

6.2 Exclusivity Contracting and Welfare

In practice, a seller sometimes hires multiple influencers and requires them not to advertise the products of rival sellers (e.g., Zietek, 2016). For example, a large survey of influencers by Mavrck (Katz, 2019) shows that the majority of influencers (61%) are receiving exclusivity requests from brands. In fact, exclusivity contracts have been prevalent in industries such as healthcare and insurance and have led to many antitrust cases (Gal-Or, 1999). However, policies have recently been introduced to better protect influencers and reduce market concentration through encouraging competition. The awareness has also grown in the industry that exclusivity should be mutual.[28] This means that either both sides can contract with multiple counterparties or both sides must exclusively collaborate, which is exactly what our balanced matching set-up aims to capture.

However, to better understand the welfare implications of exclusivity contracts in this emerging industry and to guide regulatory policies geared toward balancing the power of influencers and sellers, we extend the analysis to "unbalanced matching." In this setting, a seller can hire multiple influencers, but not the other way around, which is consistent with the contracting landscape in the early stages of the influencer industry (e.g., Zietek, 2016). Proposition 8 compares the efficiency between balanced- and unbalanced-matching to offer two key insights. First, balanced matching (mutual exclusivity contracting) is optimal under one-dimensional heterogeneity even when we allow sellers to compete for multiple

she is indifferent among all style dispersions (i.e., $\beta_S^* \in [0, \pi]$); and (iii) if her bargaining power is small (i.e., $\gamma < \frac{1}{2}$) , she prefers minimal style dispersion ($\beta_S^* = 0$). The proof is available upon request.

Now, we explain the intuition behind this result. Due to the build-up of endogenous bargaining power, the seller prefers a minimal style dispersion when her bargaining power is small and vice versa. To see it, first consider a small bargaining power parameter for the seller (i.e., γ is small). On the one hand, a large style difference implies a big market coverage and a small over-lapping in consumer base for the two influencers, leading to a large total revenue. However, a small γ also implies that only a very limited fraction of total revenue flows to the seller. On the other hand, a small β decreases market coverage and generates limited revenue, but it leads to intense competition and perfect substitution between the two influencers and leads to a more advantageous bargaining position for the seller under the bilateral Nash bargaining protocol. In the limit case where $\beta = 0$, the seller gets the total revenue of working with a single influencer on the product market. In other words, a small style difference increases the cash flow to the seller, although it decreases market coverage, and the incentive for bargaining power building dominates. In contrast, when the seller enjoys a large bargaining power (that is, a big γ), the incentive distortion by endogenous bargaining power building is minimal and thus the incentive for revenue generating dominates, and thus more style dispersion leads to a larger client base and an increase in total revenue.

[28]Influencers increasingly value long-term partnerships With brands rather than one-off exclusivity requests; they also expect to be compensated more when exclusivity is required. As early as 2008, the entertainment industry began to see the value behind full-time creators building multi-platform brands, and influencers started getting Hollywood agents to help negotiate bilateral exclusive contracts (Collectively, 2020). New York State Regulators, China's State Administration of Market Regulation, and the British Federal Trade Commissions are all increasing regulations on influencer contracting (Federal Trade Commission, 2019; Zerbo, 2017; China's State Administration for Market Regulation, 2021).

influencers. Second, unbalanced matching (unidirectional exclusivity) can be optimal when influencer style locations are sufficiently unique in uncongested influencer markets.

Proposition 8 (Exclusivity Contracting and Welfare)**.** *With exclusivity clauses:*

(i) (Congested influencer market or homogeneous product market). Unbalanced matching decreases total welfare under one-dimensional heterogeneity, including heterogeneous product quality, heterogeneous influencer power, and heterogeneous influencers' style locations.

(ii) (Uncongested influencer market). When $\beta > \frac{I_1+I_2}{2}$ *and* $y_1 > y_2$*, unbalanced matching dominates balanced matching in total welfare.*

The key messages in Proposition 8 are intuitive.[29] Both the quality gap of products and the style dispersion between the influencers affect the intensity of the seller's competition. When the market for influencers is not crowded and influencers' styles are distinct, regulation on mutual exclusivity contracting does not help encourage competition because of the inevitable local market power derived from influencer heterogeneity. Given that the economy features monopoly pricing anyway, uni-directional exclusivity improves welfare because it allows the high-quality product to dominate. However, when products are quite homogeneous or influencers are very similar in style, requiring mutual exclusivity and balanced matching can improve consumer welfare. Note that unbalanced matching always features joint profit maximization and the dominance of the higher-quality product, while balanced matching features greater price competition. Simply put, the quality improvement channel is shut down in a homogeneous product market, while regulation can improve competition when the influencer market is crowded.

Remark 5 (The general case with more than two influencers)**.** *In reality, there are not just two influencers, and there could be low entry costs into the influencer market. Indeed, imagining a situation in which a brand hires all influencers is not easy. Typically, it is feasible for a large brand to have many influencers to persuade consumers in many locations. Thus, the natural question is whether it is restrictive to have only two influencers. Proposition A.3 answers this question and extends the insights to a more general case with more than two influencers, that is, with unbalanced matching, a seller might now hire many influencers, but not necessarily all influencers. The basic insight remains. Specifically, when there is a limited supply of influencers and the quality of products differs, unbalanced matching is strictly*

[29]Under Nash bilateral bargaining, the equilibrium features the exclusivity contracting mode which maximizes welfare. Note that this result is not robust to renegotiating or introducing commitment power for sellers. However, Proposition 8, the welfare comparison of exclusivity contracting, is general and robust.

dominant in total welfare, since products with a low quality are forced out of the market. In contrast, with homogeneous products, equally distanced influencers, and a crowded market for influencers, balanced matching always weakly dominates unbalanced matching because it can enhance competition among sellers. The intuition is that the symmetry in the quality of products and the features of influencers requires identical prices among sellers. Balanced matching is always the most natural way to go (with the highest consumer welfare), although might not be the unique way sometimes. However, as long as asymmetric market power arises which generates asymmetric pricing strategies, unbalanced matching always harms total welfare. A more detailed discussion can be found in Section OA.2.

7 Conclusion

We develop a theory of the influencer economy that focuses on influencer marketing and product competition. Our model captures several salient, important, and realistic features: (i) sellers produce goods and compete for consumers through influencers, (ii) sellers and influencers are matched in influencers' labor market and engage in Nash bargaining, and (iii) influencers acquire influence to attract consumers who identify with their style in addition to value the products they promote. We derive four key insights to better understand this new business-social phenomenon, with empirical implications for future studies.

First, more powerful influencers match with sellers with better product quality and weaker direct sales capacity. Empirically, this can be verified using data from e-commerce and digital media platforms. For example, Amazon/Taobao reviews can be used to measure product quality, the gross merchandise value (GMV) of live streaming sales or the number of followers on digital media platforms for influence power, and the share of revenue from direct sales or in-store sales for direct sales capacity. Depending on the assortative matching, influencer marketing may favor large brands, but can also facilitate seller entry and revive niche markets. As larger companies hire more influencers, the market may become too saturated for small brands to differentiate and survive through influencer-intermediated sales.

Second, advances in the intermediation technology can lead to non-monotonic changes in influencers' payoff and income inequality. Empirically, digital technology advances can be measured using the average cost for influencers to connect and interact with potential consumers, or commissions/fees charged by digital platforms. Then we could possibly explore the event of the Covid-19 pandemic, which drastically made the influencer market more congested. In particular, we anticipate a positive (negative) association between intermediation

technology and influencers' payoff and income inequality before (after) the pandemic.

Third, compared to traditional economies, influencer style pluralism mitigates market concentration by effectively differentiating the consumer experience horizontally. In contrast, the dispersion in influencer power intensifies product competition and increases market concentration. Meanwhile, only intermediate style differences complement vertical differentiation. Empirically, one can measure influencers' style difference by collecting data on influencers' educational background, age, style, category of topic-based channels on social media platforms, etc. The construction of product differentiation measures is subtle but can be done through surveys, focus groups, or customer interviews.

Fourth, influencers may inefficiently under-invest in consumer outreach to avoid exacerbating price competition, and balanced matching and mutual exclusivity can be socially beneficial by fostering competition when the influencer market is congested. In fact, we expect a congested influencer market for the following two reasons. One, advances in digital technologies and platforms greatly facilitate influencer entry and allow digital creators to bypass the institutional gatekeepers in various industries. Two, machine learning and generative AI facilitate the identification and growth of influencers, which may quickly make niche markets saturated. Regulation toward balanced matching may improve welfare.

To focus on influencer marketing and its industrial organization (and for tractability), we have largely abstracted away from the inner workings of platforms and MCNs. The symbiotic relationship between influencers and platforms remains crucial in this new digital economy. The organization and heterogeneity of MCNs (e.g., TikTok and Weibo), as well as the general impact of influnencers on the society, also constitute interesting future research. In this regard, our findings establish initial benchmark results rather than foregone conclusions.

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Appendix

A Derivations and Proofs

A.1 Derivation of Equation (2)

Proof. First, when $I < \pi$, the demand $D(p) \leq 2\pi$ for all $p \geq 0$ and thus $D(p) = 2(1 - p/y) * I$, which further implies that $\Pi(p) = D(p) * p$ and thus $p^* = \frac{y}{2}$ and $\Pi(p^*) = \frac{yI}{2}$. Second, consider $I \geq \pi$. In this case, $\Pi(p) = 2\pi * p$ if $p \leq y(1 - \pi/I)$, and $\Pi(p) = 2p(1 - p/y) * I$ if $p > y(1 - \pi/I)$. Note that $\Pi(p)$ is continuous at $p = y(1 - \pi/I)$. Again, depending on the value of I, there are two cases. One, when $I \in [\pi, 2\pi]$, $y(1 - \pi/I) \leq \frac{y}{2}$. The quadratic term implies that $\Pi(p)$ is strictly increasing for all $p \in [y(1 - \pi/I), y/2]$ and strictly decreasing for $p > y/2$. Therefore, $\Pi(p)$ is maximized at $p^* = \frac{y}{2}$, which yields $\Pi(p^*) = \frac{yI}{2}$. Two, when $I > 2\pi$, we have $y(1 - \pi/I) > \frac{y}{2}$. Thus, $\Pi(p)$ is strictly increasing for $p \leq y(1 - \pi/I)$ and strictly decreasing for $p > y(1 - \pi/I)$ because the quadratic term is strictly decreasing for all $p > \frac{y}{2}$. Therefore, $\Pi(p)$ is maximized at $p = y(1 - \pi/I)$ and $\Pi(p^*) = 2\pi y(1 - \pi/I)$. Third, we can combine the two cases that $I < \pi$ and $I \in [\pi, 2\pi]$ to simplify the formula. □

A.2 Proof of Proposition 1

Proof. Let $\Pi(i, j)$ denote the group profit when Seller i works with Influencer j. The bilateral Nash bargaining further implies that $U_i(i, j) = \gamma\Pi(i, j)$ and $w_j(i, j) = (1 - \gamma)\Pi(i, j)$. Thus, it suffices to show that $U_1(1, 1) \geq U_1(1, 2)$ and $w_1(1, 1) \geq w_1(2, 1)$ to establish assortative matching (i.e., $k(j) = j$), which further reduce to

$$\Pi(1, 1) \geq \Pi(1, 2) \quad \text{and} \quad \Pi(1, 1) \geq \Pi(2, 1). \tag{A.1}$$

Now, we verify this condition case by case under balanced matching.

Case (i). When $\beta \geq \frac{I_1+I_2}{2}$, Seller $j \in \{1, 2\}$ always charges $p_j^* = \frac{y_j}{2}$, regarding of the matching outcome. Thus, $\Pi(i, j) = \frac{y_i I_j}{2}$. Obviously, equation (A.1) holds.

Case (ii). First, given that $y_1 > y_2$, Seller 1 can always get a positive profit by attracting consumers close to the influencer who works with Seller 1. Thus, when a single seller dominates, it must be Seller 1. Second, we show that negative assortative matching (i.e., $k(j) = 3 - j$ for $j = 1, 2$) never happens. If not, suppose that Seller 1 hires Influencer 2 and sets a price p_1 and beats Seller 2 who hires Influencer 1 and sets any arbitrary price p_2. Now, consider the case where Seller 1 hires Influencer 1 and use the original pricing strategy. For all consumers, they either stay outside of the reach of Seller 2, or fall within the consumer base of Seller 2, but are attracted by Seller 1. In the latter case,

$$\begin{aligned} y_1(1 - \|x - \theta_1\|/I_1) - p_1 &> y_1(1 - \|x - \theta_1\|/I_2) - p_1 \\ &\geq y_2(1 - \|x - \theta_2\|/I_1) - p_2 > y_2(1 - \|x - \theta_2\|/I_2) - p_2 \end{aligned}$$

We used the fact that $I_1 \geq I_2$ in the first and the third inequality, and the second inequality follows from the assumed market dominance. This implies that Seller 1 and Influencer 1 will jointly deviate and form a new group, and thus negative assortative matching is sub-optimal.

Case (iii). First, we compute $\Pi(i,j)$ and start with $\Pi(1,1)$. Note that $\beta = \pi$ implies that the share of consumers is symmetric around θ_j for Influencer j. Fix prices (p_1, p_2), the cutoff type is given by the indifference condition that $y_1(1 - s_1/I_1) - p_1 = y_2(1 - s_2/I_2) - p_2$ where $s_1 = \pi - s_2$ is the share of Seller 1 on one side, and we can solve it to get $s_1 = \frac{I_1(I_2(p_2-p_1)+I_2(y_1-y_2)+\pi y_2)}{y_1 I_2 + y_2 I_1}$. Thus, $\Pi(1,1) = 2s_1 p_1$ and $\Pi(2,2) = 2(\pi - s_1)p_2$. By differentiating $\Pi(1,1)$ and $\Pi(2,2)$ over p_1 and p_2, we get $\frac{\partial \Pi(1,1)}{\partial p_1} = \frac{2I_1 I_2(y_1-y_2+p_2-2p_1)+2\pi I_1 y_2}{y_1 I_2 + y_2 I_1}$ and $\frac{\partial \Pi(2,2)}{\partial p_2} = \frac{2I_1 I_2(y_2-y_1+p_1-2p_2)+2\pi I_2 y_1}{y_1 I_2 + y_2 I_1}$. Solving first-order conditions yields $p_1^* = \frac{I_1 I_2(y_1-y_2)+\pi(I_2 y_1 + 2I_1 y_2)}{3I_1 I_2}$ and $p_2^* = \frac{I_1 I_2(y_2-y_1)+\pi(I_1 y_2 + 2I_2 y_1)}{3I_1 I_2}$. Furthermore, second-order conditions can be checked directly.[30] Thus, we can compute $\Pi(1,1) = \frac{a_1}{b_1}$ and $\Pi(2,2) = \frac{a_2}{b_1}$, where $b_1 = 9I_1 I_2(y_1 I_2 + y_2 I_1)$ and

$$a_1 = 2\big(I_1 I_2(y_1 - y_2) + \pi(y_1 I_2 + 2y_2 I_1)\big)^2, \quad a_2 = 2\big(I_1 I_2(y_2 - y_1) + \pi(y_2 I_1 + 2y_1 I_2)\big)^2.$$

Similarly, we calculate $\Pi(1,2)$ and $\Pi(2,1)$ in the alternative matching $k(j) = 2 - j$. Specifically, $\Pi(1,2) = \frac{a_3}{b_2}$ and $\Pi(2,1) = \frac{a_4}{b_2}$, where $b_2 = 9I_1 I_2(y_1 I_1 + y_2 I_2)$ and

$$a_3 = 2\big(I_1 I_2(y_1 - y_2) + \pi(y_1 I_1 + 2y_2 I_2)\big)^2, \quad a_4 = 2\big(I_1 I_2(y_2 - y_1) + \pi(y_2 I_2 + 2y_1 I_1)\big)^2.$$

To verify $\Pi(1,1) \geq \Pi(1,2)$, it suffices to show that $a_1 b_2 - a_3 b_1 \geq 0$. Denote $I_1 = aI_2$, $y_1 = by_2$ and $I_2 = c\pi$, where $a \geq 1, b \geq 1$ and $c \geq 1$. Then, $a_1 b_2 - a_3 b_1 = 9ac^5\pi^7 y_2^3 * (g_0 + g_1(c-1) + g_2(c^2-1))$, where $g_2 = (a-1)a^2(b-1)^3$, $g_1 = 2a(a^2-1)(b-1)b$ and $g_0 = (4-b)b + a^2(-1+5b-2b^2+b^3) + a(4-2b+5b^2-b^3)$. Obviously, $g_1 \geq 0$ and $g_2 \geq 0$. Note that $g_0 = (a-1)(ab(b-1)^2 + b^2) + a(b(2b-1) + (4b-1) + 4) + 4b > 0$. Similarly, to verify $\Pi(1,1) \geq \Pi(2,1)$, it suffices to show that $a_1 - a_4 \geq 0$ because $b_1 \leq b_2$. Again, note that $(a_1 - a_4) \propto 2\pi(y_1 + y_2)(2I_1 + I_2)(y_1 - y_2)\big(\pi I_2 + 2I_1(I_2 - \pi)\big) \geq 0$ when $I_2 \geq \pi$. □

A.3 Proof of Proposition 2

Proof. Consider the equilibrium conjecture: (i) $k(j) = j$ for $j \in \{1,2\}$; and (ii) Seller 1, who works with Influencer 1, charges a price at $p_1^* = \frac{y_1}{2}$ and makes a total profit of $\Pi_1 = \frac{y_1 I_1}{2}$. Seller 2 sets $p_2^* = 0$ and effectively stays out of the market.

Next, we verify that this forms an equilibrium. First, given $p_1^* = \frac{y_1}{2}$, all potential consumers of Influencer 2 (that is, $\|x - \theta_2\| \leq I_2$) buy from Seller 1 for all $p_2 \geq 0$ whenever consumers with type $x = \theta_2$ and type $\|x - \theta_2\| = I_2$ & $\|x - \theta_1\| = \beta + I_2$ both choose to buy from Seller 1. This requires $y_1 * \big(1 - \beta/I_1\big) - p_1^* \geq y_2$ and $y_1 * \big(1 - (\beta + I_2)/I_1\big) - p_1^* \geq 0$. Simplifying these two equations yields $\frac{\beta}{I_1} \leq \min\{\frac{1}{2} - \frac{y_2}{y_1}, \frac{1}{2} - \frac{I_2}{I_1}\}$. Second, given the equilibrium strategies of Seller 2 and all consumers, Seller 1 always charges the monopolist price. Seller 2 cannot attract any consumer

[30] By plugging p_j^* and s_j, we can verify that the consumer indifferent between buying from Seller 1 and 2 gets a non-negative utility.

by setting $p_2^* > 0$ and it is optimal to set $p_2^* = 0$. Third, Influencer 1 gets a wage of $w_1 = \frac{(1-\gamma)y_1 I_1}{2}$ when working with Seller 1, which dominates the wage $\hat{w}_1 = \frac{(1-\gamma)y_2 I_1}{2} < w_1$ when working with Seller 2. Given Influencer 1's equilibrium matching choice, Influencer 2 can only match with Seller 2. The proof concludes. □

A.4 Proofs of Examples in Section 3.3

Here, we prove the examples in Section 3.3.

(i) Proof of Example 1

Proof. First, the assortative matching holds trivially with balanced matching, since the two influencers are identical for sellers. Second, we construct an equilibrium with $p_1^C \geq p_2^C \geq 0$ because $y_1 \geq y_2$. In particular, Seller 1 targets consumers sufficiently loyal to Influencer 1, and the cutoff type x^* is pinned down by $y_1(1 - \|x^* - \theta\|/I) - p_1 = y_2(1 - \|x^* - \theta\|/I) - p_2$. Note that all consumers with $\|x - \theta\| < \|x^* - \theta\|$ buy from Seller 1. Given this, Seller 2 serves consumers with $\|x - \theta\| \geq \|x^* - \theta\|$ and $\|x - \theta\| \leq \|x^{**} - \theta\|$ where $y_2(1 - \|x^{**} - \theta\|/I) - p_2 = 0$. Thus, the demand for Seller 1 is $q_1 = 2\|x^* - \theta\| = 2I * (1 - \frac{p_1 - p_2}{y_1 - y_2})$, and that for Seller 2 is $q_2 = 2(\|x^{**} - \theta\| - \|x^* - \theta\|) = 2I * (\frac{p_1 - p_2}{y_1 - y_2} - \frac{p_2}{y_2})$. Then, we can further calculate profits as $\Pi_1 = 2I * p_1 * (1 - \frac{p_1 - p_2}{y_1 - y_2})$ and $\Pi_2 = 2I * p_2 * (\frac{p_1 - p_2}{y_1 - y_2} - \frac{p_2}{y_2})$. Taking derivatives over Π_m with respect to p_m for $m = 1, 2$,

$$\frac{2I(-2p_1^C + p_2^C + y_1 - y_2)}{y_1 - y_2} = 0, \text{ and } \frac{I(-4p_2^C y_1 + 2p_1^C y_2)}{(y_1 - y_2)y_2} = 0$$

Solving these two equations yields the desired solution. Moreover, the second-order conditions are trivially true, and thus the F.O.C.s fully characterize the optimal solution. Inserting $(p_1^C, p_2^C) = (\frac{2y_1(y_1 - y_2)}{4y_1 - y_2}, \frac{y_2(y_1 - y_2)}{4y_1 - y_2})$ into the profits, we obtain all the profit functions after simple algebraic manipulations. Finally, we impose $I \leq \pi$ so that the market shares of both sellers are well defined (that is, $3Iy_1/(4y_1 - y_2) \leq \pi$). □

(ii) Proof of Example 2

Proof. Again, the matching part is trivial. We construct an equilibrium with $p_1^C \geq p_2^C \geq 0$. Otherwise, if $p_1 < p_2$, Seller 2 is priced out of the market because $\beta = 0$, $y_1 = y_2$ and $I_1 \geq I_2$. First, we calculate the market shares for both sellers. When $p_1 \geq p_2$, the consumer with $x \to \theta$ chooses to buy from Seller 2 because the utility gap, compared to that of Seller 1, is equal to $(p_1 - p_2) - y * \|x - \theta\| * (I_1 - I_2)/I_1 I_2$. This further implies that there exists a cutoff type x^* such that all consumers with $\|x - \theta\| < \|x^* - \theta\|$ buy from Seller 2, that is, $(p_1 - p_2) = y * \|x - \theta\| * (I_1 - I_2)/I_1 I_2$. Solving it yields $\|x^* - \theta\| = \frac{I_1 I_2 (p_1 - p_2)}{y(I_1 - I_2)}$. Meanwhile, consumers with $\|x - \theta\| > \|x^* - \theta\|$ buy from Seller 1 whenever it generates a positive utility, which gives a second cutoff type x^{**} given by $y\,(1 - \|x^{**} - \theta\|/I_1) - p_1 = 0$, and consumers with $\|x^* - \theta\| < \|x - \theta\| \leq \|x^{**} - \theta\|$ buy from Seller 1.

Second, we compute the profits as $\Pi_1 = p_1 * \big((1-p_1/y)I_1 - \frac{I_1 I_2 (p_1-p_2)}{y(I_1-I_2)}\big)$ and $\Pi_2 = p_2 * \frac{I_1 I_2 (p_1-p_2)}{y(I_1-I_2)}$. Obviously, Π_j is concave in p_j for $j = 1, 2$. Thus, the equilibrium is pinned down by first-order conditions that $\frac{I_1}{y(I_1-I_2)}\big(I_2(p_2-y) + I_1(y-2p_1)\big) = 0$ and $\frac{I_1 I_2}{y(I_1-I_2)}(p_1 - 2p_2) = 0$. By solving these two equations, we can compute (p_1^C, p_2^C) and the group profits by simple algebra. Note that market shares are well defined (that is, $\|x^{**} - \theta\| \leq \pi$) when $I_1 \leq \pi$. □

(iii) Proof of Example 3

Proof. Here, we only construct the equilibrium price. First, we calculate the group profits Π_k^C for $k = 1, 2$. Fix p_2 and consider seller 1's profit. First, if p_1 and p_2 are close and not too small (to be discussed shortly), we expect that two sellers share consumers along the short arc between θ_1 and θ_2, and the cutoff type x^* satisfies $p_2 - p_1 = (\|x^* - \theta_1\| - \|x^* - \theta_2\|)\, y/I$. Define $s_k = \|x^* - \theta_k\|$ for $j = 1, 2$. All consumers are served along the short arc between θ_1 and θ_2 only when $I * (1 - p_1/y) + I * (1 - p_2/y) \geq \|\theta_1 - \theta_2\|$ is satisfied or, equivalently, $p_1 + p_2 \leq (2 - \beta/I) * y$. When these conditions are verified, $s_1 + s_2 = \beta$ holds. Thus, we have $s_k = \frac{1}{2}(\beta + (p_{3-k} - p_k) * I/y)$ and $\Pi_1^C = p_1 * (s_1 + I(1 - p_1/y))$. However, when p_1 is sufficiently low compared to p_2, then $s_1 > \beta$ may occur, which implies $\|x - \theta_1\| - \|x - \theta_2\| = \beta$. In particular, when $p_1 \leq p_2 - y\beta/I$, Seller 1 beats Seller 2 and takes the entire market, and thus $\Pi_1^C = 2p_1 * (1 - p_1/y)I$. In contrast, when $p_1 \geq p_2 + y\beta/I$, Seller 1 loses all consumers, and thus $\Pi_1^C = 0$. Note that there are two discontinuity points for Π_1^C.

Second, we solve the equilibrium price. Obviously, it is sub-optimal for Seller 1 to set $p_1 \geq p_2 + y\beta/I$, which leads to a zero profit. Then, we start with the case in which p_1 and p_2 are sufficiently close. We can plug in the formula for s_j and derive first-order conditions: $\frac{y\beta + I(p_2 - 6p_1 + 2y)}{2y} = 0$ and $\frac{y\beta + I(p_1 - 6p_2 + 2y)}{2y} = 0$. Solving these two equations yields $p_1^* = p_2^* = \frac{y(2I+\beta)}{5I}$, and we can further plug p_j^* into Π_k^C to solve $\Pi_1^C = \frac{3y(2I+\beta)^2}{50I}$, as long as the condition $p_2^* - y\beta/I < p_1^* < p_2^* + y\beta/I$ is satisfied.

However, we also need to verify $p_1^* + p_2^* \leq (2 - \beta/I) * y$, which reduces to $\beta \leq 6I/7$. When $\beta > \frac{6}{7}I$, we define $\widehat{p}_1^* = \widehat{p}_2^* = (1 - \beta/2I) * y$. Note that given $p_2 = \widehat{p}_2^*$, the term $p_1 * ((1 - p_1/y) * I + s_1)$ is strictly increasing in p_1 for $p_1 \leq p_1^*$. Thus, the first (influencer-seller) group has no incentive to deviate downward. Meanwhile, for $p_1 > \widehat{p}_1^*$, the term $p_1 * (1 - p_1/y) * I$ is strictly decreasing for $p_1 > \widehat{p}_1^*$ since $\widehat{p}_1^* \geq \frac{y}{2}$. This implies that the first group does not have an incentive to deviate upward. Thus, $(\widehat{p}_1^*, \widehat{p}_2^*)$ constitutes an equilibrium when $\frac{6}{7}I < \beta < I$, as long as no seller is priced out of the market.

Third, we need to ensure that, for p_1 and p_2 sufficiently close, Seller 1 has no incentive to undercut Seller 2 and force Seller 2 out of the market. In this deviation, Seller 1 only needs to set $p_1 = p_2^* - y\beta/I$, which leads to a profit of $\widehat{\Pi}_1 = 2I * (p_2^* - y\beta/I) * (1 - (p_2^* - y\beta/I)/y)$. This is also the most profitable deviation, since $\Pi_1 = 2I(1 - p_1/y)p_1$ is strictly decreasing for all $p_1 \leq p_2^* - y\beta/I$. Simplifying $\widehat{\Pi}_1 \leq \Pi_1^C$ leads to the assumed condition on β_0. □

A.5 Proof of Proposition 3

Proof. The proof consists of two steps. First, we verify equilibrium matching and calculate the profits for both groups. When $\frac{I_1+I_2}{2} < \beta$, Proposition 1(i) implies $k(j) = j$ for $j \in \{1, 2\}$. Then, monopolist pricing implies that $\Pi_j = \frac{y_j I_j}{2}$ for Seller j. On the other hand, as $I_2 \to \infty$, it approaches the Bertrand price competition (that is, $p_1 = y_1 - y_2$ and $y_2 = 0$) and thus $\Pi_1 = 2\pi(y_1 - y_2)$ and $\Pi_2 = 0$, regardless of $k(j)$. This also implies $k(j) = j$ for $j \in \{1, 2\}$.

Second, we show the non-monotonicities for influencers' wages and income gap. When $I_1 < \beta$, we have $\frac{I_1+I_2}{2} < \beta$. Then, Nash bargaining implies $w_j = \frac{(1-\gamma)y_j I_j}{2}$. We can easily show that $w_1 + w_2 = \frac{(1-\gamma)(y_1 I_1 + y_2 I_2)}{2}$ and $w_1 - w_2 = \frac{(1-\gamma)(y_1 I_1 - y_2 I_2)}{2}$. Since $\Delta I_1 \geq \Delta I_2 > 0$, we have $\Delta(w_1 + w_2) > 0$. Furthermore, $\Delta(y_1 I_1 - y_2 I_2) = y_1 \Delta(I_1 - I_2) + (y_1 - y_2)\Delta I_2$ and thus $\Delta(w_1 - w_2) > 0$ also holds. Finally, given that $(I_1 - I_2)y_1 > 4\pi(y_1 - y_2)$, $w_1 + w_2 = w_1 - w_2 = 2\pi(1-\gamma)(y_1 - y_2)$ as $\Delta I_2 \to \infty$, which is smaller than $\frac{(1-\gamma)(y_1 I_1 - y_2 I_2)}{2}$. This implies that $(w_1 \pm w_2)$ indeed decreases for some ΔI_2 sufficiently large. □

A.6 Proof of Proposition 4

Proof. The proof consists of two parts.

Part (i). Calculate the equilibrium profits before differentiation.

We construct an equilibrium such that the equilibrium prices are characterized by the F.O.C.s of the profit functions. We first derive equilibrium prices and profits using the F.O.C.s and then verify the equilibrium. Note that matching is trivial because $I_1 = I_2$.

First, consider a pair of (p_1, p_2) such that no seller dominates (to be verified later). Then we have an indifference condition along the short arc whose length is β, that is, $y_1(1 - s_1/I) - p_1 = y_2(1 - s_2/I) - p_2$, where s_j is Seller k's market share along the short arc, whose identity is used to differentiate the two identical sellers. Note that s_1 can be larger than β or negative, depending on the prices p_k. By imposing $s_2 = \beta - s_1$, we can solve $s_1 = \frac{I(p_2-p_1)+I(y_1-y_2)+\beta y_2}{y_1+y_2}$ and $s_2 = \beta - s_1$. We can further set up the profit $\Pi_k = (s_k + (1 - p_k/y)I) * p_k$ and get the FOC's as below:

$$\frac{\beta y_k y_l + I(y_k p_l + 2y_k^2 - 4p_k y_k - 2p_k y_l)}{y_k(y_k + y_l)} = 0$$

Solving the FOC's yields $p_k^* = \frac{y_k(\beta y_l(3y_k+4y_l)+2I(2y_k^2+4y_k y_l+y_l^2))}{I(8y_k^2+19y_k y_l+8y_l^2)}$ and

$$\Pi_k = \frac{y_k(2y_k + y_l)(\beta y_l(3y_k + 4y_l) + 2I(2y_k^2 + 4y_k y_l + y_l^2))^2}{I(y_k + y_l)(8y_k^2 + 19y_k y_l + 8y_l^2)^2}.$$

Second, we verify the equilibrium payoffs shown above under the assumed conditions, including: (1) All consumers are served along the short arc, that is, $I(1 - p_1^*/y_1) + I(1 - p_2^*/y_2) \geq \beta$, which yields

$$\beta \leq \frac{2I(5y_1^2 + 11y_1 y_2 + 5y_2^2)}{12y_1^2 + 25y_1 y_2 + 12y_2^2} =: cI$$

Note that $I \leq \frac{\pi}{2}$ implies no competition along the long arc; and (2) Sellers have no incentive to

undercut and force competitors out of the product market. To force Seller 2 to quit, Seller 1 needs to induce the farthest consumer of Seller 2 to switch by setting $\widehat{p}_1 = y_1(1-(\beta+\widehat{s}_2)/I)$, where $\widehat{s}_2 = I(1-p_2^*/y_2)$. Meanwhile, to force out Seller 1, Seller 2 must set a price $\widehat{p}_2 = y_2(1-\beta/I)-(y_1-p_1^*)$. These deviations lead to profits given by $\widehat{\Pi}_k = 2I_j\widehat{p}_k(1-\widehat{p}_k/y_k)$. We can verify $\widehat{p}_k \leq \frac{y_k}{2}$, which constitutes the most profitable deviation because $2Ip_k(1-p_k/y_k)$ is strictly decreasing when $p_k < \frac{y_k}{2}$. Thus, the nonexistence of profitable deviations reduces to $\Pi_k \geq \widehat{\Pi}_k$ for $j = 1, 2$. In particular,

$$\Pi_1 - \widehat{\Pi}_1 = -\frac{ty_2Ih_1(b,t)}{(1+t)(8+19t+8t^2)^2} \quad \text{and} \quad \Pi_2 - \widehat{\Pi}_2 = -\frac{yI_2h_2(b,t)}{(1+t)(8+19t+8t^2)^2}$$

where

$$\begin{aligned} h_1(b,t) &= 4(1+2t)(-7-22t-18t^2-3t^3+t^4) + 4b(4+39t+110t^2+124t^3+58t^4+8t^5) \\ &\quad + b^2(144+696t+1209t^2+914t^3+288t^4+32t^5) \\ h_2(b,t) &= b^2(128+608t+1108t^2+967t^3+406t^4+66t^5) + (8+8t-55t^2-98t^3-17t^4+40t^5+16t^6) \\ &\quad \times (4+2t) + 2b(-64-216t-117t^2+322t^3+483t^4+238t^5+40t^6) \end{aligned}$$

Note that $t \geq 1$ and thus $h_k(b,t)$ is strictly increasing in b for $b > 0$. Therefore

$$\begin{aligned} h_1(b,t) &\geq h_1(a,t) = \frac{1}{100}(-1024-3456t-719t^2+7506t^3+7552t^4+2048t^5) \\ h_2(b,t) &\geq h_2(a,t) = \frac{1}{100}(512-2688t-17448t^2-22177t^3+6234t^4+27474t^5+16800t^6+3200t^7) \end{aligned}$$

Obviously, $h_k(a,t) > 0$ because $t \geq 1$. Thus, the pricing strategies and profits characterized by the F.O.C.s are optimal.

Part (ii). Verify the optimality of Sellers' differentiation choices.

We first verify statement (i). We measure the incentive to differentiate by the net return, that is, $R_k(\beta) = \frac{y_kI}{2} - \Pi_k(\beta|y_1,y_2,I) - f_E$, where $\frac{y_kI}{2}$ is the profit when $\beta^* \geq I$. Then:

$$R_1 - R_2 = \frac{(3-2b)(1-2b)I(t-1)ty2}{2(8+19t+8t^2)}$$

Thus, $R_1 - R_2 \leq 0$ if and only if $\frac{1}{2} \leq b \leq \frac{3}{2}$, which further implies that $R_1 \leq R_2$ for $\beta \in [I/2, cI]$ and $R_1 < R_2$ for $\beta \in [aI, I/2]$.

We then verify statement (ii). By the continuity and monotonicity of $H_1(b,t)$ and $H_2(b,t)$, which increases strictly with b, $R(\beta) = \max\{R_1(\beta), R_2(\beta)\}$ increases strictly with β and is continuous at $\beta = \frac{I}{2}$. Then, the condition $\frac{y_1I}{2} - H_1(c,t) * y_1I < f_E < \frac{y_2I}{2} - H_2(a,t) * y_2I$ is equivalent to $R(aI) > 0$ and $R(cI) < 0$, which further implies that there is a unique $\widetilde{\beta} \in (aI, cI)$ such that $R(\widetilde{\beta}) = 0$. Thus, for $\beta \in [aI, \widetilde{\beta})$, one seller engages in horizontal differentiation; otherwise, for $\beta \in [\widetilde{\beta}, bI]$, no seller differentiates the product horizontally.

Finally, we verify statement (iii). Denote $\widetilde{b} = \widetilde{\beta}/I$. We first verify statement (iii) for $\beta \in [\frac{I}{2}, cI]$. By (i), Seller 2 chooses horizontal differentiation. It suffices to verify $\frac{\partial\widetilde{\beta}}{\partial t} > 0$. To see it, we use

implicit function differentiation for $R_k(\widetilde{\beta},t)=\frac{y_kI}{2}-H_k(\widetilde{b},t)*y_kI-f_E$,

$$\frac{\partial\widetilde{\beta}}{\partial t}=-\frac{\partial R_k/\partial t}{\partial R_k/\partial\widetilde{\beta}}=-\frac{\partial H_2/\partial t}{\partial H_2/\partial\widetilde{b}}$$

Obviously, $\partial H_2/\partial\widetilde{b}>0$. Next, we verify $\partial H_2/\partial t<0$. Note that $\frac{\partial H_2}{\partial t}=\frac{I\left(\sum_{i=0}^6 m_it^i\right)}{(1+t)^2(8+19t+8t^2)^3}$, where $m_0=-320+384b$, $m_1=-2000+2032b+288b^2$, $m_2=-4960+4064b+1512b^2$, $m_3=-6128+3600b+3157b^2$, $m_4=-3888+1064b+3224b^2$, $m_5=-1188-236b+1576b^2$ and $m_6=-136-128b+288b^2$. It is easy to check that $m_i(b)<0$ for $1\le i\le 6$ for $b\in[0,c]\subseteq[0,6/7]$. Furthermore, since $t\ge 1$, $\sum_{i=0}^6 m_i(b)t^i<\sum_{i=0}^6 m_i(b)=245(-76+44b+41b^2)<0$ for all $b\in[0,6/7]$. All of this together implies $\frac{\partial\widetilde{\beta}}{\partial t}>0$ for $b\in[a,c]$.

Similarly, we can prove that $\frac{\partial\widetilde{\beta}}{\partial\tau}>0$ when Seller 1 chooses horizontal differentiation. Again, $\frac{\partial\widetilde{\beta}}{\partial\tau}=-\frac{\partial R_1/\partial\tau}{\partial R_1/\partial\widetilde{\beta}}=\frac{1}{y_1I}*\frac{\partial R_1/\partial\tau}{\partial H_1/\partial\widetilde{b}}$. Obviously, $\partial H_1/\partial\widetilde{b}>0$. Then, $\partial R_1/\partial\tau=-\frac{y_1I\left(H_3(b)+\sum_{i=5}^6 m_i\tau^i\right)}{(1+\tau)^2(8+19\tau+8\tau^2)^3}$. Here, $H_3(b):=\sum_{i=0}^4 m_i\tau^i=b^2\tau(288+1512\tau+3157\tau^2+3224\tau^3)+4b(96+508\tau+1016\tau^2+900\tau^3+266\tau^4)-4(80+500\tau+1240\tau^2+1532\tau^3+972\tau^4)$. Obviously, $H_3(b)$ increases strictly in $b>0$ and thus $H_3(b)\le H_3(c)<0$, where $H_3(c)=-\frac{4\tau(2496+23248\tau+93727\tau^2+214914\tau^3+299001\tau^4+254162\tau^5+125519\tau^6+27448\tau^7)}{(12+25\tau+12\tau^2)^2)}$. Thus, $\frac{\partial R_1}{\partial\tau}>0$ and $\frac{\partial\widetilde{\beta}}{\partial\tau}>0$. The proof concludes. □

A.7 Proof of Proposition 5

Proof. The proof consists of two parts.

Step (1). Compute $\Pi_{H,H}$, $\Pi_{L,L}$, $\Pi^j_{L,H}$ and $\Pi^j_{H,L}$ for $j=1,2$.

First, by Example 3 and Condition (i), we have $\Pi_{H,H}=yA(\beta)$ and $\Pi_{L,L}=A(\beta)$ for $j=1,2$, where $A(\beta)=\frac{3}{50}(2I+\beta)^2$. Second, we calculate the profits under asymmetric investments in quality, say (H,L), which means that only Group 1 chooses high quality. Given Condition (ii) assumed, Group 2 can still attract some consumers even if Group 1 sets a price at $p_1=0$. Now, denote by p_j the price charged by Group $j=1,2$. For Group 1, the size of the consumers served along the long arc is $y_1(1-\|x-\theta_1\|)-p_1\ge 0$ or equivalently $(1-p_1/y)$. Meanwhile, along the short arc, the cutoff type of the consumer x^* satisfies $y(1-\|x^*-\theta_1\|)-p_1=1-(\beta-\|x^*-\theta_2\|)-p_2$, which yields $s_1:=\|x^*-\theta_1\|=\frac{p_2-p_1+(y-1)+\beta}{y+1}$ and $s_2=\beta-s_1$. We can also calculate the group profits as $\Pi^j_{H,L}=(1-p_j/y_j)*p_j+p_1s_j$ for $j=1,2$. Using the first-order conditions, we get $p_1^*=\frac{2(1+2\beta)y+(8+3\beta)y^2+4y^3}{8+19y+8y^2}$ and $p_2^*=\frac{4+(8+3\beta)y+2(1+2\beta)y^2}{8+19y+8y^2}$. Meanwhile, we need to ensure that the cut-off type x^* gets a non-negative utility, that is, $y(1-\|x^*-\theta\|)-p_1^*\ge 0$, which reduces to $\frac{y\left(2(5-6\beta)(1+y^2)+(22-25\beta)y\right)}{(1+y)(8+19y+8y^2)}\ge 0$, which trivially holds under condition i). Furthermore, we can directly verify that the second-order conditions are satisfied. Thus, we can compute $\Pi^1_{H,L}=\frac{y(1+2y)\left(2+8y+4y^2+\beta(4+3y)\right)^2}{(1+y)(8+19y+8y^2)^2}$ and $\Pi^2_{H,L}=\frac{y(2+y)\left(4+8y+2y^2+\beta(3y+4y^2)\right)^2}{(1+y)(8+19y+8y^2)^2}$. Finally, note that $\Pi^j_{H,L}=\Pi^i_{L,H}$ for $i\ne j$.

Step (2). Nash equilibrium construction

We first state two properties as follows: (1) Fix y. $V_1(\beta,y)>V_2(\beta,y)$. Here, $V_1(\beta,y)=$

$\Pi^1_{H,L} - \Pi_{L,L}$ and $V_2(\beta, y) = \Pi_{H,H} - \Pi^1_{L,H}$. (2) Fix y. Both $V_1(\beta, y)$ and $V_2(\beta, y)$ are strictly increasing in β. Then, we verify the equilibrium under these two properties and then prove them shortly. First, by condition (iii), $V_1(\beta_0, y) < F_V < V_2(5/6, y)$ and property (1), we have

$$V_1(5/6, y) > V_2(5/6, y) > F_V > V_1(\beta_0, y) > V_2(\beta, y)$$

which, together with property (2), the strict monotonicity of $V_1(\beta, y)$ and $V_2(\beta, y)$, implies that there exists $\underline{\beta}, \overline{\beta} \in (\beta_0, 5/6)$ such that $F_V = V_1(\underline{\beta}, y) = V_2(\overline{\beta}, y)$ and $\overline{\beta} > \underline{\beta}$.

To summarize: (a) For $\beta \geq \overline{\beta}$, $V_2(\beta, y) \geq F_V$, or equivalently, $\Pi_{H,H} \geq F_V + \Pi^1_{L,H}$. Therefore, since Influencer 2 chooses high quality, it is optimal for Influencer 1 to invest. By symmetry, Influencer 2 also chooses to invest, and thus (H, H) is a Nash equilibrium.

(b) For $\underline{\beta} \leq \beta < \overline{\beta}$, we have both $V_1(\beta, y) \geq F_V$ and $V_2(\beta, y) < F_V$, that is, $\Pi^1_{H,L} - \Pi_{L,L} \geq F_V$ and $\Pi_{H,H} - \Pi^2_{H,L} < F_V$ since $\Pi^2_{H,L} = \Pi^1_{L,H}$. These two conditions read as follows. One, given that Influencer 2 chooses low quality, it is optimal for Influencer 1 to invest. Two, given that Influencer 1 chooses high quality, it is optimal for Influencer 2 to choose low quality. Thus, (H, L) is a Nash equilibrium, so is (L, H) by symmetry.

(c) For $\beta < \underline{\beta}$, $V_1(\beta, y) < F_V$, or equivalently, $\Pi^1_{H,L} - \Pi_{L,L} < F_V$, which implies that even if Influencer 2 does not invest, it is optimal for Influencer 1 not to invest. By symmetry, (L, L) is a Nash equilibrium.

Now, it suffices to verify properties (1) and (2) for $V_1(\beta, y)$ and $V_2(\beta, y)$. To this end, we write down the formulas and check them one by one.

First, with y fixed, $V_1(\beta, y)$ and $V_2(\beta, y)$ are strictly increasing in β. Note that

$$\begin{aligned} V_1(\beta, y) &= \frac{(y-1) * \left(1600y^5 + 32M_1y^4 + 4M_2y^3 + 25M_3y^2 + 8M_4y + 192(2+\beta)^2\right)}{50(y+1)(8+19y+8y^2)} \\ V_2(\beta, y) &= \frac{(y-1) * \left(1600 + 32M_1y + 4M_2y^2 + 25M_3y^3 + 8M_4y^4 + 192(2+\beta)^2y^5\right)}{50(y+1)(8+19y+8y^2)} \end{aligned}$$

where $M_1 = 251 + 51\beta - 6\beta^2$, $M_2 = 3704 + 1604\beta - 99\beta^2$, $M_3 = 500 + 340\beta + 3\beta^2$ and $M_4 = 623 + 548\beta + 62\beta^2$. After simple algebra, we can show that all quadratic terms $M_j (j = 1, 2, 3, 4)$ are positive and strictly increasing for $\beta \in (\beta_0, 5/6)$, which verifies the monotonicity of both $V_1(\beta, y)$ and $V_2(\beta, y)$.

Second, with y fixed, $V_1(\beta, y) > V_2(\beta, y)$ for all $\beta \in (\beta_0, 5/6)$.

$$V_1(\beta, y) - V_2(\beta, y) = \frac{(y-1)^2 * (64 * M_5 + 40 * M_6 y + M_7 * y^2 + 40 * M_6 y^3 + 64 * M_5 y^4)}{50(y+1)(8+19y+8y^2)}$$

where $M_5 = 13 - 12\beta - 3\beta^2, M_6 = 97 - 88\beta - 22\beta^2$, and $M_7 = 6196 - 5604\beta - 1351\beta^2$.

To see that $M_5 > 0$, note that there are two solutions $\beta_1 \approx -4.89$ and $\beta_2 \approx 0.89$. Hence, $M_5 > 0$ for all $\beta \in (-4.89, 0.89)$, and thus $M_5 > 0$ for all $\beta \in (\beta_0, 5/6)$. We can prove $M_6 > 0$ and $M_7 > 0$ in a similar way. □

A.8 Proof of Proposition 6

Proof. We construct an equilibrium such that: (1) $k(j) = 2 - j$; (2) $(p_1^d)^* = (c_2 - c_1)y$, $(p_2^d)^* = 0$, and consumers buy from Seller 1; and (3) $(p_2^i)^* = \frac{c_1 y}{2}$ and $(p_1^i)^* > (p_2^i)^*$.

First, we consider Seller 2 working with Influencer j. When $p_1^d = (c_2 - c_1)y$ and all consumers buy from Seller 1, Seller 2 is forced out of the direct sales market because $(1-c_1)y - p_1^d \geq (1-c_2)y$, which means that all consumers prefer Seller 1 even $p_2^d = 0$. Then Seller 2 makes a profit of $\Pi_2^i = 2s_2 p_2^i$ by charging p_2^i in influencer-intermediated sales, where $2s_2$ is the market share of Seller 2, which is further defined by the indifference condition $y(1 - s_2/I_j) - p_2^i = (1 - c_2)y$. Solving it yields $(p_2^i)^* = \frac{c_2 y}{2}$, $(p_2^d)^* = 0$ and $\Pi_2^i = \frac{1}{2}c_2^2 I_j y$.

Next, we consider Seller 1 working with Influnecer $i \neq j$. Given Seller 2's pricing strategies (p_2^d, p_2^i), Seller 1 makes a total profit $\Pi_1 = \Pi_1^d + \Pi_1^i$, where $\Pi_1^d = p_2^d(2\pi - 2s_2 - 2s_1)$ and $\Pi_2^i = \gamma p_1^i * 2s_1$. Here, $2s_1$ is the market share of Seller 1 in influencer-intermediated sales, which is defined by $y(1 - s_1/I_{i\neq j}) - p_1^i = (1 - c_1)y - p_1^d$. Note that $p_1^d \leq (c_2 - c_1)y$; otherwise, Seller 1 is forced out of the direct sales market by Seller 2. Furthermore, $\gamma p_1^i \geq p_1^d$, which means that Seller 1 gets more from selling through influencers compared to direct sales. Then, we get $\frac{d\Pi_1}{dp_1^d} = (2\pi - 2s_2 - 2s_2) + \frac{2I_2}{y}(\gamma p_1^i - p_1^d) \geq 0$, and thus $(p_1^d)^* = (c_2 - c_1)y$. Now, Seller 1 faces an identical problem as that of Seller 2 under the additional constraint $\gamma p_1^i \geq p_1^d$. Thus, $(p_1^i)^* = \frac{c_2 y}{2}$ if $\gamma p_1^i \geq p_1^d$ holds; otherwise, $(p_1^i)^* = (p_1^d)^*/\gamma$ since Π_1^i is a quadratic function of p_1^i. Obviously, in the latter case, $\Pi_1^i < \frac{1}{2}c_2^2 I_i y$, whenever $\gamma * \frac{c_2 y}{2} < (c_2 - c_1)y$, which is $\frac{\gamma c_2}{2} < (c_2 - c_1)$. Also note that $\beta > \frac{c_2(I_1+I_2)}{2}$ ensures that there is no overlap in the consumer base in influencer-intermediated sales.

Finally, we consider the incentive for influencers. Influencer 1 earns a salary of $w_1 = \frac{\gamma}{2}c_2^2 I_j y$ when working with Seller 2, which strictly dominates $\hat{w}_1 < \frac{\gamma}{2}c_2^2 I_j y$ when working with Seller 1. Thus, $k(j) = 2 - j$. The proof concludes. □

A.9 Proof of Proposition 7

Proof. We use superscript $l \in \{a, b\}$ to index "before" and "after" power acquisition. Let SW denote total welfare. Define $\Delta w_j = w_j^a - w_j^b$, and $\Delta SW = SW^a - SW^b$. Note that Influencer j invests if and only if $C_P < \Delta w_j$ holds. Thus, under-investment arises if $\Delta SW > \#\{j : C_P > \Delta w_j\} * C_P$, and over-investment arises if $\Delta SW < \#\{j : C_P > \Delta w_j\} * C_P$.

Case (i). The condition that $kI < \frac{\pi}{2}$ guarantees the feasibility of monopolist pricing (i.e., $p_1^* = p_2^* = \frac{y}{2}$), and thus $\Pi_j^b = \frac{yI}{2}$ and $\Pi_j^a = \frac{kyI}{2}$. Note that $w_j^l = (1 - \gamma)\Pi_j^l$ for $j \in \{1, 2\}$ and $l \in \{a, b\}$. Therefore, Influencer j will invest iff $C_P \leq w_j^a - w_j^b = \frac{(1-\gamma)(k-1)yI}{2}$. Meanwhile, note that the total welfare, before power acquisition, is $SW^b = 2\int_{y(1-\|x-\theta_1\|/I)\geq 0} y(1 - \|x - \theta_1\|)dx = 2yI$ and similarly $SW^a = 2kyI$, and thus $\Delta SW = SW^a - SW^b = 2(k-1)yI$. Thus, there exists under-investment if $\frac{(1-\gamma)(k-1)yI}{2} < C_P < 2(k-1)yI$.

Case (ii). First, note that both sellers can get positive profits by attracting consumers sufficiently loyal to their own influencers. Meanwhile, since $I \geq \frac{3}{2}\pi \geq \pi$, all consumers are served. The condition $I \geq \frac{3}{2}\pi$ ensures that the cutoff consumer who is indifferent between purchasing from the

first seller and the second one can get a positive utility. Since some influencer(s) might acquire additional power, we use I_1 and I_2 in the equilibrium construction.

Second, given prices (p_1, p_2), the cutoff type x^* is determined by the equation $y(1-s_1/I_1)-p_1 = y(1-s_2/I_2)-p_2$, where $s_j := \|x^*-\theta_j\|$ is the size of consumers attracted by Seller j on one side. We can solve it to get $s_j = \frac{\pi y + I_1 I_2 (p_{3-j}-p_j)}{y(I_1+I_2)}$ and thus Seller j's profit is given by $\Pi_j = 2 s_j p_j$. By taking derivatives w.r.t. p_j over Π_j, we get $\frac{\partial \Pi_j}{\partial p_j} = \frac{2I_1I_2(p_{3-j}-2p_j)+2I_j\pi y}{y(I_1+I_2)}$. We can solve it to get $p_1 = \frac{\beta y(2I_1+I_2)}{3I_1I_2}$ and $p_2 = \frac{\beta y(I_1+2I_2)}{3I_1I_2}$. The profits are given by

$$\Pi_j = \frac{2\pi^2 y(2I_j + I_{3-j})^2}{9I_1I_2(I_1+I_2)} \tag{A.2}$$

and the cutoff type consumer gets a utility of $u(x^*) = \frac{y\left(3I_1I_2(I_1+I_2)-2\pi(I_1^2+I_2^2)-5\pi I_1I_2\right)}{3I_1I_2(I_1+I_2)}$. We can verify that the cutoff type consumer indeed gets a positive utility. To see it, when $I_1 = I_2 = I$, it reduces to $y - \frac{3\pi y}{2I} \geq 0$, which holds if and only if $I \geq \frac{3}{2}\pi$. When $I_1 = I_2 = kI$, it reduces to $kI \geq \frac{3}{2}\pi$. When $I_1 = kI$ and $I_2 = I$, it reduces to $u(x^*) = y - \frac{\pi y(2+5k+2k^2)}{3Ik(1+k)}$. Note that $\frac{du(x^*)}{dk} = \frac{\pi y(2+4k+3k^2)}{3k^2(1+k)^2} > 0$, and thus $u(x^*) \geq 0$ for any $k \geq 1$ as long as $I \geq \frac{3}{2}\pi$.

Third, we analyze the incentive for power acquisition. Obviously, before power acquisition, $w_1^b = w_2^b = \frac{(1-\gamma)\pi^2 y}{I}$. Next, we consider the incentive for power acquisition when the opponent influencer, say Influencer 2, does not acquire additional power. This means $I_1 = kI$ and $I_2 = I$. After power acquisition by Influencer 1, by the profit formula, equation (A.2), we have $w_1^a = \frac{2\pi^2 y(1-\gamma)(1+2k)^2}{9Ik(1+k)}$ and $w_2^a = \frac{2\pi^2 y(1-\gamma)(2+k)^2}{9Ik(1+k)}$. This further implies $\Delta w_1 = \frac{\pi^2 y(1-\gamma)(2+k)(1-k)}{9Ik(1+k)} < 0$, so Influencer 1 has no incentive to acquire additional power. Finally, we consider the incentive for power acquisition when the opponent influencer acquires power. By symmetry, we consider Influencer 1. If he chooses not to acquire power, he gets $\hat{w}_1^b = \frac{2\pi^2 y(1-\gamma)(2+k)^2}{9Ik(1+k)}$. If he acquires power, he gets $\hat{w}_1^a = \frac{(1-\gamma)\pi^2 y}{kI}$. Thus, $\Delta\hat{w}_1 = \frac{\pi^2 y(1-\gamma)(1-k)(1+2k)}{9Ik(1+k)} < 0$.

Fourth, we compute the cost range of C_P where power acquisition improves welfare. Since the transfer between consumers and sellers does not affect aggregate welfare, we can measure social welfare as $SW = \sum_{j=1}^{2} \int_{\|x-\theta_j\| \leq s_j} y*(1-\|x-\theta_j\|/I_j)dx$. Then we can compute social welfare SW_0 when no influencer acquires power, SW_1 when only one influencer acquires power, and SW_2 when both influencers acquire power as follows: $SW_0 = \frac{\pi(4I-\pi)y}{2I}$, $SW_1 = \frac{1}{9}\pi y\left(18 - \frac{\pi\left(k^2+7k+1\right)}{I(k^2+k)}\right)$, and $SW_2 = \frac{\pi(4kI-\pi)y}{2kI}$. It is easy to show that $SW_2 > SW_1 > SW_0$ and $2(SW_1 - SW_0) > SW_2 - SW_0$, and thus power acquisition improves social welfare as long as $C_P < \min\{SW_1 - SW_0, \frac{1}{2}(SW_2 - SW_0)\} = \frac{\pi^2 y(7k+2)(k-1)}{18Ik(k+1)}$. The proof concludes. □

A.10 Proof of Proposition 8

Proof. Here, we first present Lemma A.1, which computes the profit when one seller hires the two influencers in a parameterized setting. The proofs can be found in Section OA.6.1.

Lemma A.1. *Assume $I_1 = I_2 = I \leq \pi$. In equilibrium, the profit $\Pi_{\{1,2\}}$ and prices satisfy: (i) when $\beta \geq I$, $\Pi_{\{1,2\}} = yI$ and $p_j^* = \frac{y}{2}$; (ii) when $2I/3 \leq \beta < I$, $\Pi_{\{1,2\}} = (2I-\beta)\beta y/I$ and*

$p_j^* = y(1-\beta/2I)$; *and (iii) when* $0 \le \beta < 2I/3$, $\Pi_{\{1,2\}} = (2I+\beta)^2 y/(8I)$ *and* $p_j^* = \frac{y}{2} + \frac{y\beta}{4I}$.

Since internal transfers do not affect total welfare, when products are homogeneous (that is, $y_1 = y_2$), efficiency only depends on the size of the consumers served, which further depends on equilibrium prices. The proof consists of two parts.

Part (i). Let W_U and W_B denote the total welfare under unbalanced matching and balanced matching. Now, it suffices to show that $W_U \le W_B$ and we have three cases:

Case (1). Heterogeneous influencer power (that is, $y_1 = y_2 = y$, $\beta = 0$ and $I_1 \ge I_2$). Under unbalanced matching, $p_1^* = p_2^* = \frac{y}{2}$, and the total size of consumers served is just I_1 (i.e., $\{x \in \mathbf{S}^1 : y\left(1 - \frac{1}{I_1}\|x-\theta\|\right) - p_1^* \ge 0\}$). In contrast, under balanced matching, the marginal consumer faces an equilibrium price given by $p_1^C = \frac{2y(I_1-I_2)}{4I_1-I_2}$ (see Example 2), which implies that the total size of consumers is greater than I_1 because $\frac{2y(I_1-I_2)}{4I_1-I_2} \le \frac{y}{2}$. Thus, $W_U \le W_B$.

Case (2). Heterogeneous influencers' style type (that is, $y_1 = y_2$, $I_1 = I_2$ and $\beta > 0$). Here, we only focus on the case in which a pure strategy equilibrium exists (that is, $\beta \ge \beta_0$) and the equilibrium prices are p_j^C in Example 3. In contrast, under unbalanced matching, the equilibrium prices are given by p_j^* in Lemma A.1. We can directly verify that $p_1^C = p_2^C \le p_1^* = p_2^*$ for $\beta \ge \beta_0$. Thus, total welfare is higher under regulated matching, because a consumer who purchases the product under unmatched matching is also willing to buy it under regulated matching. Hence, $W_U \le W_B$.

Case (3). Heterogeneous product quality (i.e., $I_1 = I_2 = I$, $\beta = 0$ and $y_1 \ge y_2$). Under unbalanced matching, seller 1 hires both influencers and sets a price at $p_j^* = \frac{y_j}{2}$. Influencer 2 is not active in the market. Therefore, $W_U = \int_{\left\{x \in \mathbf{S}^1 : \|x-\theta\| \le \frac{1}{2}I\right\}} y(1 - \|x-\theta\|/I)dx = \frac{3}{4}y_1 I$.

In contrast, under balanced matching, the equilibrium outcome is given by Example 1, and thus $W_B = \int_{R_1} y(1-\|x-\theta\|/I_1)dx + \int_{R_2} y(1-\|x-\theta\|/I_2)dx$, where $R_1 = \{x \in \mathbf{S}^1 : \|x-\theta\| \le \frac{2Iy_1}{4y_1-y_2}\}$ and $R_2 = \{x \in \mathbf{S}^1 : \frac{2Iy_1}{4y_1-y_2} < \|x-\theta\| \le \frac{3Iy_1}{4y_1-y_2}\}$. We can further compute $W_B = \frac{Iy_1(14y_1^2-4y_1y_2-y_2^2)}{(4y_1-y_2)^2} = Iy_1 * f(x)$ where $f(x) = (14x^2-x-1)/(4x-1)^2$ and $x := y_1/y_2$.[31] Note that $f'(x) = -\frac{12(x-1)}{(4x-1)^3} < 0$, and thus $f(x)$ strictly decreases for $x > 1$. Moreover, since $\lim_{x\to 1} f(x) = 1$ and $\lim_{x\to\infty} f(x) = \frac{7}{8}$, we have $W_B \ge \frac{7}{8}Iy_1 > \frac{3}{4}Iy_1 = W_U$.

Part (ii). First, note that when $\beta \ge \frac{I_1+I_2}{2}$, each influencer charges a monopolist price $p_j^* = \frac{y}{2}$ when hired by a seller with product quality y, and serves a sub-population such that $x \in R_j$ where $R_j := \left\{x \in \mathbf{S}^1 : \|x-\theta_j\| \le \frac{I}{2}\right\}$. Hence, there is zero overlap among consumers served by the two influencers.

Under unbalanced matching, both influencers are hired by Seller 1. Furthermore, similar to that of unbalanced matching in the heterogeneous product quality case, we can compute that $W_U = \frac{3}{4}y_1(I_1+I_2)$. Similarly, under balanced matching, $W_B = \frac{3}{4}y_1I_1 + \frac{3}{4}y_2I_2$. Obviously, $W_U > W_R$ as long as $y_1 > y_2$. All the proofs conclude. □

[31] The skipped algebra is available upon request.

A.11 Proof of Lemma 1

Proof. First, note that when $(1-c_1)y_1 = (1-c_2)y_2$ is satisfied, $p_k^d = 0$ for direct sales, regardless of the matches. To see it, note that if Seller 2, charges a price $p_2^d > 0$, then it is sub-optimal for Seller 1 to charge $p_1^d > p_2^d$, regardless of p_k^i. Similarly, $p_1^d = p_2^d$ is also sub-optimal unless all consumers buy from Seller 1 in direct sales. However, this implies that Seller 2 has an incentive to undercut Seller 1 if it were the case.

Second, we come to verify the optimality of monopolist pricing by both sellers. The indifference condition is given by $y_k(1 - s_j/I_j) - p_k \geq (1 - c_k)y_k$, where $s_j = \|x - \theta_j\|$. We can solve $s_j = I_j(c_k - p_k/y_k)$ and thus $\Pi_k = 2s_j * p_k = 2I_j p_k(c_k - p_k/y_k)$, which further implies that $p_k^d = \frac{c_k y_k}{2}$ and $s_j = \frac{c_k I_j}{2}$. Furthermore, we need to verify that $\beta \geq s_1 + s_2 = \frac{c_1 I_1 + c_2 I_2}{2}$. Henceforth, $\Pi_{k,j} = \frac{c^2 y_k I_j}{2}$ for any match $(j, k(j))$.

Third, we turn to equilibrium matching. Given the joint profit $\Pi_{k,j}$ and the bilateral Nash bargaining protocol, $k(j) = j$ is stable because U_1 and w_1 are maximized and therefore they have no incentive to deviate. □

A.12 Proof of Lemma 2

Proof. Note that influencers are identical for sellers under balanced matching.

Case (i). Given that Seller 2 uses influencer-intermediated sales, it is optimal for Seller 1 to use influencer-intermediated sales when $\frac{\gamma y I}{2} \geq (2\pi - I)(1 - c)y$ is satisfied.

Case (ii). When $(2\pi - I)(1 - c)y > \frac{\gamma y I}{2}$ holds, there exists no pure strategy equilibrium. We consider a symmetric mixed strategy equilibrium as decribed in Lemma 2 . Given that Seller 2 follows the prescribed strategy, the indifference condition for Seller 1 to charge a direct-sale price p_1 requires:

$$q * (2\pi - I)p_1 + (1 - q) * \int_{p_1}^{(1-c)y} 2\pi p_1 f(p_2) dp_2 = \frac{1}{2}\gamma y I \tag{A.3}$$

Here, we used the fact that $(1 - c)y \leq \frac{1}{2}y$, that is, the maximum price under direct sales is smaller than the monopolist price under influencer-intermediated sales. This further implies that Seller 2 always sells to a measure of I consumers under influencer-intermediated sales.

By plugging $p_1 = (1 - c)y$ into equation (A.3), we get $q * (2\pi - I)(1 - c)y = \frac{1}{2}\gamma y I$. Then, by plugging $p_1 = \underline{P}$ into equation (A.3), we get $q * (2\pi - I)\underline{P} + (1 - q) * 2\pi\underline{P} = \frac{1}{2}\gamma y I$, which pins down the lower bound $\underline{P}$. Note that Seller 1 gets a payoff strictly less than $\frac{1}{2}\gamma y I$ when setting $P < \underline{P}$. Finally, rearranging equation (A.3) yields $q * (2\pi - I)p_1 + (1 - q) * 2\pi p_1(1 - F(p_1)) = \frac{1}{2}\gamma y I$, where $F(\cdot)$ is the CDF of $f(\cdot)$.

Differentiating with respect to p_1, we get $q*(2\pi-I)+(1-q)*2\pi(1-F(p_1))-(1-q)*2\pi p_1 f(p_1) = 0$. We can then multiply p_1 on both sides and get $f(p)$ after simple algebra manipulation. □

Online Appendix: Extended Discussions

This online appendix presents additional results on horizontal differentiation, welfare analyses of exclusivity contracting for more than two influencers, and the labor market implications of influencer competition.

OA.1 More Results on Horizontal Differentiation

Power difference and horizontal differentiation Here, we consider identical product quality indices with heterogeneity influencer power (that is, $y_1 = y_2$, $\widehat{I}_j = \widehat{I} \in [I_2, I_1]$ and $I_1 = \kappa I_2$ with $\kappa \geq 1$). We use influencers' identities to index sellers, and Seller j refers to the one who hires Influencer j. For $j \neq i$, we define $\Pi_j^C(\beta|y, I_1, I_2) = \frac{I_j(\beta+I_j+I_i)^2(I_j+2I_i)(4I_j+3I_i)^2 y}{(I_j+I_i)(8I_j^2+19I_jI_i+8I_i^2)^2}$. Similarly, denote $b = \beta/I_2$. Then, $\Pi_j^C(\beta|y, I_1, I_2) = H_j(b, \kappa) * yI_2$, where

$$G_1(b,\kappa) = \frac{\kappa(2+\kappa)(1+b+\kappa)^2(3+4\kappa)^2}{(1+\kappa)(8+19\kappa+8\kappa^2)^2}, G_2(b,\kappa) = \frac{\kappa(1+2\kappa)(1+b+\kappa)^2(4+3\kappa)^2}{(1+\kappa)(8+19\kappa+8\kappa^2)^2},$$

Furthermore, we define

$$d_1 = \frac{2(1+\kappa)}{9} \quad \text{and} \quad d_2 = \frac{4(1+\kappa)^3 + 5\kappa(1+\kappa)}{12(1+\kappa)^2 + \kappa}$$

Obviously, $d_2 > \frac{(1+\kappa)}{3} > d_1$.

Proposition A.1. *The influencers' style dispersion is a substitute for horizontal product differentiation. Consider* $\max\{\widehat{I}, I_1, I_2\} \leq \pi/2$. *When* $\frac{y\widehat{I}}{2} - H_2(d_2, \kappa) * yI_2 < f_E < \frac{y\widehat{I}}{2} - H_2(d_1, \kappa) * yI_2$ *holds, there exists* $\widetilde{\beta} \in (d_1 I_2, d_2 I_2)$ *such that:*

(i) Seller 2 has a stronger incentive to differentiate products. Specifically, when $\widetilde{\beta} \leq \beta \leq d_2 I_2$, *Seller 2 hires Influencer 2 and accepts his style location; otherwise, when* $d_1 I_2 \leq \beta < \widetilde{\beta}$, *she pays* f_E *and selects a location such that* $\beta^* \geq \frac{I_1+\widehat{I}}{2}$.

(ii) Seller 2 is more willing to differentiate horizontally when faced with a more powerful opponent influencer, that is, given I_2, *the cutoff* $\widetilde{\beta}$ *strictly increases in* I_1 *(or* $\frac{d\widetilde{\beta}}{dk} > 0$*).*

Perfectly symmetric case Here, we consider a perfectly symmetric case with identical product quality and identical influence power (that is, $y_1 = y_2 = y$ and $I_1 = I_2 = I$). We assume $I \leq \pi$. To guarantee equilibrium existence, we assume that $\beta \geq \beta_0$ and that

$$0 < f_E < \Pi_1^C(I) - \Pi_1^C(\beta_0), \tag{A.4}$$

where $\Pi_1^C(\beta)$ is defined in Example 3. Furthermore, the monotonicity of $\Pi_1^C(\beta)$ for $\beta \in (\beta_0, I)$ implies a unique $\tilde{\beta}$ such that $\Pi_1^C(\tilde{\beta}) = \Pi_1^C(I) - f_E$ holds. Denote by α_j the location of the style chosen by Seller $j \in \{1, 2\}$.

Proposition A.2 (Horizontal Product Differentiation). *The influencers' style dispersion is a substitute for horizontal product differentiation. Given (A.4), in equilibrium we have:*
(i) When $\beta \geq \tilde{\beta}$, each seller hires one influencer and accepts his style location.
(ii) When $\beta < \tilde{\beta}$, one seller hires an influencer directly while the other pays f_E and selects a location such that $\|\alpha_1 - \alpha_2\| \geq I$.

OA.2 The General Case with More Than Two Influencers

A natural question left unanswered in Section 6.2 is whether it is restrictive to only have two influencers? To address this issue, we consider the case with more than two influencers. Specifically, there are $J \geq 3$ influencers and $K = J$ sellers. Due to the entry cost to access influencers (not modeled here), each seller can hire at most $2 \leq m \leq J$ influencers under unbalanced matching. For ease of reference, denote $0 = \theta_1 \leq \theta_2 \leq \cdots \leq \theta_J \leq \theta_{J+1} = 2\pi$ and $\beta_{j,j+1} = \|\theta_j - \theta_{j+1}\|$.

Proposition A.3 (Exclusivity and welfare). *Fix $J = K \geq 3$ and $2 \leq m \leq J$.*

(i) Assume that $\beta_{j,j+1} = \beta < I$ for $j \in \{1, \cdots, J\}$ and $y_k = y$ for $k \in \{1, \cdots, K\}$. There exists a symmetric pure-strategy equilibrium under balanced matching, which weakly dominates all equilibriums under unbalanced matching in total welfare. Furthermore, if $p_i \neq p_j$ for some $i \neq j$ under unbalanced matching, balanced matching is strictly optimal;

(ii) When $y_k \neq y_l$ for some $k \neq l$ and $\beta_{j,j+1} \geq \frac{I_j + I_{j+1}}{2}$ for all $1 \leq j \leq J$, unbalanced matching strictly dominates balanced matching.

Proposition A.3 shows that the basic insights of Proposition 8 still hold. Specifically, when there is a limited supply of influencers and the quality of products differs, unbalanced matching is strictly dominant in total welfare, since products with a low quality are forced out of the market. In contrast, with homogeneous products, equally distanced influencers, and a crowded market for influencers, balanced matching always weakly dominates unbalanced matching because it can enhance competition among sellers. The intuition is that the symmetry in the quality of products and the features of influencers requires identical prices among sellers. Then, balanced matching is always the most natural way to go, although sometimes not the unique way, as illustrated by the examples below.

Example 5 ($K = J = 3$ and $m = 2$). *In this example, the unbalanced match outcome is inefficient. Denote $\beta = 2\pi/3$. Let $K = J = 3$, $m = 2$, $\theta_j = (j-1) * \beta$, and $I_j = I \in (\beta, 2\pi)$ for $j \in \{1, 2, 3\}$, and $y_k = y$ for all $k \in \{1, 2, 3\}$. Denote by $k(j)$ the identity of the matched seller k for Influencer $j \in \{1, 2, 3\}$. Consider the following match outcome $k(1) = k(2) = 1$ and $k(3) = 2$ (with Seller 3 not in the market). Without loss of generality, suppose that all consumers are served, and Seller 1 let Influencer 1 and 2 charge the same price, say p_1, and Seller 2 charges price p_2.*[32] *The indifference condition is given by $y(1 - s_1/I) - p_1 = y\,(1 - s_2/I)) - p_2$, where s_j is the measure of consumers*

[32]Here, we assume that all consumers are served and Influencer 1 and 2 charge the same price p_1. Specifically, the result that all consumers are served comes from $\beta < I$. Furthermore, if Influencer 1 and 2 charge different prices, say $p_1^1 > p_1^2$, then Seller 1 always wants to increase p_1^2 and decrease p_1^1 by a small amount $\epsilon > 0$, that is, $(\tilde{p}_1^1, \tilde{p}_1^2) = (p_1^1 - \varepsilon, p_1^2 + \varepsilon)$. The price change can always increase the profit for Seller 1 because

for Seller j when they compete. Thus, $s_1 = \frac{1}{2}(\beta + (p_2 - p_1)I/y)$ and $s_2 = \frac{1}{2}(\beta + (p_1 - p_2)I/y)$. Then, Seller 1 and Seller 2 receive the profits given by $\Pi_1 = (2s_1 + \beta) * p_1$ and $\Pi_2 = 2s_2 * p_2$. The necessary conditions are given by $\beta + (p_1^* - 2p_2^*) * I/y = 0$ and $2\beta + (p_2^* - 2p_1^*) * I/y = 0$. Obviously, $p_1^* \neq p_2^*$, and thus the equilibrium does not maximize total welfare.

Example 6 ($K = J = 4$ and $m = 2$)**.** *In this example, we construct efficient equilibrium outcomes under unbalanced matching. Denote $\beta = \pi/2$. Let $K = J = 4$ and $m = 2$, $\theta_j = (j-1) * \beta$ and $I_j = I \in (\beta, 2\pi)$ for $1 \leq j \leq 4$, and $y_k = y > 0$ for $1 \leq k \leq 4$. Again, use the notation $k(j)$. All equilibrium matching outcomes below maximize total welfare.*

- *Case 1. $k(1) = k(3) = 1$, $k(2) = 2$ and $k(4) = 3$.*
- *Case 2. $k(1) = k(3) = 1$ and $k(2) = k(4) = 2$.*
- *Case 3. $k(j) = j$ for $1 \leq j \leq 4$.*

OA.3 Influencer Competition and Labor Market

Here, we consider a setting with one seller and many influencers and study how influencer competition affects the monopolist seller's payoff, influencers' wages, and total welfare.

Residual multilateral bargaining protocol. We propose a residual bargaining protocol to handle the multilateral bargaining problem. Specifically, given the bargaining parameter $\gamma \in [0, 1]$, when a group of influencers are hired $\mathcal{J} \in \{1, \cdots, J\}$, the seller only bargains with influencer $j \in \mathcal{J}$ for the "residual profit gap" $\Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}}$,[33] which leads to

$$w_j = (1 - \gamma)(\Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}})_+, \quad \text{and} \quad U_1 = \Pi_{\mathcal{J}} - \sum_{j \in \mathcal{J}} w_j. \tag{A.5}$$

where $\Pi_{\mathcal{J}}$ is the monopolist profit when the seller hires the influencer group $\mathcal{J}$.[34]

For simplicity, we assume that the influencers are equally distanced on the unit circle, that is,

Influencer 2 always serves more consumers than Influencer 1 and thus the decrease in the profit of lowering p_1^1 is always smaller than the increase in the profit of increasing p_1^2 as long as $p_1^2 < p_1^1$ holds. The formal proof is skipped and available upon request.

[33]The seller only shares the residual profit because it is a credible threat that if influencer j rejects the offer, the seller switch to hiring the remaining group $\mathcal{J}/\{j\}$.

[34]The residual bargaining protocol has several desirable properties: (i) the allocation is unique and feasible because $\sum_{j \in \mathcal{J}} w_j + U_1 \leq \Pi_{\mathcal{J}}$; (ii) it is efficient because $\sum_{j \in \mathcal{J}} w_j + U_1 = \Pi_{\mathcal{J}}$; (iii) it cannot be blocked by a coalition of $S \subseteq \mathcal{J}$ for any $\gamma \in [0, 1]$ because $\sum_{j \in S} w_j + U_1 \geq \Pi_S$ and $\sum_{j \in S} w_j \geq 0$; and (iv) it is consistent with the bilateral Nash bargaining in the baseline model. This can also be generated from coalitional Nash bargaining (Compte and Jehiel, 2010). In essence, the residual bargaining protocol is an equilibrium refinement. Note that property (iii) is applicable to all reasonable refinements, because otherwise the allocation is blocked by excluding the jth influencer from the group $\mathcal{J}$, which further implies that $w_j \leq \Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}}$. In this sense, the residual bargaining protocol is the natural candidate that satisfies both property (iii) and property (iv).

$\|\theta_j, \theta_{j+1}\| = \frac{2\pi}{J}$ for all $j \in \{1, \cdots, J\}$.[35] Then, the total group profit for the monopolist seller and all influencers is given by:

$$\bar{\Pi}_J = \begin{cases} \frac{yIJ}{2}, & \text{if} \quad J \leq \lfloor \frac{2\pi}{I} \rfloor \\ 2\pi y \left(1 - \frac{\pi}{IJ}\right), & \text{if} \quad J > \lfloor \frac{2\pi}{I} \rfloor \end{cases} \tag{A.6}$$

Equation (A.6) follows from that all influencers are equally distanced and that the cutoff consumer indifferent between two neighboring influencers always receives a zero utility. By the residual multilateral bargaining protocol, Influencer j bargains with the seller over the residual profit $\left(\bar{\Pi}_J - \bar{\Pi}_{J-1}\right)$, and thus

$$w_j = (1-\gamma)\Big(\bar{\Pi}_J - \bar{\Pi}_{J-1}\Big) \text{ and } U_1 = \bar{\Pi}_J - J * w_1.$$

For any $J > \lfloor \frac{2\pi}{I} \rfloor$, it is easy to verify

$$\frac{\partial \left(\sum_{j=1}^{J} w_j\right)}{\partial J} = -\frac{2\pi^2(1-\gamma)y}{I(J-1)^2} < 0, \quad \text{and} \quad \frac{\partial U_1}{\partial J} = \frac{2\pi^2 y(J^2(1-\gamma)+(J-1)^2)}{IJ^2(J-1)^2} > 0.$$

We can summarize these results as follows.

Proposition A.4. *As the number of influencers increases (that is, $J \uparrow$) and is sufficiently large (that is, $J > \lfloor \frac{2\pi}{I} \rfloor$), the total wage for all influencers is strictly decreasing, and the payoff for the seller and the total welfare are strictly increasing.*

OA.4 Monopolist and Over-investment in Influence

Proposition A.5 below formalizes the discussion in Remark 3 on how over-investment in power acquisition can arise with a monopolist seller and two influencers.

Proposition A.5. *Fix $\beta = \pi$. Suppose $2\pi \leq I$ and $k > 2$. If $\frac{\pi^2 y}{I}\frac{(k-1)}{2(k+1)} < C_P < \frac{\pi^2 y}{I}\frac{(1-\gamma)(k-1)}{(k+1)}$, there exists an overinvestment in influence power. Instead, if $\frac{\pi^2 y}{I}\frac{(1-\gamma)(k-1)}{k(k+1)} < C_P < \frac{\pi^2 y}{I}\frac{(k-1)}{2k(k+1)}$, there exists an underinvestment in influence power.*

Therefore, depending on the congestion of the influencer market, as measured by β/I, the bargaining power γ, the cost and benefit of influence training, as measured by C_P and k, both overinvestment and underinvestment can arise in power acquisition under equilibrium.

OA.5 Macro- and Micro-Influencers

In practice, influencers include celebrities, macro-influencers (with a couple hundred thousand followers), micro-influencers (with 3,000-100,000 followers) and nano-influencers (with several hundred homogeneous followers), and nowadays micro-influencers are playing an increasingly important

[35] This assumption can be justified as follows. When the seller is restricted to hiring J influencers and can freely choose style locations for all influencers, then we can show that $\bar{\Pi}_J$ is achieved when all influencers are hired and are equally distant. Here, the skipped proof is available upon request.

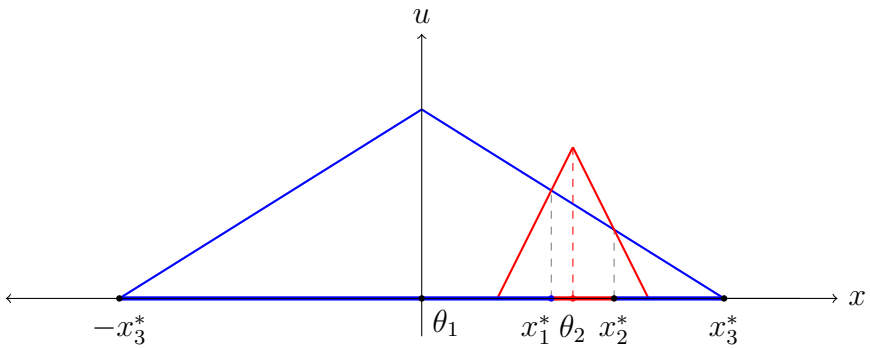


Figure OA.1: Macro-influencers and micro-influencers

role in marketing (Fainmesser and Galeotti, 2021). We analyze how both macro-influencers and micro-influencers can emerge in equilibrium. Assume $\theta_1 = 0$ and that θ_2 sits to the right of θ_1 along the short arc.

Lemma A.2. *When* $\beta < \min\left\{2\pi - I_1 - I_2, \frac{1}{3}(I_1 - I_2)\right\}$, $\exists\ 0 < x_1^* < x_2^* < x_3^* < \pi$ *such that: (i) Influencer 2 only serves consumers with* $x \in (x_2^*, x_3^*)$*; and (ii) Influencer 1 serves consumers with* $x \in [-x_3^*, x_1^*] \cup [x_2^*, x_3^*]$.

Here, $I_1 + I_2 < 2\pi - \beta$ rules out competition between influencers along the long arc, which is an artifact of the circular economy and thus not surprising. $\beta < \frac{1}{3}(I_1 - I_2)$ rules out the trivial outcomes that influencers are pure local monopolies.

Figure OA.1 illustrates two interesting patterns emerging from Lemma A.2. One, there exist two style cutoffs for which the cutoff consumer is indifferent between following either influencer. Specifically, the blue solid line is Influencer 1's consumer base and the red solid line is Influencer 2's, that is, Influencer 1 attracts consumers in $[0, x_1^*]$ (i.e., sufficiently loyal to influencer 1) and $[x_2^*, x_3*]$ (i.e., far away from Influencer 1 but not served by Influencer 2). Naturally, Influencer 1 appears as a macro-influencer, and Influencer 2 as a micro-influencer.

Two, Influencer 1 charges a monopolist price $p_1^* = \frac{y}{2}$ to serve a broad consumer base, while Influencer 2 charges a higher price $p_2^* = p_1^* + \frac{\beta}{2I_1}y$ to exploit local affinity. This is consistent with the trending phenomenon of the "rise of the micro-influencers" (Fainmesser and Galeotti, 2021), who remain credible in the eyes of their followers and charge more for the endorsements.

OA.6 Skipped Proofs

OA.6.1 Proof of Lemma A.1

Proof. First of all, when $\beta \geq I$ holds, the optimal pricing strategy is given by $p_j^* = p_2^* = \frac{y}{2}$ and each influencer serves consumers such that $\|x - \theta_j\| \leq \frac{I}{2}$ for $j = 1, 2$. Since there is no overlap in the consumer base, it yields $\Pi_{\{1,2\}} = 2\Pi_{\{1\}} = 2 * \frac{1}{2}yI = yI$.

Next, consider the case $\beta < I$. We show that in equilibrium we have: (i) consumers along the short arc between θ_1 and θ_2 are all served; (ii) both influencers get a positive share along the short arc; and (iii) $p_1 = p_2$.

First, to see (i), note that if a positive measure of consumers is not served, then $\|x-\theta_j\| < \frac{I}{2}$ is valid for some $j \in \{1,2\}$ because $\beta < I$, which implies that $p_j > I/2$. However, since $\Pi_j = p_j(1-p_j/y)I$, it is profitable to set $\hat{p}_j = p_j - \varepsilon$ for a sufficiently small $\varepsilon > 0$.

Second, to see (ii), note that if Influencer 2 is not active along the short arc, it must be $y(1-\beta/I)-p_1 \geq y-p_2$, which further implies that Influencer 2 is not actively serving any consumer, because for any consumer with $\|x-\theta_1\| = \|x-\theta_2\|+\beta$, we also have $y(1-\beta/I-\|x-\theta_2\|/I)-p_1 \geq y(1-\|x-\theta_2\|/I)-p_2$. However, this is sub-optimal because the seller's profit cannot exceed Π_1 in this case and it is dominated by the profit function proposed in the lemma.

Third, to see (iii), suppose not, assume that $p_1 < p_2$. The symmetric case $p_1 > p_2$ is proved similarly. Consider $(\hat{p}_1, \hat{p}_2) = (p_1+\varepsilon, p_2-\varepsilon)$ with $\varepsilon > 0$ sufficiently small, and we show that $(\hat{p}_1, \hat{p}_2)$ dominates (p_1, p_2). By i) and ii), all consumers between θ_1 and θ_2 on the short arc are served and denote by x^* the cutoff consumer type, and define $s_j = \|x^* - \theta_j\|$ for $j \in \{1,2\}$. Obviously, we have two conditions: $y(1-s_1/I)-p_1 = y(1-s_2/I)-p_2$ and $s_1+s_2 = \beta$. Solving them yields $s_j = \frac{1}{2}(\beta + I*(p_{3-j}-p_j)/y)$. Now, we can compare the profit under (p_1,p_2) and that under $(\hat{p}_1,\hat{p}_2)$. Specifically, $\Pi = p_1 s_1 + p_2 s_2 + p_1(1-p_1/y)I + p_2(1-p_2/y)I$ and $\hat{\Pi} = \hat{p}_1\hat{s}_1 + \hat{p}_2\hat{s}_2 + \hat{p}_1(1-\hat{p}_1/y)I + \hat{p}_2(1-\hat{p}_2/y)I$. Now, we can directly check $\hat{\Pi}-\Pi > 0$ as follows. Note that

$$\begin{aligned}\hat{p}_1(1-\hat{p}_1/y)I + \hat{p}_2(1-\hat{p}_2/y)I &= I*\left(p_1+p_2-(p_1+\varepsilon)^2/y-(p_2-\varepsilon)^2/y\right)\\ &= p_1+p_2-p_1^2/y-p_2^2/y+2(p_2-p_1)\varepsilon I/y+O(\varepsilon^2) > p_1(1-p_1/y)I+p_2(1-p_2/y)I\end{aligned}$$

Similarly, $\hat{p}_1\hat{s}_1 - p_1 s_1 = (p_1+\varepsilon)\left(s_1 - \frac{I\varepsilon}{y}\right) - p_1 s_1 = \varepsilon s_1 - \frac{I\varepsilon}{y}p_1$ and
$\hat{p}_2\hat{s}_2 - p_2 s_2 = (p_2-\varepsilon)\left(s_2 + \frac{I\varepsilon}{y}\right) - p_2 s_2 = -\varepsilon s_2 + \frac{I\varepsilon}{y}p_2$, which implies that

$$\hat{p}_1\hat{s}_1 + \hat{p}_2\hat{s}_2 - (p_1 s_1 + p_2 s_2) = (s_1-s_2)\varepsilon + (p_2-p_1)I\varepsilon/y > 0$$

Thus, $(\hat{p}_1,\hat{p}_2)$ is strictly dominant and the contradiction implies that $p_1 = p_2$.

Finally, we find the optimal pricing strategy and the profit function $\Pi_{\{1,2\}}$. By property (iii), we denote $p_1 = p_2 = p$ and the profit function is $\Pi = p*\beta + 2p(1-p/y)I$ as long as the indifferent type $\hat{x} = (\theta_1+\theta_2)/2$ gets a non-negative utility, that is , $y(1-\|x-\theta_1\|/I)-p \geq 0$. The unconstrained optimizer to Π is given by $p^\star = y\beta/(4I) + y/2$. Verification of the non-negative utility for type $\hat{x}$ yields $\beta \leq 2I/3$ and we get $\Pi_{\{1,2\}} = \frac{(2I+\beta)^2 y}{8I}$.

When $\beta \in (2I/3, I)$, $p^\star$ violates the nonnegative utility condition for type $\hat{x}$. However, note that $\Pi = p*\beta + 2p(1-p/y)I$ is strictly increasing for all $p \in (0, p^\star]$. Thus, the optimal price is given by the IR condition for the cut-off type $\hat{x}$, which yields $p^* = y*(1-\beta/(2I))$ and $\Pi_{\{1,2\}} = \frac{\beta*(2I-\beta)*y}{I}$. The proof concludes. □

OA.6.2 Proof of Proposition A.1

Proof. The proof consists of two parts.

Here, the sellers are identical and are indexed by influencers's identity j. We abuse the notation

and use k for κ within this proof.

Part (i). Calculate the equilibrium profits before differentiation.

We construct an equilibrium such that the equilibrium prices are characterized by the F.O.C.s of the profit functions. We first derive equilibrium prices and profits using the F.O.C.s and then verify the equilibrium. Note that matching is trivial because $y_1 = y_2$.

First, consider a pair of (p_1, p_2) such that no seller dominates (to be verified later). Then we have an indifference condition along the short arc whose length is β, that is, $y(1 - s_1/I_1) - p_1 = y(1 - s_2/I_2) - p_2$, where s_j is Influencer j's market share along the short arc, whose identity is used to differentiate the two identical sellers. Note that s_1 can be larger than β or negative, depending on the prices p_j. By imposing $s_2 = \beta - s_1$, we can solve $s_1 = \frac{I_1(\beta y + I_2(p_2 - p_1))}{y(I_1+I_2)}$ and $s_2 = \beta - s_1$. We can further set up the profit $\Pi_j^C = (s_j + (1 - p_j/y)I_j) * p_j$ and get the FOC's as below:

$$\frac{I_j(y\beta + I_j(y - 2p_j) + I_{i\neq j}(-4p_j + p_i + y))}{y(I_1 + I_2)} = 0$$

Solving the FOC's yields $p_j^* = \frac{y(\beta+I_1+I_2)(4I_j+3I_{i\neq j})}{8I_1^2+19I_1I_2+8I_2^2}$ and $\Pi_j^C = \frac{y*I_j(\beta+I_1+I_2)^2(I_j+2I_i)(4I_j+3I_i)^2}{(I_1+I_2)(8I_1^2+19I_1I_2+8I_2^2)}$.

Second, we verify the equilibrium payoffs shown above under the assumed conditions, including: (1) All consumers are served along the short arc, that is, $I_1(1 - p_1^*/y) + I_1(1 - p_1^*/y) \geq \beta$, which yields

$$\beta \leq \frac{4I_1^3 + 17I_1^2I_2 + 17I_1I_2^2 + 4I_2^3}{12I_1^2 + 25I_1I_2 + 12I_2^2} = cI_2$$

Note that $I_1 + I_2 \leq \pi$ implies no competition along the long arc; and (2) Sellers have no incentive to undercut and force opponents out of the product market. To force Seller 2 to quit, Seller 1 with influence power I_1 could set $\widehat{p}_1$ that satisfies $y(1 - \beta/I_1) - \widehat{p}_1 = y - p_2^*$, that is, $\widehat{p}_1 = p_2^* - \beta y/I_1$. Meanwhile, Seller 2 with power I_2 needs to induce the farthest consumer of Seller 1 to switch, that is, $y(1 - (\beta + t_1)/I_2) - \widehat{p}_2 = 0$ where $t_1 = I_1(1 - p_1^*)/y)$. These deviations lead to profits given by $\widehat{\Pi}_j = 2I_j\widehat{p}_j(1 - \widehat{p}_j/y)$. Note that these are the most profitable deviations since decreasing prices further only lowers profits, and thus the nonexistence of profitable deviations requires $\Pi_j^C \geq \widehat{\Pi}_j$ for $j = 1, 2$. In particular,

$$\Pi_1^C - \widehat{\Pi}_1 = \frac{yI_2G_1(\beta)}{k(1+k)(8+19k+8k^2)^2} \quad \text{and} \quad \Pi_2^C - \widehat{\Pi}_2 = \frac{yI_2G_2(\beta)}{k(1+k)(8+19k+8k^2)^2}$$

where

$$\begin{aligned} G_1(\beta) = {} & -7k^2(1+k)^2(2+9k+8k^2+2k^3) + 2\beta k^2(58+206k+259k^2+137k^3+26k^4) \\ & + \beta^2(128+608k+1108k^2+967k^3+406k^4+66k^5) \\ G_2(\beta) = {} & \beta^2(1+2k)(12+25k+12k^2)^2 + (1+k)^2(16+8k-103k^2-116k^3+16k^4+32k^5) \\ & + 2\beta(-48-272k-501k^2-219k^3+306k^4+344k^5+96k^6) \end{aligned}$$

We can verify that $G_1(\beta)$ and $G_2(\beta)$ are both strictly increasing for $\beta > 2(1+k)I_2/9$ and both are

strictly positive for $\beta \geq 2(1+k)I_2/9$.[36] Thus, the pricing strategies and profits characterzied by the F.O.C.s are optimal.

Part (ii). Verify the optimality of Sellers' differentiation choices.

We first verify that Seller 2 has a stronger incentive to differentiate. Note that

$$\Pi_1^C(\beta|y, I_1, I_2) - \Pi_2^C(\beta|y, I_1, I_2) = yI_2 * (H_1(b,k) - H_2(b,k)) = \frac{2yI_2(k-1)(1+b+k)^2}{8+19k+8k^2} \geq 0$$

because $k \geq 1$. Thus, since the return from horizontal differentiation is fixed at $\frac{y\widehat{I}}{2}$ for both sellers, Seller 2 benefits more $\Pi_2^C(\beta|y, I_1, I_2) \leq \Pi_1^C(\beta|y, I_1, I_2)$.

Second, by the continuity and monotonicity of $H_2(b,k)$, which strictly increases with b, and by the assumed condition that $\frac{y\widehat{I}}{2} - H_2(d_2,k) * yI_2 < f_E < \frac{y\widehat{I}}{2} - H_2(d_1,k) * yI_2$, there exists a unique $\widetilde{\beta} \in (d_1, d_2)$ satisfying $\frac{y\widehat{I}}{2} - H_2(\widetilde{\beta},k) * yI_2 = f_E$. Thus, for $\beta \in [d_1, \widehat{\beta})$, Seller 2 engages in horizontal differentiation; otherwise for $\beta \in [\widehat{\beta}, d_2]$, Seller 2 accepts Influencer 2's style location.

Third, we can find the partial derivative of $\frac{\partial\widehat{\beta}}{\partial k}$ by using the implicit function differentiation for the function $F(\widetilde{\beta}, k) = \frac{y\widehat{I}}{2} - H_2(\widetilde{\beta},k) * yI_2 - f_E$, that is,

$$\frac{\partial\widetilde{\beta}}{\partial k} = -\frac{\partial F/\partial k}{\partial F/\partial\widetilde{\beta}} = -\frac{\partial H_2/\partial k}{\partial H_2/\partial\widetilde{\beta}}$$

Note that $\partial H_2/\partial k = -\frac{yI_2G_3(b,k)}{(1+k)^2(8+19k+8k^2)^3}$, where

$$G_3(b,k) = 4b(1+k)^2(20+68k+63k^2+24k^3) - (1+k)^2(-8-12k+27k^2+28k^3) + b^2(72+340k+551k^2+376k^3+96k^4)$$

Note that $G_3(b,k)$ is strictly increasing for all $b > 0$, and thus $G_3(b,k) > G(2(1+k)/9, k)$, where $G(2(1+k)/9,k) = \frac{11}{81}(1+k)^2(216+788k+859k^2+500k^3+192k^4) > 0$. Thus, $\partial H_2/\partial k < 0$ and $\partial H_2/\partial b > 0$ together imply that $\frac{\partial\widetilde{\beta}}{\partial k} > 0$. The proof concludes. □

OA.6.3 Proof of Proposition A.2

Proof. First, by the formula of $\Pi_1^C(\beta) := \Pi_1^C(\beta, y)$ (see Example 3), we can check that $\Pi_1^C(\beta)$ is strictly increasing for all $\beta \in \big(\beta_0, I\big]$, which implies that the term, $G(\beta) := \Pi_1^C(\beta) - \big(\Pi_1^C(I) - f_E\big)$, has at most one solution for all $\beta \in \big(\beta_0, I\big]$. In fact, note that $G(I) = f_E$ and $G(\beta_0) < 0$ for $f_E < \Pi_1^C(I, y_2) - \Pi_1^C(\beta, y_2)$. Therefore, there exists a unique and well-defined $\tilde{\beta}$. Furthermore, by monotonicity, we have $G(\beta) > 0$ for $\beta > \tilde{\beta}$ and $G(\beta) < 0$ for $\beta \in [\beta_0, \tilde{\beta})$.

Second, consider $\beta > \tilde{\beta}$. By symmetry, we only need to consider the incentive for Seller 1. Given that Seller 2 hires Influencer 2, Seller 1 can choose to hire Influencer 1 and receive a payoff of $\gamma\Pi_1^C(\beta)$, or pay a fixed cost and select a location such that $\|\alpha_1 - \theta_2\| \geq I$), which yields a payoff of $\gamma(\Pi_1^C(I) - f_E)$. However, note that $G(\beta) > 0$ for $\beta > \tilde{\beta}$, or equivalently, $\gamma\Pi_1^C(\beta) \geq \gamma(\Pi_1^C(I) - f_E)$. Thus, it is optimal for Seller 1 to hire Influencer 1.

[36] We skip the algebra verification here, which is available upon request.

Third, consider $\beta \leq \tilde{\beta}$. We show that there exists an equilibrium in which Seller 1 hires Influencer 1 and Seller 2 pays the fixed cost f_E. Seller 2, given that Seller 1 hires Influencer 1, can choose to hire Influencer 2 and obtain a payoff of $\gamma\Pi_1^C(\beta)$, or pay the fixed cost f_E and select a location to avoid competition and get $\gamma(\Pi_1^C(I) - f_E)$. By the fact $G(\beta) < 0$ for $\beta \leq \tilde{\beta}$, she has an incentive to pay the fixed cost f_E. Furthermore, given that Seller 2 chooses the maximum differentiation, Seller 1 has the incentive to hire Influencer 1 and makes a profit of $\gamma\Pi_1^C(I)$. The proof concludes. □

OA.6.4 Proof of Proposition A.3

Proof. **Case (i).** First, note that total welfare is maximized only when $p_j = p \leq y(1 - \frac{\beta}{2I})$ for any j. Otherwise, if $p_i \neq p_j$ for some $i \neq j$, then the total welfare is not maximized. To see it, note that if $p_i \neq p_j$, then there exists some l such that $p_l \neq p_{l+1}$. Consider any consumer with $x \in (\theta_l, \theta_{l+1})$, the total surplus $u(x, y) = y\left(1 - \frac{\min\{\|x-\theta_l\|, \|x-\theta_{l+1}\|\}}{I}\right)$ is maximized when she purchases a unit of good from Influencer l whenever $\|x - \theta_j\| \leq \frac{I}{2}$ and from Influencer $l+1$ when $\|x - \theta_{l+1}\| < \frac{I}{2}$. In particular, if $p_l \neq p_{l+1}$, say $p_l > p_{l+1}$, then all consumers with $x \in (\theta_l + I/2 - \delta, \theta_l + I/2]$ will buy from Influencer $l+1$. Instead, they should buy from Influencer l to maximize total welfare. Furthermore, the condition that $p_l \leq y(1-\beta/(2I))$ ensures that all consumers can afford to purchase the product.

Second, we show that there exists a symmetric pure strategy equilibrium such that $p_j = p \leq y(1 - \beta/2I)$ for all j under balanced matching. The construction is similar to that of Example 3. Consider p_1 and fix $p_2 = p_{J-1} = p^*$. The indifference condition requires

$$y(1 - s_1/I) - p_1 = y(1 - (\beta - s_1)/I) - p^*$$

where $2s_1$ are the measure of consumers for influencer 1. Solving it yields $s_1 = \frac{1}{2}\beta + \frac{I}{2y}(p^* - p_1)$ and the total profit is $\pi_1 = 2p_1 s_1$. We can find the derivative and impose $p_1 = p^*$ to solve $p^* = \frac{\beta y}{I}$. Note that an implicit restriction is $y(1 - \frac{\beta}{2I}) - p^* \geq 0$, or equivalently $\beta \leq \frac{2}{3}I$. For $\beta \in (2I/3, I)$, the consumer with $x = \theta_l + \frac{I}{2}$ receives a negative utility. By symmetry, a natural conjecture is $p^* = (1 - \beta/2I)y$. For $p > p^*$, the profit is given by $2I(1 - p_1/y)p_1$, which means that the seller has no incentive to raise the price since $p^* = (1 - \beta/2I)y > \frac{y}{2}$. On the other hand, when $p < p^*$, the profit is given by $\pi_1 = 2p_1 s_1$, and $\frac{\partial \pi_1}{\partial p_1}|_{p_1 = p^*} > 0$. Last, we also need to rule out the case where Seller 1 has no incentive to deviate downward by a big price cut to fully undercut Seller 2. However, this is not viable because $p_1 < p^* - y\beta/I$, which is less than or equal to zero. This verifies the equilibrium.

Third, by the argument in Step 1, the symmetric pure-strategy equilibrium under balanced matching always maximizes total welfare, and thus always weakly dominates any equilibrium under unbalanced matching. Furthermore, when $p_i \neq p_j$ under unbalanced matching, it is strictly dominated by balanced matching, again by Step 1.

Case (ii). The condition $\beta_{j,j+1} \geq \frac{I_j + I_{j+1}}{2}$ implies sellers use monopolist pricing through influencer-intermediated sales. Therefore, only consumers whose location type x satisfies $\|x -$

$\theta_j\| \leq \frac{I_j}{2}$ buy the product from the influencer j and their total well-being depends only on the quality of the product. Furthermore, if $p_k \neq p_l$ for $k \neq l$, all sellers can survive under balanced matching. In contrast, under unbalanced matching, only sellers whose product quality ranks in the top $([J/m]+1)$ -th can survive, where $[z]$ returns the largest integer smaller than or equal to z. The proof concludes. □

OA.6.5 Proof of Proposition A.5

We first present an auxiliary result, Lemma A.3.

Lemma A.3. *Fix $\beta = \pi$. Then, if $I_1 + I_2 \geq 2\pi$, then $p_1^* = p_2^* = y(1 - \frac{\pi}{I_1+I_2})$, $s_1 = \pi - s_2 = \frac{I_1\pi}{I_1+I_2}$ and $\Pi_{\{1,2\}} = 2\pi y(1 - \pi/(I_1 + I_2))$.*

Proof. First, consider the case $I_1 + I_2 \leq 2\pi$. The monopolist seller sets prices $p_1^* = p_2^* = \frac{y}{2}$ and Influencer j attracts consumers with $x \in \mathbf{S}^1$ such that $\|x - \theta_j\| \leq \frac{I_j}{2}$ for $j = 1, 2$. Note that there exists no overlap in consumers served since $\frac{I_1}{2} + \frac{I_2}{2} \leq \pi$. Thus, $\Pi_{\{1,2\}} = \Pi_{\{1\}} + \Pi_{\{2\}} = \frac{y(I_1+I_2)}{2}$.

Second, consider the case $I_1 + I_2 > 2\pi$. Note that both influencers attract a positive share of consumers. If not, then assume that Influencer 2 is not serving consumers (and the other case is identical). For $p_1 \geq 0$, we can always charge a slightly higher price for all consumers sufficiently close to θ_2. Meanwhile, suppose that Influencer 1 serves consumers $x \in \mathbf{S}^1$ such that $\|x - \theta_1\| \leq s_1$ with $s_1 \in [0, \pi)$. Prices p_1 and p_2 are set such that the indifferent consumer is marginal (i.e., the type x such that $\|x - \theta_1\| = s_1$ gets a zero payoff), otherwise we can increase p_j at least for one influencer. Therefore, $p_1 = y\,(1 - s_1/I_1)$ and $p_2 = y\,(1 - (\pi - s_1)/I_2)$. Then, the total revenue can be written as a function of s_1 as $\Pi_{\{1,2\}} = 2p_1 * s_1 + 2p_2 * (\pi - s_1)$. Maximizing the joint revenue function, it yields $p_1^* = p_2^* = y * (1 - \pi/(I_1 + I_2))$, $s_1 = \pi I_1/(I_1 + I_2)$ and $s_2 = \pi I_2/(I_1 + I_2)$, which implies $\Pi_{\{1,2\}} = 2\pi y\left(1 - \frac{\pi}{I_1+I_2}\right)$ when $I_1 + I_2 > 2\pi$. The proof concludes. □

Then, we are ready to prove Proposition A.5.

Proof. First, note that we impose $I \geq 2\pi$ to simplify the calculations. Here, we construct an equilibrium in which only one influencer, say Influencer 1, acquires power and increases his power from I to kI. Under the conjectured equilibrium, for Influencer 1, <u>before power acquisition</u>, by equation (2), we have $\Pi_2^b = 2\pi y(1 - \pi/I)$. Meanwhile, by Lemma A.3, $\Pi_{\{1,2\}}^b = 2\pi y(1 - \pi/2I)$ and by bilateral Nash bargaining, $w_1^b = (1 - \gamma)(\Pi_{\{1,2\}}^b - \Pi_2^b) = (1 - \gamma)\pi^2 y/I$.

Second, <u>after power acquisition</u>, again by Lemma A.3, $\Pi_{\{1,2\}}^a = 2\pi y(1 - \frac{\pi}{(1+k)I})$, and thus

$$w_1^a = (1 - \gamma)(\Pi_{\{1,2\}}^a - \Pi_2^a) = (1 - \gamma)\left(2\pi y(1 - \frac{\pi}{(1 + k)I}) - 2\pi y(1 - \frac{\pi}{I})\right) = \frac{2\pi^2 y}{I}\frac{(1 - \gamma)k}{k + 1}.$$

This further implies that $\Delta w_1 = w_1^a - w_1^b = \frac{\pi^2 y}{I}\frac{(1-\gamma)(k-1)}{(k+1)}$.

Third, given that $I_1 = kI$, for Influencer 2, <u>before power acquisition</u>,

$$w_2^b = (1 - \gamma)\,(2\pi y(1 - \pi/(I + kI)) - 2\pi y(1 - \pi/(kI))) = \frac{2\pi^2 y}{I}\frac{(1 - \gamma)}{k(k + 1)}$$

Instead, given that Influencer 1 acquires power, if Influencer 2 acquires additional power, then $\hat{\Pi}_{\{1,2\}} = 2\pi y(1 - \frac{\pi}{2kI})$, $\hat{\Pi}_j = 2\pi y(1 - \frac{\pi}{kI})$, which implies

$$w_2^a = (1-\gamma)(\hat{\Pi}_{\{1,2\}} - \hat{\Pi}_1) = (1-\gamma)2\pi y\left(\frac{\pi}{kI} - \frac{\pi}{2kI}\right) = \frac{\pi^2 y}{I}\frac{(1-\gamma)}{k}.$$

and thus $\Delta w_2 = w_2^a - w_2^b = \frac{\pi^2 y}{I}\frac{(1-\gamma)}{k} - \frac{2\pi^2 y}{I}\frac{(1-\gamma)}{k(k+1)} = \frac{\pi^2 y}{I}\frac{(1-\gamma)(k-1)}{k(k+1)}$.

In short, in equilibrium, if $\Delta w_2 < C_P < \Delta w_1$, only Influencer 1 acquires additional influence power. Instead, if $C_P \leq \Delta w_2$, both influencers invest in power acquisition.

Fourth, we can compute the total welfare change when only Influencer 1 acquires additional power. By Lemma A.3, $p_1^* = p_2^* = y(1 - \frac{\pi}{2I})$, $s_1 = s_2 = \frac{\pi}{2}$, which implies that $SW^b = 4 \times \int_0^{\frac{\pi}{2}} y(1 - s/I)dx = 2 * \pi/2 * (y + y(1 - \pi/(2I))) = \pi y(2 - \pi/(2I))$.

When Influencer 1 acquires additional power, $p_1^* = p_2^* = y(1 - \frac{\pi}{(1+k)I}$, $s_1 = \pi - s_2 = \frac{k\pi}{(1+k)}$, and thus $SW^a = 2\int_0^{\frac{k\pi}{(1+k)}} y(1 - s/(kI))dx + 2\int_0^{\frac{\pi}{(1+k)}} y(1 - s/(I))dx = \pi y(2 - \pi/(I + kI))$, which implies that $\Delta SW = SW^a - SW^b = \frac{\pi^2 y}{I}\frac{(k-1)}{2(k+1)}$. Similarly, we can calculate the total welfare change by comparing the case in which only one influencer acquires power with that in which both influencers do so, which yields $\widehat{\Delta SW} = \frac{\pi^2 y}{I}\frac{(k-1)}{2k(k+1)}$.

Finally, note that when $k > 2$, $\Delta w_2 < \Delta SW$. Thus, if $\Delta SW < C_P < \Delta w_1$, it is sub-optimal for Influencer 1 to acquire power in total welfare, but only Influencer 1 to invest is indeed an equilibrium. Similarly, if $\Delta w_2 < C_P < \widehat{\Delta SW}$, it is socially optimal for both influencers to invest, but only Influencer 1 acquires additional power. □

OA.6.6 Proof of Lemma A.2

Proof. First, note that $I_1 + I_2 < 2\pi - \beta$ implies no competition along the long arc between the two influencers. Meanwhile, $\beta < \frac{1}{3}(I_1 - I_2)$ implies that $\beta < \frac{1}{2}(I_1 + I_2)$ and thus influencers compete along the short arc. This is because, if there exists a positive measure of consumers not served along the short arc, it must be case that $p_j < \frac{y}{2}$, the monopolist price, and thus it is profitable to increase the price by a small amount. Given these two observations, there exists three cases to consider: i) there exists only one cutoff type consumer, who is indifferent in purchasing between the two influencers, and receives a positive utility; ii) there exists only one cutoff type consumer who receives a zero utility; and iii) there exist two cutoff types indifferent in purchasing from either influencer.

Second, we start from case i) and identify conditions under which case iii) occurs. If there exists only one cutoff type consumer, the indifference condition is defined as equating the utility from purchasing from the two influencers, that is, $u_1 = u_2$, where $u_1 = y(1 - s_1/I_1) - p_1$ and $u_2 = y(1 - s_2/I_2) - p_2$. Here, s_j is the size of consumers served by influencer $j = 1, 2$ along the short arc. This implies that $s_1 = \frac{I_1(I2(p_2-p_1)+\beta y)}{(I_1+I_2)y}$ and $s_2 = \beta - s_1$. Next, we can compute the total profit $W_1 = p_1 * I_1(1 - p_1/y) + p_1 s_1 + p_2 s_2 + p_2 * I_2(1 - p_2/y)$. We can compute partial derivatives $\frac{\partial W_1}{\partial p_1} = \frac{I_1(\beta y + I_1(-2p_1+y) + I_2(-4p_1+2p_2+y))}{(I_1+I_2)y}$ and $\frac{\partial W_1}{\partial p_2} = \frac{I_2(\beta y + I_2(-2p_2+y) + I_1(-4p_2+2p_1+y))}{(I_1+I_2)y}$. It is easy to check the second-order conditions and thus we can use first-order conditions to jointly

solve that $p_1^* = p_2^* = \frac{y}{2} + \frac{\beta y}{2(I_1+I_2)}$ and thus $u_1 = \frac{(I_1+I_2-3\beta)y}{2(I_1+I_2)}$. We can further compute the total profit $W_1 = \frac{(\beta+I_1+I_2)^2 y}{4(I_1+I_2)}$.

Note that when $\beta < \frac{1}{3}(I_1+I_2)$, $u_1 < 0$ and case ii) ensues. In this case, we can set s_1 to be the consumer size for influencer 1 along the short arc, and then express $p_1 = y(1-s_1/I_1)$ and $p_2 = y(1-(\beta-s_1)/I_2)$ and further express the total profit as a function of s_1 and optimize over s_1 to get the optimal solution. The optimal price $p_j^* = y - \frac{\beta y}{I_1+I_2}$ and $W_2 = \frac{2\beta y((I_1+I_2)-\beta)}{(I_1+I_2)}$. Moreover, we can find another cutoff on the right side of θ_2 (recall that $\theta_1 = 0$ and θ_2 sits to the right of θ_1) by equating $\hat{u}_1 = y(1-(\beta+\hat{s}_2)/I_1) - p_1$ and $\hat{u}_2 = y(1-\hat{s}_2/I_2) - p_2$, where $\hat{s}_2$ is the distance between the far right cutoff location and θ_2. We can solve it to get $\hat{s}_2 = \frac{I_2(I_1(p_1-p_2)+\beta y)}{(I_1-I_2)y}$. Hence, we can plug in p_j^* to get $\hat{s}_2 = \frac{\beta I_2}{I_1-I_2}$. Meanwhile, note that the distance to influencer 1 from the potential consumer under price p_1^* on the right side is $\hat{s}_1 = I_1(1-p_1^*/y)$. Hence, case ii) only occurs when $\hat{s}_1 \leq \beta + \hat{s}_2$. In other words, case iii) occurs when $\hat{s}_1 > \beta + \hat{s}_2$, or equivalently, $\hat{s}_1 - \hat{s}_2 - \beta = \frac{1}{2}I_1(1-\beta(3I_1+I_2)/(I_1^2-I_2^2)) \geq 0$. We can further simplify it as $\beta < \frac{I_1^2-I_2^2}{(3I_1+I_2)} \leq \frac{(I_1-I_2)}{3}$. The proof for the lemma concludes here.

Finally, we can compute the equilibrium prices as follows. Since we already know the two cutoffs, s_1 and $\hat{s}_2$, we can define $W_3 = p_1 * 2I_1(1-p_1/y) + p_2(\beta - s_1 + \hat{s}_2) - p_1(\beta - s_1 + \hat{s}_2)$, the total profit in case iii). We can apply the F.O.C.s to solve it and get $p_1^* = \frac{y}{2}$ and $p_2^* = \frac{y}{2} + \frac{\beta y}{2I_1}$. We can further compute $\hat{s}_1 = \frac{I_1}{2}$ and $\hat{s}_2 = \frac{\beta I_2}{2(I_1-I_2)}$. Indeed, $\hat{s}_1 > \beta + \hat{s}_2$ holds when $\beta < \frac{(I_1-I_2)}{3}$. In this case, $W_3 = \frac{yI_1}{2} + \frac{\beta^2 I_2}{2(I_1^2-I_2^2)}$. □