AUCTIONS WITH WITHDRAWAL RIGHTS: A FOUNDATION FOR UNIFORM PRICE

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Abstract. When bidders in an auction have correlated private information, optimal mechanisms screen bidders based on their beliefs. By contrast, this paper shows that when bidders can withdraw from an auction *ex post*, uniform price auctions are exactly optimal in a large-market limit, and approximately optimal in finite markets. These results illustrate that withdrawal rights provides a realistic framework for studying optimal auction design with correlated information.

1. Introduction

A critical assumption in much of mechanism design theory is that private information is independent across agents. Without this assumption, optimal mechanisms generally screen agents based on their beliefs by requiring them to make side bets about others' reports in order to participate in the mechanism.[1]

In practice, however, the independence assumption is unlikely to hold, yet mechanisms that involve screening via side bets are rarely used. Instead, goods are often sold using simple mechanisms such as second price or uniform price auctions. This disconnect between optimal and practical mechanisms poses a concern for the standard Bayesian mechanism design approach to optimal auctions.

This paper proposes a foundation for uniform price auctions based on a natural restriction on the seller's ability to enforce bidders' commitments. In particular, suppose that bidders can withdraw from an auction *ex post*—i.e., after learning whether they won and how much they are supposed to pay. In this case, we show that uniform price auctions are optimal even when bidders' private information is correlated. Our optimality result holds exactly in a large market limit with many bidders and units

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[1]See, e.g., Crémer and McLean (1985, 1988), McAfee and Reny (1992), and Neeman (2004).

of a good for sale (under appropriate regularity assumptions). It also holds approximately in finite markets, where the uniform price auction corresponds to an $(M+1)$th price auction if there are M units of the good for sale. These results suggest that considering withdrawal rights provides a practical framework for studying optimal auction design without the assumption of independent private information.

Our motivation for considering withdrawal rights is two-fold. The first motivation comes from sales of major assets by auction. In such auctions, the winner often has to conduct due diligence to verify that the seller's representations regarding the asset are correct. This verification is typically done *ex post* as it is costly. Importantly, any inaccuracy in the seller's representations provides an opportunity for the winner to withdraw. For complex assets, small inaccuracies will always be present in practice, so the due diligence process provides a implicit channel for buyers to withdraw *ex post*.

A second motivation for withdrawal rights comes from consumers' ability to make returns. Sellers are often legally required to accept returns during a cooling-off period. For example, for internet sales (and, more generally, other sales that occur away from store premises), sellers in the European Union are obliged by the Consumer Rights Directive 2011 to accept returns for 14 days (see, e.g., Krähmer and Strausz (2015)).

While withdrawal rights arise commonly in practice, they turn out to restrict selling mechanisms in a fairly subtle way in theory. One implication of withdrawal rights is that the seller can only implement allocations that are *ex post* individually rational. However, withdrawal rights also open the possibility of withdrawing from the mechanism after a misreport: as Compte and Jehiel (2009) wrote, "When considering a deviation, an agent takes into account the fact that he can opt out if things turn out badly."[2] Most practical selling mechanisms—including second/uniform price auctions and first/discriminatory price auctions, as well as mixed-price auctions and Spanish auctions—are not only *ex post* individually rational, but also implementable (with standard equilibrium strategies) in the presence of withdrawal rights. Nevertheless, the strategic impact of withdrawal rights makes the resulting mechanism design problems fundamentally different from ones with *ex post* individual rationality constraints.

Our analysis of auctions with correlated information and withdrawal rights focuses on a large-market limit. For sake of tractability, in the main part of our analysis, we model large markets by supposing that there is a continuum of (unit-demand) bidders. Following Jackson and Kremer (2006), we suppose that there are enough units of a homogeneous, indivisible good to allocate to k fraction of the bidders.

[2]See also, e.g., Matthews and Postlewaite (1989), Forges (1999), and Compte and Jehiel (2007).

To isolate the role of withdrawal rights in disciplining optimal mechanisms separately from their interaction with the winner's curse, we assume that values are private. Values are i.i.d. conditional on an (aggregate) state, which is unknown to all agents; types are hence correlated through the state. We consider Bayesian mechanisms to allow, in principle, for mechanisms that depend on or elicit beliefs.

The model features a role for auction design despite the assumption that the market is large. Indeed, a uniform price auction delivers strictly higher expected revenue than a discriminatory price auction even in the large-market limit (Jackson and Kremer, 2006). In particular, this revenue difference arises even for *ex post* efficient mechanisms that are implementable with withdrawal rights—highlighting the role for the design of payment rules that results from the correlation among bidders' private information.

To isolate the impact of withdrawal rights on the payment rules of optimal mechanisms, we restrict the seller to mechanisms that implement an *ex post* efficient allocation of the good. Our main result shows that in this setting, under a regularity condition on the correlation structure, uniform price auctions maximize expected revenue over all auction mechanisms when bidders can withdraw *ex post* (Section 4).

Due to the assumptions of private values and unit-demand, uniform price auctions are dominant strategy incentive compatible and *ex post* individually rational. Thus, our result implies that when bidders can withdraw *ex post* and the seller is restricted to efficient mechanisms, there is no gain in expected revenue from using more complicated mechanisms whose design or bidding behavior depends on bidders' beliefs or their relationship with values. Additionally, the optimal efficient mechanism is prior-free—as in the independent private values model (Myerson, 1981).[3]

A simplistic intuition behind our main result is that side bets are not *ex post* individually rational, and therefore cannot be enforced when bidders can withdraw *ex post*. However, this intuition is not correct because *ex post* individually rationality does permit the seller to enforce bets of size up to a bidder's *ex post* surplus. In fact, with correlation, it turns out there are *ex post* individually rational mechanisms that deliver higher strictly expected revenue than uniform price auctions.[4]

Our analysis instead relies on more detailed understanding of the strategic implications of *ex post* participation constraints, building on Matthews and Postlewaite

[3]In the independent private values model, considering withdrawal rights does not fundamentally affect the seller's problem: it only selects among optimal mechanisms (with or without an *ex post* efficiency condition).

[4]This conclusion was shown by Lopomo (2001) in a finite-market context with two bidders and one unit of the good for sale. We show the conclusion in our setting in Section 2.

(1989), Forges (1999), and Compte and Jehiel (2007, 2009).[5] It turns out that the optimal *ex post* individually rational mechanism is not incentive compatible when bidders can withdraw *ex post*: it is vulnerable to double deviations involving the combination of a misreport and a withdrawal. Intuitively, withdrawal rights impose *ex post* individual rationality not only on the equilibrium path, but also consistently after misreports—i.e., off path in a sense (Compte and Jehiel, 2009).

For *ex post* individually rational mechanisms in general, the only deviations after which *ex post* withdrawal may be attractive is for upward deviations. It turns out that the key additional constraints entailed by withdrawal rights are non-local upward incentive constraints. In particular, the standard first-order approach is inapplicable in auction design problems, like ours, in which withdrawal rights play a role beyond *ex post* individual rationality: dropping the non-local upward incentive constraints would lead to a solution that is vulnerable to double deviations.

To prove the optimality of uniform price auctions, we consider an alternative relaxed problem that includes not only the local Bayesian incentive constraints and *ex post* individual rationality constraints, but also a double continuum of non-local incentive constraints for upward double deviations that bind for uniform price auctions (Section 5). Under our regularity condition on the correlation structure, we construct Lagrange multipliers on the constraints in the relaxed problem that, together with the uniform price auction mechanism, comprise a saddle point of the seller's Lagrangian.[6] The argument actually shows that uniform price auctions are essentially uniquely optimal among *ex post* efficient mechanisms in our setting.

The role of the large market assumption is simply to avoid complexities with uncertainty about the realized distribution of values in each state. In Section 6, we show that our main result does not rely on pathologies related to presence of a continuum of agents by connecting the continuum analysis to auctions with a finite number of potential bidders. While uniform price auctions are not exactly optimal in finite

[5]These papers formalized the strategic implications of withdrawal rights in bilateral trade settings. They called the the possibility of *ex post* withdrawal "rights of refusal," "veto incentive compatibility," and "veto rights," respectively.

[6]The role of non-local incentive constraints and double deviations in our analysis has some similarities to Carroll and Segal's (2019) analysis of optimal auctions with resale. However, only downward incentive constraints feature in Carroll and Segal's (2019) analysis, while the only non-local incentive constraints we use are upward incentive constraints. Furthermore, we construct Lagrange multipliers on these constraints as the solution of a linear integral equation, while previous work has constructed Lagrange multipliers over a double continuum of non-local incentive constraints either explicitly (Carroll and Segal, 2019; Bergemann et al., 2020), or using transport theory (Daskalakis et al., 2017).

markets, we show that uniform price auctions remain approximately optimal in a quantitative sense. More precisely, we show that the loss in expected revenue per bidder from using an $(M+1)$th price auction instead of the optimal *ex post* efficient auction mechanism is at most on the order of $\frac{1}{\sqrt{N}}$ if there are N potential bidders. The per-bidder expected revenue gap goes to 0 as N grows large—validating our analysis of the model with a continuum of bidders.

The remainder of the paper relaxes assumptions of our main result in turn. First, in Section 7.1, we relax the regularity condition on the correlation structure that our main result imposes. This regularity condition requires that the effect of the state on likelihoods of types be sufficiently small and well-behaved. In the special case in which the conditional value distributions comprise a full exponential family (such as, e.g., for truncated and untruncated normal, log-normal, Beta, and Pareto distributions), the regularity condition simply requires that bidders' values not be too informative about the state—i.e., that values are not too correlated. Without such an assumption, we can show that uniform price auctions are not exactly optimal in general. However, we provide a family of numerical examples illustrating that uniform price auctions can nevertheless deliver a large fraction of the optimal expected revenue if there are not too many units of the good for sale—in fact, at least 98% of the optimal expected revenue if at most half of the bidders can purchase the good. Thus, the uniform price auction appears to be nearly optimal even with high correlation.

Second, in Section 7.2, we allow for the possibility that withdrawal may be costly *ex post* for bidders—for example, as they may have to forfeit a deposit if they do so. We show that while uniform price auctions are no longer exactly optimal, they remain approximately optimal. More precisely, we bound above the expected revenue gain from using more complex mechanisms in terms of bidders' withdrawal costs.

For mechanism design problems with correlated information, Crémer and McLean (1988) and McAfee and Reny (1992) provided necessary and sufficient conditions for optimal mechanisms to extract the full surplus. A literature has explored the extent to which these conditions are generic (Neeman, 2004; Heifetz and Neeman, 2006; Chen and Xiong, 2013), and showed that full surplus extraction is impossible when the seller cannot pay buyers (Myerson, 1981) or when buyers have limited liability or are risk averse (Robert, 1991; Kosmopoulou and Williams, 1998).[7] However, even when full surplus extraction is impossible, optimal mechanisms generally involve side

[7]Relatedly, Compte and Jehiel (2007, 2009) studied bilateral trade with correlated values (where, generically, Crémer–McLean-type mechanisms would implement *ex post* efficient outcomes), and showed that the Myerson and Satterthwaite (1983) inefficiency result persists if withdrawal rights

bets (see, e.g., Neeman (2004)), so do not correspond to practical mechanisms. This paper goes beyond the issue of whether the seller can extract the full surplus, and shows that withdrawal rights provide a foundation for a simple, practical mechanism.

Our foundation for the uniform price auction differs from arguments that have been made for it based on robustness. Examples of such arguments include the optimality of dominant strategy mechanisms (a) among detail-free auction mechanisms (Chung and Ely, 2007; Yamashita and Zhu, 2022),[8] and (b) if it can only be ensured that bidders take weakly undominated actions (Yamashita, 2015). While our results hold exactly only in the large-market limit, we do not assume that the seller takes a worst-case perspective with respect to bidders' (higher-order) beliefs or behavior. Rather, we show that the optimality of the uniform price auction emerges from a fully Bayesian analysis of the sellers' problem when bidders have withdrawal rights.

Dynamic mechanism and information design problems with withdrawal rights have been studied by Krähmer and Strausz (2015) and Heumann (2020). Those papers used withdrawal rights (which they call "*ex post* participation constraints") to avoid full surplus extraction and obtain a realistic optimal mechanism when bidders do not have *ex ante* private information. Our setting is static, and full surplus extraction—without an *ex post* individual rationality constraint or withdrawal rights—is driven by the use of side bets rather than a lack of of private information on arrival.

2. An Illustrative Example

A seller has $\frac{1}{2}$ units of a good to sell to a unit-mass continuum of bidders. The bidders have private values whose distribution depends on a state $s \in \{0, 1, 2\}$. In state s, $\frac{8-s}{20}$ fraction of the bidders' values are 0, $\frac{8-s}{20}$ fraction of values are distributed uniformly between 0 and 84, and $\frac{2+s}{10}$ fraction of values are 84. State 1 occurs with probability $\frac{3}{5}$, and states 0 and 2 each occur with probability $\frac{1}{5}$.

Throughout this section, we restrict attention to *ex post* efficient mechanisms.

The optimal expected revenue among Bayesian incentive compatible, interim individually rational mechanisms is 31.9. Table 1 gives a menu description of a mechanism yielding this revenue.[9] Note that the mechanism involves payments of up to 180—well

are introduced. This inefficiency result applies even if withdrawal rights are weakened to an *ex post* individual rationality constraint.

[8]In related work, Lopomo (2001) and Segal (2003) showed the optimality of versions of second price and uniform price auctions among all mechanisms that are *ex post* incentive compatible and *ex post* individually rational (see also Roughgarden and Talgam-Cohen (2016)).

[9]The total surplus is 36.9. The McAfee and Reny (1992) full surplus extraction results do not apply because types with different values can share the same beliefs (Neeman, 2004).

	payment if $s = 0$	payment if $s = 1$	payment if $s = 2$
never buy	0	0	0
get good if $s = 0$	-357	144	0
get good if $s \in \{0, 1\}$	-357	180	0
get good in all states	-357	180	56

TABLE 1. Menu description of an optimal Bayesian incentive compatible, interim individually rational mechanism. Expected revenue is 31.9.

	payment if $s = 0$	payment if $s = 1$	payment if $s = 2$
never buy	0	0	0
get good if $s = 0$	21	0	0
get good if $s \in \{0, 1\}$	21	36	0
get good in all states	$-31\frac{1}{2}$	56	56

TABLE 2. Menu description of an optimal Bayesian incentive compatible, *ex post* individually rational mechanism. Expected revenue is 20.

above the highest possible value of 84. Moreover, bidders that get the good only in state 0 make payments to the seller in state 1. These payments arise due to the side bets that are central to optimal mechanisms in the presence of correlation (Crémer and McLean, 1985, 1988; McAfee and Reny, 1992; Neeman, 2004).

One might think requiring *ex post* individual rationality would discipline the design problem enough to lead to a realistic optimal mechanism. Imposing *ex post* individual rationality does lower the optimal expected revenue to 20: Table 2 gives a menu description of an optimal Bayesian incentive compatible, *ex post* individually rational mechanism. However, it does not correspond to a practical auction format.

Note that the mechanism described in Table 2 is not incentive compatible when the possibility of *ex post* withdrawal off the equilibrium path is considered. (*Ex post* individual rationality only requires that no bidder want to withdraw *ex post* on the equilibrium path—placing no restrictions off the equilibrium path beyond Bayesian incentive compatibility.) Indeed, the possibility of withdrawal effectively enlarges the menu. For example, by selecting the last option in the menu described in Table 2 and withdrawing in states 1 and 2, bidders can obtain the good in state 0 and receive $-31\frac{1}{2}$ from the seller in that state. The corresponding double deviation (involving a misreport and withdrawal) is attractive to types that do not always get the good.[10]

[10]The failure of incentive compatibility with withdrawal rights is not driven by the possibility of the seller making payments to bidders. If the seller is furthermore restricted to never make payments to bidders, then the menu description of an optimal mechanism is:

	payment if $s = 0$	payment if $s = 1$	payment if $s = 2$
never get good	0	0	0
get good if $s = 0$	21	0	0
get good if $s \in \{0, 1\}$	21	36	0
get good in all states	21	36	56

TABLE 3. Menu description of the uniform price auction, which is the optimal mechanism with withdrawal rights. Expected revenue is 18.5.

The optimal mechanism with withdrawal rights is the uniform price auction. For this mechanism, the units of the good are sold at prices of 21, 36, and 56 in the three states, respectively, yielding expected revenue of 18.5. (Table 3 gives a menu description.) The uniform price auction remains incentive compatible with withdrawal rights because withdrawal is only desirable *ex post* for a bidder if they bid above their value, which is a weakly dominated strategy.

Even with a continuum of agents, there is a role for auction design due to the correlation between bidders' values. Indeed, the discriminatory price auction yields strictly lower expected revenue than the uniform price auction (Jackson and Kremer, 2006). (In our example, the discriminatory price auction yields an expected revenue of 17.5.) The direct revelation version of the discriminatory price auction remains incentive-compatible with withdrawal rights, since—like for the uniform price auction—withdrawal is only desirable *ex post* for a bidder if they bid above their value, which is a dominated strategy. Hence, the seller faces a nontrivial mechanism design problem even in the presence of withdrawal rights.

3. MODEL

A seller has k units of a good for sale, where $0 < k < 1$. There is a unit-mass continuum of bidders who each have unit demand.

There is a state $s \in [0, 1]$ distributed according to density $f(s)$, which is continuously differentiable in s.[11] Agents have private types $\theta \in [\underline{\theta}, \overline{\theta}]$ that give their values

	payment if $s = 0$	payment if $s = 1$	payment if $s = 2$
never buy	0	0	0
get good if $s = 0$	21	0	0
get good if $s \in \{0, 1\}$	21	36	0
get good in all states	0	44	56

(This mechanism yields an expected revenue of 19.1.) The mechanism is vulnerable to double deviations. For example, types that get the good only in state 0. would prefer to select the last option in the menu and withdraw if the state is 1 or 2, thereby obtaining the good in state 0 for free.

[11]This and other differentiability assumptions are technical and only serve to simplify the proofs.

for the good, where $\underline{\theta} \geq 0$. Conditional on s, types are distributed independently according to density $f(\theta|s)$, which is twice continuously differentiable in s. We also assume throughout that the densities have full support, in that $f(s) > 0$ holds for all $s \in [0,1]$, and $f(\theta|s) > 0$ holds for all $\theta \in [\underline{\theta}, \overline{\theta}]$ and $s \in [0,1]$.

Note that the state s can be inferred by the designer from the reports of any full measure set of agents. To avoid technical issues that arise in mechanism design problems with a continuum of agents, we restrict the seller to direct mechanisms $(q,t) = (q(\theta,s), t(\theta,s)) : [\underline{\theta}, \overline{\theta}] \times [0,1] \to [0,1] \times \mathbb{R}$. In Section 6, we show that our results apply asymptotically to limits of sequences of mechanism design problems with growing, finite numbers of agents—validating our analysis of the continuum model.

The expected revenue of a mechanism (q,t) is $\mathbb{E}_s\left[\int_{\underline{\theta}}^{\overline{\theta}} t(\theta,s) f(\theta|s) d\theta\right]$.[12]

3.1. **Withdrawal-Proof Mechanisms.** We consider mechanisms that are incentive compatible (in a Bayesian sense) even when bidders can withdraw *ex post*.

Definition 1 (Matthews and Postlewaite, 1989; Forges, 1999; Compte and Jehiel, 2007, 2009)**.** A mechanism (q,t) is *withdrawal-proof* if for all types θ, θ', we have that

$$\mathbb{E}_s\left[\theta q(\theta,s) - t(\theta,s) \middle| \theta\right] \geq \mathbb{E}_s\left[\max\{\theta q(\theta',s) - t(\theta',s), 0\} \middle| \theta\right].$$

The left hand side in Definition 1 is type θ's expected utility under truthful reporting. The right hand side is type θ's expected utility from reporting θ' and withdrawing if they obtain an *ex post* utility lower than 0—the utility of the outside option.[13] Withdrawal-proofness requires that no such deviation be desirable in expectation.

We next compare withdrawal-proofness to Bayesian incentive compatibility and *ex post* individual rationality.

3.2. **Necessary Conditions for Withdrawal-Proofness.** A mechanism (q,t) is *Bayesian incentive compatible* if for all types θ and reports θ', we have that

$$\mathbb{E}_s\left[\theta q(\theta,s) - t(\theta,s) \middle| \theta\right] \geq \mathbb{E}_s\left[\theta q(\theta',s) - t(\theta',s) \middle| \theta\right].$$

Withdrawal-proofness implies Bayesian incentive compatibility because the possibility of withdrawal can only improve the payoff from misreporting.

[12]Technically, we assume that payment rules are integrable in that $\mathbb{E}_s\left[|t(\theta,s)| \middle| \theta\right] < \infty$ for all types θ, and that $\int_{\underline{\theta}}^{\overline{\theta}} \mathbb{E}_s\left[|t(\theta,s)| \middle| \theta\right] f(\theta) d\theta < \infty$.

[13]Due to the presence of a continuum of bidders in our setting and the implicit symmetry restriction we impose on the class of mechanisms, a bidder's report and the state completely determine their allocation and payment. Therefore, it is without loss of generality to assume that bidders' withdrawal decisions depend only on the state.

A mechanism (q, t) is *ex post individually rational* if for all types θ and almost all states s, we have that $\theta q(\theta, s) - t(\theta, s) \geq 0$. Withdrawal-proofness implies *ex post* individual rationality, as a failure of *ex post* individual rationality would imply that there exists a strictly profitable withdrawal strategy under truthful reporting.

Proposition 1 (Forges, 1999, Proposition 1)**.** *If a mechanism is withdrawal-proof, then it is Bayesian incentive compatible and ex post individually rational.*

Proof. Let (q, t) be a withdrawal-proof mechanism. To see that (q, t) must be Bayesian incentive compatible, note that for all types θ and reports θ', we have that

$$\mathbb{E}_s\left[\theta q(\theta, s) - t(\theta, s)\middle|\theta\right] \geq \mathbb{E}_s\left[\max\{\theta q(\theta', s) - t(\theta', s), 0\}\middle|\theta\right] \geq \mathbb{E}_s\left[\theta q(\theta', s) - t(\theta', s)\middle|\theta\right],$$

where the first inequality holds due to withdrawal-proofness.

To see that (q, t) must be *ex post* individually rational, let θ be a type. Note that the withdrawal-proofness condition for the report $\theta' = \theta$ guarantees that

$$\mathbb{E}_s\left[\theta q(\theta, s) - t(\theta, s)\middle|\theta\right] \geq \mathbb{E}_s\left[\max\{\theta q(\theta, s) - t(\theta, s), 0\}\middle|\theta\right].$$

It follows that $\theta q(\theta, s) - t(\theta, s) = \max\{\theta q(\theta, s) - t(\theta, s), 0\}$ for almost all states s, and hence that $\theta q(\theta, s) - t(\theta, s) \geq 0$ for almost all states s. □

However, withdrawal-proofness is in general more demanding than simply the conjunction of Bayesian incentive compatibility and *ex post* individual rationality. Indeed, withdrawal-proofness takes into account the strategic implications of an *ex post* participation constraint that arise with the possibility of withdrawal after a misreport (see Example 1 in Matthews and Postlewaite (1989), Compte and Jehiel (2007, 2009), or Section 2). In particular, if one attempts to microfound an *ex post* individual rationality constraint via an ability to withdraw *ex post*, then in models with adverse selection, one has actually microfounded withdrawal-proofness.[14]

3.3. **Sufficient Conditions for Withdrawal-Proofness.** However, the examples of Bayesian incentive compatible, *ex post* individually rational mechanisms that are not withdrawal-proof do not correspond to practical mechanisms. In fact, two natural properties of practical Bayesian incentive compatible mechanisms together ensure withdrawal-proofness: namely (i) that one is charged at most one's bid per unit of the good won, and (ii) that bidding above one's true value is a dominated strategy.

[14]Withdrawal-proofness does coincide with the conjunction of incentive compatibility and *ex post* individual rationality in pure moral hazard models.

Intuitively, (i) ensures that withdrawal is only desirable after bidding above one's true value, which (ii) ensures is never attractive.

To formalize this argument in the language of direct revelation mechanisms, we draw a dichotomy (for each θ) between reports that ensure a per-unit payment of at most θ and reports that are weakly dominated for type θ. We then show that any Bayesian incentive compatible mechanism for which making a report that can lead to paying above one's value is a dominated strategy is in fact withdrawal-proof.

Proposition 2. *Let (q,t) be a Bayesian incentive compatible mechanism. Suppose that for each type θ, each report θ' such that $\Pr_s\left[t(\theta',s) > \theta q(\theta',s)\middle|\theta\right] > 0$ is weakly dominated (in the game induced by (q,t) without withdrawal) by a report $\hat{\theta}$ for which $t(\hat{\theta},s) \leq \theta q(\hat{\theta},s)$ for almost all states s. Then, (q,t) is withdrawal-proof.*

Proof. Let θ be a type, and let θ' be a report. We show that θ' cannot be desirable even in the presence of withdrawal.

If $t(\theta',s) \leq \theta q(\theta',s)$ for almost all states s, then we have that

$$\max\{\theta q(\theta',s) - t(\theta',s), 0\} = \theta q(\theta',s)$$

for almost all states s. Since (q,t) is Bayesian incentive compatible, it follows that

$$\mathbb{E}_s\left[\theta q(\theta,s) - t(\theta,s)\middle|\theta\right] \geq \mathbb{E}_s\left[\theta q(\theta',s) - t(\theta',s)\middle|\theta\right] = \mathbb{E}_s\left[\max\{\theta q(\theta',s) - t(\theta',s), 0\}\middle|\theta\right].$$

If $\Pr_s\left[t(\theta',s) > \theta q(\theta',s)\middle|\theta\right] > 0$, then let $\hat{\theta}$ be a weakly dominating report with $t(\hat{\theta},s) \leq \theta q(\hat{\theta},s)$ for almost all states s. By construction, we have that

$$\theta q(\hat{\theta},s) - t(\hat{\theta},s) \geq \max\{\theta q(\theta',s) - t(\theta',s), 0\}.$$

Taking expectations and using Bayesian incentive compatibility then yields that

$$\mathbb{E}_s\left[\theta q(\theta,s) - t(\theta,s)\middle|\theta\right] \geq \mathbb{E}_s\left[\theta q(\hat{\theta},s) - t(\hat{\theta},s)\middle|\theta\right] \geq \mathbb{E}_s\left[\max\{\theta q(\theta',s) - t(\theta',s), 0\}\middle|\theta\right].$$

Since the two cases exhaust all possibilities, (q,t) must be withdrawal-proof. □

For example, it follows from Proposition 2 that the uniform price auction and the direct revelation mechanism implementing Bayesian Nash equilibrium in a discriminatory price auction are withdrawal-proof. Similarly, direct revelation versions of mixed-price auctions (see, e.g., Woodward (2021)) and Spanish auctions (Alvarez et al., 2003) are also withdrawal-proof.

Proposition 2 also implies a connection between withdrawal-proofness and dominant-strategy incentive compatibility. A mechanism (q,t) is *dominant strategy incentive*

compatible if for all types θ and reports θ', and almost all states s, we have that

$$\theta q(\theta, s) - t(\theta, s) \geq \theta q(\theta', s) - t(\theta', s).$$

Corollary 1. *If a mechanism is dominant strategy incentive compatible and ex post individually rational, then it is withdrawal-proof.*

Proof. Follows from Proposition 2 by taking $\hat{\theta} = \theta$. □

Proposition 1 and Corollary 1 show why imposing withdrawal-proofness does not affect optimal revenue in the independent private values model. Indeed, in that setting, the optimal expected revenue over Bayesian incentive compatible, interim individually rational mechanisms is the same as the optimal expected revenue over dominant strategy incentive compatible, *ex post* individually rational mechanisms;[15] withdrawal-proof mechanisms fall between these two classes, and therefore also deliver the same optimal expected revenue. Hence, it is the correlation between bidders' private information that creates a nontrivial role for withdrawal rights in our setting.

4. Optimality of the Uniform Price Auction

In this section, we provide conditions under which the uniform-price auction is optimal among *ex post* efficient, withdrawal-proof mechanisms.

Formally, an allocation rule $q(\theta, s)$ is *ex post efficient* if, for each state s, it maximizes total surplus

$$\int_{\underline{\theta}}^{\overline{\theta}} \theta q(\theta, s) f(\theta|s) d\theta$$

subject to the allocation feasibility constraint

$$\int_{\underline{\theta}}^{\overline{\theta}} q(\theta, s) f(\theta|s) d\theta \leq k.$$

A mechanism is *ex post efficient* if its allocation rule is *ex post* efficient. Letting $F(\theta|s)$ denote the cumulative distribution function of values in states s and $p(s) = F(\cdot|s)^{-1}(1-k)$ the $(1-k)$th quantile value in state s, it is easy to see that for an *ex post* efficient allocation rule $q(\theta, s)$, we must have that $q(\theta, s) = q^{\text{eff}}(\theta, s)$ for all types

[15]This equality follows from the Revenue Equivalence Theorem if the seller is restricted to *ex post* efficient mechanisms, and from the equivalence of Bayesian and dominant strategy-implementation (Manelli and Vincent, 2010; Gershkov et al., 2013) for general allocation rules.

θ and almost all states s, where q^{eff} is the allocation rule defined by

$$q^{\text{eff}}(\theta, s) = \begin{cases} 1 & \text{if } \theta \geq p(s) \\ 0 & \text{otherwise} \end{cases}.$$

The *uniform price auction* is the mechanism with allocation rule q^{eff} and payment rule

$$t^{\text{UPA}}(\theta, s) = p(s)q^{\text{eff}}(\theta, s).$$

By construction, the uniform price auction is *ex post* efficient, as well as dominant strategy incentive compatible and *ex post* individually rational. It is therefore withdrawal-proof by Corollary 1.

Our optimality result for the uniform-price auction imposes two conditions on the correlation structure. The first is the standard monotone likelihood ratio property (Milgrom, 1981). We work with the strict version to simplify the analysis.

Assumption 1 (Strict Monotone Likelihood Ratio Property)**.** For all states $s < s'$, the likelihood ratio $\frac{f(\theta|s')}{f(\theta|s)}$ is strictly increasing in θ.

Our second assumption is a novel regularity condition on the degree of affiliation between the state and bidders' values.

Assumption 2 (Regular Affiliation)**.** For all types θ and states s, we have that[16]

$$\left(\int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial^2 \log f(\theta'|s)}{\partial s \partial \theta'} d\theta'\right)\left(\int_0^1 \frac{\partial^2 \log f(\theta|s')}{\partial s' \partial \theta} ds'\right) \leq \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta}.$$

Intuitively, Assumption 2 requires that the degree of affiliation be not too high and not too irregular. We discuss the assumption in more detail in the sequel.

The following theorem is the main result of this paper.

Theorem 1. *Under Assumptions 1 and 2, the uniform price auction maximizes expected revenue over all* ex post *efficient, withdrawal-proof mechanisms.*

Due to the presence of correlation, optimal Bayesian incentive compatible, interim individually rational mechanisms generally require agents to make side bets about others' reports in order to participate in the mechanism (Crémer and McLean, 1985, 1988; McAfee and Reny, 1992; Neeman, 2004). As discussed in Section 2, even if interim individual rationality is strengthened to *ex post* individual rationality, mechanisms unlike those seen in practice remain optimal. By contrast, Theorem 1 shows

[16]In fact, a weaker condition that depends on the quantity k of the good that is for sale is sufficient for Theorem 1. See Footnote 42 in Appendix B for the details.

that when bidders can withdraw *ex post*, the uniform price auction becomes optimal. As a side benefit, this mechanism is also dominant strategy incentive compatible.

4.1. **Understanding the Regular Affiliation Assumption.** We next explain the content of the regular affiliation assumption in the case where the conditional distributions of values comprise a (full) exponential family. Formally, the family $f(\theta|s)$ of distributions is a *full exponential family* if there exist functions $A, B : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}$ and $C, D : [0,1] \to \mathbb{R}$ with $B(\theta), D(s) > 0$ such that

$$f(\theta|s) = B(\theta)D(s)\exp\left(A(\theta)C(s)\right).$$

Many common families of distributions, such as (truncated and untruncated) normal, log-normal, Beta, Gamma, and Pareto distributions, comprise full exponential families. Moreover, the exponential family assumption places no restrictions on the unconditional distribution of values.

In the case of full exponential families, it turns out that Assumption 2 can be written as a simple bound on likelihood ratios.

Assumption 2′ (Uniformly Bounded Likelihood Ratios)**.** For all types θ, θ' and states s, s', we have that

$$\frac{f(\theta'|s')}{f(\theta'|s)} \leq e\frac{f(\theta|s')}{f(\theta|s)},$$

where $e = 2.718\ldots$ denotes Euler's constant.[17]

Lemma 1. *Under Assumption 1:*

(a) Assumption 2 implies Assumption 2′.

(b) If $f(\theta|s)$ is a full exponential family, then Assumption 2′ implies Assumption 2.

The proof of Lemma 1 is a straightforward calculation—see Online Appendix E.

Hence, in the case of full exponential families, Assumption 2 simply requires that for each pair s, s' of states, the likelihood ratio statistic for a hypothesis test to distinguish states s and s' vary by at most a factor of e across types. This condition simply requires that the state not be too affiliated with values. For full exponential families, Assumption 2 can be weakened to Assumption 2′ in Theorem 1.

Corollary 2. *If $f(\theta|s)$ is a full exponential family, then under Assumptions 1 and 2′, the uniform price auction maximizes expected revenue over all* ex post *efficient, withdrawal-proof mechanisms.*

[17]Under Assumption 1, Assumption 2′ can be written equivalently as $\frac{f(\overline{\theta}|1)}{f(\overline{\theta}|0)} \leq e\frac{f(\underline{\theta}|1)}{f(\underline{\theta}|0)}$.

For general correlation structures, Assumption 2 requires more than simply the uniform bound given by Assumption 2′. Intuitively, Assumption 2 requires that that the affiliation between the state and the values be sufficiently regular, in a sense implied by the full exponential family assumption.

4.2. **The Key Intuition.** We next explain the intuition behind Theorem 1 by connecting withdrawal-proofness to Milgrom and Weber's (1982) linkage principle.

Let us restrict attention to *ex post* efficient, locally incentive compatible auctions for which losers do not pay. For such auctions, under Assumption 1, the linkage principle implies that the seller's expected revenue is higher the steeper payments of winners are in the state s—i.e., the more winners' payments are "linked" to s.

Local incentive compatibility limits how quickly expected payments can increase with type. So, in order to maximize the steepness of payments in s, the seller should offer large discounts to high types in low states and correspondingly charge premia in high states (whose size may be restricted by *ex post* individual rationality). This implication of the linkage principle drives the payment by the seller to the buyers in the optimal Bayesian incentive compatible, *ex post* individually rational mechanism in the example of Section 2 (see Table 2).

Crucially, discounts for higher types make winners' *ex post* payments decrease in their types in each state. Such discounts incentivize low types to execute "upward double deviations" in which they report high values and withdraw in states in which they do not receive discounts. Intuitively, withdrawal-proofness requires that winners' payments weakly increase in type—essentially ruling out the discounts that feature in the optimal *ex post* individually rational mechanism. This logic suggests that the most linked payments can be made to the state under withdrawal-proofness is when winners' payments are constant in type in each state—i.e., with uniform prices.

However, this intuition is incomplete, because withdrawal-proofness only requires that a type θ be unable to obtain lower *expected* payments by an upward double deviation to a report $\theta' > \theta$: in any given state, payments can decrease in type in principle. Considering the restriction withdrawal-proofness imposes on the payments of type θ', we see that when lower types' beliefs vary significantly, incentive compatibility for upward double deviations imposes a substantially different restriction than payments being increasing in type in each state. Nevertheless, under Assumption 2, we can extract a condition similar to monotonicity of payments in the state implicitly from the withdrawal-proofness. This part of our formal argument involves constructing Lagrange multipliers on a double continuum of upward incentive constraints.

5. Strategy of the Proof of Theorem 1

In this section, we summarize the proof of Theorem 1. We begin by formulating the local incentive constraint for settings with correlated information and explaining why the standard first-order approach fails in our setting. We then describe the key nonlocal incentive constraints we use in our analysis, and how we use them to derive a tight upper bound on the expected revenue of withdrawal-proof mechanisms. As a side benefit, we show that the uniform price auction is essentially uniquely optimal under the conditions of Theorem 1. The proofs are in Appendixes A and B.

5.1. The Local Incentive Constraint.

We start by deriving the local incentive constraint for settings with correlated information. Formally, we let

$$u(\theta, s) = \theta q(\theta, s) - t(\theta, s)$$

denote the *ex post* utility of type θ in state s. We also let

$$U(\theta) = \mathbb{E}_s\left[u(\theta, s)\middle|\theta\right] \quad \text{and} \quad Q(\theta) = \mathbb{E}_s\left[q(\theta, s)\middle|\theta\right]$$

denote the interim expected utility and interim allocation of type θ, respectively.

The following envelope formula for interim expected utilities formalizes the local incentive constraint in our model.

Theorem 2 (Envelope Formula with Correlated Information)**.** *If a Bayesian incentive compatible mechanism (q, t) yields nonnegative expected revenue, and the ex post utility $u(\theta, s)$ is uniformly bounded below over all types θ and almost all states s, then for all types θ, we have that*

$$U(\theta) = \underbrace{U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} Q(\theta')d\theta'}_{\text{analogous to Myerson (1981)}} + \underbrace{\int_{\underline{\theta}}^{\theta} \mathbb{E}_s\left[u(\theta', s)\frac{\partial \log f(s|\theta')}{\partial \theta'}\middle|\theta'\right] d\theta'}_{\text{correction for correlation}}. \tag{1}$$

The first two terms of (1) comprise the standard integral formula for interim expected utility with independent types from Myerson (1981) and Milgrom and Segal (2002). The last term of (1) is novel and corresponds to an adjustment for the heterogeneity of beliefs that comes from correlation. Intuitively, interim utilities change with type for two reasons: difference in values (captured by the standard Myerson (1981) term) and differences in beliefs (captured by the correlation correction term). Note that the correlation correction can be positive or negative and is not a result of side bets or belief-based screening. For example, the correlation correction is negative for the uniform price auction under Assumption 1.

The correlation correction term in (1) also quantifies the failure of payoff equivalence usually thought of in terms of the linkage principle. Indeed, Assumption 1 requires that the partial derivative $\frac{\partial \log f(s|\theta')}{\partial \theta'}$ be increasing in the state s. It follows that for a fixed interim expected utility $U(\theta') = \mathbb{E}_s\left[u(\theta', s)\middle|\theta'\right]$, the adjustment $\mathbb{E}_s\left[u(\theta', s)\frac{\partial \log f(s|\theta')}{\partial \theta'}\middle|\theta'\right]$ is smaller the more rapidly the *ex post* utility $u(\theta', s)$ decreases in the state s. This effect encapsulates the linkage principle: linking winners' payments to the state lowers correlation corrections, and hence information rents. For example, under Assumption 1, the correlation correction is negative for the discriminatory price auction, but smaller in magnitude than for the uniform price auction.

The proof of Theorem 2 is applicable well beyond the context of our model. Indeed, neither the large-market structure, nor the unit-demand assumption in the auction, nor the uni-dimensionality of the state s feature in the proof.[18,19] For the purposes of Theorem 2, the state s can be interpreted as any payoff-irrelevant information that is not known to the agent—such as the vector of other agents' types in finite markets. The critical assumptions used in the proof of Theorem 2 are the regularity conditions imposed on the density f (namely positivity and continuous differentiability), the compactness of the state space, and the regularity conditions imposed on the mechanism—namely that it yields nonnegative expected revenue and that *ex post* utilities are uniformly almost surely bounded below. The last assumption (together with incentive compatibility) ensures that interim expected utilities are not too erratic when types have different beliefs, which is important to being able to apply arguments along the lines of Milgrom and Segal (2002) in the presence of correlation. The assumption is fairly innocuous, since the theorem applies regardless of the bound.

5.2. **Failure of the First-Order Approach.** It turns out that the envelope condition implied by local incentive constraints under withdrawal-proofness is the same as the one under Bayesian incentive compatibility. Indeed, by Proposition 1, Theorem 2 applies to withdrawal-proof mechanisms that yield nonnegative expected revenue. Thus, the standard first-order approach is to maximize expected revenue subject to (1) and (*ex post*) individual rationality constraints.

[18]Furthermore, Theorem 2 does not impose Assumptions 1 or 2.

[19]The envelope formula is similar in spirit to formulations of the first-period local incentive constraint in dynamic mechanism design problems (see, e.g., Baron and Besanko (1984, Lemma A.3), Courty and Li (2000, Lemma 3.2), and Segal and Toikka (2007, Claim in Appendix A)). To understand the connection, interpret s as the realization of future private information, whose distribution may depend on first-period private information; the envelope formulas referenced above also contain a term related to the derivative of a density with respect to type. Technically, due to the difference in settings, we impose different regularity conditions to obtain an envelope formula.

However, the solution to this relaxed problem is not withdrawal-proof in general. Indeed, it is possible that the solution is not even Bayesian incentive compatible (and hence not withdrawal-proof by Proposition 1). But even if it is, it would be the optimal Bayesian incentive compatible, *ex post* individually rational mechanism—which we showed in Section 2 is not withdrawal-proof in general.[20]

Economically, the failure of the first-order approach is related to the central role of upward double deviations in the intuition described in Section 4.2. The first-order approach captures the linkage principle effect, but not the way in which withdrawal-proofness restricts how payments in each state can vary across types.

Hence, we need to take into account the nonlocal incentive constraints that do not feature in Theorem 2.

5.3. **The Relaxed Problem.** By complementary slackness, we need to focus on nonlocal incentive constraints that bind for the uniform price auction. Specifically, we focus on upward double deviations where a type θ reports $\theta' > \theta$ and withdraws *ex post* if it would be inefficient for them to get the good—that is, if the state s is such that $\theta < p(s)$. For the uniform price auction, types are indifferent between reporting truthfully and executing these upward double deviations: reporting a higher value only would only make a bidder sometimes win at a price exceeding their value—in which states it would be inefficient for the bidder to obtain the good and the bidder could withdraw. The incentive constraints for these double deviations are given by

$$U(\theta) \geq \mathbb{E}_s \left[(u(\theta', s) - (\theta' - \theta) q(\theta', s)) \, \mathbb{1}_{\theta \geq p(s)} \middle| \theta \right]. \tag{2}$$

To prove Theorem 1, we consider a relaxed problem in which we impose the nonlocal incentive constraints (2) for types $\theta \in [p(0), p(1)]$ and misreports $\theta' > \theta$ in addition to the envelope conditions (1) and the *ex post* individual rationality constraints. We construct Lagrange multipliers on these constraints that, together with the uniform price auction mechanism, comprise a saddle point of the Lagrangian of the seller's problem. We give an explicit formula for the multipliers on the envelope conditions (1); by contrast, we construct the multipliers on the nonlocal incentive constraints (2) implicitly as the solution to a family of linear Volterra integral equations.[21]

[20]The distributions of values in Section 2 comprise a natural exponential family, and satisfy Assumption 2′. Although the densities are discontinuous in that section, they can be adapted to satisfy the continuity and differentiability assumptions that we impose starting in Section 3.

[21]See Brunner (2017, Chapter 1) for an introduction to the theory of linear Volterra equations.

With independent private values, the Lagrange multiplier on the envelope condition (1) is $-f(\theta) = \frac{d}{d\theta}(1 - F(\theta))$.[22] In our case, we have to make an adjustment for correlation. Our constructed Lagrange multiplier on the envelope condition (1) is

$$\lambda(\theta) = \frac{d\Lambda(\theta)}{d\theta} \qquad \text{where} \qquad \Lambda(\theta) = \begin{cases} \frac{f(\theta)}{f(\theta|0)}(1 - F(\theta|0)) & \text{for } \theta < p(0) \\ \frac{f(\theta)}{f(\theta|s)}(1 - F(\theta|s)) & \text{for } \theta = p(s) \\ \frac{f(\theta)}{f(\theta|1)}(1 - F(\theta|1)) & \text{for } \theta > p(1) \end{cases}. \tag{3}$$

Intuitively, $\Lambda(\theta)$ is the Lagrange multiplier on the local incentive constraint of type θ. In the special case in which the conditional distribution of values is state-independent, we have that $\Lambda(\theta) = 1 - F(\theta)$, and therefore we recover the Lagrange multipliers on the envelope condition from the independent private values model.[23]

Technically, it follows from the following lemma that $\lambda(\theta)$ is well-defined except $\theta = p(0), p(1)$, where $\lambda(\theta)$ has jump discontinuities.

Lemma 2. *Under Assumption 1, the function Λ is well-defined on the interval $[\underline{\theta}, \overline{\theta}]$, and continuously differentiable on the intervals $[\underline{\theta}, p(0)]$, $[p(0), p(1)]$, and $[p(1), \overline{\theta}]$.*

We construct Lagrange multipliers $\gamma(\theta, \theta')$ on the nonlocal incentive constraints (2) as the solution to the equations

$$\int_{p(s)}^{\min\{\theta', p(1)\}} \frac{f(s|\theta)}{f(s|\theta')}\gamma(\theta, \theta')d\theta = \Lambda(\theta') \int_s^1 \frac{\partial^2 \log f(s'|\theta')}{\partial s' \partial \theta'} \mathbb{1}_{\theta' \geq p(s')} ds' \quad \text{for } \theta' \geq p(s), \tag{4}$$

which are necessary conditions for obtaining a saddle point. We need to show that (4) has a well-behaved solution that is nonnegative—the dual feasibility condition for Lagrange multipliers on inequality constraints.

The complexity in (4) comes from the factor of $\frac{f(s|\theta)}{f(s|\theta')}$ in the integrand on the left hand side, which is an adjustment for differences in beliefs about the state across types θ. If the left hand side of (4) were evaluated using any fixed set of beliefs rather than $f(s|\theta)$, (4) would simplify to an equation with an explicit solution, which would be guaranteed to be positive under Assumption 1. In terms of the intuition described in Section 4.2, the complexity therefore arises due to the fact that withdrawal-proofness

[22] This multiplier is unique for types that are allocated the good, but indeterminate for other types.

[23] Note that for $\theta = p(s)$, we have that $1 - F(\theta|s) = k$ by construction and hence that

$$\Lambda(\theta) = \frac{kf(\theta)}{f(\theta|s)}.$$

This formula also holds for the $(1 - k)$th quantile type in the independent private values model. However, it does not hold for $\theta > p(1)$ or $\theta < p(0)$.

constrains each type θ's expected payment upon executing upward double deviations to θ', which is not a well-behaved condition when beliefs vary significantly with θ.

Nevertheless, we can use Assumptions 1 and 2 to ensure that (4) has a well-behaved solution. In particular, we use Assumption 2 to ensure that the solution is nonnegative even taking into account the type-dependence of beliefs in the left hand side of (4).

Lemma 3. *Let $\mathcal{D} \subseteq [\underline{\theta}, \overline{\theta}]^2$ be the trapezoid defined by $p(0) \leq \theta \leq p(1)$ and $\theta \leq \theta' \leq \overline{\theta}$.*

(a) Under Assumption 1, (4) has a unique continuous solution $\gamma(\theta, \theta') : \mathcal{D} \to \mathbb{R}$.

(b) Under Assumptions 1 and 2, this solution is nonnegative, and positive for $\theta' \neq \overline{\theta}$.

Here, $\mathcal{D}$ is the set of upward deviations that we consider. To prove Lemma 3(a), we show that fixing a misreport θ' and letting the state s vary, (4) can be written as a linear Volterra equation of the first kind. We then show that under Assumption 1, this equation is nonsingular and hence has a unique continuous solution.[24] Combining these solutions as θ' varies, we get a solution to (4), which we show is continuous. We then prove Lemma 3(b) by direct calculation.

5.4. **Bounding Expected Revenue.** Using the Lagrange multipliers constructed in (3) and (4) yields the following formula for the revenue of any Bayesian incentive compatible mechanism, which we use to derive an upper bound on the revenue of any withdrawal-proof mechanism.

Lemma 4 (Revenue Formula with Correlated Private Values)**.** *Let λ and γ be defined by* (3) *and* (4)*. Under Assumption 1, if a Bayesian incentive compatible mechanism (q, t) yields a nonnegative expected revenue, and $u(\theta, s)$ is uniformly bounded below over all types θ and almost all states s, then (q, t) yields an expected revenue of*

$$\text{(5)} \quad \underbrace{\int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta}_{\text{(i) interim virtual values}} + \underbrace{\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} (\theta' - \theta) \gamma(\theta, \theta') \mathbb{E}_s \left[q(\theta', s) \mathbb{1}_{\theta \geq p(s)} \middle| \theta\right] d\theta' d\theta}_{\text{(ii) expected correction to interim virtual values}}$$

$$- \underbrace{\Lambda(\underline{\theta}) U(\underline{\theta})}_{\substack{\text{(iii) expected rent of lowest type} \\ \text{(slack in interim IR constraint)}}} \underbrace{- \int_{\underline{\theta}}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s \left[u(\theta, s) \mathbb{1}_{\theta \leq p(s') \leq p(s)} \middle| \theta\right] ds' d\theta}_{\substack{\text{(iv) ex post rents of other types conditional on non-allocation under } q^{\text{eff}} \\ \text{(slack in ex post IR constraints)}}}$$

[24] Here, and throughout, we focus on the unique continuous solution to (4).

$$\underbrace{-\int_{p(0)}^{p(1)}\int_{\theta}^{\overline{\theta}}\gamma(\theta,\theta')\left(U(\theta)-\mathbb{E}_s\left[(u(\theta',s)-(\theta'-\theta)q(\theta',s))\,\mathbb{1}_{\theta\geq p(s)}\middle|\theta\right]\right)d\theta' d\theta\,.}_{\text{⑦ slack in IC constraints for double deviations}}$$

The first two terms of (5) involve the allocation rule and relate to virtual values. The first term is analogous to the Myerson (1981) revenue formula for the independent private values model. Indeed, in that case, we have that $\Lambda(\theta) = 1 - F(\theta)$, so the quantity $\theta - \frac{\Lambda(\theta)}{f(\theta)}$ is a virtual value. The second term is an expected correction to interim virtual values for correlation.[25] Note that the second term involves the *ex post* allocation rule rather than merely the interim allocation rule.

The last three terms of (5) involve rents to the bidders that arise if the seller fails to use a uniform price auction. The third term is analogous to the standard effect of the interim expected utility of the lowest type on revenue from Myerson (1981). Note that this term is scaled by $\Lambda(\underline{\theta})$, which equals 1 in the independent private values model but is strictly less than 1 under Assumption 1. The fourth and fifth terms are corrections to the third term for correlation.[26] In contrast to the third term, both the fourth and fifth terms involve *ex post* utilities and consider types other than the lowest one. The fourth term only considers the *ex post* utilities of types in states in which it would be inefficient for them to be allocated the good. Thus, the fourth term captures the revenue effects of payments to types who would not be efficiently allocated the good. The fifth term considers interim utilities, as well as *ex post* utilities of types in states in which it would be efficient for them to be allocated the good. Thus, the fifth term captures the revenue effects of discriminatory (i.e., non-uniform) pricing.

The last three terms of (5) all capture slack in constraints entailed by withdrawal-proofness. Indeed, the third term captures slack in an interim individual rationality constraint, the fourth term captures slack in a family of *ex post* individual rationality constraints, and the fifth term captures slack in the withdrawal-proofness constraints incorporated in our relaxed problem. In particular, the last three terms are all non-negative under withdrawal-proofness.[27] Morevoer, the construction of the relaxed

[25]The second term of (5) vanishes in the independent private values model because for that model, the Lagrange multipliers $\gamma(\theta,\theta')$ on the nonlocal incentive constraints constructed in (4) vanish.

[26]Indeed, in the independent private values model, both the fourth and fifth terms of (5) vanish. For the fourth term, this is because the mixed partial derivative $\frac{\partial^2 \log f(s'|\theta)}{\partial s'\partial\theta}$ vanishes. For fifth term, this is because the Lagrange multipliers $\gamma(\theta,\theta')$ on the nonlocal incentive constraints vanish.

[27]Indeed, under Assumptions 1 and 2, the function Λ, the mixed partial derivative $\frac{\partial \log f(s'|\theta)}{\partial s'\partial\theta}$, and the Lagrange multipliers $\gamma(\theta,\theta')$ that feature in the last three terms of (5) are all nonnegative.

problem ensures that the third, fourth, and fifth terms of (5) all vanish for the uniform price auction. Hence, Lemma 4 yields an upper bound on the expected revenue of withdrawal-proof mechanisms that is tight for the uniform price auction.

Lemma 5. *Let λ and γ be defined by* (3) *and* (4)*. Under Assumptions 1 and 2:*

(a) If a withdrawal-proof mechanism (q,t) yields a nonnegative expected revenue, then it yields an expected revenue of at most

$$\underbrace{\int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta}_{\text{① interim virtual values}} + \underbrace{\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} (\theta' - \theta)\gamma(\theta,\theta')\mathbb{E}_s\left[q(\theta',s)\mathbb{1}_{\theta \geq p(s)}\middle|\theta\right] d\theta' d\theta}_{\text{② expected correction to interim virtual values}}. \tag{6}$$

(b) Equality holds in this bound if (q,t) is the uniform price auction.

Relative to (5), (6) preserves only the first two terms—corresponding to the interim virtual values and the expected correlation correction. The first term of (6) comes from local incentive constraints, while the second comes from nonlocal incentive constraints and captures their role in the design of withdrawal-proof auction mechanisms in the presence of correlation.

Theorem 1 follows directly from Lemma 5.

Proof of Theorem 1 assuming Lemma 5. The upper bound (6) on expected revenue given in Lemma 5(a) is constant across all *ex post* efficient mechanisms, and is the bound is achieved by the uniform price auction (by Lemma 5(b)). □

5.5. **Uniqueness of the Uniform Price Auction.** Note that the upper bound on revenue given in Lemma 5 is not tight in general—for example, it is not tight for the direct revelation version of the discriminatory price auction. In fact, it turns out that the upper bound is tight essentially only for the uniform price auction (among *ex post* efficient mechanisms). Hence, the argument actually shows that the uniform price auction is uniquely optimal among direct mechanisms, in that any other optimal mechanism must prescribe the uniform price auction payments almost surely *ex post*.

Proposition 3 (Uniqueness of the Uniform Price Auction)**.** *Under Assumptions 1 and 2, if (q,t) maximizes expected revenue over all* ex post *efficient, withdrawal-proof mechanisms, then $t(\theta,s) = t^{\mathrm{UPA}}(\theta,s)$ must hold for almost all types θ and states s.*

Proposition 3 provides a sense in which withdrawal-proofness provides a foundation specifically for the uniform price auction.

6. Finite Markets

In this section, we validate our analysis of the model with a continuum of bidders by considering markets with large finite numbers of bidders. We show that while the uniform price auction is no longer exactly optimal in finite markets, it remains approximately optimal, and asymptotically optimal in a large-market limit.

We suppose that there are $N \geq 2$ bidders, and that the seller has $1 \leq M \leq N-1$ units of a homogeneous good for sale. Values and information as are in Section 3.

For the remainder of this section, we denote the type of a particular bidder by θ_i, the vector of reports of the other bidders by $\boldsymbol{\theta}_{-i}$, and the vector of reports of all bidders by $\boldsymbol{\theta}$. By the revelation principle, we can restrict attention to direct finite-market mechanisms $(q_i, t_i)_{1\leq i\leq N}$, where

$$(q_i, t_i) = (q_i(\theta_i, \boldsymbol{\theta}_{-i}), t_i(\theta_i, \boldsymbol{\theta}_{-i})) : [\underline{\theta}, \overline{\theta}]^N \to [0,1] \times \mathbb{R}.$$

The expected revenue of a finite-market mechanism $(q_i, t_i)_{1\leq i\leq N}$ is $\sum_{i=1}^N \mathbb{E}_{\boldsymbol{\theta}}[t_i(\theta_i, \boldsymbol{\theta}_{-i})]$.[28]

Withdrawal-proofness is defined analogously to Definition 1, but where bidders can withdraw *ex post* depending on the realizations $\boldsymbol{\theta}_{-i}$ of other bidders' types.[29]

Definition 2 (Matthews and Postlewaite, 1989; Forges, 1999)**.** A finite-market mechanism $(q_i, t_i)_{1\leq i\leq N}$ is *withdrawal-proof* if for all bidders i and all types θ_i, θ_i', we have that

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q(\theta_i, \boldsymbol{\theta}_{-i}) - t(\theta_i, \boldsymbol{\theta}_{-i})|\theta_i\right] \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\max\{\theta_i q(\theta_i', \boldsymbol{\theta}_{-i}) - t(\theta_i', \boldsymbol{\theta}_{-i}), 0\}|\theta_i\right].$$

Propositions 1 and 2 and Corollary 1 carry over *verbatim* to the finite-market setting (see Appendix F). Efficiency is defined in the standard way.[30]

The finite-market analogue of the uniform price auction is the $(M+1)$th price auction. However, unlike the setting with a continuum of bidders, the $(M+1)$th price auction is not generally optimal among *ex post* efficient, withdrawal-proof, finite-market mechanisms—even under Assumptions 1 and 2. The fundamental difference

[28]Technically, we assume that payment rules are integrable in that $\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[|t_i(\theta_i, \boldsymbol{\theta}_{-i})|\middle|\theta_i\right] < \infty$ for all types bidders i and θ_i, and that $\mathbb{E}_{\boldsymbol{\theta}}\left[|t_i(\theta_i, \boldsymbol{\theta}_{-i})|\right] < \infty$ for all bidders i.

[29]In finite markets, unlike in large markets, it is not sufficient to consider withdrawal dependent on the realization of the state (compare with Footnote 13). However, the results of this section also apply if withdrawal decisions were allowed to depend on the realization of the state but not the realized distribution of types.

[30]Formally, an finite-market allocation rule $(q_i)_{1\leq i\leq N}$ is *ex post efficient* if, for each realization $\boldsymbol{\theta}$ of types, it maximizes total surplus $\sum_{i=1}^N \theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i})$ subject to the allocation feasibility constraint $\sum_{i=1}^N q_i(\theta_i, \boldsymbol{\theta}_{-i}) \leq M$. A finite market mechanism is *ex post efficient* if its allocation rule is.

between the large-market setting and the finite-market setting is that the latter features uncertainty about the realized distribution of values conditional on the state,[31] and therefore the market-clearing price is stochastic in each state. Hence, higher types make higher expected payments conditional on winning and conditional on the state. As a result, following logic along the lines of Section 4.2, there is scope for further linking the payments to the state while preserving local incentive compatibility and without incentivizing upward double deviations.

Nevertheless, we can show that $(M+1)$th price auction remains approximately optimal under the conditions of Theorem 1.

Theorem 3. *Under Assumptions 1 and 2, if $f(\theta|s) \geq \frac{1}{\alpha}$ holds for all types θ and states s, then no* ex post *efficient, withdrawal-proof, finite-market mechanism yields greater than $\frac{4\alpha}{\sqrt{N}}$ more expected revenue per bidder than the $(M+1)$th price auction.*[32]

Theorem 3 shows that a uniform price auction remains approximately optimal in finite markets. Hence, our foundation for the uniform price auction in terms of withdrawal rights persists, approximately, in finite markets.

Theorem 3 also validates our continuum analysis from Sections 3 and 4. Indeed, Theorem 3 implies that holding the ratio $\frac{M}{N}$ between the number of units for sale and the number of bidders fixed, and letting the number of bidders grow, a uniform price auction is asymptotically optimal in the sense of maximizing per-bidder expected revenue in the limit.[33] Moreover, the convergence is at a reasonable rate $\frac{1}{\sqrt{N}}$ with constants independent of fine details of the distribution of types. Hence, the exact conclusion of Theorem 1 does not rely on modeling large markets with a continuum of agents, or on the technical choices involved in the setup of the mechanism design problem with a continuum of agents in Section 3.

To prove Theorem 3, we construct a withdrawal-proof, but inefficient, large-market mechanism from any *ex post* efficient finite-market mechanism. The constructed large-market mechanism yields the same revenue per bidder as the finite-market mechanism.

[31]This uncertainty the realized distribution of values conditional on the state also prevents the seller from learning the state from bidders' reports. This force limits the sellers' ability to implement state-dependent pricing, but does not fundamentally affect the optimality of the uniform price auction.

[32]Our proof actually shows that the theorem remains true if the quantity $\frac{4\alpha}{\sqrt{N}}$ is replaced by the smaller quantity $\frac{4\alpha\sqrt{M}}{N}$.

[33]By contrast, the per-bidder expected revenue gap between the uniform price auction and the discriminatory price auction remains bounded below as the number N of bidders grows large if the number M of units for sale scales with N (Jackson and Kremer, 2006).

We then apply the revenue bound for withdrawal-proof mechanisms (Lemma 5) to bound above expected revenue in finite markets.

More precisely, we use Lemma 5 to bound the revenue that an *ex post* efficient finite-market mechanism can extract from any one bidder. Fix a bidder i and *ex post* efficient, withdrawal-proof, finite-market mechanism $(q_i, t_i)_{1\leq i\leq N}$. We can construct a large-market mechanism (q, t) by taking expectations over other players' types conditional on i's type and the state

$$q(\theta, s) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[q_i(\theta_i, \boldsymbol{\theta}_{-i})|\theta_i = \theta, s]$$
$$t(\theta, s) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[t_i(\theta_i, \boldsymbol{\theta}_{-i})|\theta_i = \theta, s]$$

Intuitively, $q(\theta, s)$ and $t(\theta, s)$ correspond to the interim allocations and payments of θ conditional on the state but not on the realization of others bidders' types. The mechanism (q, t) is withdrawal-proof, and delivers expected revenue equal to the expected revenue from bidder i under $(q_i, t_i)_{1\leq i\leq N}$. However, (q, t) is inefficient as a mechanism in large markets. Nevertheless, we can use Lemma 5 to bound above the the expected revenue of (q, t). We then bound the excess of this quantity over the per-bidder finite-market expected revenue of the $(M + 1)$th price auction in terms of the difference between $q(\theta, s)$ and the efficient large-market allocation rule $q^{\text{eff}}(\theta, s)$. The details of the argument are in Online Appendix D.

7. Extensions

We next return to the continuum model to relax two key assumptions of our main result. We first study optimal withdrawal-proof auction mechanisms when Assumption 2 fails, and then relax the assumption that withdrawal is costless for bidders.

7.1. **Allowing for Irregular Affiliation.**

To analyze optimal withdrawal-proof mechanisms when Assumption 2 fails, we numerically solve for the optimal mechanism for natural families of affiliated distributions. In this section, we focus on a family of lognormal distributions; in Online Appendix G, we consider a family of beta distributions and obtain similar results.

We suppose that (θ, s) are taken to be jointly log-normally distributed with log-variance 1 and log-correlation $\sqrt{\rho}$.[34] Here, the parameter ρ is the log-correlation between pairs of bidders' values. After discretizing the type and state spaces,[35] we

[34]We truncate values and states to be within 3 log-standard deviations of their unconditional means, and then renormalize states to be between 0 and 1.

[35]We use 251 types and 11 states.

	$\rho = 0.05$	$\rho = 0.25$	$\rho = 0.45$	$\rho = 0.65$	$\rho = 0.85$
$k = 0.1$	99.86%	99.00%	98.90%	99.21%	99.35%
$k = 0.3$	98.83%	98.68%	98.66%	98.62%	99.22%
$k = 0.5$	98.90%	98.49%	98.18%	98.06%	98.74%
$k = 0.7$	98.76%	98.30%	97.19%	96.31%	99.27%
$k = 0.9$	99.00%	95.65%	91.04%	90.24%	96.78%

TABLE 4. Percentage of Optimal Expected Revenue Achieved by the Uniform Price Auction with a Lognormal Distribution.

can write the seller's expected revenue maximization problem as a linear program, which we solve numerically using a commercial solver.

Table 4 reports the fraction of the expected revenue of the optimal withdrawal-proof mechanism that the uniform price auction achieves, for ranges of the quantity k of the good that is for sale and the correlation ρ.[36] These results show that except when more than half the bidders in the auction can buy the good ($k > 0.5$), the uniform price auction delivers upwards of 98% of expected revenue of the optimal mechanism even when correlation is very high. Thus, as long as the seller is unable to sell to the majority of bidders, the uniform price auction remains approximately optimal even when Assumption 2 is grossly violated.

7.2. **Incorporating a Withdrawal Cost.** In this section, we relax the assumption that *ex post* withdrawal is costless for bidders.

We extend our analysis to allow for a limited penalty for reneging on a commitment to purchase—for example, forfeiting a deposit. More precisely, we adapt the definition of withdrawal-proofness by supposing that withdrawing *ex post* costs bidders ε.

Definition 3. A mechanism (q, t) is *ε-withdrawal-proof* if for all types θ, θ', we have that

$$\mathbb{E}_s \left[\theta q(\theta, s) - t(\theta, s) \middle| \theta\right] \geq \mathbb{E}_s \left[\max\{\theta q(\theta', s) - t(\theta', s), -\varepsilon\} \middle| \theta\right].$$

Here, the right hand side is type θ's interim expected utility from reporting θ' and withdrawing to obtain utility $-\varepsilon$ if they would obtain a lower *ex post* utility.

We first note that ε-withdrawal-proofness implies Bayesian incentive compatibility and an approximate version of *ex post* individual rationality. Call a mechanism *ex*

[36]Note that due to the discreteness of the type space in our numerical analysis, the uniform price auction is not exactly optimal even in the independent private values model. We use a relatively large number (251) of types to mitigate this effect, but our results may understate the performance of the uniform price auction in the continuous type setting. An alternative approach would be to compare the expected revenue of the optimal withdrawal-proof mechanism to that of the optimal dominant strategy incentive compatible, *ex post* individually rational mechanism with discrete types.

post ε-individually rational if for all types θ and almost all states s, we have that $\theta q(\theta, s) - t(\theta, s) \geq -\varepsilon$.

Proposition 4. *If a mechanism is ε-withdrawal-proof, then it is Bayesian incentive compatible and* ex post *ε-individually rational.*[37]

We also impose interim individual rationality to ensure that it is in bidders' incentives to participate in the mechanism in the first place. A mechanism (q, t) is *interim individually rational* if for all types θ, we have that

$$\mathbb{E}_s \left[\theta q(\theta, s) - t(\theta, s) \middle| \theta \right] \geq 0.$$

This step is not needed in our main analysis because withdrawal-proofness implies *ex post* individual rationality (by Proposition 1), and hence interim individual rationality.

While the uniform price auction is not exactly optimal (among *ex post* efficient, interim individually rational, ε-withdrawal-proof mechanisms), we can nevertheless show that the uniform price auction remains approximately optimal.

Theorem 4. *Under Assumptions 1 and 2, no* ex post *efficient, interim individually rational, ε-withdrawal-proof mechanism can yield greater than $(2e-1)k\varepsilon < 4.5k\varepsilon$ more expected revenue than the uniform-price auction.*[38]

Theorem 4 provides a sense in which our foundation for the uniform price auction in terms of withdrawal-proofness is robust to the presence of withdrawal costs.

Note that in our model, revenue is normalized to represent revenue per bidder in a large market. Hence, a revenue gain of at most $4.5k\varepsilon$ represents a revenue gain of a multiple of 4.5ε per winning bidder. In particular, if ε is small relative to winning bidders' expected payments, then Theorem 4 implies under Assumptions 1 and 2, the revenue gain from using complex mechanisms will be small relative to the uniform price auction revenues.[39] In practice, requiring deposits on the order of winning bidders' expected payments may be infeasible when bidders are liquidity-constrained.

The strategy of the proof of Theorem 1 is to derive a version of the revenue bound of Lemma 5 for ε-withdrawal-proof mechanisms. The idea is to use a revenue formula

[37]The proof of Proposition 4 is similar to the proof of Proposition 1 and is in Online Appendix E.
[38]The bound of $(2e-1)k\varepsilon$ on the revenue gain from using the optimal mechanism is not tight. Indeed, the proof shows that equality cannot be achieved for $\varepsilon > 0$.
[39]For example, if the withdrawal penalty is $\varepsilon = \$1\text{B}$, the revenue gain from using a more complex mechanism than a uniform price auction is at most \$4.5B per winner, which may be small if revenue is on the order of \$44B.

similar to Lemma 4,[40] which applies since ε-withdrawal-proofness implies Bayesian incentive compatibility (Proposition 4). We then note that in the presence of a withdrawal cost of ε, terms in the revenue formula (5) corresponding to slack in *ex post* individual rationality constraints and incentive constraints for double deviations may not hold exactly, but nonetheless can only fail by at most ε. The details of the argument are in Online Appendix D.

8. Conclusion

This paper shows that considering the possibility of *ex post* withdrawal by bidders yields a foundation for uniform price auctions. By contrast, in the standard Bayesian mechanism design approach to optimal auctions, the assumption that bidders have independent private information is critical to obtaining optimal mechanisms that resemble auction formats that are used (or usable) in practice: without this assumption, "side bets" play a central role in optimal mechanisms (see, e.g., Crémer and McLean (1985, 1988), McAfee and Reny (1992), and Neeman (2004)). The idea is that bidders' withdrawal rights make it impossible to raise revenue using side bets, substantively disciplining the mechanism design problem in the presence of correlation.

In our analysis, we restricted the seller to mechanisms that implement an *ex post* efficient allocation of the good. In a follow-up paper (Haberman and Jagadeesan, 2023), we relax this assumption and show that under appropriate regularity conditions, a uniform price auction with reserve price calculated from the distribution of bids (Baliga and Vohra, 2003; Segal, 2003; Roughgarden and Talgam-Cohen, 2016) maximizes expected revenue. When enough units of the good can be produced to satisfy demand, it follows that withdrawal-proofness also provides a foundation for pricing mechanisms that are used in practice.

More broadly, we hope that withdrawal rights may provide a practical modeling framework for investigating mechanism design problems without the assumption of independent private values.

[40]Technically, we need to use a version of Lemma 5 that accounts differently for types that are never allocated the good under *ex post* efficient mechanisms, which we develop in Online Appendix D.

Appendix A. Proof of Theorem 2

The proof is an application of the Envelope Theorem (Milgrom and Segal, 2002). To be able to apply the theorem, we need to prove that interim expected utilities satisfy appropriate regularity conditions.

We first derive the local incentive constraint. Let

$$\mathcal{U}(\theta', \theta) = \int_0^1 (\theta q(\theta', s) - t(\theta', s))\, f(s|\theta) ds$$

denote the interim expected utility that type θ obtains from reporting θ'. Let

$$M = \max\left\{\sup_{s,\theta}\left\{\frac{1}{f(s|\theta)}\right\}, \sup_{s,\theta}\left\{\left|\frac{\partial f(s|\theta)}{\partial \theta}\right|\right\}, \sup_{\theta}\left\{\operatorname*{ess\,sup}_{s}\{u(\theta, s)\}\right\}\right\}, \tag{A.1}$$

which is finite because $f(s)$ and $f(\theta|s)$ are positive and continuously differentiable, and $u(\theta, s)$ is uniformly bounded below over all types θ and almost all states s.

Claim A.1. *If a Bayesian incentive compatible mechanism (q, t) yields nonnegative expected revenue and M as defined by* (A.1) *is finite, then for all types θ, θ', we have that*

$$\frac{\partial \mathcal{U}(\theta', \theta)}{\partial \theta} = \int_0^1 \left[q(\theta', s) f(s|\theta) + (u(\theta', s) - (\theta' - \theta) q(\theta', s))\, \frac{\partial f(s|\theta)}{\partial s}\right] ds$$

and that

$$\left|\frac{\partial \mathcal{U}(\theta', \theta)}{\partial \theta}\right| \leq 1 + (2M + 1) M \overline{\theta} + M^3.$$

Theorem 2 follows straightforwardly from Claim A.1 and the Envelope Theorem.

Proof of Theorem 2 assuming Claim A.1. Bayesian incentive compatibility requires that for all types θ, we have that

$$\theta \in \arg\max_{\theta' \in [\underline{\theta}, \overline{\theta}]} \mathcal{U}(\theta', \theta).$$

Moreover, Claim A.1 implies that $\frac{\partial \mathcal{U}(\theta', \theta)}{\partial \theta}$ is uniformly bounded above in magnitude. By Theorem 2 in Milgrom and Segal (2002), it follows that

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left.\frac{\partial \mathcal{U}(\theta', \hat{\theta})}{\partial \hat{\theta}}\right|_{\theta' = \hat{\theta}}.$$

Applying Claim A.1 again then yields that

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \int_0^1 \left[q(\hat{\theta}, s) f(s|\hat{\theta}) + u(\hat{\theta}, s) \frac{\partial f(s|\theta)}{\partial s}\right] ds d\hat{\theta}$$

$$= U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} Q(\hat{\theta})d\hat{\theta} + \int_{\underline{\theta}}^{\theta}\int_0^1 u(\hat{\theta}, s)\frac{\partial \log f(s|\theta)}{\partial s} f(s|\hat{\theta})dsd\hat{\theta},$$

as claimed. □

It remains to prove Claim A.1. We first prove an averaged upper bound on *ex post* utilities, which we use the exchange the order of integration with respect to s and differentation with respect to θ.

Claim A.2. *If a Bayesian incentive compatible mechanism (q, t) yields nonnegative expected revenue and M is defined by* (A.1)*, then for all types θ, we have that*

$$\int_0^1 u(\theta, s)ds \leq 2M\overline{\theta} + M(M-1).$$

Proof. Suppose for sake of deriving a contradiction that

$$\int_0^1 u(\theta, s)ds > 2M\overline{\theta} + M(M-1).$$

We show that expected revenue must be negative.

Let $\hat{\theta}$ be any type. Bayesian incentive compatibility implies that

$$\begin{aligned} U(\hat{\theta}) &\geq \int_0^1 \left(u(\theta, s) - (\theta - \hat{\theta})q(\hat{\theta}, s)\right) f(s|\hat{\theta})ds \\ &\geq \int_0^1 \left(u(\theta, s) - \overline{\theta}\right) f(s|\hat{\theta})ds \\ &= \frac{1}{M}\int_0^1 u(\theta, s)ds - \frac{\overline{\theta}}{M} + \int_0^1 \left(u(\theta, s) - \overline{\theta}\right)\left(f(s|\hat{\theta}) - \frac{1}{M}\right)ds \end{aligned}$$

By the definition of M, we have that $f(s|\hat{\theta}) - \frac{1}{M} \geq 0$ and that $u(\theta, s) \geq -M$ for almost all s. It therefore follows that

$$U(\hat{\theta}) > \frac{2M\overline{\theta} + M(M-1) - \overline{\theta}}{M} - \int_0^1 (\overline{\theta} + M)\left(f(s|\hat{\theta}) - \frac{1}{M}\right)ds = \overline{\theta}.$$

As this inequality holds for all types $\hat{\theta}$, and total surplus is at most $\overline{\theta}$, expected revenue must be negative—a contradiction. □

Proof of Claim A.1. By construction, we have that $\mathcal{U}(\theta', \theta) = \int_0^1 \mathcal{U}(\theta', \theta, s)ds$, where

$$\mathcal{U}(\theta', \theta, s) = \left(u(\theta', s) - (\theta' - \theta)q(\theta', s)\right) f(s|\theta)ds.$$

Note that $\mathcal{U}(\theta', \theta, s)$ is continuously partially differentiable in θ for all θ', s.

Consider the difference quotient

$$\frac{\mathcal{U}(\theta', \theta + \Delta\theta) - \mathcal{U}(\theta', \theta)}{\Delta\theta} = \int_0^1 \frac{\mathcal{U}(\theta', \theta + \Delta\theta, s) - \mathcal{U}(\theta', \theta, s)}{\Delta\theta} ds.$$

By the Mean Value Theorem, there exists $\tilde{\theta}(\theta', \theta, \Delta\theta, s)$ with

$$\theta < \tilde{\theta}(\theta', \theta, \Delta\theta, s) < \theta + \Delta\theta \quad \text{if } \Delta\theta > 0$$
$$\theta > \tilde{\theta}(\theta', \theta, \Delta\theta, s) > \theta + \Delta\theta \quad \text{if } \Delta\theta < 0$$

such that

$$\frac{\mathcal{U}(\theta', \theta + \Delta\theta) - \mathcal{U}(\theta', \theta)}{\Delta\theta} = \int_0^1 \frac{\partial \mathcal{U}(\theta', \tilde{\theta}, s)}{\partial \tilde{\theta}} ds. \tag{A.2}$$

We have that $\lim_{\Delta\theta \to 0} \tilde{\theta}(\theta', \theta, \Delta\theta, s) = \theta$ by construction, and due to the continuous partial differentiability of $\mathcal{U}(\theta', \theta, s)$ in θ, it follows that

$$\lim_{\Delta\theta \to 0} \frac{\partial \mathcal{U}(\theta', \tilde{\theta}(\theta', \theta, \Delta\theta, s), s)}{\partial \tilde{\theta}} = \frac{\partial \mathcal{U}(\theta', \theta, s)}{\partial \theta}. \tag{A.3}$$

To calculate $\frac{\partial \mathcal{U}(\theta', \theta)}{\partial \theta}$ using the Dominated Convergence Theorem, we need to derive an upper bound on $\left| \frac{\partial \mathcal{U}(\theta', \tilde{\theta}(\theta', \theta, \Delta\theta, s), s)}{\partial \tilde{\theta}} \right|$. Note that

$$\frac{\partial \mathcal{U}(\theta', \tilde{\theta}, s)}{\partial \tilde{\theta}} = q(\theta', s) f(s|\tilde{\theta}) + \Big(u(\theta', s) - (\theta' - \tilde{\theta}) q(\theta', s) \Big) \frac{\partial f(s|\tilde{\theta})}{\partial \tilde{\theta}}. \tag{A.4}$$

Since $u(\theta', s) \geq -M$ (by the definition of M), it follows that

$$\left| \frac{\partial \mathcal{U}(\theta', \tilde{\theta}, s)}{\partial \tilde{\theta}} \right| \leq f(s|\tilde{\theta}) + (u(\theta', s) + \overline{\theta}) \frac{\partial f(s|\tilde{\theta})}{\partial \tilde{\theta}} \leq f(s|\tilde{\theta}) + M u(\theta', s) + M\overline{\theta} + M^2 \tag{A.5}$$

The right hand side of (A.5) does not depend on $\tilde{\theta}$, and is integrable by Claim A.2. In light of (A.3) and (A.5), we can therefore apply the Dominated Convergence Theorem to (A.2) to obtain that

$$\lim_{\Delta\theta \to 0} \frac{\mathcal{U}(\theta', \theta + \Delta\theta) - \mathcal{U}(\theta', \theta)}{\Delta\theta} = \int_0^1 \frac{\partial \mathcal{U}(\theta', \theta, s)}{\partial \theta} ds$$

The first part of the claim follows by the definition of the partial derivative and (A.4). The second part of the claim follows by (A.5) and Claim A.2. □

Appendix B. Other Proofs Omitted from Section 5

We first state a preliminary claim and introduce a piece of notation that we use throughout this appendix.

Claim B.1. *Under Assumption 1, the function $p(s)$ is continuously differentiable and has positive derivative.*

The proof of Claim B.1 is straightforward: the details of the argument are in Online Appendix C. Note that Claim B.1 implies that $p(s)$ is continuous and strictly increasing, and therefore bijective between $[0,1]$ and $[p(0),p(1)]$. We can therefore introduce a left-inverse $b : [\underline{\theta},\overline{\theta}] \to [0,1]$ defined by

$$b(\theta) = \begin{cases} 0 & \text{if } \theta \leq p(0) \\ p^{-1}(\theta) & \text{if } p(0) < \theta < p(1) \\ 1 & \text{if } \theta \geq p(1) \end{cases}.$$

B.1. **Proof of Lemma 2.** The function $\Lambda(\theta)$ is well-defined on $[\underline{\theta},\overline{\theta}]$ as Claim B.1 implies that $p(s)$ gives a bijection between $[0,1]$ and $[p(0),p(1)]$.

It remains to prove that $\Lambda(\theta)$ is continuously differentiable on the intervals $[\underline{\theta},p(0)]$, $[p(0),p(1)]$, and $[p(1),\overline{\theta}]$. Note that

$$\Lambda(\theta) = \frac{f(\theta)}{f(\theta|b(\theta))}(1 - F(\theta|b(\theta)))$$

Furthermore, note that $b(\theta)$ is continuously differentiable on the intervals $[\underline{\theta},p(0)]$ and $\overline{\theta},p(1)]$ since it is constant on those intervals, and on the interval $[p(0),p(1)]$ by Claim B.1 and the Inverse Function Theorem. Last, since $f(\theta|s)$ is continuous, $F(\theta|s)$ is continuously differentiable, as can be seen by differentiating inside the integral sign. The lemma follows due to the positivity and continuous differentability of f.

B.2. **Proof of Lemma 3.** Throughout this section, we impose Assumption 1.

We reparameterize the nonlocal incentive constraints by (s,θ'), where $\theta = p(s)$. The space of constraints is then transformed to $\check{\mathcal{D}} = \{(s,\theta') \in [0,1] \times [\underline{\theta},\overline{\theta}] \mid \theta' \geq p(s)\}$.

We write $\check{\gamma}(s,\theta') = \gamma(p(s),\theta')$. By change of variables, noting that $p(s)$ is increasing and differentiable (by Claim B.1), (4) can be written as

(B.1)
$$\int_s^{b(\theta')} \frac{f(s|p(s'))}{f(s|\theta')}\frac{dp(s')}{ds'}\check{\gamma}(s',\theta')ds' = \Lambda(\theta')\int_s^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s' \partial \theta'}ds' \quad \text{for } \theta' \geq p(s).$$

For fixed θ', (B.1) is a linear Volterra equation of the first kind (with time flowing backwards from $b(\theta')$) with kernel

$$H_{\theta'}(s,s') = \frac{f(s|p(s'))}{f(s|\theta')}\frac{dp(s')}{ds'}$$

and free term

$$g_{\theta'}(s) = \Lambda(\theta') \int_s^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s' \partial \theta'} ds'.$$

We next check that the standard conditions for existence and uniqueness of solutions to equations of this type (see, e.g., Theorem 1.4.1 in Brunner (2017)), and the continuity of the relevant data in θ'.

- $H_{\theta'}(s, s')$ is continuous in (s, s', θ'). This assertion follows from the continuous differentiability of $p(s')$ (Claim B.1) and the continuity and positivity of f.
- $H_{\theta'}(s, s) \neq 0$ for all $0 \leq s \leq b(\theta')$. This assertion follows from the positivity of $p(s')$ (Claim B.1) and the positivity of f.
- $\frac{\partial H_{\theta'}(s,s')}{\partial s}$ is defined and continuous in (s, s', θ'). This assertion follows from the continuous differentiability of $p(s')$ (Claim B.1) and the continuous differentiability and positivity of f.
- $\frac{\partial g_{\theta'}(s)}{\partial s}$ is continuous in (s, θ'). This assertion follows from the twice continuous differentiability and positivity of f, the Fundamental Theorem of Calculus, and the continuity of Λ.
- $g_{\theta'}(b(\theta')) = 0$ for all θ'. This assertion holds due to the bounds of integration in the definition of $g_{\theta'}$.

Hence, Theorem 1.4.1 in Brunner (2017)) guarantees that for each θ', (B.1) has a unique continuous solution $\check{\gamma}(\cdot, \theta') : [\underline{\theta}, b(\theta')] \to \mathbb{R}$.

There is also a related integral equation obtained from (B.1) that we use in the remainder of the proof. Indeed, as discussed in Section 1.4 of Brunner (2017), by differentiating (B.1) with respect to s, we obtain that[41]

$$H_{\theta'}(s, s)\check{\gamma}(s, \theta') = -\frac{\partial g_{\theta'}(s)}{\partial s} + \int_s^{b(\theta')} \frac{\partial H_{\theta'}(s, s')}{\partial s} \check{\gamma}(s', \theta') ds' \quad \text{for } \theta' \geq p(s). \tag{B.2}$$

Proof of Part (a). It suffices to prove that the unique solution $\check{\gamma}(s, \theta') : \check{\mathcal{D}} \to \mathbb{R}$. to (B.1) that is continuous in s is in fact jointly continuous in (s, θ'). We do so by working with a formula for the solution to a linear Volterra equation of the first kind (namely (B.1)) in terms of the resolvent of the corresponding linear Volterra equation of the second kind (namely (B.2)).

Indeed, for fixed θ', (B.2) is a linear Volterra equation of the second kind (with time flowing backwards from $b(\theta')$) with kernel $K^{(1)}_{\theta'}(s, s')$ and free term $\frac{-1}{H_{\theta'}(s,s)} \frac{\partial g_{\theta'}(s)}{\partial s}$.

[41]Here, the regularity conditions on $H_{\theta'}$ and $g_{\theta'}$ and the continuity of $\check{\gamma}(\cdot, \theta')$ for fixed θ' allow us to differentiate inside the integral sign.

Defining the iterated kernels $K_{\theta'}^{(n)}(s,s')$ by

$$K_{\theta'}^{(1)}(s,s') = \frac{1}{H_{\theta'}(s,s)}\frac{\partial H_{\theta'}(s,s')}{\partial s} \quad \text{and} \quad K_{\theta'}^{(n)}(s,s') = \int_s^{s'} K_{\theta'}^{(1)}(s,t)K_{\theta'}^{(n-1)}(t,s')dt,$$

note that $K_{\theta'}^{(n)}$ is continuous in (s,s',θ') for each n by construction. Letting

$$\bar{K} = \max_{\theta'} \max_{0\leq s\leq s'\leq b(\theta')} \left|K_{\theta'}^{(1)}(s,s')\right|,$$

which exists due to the continuity of $K^{(1)}$ in (s,s',θ), Brunner (2017, p. 4) shows that

$$\left|K_{\theta'}^{(n)}(s,s')\right| \leq \frac{\bar{K}^n(s'-s)^n}{n!} \leq \frac{\bar{K}^n}{n!}.$$

Hence, the series

$$\sum_{n=1}^{\infty} K_{\theta'}^{(n)}(s,s')$$

converges absolutely and uniformly in (s,s',θ) to a continuous function—called the resolvent—which we denote by $R_{\theta'}(s,s')$. Applying Theorem 1.2.3 in Brunner (2017) to (B.2), we have that

$$\check{\gamma}(s,\theta') = -\frac{1}{H_{\theta'}(s,s)}\frac{\partial g_{\theta'}(s')}{\partial s'} - \int_s^{b(\theta')} \frac{R_{\theta'}(s,s')}{H_{\theta'}(s',s')}\frac{\partial g_{\theta'}(s')}{\partial s'}ds'. \tag{B.3}$$

The continuity of $R_{\theta'}(s,s')$, $H_{\theta'}(s,s')$, and $\frac{\partial g_{\theta'}(s)}{\partial s}$ and the nonvanishing of $H_{\theta'}(s',s')$ together imply the continuity of $\check{\gamma}(s,\theta')$ on $\check{\mathcal{D}}$.

Proof of Part (b). It suffices to prove that $\check{\gamma}(s,\theta') > 0$ for all $p(s) \leq \theta' < \overline{\theta}$.

Fix $\theta' \in [p(0),\overline{\theta})$; we prove that $\check{\gamma}(s,\theta') > 0$ for all $0 \leq s \leq b(\theta)$. First, Theorem 1.4.1 in Brunner (2017) also implies that

$$\check{\gamma}(b(\theta'),\theta') = \frac{-1}{H_{\theta'}(b(\theta'),b(\theta'))}\frac{\partial g_{\theta'}(s')}{\partial s'}\bigg|_{s'=b(\theta')}.$$

Hence, we have that

$$\check{\gamma}(b(\theta'),\theta') = \Lambda(\theta')\frac{f(b(\theta')|\theta')}{f(b(\theta')|p(b(\theta')))}\frac{\frac{\partial^2 \log f(s'|\theta')}{\partial s'\partial\theta'}\Big|_{s'=b(\theta')}}{\frac{dp(s')}{ds'}\Big|_{s'=b(\theta')}}.$$

Note that $\Lambda(\theta')$ is positive since $\theta' < \overline{\theta}$, the density f is positive by hypothesis, the mixed partial derivative $\frac{\partial^2 \log f(s'|\theta')}{\partial s'\partial\theta'}$ is positive by Assumption 1, and $\frac{dp(s')}{ds'}$ is positive by Claim B.1. Hence, we can conclude that $\check{\gamma}(b(\theta'),\theta') > 0$.

Now suppose for sake of deriving a contradiction that $\check{\gamma}(s,\theta')$ is not globally positive for $\theta' < M$. By continuity, there exists s such that $\check{\gamma}(s,\theta') = 0$ and $\check{\gamma}(s',\theta') > 0$ for all $s' \in (s, b(\theta')]$. (B.2) then implies that

$$\begin{aligned} 0 = H_{\theta'}(s,s)\check{\gamma}(s,\theta') &= -\frac{\partial g_{\theta'}(s)}{\partial s} + \int_s^{b(\theta')} \frac{\partial H_{\theta'}(s,s')}{\partial s}\check{\gamma}(s',\theta')ds' \\ &= \Lambda(\theta')\frac{\partial^2 \log f(s|\theta')}{\partial s \partial \theta'} + \int_s^{b(\theta')} \frac{\partial \log H_{\theta'}(s,s')}{\partial s} H_{\theta'}(s,s')\check{\gamma}(s',\theta')ds'. \end{aligned}$$

Note also that for $s < s'$, we have that

$$\begin{aligned} \frac{\partial \log H_{\theta'}(s,s')}{\partial s} = \frac{\partial \log f(s|p(s'))}{\partial s} - \frac{\partial \log f(s|\theta')}{\partial s} &= -\int_{p(s')}^{\theta'} \frac{\partial^2 \log f(s|\theta)}{\partial s \partial \theta} d\theta \\ &> -\int_{p(s)}^{\theta'} \frac{\partial^2 \log f(s|\theta)}{\partial s \partial \theta} d\theta, \end{aligned}$$

where the inequality follows from Assumption 1 since $p(s) \leq p(s')$ (by Claim B.1 and the positivity of f). Combining the hypothesis that $\check{\gamma}(s',\theta') > 0$ for all $s' \in (s, b(\theta')]$ with the observation that $H_{\theta'}(s,s') > 0$ (by Claim B.1), it follows that

$$\Lambda(\theta')\frac{\partial^2 \log f(s|\theta')}{\partial s \partial \theta'} < \left(\int_{p(s)}^{\theta'} \frac{\partial^2 \log f(s|\theta)}{\partial s \partial \theta} d\theta\right)\left(\int_s^{b(\theta')} H_{\theta'}(s,s')\check{\gamma}(s',\theta')ds'\right).$$

Substituting on the right hand side using (B.1), we have that

$$\Lambda(\theta')\frac{\partial^2 \log f(s|\theta')}{\partial s \partial \theta'} < \left(\int_{p(s)}^{\theta'} \frac{\partial^2 \log f(s|\theta)}{\partial s \partial \theta} d\theta\right)\Lambda(\theta')\left(\int_s^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s' \partial \theta'} ds'\right).$$

Since $\theta' < \overline{\theta}$, we have that $\Lambda(\theta') > 0$, and it follows that

$$\frac{\partial^2 \log f(s|\theta')}{\partial s \partial \theta'} < \left(\int_{p(s)}^{\theta'} \frac{\partial^2 \log f(s|\theta)}{\partial s \partial \theta} d\theta\right)\left(\int_s^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s' \partial \theta'} ds'\right).$$

In light of Assumption 1, this contradicts Assumption 2.[42]

[42]In fact, for the purposes of Theorem 1, Lemmata 3(b) and 5, and Proposition 3, Assumption 2 can be weakened to the assumption that

$$\left(\int_{p(s)}^{\theta} \frac{\partial^2 \log f(\theta'|s)}{\partial s \partial \theta'} d\theta\right)\left(\int_s^1 \frac{\partial^2 \log f(\theta|s')}{\partial s' \partial \theta} \mathbb{1}_{\theta \geq p(s')} ds'\right) \leq \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta}$$

holds for all $\theta \geq p(s)$. However, as discussed in Footnote 16, this assumption depends on the number k of units of the good that are for sale.

B.3. **Proof of Lemma 4.** The proof of the lemma, as well as the proof of Theorem 4, uses the following analytical property of the constructed Lagrange multipliers, which we prove in Online Appendix C.

Claim B.2. *Let λ be defined by (4). Under Assumption 1, for all $p(0) < \theta < p(1)$, we have that*

$$\int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta')d\theta' = f(\theta) - \lambda(\theta) + \Lambda(\theta)\frac{\partial \log f(s|\theta)}{\partial \theta}\bigg|_{s=b(\theta)}. \tag{B.4}$$

To prove the lemma, note that revenue can be written as

$$\text{Rev} = \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta Q(\theta) - U(\theta)\right) f(\theta)d\theta.$$

Combining Theorem 2 and Lemma 2, we have that

$$\int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta)U(\theta)d\theta = U(\underline{\theta})\int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta)\int_{\underline{\theta}}^{\theta} Q(\theta')d\theta' d\theta + \int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta)\int_{\underline{\theta}}^{\theta} \mathbb{E}_s\left[u(\theta', s)\frac{\partial \log f(s|\theta')}{\partial \theta'}\bigg|\theta'\right] d\theta' d\theta.$$

Applying Dirichlet's formula (which is possible due to Lemma 2) then yields that

$$\int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta)U(\theta)d\theta = \Lambda(\underline{\theta})U(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \Lambda(\theta)Q(\theta)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} \Lambda(\theta)\mathbb{E}_s\left[u(\theta, s)\frac{\partial \log f(s|\theta)}{\partial \theta}\bigg|\theta\right] d\theta.$$

It follows that

$$\text{Rev} = \int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta)f(\theta)d\theta - \Lambda(\underline{\theta})U(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} (f(\theta) - \lambda(\theta))U(\theta)d\theta - \int_{\underline{\theta}}^{\overline{\theta}} \Lambda(\theta)\mathbb{E}_s\left[u(\theta, s)\frac{\partial \log f(s|\theta)}{\partial \theta}\bigg|\theta\right] d\theta.$$

Adding $\int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial \log f(s|\theta)}{\partial \theta}\Big|_{s=b(\theta)} U(\theta)d\theta$ to the third term and subtracting it from the fourth yields that

$$\text{Rev} = \int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta)f(\theta)d\theta - \Lambda(\underline{\theta})U(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} \left(f(\theta) - \lambda(\theta) + \frac{\partial \log f(s|\theta)}{\partial \theta}\bigg|_{s=b(\theta)}\right) U(\theta)d\theta$$

$$- \int_{\underline{\theta}}^{\overline{\theta}} \Lambda(\theta) \mathbb{E}_s \left[u(\theta, s) \left(\frac{\partial \log f(s|\theta)}{\partial \theta} - \frac{\partial \log f(s|\theta)}{\partial \theta} \bigg|_{s=b(\theta)} \right) \bigg| \theta \right] d\theta.$$

It is straightforward to verify that

$$f(\theta) - \lambda(\theta) + \frac{\partial \log f(s|\theta)}{\partial \theta} \bigg|_{s=b(\theta)} = 0 \quad \text{for } \theta \notin [p(0), p(1)].$$

Applying Claim B.2, we therefore have that

$$\begin{aligned} \text{Rev} = &\int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)} \right] Q(\theta) f(\theta) d\theta - \Lambda(\underline{\theta}) U(\underline{\theta}) - \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') U(\theta) d\theta' d\theta \\ &- \int_{\underline{\theta}}^{\overline{\theta}} \Lambda(\theta) \mathbb{E}_s \left[u(\theta, s) \left(\frac{\partial \log f(s|\theta)}{\partial \theta} - \frac{\partial \log f(s'|\theta)}{\partial \theta} \bigg|_{s'=b(\theta)} \right) \bigg| \theta \right] d\theta. \end{aligned}$$

Using the partition of unity $1 = \mathbb{1}_{\theta \leq p(s)} + \mathbb{1}_{\theta > p(s)}$ and the Fundamental Theorem of Calculus, we have that

$$\begin{aligned} &\frac{\partial \log f(s|\theta)}{\partial \theta} - \frac{\partial \log f(s'|\theta)}{\partial \theta} \bigg|_{s'=b(\theta)} \\ &= \left(\int_{b(\theta)}^{s} \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} ds' \right) \mathbb{1}_{\theta \leq p(s)} - \left(\int_{s}^{b(\theta)} \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} ds' \right) \mathbb{1}_{\theta > p(s)} \\ &= \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{1}_{\theta \leq p(s') \leq p(s)} ds' - \int_s^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{1}_{\theta \geq p(s')} ds'. \end{aligned}$$

Noting that the first term vanishes for all s if $\theta > p(1)$ and the second term always vanishes for all s if $\theta < p(0)$, it follows that

$$\begin{aligned} \text{Rev} = &\int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)} \right] Q(\theta) f(\theta) d\theta - \Lambda(\underline{\theta}) U(\underline{\theta}) - \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') U(\theta) d\theta' d\theta \\ &- \int_{\underline{\theta}}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s \left[u(\theta, s) \mathbb{1}_{\theta \leq p(s') \leq p(s)} \big| \theta \right] ds' d\theta \\ &+ \int_{p(0)}^{\overline{\theta}} \mathbb{E}_s \left[u(\theta, s) \Lambda(\theta) \int_s^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{1}_{\theta \geq p(s')} ds' \bigg| \theta \right] d\theta. \end{aligned}$$

Substituting using (4) into the last term, we have that

$$\begin{aligned}\text{Rev} = \int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta - \Lambda(\underline{\theta}) U(\underline{\theta}) - \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') U(\theta) d\theta' d\theta \\ - \int_{\underline{\theta}}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s \left[u(\theta, s) \mathbb{1}_{\theta \leq p(s') \leq p(s)} \middle| \theta\right] ds' d\theta \\ + \int_{p(0)}^{\overline{\theta}} \int_{p(0)}^{\min\{\theta', p(1)\}} \gamma(\theta, \theta') \mathbb{E}_s \left[u(\theta', s) \frac{f(s|\theta)}{f(s|\theta')} \mathbb{1}_{\theta \geq p(s)} \middle| \theta'\right] d\theta d\theta'.\end{aligned}$$

Applying change of measure and Dirichlet's formula to the last term yields that

$$\begin{aligned}\text{Rev} = \int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta - \Lambda(\underline{\theta}) U(\underline{\theta}) - \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') U(\theta) d\theta' d\theta \\ - \int_{\underline{\theta}}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s \left[u(\theta, s) \mathbb{1}_{\theta \leq p(s') \leq p(s)} \middle| \theta\right] ds' d\theta \\ + \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') \mathbb{E}_s \left[u(\theta', s) \mathbb{1}_{\theta \geq p(s)} \middle| \theta\right] d\theta' d\theta.\end{aligned}$$

Grouping the fourth and sixth terms yields that

$$\begin{aligned}\text{Rev} = \int_{\underline{\theta}}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta - \Lambda(\underline{\theta}) U(\underline{\theta}) \\ - \int_{\underline{\theta}}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s \left[u(\theta, s) \mathbb{1}_{\theta \leq p(s') \leq p(s)} \middle| \theta\right] ds' d\theta \\ - \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') \left(U(\theta) - \mathbb{E}_s \left[u(\theta', s) \mathbb{1}_{\theta \geq p(s)} \middle| \theta\right]\right) d\theta' d\theta.\end{aligned}$$

Adding the quantity

$$\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} (\theta' - \theta) \gamma(\theta, \theta') \mathbb{E}_s \left[q(\theta', s) \mathbb{1}_{\theta \geq p(s)} \middle| \theta\right] d\theta' d\theta$$

and subtracting it from the final term yields (D.8).

B.4. **Proof of Lemma 5 and Proposition 3.** We first prove Lemma 5(a), and then extend the argument to prove Lemma 5(b) and Proposition 3.

Proof of Lemma 5(a). Proposition 1 guarantees that (q, t) is Bayesian incentive compatible and *ex post* individually rational. By Lemma 4, it suffices to show that the last three terms of (5) are nonnegative—i.e.,

$$\Lambda(\underline{\theta}) U(\underline{\theta}) \geq 0 \tag{B.5}$$

$$\int_{\underline{\theta}}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s \left[u(\theta, s) \mathbb{1}_{\theta \leq p(s') \leq p(s)} \middle| \theta \right] ds' d\theta \geq 0 \tag{B.6}$$

$$\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') \left(U(\theta) - \mathbb{E}_s \left[(u(\theta', s) - (\theta' - \theta) q(\theta', s)) \, \mathbb{1}_{\theta \geq p(s)} \middle| \theta \right] \right) d\theta' d\theta \geq 0 \tag{B.7}$$

—with equality for $(q, t) = (q^{\text{eff}}, t^{\text{UPA}})$. (B.5) and (B.6) follow from the *ex post* individual rationality of (q, t) (Proposition 1) because $\Lambda(\theta) \geq 0$ (by construction) and $\frac{\partial \log f(s'|\theta)}{\partial s' \partial \theta} \geq 0$ (by Assumption 1). (B.7) follows from the withdrawal-proofness of (q, t) (in particular, (2)) because $\gamma(\theta, \theta') \geq 0$ (by Lemma 3).

Proof of Lemma 5(b). We next consider $(q, t) = (q^{\text{eff}}, t^{\text{UPA}})$ and show that equality holds in (B.5)–(B.7). Note that $q^{\text{eff}}(\theta, s) = t^{\text{UPA}}(\theta, s) = 0$ for $\theta < p(s)$ by construction, so $u(\theta, s) = 0$ for $\theta < p(s)$. It follows that (B.6) holds with equality. Since $k < 1$, we have that $p(s) > \underline{\theta}$ for all states s, and hence that $U(\underline{\theta}) = 0$ for all states s. In particular, (B.5) holds with equality. Last, note that since $(q^{\text{eff}}(\theta, s), t^{\text{UPA}}(\theta, s)) = (1, p(s))$ for $\theta \geq p(s)$, we also have that

$$u(\theta', s) = u(\theta, s) + (\theta' - \theta) q(\theta', s) \quad \text{for } \theta' \geq \theta \geq p(s),$$

and hence that

$$U(\theta) = \mathbb{E}_s \left[u(\theta, s) | \theta \right] = \mathbb{E}_s \left[u(\theta, s) \mathbb{1}_{\theta \geq p(s)} \middle| \theta \right] = \mathbb{E}_s \left[(u(\theta', s) - (\theta' - \theta) q(\theta', s)) \mathbb{1}_{\theta \geq p(s)} \middle| \theta \right]$$

for $\theta' \geq \theta$. In particular, (B.7) holds with equality.

Proof of Proposition 3. By hypothesis, we must have that $q(\theta, s) = q^{\text{eff}}(\theta, s)$ for almost all types θ and states s. It follows from Lemma 5, the revenue of (q, t) must be given by (6). Since (B.5)–(B.7) hold with inequality, all three inequalities must therefore hold with equality.

Since $\Lambda(\theta) > 0$ for $\theta < \overline{\theta}$ (by construction) and $\frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} > 0$ (by Assumption 1), equality in (B.6) and *ex post* individual rationality (which must hold by Proposition 1) together imply that $u(\theta, s) = 0$ must hold for almost all $\theta \leq p(s)$. In particular, we must have that $t(\theta, s) = 0$ for almost all $\theta \leq p(s)$.

Since $\gamma(\theta, \theta') > 0$ for $\theta' < \overline{\theta}$ (by Lemma 3(b)), equality in (B.7) and withdrawal-proofness together imply that

$$U(\theta) = \mathbb{E}_s \left[(u(\theta', s) - (\theta' - \theta) q(\theta', s)) \, \mathbb{1}_{\theta \geq p(s)} \middle| \theta \right] \quad \text{for almost all } (\theta, \theta') \in \mathcal{D}.$$

Since $t(\theta, s) = 0$ holds for almost all $\theta \leq p(s)$ and q is *ex post* efficient, it follows that

$$\mathbb{E}_s \left[(t(\theta, s) - t(\theta', s)) \, \mathbb{1}_{\theta \geq p(s)} \middle| \theta \right] = 0 \quad \text{for almost all } (\theta, \theta') \in \mathcal{D}.$$

In particular, writing $\theta = p(s')$, for almost all $\theta' > p(0)$, we must have that

$$\int_0^{s'} (t(p(s), s) - t(\theta', s))\, f(s|p(s'))ds = 0 \quad \text{for almost all } s \text{ with } p(s) \leq \theta'. \tag{B.8}$$

Let $\Theta \subseteq [p(0), \overline{\theta}]$ be the (full measure) set of types θ' for which (B.8) holds.

We claim that for all $\theta', \theta'' \in \Theta$, we must have that $t(\theta', s) = t(\theta'', s)$ for almost all s with $p(s) \leq \min\{\theta', \theta''\}$. Letting $\theta', \theta'' \in \Theta$, it follows from (B.8) that

$$\int_0^{s'} (t(\theta', s) - t(\theta'', s))\, f(s|p(s'))ds = 0 \quad \text{for almost all } s \text{ with } p(s) \leq \min\{\theta', \theta'\}.$$

Define a function $T_{\theta',\theta''}(s) = \int_0^s (t(\theta', s) - t(\theta'', s))ds$, which is well-defined since $t(\theta', \cdot)$ and $t(\theta'', s)$ are assumed integrable and continuous by the Fundamental Theorem of Calculus. Integrating by parts then yields that

$$T_{\theta',\theta''}(s')f(s'|p(s)) = \int_0^{s'} T_{\theta',\theta''}(s)\frac{\partial f(s|p(s'))}{\partial s}ds \quad \text{for almost all } s \text{ with } p(s) \leq \min\{\theta', \theta'\}.$$

Due to the continuity of $T_{\theta,'\theta'}$ and the continuous differentiability of f, we must have that

$$T_{\theta',\theta''}(s') = \int_0^{s'} \frac{T_{\theta',\theta''}(s)}{f(s'|p(s))}\frac{\partial f(s|p(s'))}{\partial s}ds \quad \text{for all } s \text{ with } p(s) \leq \min\{\theta', \theta'\}.$$

This equation is a linear Volterra equation of the second kind with kernel $\frac{1}{f(s'|p(s))}\frac{\partial f(s|p(s'))}{\partial s}$ and free term 0. The kernel is continuous due to the positivity and continuous differentiability of f. Thus, Theorem 1.2.3 in Brunner (2017) guarantees that the equation

$$j(s') = \int_0^{s'} \frac{j(s)}{f(s'|p(s))}\frac{\partial f(s|p(s'))}{\partial s}ds \quad \text{for all } s \text{ with } p(s) \leq \min\{\theta', \theta'\}$$

has a unique continuous solution $j : [0, \min\{\theta, \theta'\}] \to \mathbb{R}$. Note that $j(s) = 0$ is obviously a solution. Therefore, we must have that $T_{\theta',\theta''}(s) = 0$ for all s with $p(s) \leq \min\{\theta', \theta''\}$. By the Fundamental Theorem of Calculus, it follows that $t(\theta', s) = t(\theta'', s)$ for almost all s with $p(s) \leq \min\{\theta', \theta''\}$.

Hence, there must exist a function $\hat{p} : [0, 1] \to \mathbb{R}$ such that $t(\theta, s) = \hat{p}(s)$ for all $\theta \in \Theta$ and almost all s with $p(s) \leq \theta$. As the uniform price auction is *ex post* efficient and withdrawal-proof, (q, t) must deliver at least the expected revenue of the uniform price auction. Since $t(\theta, s) = 0$ for almost all $\theta \leq p(s)$, we must have that

$$\int_0^1 \int_{p(s)}^1 \hat{p}(s)f(\theta|s)d\theta ds \geq \int_0^1 \int_{p(s)}^1 p(s)f(\theta|s)d\theta ds.$$

By *ex post* individual rationality, we must have that $\hat{p}(s) \leq p(s)$ for almost all s. As a result, it follows that $\hat{p}(s) = p(s)$ must hold for almost all s.

Therefore, we have that $t(\theta, s) = t^{\text{UPA}}(\theta, s)$ for almost all $\theta > p(s)$. As $t(\theta, s) = 0 = t^{\text{UPA}}(\theta, s)$ for almost all $\theta \leq p(s)$, we have shown that $t(\theta, s) = t^{\text{UPA}}(\theta, s)$ for almost all types θ and states s.

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ONLINE APPENDIX

Online Appendix C presents proofs that are omitted from Appendix B. Online Appendix D proves Theorems 3 and 4. Online Appendix E presents the other proofs omitted from the main text, namely the proofs of Lemma 1 and Proposition 4. Online Appendix F presents extensions of Propositions 1 and 2 and Corollary 1 to the finite-market setting of Section 6. Online Appendix G presents another family of numerical examples in which Assumption 2 is not satisfied to complement Section 7.1.

Appendix C. Proofs Omitted from Appendix B

C.1. **Proof of Claim B.1.** Recall that $p(s)$ is defined by $1-k=F(p(s)|s)$. Note that

$$\frac{\partial F(\theta|s)}{\partial \theta}=f(\theta|s)>0.$$

Noting that $F(\theta|s)=\int_{\underline{\theta}}^{\theta} f(\theta'|s)d\theta'$ and that $f(\theta|s)$ is continuously differentiable, we can differentiate inside the integral sign to obtain

$$\frac{\partial F(\theta|s)}{\partial s}=\int_{\underline{\theta}}^{\theta}\frac{\partial f(\theta'|s)}{\partial s}d\theta'=\int_{\underline{\theta}}^{\theta} f(\theta'|s)\frac{\partial \log f(\theta'|s)}{\partial s}d\theta', \tag{C.1}$$

which is continuous by the Fundamental Theorem of Calculus.

We next show that $\frac{\partial F(\theta|s)}{\partial s}<0$ for $\underline{\theta}<\theta<\overline{\theta}$, by dividing into cases based on the sign of $\frac{\partial \log f(\theta|s)}{\partial s}$.

Case 1: $\frac{\partial \log f(\theta|s)}{\partial s}\leq 0$. By Assumption 1, it follows that $\frac{\partial \log f(\theta'|s)}{\partial s}<0$ for all $\theta'<\theta$. For $\theta>\underline{\theta}$, we therefore have that

$$\frac{\partial F(\theta|s)}{\partial s}=\int_{\underline{\theta}}^{\theta} f(\theta'|s)\frac{\partial \log f(\theta'|s)}{\partial s}d\theta'<0$$

Case 2: $\frac{\partial \log f(\theta|s)}{\partial s}>0$. By Assumption 1, it follows that $\frac{\partial \log f(\theta'|s)}{\partial s}>0$ for all $\theta'>\theta$. Noting that $F(\overline{\theta}|s)=1$ and applying (C.1) for $\theta=\overline{\theta}$ yields that

$$0=\frac{\partial F(\overline{\theta}|s)}{\partial s}=\int_{\underline{\theta}}^{\overline{\theta}} f(\theta'|s)\frac{\partial \log f(\theta'|s)}{\partial s}d\theta'.$$

Hence, we have that

$$\frac{\partial F(\theta|s)}{\partial s}=-\int_{\theta}^{\overline{\theta}} f(\theta'|s)\frac{\partial \log f(\theta'|s)}{\partial s}d\theta'.$$

For $\theta M\overline{\theta}$, it follows that $\frac{\partial F(\theta|s)}{\partial s}<0$.

Hence, the Implicit Function Theorem implies that $p(s)$ is differentiable and that

$$\frac{dp(s)}{ds} = -\frac{\left.\frac{\partial F(\theta|s)}{\partial s}\right|_{\theta=p(s)}}{\left.\frac{\partial F(\theta|s)}{\partial \theta}\right|_{\theta=p(s)}}.$$

Since $0 < k < 1$, we have that $\underline{\theta} < p(s) < \overline{\theta}$, and hence that $\frac{dp(s)}{ds} > 0$. The continuous differentiability of $p(s)$ follows from the continuity of the partial derivatives of $F(\theta|s)$.

C.2. **Proof of Claim B.2.** This proof uses the notation of Appendix B. To prove the claim, We show that both sides of (B.4) satisfy the same equation, which the following claim shows has a unique solution.

Claim C.1. *Under Assumption 1, the system of equations*

(C.2)
$$\int_{p(s)}^{p(1)} \Gamma(\theta) f(s|\theta) d\theta = \int_{p(s)}^{\overline{\theta}} \Lambda(\theta') f(s|\theta') \int_{s}^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s \partial \theta'} ds' d\theta' \quad \textit{for } 0 \leq s \leq 1$$

has a unique continuous solution $\Gamma : [p(0), p(1)] \to \mathbb{R}$.

Proof. As in the proof of Lemma 3, we first reparameterize by $\theta = p(s)$. We write $\check{\Gamma}(s) = \Gamma(p(s))$. By change of variables, noting that $p(s)$ is increasing and differentiable (by Claim B.1), (C.2) can be written as

$$\int_{s}^{1} \check{\Gamma}(s') f(s|p(s')) \frac{dp(s')}{ds'} ds' = \int_{p(s)}^{\overline{\theta}} \Lambda(\theta') f(s|\theta') \int_{s}^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s \partial \theta'} ds' d\theta'. \tag{C.3}$$

It suffices to prove that (C.3) has a unique continuous solution $\check{\Gamma} : [0, 1] \to \mathbb{R}$.

(C.3) is a linear Volterra equation of the first kind (with time flowing backwards from 1) with kernel

$$\mathcal{H}(s, s') = f(s|p(s')) \frac{dp(s')}{ds'}$$

and free term

$$\mathcal{g}(s) = \int_{p(s)}^{\overline{\theta}} \Lambda(\theta') f(s|\theta') \int_{s}^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s \partial \theta'} ds' d\theta'.$$

Analogous arguments to the proof of Lemma 3 show that the standard conditions for existence and uniqueness of solutions to equations of this type (see, e.g., Theorem 1.4.1 in Brunner (2017)) are satisfied. Hence, Theorem 1.4.1 in Brunner (2017)) guarantees that (C.3) has a unique continuous solution $\check{\Gamma} : [0, 1] \to \mathbb{R}$. □

To complete the proof of Claim B.2, we first show that

$$\Gamma_1(\theta) = \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta')d\theta'$$

is a continuous solution to (C.2). The continuity of Γ_1 follows from the continuity of γ on $\mathcal{D}$ (Lemma 3(a)). By the definition of Γ_1 and Dirichlet's formula, we have that

$$\begin{aligned}\int_{p(s)}^{p(1)} \Gamma(\theta) f(s|\theta)d\theta &= \int_{p(s)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta')d\theta' f(s|\theta)d\theta \\ &= \int_{p(s)}^{\overline{\theta}} \int_{p(s)}^{\min\{\theta', p(1)\}} \gamma(\theta, \theta') f(s|\theta)d\theta d\theta'.\end{aligned}$$

It follows from (4) that

$$\int_{p(s)}^{\min\{\theta', p(1)\}} \gamma(\theta, \theta') f(s|\theta)d\theta = \Lambda(\theta') f(s|\theta') \int_{s}^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s' \partial \theta'} ds' \quad \text{for } \theta' \geq p(s).$$

Integrating from $\theta' = p(s)$ to $\theta' = \overline{\theta}$, we see that Γ_1 is a solution to (C.2).

We next show that the function $\Gamma_2 : [p(0), p(1)] \to \mathbb{R}$ defined by

$$\Gamma_2(\theta) = f(\theta) - \lambda(\theta) + \Lambda(\theta) \frac{\partial \log f(s|\theta)}{\partial \theta}\bigg|_{s=b(\theta)} \quad \text{for } \theta \in (p(0), p(1)),$$

and extended to $[p(0), p(1)]$ by continuity, is a continuous solution to (C.2). The continuity of $\Gamma_2(\theta)$ follows from Lemma 2 and the positivity and continuous differentiability of f.

Note that

$$\int_{p(s)}^{p(1)} f(\theta) f(s|\theta)d\theta = \int_{p(s)}^{p(1)} f(s) f(\theta|s)d\theta = f(s)\big(k - (1 - F(p(1)|s))\big),$$

where the first equality follows from Bayes' rule and the second equality follows from the definition of $p(s)$. Separately, integrating by parts yields that

$$\int_{p(s)}^{p(1)} \lambda(\theta) f(s|\theta)d\theta = -\Lambda(p(1)) f(s|p(1)) + kf(s) + \int_{p(s)}^{p(1)} \Lambda(\theta) \frac{\partial \log f(s|\theta)}{\partial \theta} f(s|\theta)d\theta.$$

Putting these two pieces together and using the definition of Γ_2, we have that

$$\int_{p(s)}^{p(1)} \Gamma_2(\theta) f(s|\theta) d\theta = \Lambda(p(1)) f(s|p(1)) - f(s)(1 - F(p(1)|s)) + \int_{p(s)}^{p(1)} \Lambda(\theta) \left(\left. \frac{\partial \log f(s|\theta)}{\partial \theta} \right|_{s=b(\theta)} - \frac{\partial \log f(s|\theta)}{\partial \theta} \right) f(s|\theta) d\theta. \tag{C.4}$$

Note also that as $b(\theta') = 1$ for $\theta' \geq p(1)$, we have that

$$\begin{aligned}
&\int_{p(1)}^{\overline{\theta}} \Lambda(\theta') f(s|\theta') \int_s^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s \partial \theta'} ds' d\theta' \\
&= \int_{p(1)}^{\overline{\theta}} \Lambda(\theta') f(s|\theta') \left(\frac{\partial \log f(1|\theta')}{\partial \theta'} - \frac{\partial \log f(s|\theta')}{\partial \theta'} \right) d\theta' && \text{evaluating inner integral} \\
&= - \int_{p(1)}^{\overline{\theta}} \Lambda(\theta') f(1|\theta') \frac{\partial \frac{f(s|\theta')}{f(1|\theta')}}{\partial \theta'} d\theta' && \text{properties of log} \\
&= \Lambda(p(1)) f(s|p(1)) + \int_{p(1)}^{\overline{\theta}} \frac{d\left(\Lambda(\theta') f(1|\theta')\right)}{d\theta'} \frac{f(s|\theta')}{f(1|\theta')} d\theta' && \text{by parts} \\
&= \Lambda(p(1)) f(s|p(1)) - \int_{p(1)}^{\overline{\theta}} f(1) f(\theta'|1) \frac{f(s|\theta')}{f(1|\theta')} d\theta' && \text{definition of } \Lambda \\
&= \Lambda(p(1)) f(s|p(1)) - \int_{p(1)}^{\overline{\theta}} f(s) f(\theta'|s) d\theta' && \text{Bayes' rule} \\
&= \Lambda(p(1)) f(s|p(1)) - f(s)(1 - F(p(1)|s)).
\end{aligned}$$

It follows that

$$\int_{p(s)}^{\overline{\theta}} \Lambda(\theta') f(s|\theta') \int_s^{b(\theta')} \frac{\partial^2 \log f(s'|\theta')}{\partial s \partial \theta'} ds' d\theta' = \Lambda(p(1)) f(s|p(1)) - f(s)(1 - F(p(1)|s)) + \int_{p(s)}^{p(1)} \Lambda(\theta) \left(\left. \frac{\partial \log f(s|\theta)}{\partial \theta} \right|_{s=b(\theta)} - \frac{\partial \log f(s|\theta)}{\partial \theta} \right) f(s|\theta) d\theta. \tag{C.5}$$

Combining (C.4) and (C.5), we see that Γ_2 satisfies (C.2).

Claim C.1 then guarantees that $\Gamma_1(\theta) = \Gamma_2(\theta)$ for $p(0) \leq \theta \leq p(1)$, as desired.

Appendix D. Proofs of Theorems 3 and 4

Throughout this appendix, we use the notation of Appendix B.

Both proofs also use the following claim to bound Lagrange multipliers.

Claim D.1. *Under Assumptions 1 and 2, we have that*

$$\frac{f(\theta|s')}{f(\theta|s)} \cdot \frac{1-F(\theta'|s)}{1-F(\theta'|s')} \leq e$$

for all types θ, θ' and states s, s'. In particular, taking $\theta' = \underline{\theta}$, we have that

$$\frac{f(\theta|s')}{f(\theta|s)} \leq e$$

for all types θ and states s, s'.

Proof. By Lemma 1(a), Assumption 2′ must hold. It follows that

$$\frac{f(\theta|s')}{f(\theta|s)} f(\theta''|s) \leq e f(\theta''|s')$$

for all types θ''. Integrating then yields that

$$\frac{f(\theta|s')}{f(\theta|s)}(1-F(\theta'|s)) = \frac{f(\theta|s')}{f(\theta|s)} \int_{\theta'}^{\overline{\theta}} f(\theta''|s) d\theta'' \leq e \int_{\theta'}^{\overline{\theta}} f(\theta''|s') d\theta'' = e(1-F(\theta'|s')),$$

as claimed. □

D.1. **Proof of Theorem 3.** Let $(q_i, t_i)_{1 \leq i \leq N}$ be an *ex post* efficient, withdrawal-proof, finite-market mechanism. Let $k = \frac{M}{N}$, and define a corresponding (large-market) mechanism by

$$q(\theta, s) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[q_i(\theta_i, \boldsymbol{\theta}_{-i})|\theta_i = \theta, s]$$
$$t(\theta, s) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[t_i(\theta_i, \boldsymbol{\theta}_{-i})|\theta_i = \theta, s].$$

We first show that (q, t) is withdrawal-proof. Consider a type θ and a report θ', and note that

$$\begin{aligned}
&\mathbb{E}_s\left[\max\{\theta q(\theta', s) - t(\theta', s), 0\}\middle|s\right] \\
=&\mathbb{E}_s\left[\max\left\{\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta q_i(\theta', \boldsymbol{\theta}_{-i}) - t_i(\theta', \boldsymbol{\theta}_{-i})\middle|\theta_i = \theta', s\right], 0\right\}\middle|s\right] \\
=&\mathbb{E}_s\left[\max\left\{\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta q_i(\theta', \boldsymbol{\theta}_{-i}) - t_i(\theta', \boldsymbol{\theta}_{-i})\middle|\theta_i = \theta, s\right], 0\right\}\middle|s\right] \\
\leq&\mathbb{E}_s\left[\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\max\{\theta q_i(\theta', \boldsymbol{\theta}_{-i}) - t_i(\theta', \boldsymbol{\theta}_{-i}), 0\}\middle|\theta_i = \theta, s\right]\middle|s\right] \\
=&\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\max\{\theta q_i(\theta', \boldsymbol{\theta}_{-i}) - t_i(\theta', \boldsymbol{\theta}_{-i}), 0\}\middle|\theta_i = \theta\right] \\
\leq&\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta q_i(\theta, \boldsymbol{\theta}_{-i}) - t_i(\theta, \boldsymbol{\theta}_{-i})\middle|\theta_i = \theta\right] \\
=&\theta q(\theta, s) - t(\theta, s),
\end{aligned}$$

as desired, where the second equality holds due to the independence of θ_i and $\boldsymbol{\theta}_{-i}$ conditional on s, and the inequalities hold by Jensen's inequality and due to the withdrawal-proofness of $(q_i, t_i)_{1\leq i\leq N}$, respectively.

It follows from Lemma 5(a) that the expected revenue from bidder i from finite-market mechanism $(q_i, t_i)_{1\leq i\leq N}$ is at most

$$\text{(D.1)}\quad \mathrm{Rev}_i^{\mathrm{fin}} \leq \mathbb{E}_s\left[\int_{\underline{\theta}}^{\overline{\theta}}\left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] q(\theta,s) f(\theta|s) d\theta\right] + \int_{p(0)}^{p(1)}\int_{\theta}^{\overline{\theta}} (\theta'-\theta)\gamma(\theta,\theta')\mathbb{E}_s\left[q(\theta',s)\mathbb{1}_{\theta\geq p(s)}\middle|\theta\right] d\theta' d\theta.$$

Applying the standard Myerson (1981) revenue formula state by state, we see that the expected revenue per bidder of the $(M+1)$th auction in the finite market can be written as

$$\text{(D.2)}\qquad \mathrm{Rev}_{\mathrm{UPA}}^{\mathrm{fin}} = \mathbb{E}_s\left[\int_{\underline{\theta}}^{\overline{\theta}}\left(\theta - \frac{1-F(\theta|s)}{f(\theta|s)}\right) q(\theta,s) f(\theta|s) ds\right].$$

and that the expected revenue of the uniform price auction in the large market can be written as

$$\text{(D.3)}\qquad \mathrm{Rev}_{\mathrm{UPA}}^{\mathrm{lg}} = \mathbb{E}_s\left[\int_{\underline{\theta}}^{\overline{\theta}}\left(\theta - \frac{1-F(\theta|s)}{f(\theta|s)}\right) q^{\mathrm{eff}}(\theta,s) f(\theta|s) ds\right].$$

It also follows from Lemma 5(b) that the expected revenue of the uniform price auction in the large market can be written as

$$\text{(D.4)}\quad \mathrm{Rev}_{\mathrm{UPA}}^{\mathrm{lg}} = \mathbb{E}_s\left[\int_{\underline{\theta}}^{\overline{\theta}}\left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] q^{\mathrm{eff}}(\theta,s) f(\theta|s) d\theta\right] + \int_{p(0)}^{p(1)}\int_{\theta}^{\overline{\theta}} (\theta'-\theta)\gamma(\theta,\theta')\mathbb{E}_s\left[q^{\mathrm{eff}}(\theta',s)\mathbb{1}_{\theta\geq p(s)}\middle|\theta\right] d\theta' d\theta.$$

Adding (D.1) and (D.3) and subtracting the sum of (D.2) and (D.4) yields that

$$\mathrm{Rev}_i^{\mathrm{fin}} - \mathrm{Rev}_{\mathrm{UPA}}^{\mathrm{fin}} \leq \mathbb{E}_s\left[\int_{\underline{\theta}}^{\overline{\theta}}\left[\frac{1-F(\theta|s)}{f(\theta|s)} - \frac{\Lambda(\theta)}{f(\theta)}\right]\left(q(\theta,s) - q^{\mathrm{eff}}(\theta,s)\right) f(\theta|s) d\theta\right] + \int_{p(0)}^{p(1)}\int_{\theta}^{\overline{\theta}} (\theta'-\theta)\gamma(\theta,\theta')\mathbb{E}_s\left[\left(q(\theta,s) - q^{\mathrm{eff}}(\theta',s)\right)\mathbb{1}_{\theta\geq p(s)}\middle|\theta\right] d\theta' d\theta.$$

Noting that $\gamma(\theta, \theta') \geq 0$ (by Lemma 3(b)) and that $q(\theta, s) < q^{\text{eff}}(\theta, s) = 1$ for $\theta \geq p(s)$ (by construction), it follows that

(D.5)

$$\text{Rev}_i^{\text{fin}} - \text{Rev}_{\text{UPA}}^{\text{fin}} < \mathbb{E}_s \left[\int_{\underline{\theta}}^{\overline{\theta}} \left[\frac{1 - F(\theta|s)}{f(\theta|s)} - \frac{\Lambda(\theta)}{f(\theta)} \right] \left(q(\theta, s) - q^{\text{eff}}(\theta, s) \right) f(\theta|s) d\theta \right].$$

We next bound the quantity $\frac{1-F(\theta|s)}{f(\theta|s)} - \frac{\Lambda(\theta)}{f(\theta)}$. Noting that $\frac{\Lambda(\theta)}{f(\theta)} = \frac{1-F(\theta|p(\theta))}{f(\theta|p(\theta)}$, we have that

$$\begin{aligned} \frac{1 - F(\theta|s)}{f(\theta|s)} - \frac{\Lambda(\theta)}{f(\theta)} &= \frac{1 - F(\theta|p(\theta))}{f(\theta|p(\theta)} \left(\frac{f(\theta|b(\theta))}{f(\theta|s)} \frac{1 - F(\theta|s)}{1 - F(\theta|b(\theta))} - 1 \right) \\ &\geq \left(\frac{1}{e} - 1 \right) \frac{1 - F(\theta|s)}{f(\theta|s)} \geq \left(\frac{1}{e} - 1 \right) \alpha, \end{aligned}$$

where the first inequality is due to Claim D.1 and the second inequality follows from the hypothesis of the theorem. Applying similar logic with the roles of s and $b(\theta)$ reversed yields that

$$\frac{1 - F(\theta|s)}{f(\theta|s)} - \frac{\Lambda(\theta)}{f(\theta)} \leq \left(1 - \frac{1}{e} \right) \alpha.$$

It follows that

$$\begin{aligned} &\int_{\underline{\theta}}^{\overline{\theta}} \left[\frac{1 - F(\theta|s)}{f(\theta|s)} - \frac{\Lambda(\theta)}{f(\theta)} \right] \left(q(\theta, s) - q^{\text{eff}}(\theta, s) \right) f(\theta|s) d\theta \\ &\qquad \leq \left(1 - \frac{1}{e} \right) \alpha \int_{\underline{\theta}}^{\overline{\theta}} \left| q(\theta, s) - q^{\text{eff}}(\theta, s) \right| f(\theta|s) d\theta \end{aligned}$$

By construction, we have that $q(\theta, s) < 1 = q^{\text{eff}}(\theta, s)$ for $\theta < p(s)$, that $q(\theta, s) > 0 = q^{\text{eff}}(\theta, s)$ for $\theta > p(s)$, and that

$$\int_{\underline{\theta}}^{\overline{\theta}} q(\theta, s) f(\theta|s) d\theta = k = \int_{\underline{\theta}}^{\overline{\theta}} q^{\text{eff}}(\theta, s) f(\theta|s) d\theta.$$

We therefore have that

$$\int_{\underline{\theta}}^{\overline{\theta}} \left| q(\theta, s) - q^{\text{eff}}(\theta, s) \right| f(\theta|s) d\theta = 2 \int_0^{p(s)} q(\theta, s) f(\theta|s) d\theta$$

and hence that

(D.6)

$$\begin{aligned} &\int_{\underline{\theta}}^{\overline{\theta}} \left[\frac{1 - F(\theta|s)}{f(\theta|s)} - \frac{\Lambda(\theta)}{f(\theta)} \right] \left(q(\theta, s) - q^{\text{eff}}(\theta, s) \right) f(\theta|s) d\theta \\ &\qquad \leq 2 \left(1 - \frac{1}{e} \right) \alpha \int_0^{\underline{\theta}} q(\theta, s) f(\theta|s) d\theta. \end{aligned}$$

To bound the right hand side, we use the following claim.

Claim D.2. *If $(q_i)_{1\le i\le N}$ is an* ex post *efficient finite-market allocation rule, then for all states s, we have that*

$$\int_{\underline{\theta}}^{p(s)} q(\theta,s)f(\theta|s)d\theta \le \frac{e^{\frac{1}{6}}\sqrt{2\pi M}}{N}.$$

To complete the proof of the theorem assuming Claim D.2, note that combining (D.5) and (D.6) with Claim D.2 yields that

$$\begin{aligned}\mathrm{Rev}_i^{\mathrm{fin}} - \mathrm{Rev}_{\mathrm{UPA}}^{\mathrm{fin}} &< 2\left(1-\frac{1}{e}\right)\alpha\mathbb{E}_s\left[\int_0^{\underline{\theta}} q(\theta,s)f(\theta|s)d\theta\right]\\ &\le 2\left(1-\frac{1}{e}\right)\alpha\frac{e^{\frac{1}{6}}\sqrt{2\pi M}}{N} < \frac{4\alpha\sqrt{M}}{N} < \frac{4\alpha}{\sqrt{N}}.\end{aligned}$$

Since this bound applies for all agents i, excess per-bidder expected revenue of $(q_i,t_i)_{1\le i\le N}$ over the $(M+1)$th price auction must be at most $\frac{4\alpha}{\sqrt{N}}$.

It remains to prove Claim D.2.

Proof of Claim D.2. Note that

$$\int_{\underline{\theta}}^{p(s)} q(\theta,s)f(\theta|s) = \mathbb{E}_{\boldsymbol{\theta}}\left[q_i(\theta_i,\boldsymbol{\theta}_{-i})\mathbb{1}_{\theta_i\le p(s)}\middle|s\right].$$

It follows that

$$N\int_{\underline{\theta}}^{p(s)} q(\theta,s)f(\theta|s) = \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i=1}^N q_i(\theta_i,\boldsymbol{\theta}_{-i})\mathbb{1}_{\theta_i\le p(s)}\middle|s\right].$$

Let $\theta^{(1)},\dots,\theta^{(M)}$ denote the highest M bidders' values in decreasing order. Due to *ex post* efficiency, it follows that

$$\begin{aligned}N\int_{\underline{\theta}}^{p(s)} q(\theta,s)f(\theta|s)d\theta = \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{j=1}^M \mathbb{1}_{\theta^{(j)}\le p(s)}\middle|s\right] &= \sum_{j=1}^M \Pr_{\boldsymbol{\theta}}\left[\theta^{(j)}\le p(s)\middle|s\right]\\ &= \sum_{j=1}^M \Pr_{\boldsymbol{\theta}}\left[F(\theta^{(j)}|s)\le 1-k\middle|s\right],\end{aligned}$$

where the last equality is due to the definition of $p(s)$.

By the Probability Integral Transform, the distribution of $F(\theta^{(j)}|s)$ conditional on s is the distribution of the jth largest of N standard uniform random variables, which is a Beta distribution with shape parameters $\alpha = N+1-j$ and $\beta = j$. Noting that the distribution of logit $F(\theta^{(j)}|s)$ conditional on s is then a type IV generalized logistic

distribution with shape parameters $\alpha = N + 1 - j$ and $\beta = j$,[43] a Chernoff bound with parameter $j - M$ then shows that for $1 \leq j \leq M$, we have that

$$\begin{aligned}\Pr_{\boldsymbol{\theta}}\left[F(\theta^{(j)}|s) \leq 1-k\middle|s\right] &= \Pr_{\boldsymbol{\theta}}\left[\operatorname{logit} F(\theta^{(j)}|s) \leq \operatorname{logit}(1-k)\middle|s\right] \\ &\leq \frac{\Gamma(N+1-M)\Gamma(M)}{\Gamma(N+1-j)\Gamma(j)}\left(\frac{1-k}{k}\right)^{M-j},\end{aligned}$$

where Γ denotes the Gamma function. It follows that

$$\begin{aligned}N\int_{\underline{\theta}}^{p(s)} q(\theta,s)f(\theta|s)d\theta &\leq \sum_{j=1}^{M}\frac{\Gamma(N+1-M)\Gamma(M)}{\Gamma(N+1-j)\Gamma(j)}\left(\frac{1-k}{k}\right)^{M-j} \\ &= \frac{\left(\frac{1-k}{k}\right)^{M-1}}{\binom{N-1}{M-1}}\sum_{j=1}^{M}\binom{N-1}{j-1}\left(\frac{k}{1-k}\right)^{j-1} \\ &\leq \frac{\left(\frac{1-k}{k}\right)^{M-1}}{\binom{N-1}{M-1}}\sum_{j=1}^{N}\binom{N-1}{j-1}\left(\frac{k}{1-k}\right)^{j-1} \\ &= \frac{\left(\frac{1-k}{k}\right)^{M-1}}{\binom{N-1}{M-1}}\left(1+\frac{k}{1-k}\right)^{N-1} \\ &= \frac{(N-M)!(M-1)!}{(N-1)!k^{M-1}(1-k)^{N-M}},\end{aligned}$$

where the first equality is a consequence of the formula $\Gamma(n+1) = n!$ for nonnegative integers n, and the second equality follows from the Binomial Theorem. Since $k = \frac{M}{N}$, it follows that

$$N\int_{\underline{\theta}}^{p(s)} q(\theta,s)f(\theta|s)d\theta \leq \frac{(N-M)!M!}{N!k^{M}(1-k)^{N-M}} = \frac{(N-M)!(M)!}{N!\left(\frac{M}{N}\right)^{M}\left(\frac{N-M}{N}\right)^{N-M}}.$$

Applying Stirling's Approximation to bound the factorials,[44] we therefore have that

$$N\int_{\underline{\theta}}^{p(s)} q(\theta,s)f(\theta|s)d\theta \leq e^{\frac{1}{6}}\sqrt{2\pi\frac{M(N-M)}{N}} \leq e^{\frac{1}{6}}\sqrt{2\pi M},$$

and the claim follows. □

[43]Here, we write $\operatorname{logit} p = \log\frac{p}{1-p}$.

[44]Stirling's Approximation states that

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} e^{\frac{1}{12n+1}} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} e^{\frac{1}{12n}}.$$

We use a slightly weaker version

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n} e^{\frac{1}{12}}.$$

D.2. **Proof of Theorem 4.** The key to the proof of Theorem 1 is the following approximate version of Lemma 5.

Lemma D.1. *Let λ and γ be defined by* (3) *and* (4)*. Under Assumptions 1 and 2, if (q,t) is an ε-withdrawal-proof, interim individually rational mechanism that yields a nonnegative expected revenue, then it yields an expected revenue of at most*

(D.7)
$$\underbrace{\int_{p(0)}^{\overline{\theta}}\left[\theta-\frac{\Lambda(\theta)}{f(\theta)}\right]Q(\theta)f(\theta)d\theta}_{\text{interim virtual values}}+\underbrace{\int_{p(0)}^{p(1)}\int_{\theta}^{\overline{\theta}}(\theta'-\theta)\gamma(\theta,\theta')\mathbb{E}_s\left[q(\theta',s)\mathbb{1}_{\theta\geq p(s)}\middle|\theta\right]d\theta'd\theta}_{\text{expected correction to interim virtual values}}$$
$$+\underbrace{\int_{\underline{\theta}}^{p(0)}\mathbb{E}_s\left[t(\theta,s)\middle|\theta\right]d\theta}_{\text{revenue from types below } p(0)}+(2e-1)k\varepsilon,$$

with strict inequality for $\varepsilon>0$.

Compared to (6), (D.7) ignores the interim virtual values of types $\theta<p(0)$, but instead accounts for the revenue explicitly in the third term. Since Lemma D.1 assumes ε-withdrawal-proofness and interim individual rationality rather than withdrawal-proofness, there is also an error term of $(2e-1)k\varepsilon$ that comes from the possibility of desirable double deviations.

Theorem 4 follows straightforwardly from Lemma D.1.

Proof of Theorem 4 assuming Lemma D.1. Let (q,t) be an *ex post* efficient, interim individually rational, ε-withdrawal-proof mechanism. *Ex post* efficiency and interim individual rationality for types $\theta<p(0)$ together imply that $\mathbb{E}_s[t(\theta,s)|\theta]\leq 0$ for $\theta<p(0)$. Lemma D.1 therefore implies that expected revenue is at most

$$\int_{p(0)}^{\overline{\theta}}\left[\theta-\frac{\Lambda(\theta)}{f(\theta)}\right]Q(\theta)f(\theta)d\theta+\int_{p(0)}^{p(1)}\int_{\theta}^{\overline{\theta}}(\theta'-\theta)\gamma(\theta,\theta')\mathbb{E}_s\left[q(\theta',s)\mathbb{1}_{\theta\geq p(s)}\middle|\theta\right]d\theta'd\theta$$
$$+(2e-1)k\varepsilon.$$

Note that $Q(\theta)=0$ for $\theta<p(0)$ by *ex post* efficiency. Therefore, since the uniform price auction is also *ex post* efficient, it follows from Lemma 5(b) that the revenue of the uniform price auction is given by

$$\int_{p(0)}^{\overline{\theta}}\left[\theta-\frac{\Lambda(\theta)}{f(\theta)}\right]Q(\theta)f(\theta)d\theta+\int_{p(0)}^{p(1)}\int_{\theta}^{\overline{\theta}}(\theta'-\theta)\gamma(\theta,\theta')\mathbb{E}_s\left[q(\theta',s)\mathbb{1}_{\theta\geq p(s)}\middle|\theta\right]d\theta'd\theta,$$

and the theorem follows. □

The remainder of this section is devoted to the proof of Lemma D.1. The proof relies on an extension of Lemma 4 that separately accounts for the revenue of types below a cutoff θ^*.

Lemma D.2. *Let λ and γ be defined by (3) and (4), and let $\theta^* \leq p(0)$ be a type. Under Assumption 1, if a Bayesian incentive compatible mechanism (q,t) yields a nonnegative expected revenue, and $u(\theta,s)$ is uniformly bounded below over all types θ and almost all states s, then (q,t) yields an expected revenue of*

(D.8)

$$\underbrace{\int_{\theta^*}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta}_{\text{(i) interim virtual values}} + \underbrace{\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} (\theta' - \theta)\gamma(\theta,\theta') \mathbb{E}_s\left[q(\theta',s)\mathbb{1}_{\theta \geq p(s)} \middle| \theta\right] d\theta' d\theta}_{\text{(ii) expected correction to interim virtual values}}$$

$$- \underbrace{\Lambda(\theta^*)U(\theta^*)}_{\text{(iii) expected rent of } \theta^*} - \underbrace{\int_{\theta^*}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s\left[u(\theta,s)\mathbb{1}_{\theta \leq p(s') \leq p(s)} \middle| \theta\right] ds' d\theta}_{\text{(iv) ex post rents of other types conditional on non-allocation under } q^{\text{eff}}}$$

$$- \underbrace{\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta,\theta') \left(U(\theta) - \mathbb{E}_s\left[\left(u(\theta',s) - (\theta' - \theta)q(\theta',s)\right)\mathbb{1}_{\theta \geq p(s)} \middle| \theta\right]\right) d\theta' d\theta}_{\text{(v) slack in IC constraints for double deviations}}$$

$$+ \underbrace{\int_{\underline{\theta}}^{\theta^*} \mathbb{E}_s\left[t(\theta,s) \middle| \theta\right] d\theta}_{\text{(vi) revenue from types below } \theta^*}.$$

Lemma 4 is the special case of Lemma D.2 for $\theta^* = \underline{\theta}$. Indeed, for $\theta^* = \underline{\theta}$, the first five terms of (D.8) coincide with those of (5), and the sixth term of (D.8) vanishes.

Intuitively, Lemma D.2 exploits the indeterminacy of the Lagrange multipliers on the local incentive constraint for types below $p(0)$ (see Footnote 22). An alternative choice for these Lagrange multipliers is to set them equal to 0 for types below any cutoff $\theta^* \leq p(0)$. Lemma D.2 corresponds to this choice of Lagrange multipliers, rather than the one used in the text. This alternative choice is useful for the proof of Lemma D.1 because it places more emphasis on the interim individual rationality constraint for types below θ^*, which is assumed to hold exactly, than the *ex post* individual rationality constraints for such types, which is we only know holds up to ε.

The proof of Lemma D.2 is similar to the proof of Lemma 4, but takes a different combination of local incentive constraints.

Proof of Lemma D.2. Note that revenue can be written as

$$\text{Rev} = \int_{\theta^*}^{\overline{\theta}} \left(\theta Q(\theta) - U(\theta)\right) f(\theta) d\theta + \int_{\underline{\theta}}^{\theta^*} \mathbb{E}_s \left[t(\theta, s) \middle| \theta\right] d\theta.$$

Combining Theorem 2 and Lemma 2, we have that

$$\int_{\theta^*}^{\overline{\theta}} \lambda(\theta) U(\theta) d\theta = U(\theta^*) \int_{\theta^*}^{\overline{\theta}} \lambda(\theta) d\theta + \int_{\theta^*}^{\overline{\theta}} \lambda(\theta) \int_{\theta^*}^{\theta} Q(\theta') d\theta' d\theta + \int_{\theta^*}^{\overline{\theta}} \lambda(\theta) \int_{\theta^*}^{\theta} \mathbb{E}_s \left[u(\theta', s) \frac{\partial \log f(s|\theta')}{\partial \theta'} \middle| \theta' \right] d\theta' d\theta.$$

Applying Dirichlet's formula (which is possible due to Lemma 2) then yields that

$$\int_{\theta^*}^{\overline{\theta}} \lambda(\theta) U(\theta) d\theta = \Lambda(\theta^*) U(\theta^*) + \int_{\theta^*}^{\overline{\theta}} \Lambda(\theta) Q(\theta) d\theta + \int_{\theta^*}^{\overline{\theta}} \Lambda(\theta) \mathbb{E}_s \left[u(\theta, s) \frac{\partial \log f(s|\theta)}{\partial \theta} \middle| \theta \right] d\theta.$$

It follows that

$$\text{Rev} = \int_{\underline{\theta}}^{\theta^*} \mathbb{E}_s \left[t(\theta, s) \middle| \theta\right] d\theta + \int_{\theta^*}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta - \Lambda(\theta^*) U(\theta^*) - \int_{\theta^*}^{\overline{\theta}} (f(\theta) - \lambda(\theta)) U(\theta) d\theta - \int_{\theta^*}^{\overline{\theta}} \Lambda(\theta) \mathbb{E}_s \left[u(\theta, s) \frac{\partial \log f(s|\theta)}{\partial \theta} \middle| \theta \right] d\theta.$$

Adding $\int_{\theta^*}^{\overline{\theta}} \frac{\partial \log f(s|\theta)}{\partial \theta}\Big|_{s=b(\theta)} U(\theta) d\theta$ to the third term and subtracting it from the fourth yields that

$$\text{Rev} = \int_{\underline{\theta}}^{\theta^*} \mathbb{E}_s \left[t(\theta, s) \middle| \theta\right] d\theta + \int_{\theta^*}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta - \Lambda(\theta^*) U(\theta^*) - \int_{\theta^*}^{\overline{\theta}} \left(f(\theta) - \lambda(\theta) + \frac{\partial \log f(s|\theta)}{\partial \theta}\Big|_{s=b(\theta)} \right) U(\theta) d\theta - \int_{\theta^*}^{\overline{\theta}} \Lambda(\theta) \mathbb{E}_s \left[u(\theta, s) \left(\frac{\partial \log f(s|\theta)}{\partial \theta} - \frac{\partial \log f(s|\theta)}{\partial \theta}\Big|_{s=b(\theta)} \right) \middle| \theta \right] d\theta.$$

It is straightforward to verify that

$$f(\theta) - \lambda(\theta) + \frac{\partial \log f(s|\theta)}{\partial \theta}\Big|_{s=b(\theta)} = 0 \quad \text{for } \theta \notin [p(0), p(1)].$$

Applying Claim B.2, we therefore have that

$$\begin{aligned}\text{Rev} = \int_{\underline{\theta}}^{\theta^*} \mathbb{E}_s\left[t(\theta,s)\middle|\theta\right] d\theta + \int_{\theta^*}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta - \Lambda(\theta^*) U(\theta^*) \\ - \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta,\theta') U(\theta) d\theta' d\theta \\ - \int_{\theta^*}^{\overline{\theta}} \Lambda(\theta) \mathbb{E}_s \left[u(\theta,s) \left(\frac{\partial \log f(s|\theta)}{\partial \theta} - \left.\frac{\partial \log f(s'|\theta)}{\partial \theta}\right|_{s'=b(\theta)} \right) \middle| \theta \right] d\theta.\end{aligned}$$

Using the partition of unity $1 = \mathbb{1}_{\theta \leq p(s)} + \mathbb{1}_{\theta > p(s)}$ and the Fundamental Theorem of Calculus as in the proof of Lemma 4, we have that

$$\begin{aligned}&\frac{\partial \log f(s|\theta)}{\partial \theta} - \left.\frac{\partial \log f(s'|\theta)}{\partial \theta}\right|_{s'=b(\theta)} \\ &\qquad = \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{1}_{\theta \leq p(s') \leq p(s)} ds' - \int_s^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{1}_{\theta \geq p(s')} ds'.\end{aligned}$$

Noting that the first term vanishes for all s if $\theta > p(1)$ and the second term always vanishes for all s if $\theta < p(0)$, it follows that

$$\begin{aligned}\text{Rev} = \int_{\underline{\theta}}^{\theta^*} \mathbb{E}_s\left[t(\theta,s)\middle|\theta\right] d\theta + \int_{\theta^*}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta - \Lambda(\underline{\theta}) U(\underline{\theta}) \\ - \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta,\theta') U(\theta) d\theta' d\theta \\ - \int_{\theta^*}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s\left[u(\theta,s) \mathbb{1}_{\theta \leq p(s') \leq p(s)} \middle| \theta\right] ds' d\theta \\ + \int_{p(0)}^{\overline{\theta}} \mathbb{E}_s \left[u(\theta,s) \Lambda(\theta) \int_s^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{1}_{\theta \geq p(s')} ds' \middle| \theta \right] d\theta.\end{aligned}$$

We can then simplify the last term as in the proof of Lemma 4 to complete the proof of the lemma. □

Proof of Lemma D.1. Proposition 4 guarantees that (q,t) is Bayesian incentive compatible and *ex post* ε-individually rational. Applying Lemma D.2 for $\underline{\theta} = p(0)$ shows that expected revenue is

$$\int_{p(0)}^{\overline{\theta}} \left[\theta - \frac{\Lambda(\theta)}{f(\theta)}\right] Q(\theta) f(\theta) d\theta + \int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} (\theta' - \theta) \gamma(\theta,\theta') \mathbb{E}_s\left[q(\theta',s) \mathbb{1}_{\theta \geq p(s)} \middle| \theta\right] d\theta' d\theta$$

$$- \Lambda(p(0))U(p(0)) - \int_{p(0)}^{p(1)} \Lambda(\theta) \int_0^1 \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} \mathbb{E}_s \left[u(\theta, s) \mathbb{1}_{\theta \leq p(s') \leq p(s)} \middle| \theta \right] ds' d\theta$$
$$- \int_{p(0)}^{p(1)} \int_\theta^{\overline{\theta}} \gamma(\theta, \theta') \left(U(\theta) - \mathbb{E}_s \left[(u(\theta', s) - (\theta' - \theta) q(\theta', s)) \, \mathbb{1}_{\theta \geq p(s)} \middle| \theta \right] \right) d\theta' d\theta$$
$$+ \int_{\underline{\theta}}^{p(0)} \mathbb{E}_s \left[t(\theta, s) \middle| \theta \right] d\theta.$$

The first two terms and the last term appear in (D.7). For the third term, note that $\Lambda(\theta) \geq 0$ by construction and that $U(p(0)) \geq 0$ by interim individual rationality. For the fourth term, note also that $\frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} > 0$ (by Assumption 1) and that $u(\theta, s) \geq -\varepsilon$ for almost all (θ, s) by *ex post* ε-individual rationality. For the fifth term, note that $\gamma(\theta, \theta') \geq 0$ (by Lemma 3(b)), and that considering the double deviation of misreporting $\theta' > \theta$ and withdrawing if $p(s) > \theta$, ε-withdrawal-proofness implies that

$$U(\theta) = \mathbb{E}_s \left[(u(\theta', s) - (\theta' - \theta) q(\theta', s)) \, \mathbb{1}_{\theta \geq p(s)} \middle| \theta \right] - \varepsilon \Pr_s \left[\theta \leq p(s) \middle| \theta \right]$$

It therefore suffices to prove that

$$\int_{p(0)}^{p(1)} \Lambda(\theta) \int_0^{b(\theta)} \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} ds' d\theta + \int_{p(0)}^{p(1)} \int_\theta^{\overline{\theta}} \gamma(\theta, \theta') \Pr_s[\theta \leq p(s) \big| \theta] d\theta' d\theta < (2e - 1)k. \tag{D.9}$$

To bound the second term of the left hand side of (D.9), note that Claim B.2 implies that

$$\int_{p(0)}^{p(1)} \int_\theta^{\overline{\theta}} \gamma(\theta, \theta') \Pr_s[\theta \leq p(s) \big| \theta] d\theta' d\theta = \int_{p(0)}^{p(1)} \left(f(\theta) - \lambda(\theta) + \Lambda(\theta) \frac{\partial \log f(s|\theta)}{\partial \theta} \bigg|_{s=b(\theta)} \right) \Pr_s[\theta \leq p(s) \big| \theta] d\theta \tag{D.10}$$

For the first term of the integrand, note that

$$\int_{p(0)}^{p(1)} f(\theta) \Pr_s[\theta \leq p(s) \big| \theta] d\theta = \mathbb{E}_s \left[F(p(s)|s) - F(p(0)|s) \right] = \mathbb{E}_s \left[(1 - k) - F(p(0)|s) \right],$$

where the second equality holds by the definition of $p(s)$. Separately, it follows from Claim D.1 that

$$1 - F(p(0)|s) = \int_{p(0)}^{\overline{\theta}} f(\theta|s) d\theta \leq e \int_{p(0)}^{\overline{\theta}} f(\theta|0) d\theta = ke$$

for all states s. Hence, we have that

$$\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} f(\theta) \Pr_s[\theta \leq p(s) \big| \theta] d\theta' d\theta \leq k(e-1). \tag{D.11}$$

To bound the last two terms of the integrand of (D.10), note that for $p(0) < \theta < p(1)$, we have that $\Lambda(\theta) = \frac{kf(\theta)}{f(\theta|b(\theta))}$ and hence that

$$-\lambda(\theta) + \Lambda(\theta) \frac{\partial \log f(s|\theta)}{\partial \theta}\Big|_{s=b(\theta)} = -\Lambda(\theta) \frac{\partial \log f(\theta|s)}{\partial s}\Big|_{s=p(\theta)} \frac{db(\theta)}{d\theta}.$$

Note furthermore that $\Lambda(\theta) > 0$ for $\theta < \overline{\theta}$ by construction, and that $\frac{dp(\theta)}{d\theta} > 0$ for $\theta \in (p(0), p(1))$ in light of Claim B.1 and the Inverse Function Theorem. Moreover, it follows from Assumption 1 that $\frac{\partial \log f(\overline{\theta}|s)}{\partial s} > 0$. Therefore, for $p(0) < \theta < p(1)$, we have that $\Lambda(\theta) \frac{\partial \log f(\overline{\theta}|s)}{\partial s}\big|_{s=b(\theta)} \frac{db(\theta)}{d\theta} > 0$, and hence that

$$-\lambda(\theta) + \Lambda(\theta) \frac{\partial \log f(s|\theta)}{\partial \theta}\Big|_{s=b(\theta)} < \Lambda(\theta) \left(\frac{\partial \log f(\overline{\theta}|s)}{\partial s}\Big|_{s=b(\theta)} - \frac{\partial \log f(\theta|s)}{\partial s}\Big|_{s=b(\theta)} \right) \frac{db(\theta)}{d\theta}. \tag{D.12}$$

Combining (D.10), (D.11), and (D.12) yields that

$$\begin{aligned} &\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') \Pr_s[\theta \leq p(s) \big| \theta] d\theta' d\theta < k(e-1) \\ &\quad + \int_{p(0)}^{p(1)} \Lambda(\theta) \left(\frac{\partial \log f(\overline{\theta}|s)}{\partial s}\Big|_{s=b(\theta)} - \frac{\partial \log f(\theta|s)}{\partial s}\Big|_{s=b(\theta)} \right) \Pr_s[\theta \leq p(s) \big| \theta] \frac{db(\theta)}{d\theta} d\theta. \end{aligned}$$

Noting that $\Lambda(\theta) \geq 0$ (by construction) and that $\frac{\partial \log f(\overline{\theta}|s)}{\partial s}\big|_{s=b(\theta)} - \frac{\partial \log f(\theta|s)}{\partial s}\big|_{s=b(\theta)} \geq 0$ (by Assumption 1) and applying the change of variables $s = p(\theta)$ then yields that

$$\begin{aligned} &\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') \Pr_s[\theta \leq p(s)] d\theta' d\theta \\ &\qquad < k(e-1) + \int_0^1 \Lambda(p(s)) \left(\frac{\partial \log f(\overline{\theta}|s)}{\partial s} - \frac{\partial \log f(\theta|s)}{\partial s}\Big|_{\theta=p(s)} \right) ds. \end{aligned}$$

By the Fundamental Theorem of Calculus and Dirichlet's formula, it follows that

$$\int_{p(0)}^{p(1)} \int_{\theta}^{\overline{\theta}} \gamma(\theta, \theta') \Pr_s[\theta \leq p(s)] d\theta' d\theta < k(e-1) + \int_{p(0)}^{\overline{\theta}} \int_{b(\theta)}^{1} \Lambda(p(s)) \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} ds d\theta.$$

We therefore have that

$$\int_{p(0)}^{p(1)} \Lambda(\theta) \int_0^{b(\theta)} \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} ds' d\theta + \int_{p(0)}^{p(1)} \int_\theta^{\overline{\theta}} \gamma(\theta, \theta') \Pr_s[\theta \leq p(s) \big| \theta] d\theta' d\theta$$
$$< k(e-1) + \int_{p(0)}^{p(1)} \Lambda(\theta) \int_0^{b(\theta)} \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} ds d\theta + \int_{p(0)}^{\overline{\theta}} \int_{b(\theta)}^1 \Lambda(p(s)) \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} ds d\theta.$$

Note that for $p(0) \leq \theta \leq p(1)$, we have that

$$\Lambda(\theta) = \frac{k f(\theta)}{f(\theta|b(\theta))} = k \int_0^1 \frac{f(\theta|s)}{f(\theta|b(\theta))} f(s) ds \leq ke \int_0^1 f(s) ds = ke,$$

where the inequality is due to Claim D.1. Since $\frac{\partial^2 \log f(\theta|s)}{\partial \theta \partial s} \geq 0$ (by Assumption 1), it follows that

$$\begin{aligned}
&\int_{p(0)}^{p(1)} \Lambda(\theta) \int_0^{b(\theta)} \frac{\partial^2 \log f(s'|\theta)}{\partial s' \partial \theta} ds' d\theta + \int_{p(0)}^{p(1)} \int_\theta^{\overline{\theta}} \gamma(\theta, \theta') \Pr_s[\theta \leq p(s) \big| \theta] d\theta' d\theta \\
<& k(e-1) + ke \left[\int_{p(0)}^{p(1)} \int_0^{b(\theta)} \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} ds d\theta + \int_{p(0)}^{\overline{\theta}} \int_{b(\theta)}^1 \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} ds d\theta \right] \\
=& k(e-1) + ke \left[\int_{\underline{\theta}}^{\overline{\theta}} \int_0^{b(\theta)} \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} ds d\theta + \int_{\underline{\theta}}^{\overline{\theta}} \int_{b(\theta)}^1 \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} ds d\theta \right] \\
=& k(e-1) + ke \log \frac{f(\overline{\theta}|1)}{f(\overline{\theta}|0)} \Big/ \frac{f(\underline{\theta}|1)}{f(\underline{\theta}|0)} \\
\leq& k(e-1) + ke
\end{aligned}$$

where the last inequality holds due to Lemma 1(b)—yielding (D.9). □

Appendix E. Other Proofs Omitted from the Text

E.1. **Proof of Lemma 1.** We prove the two parts of the lemma separately.

Proof of Part (a). Assumption 2 implies that

$$\int_0^1 \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} d\theta ds \geq \int_0^1 \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial^2 \log f(\theta'|s)}{\partial s \partial \theta'} d\theta' \right) \left(\int_0^1 \frac{\partial^2 \log f(\theta|s')}{\partial s' \partial \theta} ds' \right) d\theta ds$$
$$= \left(\int_0^1 \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} d\theta ds \right)^2.$$

Hence, we must have that

$$\int_0^1 \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} d\theta ds \leq 1.$$

By Assumption 1, it follows that for all types $\theta \leq \theta'$ and states $s \leq s'$, we have that

$$1 \geq \int_s^{s'} \int_{\theta}^{\theta''} \frac{\partial^2 \log f(\theta''|s'')}{\partial s'' \partial \theta''} d\theta'' ds'' = \log \frac{f(\theta'|s')}{f(\theta'|s)} \Big/ \frac{f(\theta|s')}{f(\theta|s)}.$$

A similar conclusion holds if $\theta \geq \theta'$ and $s \geq s'$. If $\theta \leq \theta'$ and $s \geq s'$ (and similarly if $\theta \geq \theta'$ and $s \leq s'$), then Assumption 1 implies that

$$\frac{f(\theta'|s')}{f(\theta'|s)} \leq \frac{f(\theta|s')}{f(\theta|s)}.$$

Proof of Part (b). Suppose that

$$f(\theta|s) = B(\theta)D(s)\exp\left(A(\theta)C(s)\right),$$

where $B(\theta), D(s) > 0$. Assumption 2′ implies that

$$1 \geq \log \frac{f(\overline{\theta}|1)}{f(\overline{\theta}|0)} \Big/ \frac{f(\underline{\theta}|1)}{f(\underline{\theta}|0)} = (A(\overline{\theta}) - A(\underline{\theta}))(C(1) - C(0)).$$

Note that

$$\frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta} = \frac{dA(\theta)}{d\theta}\frac{dC(s)}{ds}$$

for all types θ and states s. Hence, for all types θ and states s, we have that

$$\begin{aligned}
&\left(\frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta}\right)^{-1} \left(\int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial^2 \log f(\theta'|s)}{\partial s \partial \theta'} d\theta'\right) \left(\int_0^1 \frac{\partial^2 \log f(\theta|s')}{\partial s' \partial \theta} ds'\right) \\
=&\left(\frac{dA(\theta)}{d\theta}\frac{dC(s)}{ds}\right)^{-1} \left(\int_{\underline{\theta}}^{\overline{\theta}} \frac{dA(\theta')}{d\theta'}\frac{dC(s)}{ds} d\theta'\right) \left(\int_0^1 \frac{dA(\theta)}{d\theta}\frac{dC(s')}{ds'} ds'\right) \\
=&(A(\overline{\theta}) - A(\underline{\theta}))(C(1) - C(0)) \leq 1.
\end{aligned}$$

By Assumption 1, it follows that

$$\left(\int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial^2 \log f(\theta'|s)}{\partial s \partial \theta'} d\theta'\right) \left(\int_0^1 \frac{\partial^2 \log f(\theta|s')}{\partial s' \partial \theta} ds'\right) \leq \frac{\partial^2 \log f(\theta|s)}{\partial s \partial \theta}.$$

E.2. **Proof of Proposition 4.** Let (q, t) be a withdrawal-proof mechanism. Similarly to the proof of Proposition 1, to see that (q, t) must be Bayesian incentive

compatible, note that for all types θ and reports θ', we have that

$$\begin{aligned}\mathbb{E}_s\left[\theta q(\theta,s)-t(\theta,s)\middle|\theta\right] &\geq \mathbb{E}_s\left[\max\{\theta q(\theta',s)-t(\theta',s),-\varepsilon\}\middle|\theta\right]\\ &\geq \mathbb{E}_s\left[\theta q(\theta',s)-t(\theta',s)\middle|\theta\right],\end{aligned}$$

where the first inequality holds due to withdrawal-proofness.

To see that (q,t) must be *ex post* ε-individually rational, let θ be a type. Note that the ε-withdrawal-proofness condition for the report $\theta'=\theta$ guarantees that

$$\mathbb{E}_s\left[\theta q(\theta,s)-t(\theta,s)\middle|\theta\right] \geq \mathbb{E}_s\left[\max\{\theta q(\theta,s)-t(\theta,s),-\varepsilon\}\middle|\theta\right].$$

It follows that $\theta q(\theta,s)-t(\theta,s)=\max\{\theta q(\theta,s)-t(\theta,s),-\varepsilon\}$ for almost all states s, and hence that $\theta q(\theta,s)-t(\theta,s)\geq -\varepsilon$ for almost all states s.

Appendix F. Understanding Withdrawal-Proofness in Finite Markets

In this appendix, we extend Proposition 1 and 2 and Corollary 1 to finite-market mechanisms. The statements and proofs are similar to the large-market case.

Recall that a finite-market mechanism $(q_i,t_i)_{1\leq i\leq N}$ is *Bayesian incentive compatible* if for all bidders i, types θ_i, and reports θ_i', we have that

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q_i(\theta_i,\boldsymbol{\theta}_{-i})-t_i(\theta_i,\boldsymbol{\theta}_{-i})\middle|\theta_i\right] \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q_i(\theta_i',\boldsymbol{\theta}_{-i})-t_i(\theta_i',\boldsymbol{\theta}_{-i})\middle|\theta_i\right].$$

Recall also that a finite-market mechanism $(q_i,t_i)_{1\leq i\leq n}$ is *ex post individually rational* if for all bidders i and types θ_i, and almost all type profiles $\boldsymbol{\theta}_{-i}$, we have that $\theta_i q_i(\theta_i,\boldsymbol{\theta}_{-i})-t_i(\theta_i,\boldsymbol{\theta}_{-i})\geq 0$.

The following result is the analogue of Proposition 1 for finite-market mechanisms.

Proposition F.1 (Forges, 1999, Proposition 1)**.** *If a finite-market mechanism is withdrawal-proof, then it is Bayesian incentive compatible and ex post individually rational.*

Proof. Let $(q_i,t_i)_{1\leq i\leq N}$ be a withdrawal-proof mechanism. To see that $(q_i,t_i)_{1\leq i\leq N}$ must be Bayesian incentive compatible, note that for all bidders i, types θ_i, and reports θ_i', we have that

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q_i(\theta_i,\boldsymbol{\theta}_{-i})-t_i(\theta_i,\boldsymbol{\theta}_{-i})\middle|\theta_i\right] &\geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\max\{\theta_i q_i(\theta_i',\boldsymbol{\theta}_{-i})-t_i(\theta_i',\boldsymbol{\theta}_{-i}),0\}\middle|\theta_i\right]\\ &\geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q(\theta_i',\boldsymbol{\theta}_{-i})-t(\theta_i',\boldsymbol{\theta}_{-i})\middle|\theta_i\right],\end{aligned}$$

where the first inequality holds due to withdrawal-proofness.

To see that $(q_i, t_i)_{1 \leq i \leq N}$ must be *ex post* individually rational, let θ_i be a type. Note that the withdrawal-proofness condition for the report $\theta_i' = \theta_i$ guarantees that

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i}) - t_i(\theta_i, \boldsymbol{\theta}_{-i}) \middle| \theta_i\right] \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\max\{\theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i}) - t_i(\theta_i, \boldsymbol{\theta}_{-i}), 0\} \middle| \theta_i\right].$$

It follows that $\theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i}) - t_i(\theta_i, \boldsymbol{\theta}_{-i}) = \max\{\theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i}) - t_i(\theta_i, \boldsymbol{\theta}_{-i}), 0\}$ for almost all type profiles $\boldsymbol{\theta}_{-i}$, and hence that $\theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i}) - t_i(\theta_i, \boldsymbol{\theta}_{-i}) \geq 0$ for almost all type profiles $\boldsymbol{\theta}_{-i}$. □

The following result is the analogue of Proposition 2 for finite-market mechanisms.

Proposition F.2. *Let $(q_i, t_i)_{1 \leq i \leq N}$ be a Bayesian incentive compatible finite-market mechanism. Suppose that for each bidder i and type θ_i, each report θ_i' such that $\Pr_{\boldsymbol{\theta}_{-i}}\left[t_i(\theta_i', \boldsymbol{\theta}_{-i}) > \theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i}) \middle| \theta_i\right] > 0$ is weakly dominated (in the game induced by $(q_i, t_i)_{1 \leq i \leq N}$ without withdrawal) by a report $\hat{\theta}_i$ for which $t(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) \leq \theta_i q_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$ for almost all states s. Then, $(q_i, t_i)_{1 \leq i \leq N}$ is withdrawal-proof.*

Proof. Let i be a bidder, let θ_i be a type, and let θ_i' be a report. We show that θ_i' cannot be desirable even in the presence of withdrawal.

If $t_i(\theta_i', \boldsymbol{\theta}_{-i}) \leq \theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i})$ for type profiles $\boldsymbol{\theta}_{-i}$, then we have that

$$\max\{\theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i}) - t_i(\theta_i', \boldsymbol{\theta}_{-i}), 0\} = \theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i})$$

for almost all type profiles $\boldsymbol{\theta}_{-i}$. Since $(q_i, t_i)_{1 \leq i \leq N}$ is Bayesian incentive compatible, it follows that

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i}) - t_i(\theta_i, \boldsymbol{\theta}_{-i}) \middle| \theta_i\right] &\geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i}) - t_i(\theta_i', \boldsymbol{\theta}_{-i}) \middle| \theta_i\right] \\ &= \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\max\{\theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i}) - t_i(\theta_i', \boldsymbol{\theta}_{-i}), 0\} \middle| \theta_i\right].\end{aligned}$$

If $\Pr_{\boldsymbol{\theta}_{-i}}\left[t_i(\theta_i', \boldsymbol{\theta}_{-i}) > \theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i}) \middle| \theta_i\right] > 0$, then let $\hat{\theta}_i$ be a weakly dominating report with $t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) \leq \theta_i q_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})$ for almost all type profiles $\boldsymbol{\theta}_{-i}$. By construction, we have that

$$\theta_i q_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) \geq \max\{\theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i}) - t_i(\theta_i', \boldsymbol{\theta}_{-i}), 0\}.$$

Taking expectations and using Bayesian incentive compatibility yields that

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i}) - t_i(\theta_i, \boldsymbol{\theta}_{-i}) \middle| \theta_i\right] &\geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\theta_i q_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) - t_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}) \middle| \theta_i\right] \\ &\geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[\max\{\theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i}) - t_i(\theta_i', \boldsymbol{\theta}_{-i}), 0\} \middle| \theta_i\right].\end{aligned}$$

Since the two cases exhaust all possibilities, $(q_i, t_i)_{1 \leq i \leq N}$ must be withdrawal-proof. □

Recall that a finite-market mechanism $(q_i, t_i)_{1\leq i\leq N}$ is *dominant strategy incentive compatible* if for all bidders i, types θ_i, and reports θ_i', and almost all type profiles $\boldsymbol{\theta}_{-i}$, we have that

$$\theta_i q_i(\theta_i, \boldsymbol{\theta}_{-i}) - t_i(\theta_i, \boldsymbol{\theta}_{-i}) \geq \theta_i q_i(\theta_i', \boldsymbol{\theta}_{-i}) - t_i(\theta_i', \boldsymbol{\theta}_{-i}).$$

The following result is the analogue of Corollary 1 for finite-market mechanisms.

Corollary F.1. *If a finite-market mechanism is dominant strategy incentive compatible and ex post individually rational, then it is withdrawal-proof.*

Proof. Follows from Proposition 2 by taking $\hat{\theta}_i = \theta_i$. □

Appendix G. Approximate Optimality of the Uniform Price Auction: Further Numerical Examples

In this appendix, we study optimal mechanisms for a family of beta distributions that violate Assumption 2. We suppose that the state uniformly distributed between $[0, \overline{s}]$, and that values are supported on $[0, 1]$ with conditional distribution

$$f(\theta|s) = (s+1)x^s.$$

The parameter $\overline{s}$ measures the strength of the affiliation between types and states. As in Section 7.1, we can solve the seller's expected revenue maximization problem numerically after discretization.

Table G.1 reports the fraction of the expected revenue of the optimal withdrawal-proof mechanism that the uniform price auction achieves, for ranges of the quantity k of the good that is for sale and the value of the parameter $\overline{s}$. These results show that the uniform price auction delivers upwards of 98% of expected revenue of the optimal mechanism even when affiliation is very high. Thus, the uniform price auction remains approximately optimal even when Assumption 2 is grossly violated.

	$\overline{s}=1$	$\overline{s}=3$	$\overline{s}=5$	$\overline{s}=7$	$\overline{s}=9$
$k=0.1$	99.78%	99.78%	99.82%	99.79%	99.78%
$k=0.3$	99.66%	99.73%	99.70%	99.78%	99.85%
$k=0.5$	99.72%	99.75%	99.77%	99.84%	99.80%
$k=0.7$	99.45%	99.68%	99.73%	99.74%	99.71%
$k=0.9$	99.18%	99.48%	99.58%	99.65%	99.54%

Table G.1. Percentage of Optimal Expected Revenue Achieved by the Uniform Price Auction with a Beta Distribution.