Constrained efficient borrowing with sovereign default risk*

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Abstract

We solve a quantitative Eaton-Gersovitz sovereign default model with long-term debt and constrained efficient borrowing, i.e., with a Ramsey government that decides the future borrowing path taking as given future (ex-post optimal) default decisions. The Ramsey government improves upon the Markov government because the Ramsey government takes into account how borrowing decisions in period t affect the borrowing sets in periods before t. We use the optimality condition of the Ramsey government to solve for the constrained efficient borrowing path. We show that compared with the Markov condition, the Ramsey condition only features one additional single dimensional state variable. This enables us to propose a tractable algorithm for solving the Ramsey problem. Using Mexico as a case study, we find that more than 80% of the default risk would be eliminated by a Ramsey government. A simple debt ceiling rule could be as successful in reducing the default risk and would achieve 60% of the welfare gain that the Ramsey planner achieves. In addition, starting with high initial debt, compared with a Markov government, a Ramsey government has a higher probability of completing a successful deleveraging (without defaulting) even though it chooses weaker austerity. These findings underscore the importance of governments' efforts to constrain fiscal policies with fiscal rules.

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1 Introduction

As evidenced by the widespread use and discussion of fiscal rules, the overindebtedness or deficit bias of many governments is often at the center of policy debates.[1] Following previous studies, we focus on overindebtedness that results from the government's lack of commitment to future borrowing in the presence of default risk. While some studies of overindebtedness present stylized models in which efficient borrowing can be characterized analytically, quantitative studies do not characterize efficient borrowing.

We study a standard quantitative sovereign default framework à la Eaton and Gersovitz (1981), and we charachterize borrowing by both an inefficient Markov government and a constrained efficient Ramsey government. At the beginning of each period, the government first observes its endowment and continuation value under defaulting. Both variables are stochastic. Second, if the government is not in default, it decides whether to default on its debt. Third, a government not in default issues long-term bonds that are priced by competitive risk-neutral foreign investors. We work with two equilibrium concepts. In the Markov Perfect Equilibrium that the quantitative literature has focused on, in each period, the Markov government chooses it default and borrowing actions taking as given the optimal future default and borrowing strategies.[2] In the Ramsey equilibrium, we focus on the constrained efficient borrowing path, i.e., we assume that the Ramsey government can commit to a history-dependent borrowing policy but takes as given (ex-post optimal) default decisions. Thus, in period zero (or in the first period after a default), the Ramsey government chooses the income-history-contingent borrowing path that it will follow in every future period (until it defaults).

The Ramsey government improves upon the Markov government because the Ramsey government takes into account how borrowing decisions in period t affect the borrowing sets in periods before t. Borrowing more in t lowers the price of bonds issued in every period before t, thus shrinking the borrowing sets in those periods. The Ramsey government takes these borrowing costs into account when choosing the constrained efficient borrowing plan. The Markov gov-

[1] As defined by the IMF (2017), "A fiscal rule is a long-lasting constraint on fiscal policy through numerical limits on budgetary aggregates."

[2] Aguiar and Amador (2019) show there may be multiple equilibria in an Eaton-Gersovitz model with long-term debt. We rule out the bad equilibria by focusing on the equilibrium is the limit of the finite horizon economy.

ernment's ex-post optimal borrowing policy ignores these borrowing costs and thus the Markov government overborrows from an ex-ante perspective.

Using the optimality condition of the Ramsey government, we show that the effects of borrowing in period-t on utility flows prior to period t can be encapsulated in a single dimensional state variable. This result enables us to propose an algorithm to solve for the Ramsey government's constrained efficient borrowing plan.

In the optimality condition for the Markov government, part of the marginal cost of borrowing is given by the negative effect of borrowing on current consumption that arises because borrowing lowers the price at which the government sells bonds. Thus, in this optimality condition, the derivative of the bond price appears weighted by the marginal utility of consumption and the level of issuances in the current period.

We show that the optimality condition for the Ramsey problem encompasses the one for the Markov government. The difference is that in the Ramsey condition, the derivative of the bond price appears weighted not only by the marginal utility and debt issuances in the current period but also by a history variable that summarizes marginal utilities and issuances in previous periods. That is, for the same level of initial income and debt, the Ramsey optimality condition is the same for all income histories that imply the same weighted sum of previous periods marginal utilities and issuances (which we summarize in the history state variable).

We impose discipline on our quantitative exercise by calibrating the baseline model without commitment to match data from Mexico, a representative economy with sovereign risk and a standard reference in the literature. The overall match between the simulations for the Markov government and the data makes the model a plausible laboratory for the quantitative exercises we conduct in this paper. We measure the effects of commitment to future borrowing by comparing simulations for the Markov and Ramsey governments.

We find that the overindebtedness resulting from the government's lack of commitment is modest: the average debt ratio is 44% for the Markov government and 40% for the Ramsey government. However, the Markov government's inability to commit to a borrowing policy accounts for the majority of the default risk in the baseline simulations, as reflected in sovereign spreads. In the Ramsey economy, the mean and the standard deviation of the sovereign spread

are only 0.5% and 0.4%, respectively. This contrasts with a mean and standard deviation of the spread of 3.3% and 2.4% with the Markov government (both matching calibration targets and thus, representative of the data). Compared to the Markov Perfect Equilibrium, starting in a state with no debt, and depending on the initial level of income, the Ramsey government would generate welfare gains equivalent to a permanent consumption increase between 0.3% and 0.7%. We also show that a simple debt-ceiling rule could be as successful in reducing the default risk and would achieve 60% of the welfare gain that generated by the Ramsey government.

We also quantify the importance of commitment to a deleveraging plan. In our benchmark model, we assume the Ramsey plan is chosen for an initial state without debt. We study the benefits of commitment to deleveraging plans by computing the Ramsey plan starting from an initial state with high debt. In particular, we consider an initial state for which the Markov government faces a probability of default of 32% within the following six years. We show that implementing the optimal Ramsey plan would lower this probability to 22%. Furthermore, we do not find any income path for which the Ramsey government would choose to default and the Markov government would choose to repay. The Ramsey government achieves a higher probability of a successful deleveraging (without default) with less austerity. During the first six years of deleveraging, while the Markov government starts with a median saving ratio of 5% that is eventually reduced to 3%, the Ramsey government starts with a median saving ratio of about 3% that is only reduced slightly over time. The Ramsey government is able to implement a more successful deleveraging with weaker austerity in part because it has to pay a lower interest rate bill (because of the lower spreads that reflect the lower default probability). For example, in the first period of the exercise, the spread is 6.5% for the Markov government and 4.0% for the Ramsey government. The welfare gains are of similar magnitudes to the ones observed when the government starts without debt (with positive debt, on top of the welfare gains, there are capital gains for bondholders).

Overall, these results are indicative of the quantitative importance of enhancing long-term fiscal discipline for reducing sovereign risk and for the success of fiscal consolidation programs. This is consistent with ongoing governments' efforts to gain commitment to future borrowing policies (fiscal deficits) through fiscal rules and independent fiscal councils.

We also find that the government does not want to commit to a more countercyclical fiscal policy. The Ramsey government commits to issuing even more debt in good times and less debt in bad times.[3] Thus, our result highlights that the conventional wisdom on the desirability of fiscal rules allowing for countercyclical fiscal policy (IMF, 2017) may come at the cost of increasing default risk (it should be noted that we study a setup in which fiscal choices do not affect aggregate income nor the distribution of income).

We also show that the Ramsey government may even want to commit to buying back debt when income is low. In contrast, the Markov government never buys back debt (consistently with the findings presented by Aguiar et al., 2019, and Bulow and Rogoff, 1988, 1991). The Markov government does not benefit from the increase in bond prices implied by a buyback in the current period. The Ramsey government that wants to issue debt may benefit from the increase in bond prices that would result from committing to a debt buyback in the future.

1.1 Related literature

Following Aguiar and Gopinath (2006) and Arellano (2008), the sovereign default framework à la Eaton and Gersovitz (1981) is being widely used for quantitative studies of sovereign risk. For instance, Bocola and Dovis (2019) analyze self-fulfilling crises, Sanchez et al. (2018) discuss the optimal maturity of sovereign debt, Hatchondo et al. (2017) characterize gains from introducing non-defaultable debt, Cuadra and Sapriza (2008) study the role of political uncertainty, Durdu et al. (2013) study the effects of news on sovereign risk, Boz (2011) introduces international financial institutions, Lizarazo (2005) emphasizes risk premia, Pouzo and Presno (2016) study uncertainty premia, Roch and Uhlig (2018) study bailouts, and Asonuma and Joo (2019), Asonuma and Trebesch (2016), and Yue (2010) focus on debt renegotiation.[4]

Chatterjee and Eyigungor (2012) and Hatchondo and Martinez (2009) introduce long-term debt into the sovereing default model, creating overindebtedness due to a time-inconsistency

[3]However, the effect on consumption of the procyclicality of Ramsey issuances is mitigated because compared with the Markov government, the Ramsey government faces higher debt prices in bad times (as evidenced by the lower volatility of countercyclical spreads).

[4]This framework is also used in studies of household default—see, for example, Athreya et al. (2007), Chatterjee et al. (2007), Hatchondo et al. (2015a), Li and Sarte (2006), Livshits et al. (2008), and Sanchez (2017). Bi (2012) and Bi and Leeper (2012) present models of non-strategic default.

(debt dilution) problem.[5] They show that this feature appears essential for generating plausible quantitative implications for the sovereign default premium. Previous quantitative studies present modifications to debt contracts that mitigate this overindebtedness but do not characterize efficient borrowing: Chatterjee and Eyigungor (2012) and Hatchondo and Martinez (2009) study one-period bonds, Chatterjee and Eyigungor (2015) focus on seniority, and Hatchondo et al. (2016) study debt covenants that penalize the government for future borrowing.[6] While these modifications of debt contracts mitigate overindebtedness, they also have other effects. For example, they affect the government's exposure to changes in the borrowing cost. For instance, in our quantitative exercise, while Ramsey borrowing is superior to Markov borrowing, for a Markov government, using one-period bonds (without overindebtedness) instead of long-term bonds (with overindebtedness) produces welfare losses. We also show that in contrast with the no-dilution policy studied by Hatchondo et al. (2016), the efficient borrowing policy implies debt dilution (i.e., a reduction in the value of existing debt triggered by the issuance of new debt).

While we focus on constrained efficient borrowing, i.e., we assume the Ramsey government can commit to a history-dependent borrowing policy but takes as given (ex-post optimal) default decisions, Adam and Grill (2017), Mateos-Planas and Ríos-Rull (2016), and Rottger (2019) study commitment to a default policy. Our focus on constrained efficient borrowing is motivated in part by existing fiscal rules, in which governments typically attempt to commit to future borrowing and not to future default decisions (IMF, 2017). In Hatchondo et al. (2015b), we argue that the focus of existing fiscal rules on restricting borrowing and not default is consistent with the commitment to default policies being more costly than the commitment to borrowing limits. Our focus on constrained efficient borrowing is also consistent with the focus of the literature that study the effect of constraining borrowing with different debt instruments while assuming that

[5]This overindebtedness is also a relevant concern in other credit markets. For example, corporate debt contracts often include covenants intended to limit overindebtedness (Asquith et al., 2005; Smith and Warner, 1979; Rodgers, 1965; and Carey et al., 1993). Corporate debt and mortgage loans typically feature a seniority structure that mitigates this problem. In contrast, sovereign bonds present a weaker protection against overindebtedness (due in part to the weak enforcement of sovereign debt claims). On top of the mechanism studied in this paper, sovereign overindebtedness could also arise, for example, from externalities in a monetary union (Beetsma and Uhlig, 1999; Aguiar et al., 2015).

[6]Bizer and DeMarzo (1992), Bolton and Jeanne (2009), and Kletzer (1984) present stylized models in which efficient borrowing can be characterized analytically.

default decisions continue to be decided by a Markov government (Chatterjee and Eyigungor, 2012, 2015; Hatchondo and Martinez, 2009; Hatchondo et al., 2016).

Of course, implementing other algorithms for finding the constrained efficient borrowing policy may be possible. We study an optimization problem with forward-looking constraints, where actions in periods $t+s>t$ constrain the feasible set of at t (in our case, borrowing at $t+s>t$ affect the price of bonds issued at t and thus the feasible set at t). The seminal work by Marcet and Marimon (2019) present a recursive formulation for this class of problems that have been widely used for several applications. This recursive formulation requires one state variable for each future period in which the action affects the current-period feasible set. Thus, a literal application of Marcet and Marimon's (2019) approach is not feasible for the model we study with infinitively lived bonds: since the current-period bond price depends on the issuances in every future period, one would need an infinite number of co-state variables. For example, Faraglia et al. (2016, 2019) present recursive formulations for models with M-period bonds using M co-state variables. In contrast, we show that with the long-term bonds commonly used in the default literature for presenting a tractable Markov problem (Hatchondo and Martinez, 2009), the effects of borrowing in period-t on utility flows in previous periods can be summarized using in a single dimensional state variable, allowing for a tractable algorithm for solving the Ramsey problem. Presenting alternative algorithms and comparing their computational efficiency with the one of the algorithm we propose is an interesting avenue for future research but is beyond the scope of this paper.

The rest of the article proceeds as follows. Section 2 introduces the model. Section 3 presents the optimality conditions. Section 4 discusses the calibration. Section 5 discusses our computation algorithm. Section 6 presents the quantitative results. Section 7 concludes. The Appendix presents the derivation of the simplified optimality condition for the Ramsey government.

2 The model

We study an infinite-horizon small open economy with $t \in \{0,1,2,...\}$ and a single tradable good. The domestic economy receives a stochastic endowment stream y of this good, where $y \in$

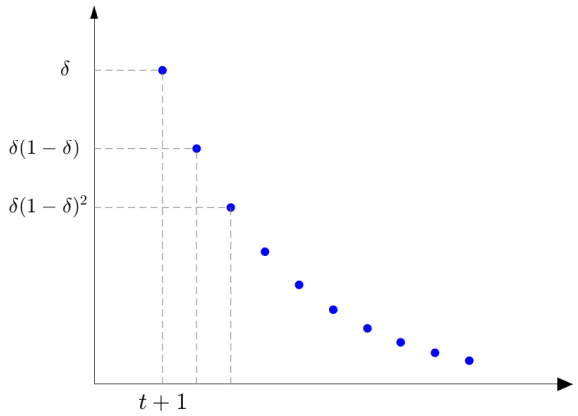


Figure 1: Coupon payments of a bond issued at t.

$\mathcal{Y} = \{y_1, ..., y_J\}$ follows a Markov process. Let U denote the continuation utility from defaulting. U takes values in $(-\infty, \infty)$ and has pdf f_j and cdf F_j.[7] At the beginning of each period, the government observes y and U and chooses whether to default. After that, a government not in default borrows using a long-term bond. A bond issued at t pays $\{\delta, \delta(1-\delta), \delta(1-\delta)^2, ..., \}$ at $t+1, t+2, ...$ (Hatchondo and Martinez, 2009). Figure 1 illustrates this profile of payments. When the government defaults, it does so on all current and future debt obligations. Bonds are priced by competitive risk-neutral lenders that discount future payoffs at the rate r. The government's objective is to maximize the present expected discounted value of future utility flows of the representative agent in the economy.

2.1 Markov government

We first assume that (as is standard in the literature), each period, the Markov government chooses how much to borrow. This is, each period t, the government maximizes

$$\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s\right),$$

[7] Assuming the distribution of U depends on income allows us to make the mean continuation value of defaulting endogenous, as it is usually done in the quantitative default literature (see Section 4). Shocks to the utility cost of defaulting around this mean endogenous value can be motivated, for example, by political turnover (Hatchondo and Martinez, 2010; Hatchondo et al., 2009).

where $\mathbb{E}$ denotes the expectation operator, β denotes the subjective discount factor, and c_t represents consumption of private agents. Thus, the government cannot commit to future (default and borrowing) decisions, and one may interpret this environment as a game in which the government making decisions in period t is a player who takes as given the (default and borrowing) strategies of other players (governments) who will decide after t.

2.1.1 Recursive formulation for the Markov government

We denote with x' the next-period value of a variable x. Let $\pi_j(y)$ denote the probability of $y' = y_j$ given y. Let b denote the quantity of coupon obligations that mature in the current period, and q denote the bond price function. The expected utility of a government not in default with income y and debt b is given by

$$V(b,y) = \underset{b'}{Max} \left\{ u(c) + \beta \sum_{j=1}^{J} \pi_j(y) \left[\; F_j\left(V(b',y_j)\right) V(b',y_j) + \int_{V(b',y_j)}^{\infty} U \; f_j(U) dU \; \right] \right\} \quad (1)$$

$$s.t. \quad c = y - \delta b + q(b',y)\left[b' - (1-\delta)b\right], \quad (2)$$

$$q(b',y) = \frac{1}{1+r} \sum_{j=1}^{J} \pi_j(y) F_j\left(V(b',y_j)\right) \left[\delta + (1-\delta) q\left(\hat{b}(b,y_j), y_j\right)\right],$$

where $\hat{b}$ denotes the Markov Equilibrium borrowing policy.

2.2 Ramsey government

For any t, let $y^t = \{y_0, ..., y_t\}$ denote the history of income until period t. Let $\mathcal{Y}^t$ denote the set of all possible histories until period t. In period 0, the Ramsey government chooses income-history-contingent borrowing policies for every future period. That is, at $t = 0$, the Ramsey government chooses $b_{t+1}(y^t)$, for all $y^t = \{y_0, ..., y_t\} \in \mathcal{Y}^t$ and all t, by maximizing

$$\underset{\left\{\{b_{t+1}(y^t)\}_{\forall y^t\in\mathcal{Y}^t}\right\}_{t=0}^{\infty}}{Max}\left\{u(c_0)+\sum_{t=1}^{\infty}\beta^t\sum_{\forall y^t\in\mathcal{Y}^t}Pr(y^t)\prod_{n=1}^{t-1}F_{I(n)}(V_n(y^n))\left[\begin{array}{l}F_{I(t)}(V_t(y^t))u(c_t(y^t))+\\ \int_{V_t(y^t)}^{\infty}U\ f_{I(t)}(U)dU\end{array}\right]\right\} \quad (3)$$

$$s.t.\quad c_t(y^t)=y_t-\delta b_t(y^{t-1})+q_t(y^t)\left[b_{t+1}(y^t)-(1-\delta)b_t(y^{t-1})\right], \quad (4)$$

$$V_t(y^t)=u(c_t(y^t))+\beta\sum_{j=1}^{J}\pi_j(y_t)\left[\ F_j(V_{t+1}(y^t,y_j))V_{t+1}(y^t,y_j)+\int_{V_{t+1}(y^t,y_j)}^{\infty}Uf_j(U)dU\ \right], \quad (5)$$

$$q_t(y^t)=\tfrac{1}{1+r}\sum_{j=1}^{J}\pi_j(y_t)F_j(V_{t+1}(y^t,y_j))\left[\delta+(1-\delta)q_{t+1}(y^t,y_j)\right], \quad (6)$$

where for all $y^t=\{y_0,...,y_t\}\in\mathcal{Y}^t$, $I(t)$ denotes the income realization index for y_t.

3 Optimality conditions

This section presents the optimality conditions for both the Markov and the Ramsey governments.

3.1 Optimality condition for the Markov government

The next equation presents the standard optimality condition for the Markov government:

$$\begin{aligned}u'(c)q(b',y) &= \beta\sum_{j=1}^{J}\pi_j(y)\underbrace{F_j(V(b',y_j))}_{t+1\text{ Repayment prob}}\frac{\partial V(b',y_j)}{\partial b'}-\frac{\partial q(b',y)}{\partial b'}u'(c)\left[b'-(1-\delta)b\right]\\ &= \beta\sum_{j=1}^{J}\pi_j(y)F_j(V(b',y_j))u'(c')\left[\delta+(1-\delta)q\left(\hat{b}(b,y_j),y_j\right)\right]\\ &\quad -\frac{\partial q(b',y)}{\partial b'}u'(c)\left[b'-(1-\delta)b\right]. \quad (7)\end{aligned}$$

The second term of the right-hand side of this equation shows how it is costly for the government to lower the bond price. A lower bond price lowers the proceeds the government obtains from issuing bonds, and thus it lowers consumption. Consequently, in equation (7), the marginal cost of borrowing is given in part by the derivative of the bond price weighted by the marginal utility of consumption and the level of issuances of long-term debt $(b'-(1-\delta)b)$.

3.2 Optimality condition for the Ramsey government

For any income history $\tilde{y}^t$, the next equation presents the optimality condition for the Ramsey problem in equation (3):

$$\overbrace{\frac{\partial}{\partial b_{t+1}(\tilde{y}^t)}\left[u(c_0)+\sum_{k=1}^{t-1}\beta^k Pr(\tilde{y}^k)\prod_{n=1}^{k-1}F_{I(n)}(V_n(\tilde{y}^n))\left[\begin{array}{l}F_{I(k)}(V_k(\tilde{y}^k))u(c_k(\tilde{y}^k))+\\ \int_{V_k(\tilde{y}^k)}^{\infty}U\ f_k(U)dU\end{array}\right]\right]}^{\text{Effect of } b_{t+1}(\tilde{y}^t) \text{ on the utility in periods before } t}$$

$$+\quad \beta^t Pr(\tilde{y}^t)\prod_{n=1}^{t}F_{I(n)}(V_n(\tilde{y}^n))\times$$

$$\underbrace{\left[u'(c_t(\tilde{y}^t))\left[q_t+\frac{\partial q_t}{\partial b_{t+1}(\tilde{y}^t)}\right]\iota_t(\tilde{y}^t)-\beta E_{\tilde{y}_t}\left[F_{I(t+1)}(V_{t+1})u'(c_{t+1})\left[\delta+(1-\delta)q_{t+1}\right]\right]\right]}_{\text{Resemble optimality condition for the Markov government}}=0 \quad (8)$$

$$s.t.\quad c_t(\tilde{y}^t)=\tilde{y}_t-\delta b_t(\tilde{y}^{t-1})+q_t(\tilde{y}^t)\left[b_{t+1}(\tilde{y}^t)-(1-\delta)b_t(\tilde{y}^{t-1})\right],$$

where $\iota_t(\tilde{y}^t)=b_{t+1}(\tilde{y}^t)-(1-\delta)b_t(\tilde{y}^{t-1})$ denotes the level of issuances of long-term debt. Comparing equations (7) and (8) illustrates the time inconsistency problem in the standard default model with long-term debt. The last term in equation (8) shows that the Ramsey government considers the same trade-off considered by the Markov government: borrowing allows for more consumption today at the expense of less consumption next period. However, the first term in equation (8) shows that the Ramsey government also considers the effect of period t borrowing ($b_{t+1}(\tilde{y}^t)$) on the utility of every previous period (which the Markov government does not do). Intuitively, the Ramsey government considers that by credibly promising to borrow more in period t and thus increasing the default probability in period $t+1$, it would lower the price of bonds issued in every period before t and thus it would lower consumption in every period before t. In contrast, as illustrated in equation (7), the Markov government is only concerned about the effect of issuances in the bond price of new issuances $b'-(1-\delta)b$, and ignores the effect on the price of bonds issued in previous periods.

3.2.1 Differentiability

For the Markov government, Clausen and Strub (2017) show that in the default model, the objective function is continuously differentiable at the optimum and a version of the envelope theorem applies. Thus, the optimality condition can be used and has been used for solving this model (Arellano et al., 2017; Hatchondo et al., 2010).

For differentiability of the Ramsey problem (3), note first that $b_s(y^{s-1})$ affects c_s and c_{s-1} (equation 15), affects V_t for all $t \leq s$ (equation 5), and affects q_t for all $t < s$ (equation 6). Since c_s is differentiable in $b_s(y^{s-1})$ and u is differentiable, V_s is differentiable. Since V_s is differentiable and F is differentiable, q_{s-1} is differentiable.[8] Therefore, c_{s-1} is differentiable, V_{s-1} is differentiable, and V_t is differentiable for all $t \leq s$.

3.2.2 Simpler optimality condition for the Ramsey government

The following proposition presents a simpler recursive optimality condition for the Ramsey government, which we use to solve the Ramsey problem (see the Appendix for the proof).

Proposition 1 *In any period, for any history of income that give us y, b, A, and B, the optimality condition for the Ramsey government in equation (8) can be written as*

$$-A\underbrace{\frac{\sum_j \pi_j(y) f_j(V(b',y',A',B',b'')))u'(c')\left[\delta+(1-\delta)q(\hat{b}(b',y_j,A,B),y_j,h')\right]^2}{1+r}}_{\frac{\partial q_t(y^t)}{\partial b_{t+1}(y^t)}}+B\frac{\partial V(b,y,A,B,b')}{\partial b'}=0, \tag{9}$$

where the initial values of A and B are $A_0 = 0$ and $B_0 = 1$, respectively,

$$\begin{aligned} V(b,y,A,B,b') &= u(c)+\beta\sum_j \pi_j(y)\left[F_j\left(V(b',y_j,A',B',b'')\right)V(b',y_j,A',B',b'')\right. \\ &\quad \left.+\int_{V(b',y_j,A',B',b'')} U f_j(dU)\right], \end{aligned}$$

[8] Assuming a stochastic continuation value of defaulting (as, for example, Aguiar et al., 2019) plays a key role here. It implies that when the Ramsey government chooses $b_s(y^{s-1})$, it is choosing a probability of default (as the Markov government does) and not whether to default.

$$c = y - \delta b + q(b', y, A', B')\iota,$$

$$q(b', y, A, B) = \frac{1}{1+r}\sum_j \pi_j(y) F_j(V(b', y_j, A', B', b'')) \left[\delta + (1-\delta) q(\hat{b}(b', y_j, A', B'), y_j, A', B')\right],$$

$$A' = \frac{(1-\delta) F_{y'}(V(b', y', A', B', b''))}{1+r}\left[A + u'(c)\iota B\right], \tag{10}$$

$$B' = \frac{f_{y'}(V(b', y', A', B', b''))\left[\delta + (1-\delta) q(b'', y', A', B')\right]}{1+r}\left[A + u'(c)\iota B\right] + \beta F_{y'}(V(b', y', A', B', b''))\tag{11}$$

and $\hat{b}$ denotes the Ramsey equilibrium borrowing policy.

The formulation of the Ramsey government's optimality condition can be simplified even further. Instead of solving the Ramsey problem with two additional state variables, A and B, we use one additional state variable h. As hinted by equation (9), what matters for the optimality condition is the relative weight of the derivatives of q and V. If $B \neq 0$, we can define the history variable $h' = A/B + u'(c)\iota$. We solve the Ramsey problem using this definition, we find well defined values of h' (indicating that $B \neq 0$), and we check that our solution satisfies equation (9).

Using equations (10) and (11), we can write the law of motion for h' as

$$h' = \frac{F_y(V(b, y, h))(1-\delta)}{f_y(V(b, y, h))\left[\delta + (1-\delta) q(b', y, h')\right] h + \beta(1+r) F_y(V(b, y, h))} h + u'(c)\iota. \tag{12}$$

Therefore, developing $\partial V / \partial b'$, we can write the Ramsey government's optimality condition as

$$\begin{aligned} u'(c) q(b', y, h') &= \beta \sum_j \pi_j(y) F_j(V(b', y_j, h')) u'(c') \left[\delta + (1-\delta) q(\hat{b}(b', y_j, h'), y_j, h')\right] \\ &- \underbrace{\frac{\sum_j \pi_j(y) f_j(V(b', y_j, h')) u'(c') \left[\delta + (1-\delta) q(\hat{b}(b', y_j, h'), y_j, h')\right]^2}{1+r}}_{\frac{\partial q_t(y^t)}{\partial b_{t+1}(y^t)}} h', \end{aligned} \tag{13}$$

where

$$V(b, y, h) = u(c) + \beta \sum_j \pi_j(y) \left[F_j\left(V(b', y_j, h')\right) V(b', y_j, h') + \int_{V(b', y_j, h')} U f_j(dU)\right], \tag{14}$$

$$c = y - \delta b + q(b', y, h')\iota, \tag{15}$$

$$q(b', y, h') = \frac{1}{1+r}\sum_j \pi_j(y) F_j(V(b', y_j, h')) \left[\delta + (1-\delta) q(\hat{b}(b', y_j, h'), y_j, h'')\right]. \tag{16}$$

History variable h. Equation (12) shows that the history variable h can be thought of as an accumulation of the product of marginal utilities and issuances. Note that if h is positive, h' is well defined (i.e., the denominator in the first term of equation 12 is different from 0). In our quantitative exercises, h is always positive, indicating a history of positive issuances on (weighted) average. This is not surprising with the standard assumption in the literature of relatively impatient governments.

Equation (12) shows that on top of being higher for histories with higher marginal utilities and higher levels of issuances, h' is higher (and thus the Ramsey government's marginal cost of borrowing is higher) for histories with a higher probability of repayment F. Intuitively, it is optimal for the Ramsey government to borrow less for histories that affect bond prices in previous periods more, and these are histories with a higher probability of repayment (recall that a default ends the Ramsey promises).

In equation (12), h' is also higher when f is lower, i.e., when issuing less debt would increase less the probability of default. Recall that when a Ramsey government enters a period committed to borrow less, it is more likely to choose to default (depending on the utility of defaulting U). Therefore, committing to borrow less for some history increases the default probability for that history and thus it lowers bond prices in previous periods (for that history). Consequently, it is optimal for the Ramsey government to commit to borrow less when this effect is weaker, i.e., when f is lower. This is reflected in h' being a decreasing function of f.

Ramsey optimality condition. Note first that in the second term of the right-hand side of the Ramsey optimality condition in equation (13), h' is multiplied by the derivative of the sequential Ramsey bond price with respect to the end-of-period debt level ($\frac{\partial q_t(y^t)}{\partial b_{t+1}(y^t)}$). This derivative (in contrast with the derivative of the Markov bond price of of the recursive Ramsey bond price) does not present an effect of current borrowing on future borrowing.

Recall now that equation (8) shows that the Ramsey government considers (i) the same trade-off between current-period and next-period consumption considered by the Markov government, and (ii) the effect of period t borrowing on the utility of every previous period. Comparing equations (8) and (13) shows that the latter effect can be summarized with the effect of borrowing

on the (sequential Ramsey) bond price multiplied by the first term of the history variable h' (equation 12), which represents an accumulation of the product of marginal utilities and issuances in previous periods weighted by the effect of changes in the current-period bond price in previous-periods bond prices.

Note also that if $h = 0$ (for instance, in period 0, or for histories for which the government chooses to not issue debt), the Ramsey optimality condition in equation (13) resemble the Markov optimality condition in equation (7), because the derivative of the bond price is only weighted by the marginal utility and issuances in the current period. In contrast, if $h > 0$ and thus $h' > u'(c)\iota$, the derivative of the bond price is weighted by a higher number in the Ramsey condition, indicating that the marginal cost of issuing debt would be higher for the Ramsey government than for the Markov government (everything else equal, and assuming that issuing more debt lowers the bond price, as it is always the case in default models). In this sense, overindebtedness is typically chosen by Markov governments.

Comparing equations (7) and (13) also shows that with one-period debt ($\delta = 1$), the incentives of the Ramsey and Markov governments coincide. If $\delta = 1$, $h' = u'(c)b'$ and, therefore, h is no longer a relevant state variable, and the optimality conditions for the Ramsey and Markov governments in equations (7) and (13) coincide (note that with one-period debt, bond prices do not depend on bond issuances in future periods).

4 Calibration

We present a standard calibration such that the simulations for the Markov government match data from Mexico. Mexico is a common reference in the default literature because its business cycle displays the same properties that are observed in other economies with sovereign risk (Aguiar and Gopinath, 2007; Neumeyer and Perri, 2005). Unless specified otherwise, we use data from 1993 to 2014.

The utility function displays a constant coefficient of relative risk aversion, that is,

$$u\left(c\right) = \frac{c^{1-\gamma} - 1}{1-\gamma}, \text{ with } \gamma \neq 1.$$

The income process is a discretization of $log(y_t) = \rho log(y_{t-1}) + (1-\rho)\mu + \varepsilon_t$.

For the continuation utility of defaulting, we assume that $U \sim N(V^D(y), \sigma_U)$, where

$$V^D(y) = u(y(1 - d_0 - d_1 y)) + \beta \sum_j \pi_j(y) \left[\psi V(0, y_j, 0) + (1 - \psi) V^D(y_j)\right], \tag{17}$$

and V denotes the expected utility of repayment in equations (1) and (14) for the Markov and Ramsey governments, respectively (and the history state variable h is irrelevant for the Markov government). This formulation endogenizes the continuation utility of defaulting, incorporating the standard assumptions on the cost of defaulting in the quantitative default literature: defaults trigger exclusion from debt markets and a loss of income. As it is standard in the literature, the duration of the exclusion from debt markets is stochastic, with ψ denoting the probability of exiting the default state. During defaults, the government looses a proportion $d_0 + d_1 y$ of its income. As in Chatterjee and Eyigungor (2012), having two parameters in the cost of defaulting gives us the flexibility to match the levels of debt and spread in the data. Note that in equation (17), the history variable h resets to 0 after a default. This assumption rules out the possibility of the Ramsey government manipulating the cost of defaulting with its post-default borrowing promises.

Table 1 presents the values given to all parameters in the model. We first pin down a subset of parameters values without using the model simulations. A period in the model refers to a quarter. The value of the domestic discount factor is standard in quantitative business cycle and sovereign default studies. The parameter values that govern the endowment process are chosen so as to mimic the behavior of logged and linearly detrended GDP in Mexico from 1980 to 2014. The estimation of the AR(1) process for the cyclical component of GDP yields $\rho = 0.94$ and $\sigma_\varepsilon = 0.015$. We set $\delta = 0.035$, which, with the targeted level of sovereign spreads, implies an average debt duration in the simulations of 4.8 years, roughly the average duration of public debt in Mexico.[9]

Our modeling of the continuation utility of defaulting assumes that on top of the most common cost of defaulting in the quantitative literature, there is an stochastic utility cost (as

[9]We use the Macaulay definition of duration that, with the coupon structure in this paper, is given by $D = \frac{1+i_b}{\delta+i_b}$, where i_b denotes the constant per-period yield delivered by the bond. The data for duration corresponds to the average Modified Duration for Mexican government bonds computed by J.P. Morgan between January 2002 and March 2018.

r	0.01	Standard
β	0.97	Standard
ρ	0.94	Mexico GDP
σ_ε	1.5%	Mexico GDP
ψ	0.083	E(exclusion duration) = 3 years
δ	0.035	Debt duration = 4.8 years
σ_U	0.1	Std spread = 2.4%
d_0	0.17	Avg debt = 44.2%
d_1	1.2	Avg spread = 3.3%
γ	4.2	$\sigma(c)/\sigma(y) = 1.1$

Table 1: Parameter values.

presented by Aguiar et al., 2019). Compared with standard calibrations in the literature, this necessitates the calibration of the value of an additional parameter for the volatility of the utility cost, σ_U. In order to calibrate this parameter value, we choose to add as a target the volatility of the sovereign spread, which is a statistic of interest and is strongly affected by σ_U (Section 6.7).[10] The targets for the mean and the standard deviation of the spread are 3.3% and 2.3%, respectively, and corresponds to the J.P. Morgan EMBI spread from the first quarter of 1994 to the first quarter of 2018.

Following Bianchi et al. (2018), we also choose to make the domestic risk aversion part of the calibration. This is a key parameter determining the government's willingness to tolerate fluctuations of consumption and thus the optimal cyclicality of fiscal policy. The value of the risk aversion parameter that results from the calibration ($\gamma = 4.2$) is well within the range of values used for macro models.

[10]To compute the sovereign spread that is implicit in a bond price, we first compute the yield i_b, defined as the return an investor would earn if he holds the bond to maturity (forever) and no default is declared. This yield satisfies

$$q_t = \sum_{j=1}^{\infty} \delta(1-\delta)^{j-1} e^{-j i_b}.$$

The sovereign spread, r_t^s, is then computed as the difference between the yield i_b and the risk-free rate, which we assume is equal to zero.

Overall, we use the simulations to calibrate the value of four parameters: the two parameters of the utility cost of defaulting d_0 and d_1, the parameter σ_U determining the volatility of the income cost of defaulting, and the government's risk aversion γ. We choose these four parameters to match four targets in the data: (i) a public debt-to-income ratio of 44.2 percent, (ii) a mean level of spreads of 330 basis points, (iii) a spread volatility of 240 basis points, and (iv) a volatility of consumption relative to output of 1.1. The values for the default cost d_0 and d_1 mainly determine the average debt and spread levels (Hatchondo and Martinez, 2017), σ_U mainly determines the spread volatility, and γ is determined mainly by the consumption-volatility target.

5 Computation

We solve the model using the optimality conditions in equations (7) and (13) for the Markov and Ramsey governments, respectively. As in previous studies (Hatchondo et al., 2010), we confirm that we can obtain the same results for the Markov government using optimality conditions or using value function iteration.

For the Ramsey government, we cannot solve the model with value function iteration, but we perform other exercises that give us confidence in our algorithm. First, we study a five-period version of the model with three income levels. Thus, in this model, the Ramsey government chooses 40 optimal policies. We solve for the optimal policies for all possible income histories using the finite-horizon version of problem (3). Let $\hat{c}(y^t)$ denote the implied optimal consumption for all possible income histories. We find that $\hat{c}(y^t)$ is almost identical to the optimal consumption found using our algorithm, $\hat{c}(b, y, h)$, for which income histories are summarized by the initial levels of debt and income (b and y) and the history variable h. The average and maximum differences between the optimal consumption levels found with the two algorithms are 0.0005% and 0.0027%, respectively.

We perform a similar exercise with the infinite-horizon version of the model but without income uncertainty ($\sigma_\varepsilon = 0$, and all other parameter values as in the benchmark calibration). In this case, there is no hedging motive for issuing long-term debt, and it is optimal to issue one-period bonds (Aguiar et al., 2019), for which there is no time inconsistency problem (as mentioned

before, the optimality condition for the Markov and Ramsey governments coincide when $\delta = 1$). We find that the optimal paths of consumption with one-period bonds, $\hat{c}(b^1)$, and the optimal paths of consumption implied by our algorithm (with long-term bonds and commitment to future borrowing), $\hat{c}(b, h)$, are almost identical. That is, our algorithm for solving the problem of the Ramsey government eliminates the time inconsistency problem generated by long-term debt and thus, in a version of the model without a hedging motive, our algorithm makes one-period and long-term debt equivalent. In contrast, consumption path $\hat{c}(b)$ chosen by a Markov government with long-term debt implies lower welfare than the consumption path $\hat{c}(b^1)$ generated with one-period debt.

We also show that with the benchmark calibration with income uncertainty, the Ramsey policy $\hat{b}_i(b, h)$ generates higher welfare than the Markov policy $\hat{b}_i(b)$, than issuing one-period bonds, and than the optimal debt limit the government could commit to (Section 6). In addition, we find that equilibrium functions for the Ramsey government are well behaved. Overall, we are confident the algorithm we propose is useful for solving the Ramsey problem and thus, to quantify time inconsistency in default models.

It should be noted that when solving the Markov problem using the optimality condition, equation (2) gives us the level of consumption implied by any b'. In contrast, when solving the Ramsey problem using the optimality condition, for any b', equations (15) and (12) give us simultaneously the levels of consumption and the next-period history state h'. We find this is not a problem, as the algorithm finds unique levels of consumption and h' that satisfy these equations simultaneously (the algorithm produces the same results if we start the search for the levels of c and h' that satisfy simultaneously both equations either with the minimum or the maximum value of h' in the grid). This is not surprising. In equation (12), the marginal utility of consumption and then h' goes to infinity (zero) when consumption goes to zero (infinity). In equation (15), a higher h' increases q and then consumption (thus possible levels of consumption for different levels of h' are bounded by possible levels of q). Figure 2 illustrates the solution of this system of two equations as a fixed point problem in h'.

For all (b, h, y), we use the derivative of the Ramsey government's objective $\partial\mathcal{U}/\partial b'$ to approximate the shape of $\mathcal{U}$ and search globally for the optimal b' (that maximizes the approximated

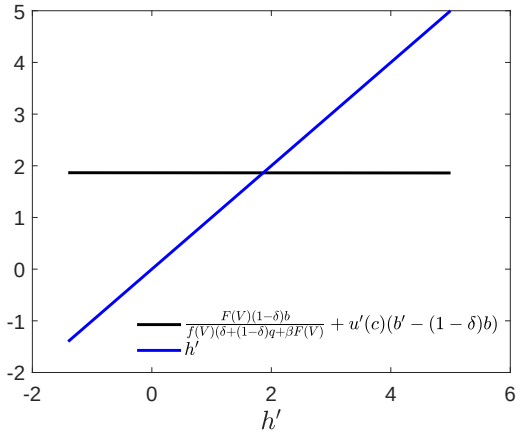


Figure 2: Finding the equilibrium h'. The figure assumes the average levels of b, h, and y in the simulations. The black line presents h' as a function of c (equation 12), where c is function of h' (equation 15). The black line seems flat because the bond price q is not very sensitive to changes in h'.

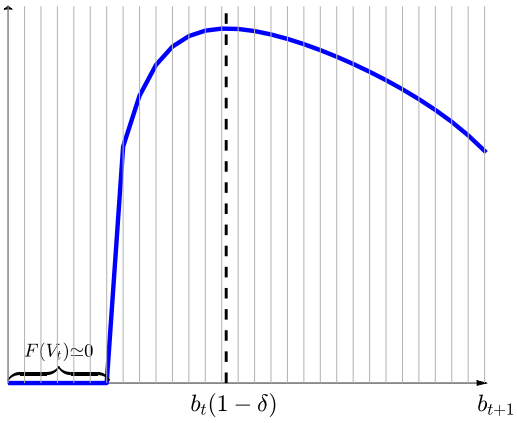


Figure 3: Approximated shape of the objective function of the Ramsey government. The figure assumes the average levels of b, h, and y in the simulations.

function). Figure 3 illustrates the approximated shape of the objective function of the Ramsey government.

6 Quantitative results

Section 6.1 shows that the overindebtedness of the Markov government (compared to the constrained efficient debt levels chosen by the Ramsey government) is small but accounts for the majority of sovereign risk. Section 6.2 shows that the Ramsey government does not choose a countercyclical fiscal policy, and that it may choose to commit to a debt buyback. Sections 6.3

	Data	Markov	Ramsey	One-period debt	Debt limit
Mean debt-to-income ratio (in %)	44.2	44.3	39.5	26.6	32.4
Mean spread (in %)	3.3	3.3	0.5	0.2	0.6
Std dev spread	2.4	2.4	0.4	0.7	0.2
$\sigma(c)/\sigma(y)$	1.1	1.1	1.4	1.4	1.2

Table 2: Data and simulations. For the moments in the simulations, we present the average of 1000 samples of 30 periods without default. The debt level in the simulations is calculated as the present value of future payment obligations, i.e., b/δ. We report the annualized spread.

and 6.4 show the Ramsey allocation is superior to the allocation obtained with one-period debt, and to the one obtained with the optimal debt limit. Sections 6.5 shows that compared with a Markov government, the Ramsey government has a higher probability of completing a successful deleveraging (without default), even with weaker austerity. Section 6.6 shows how Ramsey deleveraging outperforms the optimal sequence of debt limits. Section 6.7 discusses the role of the shocks to the utility of defaulting.

6.1 Overindebtedness and sovereign risk

Table 2 reports long-run moments in the data and in the model simulations. The table shows that the simulations of the Markov government match the calibration targets.

Table 2 also shows that the overindebtedness chosen by the Markov government is, on average, small. The average debt ratio is 44% for the Markov government and 40% for the Ramsey government.

However, this small overindebtedness accounts for a majority of the sovereign risk faced by the Markov government (calibrated to match sovereign risk in the data). Table 2 shows that compared with the Markov government, the Ramsey government would eliminate more than 80% of both the level and the volatility of sovereign risk (as measured by the sovereign spread). The average sovereign spread would decline from 3.3% to 0.5% and the standard deviation of the spread would decline from 2.4% to 0.4%.

It should be noticed that, since the Ramsey government faces higher bond prices (as reflected

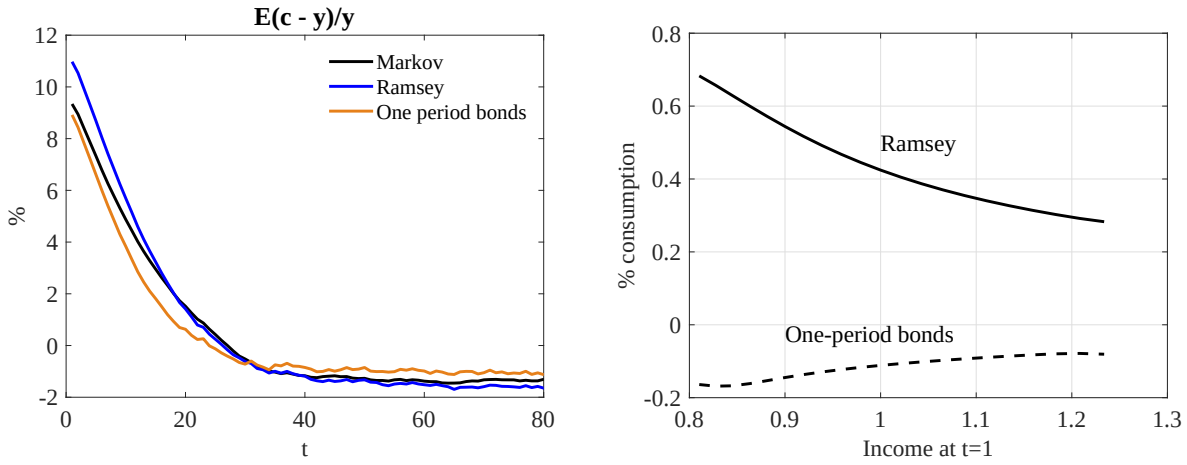


Figure 4: Borrowing and welfare. The left panel presents average borrowing in the simulations. The right panel presents welfare gains measured as the constant proportional change in consumption that would leave domestic consumers indifferent between staying in the benchmark economy with a Markov government and moving either to an economy with a Ramsey government or to an economy in which the government only issues one-period debt ($\delta = 1$ and all other parameter values as in the benchmark calibration). The initial levels of debt and the history variable h are equal to zero.

by the lower spreads it pays), for the same debt levels, the Ramsey government borrows significantly more than the Markov government. Thus, the small overindebteness in the simulations does not imply overborrowing, and it may even be consistent with underborrowing. Figure 4 shows that in fact, on average, and even more so in the initial periods, the Ramsey government borrows more than the Markov government, generating welfare gains.[11]

Summing up, compared with the constrained efficient borrowing policy that a Ramsey government would choose, the Markov government chooses a small overindebtedness that results in substantial and volatile sovereign risk. These results underscore the importance of ongoing efforts by many governments in establishing credible fiscal rules and independent fiscal councils that would allow them to commit to more sensible future debt policies (as the Ramsey government does).

[11]Our model economy without production is likely to severely underestimate the welfare gains from almost eliminating sovereign risk. Bocola (2016) documents sizable effects of sovereign risk on aggregate income, which we do not model. Mendoza and Yue (2012) present a quantitative sovereign default model in which the sovereign spread affects aggregate income.

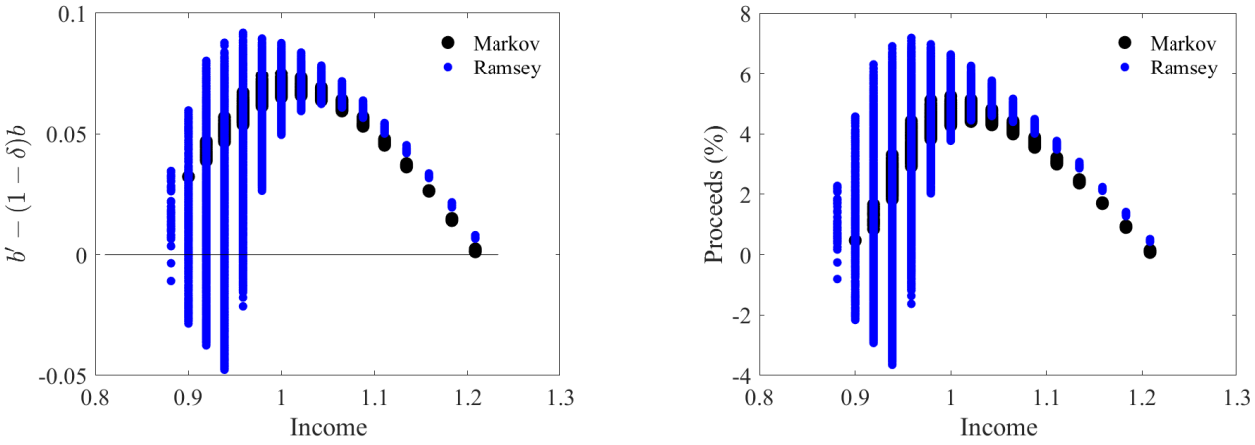


Figure 5: Debt issuances $(b' - (1-\delta)b)$ and issuance proceeds $(q[b' - (1-\delta)b])$ in the simulations.

6.2 Cyclicality of fiscal policy and debt buybacks

The left panel of Figure 5 shows that compared with Markov borrowing, constrained efficient borrowing implies issuing on average more debt when income is high and less debt when income is low. This is, governments would like to commit to a more procyclical debt issuance policy.

Note however, that the Ramsey government faces a much smaller decline in bond prices when income is low (as reflected in the much lower spread volatility faced by the Ramsey government, reported in Table 2). Consequently, lower debt issuances by the Ramsey government in bad times (left panel of Figure 5) do not imply lower borrowing. In fact, as illustrated in the right panel of Figure 5, in bad times, the proceeds from debt issuances are not significantly smaller on average for the Ramsey government.

Overall, with find that the Ramsey government does not want to commit to a fiscal policy that is less procyclical than the one chosen by the Markov government. This casts doubts on the emphasis of existing fiscal rules in promoting countercyclical fiscal policy, including with the presence of escape clauses for the limits imposed by the rules (IMF, 2017). This emphasis is motivated in part by the procyclicality of fiscal policy observed in economies with sovereign risk (Végh and Vuletin, 2011). Cuadra et al. (2010) show that in a sovereign default model the optimal fiscal policy is procyclical. We show that when the government can commit to future borrowing policies that eliminate most default risk, it may still want to commit to a procyclical

policy.

Figure 5 also shows that in contrast with the Markov government, the Ramsey government may choose significant debt buybacks of up to 4% of average annual income. Note that while the Markov government does not benefit from the increase in bond prices implied by a buyback in the current period (Aguiar et al., 2019; Bulow and Rogoff, 1988, 1991), the Ramsey government may benefit from the increase in bond prices that would result from committing to a debt buyback in the future.

6.3 One-period debt

Recall that assuming one-period debt ($\delta = 1$) eliminates the time inconsistency (debt dilution) problem the Ramsey government deals with. Comparing equations (7) and (13) shows that with one-period bonds, the incentives to borrow are identical for the Markov and the Ramsey governments.

However, one-period debt does not only eliminate the time inconsistency problem, but also forces the government to use an inferior debt instrument. Therefore, comparing the economy with one-period debt and the benchmark Markov economy does not allows us to measure the effect of the time inconsistency problem, as comparing the Markov and Ramsey economies does.[12] For example, Figure 4 show that compared with the Ramsey economy, the one-period-debt economy features lower consumption in earlier periods (that are more valuable for the government) and welfare losses instead of welfare gains when compared to the benchmark Markov economy. This is the case because one-period debt creates more rollover needs that could expose the government to more rollover risk, and the government chooses lower debt levels to mitigate these concerns (Table 2).[13]

[12]The same problem appears with the comparison of the benchmark Markov economy and a Markov economy with a different debt instrument that helps mitigate the time inconsistency problem, such as bonds with seniority (Chatterjee and Eyigungor, 2015) or bonds with price-contingent debt covenants (Hatchondo et al., 2016).

[13]The analysis of endogenous maturities is beyond the scope of this paper. Arellano and Ramanarayanan (2012) and Hatchondo et al. (2016) show that with plausible calibrations of the government's incentives to hedge against changes in the borrowing cost, the default model can generate plausible endogenous debt maturities. Hatchondo et al. (2016) also show that mitigating the government's time inconsistency problem with debt covenants would lead the Markov government to choose significantly longer maturities. Hatchondo and Martinez (2013) discuss the time inconsistency problem in the Markov government's maturity choice. Extending our approach for computing Ramsey policies to study time inconsistency problems in the government's maturity choice and other debt

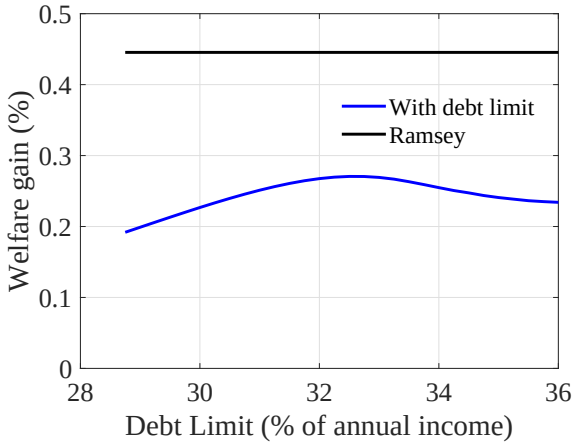


Figure 6: Welfare attained with debt limits and with the Ramsey government. The figure assumes income is at its average level and the initial debt level is zero.

6.4 Optimal debt limit

In this section, we solve the model with the Markov government and the additional constraint $b' \leq \bar{b}$. We find the optimal $\bar{b}$ when the first-period income is at its average level and the initial debt level is zero. As shown in Figure 6, this optimal $\bar{b}$ is 32.5% of mean annual income. Figure 6 also shows that, as it should be, the welfare gain attained with the optimal debt limit is lower than the one attained with the Ramsey government.

6.5 Deleveraging

While previous sections focus on Ramsey plans for a initial state without debt, this section focuses on Ramsey plans for a initial state with high debt and compares the simulations that result from this plan with the ones implied by Markov policies. Thus, comparing outcomes for the Ramsey and Markov governments allows us to quantify the importance of commitment to a long-term deleveraging plan in a crisis.

Figure 7 presents simulations for the Markov and Ramsey government for a initial state with average income and a debt-to-income ratio of 50%, which for the Markov government implies an initial spread of 6.5% and a probability of default during the first six years of deleveraging of 32%.

management policies is an interesting avenue for future research, but is beyond the scope of this paper.

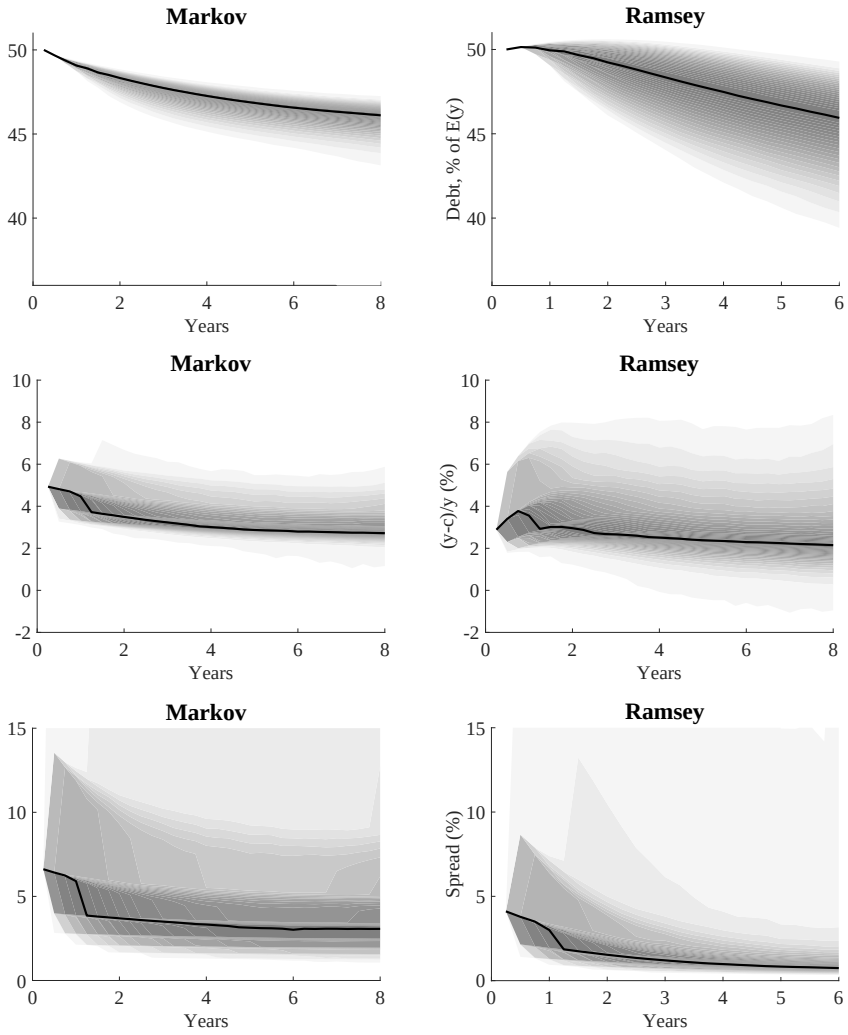


Figure 7: Deleveraging. The initial state features the average level of income and debt-to-income ratio of 50%. To construct the figure, 1000 income path are generated. Only the paths for which the governments choose not to default are presented in the figure. The black line presents the median value.

The Ramsey government achieves a significantly higher probability of a successful deleveraging, reducing the probability of default during the first six years of deleveraging to 22%. Furthermore, none of the income paths in the simulations leads to a default of the Ramsey government without leading to a default of the Markov government.

The Ramsey implements a more successful deleveraging even with weaker austerity. The top panels of Figure 7 show that compared with the Markov government, the Ramsey government chooses a slower deleveraging. This is consistent with the lower savings chosen by the Ramsey government and presented in the middle panels of Figure 7.

The bottom panels of Figure 7 show that the lower probability of default with the Ramsey deleveraging is reflected immediately in lower sovereign spreads, which in turn facilitate the deleveraging. The Ramsey government also reduces greatly the spread volatility (this is in spite of the greater dispersion of Ramsey debt levels).

In sum, our results indicate that the ability to commit to a long-term deleveraging plan allows for a smaller consumption sacrifice while greatly increasing the probability of a successful deleveraging that avoids a default. The implied lower default probability reflected in lower interest rates allows for higher consumption during the Ramsey plan. It should also be emphasized that our model underestimates the gains from having commitment to a long term deleveraging plan (with a consumption equivalent welfare gain of 0.44%). First, we are not considering the gains of bondholders, who benefit greatly from the increase in bond prices implied by commitment (and reflected in the lower spreads paid by the Ramsey government since the first period of consolidation). Second, the lower spreads implied by commitment are not reflected in higher aggregate income.[14]

6.6 Deleveraging with debt limits

In this section, we study deleveraging from the same initial point studied before, but with a sequence of debt limits. We find the optimal sequence of debt limits $\bar{b}(t)$ with $\bar{b}'(t) = a + bt$ and $\bar{b}(t) = \bar{b}(\infty) \forall t \geq T$. To that end, we solve the problem of a Markov government with the debt limit constraint $b' \leq \bar{b}(t)$ and optimize over $\left\{T, \bar{b}(\infty), \bar{b}(0)\right\}$.

The left panel of Figure 8 presents the optimal sequence of debt limits, which are almost always binding in the simulations. This optimal sequence features an initial limit of 50% (equal to the initial level of debt), a transition of 8 years (T=32), and a final debt limit of 38.25%. The optimal sequence achieves a welfare gain of 0.26%, equivalent to 59% of the welfare gain a Ramsey government achieves. The left panel of Figure 8 shows that compared to the median debt levels for the Ramsey deleveraging presented in Figure 7, the optimal sequence of debt limits imposes a much faster deleveraging with a debt reduction three times larger than the Ramsey

[14]Bi et al. (2013) show that expectations about future fiscal consolidations are an important determinant of the success of fiscal adjustments.

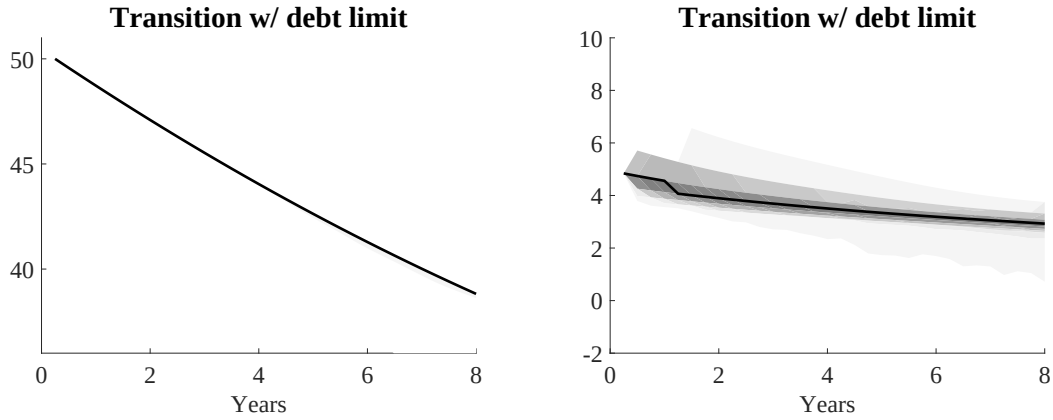


Figure 8: Optimal sequence of debt limits and implied savings for deleveraging. The initial state features the average level of income and debt-to-income ratio of 50%.

median debt reduction. Consistently, comparing Ramsey savings in Figure 7 with the right panel of Figure 8 shows that the optimal sequence of debt limits typically implies stronger austerity than Ramsey deleveraging.

6.7 Shocks to the utility cost of defaulting

Introducing shocks to the utility of defaulting allows us to decouple defaults in model simulations from the income shock, thus bringing the predictions of the baseline model closer to the data. Tomz and Wright (2007) argue that in the data, sovereign defaults do not seem to be explained only by low income, and that political shocks (a possible interpretation for the shocks to the utility of defaulting in the model) may be important. This is in sharp contrast with the predictions of models with only income shocks. Figure 9 shows that in the simulations of our Markov economy with both income shocks and shocks to the utility of defaulting, both shocks are important for accounting for defaults. Most defaults occur for shocks to the utility of defaulting that are between 1.5 and 3 standard deviations above the mean, and there is no default for shocks below the mean. Furthermore, there are almost no defaults for income levels above the mean.

The importance of the shock to the utility cost of defaulting is given by the dispersion of this shock σ_U, which in our benchmark calibration is determined by adding as a target the dispersion of the sovereign spread in the data. Table 3 presents simulation results and Ramsey welfare gains for different values of σ_U (all other parameter values are the ones in our benchmark calibration).

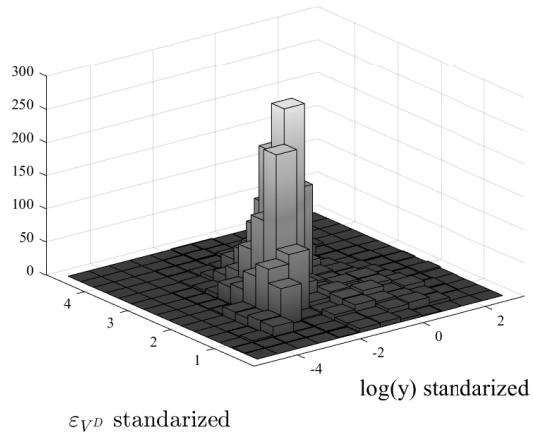


Figure 9: Distribution of defaults in the simulations with the Markov government. The levels of logged income and the shock to the value of defaulting are presented as the number of standard deviations above the mean.

The table shows that for both the Ramsey and the Markov governments, a higher value of σ_U implies lower debt levels and a higher mean and standard deviation of sovereign spreads in the simulations (the latter is consistent with our calibration strategy). Intuitively, a higher value of σ_U implies less control over the cost of debt, making debt more risky (and thus expensive) and less attractive. The government reacts to this by issuing less debt and accepting to pay a higher and more volatile spread.

Table 3 also shows that across different values of σ_U the Ramsey government achieves significant welfare gains and significant reductions of sovereign risk. However, when the shock to the utility cost of defaulting is more extreme (i.e., when σ_U is higher implying extreme values of the spread volatility), compared with the Markov government, the Ramsey government chooses a larger reduction of debt that still implies a relatively high levels of sovereign risk (although significantly lower than in the Markov benchmark). Thus, the Ramsey government achieves lower welfare gains.

7 Conclusions

We show that in quantitative models of sovereign default, compared with the Markov optimality condition, the Ramsey optimality condition only features one additional single dimensional state

Std. dev. shock to V^D (σ_U)	Mean debt (%)	Mean spread (%)	Std dev spread	$\sigma(c)/\sigma(y)$	Welfare gain (%)
Markov					
0.07	44.3	3.0	2.2	1.1	
0.10	**44.3**	**3.3**	**2.4**	**1.1**	
0.25	40.7	3.9	2.8	1.1	
0.75	26.4	8.9	4.7	1.0	
1.00	22.5	12.1	5.2	1.0	
Ramsey					
0.075	40.4	0.5	0.4	1.4	0.43
0.10	**39.5**	**0.5**	**0.4**	**1.4**	**0.45**
0.25	34.5	0.9	0.6	1.3	0.41
0.75	17.8	4.4	2.1	1.1	0.17
1.00	13.8	8.4	3.5	1.0	0.13

Table 3: Importance of the shock to the utility cost of defaulting.

variable. This allows us to propose a tractable algorithm for finding the constrained efficient borrowing policy (with commitment to future borrowing strategies but not to a default policy).

Comparing the simulations of the model with and without commitment allows us to present a quantitative assessment of the importance of the government's time inconsistency problem. We find that without commitment, there is a small overindebtedness that, however, accounts for more than 80% of the default risk. Furthermore, commitment can increase significantly the probability of a successful deleveraging, even when committing to weaker austerity. These results underscore the importance of governments' efforts to limit their future policies with fiscal rules and independent fiscal councils. We also find that commitment does not affect significantly the procyclicality of fiscal policy, which questions the emphasis on countercyclical fiscal policy in existing fiscal rules.

Our algorithm could be extended to study other aspects of debt management in which time inconsistency plays a role. Overindebtedness arises because Markov governments do not internalize the losses they impose to debt holders by borrowing more. Similar inefficiencies arise because Markov governments do not internalize the losses of debt holders when they issue debt to ac-

cumulate assets (Bianchi et al., 2018), when they issue debt with longer maturities (Hatchondo and Martinez, 2013), or when they issue more foreign currency debt. Extending our analysis to study and quantify these issues and thus inform the limits fiscal rules should impose on other debt management aspects is an interesting avenue for future research.

Our framework could also be extended to study weaker commitment to Ramsey plans. In this paper, the Ramsey government can only abandon a plan (and star a new one) when defaulting on its debt (and defaulting forces the government to abandon the Ramsey plan). Future research could decouple the decisions of defaulting on debt and of defaulting on the Ramsey plan.

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A Proof of Proposition 1

Proof. First note that the derivative of the Ramsey government's objective $\mathcal{U}$ with respect to the debt outstanding at the end of any period t and after any income history $\tilde{y}^t$, $b_{t+1}(\tilde{y}^t)$, can be written as:

$$\frac{\partial \mathcal{U}}{\partial b_{t+1}(\tilde{y}^t)} = \frac{\partial \left[u(c_0) + \beta Pr(\tilde{y}_1) \left[F\left(V_1(\tilde{y}^1)\right) V_1(\tilde{y}^1) + \int_{V_1(\tilde{y}^1)}^{\infty} U f(U) dU \right] \right]}{\partial b_{t+1}(\tilde{y}^t)}.$$

Since the continuation values under repayment and default are identical at $V_1(\tilde{y}^1)$, changing the repayment probability $F\left(V_1(\tilde{y}^1)\right)$ does not affect the expected continuation value. Therefore,

$$\begin{aligned}
\frac{\partial \mathcal{U}}{\partial b_{t+1}(\tilde{y}^t)} &= u'(c_0)\iota_0 \frac{\partial q_0}{\partial b_{t+1}(\tilde{y}^t)} + \beta Pr(\tilde{y}_1) F\left(V_1(\tilde{y}^1)\right) \frac{\partial V_1(\tilde{y}^1)}{\partial b_{t+1}(\tilde{y}^t)} \\
&= u'(c_0)\iota_0 \frac{\partial q_0}{\partial b_{t+1}(\tilde{y}^t)} + \beta Pr(\tilde{y}_1) F\left(V_1(\tilde{y}^1)\right) \left[u'\left(c_1(\tilde{y}^1)\right) \iota_1\left(\tilde{y}^1\right) \frac{\partial q_1\left(\tilde{y}^1\right)}{\partial b_{t+1}(\tilde{y}^t)} \right. \\
&+ \quad \left. \beta^2 Pr(\tilde{y}^2) F\left(V_1(\tilde{y}^1)\right) F\left(V_2(\tilde{y}^2)\right) \frac{\partial V_2(\tilde{y}^2)}{\partial b_{t+1}(\tilde{y}^t)} \right]. \qquad (18)
\end{aligned}$$

Continuing developing the derivatives $\frac{\partial V_{t-n}(\tilde{y}^{t-n})}{\partial b_{t+1}(\tilde{y}^t)}$ for all $0 < n < t$ we obtain

$$\begin{aligned}
\frac{\partial \mathcal{U}}{\partial b_{t+1}(\tilde{y}^t)} = \sum_{n=1}^{t} \beta^{t-n} Pr(\tilde{y}^{t-n}) \prod_{m=1}^{t-n} F\left(V_m(\tilde{y}^m)\right) u'\left(c_{t-n}(\tilde{y}^{t-n})\right) \iota_{t-n}\left(\tilde{y}^{t-n}\right) \frac{\partial q_{t-n}\left(\tilde{y}^{t-n}\right)}{\partial b_{t+1}} + \\
\beta^t Pr(\tilde{y}^t) \prod_{m=1}^{t} F\left(V_m(\tilde{y}^m)\right) \frac{\partial V_t(\tilde{y}^t)}{\partial b_{t+1}}, \qquad (19)
\end{aligned}$$

where we assume $F(V_0) = 1$.

Bond price derivatives. We next show that the effect of bond issuances on the bond prices of previous periods ($\frac{\partial q_{t-n}(\tilde{y}^{t-n})}{\partial b_{t+1}(\tilde{y}^t)}$) can be written as a liner function of the effect of bond issuances on the bond price in the issuance period ($\frac{\partial q_t(\tilde{y}^t)}{\partial b_{t+1}(\tilde{y}^t)}$), and of the effect of bond issuances on the expected utility in the issuance period ($\frac{\partial V_t(\tilde{y}^t)}{\partial b_{t+1}(\tilde{y}^t)}$). The latter affects the probability of a default in that period, which is not captured in the bond price for that period, but affects bond prices in previous periods.

Lemma 2

$$\frac{\partial q_{t-n}(\tilde{y}^{t-n})}{\partial b_{t+1}(\tilde{y}^t)} = Pr(\tilde{y}_{t-n+1}, ...\tilde{y}_t \mid \tilde{y}_{t-n}) \left[A^q_{t,t-n}(\tilde{y}^t) \frac{\partial q_t(\tilde{y}^t)}{\partial b_{t+1}(\tilde{y}^t)} + B^q_{t,t-n}(\tilde{y}^t) \frac{\partial V_t(\tilde{y}^t)}{\partial b_{t+1}(\tilde{y}^t)} \right]. \tag{20}$$

Proof. First note that we can write the derivative of a bond price $q_{t-n}(\tilde{y}^{t-n})$ with respect to issuances in a future period $b_{t+1}(\tilde{y}^t)$ as a linear function of the derivative of the next-period expected utility $V_{t-n+1}(\tilde{y}^{t-n+1})$ and the next-period bond price $q_{t-n+1}(\tilde{y}^{t-n+1})$. Recall

$$q_t(y^t) = \frac{1}{1+r} \sum_{j=1}^{J} \pi_j(y_t) F_j(V_{t+1}(y^t, y_j)) \left[\delta + (1-\delta) q_{t+1}(y^t, y_j) \right].$$

Therefore,

$$\begin{aligned} \frac{\partial q_{t-n}(\tilde{y}^{t-n})}{\partial b_{t+1}(\tilde{y}^t)} = \frac{Pr\left(\tilde{y}_{t-n+1} \mid \tilde{y}_{t-n}\right)}{1+r} \Bigg[& f(V_{t-n+1}(\tilde{y}^{t-n+1})) \frac{\partial V_{t-n+1}(\tilde{y}^{t-n+1})}{\partial b_{t+1}(\tilde{y}^t)} \left[\delta + (1-\delta) q_{t-n+1}(\tilde{y}^{t-n+1}) \right] \\ & + F(V_{t-n+1}(\tilde{y}^{t-n+1}))(1-\delta) \frac{\partial q_{t-n+1}(\tilde{y}^{t-n+1})}{\partial b_{t+1}(\tilde{y}^t)} \Bigg] \end{aligned} \tag{21}$$

As illustrated in equation (18), the derivative of the expected utility in any period $t-n$ can be written as a linear function of the derivative of the bond price in that period and the derivative of the expected utility next period. Therefore, we can continue to substitute the derivatives of the expected utility and the bond price in equation (21) for linear functions of the derivative of the next-period expected utility and bond price, until we write $\frac{\partial q_{t-n}(\tilde{y}^{t-n})}{\partial b_{t+1}(\tilde{y}^t)}$ as a liner function of the derivative of the bond price and the expected utility in period t. ■

The next lemma shows how the coefficients that determine the effect of issuances in $t+1$ in the $t-n$ bond price, $A^q_{t+1,t-n}$ and $B^q_{t+1,t-n}$, can be written as functions of $A^q_{t,t-n}$ and $B^q_{t,t-n}$. To simplify notation, we drop the arguments $\tilde{y}$ and use the notation $u'_t = u'(c_t(\tilde{y}^t))$.

Lemma 3

$$A^q_{t+1,t-n} = \left(A^q_{t,t-n} + B^q_{t,t-n} u'_t \iota_t \right) (1-\delta) F(V_{t+1})/(1+r), \tag{22}$$

$$B^q_{t+1,t-n} = A^q_{t,t-n} \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1} \right]}{1+r} + B^q_{t,t-n} \left[u'_t \iota_t \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1} \right]}{1+r} + \beta F(V_{t+1}) \right]. \tag{23}$$

Proof. Since $\frac{\partial V_t}{\partial b_{t+2}(\tilde{y}^{t+1})} = u'_t \iota_t \frac{\partial q_t}{\partial b_{t+2}(\tilde{y}^{t+1})} + \beta Pr(\tilde{y}_{t+1} \mid \tilde{y}_t) F(V_{t+1}) \frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})}$,

$$\begin{aligned}\frac{\partial q_{t-n}}{\partial b_{t+2}(\tilde{y}^{t+1})} &= Pr(\tilde{y}_{t-n+1},...,\tilde{y}_t \mid \tilde{y}_{t-n}) \left[A^q_{t,t-n} \frac{\partial q_t}{\partial b_{t+2}(\tilde{y}^{t+1})} \right. \\ &\left. + B^q_{t,t-n} \left[u_t' \iota_t \frac{\partial q_t}{\partial b_{t+2}(\tilde{y}^{t+1})} + \beta Pr(\tilde{y}_{t+1} \mid \tilde{y}_t) F(V_{t+1}) \frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} \right] \right] \\ &= Pr(\tilde{y}_{t-n+1},...,\tilde{y}_t \mid \tilde{y}_{t-n}) \left[\left(A^q_{t,t-n} + B^q_{t,t-n} u_t' \iota_t \right) \frac{\partial q_t}{\partial b_{t+2}(\tilde{y}^{t+1})} \right. \\ &\left. + B^q_{t,t-n} \beta Pr(\tilde{y}_{t+1} \mid \tilde{y}_t) F(V_{t+1}) \frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} \right].\end{aligned}$$

Recall that

$$\frac{\partial q_t}{\partial b_{t+2}(\tilde{y}^{t+1})} = \frac{Pr(\tilde{y}_{t+1} \mid \tilde{y}_t)}{1+r} \left[f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1} \right] \frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} + (1-\delta) F(V_{t+1}) \frac{\partial q_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} \right].$$

Therefore, we can write $\frac{\partial q_{t-n}}{\partial b_{t+2}(\tilde{y}^{t+1})}$ as a function of $\frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})}$ and $\frac{\partial q_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})}$:

$$\begin{aligned}\frac{\partial q_{t-n}}{\partial b_{t+2}(\tilde{y}^{t+1})} &= Pr(\tilde{y}_{t-n+1},...,\tilde{y}_t \mid \tilde{y}_{t-n}) \left[\left(A^q_{t,t-n} + B^q_{t,t-n} u_t' \iota_t \right) \right. \\ &\frac{Pr(\tilde{y}_{t+1} \mid \tilde{y}_t)}{1+r} \left[f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1} \right] \frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} + (1-\delta) F(V_{t+1}) \frac{\partial q_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} \right] \\ &\left. + B^q_{t,t-n} \beta Pr(\tilde{y}_{t+1} \mid \tilde{y}_t) F(V_{t+1}) \frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} \right].\end{aligned}$$

Since $Pr(\tilde{y}_{t-n+1},...,\tilde{y}_{t+1} \mid \tilde{y}_{t-n}) = Pr(\tilde{y}_{t-n+1},...,\tilde{y}_t \mid \tilde{y}_{t-n}) Pr(\tilde{y}_{t+1} \mid \tilde{y}_t)$,

$$\begin{aligned}\frac{\partial q_{t-n}}{\partial b_{t+2}(\tilde{y}^{t+1})} &= Pr(\tilde{y}_{t-n+1},...,\tilde{y}_{t+1} \mid \tilde{y}_{t-n}) \left[\left(A^q_{t,t-n} + B^q_{t,t-n} u_t' \iota_t \right) \frac{(1-\delta) F(V_{t+1})}{1+r} \frac{\partial q_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} \right. \\ &+ \left[A^q_{t,t-n} \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1} \right]}{1+r} \right. \\ &\left. \left. + B^q_{t,t-n} \left[\frac{u_t' \iota_t f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1} \right]}{1+r} + \beta F(V_{t+1}) \right] \right] \frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} \right]. \qquad (24)\end{aligned}$$

Equations (20) and (24) imply equations (22) and (23). ■

Note that $A^q_{t,t} = 1$ and $B^q_{t,t} = 0$ (equation 20). Therefore, equations (22) and (23) imply

$$A^q_{t+1,t} = (1-\delta) F(V_{t+1}) / (1+r), \qquad (25)$$

$$B^q_{t+1,t} = f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1} \right] / (1+r). \qquad (26)$$

Simplified optimality condition for the Ramsey government. Plugging in equation (20) into equation (19), we can write the derivative of the Ramsey's objective as a linear function of the effect of bond issuances on the bond price in the issuance period ($\frac{\partial q_t(\tilde{y}^t)}{\partial b_{t+1}(\tilde{y}^t)}$), and of the effect of bond issuances on the expected utility in the issuance period ($\frac{\partial V_t(\tilde{y}^t)}{\partial b_{t+1}(\tilde{y}^t)}$). Therefore, we can write the optimality condition for the Ramsey planner as

$$Pr(\tilde{y}^t)\left[A_t(\tilde{y}^t)\frac{\partial q_t(\tilde{y}^t)}{\partial b_{t+1}(\tilde{y}^t)} + B_t(\tilde{y}^t)\frac{\partial V_t(\tilde{y}^t)}{\partial b_{t+1}(\tilde{y}^t)}\right] = 0, \tag{27}$$

with

$$A_t(\tilde{y}^t) = \sum_{n=1}^{t}\beta^{t-n}\prod_{m=1}^{t-n}F(V_m(\tilde{y}^m))u'(c_{t-n}(\tilde{y}^{t-n}))\iota_{t-n}(\tilde{y}^{t-n})A^q_{t,t-n},$$
$$B_t(\tilde{y}^t) = \sum_{n=1}^{t}\beta^{t-n}\prod_{m=1}^{t-n}F(V_m(\tilde{y}^m))u'(c_{t-n}(\tilde{y}^{t-n}))\iota_{t-n}(\tilde{y}^{t-n})B^q_{t,t-n} + \beta^t\prod_{m=1}^{t}F(V_m(\tilde{y}^m)).$$

The next lemma shows how the coefficients of for the effect of $t+1$ issuances on the bond price and the expected utility in the optimality condition of the Ramsey government, $A_{t+1}(\tilde{y}^{t+1})$ and $B_{t+1}(\tilde{y}^{t+1})$, can be written as functions of $A_t(\tilde{y}^t)$ and $B_t(\tilde{y}^t)$.

Lemma 4

$$A_{t+1} = \frac{(1-\delta)F(V_{t+1})}{1+r}\left(A_t + u'_t\iota_t B_t\right), \tag{28}$$

$$B_{t+1} = \frac{f(V_{t+1})\left[\delta+(1-\delta)q_{t+1}\right]}{1+r}A_t + \left[u'_t\iota_t\frac{f(V_{t+1})\left[\delta+(1-\delta)q_{t+1}\right]}{1+r} + \beta F(V_{t+1})\right]B_t. \tag{29}$$

Proof.

Since $Pr(\tilde{y}^t) > 0$, the optimality condition (27) requires that the term in brackets equals zero. Therefore, the optimality condition for $b_{t+2}(\tilde{y}^{t+1})$ implies that

$$A_{t+1}(\tilde{y}^{t+1})\frac{\partial q_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} + B_{t+1}(\tilde{y}^{t+1})\frac{\partial V_{t+1}}{\partial b_{t+2}(\tilde{y}^{t+1})} = 0 \tag{30}$$

where

$$A_{t+1}(\tilde{y}^{t+1}) = \sum_{n=1}^{t}\beta^{t-n}\prod_{m=1}^{t-n}F(V_m(\tilde{y}^m))u'(c_{t-n}(\tilde{y}^{t-n}))\iota_{t-n}(\tilde{y}^{t-n})A^q_{t+1,t-n}$$
$$+\beta^t\prod_{m=1}^{t}F(V_m(\tilde{y}^m))u'(c_t(\tilde{y}^t))\iota_t(\tilde{y}^t)A^q_{t+1,t} \tag{31}$$

$$B_{t+1}(\tilde{y}^{t+1}) = \sum_{n=1}^{t} \beta^{t-n} \prod_{m=1}^{t-n} F(V_m(\tilde{y}^m)) u'(c_{t-n}(\tilde{y}^{t-n})) \iota_{t-n}(\tilde{y}^{t-n}) B^q_{t+1,t-n} + \beta^t \prod_{m=1}^{t} F(V_m(\tilde{y}^m)) u'(c_t(\tilde{y}^t)) \iota_t(\tilde{y}^t) B^q_{t+1,t} + \beta^{t+1} \prod_{m=1}^{t+1} F(V_m(\tilde{y}^m)) \tag{32}$$

To simplify notation, we drop the income history arguments $\tilde{y}^t$ and use the notation $u'_t = u'(c_t(\tilde{y}^t))$. Plugging in equations (22) and (25) into equation (31) we obtain equation (28):

$$\begin{aligned}
A_{t+1} &= \sum_{n=1}^{t} \beta^{t-n} \prod_{m=1}^{t-n} F(V_m) u'_{t-n} \iota_{t-n} \left(A^q_{t,t-n} + B^q_{t,t-n} u'_t \iota_t \right) \frac{(1-\delta) F(V_{t+1})}{1+r} \\
&+ \beta^t \prod_{m=1}^{t} F(V_m) u'_t \iota_t \frac{(1-\delta) F(V_{t+1})}{1+r} \\
&= \frac{(1-\delta) F(V_{t+1})}{1+r} \left[\sum_{n=1}^{t} \beta^{t-n} \prod_{m=1}^{t-n} F(V_m) u'_{t-n} \iota_{t-n} A^q_{t,t-n} \right. \\
&\left. + u'_t \iota_t \left[\sum_{n=1}^{t} \beta^{t-n} \prod_{m=1}^{t-n} F(V_m) u'_{t-n} \iota_{t-n} B^q_{t,t-n} + \beta^t \prod_{m=1}^{t} F(V_m) \right] \right] \\
&= \frac{(1-\delta) F(V_{t+1})}{1+r} \left[A_t + u'_t \iota_t B_t \right].
\end{aligned}$$

Plugging in equations (23) and (26) into equation (32) we obtain equation (29):

$$
\begin{aligned}
B_{t+1} =& \sum_{n=1}^{t} \beta^{t-n} \prod_{m=1}^{t-n} F(V_m) u'_{t-n} \iota_{t-n} \left[A^q_{t,t-n} \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1}\right]}{1+r} \right. \\
& \left. + B^q_{t,t-n} \left[u'_t \iota_t \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1}\right]}{1+r} + \beta F(V_{t+1}) \right] \right] \\
& + \beta^t \prod_{m=1}^{t} F(V_m) u'_t \iota_t \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1}\right]}{1+r} + \beta^{t+1} \prod_{m=1}^{t+1} F(V_m) \\
=& \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1}\right]}{1+r} \sum_{n=1}^{t} \beta^{t-n} \prod_{m=1}^{t-n} F(V_m) u'_{t-n} \iota_{t-n} A^q_{t,t-n} \\
& + \left[u'_t \iota_t \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1}\right]}{1+r} + \beta F(V_{t+1}) \right] \sum_{n=1}^{t} \beta^{t-n} \prod_{m=1}^{t-n} F(V_m) u'_{t-n} \iota_{t-n} B^q_{t,t-n} \\
& + \left[u'_t \iota_t \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1}\right]}{1+r} + \beta F(V_{t+1}) \right] \beta^t \prod_{m=1}^{t-n} F(V_m) \\
=& \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1}\right]}{1+r} A_t + \left[u'_t \iota_t \frac{f(V_{t+1}) \left[\delta + (1-\delta) q_{t+1}\right]}{1+r} + \beta F(V_{t+1}) \right] B_t
\end{aligned}
$$

■

Lemma 4 shows that there is a recursive law of motion for A' and B' that allows us to present a recursive formulation of the Ramsey government's optimality condition using A and B as additional state variables to the Markov state variables b, and y. The Ramsey equilibrium borrowing policy $\hat{b}(b, y, A, B)$ solves

$$
-A \frac{\sum_j \pi_j(y) f_j(V(b', y', A', B'))) u'(c') \left[\delta + (1-\delta) q(\hat{b}(b', y_j, A, B), y_j, h')\right]^2}{1+r} + B \frac{\partial V(b, y, A, B)}{\partial b'} = 0,
$$

where

$$
\begin{aligned}
V(b, y, A, B) \;=\; & u(c) + \beta \sum_j \pi_j(y) \left[F_j\left(V(b', y_j, A', B')\right) V(b', y_j, A', B') \right. \\
& \left. + \int_{V(b', y_j, A', B')} U f_j(dU) \right],
\end{aligned}
$$

$$
\begin{aligned}
c &= y - \delta b + q(b', y, A', B')\iota, \\
q(b', y, A, B) &= \frac{1}{1+r} \sum_j \pi_j(y) F_j(V(b', y_j, A', B')) \left[\delta + (1-\delta) q(\hat{b}(b', y_j, A', B'), y_j, A', B')\right], \\
A' &= \frac{(1-\delta) F_{y'}(V(b', y', A', B'))}{1+r} \left[A + u'(c)\iota B\right], \\
B' &= \frac{f_{y'}(V(b', y', A', B'))\left[\delta + (1-\delta) q(b'', y', A', B')\right]}{1+r} \left[A + u'(c)\iota B\right] + \beta F_{y'}(V(b', y', A', B'))B,
\end{aligned}
$$

and $\hat{b}$ denotes the Ramsey equilibrium borrowing policy. ■