Strategic Complementarity in a Dynamic Model of Technology Adoption: Evidence from Mobile Payments*

Fernando Alvarez[†] University of Chicago

David Argente[‡] Pennsylvania State University

Francesco Lippi[§] EIEF and LUISS

Esteban Méndez-Chacón[¶] Central Bank of Costa Rica

Diana Van Patten[‖] Yale University

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Abstract

We study the network externalities that arise when introducing electronic payment methods into an economy, and how these evolve as the adoption of these payment methods increases in the population. We develop a theoretical model of technology adoption that features users' networks and their impact on the dynamics of adoption patterns through strategic complementarities. Our model allows for both stochastic network connections and arbitrary paths of the distribution of adopters. The adopter's decision problem and the aggregate adoption outcome in our model are cast as a Mean Field Game. We apply the model to study SINPE Mobile, an electronic method of payment developed by the Central Bank of Costa Rica. SINPE Mobile was launched in 2015. Since then, the technology has been adopted by over 70% of the adult population, and transactions via SINPE accounted for more than 10% of GDP in 2021. We use detailed individual-level data on the diffusion of electronic payment methods and user networks to quantify strategic complementarities through the lens of the model.

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[†]Email: f-alvarez1@uchicago.edu. Address: 1126 E. 59th St., Chicago, IL 60637

[‡]Email: dargente@psu.edu. Address: 403 Kern Building, University Park, PA 16801.

[§]Email: francescolippi@gmail.com. Address: Via Sallustiana, 62, 00187 Rome, Italy.

[¶]Email: mendezce@bccr.fi.cr. Address: Avenida Central y Primera, Calles 2 y 4, San José, Costa Rica 10101.

[‖]Email: diana.vanpatten@yale.edu. Address: 165 Whitney Avenue 3479 Evans Hall, New Haven, CT 06511.

1 Introduction

The diffusion of new technologies is critical for economic growth. The importance of this topic has long been recognized and studied in economics. However, progress in this area of research has been hindered by the lack of detailed data on diffusion and by the challenges that arise when modelling adoption dynamically.

In this project, we study the network externalities that arise when introducing electronic payment methods into an economy, and how these evolve as the adoption of these payment methods increases. The field of payment methods is experiencing a digital transformation. There is a growing interest among central banks to get involved in the promotion of digital payment platforms, disincentivizing the use of cash, and in analysing the convenience of issuing central banking digital currencies. Behind mobile payments there is the promise of reducing costs associated with cash, increasing financial inclusion, and encouraging innovation. This transformation has been particularly relevant in developing countries, given the scarce infrastructure in terms of physical transportation and other types of communication, like fixed lines.[1] Evidence of these benefits, however, is scarce and mostly limited to RCTs or surveys on a small share of the population.

First, we develop a dynamic model of technology adoption, which leverages on new theory on Mean Field Games (MFG), and seeks to understand the role of strategic complementarities in the diffusion process of electronic methods of payment. The model features users' networks and tracks their impact on the dynamics of adoption patterns through strategic complementarities while remaining tractable, and it allows us to characterize the process of learning and derive testable predictions.

We then use data on the diffusion of electronic payment methods and user networks to quantify these strategic complementarities. In particular, we leverage detailed data on Costa Rica's National Electronic Payment System (SINPE, by its initials in Spanish).[2] SINPE is a digital platform developed and administered by the Central Bank of Costa Rica, which connects users in the country and allows them to mobilize funds between their bank accounts. SINPE clients include both firms and individuals. The platform was launched in May 2015 and now has over 75% of the adult population. By 2021, over 10% of the country's GDP was transacted through SINPE, with over 90% of the transactions being between individuals.

During the time SINPE has been operating, information on all transactions has been collected, which allows us to analyse this system of national payments in great detail. These

[1] Suri (2017) summarizes numerous studies documenting how RCTs and surveys involving digital payments in sub-Saharan African countries like Kenya have led to large benefits to the subset of users involved in the study.

[2] More precisely, the app is called "SINPE móvil," although throughout we will be referring to it as "SINPE."

features are, to the best of our knowledge, unique, and is one of the main reasons to choose Costa Rica as our setting, and would allow us to speak to general equilibrium effects by design. In turn, data on users—both receivers and senders—can be linked to several relevant networks, including: the input-output firm network that tracks transactions between firms, the employer-employee network, family networks, and spatial networks to explore local diffusion and the role of neighbors.

The network data from the platform allows us to study network externalities and model predictions empirically, by exploiting changes in the relevant networks of users. We do so both naively and through an IV strategy that relies on exogenous changes on transactions between individuals—which are an overwhelming majority of all transactions and the main focus of our study—due to firms' adoption of the app. We also exploit the dynamic nature of our data to conduct several mover designs, that rely on individuals changing their job or neighborhood. Across specifications, we document that the technology features robust strategic complementarities.

Contribution to the literature In contrast to the previous literature which has studied *deterministic* problems (e.g. Stokey 2020), our model allows for both stochastic network connections and arbitrary paths of the distribution of adopters. The adopter's decision problem and the aggregate adoption outcome in our model are cast as a MFG. The stochastic nature of the network benefits is both important to link to the estimates of the effect on exogenous individual changes on adoption probabilities, as well as to produce well behaved equilibrium outcomes. The solution of this MFG is the first one in the literature to characterize the full path of the dynamic adoption patterns of a technology, while allowing for network externalities to play a role.

Our theoretical approach has three advantages relative to the previous literature studying adoption *dynamically*. First, it allows for dynamics in technology adoption as observed in the data. In the stochastic version of the model, our model has dynamics even without the inclusion of frictions to delay the transition to steady state. Second, our model allows for multiple equilibria. As a result, we can consider equilibria with low adoption rates due to coordination failures, a feature that is very relevant in low income countries. Lastly, and more importantly, casting the problem as a Mean Field Game allows us to solve the planning problem. This is relevant for policy since the presence of network externalities implies that the solution of the decentralized problem deviates from the first best. The solution of this problem is non-trivial since the planner needs to account for the law of motion of the density of non-adopters at each point of the state space and each time period. Our framework allows us to compute the optimal subsidy, which equates the solution of the decentralized problem with

that of the planner and depends positively on the importance of strategic complementarities, which can be estimated directly in the data. Relevant studies include Benhabib et al. (2021), who model firms that can endogenously innovate and adopt at technology and the effect of these choices on productivity and balanced growth, but without conducting an analysis of the transitions between steady states; Crouzet et al. (2020), who develop a model with a unique equilibrium where the rate of adoption increases given a shock due to complementarities and where dynamics come from a sluggish adjustment a la Calvo (1983); and Buera et al. (2021), who study policies that can coordinate technology adoption across firms.

On the empirical front, SINPE data spans from 2015 to 2021. During this time, it went from zero adoption to over 75% of *national* adoption. These features allow us to study the general equilibrium effects of adoption across a long time period and complements previous studies, summarized by Suri (2017), that have relied on RCTs or shorter periods of time to analyze the patters of adoption of electronic methods of payment. In contrast to other large-scale studies, of which the closest one to our work is Crouzet et al. (2020), who rely on variation in the intensity with which Indian districts were exposed to the cash contraction induced by the 2016 Indian Demonetization, we are able to use individual-level data on adoption and SINPE usage, and on each person's (and firm's) network of SINPE users, relatives, neighbors, and coworkers. This provides an opportunity to understand the relevant networks for each user, identify the strength of complementarities and how they vary across networks and time, and the dynamics of adoption over a long time period.

2 Model setup

We setup a model to study the adoption of a new technology. The economy is populated by a continuum of agents that differ in the potential benefits from adopting the technology. Let $N(t)$ denote the number of agents that have adopted the technology at time t and x be the idiosyncratic potential benefit of adopting related to, e.g. the agent's strength of connections. We assume that if an agent adopts the technology it has a flow benefit

$$x(\theta_0 + \theta_n N(t)) \tag{1}$$

at time t, where $\theta_0, \theta_n > 0$ are parameters. The idiosyncratic benefit x follows a Brownian motion, independent across agents, with variance per unit of time σ, no drift, and reflecting barriers at $x = 0$ and $x = U$, so that $dx = \sigma dW$ where W is a standardized Brownian motion. We let $c > 0$ be the fixed cost of adopting the technology. The time discount rate is $r > 0$, and we assume that with probability ν per unit of time agents die, so that the agents

discount at rate $\rho \equiv r + \nu$. Agents that die are replaced by newborns without the technology and are given a random draw x from the invariant density f on $[0, U]$ which is uniformly distributed due to our reflecting barriers assumption.

2.1 The Decentralized Problem

Value of Adoption: Let $a(x,t)$ be the value function of an agent that has adopted:

$$a(x,t) = \int_t^\infty e^{-\rho(s-t)} \left(\theta_0 + \theta_n N(s)\right) \mathbb{E}\Big[x(s) | x(t) = x\Big] ds$$

for all $t \geq 0$ and $x \in [0, U]$. Note that the individual agent takes the path of $N(t)$ as given. Moreover, $a(x,t)$ solves the p.d.e. and boundary conditions for all $t \geq 0$:

$$\begin{aligned} \rho a(x,t) &= x(\theta_0 + \theta_n N(t)) + \frac{\sigma^2}{2} a_{xx}(x,t) + a_t(x,t) \text{ if } x \in [0,U] \\ a_x(0,t) &= a_x(U,t) = 0 \end{aligned}$$

where the boundary conditions arise from our assumption of reflecting barriers. Throughout, we assume $0 \leq a(x,t) \leq \frac{U(\theta_0+\theta_n)}{\rho}$ for all x, t, and $0 < c < \frac{U(\theta_0+\theta_n)}{\rho}$.

Adoption Decision: The value function of an agent that has not adopted solves the following variational inequality:

$$\rho v(x,t) = \max\left\{ \frac{\sigma^2}{2} v_{xx}(x,t) + v_t(x,t)\,,\; \rho\left(-c + a(x,t)\right) \right\}$$

for all $t \geq 0$ and $x \in [0, U]$. We conjecture that the optimal decision rule is given by a path for the threshold $\bar{x}(t) \in (0, U)$ such so that, for each $t \geq 0$, the following holds

$$\begin{aligned} \rho v(x,t) &= \frac{\sigma^2}{2} v_{xx}(x,t) + v_t(x,t) \text{ if } 0 \leq x \leq \bar{x}(t) \\ v(x,t) &= -c + a(x,t) \text{ if } \bar{x}(t) \leq x \leq U \end{aligned}$$

If $v(\cdot, t)$ is C^1 we have the following boundary conditions for all $t \geq 0$:

$$\begin{aligned} v(\bar{x}(t), t) &= a(\bar{x}(t), t) - c && \text{Value Matching} \\ v_x(\bar{x}(t), t) &= a_x(\bar{x}(t), t) && \text{Smooth Pasting} \\ v_x(0, t) &= 0 && \text{Reflecting} \end{aligned}$$

where the first one is the value matching condition, the second the smooth pasting condition, and the last one arises from the reflecting barrier at $x = 0$.[3]

Invariant Distribution: Let $m(x,t)$ denote the density of the agents with x that have not adopted at t. The law of motion of m for all $t \geq 0$ is:

$$\begin{aligned} m_t(x,t) &= \nu\left(f(x) - m(x,t)\right) + \frac{\sigma^2}{2} m_{xx}(x,t) \text{ if } 0 \leq x \leq \bar{x}(t) \\ m(x,t) &= 0 \quad \text{for } x \in [\bar{x}(t), U] \\ m_x(0,t) &= 0 \end{aligned} \tag{2}$$

The p.d.e. is the standard KFE. The density of non-adopters is zero to the right of $\bar{x}(t)$, since this is an exit point. The last boundary condition is obtained from our assumption that x reflects at $x = 0$.

The Decentralized Dynamic Equilibrium as a Mean Field Game. Let the initial condition $m(x,0) = m_0(x)$ for $x \in [0,U]$ be given. The fraction of adopters $N(t)$ is:

$$N(t) = 1 - \int_0^{\bar{x}(t)} m(x,t)dx$$

Mean Field Game (MFG): Given parameters $c, \rho, \nu, \theta_0, \theta_n, U$ a function $f(x)$, and initial condition $m_0(x)$, a MFG equilibrium is given by:

1. A path of the fraction of adopters $N : [0,\infty) \to [0,1]$
2. A path of the optimal threshold $\bar{x} : [0,\infty) \to (0,U)$
3. A value for adoption $a : [0,U] \times [0,\infty) \to [0,(\theta_0+\theta_n)/\rho]$
4. A value function $v : [0,U] \times [0,\infty) \to [0,(\theta_0+\theta_n)U/\rho]$
5. A density of non-adopters $m : [0,\bar{x}(t)] \to [0,\infty)$ for each $t \geq 0$

where

[3]Equivalently, the problem can be cast as a stopping time problem:

$$v(x,t) = \max_{\tau \geq t} \mathbb{E}\left[e^{-\rho(\tau - t)}\left(a\left(x\left(\tau\right),\tau\right) - c\right) \mid x(t) = x\right]$$

where τ denotes the time of the adoption and depends only on the information generated by the process for x's and on calendar time.

1. the function a solves its p.d.e. and boundary given the path N
2. the function v and path $\bar{x}$ solves its p.d.e. and boundary given a
3. the function m solves its p.d.e. and boundary given path $\bar{x}$
4. the path N is consistent with the measure m at all times.

3 The Deterministic Case ($\sigma = 0$)

We begin by studying the deterministic case where $\sigma = 0$, so that the agent's valuation x does not change. This case is useful to relate to the existing literature studying technology diffusion (e.g. Stokey, 2020, Buera, et al. 2021, Crouzet, et al. 2020), and it unveils the basic forces at work in the adoption problem. We will show that under a natural initial condition, equivalent to assuming that at time zero no agent with low valuation has adopted the technology (while high valuation agents may have done so), the equilibrium of the deterministic problem has no dynamics. This implies that adoption occurs instantaneously and that the fraction of adopters is a constant $N(t) = N$. Interestingly, the stochastic version of the model will instead feature dynamics with a gradual adoption of the technology and a time evolution for the variable $N(t)$.

For each x, let $a(x,t) = x\,\alpha(t)$ where $\rho\alpha(t) = \theta_0 + \theta_n N(t) + \alpha_t(t)$. We fix an x and a path $\alpha(t)$ for $t \geq 0$. Let $t^*(x)$ be the optimal t that solves the adoption problem:

$$t^*(x) = \arg\max_{t\geq 0} G(t,x) \text{ with } G(x,t) \equiv e^{-\rho t}(\alpha(t)x - c)$$

The necessary first order conditions if $\alpha(t)$ is differentiable at $t = t^*(x) < \infty$ are:[4]

$$-\rho(\alpha(t^*)x - c) + x\alpha_t(t^*) = 0 \text{ if } t^*(x) > 0$$
$$-\rho(\alpha(0)x - c) + x\alpha_t(0) \leq 0 \text{ if } t^*(x) = 0$$

Furthermore, since not adopting is feasible (i.e. $t = \infty$) and yields a zero payoff, then

$$\alpha(t^*(x))x \geq c \text{ for all } x.$$

[4] If $\alpha(t)$ is not differentiable at $0 < t = t^*(x) < \infty$:

$$-\rho(\alpha(t^*)x - c) + x\alpha_t^+(t^*) \leq 0 \leq -\rho(\alpha(t^*)x - c) + x\alpha_t^-(t^*)$$

where $\alpha_t^-(t^*)$ and $\alpha_t^+(t^*)$ are the left and right derivatives of $\alpha(t)$ at $t = t^*(x)$. Note that α is differentiable at t provided that $N(t)$ does not jump at t. If $N(t)$ jumps at t, then α has right and left derivatives.

Given $t^*(x)$ we can define $\bar{x}(t)$ as the smallest value of x that makes $t = t^*(x)$ optimal for any $t \geq 0$.[5] We will look for an equilibrium where at any $t \geq 0$ someone adopts, so

$$\rho(\alpha(t)\bar{x}(t) - c) = \bar{x}(t)\alpha_t(t)$$

provided α is differentiable at t.[6] The following two lemmas are useful to characterize the solution for the deterministic case. The proof of both lemmas can be found in Appendix A.

LEMMA 1. Assume that for $t > 0$ there is some x for which $0 < t^*(x) < \infty$, we denote the smallest of such x as $\bar{x}(t)$. Then $\bar{x}(t)$ is weakly decreasing in time, and $\alpha(t)$ is weakly increasing in time.

The fact that the threshold $\bar{x}(t)$ is weakly decreasing rules out a solution where the number of adopters is decreasing through time.

LEMMA 2. Consider the initial condition $m_0(x) = f(x)$ holding for all $x < \bar{x}(0)$ where the latter the adoption threshold at time zero. If $\bar{x}(t)$ is weakly decreasing in t then $m(x,t) = f(x)$ and $N(t) = 1 - F(\bar{x}(t))$.

The second lemma states that if we start the economy with a threshold $\bar{x}(0)$ and no agent below that threshold has adopted, then all agents with $x > \bar{x}(0)$ will immediately adopt and the distribution of the non adopters is time invariant. This result is intuitive and it is at the heart of the lack of dynamics in the equilibrium of the deterministic problem. It rules out "patological" initial conditions in which some agents below the threshold $\bar{x}(0)$ have (for some reason) adopted the technology.

Assuming Lemma 2 holds and combining the first order conditions with the law of motion for $\alpha(t)$, $\rho\alpha(t) - \alpha_t(t) = \theta_0 + \theta_n(1 - F(\bar{x}(t)))$, we get

$$\bar{x}(t)\left[\theta_0 + \theta_n(1 - F(\bar{x}(t)))\right] - \rho c = 0 \tag{3}$$

Note that in equation (3) the solution for $\bar{x}(t)$ does not depend on t. Thus, we can construct equilibrium where $\bar{x}(0) = \bar{x}(t)$ for all $t \geq 0$. We summarize this result in the following proposition

PROPOSITION 1. Consider the initial condition $m_0(x) = f(x)$ for all $x < \bar{x}(0)$. Then the solution implies a time invariant threshold $\bar{x}(t) = \bar{x}$ solving equation (3), immediate adoption

[5] Since in equilibrium $0 \leq N(t) \leq 1$, then $\frac{\theta_0}{\rho} \leq \alpha(t) \leq \frac{\theta_0+\theta_n}{\rho}$ and $0 < \frac{\rho c}{\theta_0+\theta_n} \leq \bar{x}(t) \leq \frac{\rho c}{\theta_0}$. Note that we allow $\bar{x}(t) > U$, but in this case everybody is adopting at t. Thus, we assume that $\frac{(\theta_0+\theta_n)}{\rho} > \frac{c}{U}$, otherwise nobody can ever adopt, and $c > 0$, so that some type will never adopt.

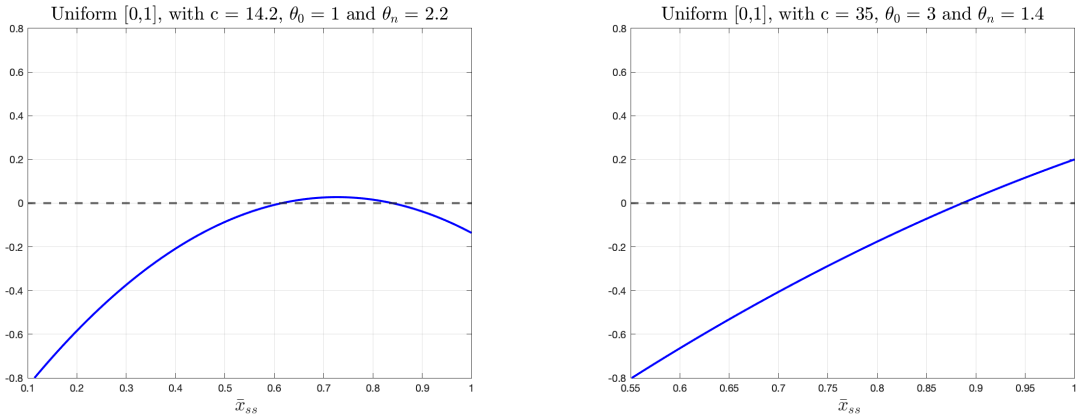
[6] If α is not differentiable at t, then: $\bar{x}(t)\alpha_t^+(t) \leq \rho(\alpha(t)\bar{x}(t) - c) \leq \bar{x}(t)\alpha_t^-(t)$.

for all agents with $x \geq \bar{x}$, and a time-invariant fraction of adopters $N(t) = N = 1 - F(\bar{x})$.

Note that equation (3) may have multiple solutions. Given that from Lemma 1 we know that $\bar{x}(t)$ is weakly decreasing and $\alpha(t)$ must be strictly increasing in time, then the lower root is the stable solution in the sense that, if the economy is at that point it will remain there forever. We show below that other paths are also possible in the presence of multiple solutions, with the fraction of adopters $N(t)$ ratcheting up at discrete moments in time.

For illustration purposes, we consider two cases. Recall that the unconditional distribution is uniform so $F(\bar{x}) = \bar{x}/U$. Consider first the case where $\rho\frac{c}{U} > \theta_0$. We note that if θ_n is large enough, there are at least two steady states. Next, consider the case where $\rho\frac{c}{U} < \theta_0$. In this case there is always a steady state, and in general an odd number of steady states. Both cases are shown in Figure 1, which shows the solutions of equation (3). In Panel (a) the solution with low $\bar{x}_{ss}$ is the stable solution, since $\bar{x}(t)$ is weakly decreasing from Lemma 1, and the one with higher adoption, since $N(t) = 1 - F(\bar{x}(t))$. Panel (b) shows that with low strategic complementarities (i.e. low θ_n), there is only one steady state.

Figure 1: Steady State Solution



(a) $\theta_0 < \rho\frac{c}{U}$ and θ_n large

(b) $\rho\frac{c}{U} < \theta_0$

Let us consider the case with two possible stationary equilibria, denoted by $\bar{x}_H > \bar{x}_L$, with associated adoption rates $N_H < N_L$, and the stationary value function $\rho\alpha(t) = \theta_0 + \theta_n N(t)$ (recall $\alpha_t = 0$) with solution $\bar{\alpha}_H = (\theta_0 + \theta_n N_H)/\rho$ and $\bar{\alpha}_L = (\theta_0 + \theta_n N_L)/\rho$, where $\alpha_H < \alpha_L$ since a low threshold yields higher utility due to the larger adoption rate.

We can now check that indeed $t^*(x)$ are optimal for a steady state equilibrium. Since $\alpha(t) = \bar{\alpha}_i$ for $i = L, H$, then the adoption problem becomes:

$$t^*(x) = \arg\max_{t \geq 0} e^{-\rho t}\left(\bar{\alpha}_i x - c\right)$$

and the solution is:

$$t^*(x) = \begin{cases} \infty & \text{if } x < \frac{c}{\bar{\alpha}_i} \\ 0 & \text{if } x \geq \frac{c}{\bar{\alpha}_i} \end{cases}$$

Hence, there are no dynamics in the deterministic case. Nonetheless, an equilibrium can be constructed where $\bar{x}(t)$ is piecewise continuous and jumps from $\bar{x}_H$ to $\bar{x}_L$ at some arbitrary time T and where the value of $N(t)$ also jumps. For instance, let $\bar{x}(t) = \bar{x}_H$ for $t \in [0, T)$ and let $\bar{x}(t) = \bar{x}_L$ for $t \in [T, \infty)$, where $T > 0$ is arbitrary. For $t \geq T$, set $\alpha(t) = \bar{\alpha}_L$ and for $t \in [0, T)$, solve $\alpha_t(t) = \rho(\alpha(t) - \alpha_H)$ with boundary condition $\alpha(T) = \bar{\alpha}_L$. This gives

$$\alpha(t) = \bar{\alpha}_H + (\bar{\alpha}_L - \bar{\alpha}_H)\ e^{-\rho(T-t)}$$

Note that $\alpha'(t) > 0$ for $t \in [0, T)$ and $\alpha(0) > \bar{\alpha}_H$. The equilibrium so constructed satisfies the first and second order condition for $t^*(x)$. The following proposition describes such "ratcheting" equilibria:

PROPOSITION 2. Let $\bar{X}$ be the set of steady state equilibria, i.e.

$$\bar{X} \equiv \{0 < \bar{z}_i \leq U : \ \rho c = \bar{z}_i \left[\theta_0 + \theta_n \left(1 - F(\bar{z}_i)\right)\right] \}$$

An equilibrium is described by a path $\bar{x}(t)$ that at times $0 = t_0 \leq t_1 < t_2 < t_m < \infty$:

$$\bar{x}(t) = \bar{z}_i \in \bar{X} \text{ for } t_{i-1} \leq t < t_i \text{ for all } i = 1, 2, \ldots, m$$

and where $\bar{z}_i > \bar{z}_{i+1}$ for all $i = 1, 2, \ldots, m$.

In words, an equilibrium is given by a piece-wise constant path for $\bar{x}(t)$, such that at each discontinuity point $\bar{x}(t)$ jumps down to a value that is one of the steady-state solutions of equation (3). The set of equilibria thus includes the fully static one where $\bar{x}(0) = \bar{z}_m$, as well as several other time-varying paths where the elements of $\bar{X}$ (the steady state solutions) and the time sequence t_i are arbitrarily selected subject to the the constraint that the path for $\bar{x}(t)$ must be weakly decreasing.

3.1 Planning Problem

In this section we solve the planning problem for the deterministic case. We set $\bar{x}(t) = \bar{x}$ so that adoption occurs for $x \geq \bar{x}$. The planner discounts the benefit of using the technology at different times using the rate r. We assume that at time $t = 0$ nobody as adopted the

technology yet, and that $m(x,0) = f(x)$. The planning problem solves $\max_{\bar{x}} \mathcal{L}(\bar{x})$ where

$$\mathcal{L}(\bar{x}) \equiv \frac{1}{r}\int_{\bar{x}}^{U} xf(x)\left[\theta_0 + \theta_n(1-F(\bar{x}))\right]dx - c(1-F(\bar{x})) - \frac{\nu c}{r}(1-F(\bar{x})) \tag{4}$$

The planner chooses the optimal adoption threshold $\bar{x}$ to maximize the present discounted value $\mathcal{L}(\bar{x})$. The term $\int_{\bar{x}}^{U} xf(x)\left[\theta_0 + \theta_n(1-F(\bar{x}))\right]dx$ denotes the flow benefit accruing from those that have adopted. The term $c(1-F(\bar{x}))$ is the cost born at $t=0$ for the mass of agents who adopt the technology. The term $c\nu(1-F(\bar{x}))$ is the flow cost or re-adopting the technology for agents with $x > \bar{x}$ who die and are replaced by new agents. Using $\rho \equiv r + \nu$ as above we can write $r\mathcal{L}(\bar{x}) = \int_{\bar{x}}^{U} xf(x)\left[\theta_0 + \theta_n(1-F(\bar{x}))\right]dx - \rho c(1-F(\bar{x}))$. Taking the first order condition with respect to $\bar{x}$ we get:

$$\underbrace{\bar{x}\left[\theta_0 + \theta_n(1-F(\bar{x}))\right] + \theta_n \overbrace{\int_{\bar{x}}^{U} xf(x)dx}^{\mathcal{E}(\bar{x})}}_{Q(\bar{x})} - \rho c = 0 \tag{5}$$

Although the solution is similar to the one of the decentralized problem in equation (3), the planner takes into account the benefits of inframarginal adopters, represented by the term $\mathcal{E}(\bar{x})$, which does not appear in equation (3). In the case of a uniform distribution $f(x)$, there are two roots, which are more polarized than in the decentralized steady state solution.[7] Furthermore, it can be shown that the global maximum is at the lower root of $\bar{x}$ (i.e. $\frac{\partial^2 r\mathcal{L}}{\partial \bar{x}^2} < 0$).

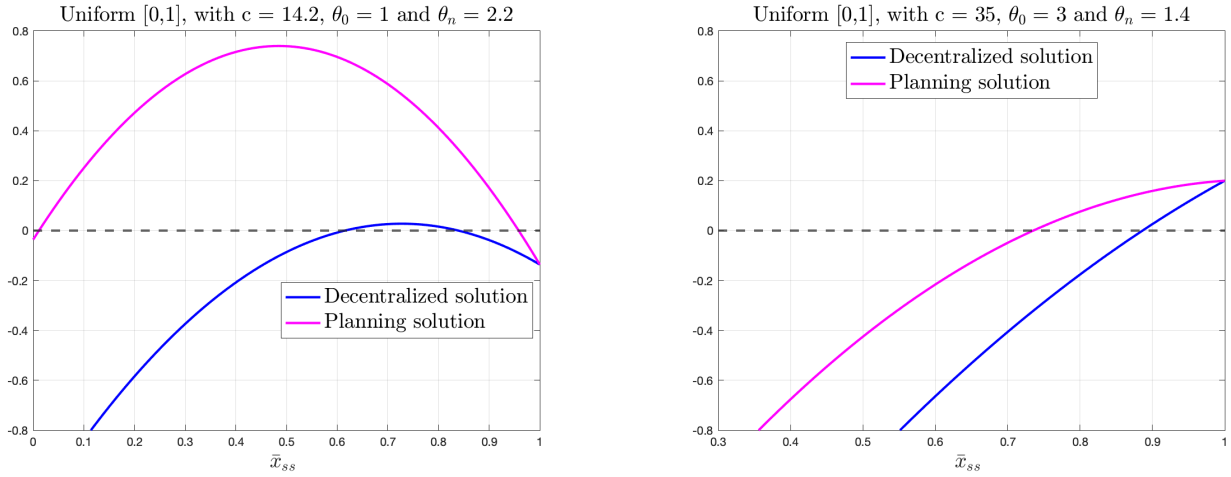
An intuitive way of understanding this result is the following. Let $\bar{\theta}_{ss} \equiv \frac{1}{\rho}(\theta_0 + \theta_n(1 - F(\bar{x}_{ss})))$. Then, the lower root of $\bar{x}$ can be written as

$$\bar{x}_{ss} = \frac{c}{\bar{\theta}_{ss}} - \frac{\theta_n \mathcal{E}_{ss}}{\rho \bar{\theta}_{ss}}$$

where the first term is exactly the solution of the decentralized problem and the second term accounts for the externality as the planner accounts for the inframarginal adopters when $\theta_n > 0$. As a result, the lower root of the planning problem is more extreme than in the decentralized problem, as depicted in Figure 2.

[7] Note that $Q'(0) = \theta_0 + \theta_n > 0$ and $Q''(x) < 0$, then, assuming $Q(0) = \frac{\theta_n U}{2} < c\rho$ and $Q(U) = U\theta_0 < c\rho$ the two roots are in $(0, U)$. The condition on $Q(0)$ is equivalent to a positive discriminant. Note also that $\mathcal{E}(x) \geq 0$ and $\frac{\partial^2 \mathcal{E}(\bar{x})}{\partial \bar{x}^2} < 0$ so that the zeros of $Q(\bar{x})$ are more extreme than the ones for the steady state equilibria.

Figure 2: Steady State Solution: Planning Problem



(a) Two steady states

(b) One steady State

Furthermore, let s be the optimal subsidy (i.e. the reduction in the cost of adopting that equalizes the smallest root of the decentralized problem with that of the planner). It can be shown that the optimal subsidy increases with the strength of the strategic complementarities (i.e. $\frac{\partial s}{\partial \theta_n} > 0$).

4 Stochastic Model ($\sigma > 0$)

4.1 Planning Problem

The planner solves the following problem:

$$\max_{\{\bar{x}(t)\}} \Bigg\{ \int_0^\infty e^{-rt} \int_0^U \underbrace{(f(x) - m(x,t))}_{\text{Density of adopters}} \underbrace{x\,(\theta_0 + \theta_n N(t))}_{\text{Flow benefit}} dx\, dt - \underbrace{\int_0^\infty e^{-rt} c\,(N_t(t) + \nu N(t))\, dt}_{\text{Flow of adoption cost: gross new adoptions}} \Bigg\}$$

subject to

$$N(t) = 1 - \int_0^{\bar{x}(t)} m(z,t)dz \quad \text{for all } t$$

$$m_t(x,t) = -\nu\left(m(x,t) - f(x)\right) + \frac{\sigma^2}{2} m_{xx}(x,t) \quad \text{for } x \in (0, \bar{x}(t)) \text{ and all } t \geq 0 \quad \text{KFE}$$

$$m(x,t) = 0 \quad \text{for } x \in [\bar{x}(t), U] \text{ and all } t \geq 0 \quad \text{Adoption}$$

$$m_x(0,t) = 0 \quad \text{for all } t \geq 0 \quad \text{Reflecting}$$

The first constraint is the consistency condition. The second is the KFE of the density of non-adopters, we denote the lagrange multiplier of this constraint as $\lambda(x,t)$. As before, the density of non-adopters is zero to the right of $\bar{x}(t)$, there is an exit point at $\bar{x}(t)$, and there is a boundary conditions from reflection at zero. To derive the p.d.e's for non-adopters, we first adopt the planning problem to discrete-time discrete-state using a finite-difference approximation. We obtain the first order conditions for a problem in finite dimensions and take limits to find the corresponding p.d.e's. We provide details of this derivation in Appendix B. The p.d.e's corresponding to the planning problem are summarized in the following proposition.

PROPOSITION 3. A planner problem is given by $\{\bar{x}(t), \lambda(x,t), m(x,t)\}$ the path of optimal threshold so that adoption occurs for $x \geq \bar{x}(t)$, the Lagrange multiplier λ, and the density of non-adopters m, respectively, such that the p.d.e. for non-adopters:

$$\begin{aligned}
\rho\lambda(x,t) &= x\big(\theta_0 + \theta_n[1 - \int_0^{\bar{x}(t)} m(z,t)dz]\big) + \theta_n\big(\tfrac{U}{2} - \int_0^{\bar{x}(t)} m(z,t)z\,dz\big) \\
&\quad + \tfrac{\sigma^2}{2}\lambda_{xx}(x,t) + \lambda_t(x,t) \text{ for } x \leq \bar{x}(t) \text{ and } t \geq 0 \\
\lambda(x,t) &= c \text{ for } x \geq \bar{x}(t) \text{ and } t \geq 0 \\
\lambda_x(\bar{x}(t),t) &= 0 \text{ for } t \geq 0 \\
\lambda_x(0,t) &= 0 \text{ for } t \geq 0
\end{aligned}$$

subject to

$$\begin{aligned}
m_t(x,t) &= \nu\big(1/U - m(x,t)\big) + \tfrac{\sigma^2}{2} m_{xx}(x,t) \text{ for } x < \bar{x}(t) \text{ and } t \geq 0 \\
m(x,t) &= 0 \text{ for } x \geq \bar{x}(t) \text{ and } t \geq 0 \\
m_x(0,t) &= 0 \text{ for } t \geq 0
\end{aligned}$$

4.2 Steady State: Decentralized Problem

A steady state is given by two constant path N_{ss} and $\bar{x}_{ss}$ that solve the time invariant version of the partial differential equations stated in Section 2.1. Given N_{ss} we obtain $a(x,t) = \tilde{a}(x)$, $v(x,t) = \tilde{v}(x)$ and $m(x,t) = \tilde{m}(x)$ solve the o.d.e.'s and boundary conditions: Given N_{ss}, we find $\tilde{a}, \tilde{b}$ and $\bar{x}_{ss}$:

$$\begin{aligned}
\rho\tilde{a}(x) &= x(\theta_0 + \theta_n N_{ss}) + \frac{\sigma^2}{2}\tilde{a}_{xx}(x) \text{ if } x \in [0, U] && \text{Value of Adoption} \\
\tilde{a}_x(0) &= \tilde{a}_x(U) = 0 && \text{Reflecting} \\
\rho\tilde{v}(x) &= \frac{\sigma^2}{2}\tilde{v}_{xx}(x) \text{ if } x \leq \bar{x}_{ss} && \text{Adoption Decision} \\
v(\bar{x}_{ss}) &= \tilde{a}(\bar{x}_{ss}) - c && \text{Value Matching} \\
v_x(\bar{x}_{ss}) &= \tilde{a}_x(\bar{x}_{ss})\,,\ v_x(0) = 0 && \text{Smooth Pasting}
\end{aligned}$$

Given $\bar{x}_{ss}$ solve for $\tilde{m}$

$$\begin{aligned}
0 &= -\nu\tilde{m}(x) + \nu f(x) + \frac{\sigma^2}{2}\tilde{m}_{xx}(x) && \text{KFE if } x \leq \bar{x}_{ss} \\
\tilde{m}(\bar{x}_{ss}) &= 0 \text{ and } \tilde{m}_x(0) = 0 && \text{Exit and Reflecting}
\end{aligned}$$

and given $\tilde{m}(x)$ and $\bar{x}_{ss}$, we define the fixed point

$$N_{ss} = 1 - \int_0^{\bar{x}_{ss}} \tilde{m}(s)ds$$

We begin by solving $\tilde{a}(x)$, $\tilde{v}(x)$ and $\bar{x}$ given a value for N. The boundary conditions for $\tilde{a}(x)$ come from our assumption of reflecting barriers. The boundary conditions for $\tilde{v}(x)$ are the standard value matching, and smooth pasting. The details of the solution can be found in Appendix C.1. Combining the solutions for $\tilde{a}(x)$ and $\tilde{v}(x)$ we can solve for $\bar{x}^o(N)$, a function that gives the *optimal* threshold as a function of the given value N. The monotonicity properties of the function $\tilde{a}$ and $\tilde{v}$ on the parameters N, θ_n, c and θ_0 give the following characterization of the threshold $\bar{x}^0$:

LEMMA 3. The function $\bar{x}^o$ is decreasing in N, strictly so at the points where $0 < \bar{x} < U$. Fixing a value of N, the function $\bar{x}^o$ is strictly increasing in c, strictly so at the points where $0 < \bar{x} < U$. Fixing a value of N, the function $\bar{x}^o$ is strictly increasing decreasing in θ_0 and θ_n at the points where $0 < \bar{x} < U$. Moreover we have the following expansion: $\bar{x}^o(N) = \frac{\rho c}{\theta_0 + \theta_n N} + \frac{\sigma}{\sqrt{2\rho}} + o(\sigma)$.

Since the function $\bar{x}^o(N)$ is decreasing in N, it has an inverse, which we denote by $N^o(\bar{x})$, and it is given by:

$$N^o(\bar{x}) = \frac{1}{\theta_n}\left[\frac{\rho c}{\left(\bar{x} + \bar{A}_1 e^{\eta \bar{x}_{ss}} + \bar{A}_2 e^{-\eta \bar{x}}\right) - \frac{\left(1+\eta\left(\bar{A}_1 e^{\eta\bar{x}} - \bar{A}_2 e^{-\eta\bar{x}}\right)\right)\left(e^{\eta\bar{x}} + e^{-\eta\bar{x}}\right)}{\eta\left(e^{\eta\bar{x}} - e^{-\eta\bar{x}}\right)}} - \theta_0\right] \text{ where}$$

$$\bar{A}_1 \equiv \frac{1}{\eta}\frac{\left(1 - e^{-\eta U}\right)}{\left(e^{-\eta U} - e^{\eta U}\right)}, \; \bar{A}_2 \equiv \frac{1}{\eta}\frac{\left(1 - e^{\eta U}\right)}{\left(e^{-\eta U} - e^{\eta U}\right)} \text{ and } \eta \equiv \sqrt{2\rho/\sigma^2} \tag{6}$$

Next we can solve the Kolgomorov forward equations for $\tilde{m}(x)$, given a barrier $\bar{x}$ subject to an exit point and to the conditions coming from the reflecting barriers. We denote the corresponding value of the number that have adopted as $N^a(\bar{x})$, where the superscript a stands for *aggregation*. The details of the solutions can be found in Appendix C.2. Solving this equation we obtain

$$N^a(\bar{x}) = 1 - \frac{\bar{x}}{U} + \frac{\left(e^{-\gamma\bar{x}} - e^{\gamma\bar{x}}\right)}{\gamma U\left(e^{\gamma\bar{x}} + e^{-\gamma\bar{x}}\right)} \equiv 1 - \frac{\bar{x}}{U} - \frac{\tanh\left(\gamma\bar{x}\right)}{U\gamma} \text{ where } \gamma \equiv \sqrt{2\nu/\sigma^2} \tag{7}$$

As it is intuitive, the value of $N^a(\bar{x})$ is *decreasing* in the level of the barrier $\bar{x}$. The next lemma, obtained by analyzing equation (7) gives a characterization of N^a.

LEMMA 4. Fix $\gamma > 0$, then, $N^a(\bar{x})$ is strictly decreasing in $\bar{x}$. Fixing $\bar{x} > 0$, then N^a is strictly increasing in γ, and hence strictly decreasing in σ. Moreover, we have the expansion: $N^a(\bar{x}) = 1 - \frac{\bar{x}}{U} - \frac{\sigma}{U\sqrt{2\nu}} + o(\sigma)$.

Thus, together equation (6) and equation (7) determine $\bar{x}_{ss}$ and N_{ss}. In particular, a steady state, described by the pair $\bar{x}_{ss}, N_{ss}$, which solve

$$N_{ss} \equiv N^a(\bar{x}_{ss}) = N^o(\bar{x}_{ss})$$

The next proposition summarizes the behaviour of the steady states for small values of σ.

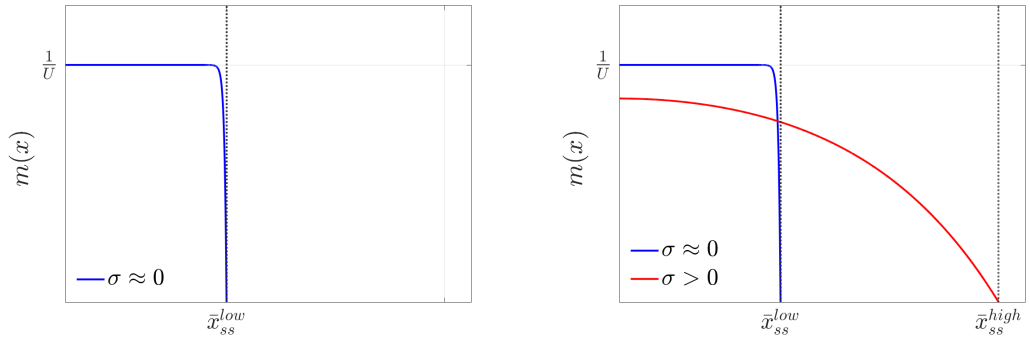
PROPOSITION 4. Assume that $\nu > 0$. Assume that the parameters θ_0, θ_n, c and ρ are such that there are two steady states in the deterministic case of $\sigma = 0$, and label them as $\bar{x}^1_{ss} < \bar{x}^2_{ss}$. Then, (i) there exists a $\bar{\sigma} > 0$ such that for all $\sigma \in (0, \bar{\sigma})$ there are two steady states $\bar{x}^1_{ss} < \bar{x}^2_{ss}$. (ii) Each steady state is continuous with respect to σ at $\sigma = 0$. (iii) The steady states of the model with idiosyncratic shocks are less extremes, i.e. $\frac{\partial \bar{x}^2_{ss}}{\partial \sigma} > 0 > \frac{\partial \bar{x}^1_{ss}}{\partial \sigma}$.

(iv) The sign of the comparative static differ across steady states, with

$$\frac{\partial \bar{x}^2_{ss}}{\partial c} > 0 > \frac{\partial \bar{x}^1_{ss}}{\partial c} \text{ and } \frac{\partial \bar{x}^1_{ss}}{\partial \theta_0} > 0 > \frac{\partial \bar{x}^2_{ss}}{\partial \theta_0}$$

Importantly, the last term of equation (7) shows that there is a smaller density of non-adopters for $x \in [0, \bar{x}_{ss}]$ in the stochastic case relative to the deterministic case. This observations is key to understand why the stochastic case has dynamics. Panel (a) of Figure 3 shows the density of non-adopters around $\bar{x}$ for $\sigma \approx 0$. In this case, the density of adopters is very close to zero so that the distribution looks almost uniform as in the deterministic case. Panel (b) shows the same figure for $\sigma > 0$. First, $\bar{x}_{ss}$ is larger due to the option value that is present in the stochastic model. More importantly, for $\sigma > 0$, the density of adopters below $\bar{x}_{ss}$ is non-zero, as a result the density of non-adopters is far from being uniform.[8] Given that the density takes time to adjust, the stochastic model features the presence of dynamics in the adoption of new technology as the optimal value of $\bar{x}$ is not independent of time.

Figure 3: Stochastic Steady State: Density of non-adopters: $m(x)$



(a) $\sigma \approx 0$ (b) $\sigma > 0$

Note: The figure plots the distribution of non-adopters in the stochastic steady state. Panel (a) shows an example with σ close to zero (i.e. $\sigma = 0.00003$) , as in the deterministic case. Panel (b) shows an example where σ is larger than zero (i.e $.\sigma = 0.002$). The parameters used in the example are $r = 0.03$, $\nu = 0.05$, $U = 0.05$, $c = 0.1$, $\theta_0 = 0.01$, and $\theta_n = 1.2$.

[8] More details of this argument can be found in Appendix E.

4.3 Steady State: Planning Problem

A steady state is given by two constant path N_{ss} and $\bar{x}_{ss}$ that solve the time invariant version of the p.d.e. stated in Section 4.1. The p.d.e. for non-adopters in steady state is

$$\begin{aligned}
\rho\tilde{\lambda}(x) &= x\big(\theta_0 + \theta_n N_{ss}\big) + \theta_n \mathcal{E}_{ss} + \tfrac{\sigma^2}{2}\tilde{\lambda}_{xx}(x) \text{ if } x \leq \bar{x}_{ss} && \text{KBE}\\
\tilde{\lambda}(\bar{x}_{ss}) &= c && \text{FOC}\\
\tilde{\lambda}_x(\bar{x}_{ss}) &= 0 && \text{Smooth Pasting}\\
\tilde{\lambda}_x(0) &= 0 && \text{Reflecting}\\
0 &= -\nu\tilde{m}(x) + \nu f(x) + \frac{\sigma^2}{2}\tilde{m}_{xx}(xx) \text{ if } x \leq \bar{x}_{ss} && \text{KFE}\\
\tilde{m}(\bar{x}_{ss}) &= 0 \text{ and } \tilde{m}_x(0) = 0
\end{aligned}$$

and given $\tilde{m}$ and $\bar{x}_{ss}$, N_{ss} and $\mathcal{E}_{ss}$ are defined as:

$$\begin{aligned}
N_{ss} &= 1 - \int_0^{\bar{x}_{ss}} \tilde{m}(x)dx\\
\mathcal{E}_{ss} &= U/2 - \int_0^{\bar{x}_{ss}} x\tilde{m}(x)dx
\end{aligned}$$

Recall that $\tilde{\lambda}(\bar{x}_{ss})$ is the Lagrange multiplier of the law of motion of the density of agents that have not adopted in steady state. The details of the solution can be found in Appendix D.1. Let $\tilde{\theta}_{ss} \equiv \frac{1}{\rho}(\theta_0 + \theta_n N_{ss})$. The following proposition summarizes the solution of stochastic steady state of the planning problem.

PROPOSITION 5. For fixed $0 < \eta < \infty$ and small c, $\bar{x}_{ss} = 2\left(\frac{c}{\tilde{\theta}_{ss}} - \frac{\theta_n \mathcal{E}_{ss}}{\rho\tilde{\theta}_{ss}}\right)$. For the case when σ is small (i.e. η is large), $\bar{x}_{ss} = \frac{c}{\tilde{\theta}} - \frac{\theta_n \mathcal{E}_{ss}}{\rho\tilde{\theta}_{ss}} + \frac{\sigma}{\sqrt{2\rho}}$

Proposition 5 indicates that the solution of the stochastic version of the planning problem also has the option value present in the decentralized version. In addition, the roots of the planning problem are more extreme due to the externality that the planner is solving. As before the optimal subsidy that equalizes the smaller root of decentralize problem with that of the planner increases in θ_n.

5 A Small Shock

5.1 The Decentralized Problem

We study the evolution of the MFG where the initial condition is given by small perturbation ϵ on the steady state distribution: $m_0(x) = \tilde{m}(x) + \epsilon\omega(x)$. A useful preliminary state is to define $D(x,t) \equiv a(x,t) - v(x,t)$, which satisfies:

$$\rho D(x,t) = x(\theta_0 + \theta_n N(t)) + \frac{\sigma^2}{2} D_{xx}(x,t) + D_t(x,t) \tag{8}$$

for $x \in [0, \bar{x}(t)]$ and $t \in [0,T]$. The boundary conditions are:

$$\begin{aligned} D(\bar{x}(t),t) &= c \\ D_x(\bar{x}(t),t) &= 0 \\ D_x(0,t) &= 0 \end{aligned}$$

for $t \in [0,T]$ and terminal condition $D(x,T) = \tilde{a}(x) - \tilde{v}(x)$ for $x \in [0, \bar{x}(T)]$.

We then consider an equilibrium with $\{\bar{x}(t,\epsilon), N(t,\epsilon), D(x,t,\epsilon), m(x,t,\epsilon)\}$. We will linearize this equilibrium with respect to ϵ and evaluate it at $\epsilon = 0$. For all $t \in [0,T]$, we denote these derivatives as follows:

$$\begin{aligned} p(x,t) &\equiv \frac{\partial}{\partial \epsilon} m(x,t,\epsilon)\Big|_{\epsilon=0} \\ d(x,t) &\equiv \frac{\partial}{\partial \epsilon} D(x,t,\epsilon)\Big|_{\epsilon=0} \\ n(t) &\equiv \frac{\partial}{\partial \epsilon} N(t,\epsilon)\Big|_{\epsilon=0} \\ \bar{y}(t) &\equiv \frac{\partial}{\partial \epsilon} \bar{x}(t,\epsilon)\Big|_{\epsilon=0} \end{aligned}$$

5.1.1 Linearization and Solution of the KB Equation

We differentiate $D(x,t,\epsilon)$ with respect to ϵ at each (x,t) to obtain $d(x,t)$ which solves the following p.d.e

$$\rho d(x,t) = x\theta_n n(t) + \frac{\sigma^2}{2} d_{xx}(x,t) + d_t(x,t) \tag{9}$$

for $x \in [0, \bar{x}_{ss}]$ and $t \in [0, T]$. The boundary conditions are:

$$\begin{aligned} d_x(0,t) &= 0 \\ d(\bar{x}_{ss}, t) &= 0 \\ \tilde{D}_{xx}(\bar{x}_{ss})\bar{y}(t) + d_x(\bar{x}_{ss}, t) &= 0 \end{aligned} \tag{10}$$

for $t \in [0, T]$ and $d(x, T) = 0$ for $x \in [0, \bar{x}_{ss}]$. Note that equation (10) defines $\bar{y}(t)$ and that $\tilde{D}_{xx}(\bar{x}_{ss}) = \tilde{a}_{xx}(\bar{x}_{ss}) - \tilde{v}_{xx}(\bar{x}_{ss}) < 0$.

Taking the derivative of the solution for $d(x, t)$ with respect to x and combining it with equation (10) we find

$$\bar{y}(t) = \frac{\theta_n}{\tilde{D}_{xx}(\bar{x}_{ss})} \int_t^T G(\tau - t) n(\tau) d\tau \tag{11}$$

where $G(s) \equiv \sum_{j=0}^{\infty} c_j e^{-\psi_j s}$ for $s \geq 0$, $\psi_j \equiv \rho + \frac{\sigma^2}{2}\left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2$, and $c_j = 2\left(1 - \frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right)$. We provide details of the solution of the p.d.e. for d in Appendix F.

5.1.2 Linearization and Solution of the KF Equation

We differentiate the KFE for $m(x, t, \epsilon)$ with respect to ϵ at each (x, t) to obtain:

$$p_t(x,t) = \frac{\sigma^2}{2} p_{xx}(x,t) - \nu p(x,t) \tag{12}$$

for $x \in [0, \bar{x}_s s]$ and $t \in [0, T]$.

Differentiating the boundary conditions $m(\bar{x}(t, \epsilon), t, \epsilon) = 0$ and $m_x(0, t, \epsilon) = 0$ with respect to ϵ we get

$$\begin{aligned} \tilde{m}_x(\bar{x})\bar{y}(t) + p(\bar{x}, t) &= 0 \\ p_x(0,t) &= 0 \end{aligned} \tag{13}$$

The initial condition comes from differentiating $m_0(x)$ with respect to ϵ

$$p(0, x) = \omega(x) \tag{14}$$

The solution for p satisfies the p.d.e given in equation (12), its boundary conditions in equation (13), and the initial condition in equation (14). Details of the derivation can be

found in Appendix G. Given $p(x,t)$ we can compute $n(t)$ as:

$$\begin{aligned} n(t) &= -\int_0^{\bar{x}_{ss}} p(x,t)dx \\ &= n_0(t) + \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2}{\bar{x}_{ss}} \int_0^t J(t-\tau)\bar{y}(\tau)d\tau \end{aligned} \tag{15}$$

where $J(s) = \sum_{j=0}^{\infty} e^{-\mu_j s}$ with $\mu_j = \nu + \frac{1}{2}\sigma^2 \left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2$ and $n_0(t) \equiv -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+j)} \frac{\langle \varphi_j, \epsilon \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$.

5.1.3 Deriving the fixed point

In this section, we put together the solutions for the KBE and KFE. We combine equations equation (11) and equation (15) to arrive to a single linear equation that $n(t)$ must solve. The equilibrium of a MFG must solve

$$n(t) = n_0(t) + \Theta(\bar{x}_{ss}) \int_0^T K(t,s)n(s)ds \tag{16}$$

where $n_0(t)$ is given and is the solution of the MFG when there are no strategic complementarities, $\Theta(\bar{x}_{ss}) \equiv \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2\theta_n}{\bar{x}_{ss}\tilde{D}_{xx}(\bar{x}_{ss})} > 0$ captures the degree of strategic complementarities, and the kernel K is

$$K(t,s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j e^{-\mu_i t - \psi_j s} \left[\frac{e^{(\mu_i+\psi_j)\min\{t,s\}} - 1}{\mu_i + \psi_j}\right] \tag{17}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j \left(\frac{e^{-\mu_i t - \psi_j s + (\mu_i+\psi_j)\min\{t,s\}} - e^{-\mu_i t - \psi_j s}}{\mu_i + \psi_j}\right) \tag{18}$$

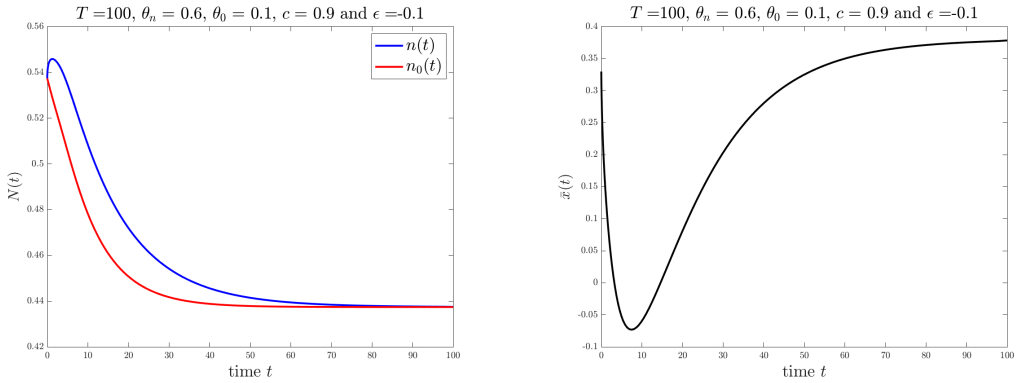
The kernel K is independent of $\Theta(\bar{x}_{ss})$ as well as the initial perturbation so that the effect of strategic complementarities on the equilibrium path of n depends on $\Theta(\bar{x}_{ss}$ only as a scalar multiplying the kernel K. More details of the equilibrium for the Perturbed MFG can be found in Appendix H. Furthermore, we derive the Lipschitz bound $\text{Lip}_K \equiv \sup_{t\in[0,T]} \int_0^T |K(t,s)|ds \leq \left(\frac{\bar{x}_{ss}^2}{\sigma^2}\right)^2$, which we use to find values of $\Theta(\bar{x}_{ss})$ for which the right hand side of equation (16) is a contraction in the case where T is unbounded.

5.1.4 Perturbation

As initial condition we consider a perturbation to the steady state density of non-adopters. In particular, $m_0(x) = \tilde{m}(x) + \epsilon\omega(x)$ where $\omega(x) = \delta_{\hat{x}}(x)$ and $\delta_{\hat{x}}(x)$ is a Dirac delta function at $\hat{x}$. For $\epsilon < 0$ this is equivalent to increasing the mass of adopters at $\hat{x}$.

In Panel (a) of Figure 4 we display the time path of the equilibrium $n(t)$ based on the linear approximation. The figure considers the case of $\epsilon = -0.1$ and $\hat{x} = 0.01 \times \bar{x}_{ss}$, which consist of increasing the mass of adopters with low values of x. The red line shows the path for $n_0(t)$, which is the path of adopters using steady state decision rule. The blue line is the equilibrium path of $n(t)$. The presence of strategic complementarities lowers $\bar{x}(t)$, as shown in Panel (b) of Figure 4, and makes the impulse response larger. Importantly, $n_0(t)$ decreasing in $\hat{x}$ by monotonicity of Brownian (Feller Property), so that the propagation of the impulse is larger if the perturbation is farther away from the threshold $\bar{x}_{ss}$.

Figure 4: Perturbation: $m_0(x) = \tilde{m}(x) + \epsilon\delta_{\hat{x}}(x)$



(a) $N(t)$ (b) $\bar{x}(t)$

5.2 The Planning Problem

We then consider an equilibrium of the planning problem with $\{\bar{x}(t,\delta), N(t,\delta), \lambda(x,t,\delta), m(x,t,\delta)\}$. We again linearize this equilibrium with respect to δ and evaluate it at $\delta = 0$. We differentiate $\lambda(x,t,\delta)$ with respect to δ at each (x,t) to obtain $\ell(x,t)$ which solves the following p.d.e

$$\rho\ell(x,t) = x\theta_n n(t) + \theta_n \mathbf{e}(t) + \frac{\sigma^2}{2}\ell_{xx}(x,t) + \ell_t(x,t) \tag{19}$$

for $x \in [0, \bar{x}_{ss}]$ and $t \in [0,T]$ and where $\mathbf{e}(t) \equiv \frac{\partial}{\partial\delta}\mathcal{E}(t,\delta)\Big|_{\delta=0}$. The boundary conditions are:

$$\begin{aligned} \ell(x,T) &= 0 \\ \ell_x(0,t) &= 0 \\ \ell(\bar{x}_{ss},t) &= 0 \\ \tilde{\lambda}_{xx}(\bar{x}_{ss})\bar{y}(t) + \ell_x(\bar{x}_{ss},t) &= 0 \end{aligned} \tag{20}$$

5.2.1 Deriving the fixed point

The solution for ℓ satisfies the p.d.e given in equation (19), its boundary conditions and its condition. Details of the derivation can be found in Appendix I. Given $l(x,t)$ we can compute $\bar{y}(t)$ using equation (20) as follows. From equation (20), we know that

$$\bar{y}(t) = \frac{-\ell_x(\bar{x}_{ss},t)}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \tag{21}$$

$$= \int_t^T G_{yn}(\tau - t)n(\tau)d\tau + \int_t^T G_{ye}(\tau - t)\mathbf{e}(\tau)d\tau \tag{22}$$

The expression for $n(t)$ is given by equation (15). We obtain a similar expression for $\mathbf{e}(t)$ using the solution for $p(x,t)$ as

$$\mathbf{e}(t) = -\int_0^{\bar{x}_{ss}} xp(x,t)dx \tag{23}$$

$$= \mathbf{e}_0(t) + \int_0^t H_{ey}(t-s)\bar{y}(s)ds \tag{24}$$

where $\mathbf{e}_0(t) \equiv -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}^2(\pi j + \frac{1}{2} - \cos(j\pi))}{\pi(\frac{1}{2}+j)} \frac{\langle \varphi_j, \epsilon \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$ and $H_{ey}(q) = \tilde{m}_x(\bar{x}_{ss})\sigma^2 \sum_{j=0}^{\infty} \eta_j e^{-\mu_j(q)}$. Using equation (15) and equation (23), in Appendix J we show that equation (21) can be written as

$$\bar{y}(t) = \bar{y}_0(t) + \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \tilde{K}(t,s)\bar{y}(s)ds$$

where is given in equation (30), $\tilde{\Theta}(\bar{x}_{ss}) \equiv \frac{\theta_n \tilde{m}_x(\bar{x}_{ss})\sigma^2}{\tilde{\lambda}_{xx}(\bar{x}_{ss})\bar{x}_{ss}}$, and $\tilde{K}$ is

$$\tilde{K}(t,s) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(c_j + c_i)e^{\psi_j t + \mu_i s}\left(\frac{e^{-(\psi_j+\mu_i)\max\{t,s\}} - e^{-(\psi_j+\mu_i)T}}{\psi_j + \mu_i}\right)$$

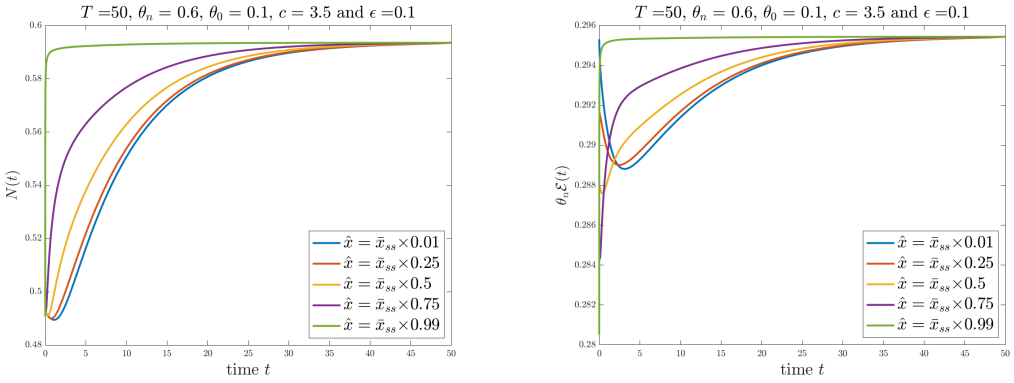
5.2.2 Perturbation

We again consider a perturbation to the steady state density of non-adopters so that $m_0(x) = \tilde{m}(x) + \epsilon\delta_{\hat{x}}(x)$ for several values of $\hat{x}$. In this case, we consider the case of $\epsilon > 0$, so that for different values of $\hat{x}$ the shock resembles a starting equilibrium with lower adoption.

In Panel (a) of Figure 5 we display the time path of the equilibrium $n(t)$ based on the linear approximation. The figure considers the case of $\epsilon = 0.1$ for several values of $\hat{x}$. The lines are the equilibrium path of $n(t)$, which are monotone in $\hat{x}$ due to the Feller property of the Brownian motion. Importantly, Panel (b) shows the equilibrium path of the optimal

subsidy $\theta\mathcal{E}(t)$ also depends on $\hat{x}$ and its equilibrium path is not monotone.

Figure 5: Planning Problem Perturbation: $m_0(x) = \tilde{m}(x) + \epsilon\delta_{\hat{x}}(x)$



(a) $N(t)$ (b) $\bar{x}(t)$

6 Application: Background on SINPE

On May 2015, the Central Bank of Costa Rica launched SINPE Móvil (hereafter, SINPE), a digital platform that allows users to make money transfers between each other using their mobile phones.[9] To use SINPE, users must have a bank account at a financial entity, and link the account to their mobile.

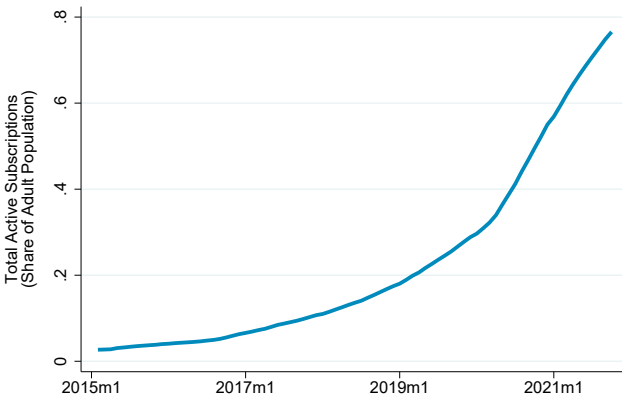
According to the Central Bank of Costa Rica, SINPE's main goal was becoming a mass-market payment mechanism that could reduce demand for cash as a method of payment. As such, SINPE was designed to be used for relatively small transfers, which are free and not subject to any fee as long as they do not exceed a daily amount. The maximum daily amount transferred without a fee varies by bank; for most users, it is approximately $310, although some banks have lower limits of $233 and $155.[10]

Adoption has increased dramatically over time, as shown in Figure 6. By 2021, close to 79% of the adult population in the country owned a bank account, and over 75% of adults had adopted SINPE. Thus, this setting has the unique feature of allowing us to study the adoption of mobile payments in the entire population of the country, and across many years since the inception of the technology until it reached almost all the adult population of the country. These features will also allow us to study the general equilibrium effects of adoption across a long time period.

[9] SINPE is an acronym for the initials of "National Electronic Payment System", in Spanish.

[10] Respectively, these limits in dollars correspond with approximately 200,000; 150,000; and 100,000 Costa Rican colones.

Figure 6: SINPE Users as a Share of Adult Population



Notes: The figure shows total SINPE subscriptions by individuals as a share of the total adult population in Costa Rica. We include only active subscriptions, as users have the option of deactivating their account.

7 Data

SINPE Transactions Our data on SINPE usage is comprehensive: For each user in the country, we have official records on *when* they adopted the technology (exact date), and we have individual records on each transaction made using accounts across different banks. In particular, for each transaction, the data records the *amount transacted* along with the individual identifier of *the sender and receiver* of the money. We also know the sender's and receiver's bank. Importantly, this information is available, not only for individual people, but also for firms.

Family Networks and Demographics Data on nation-wide family networks is available from Costa Rica's National Register. In particular, this allows us to know, for each citizen, if he or she is married, to whom, and who their children are. Thus, we know each person's first-degree relatives. The data includes individual identifiers that can be linked to SINPE. The data is dynamic, meaning that we can see how family networks are changing over time between 2015 and 2021. The same data source provides details on demographics.

Networks of Coworkers, Income, and Occupation Matched employer-employee data was obtained from the Registry of Economic Variables of the Central Bank of Costa Rica, which tracks the universe of formal employment and labor earnings. This data includes *monthly* details on each employee, including her occupation, earnings, and employment his-

tory between 2006 and 2021.[11] Using this data, we can identify which people are working at the same firm in a given month to construct networks of coworkers that can be matched to SINPE records. Networks change across time as people change their employers.

Networks of Neighbors and Residential Location We construct networks of neighbors for all adult citizens in the country leveraging data from the National Registry and the Supreme Court of Elections. Every two years since 2014, we have official records on the residence of each citizen, along with his or her identifier. While the records do include the district of residence (488 in total across the country), they also include the voting center which is closest to the citizen's house (2,059 centers in total), so we use the latter to get a more precise notion of a person's neighborhood. There are approximately 700 adults assigned to each voting center, on average.

Firm-Level Data We leverage data on corporate income tax returns from the Ministry of Finance, which cover the universe of formal firms in the country and contain typical balance sheet variables, including sales, input costs, and net assets. The data spans 2005 to 2021, includes details on each firm's sector and location, and can be linked to SINPE.

7.1 Anatomy of SINPE Transactions

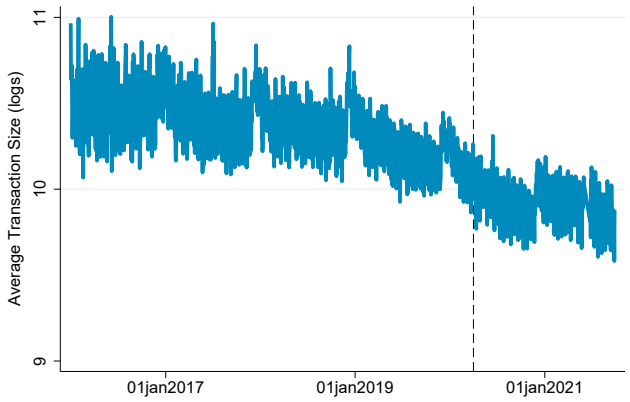
The average size of transactions in SINPE is about $50, and has slowly decreased across time as shown is Figure 7. Most transactions in SINPE occur between individuals in their 30s (see Figure 8). In fact, Figure 7 shows that individual-to-individual transactions account for over 90% of all transactions, and that is true across time. While firms can adopt SINPE, they represent a small fraction of all transactions. We will then look at adoption focusing on individual-to-individual transactions only, through the lens of our model.

Moreover, users tend to have relatively few peers with whom they transact. Figure 8 is the distribution of connections for users across time, and shows that the median user had 12 distinct peers in 2021, and that this number was less than half two years before. The distributions of distinct receivers per sender and senders per receiver, in Figures K2 and K1 respectively, exhibit similar patters.

Relevant Networks for SINPE Transactions Who are users transacting with via SINPE? To answer this question, we consider several types of networks, and examine their importance within SINPE. First, we construct the network of neighbors for each user and

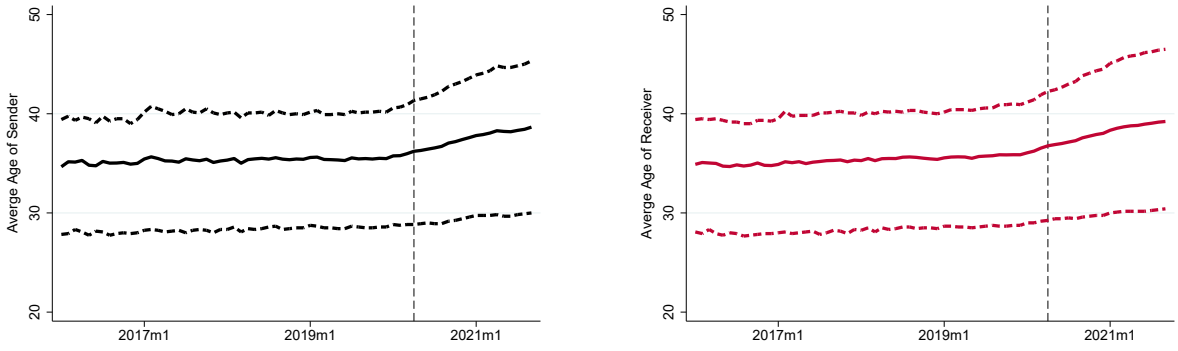
[11] It is worth noting that informal workers are a relatively small share of all workers in Costa Rica (27.4%), which is significantly below the Latin American average of 53.1% (ILO, 2018).

Figure 7: Average Transaction Size



Notes: The figure shows distinct number of peers that transacted with each user (either as senders or receivers). Year 2021 only includes information up to June.

Figure 8: Average Age of Senders and Receivers

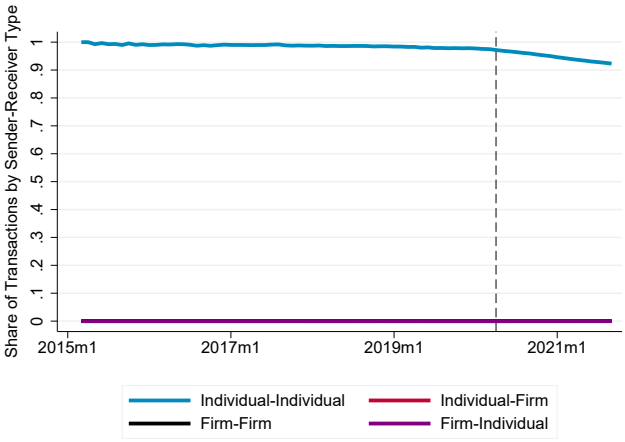


(a) Senders

(b) Receivers

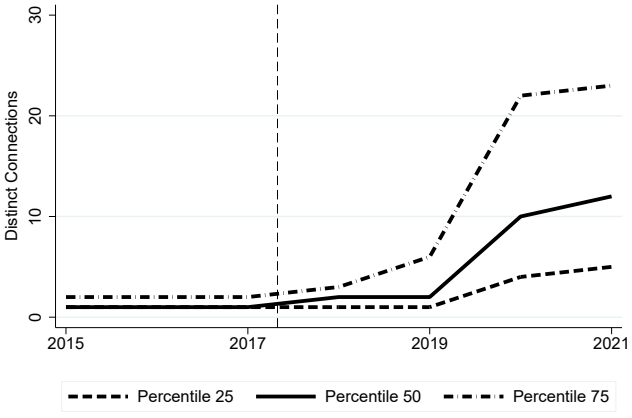
Notes: The figure shows the average age of senders and receivers across time.

Figure 7: Share of Transactions Between Types of Users Agents and Firms (Share)



Notes: The figure shows total number of SINPE transactions between types of users, as a share of all transactions.

Figure 8: Distribution of Connections with Distinct Peers (Yearly)

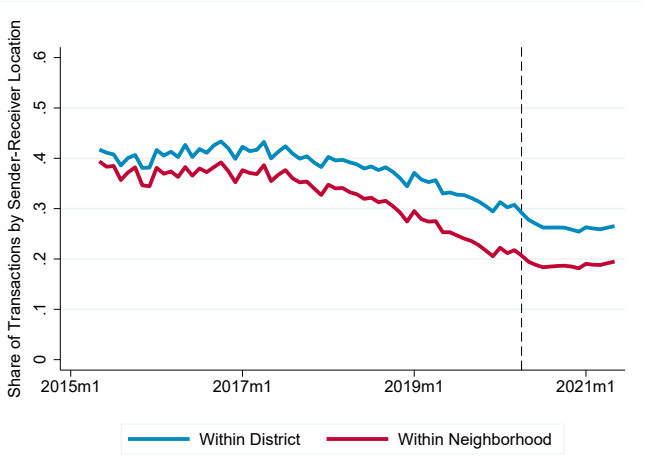


Notes: The figure shows distinct number of peers that transacted with each user (either as senders or receivers). Year 2021 only includes information up to June.

calculate the number of SINPE transactions that have occur with another user that also resides in the same neighborhood. The results are presented in Figure K3. We document that between 20% and 40% of all transactions happen between neighbors. This number increases to up to 52% if the transactions are weighted by the amount being transacted (see Figure K3).[12]

Second, we construct the network of coworkers for each employed use. We find that between 40% and 57% of all SINPE transactions are conducted with someone working in the same firm as the user, and in a similar broadly defined occupation. Results are reported in Figure 10. The share of transactions sent or received from a coworker, weighted by amount transacted, is slightly higher, as shown in Figure K4.

Figure 9: Number of Transactions Between Users Living in the Same Location (Share)



Notes: The figure shows total number of SINPE transactions between users living in the, as a share of all transactions.

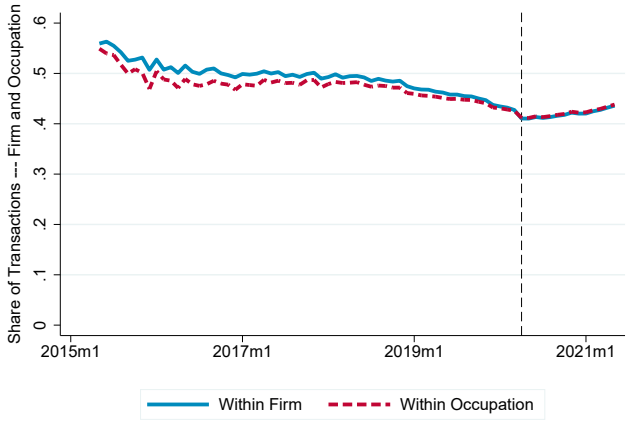
8 Empirical Strategy

8.1 Interpretation of the Model

We bring the model to the data by interpreting the flow benefit of agents that adopt the technology as being proportional to how intensively they use SINPE. Specifically, suppose SINPE users choose the intensity in which they use the application, ξ, maximizing the fol-

[12]We verify this number with data on location coming from a different source—namely, the employer-employee data instead of the National Registrar—and find very similar results, which are shown in Figures K5 and K6.

Figure 10: Number of Transactions Between Users Working in the Same Firm (Share)



Notes: The figure shows total number of SINPE transactions between users working at the same firm in 2018, as a share of all transactions.

lowing expression:

$$\max_{\xi} F(\xi; a_1) \equiv a_1 \xi - k \frac{\xi^{1+p}}{1+p}$$

where $p > 0$ and $k > 0$ are parameters and a_1 parameterizes the benefits from using SINPE. The first order condition describes the optimal intensity in which the technology is used:

$$\xi = a_1^{1/p} k^{-1/p}. \tag{25}$$

Using equation (25) we can let $F^{\star}(\xi; a_1)$ be equal to the flow benefit in the model, $x(\theta_0+\theta_n)N$, so that

$$a_1 = k^{\frac{1}{1+p}} \left(\frac{1+p}{p} \right)^{\frac{p}{p+1}} [(\theta_0 + \theta_n N)x]^{\frac{p}{p+1}} \tag{26}$$

Combining equation (25) and equation (26), and taking logs

$$\ln \xi_t = \alpha + \beta \ln [(\theta_0 + \theta_n N_t) x_t] \tag{27}$$

where α is a constant and $\beta \equiv \frac{1}{1+p} < 0$.[13] Under this interpretation of the model, the intensity in which the application is used, which is observable in the data (e.g. number of

[13] $\alpha \equiv \frac{1}{p} \ln k - \frac{1}{p(1+p)} \ln k + \frac{1}{1+p} \ln \left(\frac{1+p}{p} \right)$

transactions), is proportional to the flow benefit of adopting the application as described in the model.

We can separate the effect of N_t and x_t by taking differences and using that x_t follows a Brownian motion without drift (e.g. $\Delta \ln x_t \approx 0$) when is away from the boundaries

$$\Delta \ln \xi_t = \frac{1}{1+p} \Delta \ln(\theta_0 + \theta_n N_t)$$

which can be estimated using non-linear least squares. Alternatively, an approximation around $\theta_n = 0$ yields

$$\Delta \ln \xi_t \approx \gamma + \widetilde{\theta} \Delta N_t \tag{28}$$

where γ is a constant and $\widetilde{\theta} \equiv \frac{\beta \theta_n}{\theta_0}$. We can estimate equation (28) using a linear specification. The coefficient of interest is $\widetilde{\theta}$ since strategic complementarities in the adoption of the technology exist if $\widetilde{\theta} > 0$ (i.e. $\theta_0 > 0$ and $\theta_n > 0$).

8.2 Implementation Using Network Data

Leveraging data on networks, we can run the following specification from our derivation in equation 28:

$$\Delta \ln \xi_{it}^n = \gamma + \widetilde{\theta} \Delta N_t^n + \nu_{it}^n,$$

where $\ln \xi_{it}^n$, the intensity with which individual i uses the technology, can be interpreted as either the value or the number of SINPE transactions in a given time period. Tables 1 and 2 show, respectively, the results when considering n as a user's network of neighbors and a user's network of coworkers. Across specifications, we find that $\widetilde{\theta}$ remains positive and statistically significant.

Table 1: Intensity of Usage and Changes in N Defined as Network of Neighbors

Dependent variable: $\Delta \ln$ *Value of Transactions*

	All sample			Drop top/bottom 1%			Drop top/bottom 5%		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(6)	(7)
$\widetilde{\theta}$	3.392***	0.903***	0.810***	3.431***	1.020***	0.933***	2.995***	0.827***	0.790***
	(0.055)	(0.118)	(0.129)	(0.049)	(0.107)	(0.117)	(0.047)	(0.105)	(0.115)
Obs.	108,447	108,447	90,623	92,367	84,620	86,946	74,503	74,503	61,542
Time FE	No	Yes	No	No	Yes	No	No	Yes	No
COVID	No	No	Yes	No	No	Yes	No	No	Yes

Dependent variable: $\Delta \ln$ *Number of Transactions*

	All sample			Drop top/bottom 1%			Drop top/bottom 5%		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(6)	(7)
$\widetilde{\theta}$	3.392***	0.805***	0.749***	3.268***	0.679***	0.617***	3.088***	0.520***	0.482***
	(0.038)	(0.082)	(0.090)	(0.038)	(0.082)	(0.090)	(0.038)	(0.086)	(0.095)
Obs.	108,447	108,447	90,623	101,399	101,399	86,946	90,205	90,205	73,777
Time FE	No	Yes	No	No	Yes	No	No	Yes	No
COVID	No	No	Yes	No	No	Yes	No	No	Yes

Notes: The unit of observation is the individual. Time runs biyearly 2016 through 2021. Standard errors are in parentheses. COVID refers to a control using a COVID index equal to $\frac{1}{2}\ln(cases_k t) + \frac{1}{2}\ln(deaths_k t)$ for each district k at time t.

Table 2: Intensity of Usage and Changes in N Defined as Network of Coworkers

Dependent variable: $\Delta \ln$ *Value of Transactions*

	All sample			Drop top/bottom 1%			Drop top/bottom 5%		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(6)	(7)
$\widetilde{\theta}$	1.821***	0.079***	1.832***	1.792***	0.077***	1.801***	1.578***	0.062***	1.587***
	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.008)	(0.007)	(0.007)	(0.007)
Obs.	4,881,791	4,881,791	4,881,791	4,761,645	4,761,645	4,761,645	4,176,231	4,176,231	4,176,231
Time FE	No	Yes	No	No	Yes	No	No	Yes	No
Control wage	No	No	Yes	No	No	Yes	No	No	Yes

Dependent variable: $\Delta \ln$ *Number of Transactions*

	All sample			Drop top/bottom 1%			Drop top/bottom 5%		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(6)	(7)
$\widetilde{\theta}$	1.568***	0.045***	1.578***	1.563***	0.050***	1.572***	1.518***	0.061***	1.527***
	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)
Obs.	4,881,791	4,881,791	4,881,791	4,796,864	4,796,864	4,796,864	4,476,608	4,476,608	4,476,608
Time FE	No	Yes	No	No	Yes	No	No	Yes	No
Control wage	No	No	Yes	No	No	Yes	No	No	Yes

Notes: The unit of observation is the individual. Time runs monthly Jan 2015 through May 2021. Standard errors are in parentheses.

9 Conclusion

Understanding the process of adoption of a technology, and the transition from low to high adoption is challenging, especially in the presence of strategic complementarities. This paper develops a new dynamic model of technlogy adoption that allows us to model this transition, solve for the social planner's problem and provides a framework to characterize the process of learning and derive predictions that can be tested empirically.

While the model could be applied to studying any technology that features strategic complementarities, we use it to analyze electronic methods of payment. This application is particularly relevant today, given that payment methods are undertaking a digital transformation, that has been echoed by growing interest by monetary authorities to promote and develop digital payment platforms, both in developed and developing countries. Using SINPE, a National Electronic Payment System adopted by most of the adult population in Costa Rica, we estimate the importance of strategic complementarities across networks of users, and document that strategic complementarities play an important role in the adoption of this technology.

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APPENDIX

A Proofs

Proof. (of Lemma 1) The necessary second order condition for $0 < t^*(x) < \infty$ is:

$$G_{tt}(t,x)|_{t=t^*(x)} = e^{-\rho t^*}\left(-\rho\alpha_t(t^*)x + \alpha_{tt}(t^*)\right)$$

Differentiating with respect to time the definition of $\bar{x}(t)$ given above, which is the first order condition $\rho(\alpha(t)\bar{x}(t) - c) = \bar{x}(t)\alpha_t(t)$ we have:

$$\alpha_{tt}(t) = \rho\alpha_t(t) + \rho\frac{c}{(\bar{x}(t))^2}\bar{x}_t(t)$$

Evaluating the second order condition at $(t, \bar{x}(t))$ and replacing $\alpha_{tt}(t)$:

$$\begin{aligned} G_{tt}(t,\bar{x}(t)) &= e^{-\rho t}\left(-\rho\alpha_t(t)\bar{x}(t) + \bar{x}(t)\alpha_{tt}(t)\right) \\ &= e^{-\rho t}\left(-\rho\alpha_t(t)\bar{x}(t) + \bar{x}(t)\rho\alpha_t(t) + \rho\frac{c}{(\bar{x}(t))}\bar{x}_t(t)\right) = e^{-\rho t}\rho\frac{c}{(\bar{x}(t))}\bar{x}_t(t) \end{aligned}$$

Thus, if the necessary first order condition holds, i.e if $G_{tt}(t,\bar{x}(t)) \leq 0$, then $\bar{x}_t(t) \leq 0$ and hence it is weakly decreasing.

Furthermore, using the first order condition at $t > 0$

$$\rho(\alpha(t)\bar{x}(t) - c) = \alpha_t(t)\bar{x}(t)$$

Note that if $\alpha_t(t)$ is strictly decreasing then $\alpha(t)\bar{x}(t) - c < 0$, which is a contradiction with $\alpha(t)\bar{x}(t) - c \geq 0$. Thus, for a t where α is differentiable (no jump on N), then $\alpha(t)$ must be weakly increasing. □

Proof. (of Lemma 2). Recall from Lemma 1 that $\bar{x}(t)$ is weakly decreasing and $\alpha(t)$ weakly increasing in t. Hence if $x \leq \bar{x}(t)$ then $x \leq \bar{x}(s)$ for all $s \leq t$. Then equation (2), with $\sigma = 0$, gives

$$m_t(x,s) = \nu(f(x) - m(x,s)) \text{ for all } s \in [0,t]$$

solving the o.d.e. for each $x \leq \bar{x}(t)$:

$$m(x,t) = m_0(x)e^{-\nu t} + f(x)\left(1 - e^{-\nu t}\right) = f(x) + e^{-\nu t}\left(m_0(x) - f(x)\right)$$

We can write: $N(t) = 1 - \int_0^{\bar{x}(t)} m(z,t)dz$. It is immediate that if $m_0 = f$ then $N(t) = 1 - F(\bar{x}(t))$. Note that if $\bar{x}$ is decreasing then N is increasing in time. $\square$

Proof. (of Lemma 3) The monotonicity of $\bar{x}^o$ with respect to the parameters $\bar{\theta}_{ss} \equiv (\theta_0 + \theta_n N)/\rho$ is established in Appendix C.1. It is obtained by solving the o.d.e. for the value functions, and using the boundary conditions. It is clear that the optimal threshold, fixing η, solves an implicit equation $\psi(\gamma \bar{x}^o) = \eta c/\bar{\theta}_{ss}$, where the function ψ is explicitly derived in the appendix. This function is strictly increasing, and satisfies $\psi(0) = 0$. Thus $\bar{x}^o$ is strictly decreasing in $\bar{\theta}_{ss}$ and strictly increasing in c. A first order approximation of ψ gives the expansion used in the lemma.

□

Proof. (of Lemma 4)

That N^a is decreasing in $\bar{x}$ follows immediately since $\tanh(\cdot)$ is an increasing function. That N^a is strictly increasing in γ follows from differentiating $-\tanh(\bar{x}\gamma)/\gamma$ with respect to γ. This derivative is proportional to $\tanh(\bar{x}\gamma) - \bar{x}\gamma \text{sech}^2(\bar{x}\gamma) = \tanh(\bar{x}\gamma) - \bar{x}\gamma \tanh'(\bar{x}\gamma) > 0$, where we used that $\tanh(z)$ is strictly concave for $z > 0$.

□

Proof. (of Proposition 4). In the deterministic case, i.e. when $\sigma = 0$, there are at most two steady states. And by hypothesis, we have two steady states. In each of them the functions N^a and N^o have strictly different slopes, i.e. the derivative of $N^a - N^o$ with respect to $\bar{x}$ is positive for the intersection for small $\bar{x}$ and negative for the one with large $\bar{x}$. The functions N^a and N^o and they derivatives are continuously functions of $\bar{x}, \sigma, c, \theta_0$. Thus, for all value of $\sigma \in (0, \bar{\sigma})$, the functions N^a, N^o intersect twice, and he derivative of $N^a - N^o$ with respect to $\bar{x}$ is positive for the intersection for small $\bar{x}$ and negative for the one with large $\bar{x}$.

Fix one steady state, and to simplify the notation omit the subindex ss so we write $\bar{x}(\sigma) = \bar{x}^j_{ss}(\sigma)$. Then we must have

$$N^a\left(\bar{x}(\sigma), \sigma\right) = N^o\left(\bar{x}(\sigma), \sigma\right)$$

Differentiating it with respect to σ we get:

$$\left[N^a_{\bar{x}}\left(\bar{x}(\sigma), \sigma\right) - N^o_{\bar{x}}\left(\bar{x}(\sigma), \sigma\right)\right] \frac{\partial \bar{x}(\sigma)}{\partial \sigma} = N^o_{\sigma}\left(\bar{x}(\sigma), \sigma\right) - N^a_{\sigma}\left(\bar{x}(\sigma), \sigma\right) > 0$$

where subindices of N^a and N^o denote partial derivatives with respect to $\bar{x}$ and σ, and where the inequality holds for $\sigma \in (0, \bar{\sigma})$ and small enough using the lemmas stated above. In this range

$$\frac{\partial \bar{x}(\sigma)}{\partial \sigma} = \frac{N^o_{\sigma}\left(\bar{x}(\sigma), \sigma\right) - N^a_{\sigma}\left(\bar{x}(\sigma), \sigma\right)}{N^a_{\bar{x}}\left(\bar{x}(\sigma), \sigma\right) - N^o_{\bar{x}}\left(\bar{x}(\sigma), \sigma\right)}$$

So the sign of $\frac{\partial \bar{x}(\sigma)}{\partial \sigma}$ depends on whether slope of N^a with respect to $\bar{x}$ is higher than the one

for N^o. This, in terms, depends on which steady state is being considered. For the smaller steady state $N^o(\bar{x}^1_{ss}(\sigma)) < N^o(\bar{x}^1_{ss}(\sigma))$, and the opposite holds for the largest one. This shows that the steady states become less extreme,

$$\frac{\partial \bar{x}^2_{ss}(0)}{\partial \sigma} > 0 > \frac{\partial \bar{x}^1_{ss}(0)}{\partial \sigma}$$

In each of the steady states we write

$$N^a\left(\bar{x}^j(c)\right) = N^o\left(\bar{x}^j(c), c\right)$$

where we note that c does not enter in N^a. Differentiating with respect to c, we have:

$$\left[N^a_{\bar{x}}\left(\bar{x}(c)\right) - N^o_{\bar{x}}\left(\bar{x}(c), c\right)\right] \frac{\partial \bar{x}(c)}{\partial c} = N^o_c\left(\bar{x}(c), c\right) > 0$$

and again using the properties of each steady state:

$$\frac{\partial \bar{x}^2_{ss}}{\partial c} > 0 > \frac{\partial \bar{x}^1_{ss}}{\partial c}$$

Following exactly the same steps we get:

$$\frac{\partial \bar{x}^1_{ss}}{\partial \theta_0} > 0 > \frac{\partial \bar{x}^2_{ss}}{\partial \theta_0}$$

□

B Derivation Planning Problem: Finite Differences

To derive the problem in continuous time, we write the adopting the model to discrete time-discrete state. We do so by using finite-difference approaximation and then we consider the planning problem in that set-up. We obtain the first order conditions for a problem in finite dimensions. Lastly, we take the limit to develop the corresponding p.d.e's.

First we derive the finite difference approximation for a Brownian motion reflected between two barriers. The time step Δ so that times are between $t = 0, \Delta, 2\Delta, \ldots$. The space step is Δ_x so that $x \in \{x_1, x_2, \ldots, x_I\}$, where $x_1 = 0, x_J = U$ and $x_{i+1} - x_i = \Delta_x$. The p.d.e. inside the barriers is

$$m_t(x,t) = -\nu m(x,t) + \nu f(x) + \frac{\sigma^2}{2} m_{xx}(x,t)$$

Its finite difference approximation is:

$$\frac{m_{i,t+\Delta} - m_{i,t}}{\Delta} = -\nu m_{i,t} + \nu f_i + \frac{\sigma^2}{2}\frac{(m_{i+1,t} - 2m_{i,t} + m_{i-1,t})}{(\Delta_x)^2}$$

for $i = 2, \ldots, I-1$. We can write the finite difference approximation as:

$$\begin{aligned} m_{i,t+\Delta} = m_{i,t}\left(1 - \nu\Delta - \sigma^2\frac{\Delta}{(\Delta_x)^2}\right) + f_i\nu\Delta \\ + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{i+1,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}\Delta m_{i-1,t} \end{aligned}$$

For the finite approximation, we have that since the law of motion must be local, and mean preserving:

$$\begin{aligned} m_{1,t+\Delta} = m_{1,t}\left(1 - \nu\Delta - \sigma^2\frac{\Delta}{(\Delta_x)^2}\right) + f_1\nu\Delta \\ + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{2,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{1,t} \\ m_{I,t+\Delta} = m_{I,t}\left(1 - \nu\Delta - \sigma^2\frac{\Delta}{(\Delta_x)^2}\right) + f_I\nu\Delta \\ + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{I-1,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{I,t} \end{aligned}$$

We can write the l.o.m. at the boundaries as:

$$\begin{aligned} m_{1,t+\Delta} = m_{1,t}\left(1 - \nu\Delta\right) + f_1\nu\Delta + \frac{\sigma^2}{2}\frac{\Delta}{\Delta_x}\frac{(m_{2,t} - m_{1,t})}{\Delta_x} \\ m_{I,t+\Delta} = m_{I,t}\left(1 - \nu\Delta\right) + f_I\nu\Delta + \frac{\sigma^2}{2}\frac{\Delta}{\Delta_x}\frac{(m_{I-1,t} - m_{I,t})}{\Delta_x} \end{aligned}$$

At the reflecting boundaries $x = 0$ and $x = U$, the boundary conditions is $m_x(x,t) = 0$. Note that as $\Delta_x \to 0$ we require that

$$\frac{(m_{I-1,t} - m_{I,t})}{\Delta_x} = \frac{(m_{2,t} - m_{1,t})}{\Delta_x} \to 0$$

Now we get back to the planning problem. We will have two measures, $\{m_{i,t}\}$ and $\{g_{i,t}\}$. $m_{i,t}$ is the measures of those that have not adopted and $g_{i,t}$ the measure of those that have adopted. Let $\alpha_{it} \geq 0$ be the measure of adopting at t with $x = x_i$ at t. Thus at time t, the measure $\alpha_{i,t}$ is transferred from measure $m_{i,t}$ to measure $g_{i,t}$ Note that $m_{i,t} + g_{i,t} = \frac{1}{I}$ since the sum of the two is the invariant distribution. The initial condition are $g_{i,0} = 0\ \forall i$ and

$m_{i,0} = \frac{1}{I}$ all non-adopters. The law of motion of the state is then:

$$\begin{aligned}
0 \leq m_{1,t+\Delta} &= m_{1,t}\left(1 - \nu\Delta - \sigma^2\frac{\Delta}{(\Delta_x)^2}\right) + f\nu\Delta \\
&\quad + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{2,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{1,t} - \alpha_{1,t} \\
0 \leq m_{i,t+\Delta} &= m_{i,t}\left(1 - \nu\Delta - \sigma^2\frac{\Delta}{(\Delta_x)^2}\right) + f\nu\Delta \\
&\quad + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{i+1,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}\Delta m_{i-1,t} - \alpha_{i,t} \text{ for } i = 2, \ldots, I-1 \\
0 \leq m_{I,t+\Delta} &= m_{I,t}\left(1 - \nu\Delta - \sigma^2\frac{\Delta}{(\Delta_x)^2}\right) + f\nu\Delta \\
&\quad + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{I-1,t} + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}m_{I,t} - \alpha_{I,t}
\end{aligned}$$

which can be written in vector notation as:

$$m_{t+1} = L\, m_t - \alpha_t \geq 0$$

where L is an $I \times I$ stochastic matrix which depends on I, ν, σ^2, Δ and Δ_x. We assume that $\Delta\left(\nu + (\sigma/\Delta_x)^2\right) < 1$ so that all implied probabilities are positive.

$$\max_{\{\alpha_t, m_{t+\Delta}\}_{t=0}^{\infty}} \sum_{\{t=0,\Delta,2\Delta,\ldots\}} \left(\frac{1}{1+\Delta r}\right)^t \left\{\mathcal{U}(m_t)\,\Delta - \sum_{i=1}^{I}\alpha_{it}c\right\}$$

where

$$\mathcal{U}(m_t) \equiv \sum_{i=1}^{I}\left(\frac{1}{I} - m_{it}\right)\left(\theta_0 + \theta_n\left[1 - \sum_{j=1}^{I} m_{j,t}\right]\right)x_i$$

subject to the law of motion:

$$m_{t+1} = L\, m_t - \alpha_t \text{ for all } t = 0, \Delta, 2\Delta, \ldots$$

and subject to non-negativity:

$$m_{j,t+1} \geq 0 \text{ and } \alpha_{j,t} \geq 0 \text{ for all } j = 1, \ldots, I, \text{ and for all } t = 0, \Delta, 2\Delta, \ldots$$

Let $\left(\frac{1}{1+\Delta r}\right)^t \lambda_{it}$ be Lagrange multiplier of the law of motion for m_{it}. Let L_i be the i^{th} row

vector of the matrix L . The Lagrangian $\mathcal{L}$ becomes:

$$\begin{aligned}\mathcal{L} = & \sum_{\{t=0,\Delta,\dots\}} \left(\frac{1}{1+\Delta r}\right)^t \left\{\mathcal{U}\left(m_t\right)\Delta - \sum_{i=1}^{I} \alpha_{it} c\right\} \\ & + \sum_{\{t=0,\Delta,\dots\}} \left(\frac{1}{1+\Delta r}\right)^t \left\{\sum_{i=1}^{I} \lambda_{it}\left(m_{i,t+\Delta} - L_i \cdot m_t + \alpha_{it}\right)\right\}\end{aligned}$$

Derivative of Lagrangian with respect to α_{it}:

$$\frac{\partial \mathcal{L}}{\partial \alpha_{jt}} = \left(\frac{1}{1+\Delta r}\right)^t \left[\lambda_{j,t} - c\right]$$

Derivative of Lagrangian with respect to m_{jt} for $2 \leq j \leq I-1$:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial m_{j,t}} = & \left(\frac{1}{1+\Delta r}\right)^t \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{j,t}}\Delta \\ & + \left(\frac{1}{1+\Delta r}\right)^t \left[\lambda_{j,t-\Delta}(1+\Delta r) - \lambda_{j,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right)\right] \\ & - \left(\frac{1}{1+\Delta r}\right)^t \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}\left[\lambda_{j+1,t} + \lambda_{j-1,t}\right]\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} = & - x_j\left(\theta_0 + \theta_n\left(1 - \sum_{i=1}^{I} m_{i,t}\right)\right) - \theta_n \sum_{i=1}^{I}\left(\frac{1}{I} - m_{it}\right) x_i \\ = & -x_j\left(\theta_0 + \theta_n N_t\right) - \theta_n\left(\frac{U}{2} - \sum_{i=1}^{I} m_{it} x_i\right)\end{aligned}$$

We can write m_{jt} for $2 \leq j \leq I-1$:

$$\begin{aligned}\left(\frac{1}{1+\Delta r}\right)^{-t} \frac{\partial \mathcal{L}}{\partial m_{jt}} = & \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}}\Delta + \lambda_{j,t-\Delta}(1+\Delta r) \\ & - \lambda_{j,t}\left(1 - \nu\Delta - \sigma^2 \frac{\Delta}{(\Delta_x)^2}\right) - \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}\left[\lambda_{j+1,t} + \lambda_{j-1,t}\right]\end{aligned}$$

and rearranging:

$$(1+\Delta r)\lambda_{j,t-\Delta} = \left(\frac{1}{1+\Delta r}\right)^{-t}\frac{\partial \mathcal{L}}{\partial m_{jt}} - \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}}\,\Delta + \lambda_{j,t}\left(1-\nu\Delta-\sigma^2\frac{\Delta}{(\Delta_x)^2}\right) + \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}\left[\lambda_{j+1,t}+\lambda_{j-1,t}\right]$$

dividing by Δ and further rearranging the expressions:

$$(r+\nu)\lambda_{j,t-\Delta} = \left(\frac{1}{1+\Delta r}\right)^{-t}\frac{1}{\Delta}\frac{\partial \mathcal{L}}{\partial m_{jt}} - \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} - \nu\left(\lambda_{j,t}-\lambda_{j,t-\Delta}\right) + \left(\frac{\lambda_{j,t}-\lambda_{j,t-\Delta}}{\Delta}\right) + \frac{\sigma^2}{2}\left(\frac{\lambda_{j+1,t}-2\lambda_{j,t}+\lambda_{j-1,t}}{(\Delta_x)^2}\right)$$

For the bottom boundary $j=1$ we have:

$$\left(\frac{1}{1+\Delta r}\right)^{-t}\frac{\partial \mathcal{L}}{\partial m_{1t}} = \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{1t}}\,\Delta + \lambda_{1,t-\Delta}(1+\Delta r) - \lambda_{1,t}\left(1-\nu\Delta-\sigma^2\frac{\Delta}{(\Delta_x)^2}\right) - \frac{\sigma^2}{2}\frac{\Delta}{(\Delta_x)^2}\left[\lambda_{1,t}+\lambda_{2,t}\right]$$

$$(r+\nu)\lambda_{1,t-\Delta} = \left(\frac{1}{1+\Delta r}\right)^{-t}\frac{1}{\Delta}\frac{\partial \mathcal{L}}{\partial m_{1t}} - \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{1t}} - \nu\left(\lambda_{1,t}-\lambda_{1,t-\Delta}\right) + \left(\frac{\lambda_{1,t}-\lambda_{1,t-\Delta}}{\Delta}\right) + \frac{\sigma^2}{2}\frac{1}{\Delta_x}\left(\frac{\lambda_{2,t}-\lambda_{1,t}}{\Delta_x}\right)$$

For the top boundary $j=I$:

$$(r+\nu)\lambda_{I,t-\Delta} = \left(\frac{1}{1+\Delta r}\right)^{-t}\frac{1}{\Delta}\frac{\partial \mathcal{L}}{\partial m_{It}} - \frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{It}} - \nu\left(\lambda_{I,t}-\lambda_{I,t-\Delta}\right) + \left(\frac{\lambda_{I,t}-\lambda_{I,t-\Delta}}{\Delta}\right) + \frac{\sigma^2}{2}\frac{1}{\Delta_x}\left(\frac{\lambda_{I-1,t}-\lambda_{I,t}}{\Delta_x}\right)$$

Thus the limit as $\Delta \downarrow 0$ and $\Delta_x \downarrow 0$ is that

$$\lambda_x(0,t) = \lambda_x(U,t) = 0$$

First order condition with respect to α_{it} for $t = 0, \Delta, \ldots$ and $j = 1, \ldots, I$::

$$\begin{aligned} \lambda_{j,t} - c \leq 0\,,\ \alpha_{jt} \geq 0 \text{ and} \\ \alpha_{j,t}\left[\lambda_{j,t} - c\right] = 0 \end{aligned}$$

First order condition with respect to m_{jt} for $t = \Delta, 2\Delta, \ldots$ and $j = 1, \ldots, I$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial m_{jt}} \leq 0\,,\ m_{jt} \geq 0 \text{ and} \\ m_{jt}\frac{\partial \mathcal{L}}{\partial m_{jt}} = 0 \end{aligned}$$

Note that as $\Delta \downarrow 0$ and $\Delta_x \downarrow 0$ and $x = x_j$ we have

$$\frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} \to x(\theta_0 + \theta_n N(t)) + \theta_n\left(\frac{U}{2} - \int_0^U m(z,t) z\, dz\right)$$

Consider a $x_j = x$ for $j = 2, \ldots, I-1$ or $0 < x < U$. Take the f.o.c. for $m_{j,t}$ derived above and assume that $\frac{\partial \mathcal{L}}{\partial m_{jt}} = 0$. Take the limit as $\Delta \downarrow 0$ and $\Delta_x \downarrow 0$:

$$\begin{aligned} (r+\nu)\lambda(x,t) = x(\theta_0 + \theta_n N(t)) + \theta_n\left(\frac{U}{2} - \int_0^U m(z,t) z\, dz\right) \\ + \lambda_t(x,t) + \frac{\sigma^2}{2}\lambda_{xx}(x,t) \end{aligned}$$

If instead $\frac{\partial \mathcal{L}}{\partial m_{jt}} \leq 0$, then

$$\begin{aligned} (r+\nu)\lambda(x,t) \leq x(\theta_0 + \theta_n N(t)) + \theta_n\left(\frac{U}{2} - \int_0^U m(z,t) z\, dz\right) \\ + \lambda_t(x,t) + \frac{\sigma^2}{2}\lambda_{xx}(x,t) \end{aligned}$$

We derive **smooth pasting** here. Suppose that at t we have $\lambda_{i,t} = c$ for all $i \geq j$, i.e. for all $x \geq \bar{x}(t)$, or $\lambda(x,t) < c$ for $x < \bar{x}(t)$ and $\lambda(x,t) = c$ for $x \geq \bar{x}(t)$. Assume also $m_{j,t} > 0$ and $m_{j-1,t} > 0$, so that $\partial \mathcal{L}/\partial m = 0$ for both. Then we can write the f. o.c. as:

$$\begin{aligned} (r+\nu)c = -\frac{\partial \mathcal{U}\left(m_t\right)}{\partial m_{jt}} - \nu\left(c - \lambda_{j,t-\Delta}\right) \\ + \left(\frac{c - \lambda_{j,t-\Delta}}{\Delta}\right) + \frac{\sigma^2}{2}\frac{1}{\Delta_x}\left(\frac{c - 2c + \lambda_{j-1,t}}{\Delta_x}\right) \end{aligned}$$

Taking the limit as $\Delta_x \downarrow 0$ we have: $\lambda_x(\bar{x}(t), t) = 0\,.$

In summary, a planner problem is given by $\{\bar{x}(t), \lambda(x,t), m(x,t)\}$ the path of optimal threshold so that adoption occurs for $x \geq \bar{x}(t)$, the Lagrange multiplier V, and the density of non-adopters m, respectively, such that the p.d.e. for the non-adopters is:

$$\begin{aligned} m_t(x,t) &= \nu\big(1/U - m(x,t)\big) + \tfrac{\sigma^2}{2} m_{xx}(x,t) \text{ for } x < \bar{x}(t) \text{ and } t \geq 0 \\ m(x,t) &= 0 \text{ for } x \geq \bar{x}(t) \text{ and } t \geq 0 \\ m_x(0,t) &= 0 \text{ for } t \geq 0 \end{aligned}$$

The p.d.e. for the non-adopters:

$$\begin{aligned} \rho\lambda(x,t) &= x\big(\theta_0 + \theta_n[1 - \int_0^{\bar{x}(t)} m(z,t)dz]\big) + \theta_n\big(\tfrac{U}{2} - \int_0^{\bar{x}(t)} m(z,t) z\, dz\big) \\ &\quad + \tfrac{\sigma^2}{2}\lambda_{xx}(x,t) + \lambda_t(x,t) \text{ for } x \leq \bar{x}(t) \text{ and } t \geq 0 \\ \lambda(x,t) &= c \text{ for } x \geq \bar{x}(t) \text{ and } t \geq 0 \\ \lambda_x(\bar{x}(t),t) &= 0 \text{ for } t \geq 0 \\ \lambda_x(0,t) &= 0 \text{ for } t \geq 0 \end{aligned}$$

The conditions for $\bar{x}$ are:

- We look for $\bar{x}(\cdot)$ to be continuous $t \geq 0$.

Conditions for m:

- We look for $m(\cdot, t)$ to be continuous for all $x \in [0, U]$ and $t \geq 0$.
- We look for $m(\cdot, t)$ to be C^2 for all $x \in [0, \bar{x}(t)]$, and $t \geq 0$.
- We look for $m(x, \cdot)$ to be C^1 for all $x \in [0, \bar{x}(t)]$, and $t \geq 0$.
- The initial boundary condition for m is $m(x, 0) = 0$ for all $x \in [0, U]$

Conditions for λ:

- We look for $\lambda(\cdot, t)$ to be C^1 for all $x \in [0, U]$.
- We look for $\lambda(\cdot, t)$ to be C^2 for all $x \in [0, \bar{x}(t)]$, and $t \geq 0$.
- We look for $\lambda(x, \cdot)$ to be C^1 for all $x \in [0, \bar{x}(t)]$, and $t \geq 0$.
- The final boundary for λ is $\lambda(x, T) = 0$ for all $x \in [0, U]$ (T may be $+\infty$).

C Solution Steady State Problem

C.1 Solution for $\tilde{a}(x)$ and $\tilde{v}(x)$

The solution to $\tilde{a}$ is of the form:

$$\tilde{a}(x) = x\frac{\theta_0 + \theta_n N_{ss}}{\rho} + A_1 e^{\eta x} + A_2 e^{-\eta x}$$

for $\eta = \sqrt{2\rho/\sigma^2}$, and

$$0 = \frac{\theta_0 + \theta_n N_{ss}}{\rho} + \eta(A_1 - A_2) = \frac{\theta_0 + \theta_n N_{ss}}{\rho} + \eta(A_1 e^{\eta U} - A_2 e^{-\eta U})$$

Thus, given $\theta_0 + \theta_n N_{ss}$, the constants (A_1, A_2) are the solution of two linear equations. Moreover, the values of A_1, A_2 are proportional to $\bar{\theta}_{ss}$ given by

$$\bar{\theta}_{ss} \equiv \frac{\theta_0 + \theta_n N_{ss}}{\rho} = \eta(A_2 - A_1) = \eta(A_2 e^{-\eta U} - A_1 e^{\eta U})$$

Let $\bar{A}_i \equiv A_i/\bar{\theta}_{ss}$, we can write:

$$1 = \eta(\bar{A}_2 - \bar{A}_1) = \eta(\bar{A}_2 e^{-\eta U} - \bar{A}_1 e^{\eta U})$$

which has solution:

$$\bar{A}_1 = \frac{1}{\eta}\frac{\left(1 - e^{-\eta U}\right)}{(e^{-\eta U} - e^{\eta U})}$$
$$\bar{A}_2 = \frac{1}{\eta}\frac{\left(1 - e^{\eta U}\right)}{(e^{-\eta U} - e^{\eta U})}$$

The solution for $\tilde{v}$ for $x \in [0, \bar{x}_{ss}]$ is of the form

$$\tilde{v}(x) = B_1 e^{\eta x} + B_2 e^{-\eta x}$$

Given the solution for $\tilde{a}$, then $B_1, B_2, \bar{x}_{ss}$ solve:

$$0 = \eta(B_1 - B_2)$$
$$\tilde{a}_x(\bar{x}_{ss}) = \eta(B_1 e^{\eta \bar{x}_{ss}} - B_2 e^{-\eta \bar{x}_{ss}})$$
$$\tilde{a}(\bar{x}_{ss}) - c = B_1 e^{\eta \bar{x}_{ss}} + B_2 e^{-\eta \bar{x}_{ss}}$$

Thus, using the first equation $B_1 = B_2 = B$ and taking the ratio of these equations:

$$\frac{\tilde{a}(\bar{x}_{ss}) - c}{\tilde{a}_x(\bar{x}_{ss})} = \frac{1}{\eta}\frac{e^{\eta\bar{x}_{ss}} + e^{-\eta\bar{x}_{ss}}}{(e^{\eta\bar{x}_{ss}} - e^{-\eta\bar{x}_{ss}})}$$

Replacing the expressions for $\tilde{a}(\bar{x}_{ss})$ and $\tilde{a}'(\bar{x}_{ss})$, we obtain:

$$\frac{\bar{x}_{ss} + \bar{A}_1 e^{\eta\bar{x}_{ss}} + \bar{A}_2 e^{-\eta\bar{x}_{ss}} - c/\bar{\theta}_{ss}}{1 + \eta\left(\bar{A}_1 e^{\eta\bar{x}_{ss}} - \bar{A}_2 e^{-\eta\bar{x}_{ss}}\right)} = \frac{1}{\eta}\frac{e^{\eta\bar{x}_{ss}} + e^{-\eta\bar{x}_{ss}}}{(e^{\eta\bar{x}_{ss}} - e^{-\eta\bar{x}_{ss}})}$$

Note that this is one equation for $\bar{x}_{ss}$ as a function of $\bar{\theta}_{ss}$ (recall that $\bar{A}_1, \bar{A}_2$ are known constants). The last expression can be written as

$$\eta\bar{x}_{ss} + \eta\bar{A}_1 e^{\eta\bar{x}_{ss}} + \eta\bar{A}_2 e^{-\eta\bar{x}_{ss}} - \frac{e^{\eta\bar{x}_{ss}} + e^{-\eta\bar{x}_{ss}}}{(e^{\eta\bar{x}_{ss}} - e^{-\eta\bar{x}_{ss}})}\left(1 + \eta\left(\bar{A}_1 e^{\eta\bar{x}_{ss}} - \bar{A}_2 e^{-\eta\bar{x}_{ss}}\right)\right) = \frac{\eta}{\bar{\theta}_{ss}}c$$

or letting $y \equiv \eta\bar{x}_{ss}$ and defining $\psi(y)$ we can write

$$\begin{aligned}\psi(y) &\equiv y + \eta\left(\bar{A}_1 e^{y} + \bar{A}_2 e^{-y}\right) - \frac{e^{y} + e^{-y}}{(e^{y} - e^{-y})}\left(1 + \eta\left(\bar{A}_1 e^{y} - \bar{A}_2 e^{-y}\right)\right) \\ &= \frac{\eta}{\bar{\theta}_{ss}}c\end{aligned}$$

We can approximate the left hand side around $\bar{x}_{ss} = 0$, which corresponds to $c = 0$. Using that $\eta\bar{A}_2 = \eta\bar{A}_1 + 1$, we have the following properties.

1. $\psi(0) = 0$, $\psi(y) > 0$ if $y > 0$
2. $\psi'(y) = \frac{e^{2y}+1}{(e^y+1)^2}$ so $\psi'(0) = \frac{1}{2}, \psi'(\infty) = 1$, and $\psi''(y) > 0$,
3. $\psi(y) = \frac{y}{2} + \frac{y^3}{24} + o(y^4)$ and $\lim_{y\to\infty}\frac{\psi(y)-y}{y} = 0$

Now we use ψ to solve for $\bar{x}_{ss} = \chi(\eta, c/\bar{\theta}_{ss})$ i.e. $\frac{\eta c}{\bar{\theta}_{ss}} = \psi(\eta\chi(\eta, c/\bar{\theta}_{ss}))$. $\bar{x}_{ss}$ is the unique solution of $\frac{\psi(\eta\bar{x}_{ss})}{\eta} = \frac{c}{\bar{\theta}_{ss}}$, which always exists. For fixed $0 < \eta < \infty$ and small c using the first order approximation:

$$y = \eta\bar{x}_{ss} = 2\frac{\eta c}{\bar{\theta}_{ss}} \text{ or } \bar{x}_{ss} = 2\frac{c}{\bar{\theta}_{ss}}$$

since $\eta = \frac{\sqrt{2\rho}}{\sigma}$ the option value for a fixed $\bar{\theta}$ is given by:

$$\lim_{c\to 0}\frac{\chi(\eta, c/\bar{\theta}_{ss})}{\chi(\infty, c/\bar{\theta}_{ss})} = 2$$

For fixed $0 < \eta < \infty$ and small c, using the third order approximation $y^3 + 12y = \kappa \equiv \frac{24\eta c}{\bar{\theta}_{ss}}$ or:

$$\begin{aligned}\bar{x}_{ss} &= \frac{1}{\eta}\left(\frac{1}{2}\kappa + \sqrt{\frac{1}{4}\kappa^2 + \frac{12^3}{27}}\right)^{1/3} + \frac{1}{\eta}\left(\frac{1}{2}\kappa - \sqrt{\frac{1}{4}\kappa^2 + \frac{12^3}{27}}\right)^{1/3} \\ &= \frac{1}{\eta}\left(\frac{1}{2}\right)^{1/3}\left[\left(\kappa + \sqrt{\kappa^2 + 16}\right)^{1/3} + \left(\kappa - \sqrt{\kappa^2 + 16}\right)^{1/3}\right]\end{aligned}$$

For the case when σ is small (i.e. η is large), let $S(y) \equiv y - \psi(y) + 1$ and recall that $\lim_{y\to\infty} S(y) = 0$. Then, using the definitions of y and $\psi(y)$, this implies

$$\lim_{\sigma\to 0} \frac{\sqrt{2\rho}}{\sigma}\left(\chi\left(\infty, \frac{c}{\bar{\theta}_{ss}}\right) - \frac{c}{\bar{\theta}_{ss}} - \frac{\sigma}{\sqrt{2\rho}}\right) = 0$$

Thus, for σ is small we can use:

$$\bar{x}_{ss} = \frac{c}{\bar{\theta}_{ss}} + \frac{\sigma}{\sqrt{2\rho}} + o(\sigma)$$

Alternatively, note that $\bar{x}_{ss} - \frac{\sigma}{\sqrt{2\rho}}$ is the derivative of $\frac{\psi(\eta\bar{x}_{ss})}{\eta}$ with respect to σ evaluated at $\sigma = 0$.

C.2 Solution for $\tilde{m}(x)$

We can write the solution of the KFE as the sum the two homogeneous and the particular solution m_p, given $\bar{x}_{ss}$, i.e.

$$\tilde{m}(x) = C_1 e^{\gamma x} + C_2 e^{-\gamma x} + m_p(x)$$

where $\gamma = \sqrt{2\nu/\sigma^2}$. The particular solution solves:

$$m_p(x) = f(x) + \frac{\sigma^2}{2\nu} m_p''(x) \text{ for all } x \in [0, \bar{x}_{ss}]$$

The constants C_1, C_2 are given by:

$$\begin{aligned}0 &= \gamma(C_1 - C_2) + m_p'(0) \\ 0 &= m_p(\bar{x}_{ss}) + C_1 e^{\gamma\bar{x}_{ss}} + C_2 e^{-\gamma\bar{x}_{ss}}\end{aligned}$$

Assume that f is given by an exponential distribution on $[0, U]$, so that for some $\kappa \geq 0$:

$$f(x) = \frac{\kappa}{1 - e^{-\kappa U}} e^{-\kappa x}$$

and if $\kappa = 0$ we have $f(x) = 1/U$. In this case the particular solution is given by:

$$m_p(x) = \frac{\kappa}{(1 - e^{-\kappa U})} \frac{e^{-\kappa x}}{\left(1 - \frac{\sigma^2 \kappa^2}{2\nu}\right)} \text{ for all } x \in [0, \bar{x}_{ss}]$$

and if $\kappa = 0$:

$$m_p(x) = \frac{1}{U} \text{ for all } x \in [0, \bar{x}_{ss}]$$

For instance, for the uniform case we have: $m_p(x) = 1/U$ and thus $C_1 = C_2 = C$. Then

$$C = -\frac{1}{U} \frac{1}{(e^{\gamma \bar{x}_{ss}} + e^{-\gamma \bar{x}_{ss}})}$$

Thus,

$$\tilde{m}(x) = \frac{1}{U} \left[1 - \frac{(e^{\gamma x} + e^{-\gamma x})}{(e^{\gamma \bar{x}_{ss}} + e^{-\gamma \bar{x}_{ss}})} \right] \text{ for } x \in [0, \bar{x}_{ss}]$$

Finally, we want to compute:

$$1 - N_{ss} = \int_0^{\bar{x}_{ss}} \tilde{m}(x) dx = \int_0^{\bar{x}_{ss}} \frac{1}{U} \left[1 - \frac{(e^{\gamma x} + e^{-\gamma x})}{(e^{\gamma \bar{x}_{ss}} + e^{-\gamma \bar{x}_{ss}})} \right] dx$$

This gives another equation for $\bar{x}_{ss}$ as function of $\bar{\theta}$.

D Solution Steady State Planning Problem

D.1 Solution for $\tilde{\lambda}(x)$

The solution for $\tilde{\lambda}$ of the form

$$\tilde{\lambda}(x) = x\frac{\theta_0 + \theta_n N_{ss}}{\rho} + \frac{\theta_n}{\rho}\mathcal{E}_{ss}x + C_1 e^{\eta x} + C_2 e^{-\eta x}$$

for $\eta = \sqrt{2\rho/\sigma^2}$, and

$$\frac{\theta_0 + \theta_n N_{ss}}{\rho} + \eta(C_1 e^{\eta \bar{x}_{ss}} - C_2 e^{-\eta \bar{x}_{ss}}) = 0$$
$$\frac{\theta_0 + \theta_n N_{ss}}{\rho} + \eta(C_1 - C_2) = 0$$

Thus, given $\theta_0 + \theta_n N_{ss}$, and $\bar{x}_{ss}$, the constants (C_1, C_2) are the solution of two linear equations. Moreover, the values of A_1, A_2 are proportional to $\tilde{\theta}_{ss}$ given by

$$\tilde{\theta}_{ss} \equiv \frac{\theta_0 + \theta_n N_{ss}}{\rho} = \eta(C_2 - C_1) = \eta(C_2 e^{-\eta \bar{x}_{ss}} - C_1 e^{\eta \bar{x}_{ss}})$$

Let $\tilde{C}_i \equiv C_i/\tilde{\theta}_{ss}$. We can write:

$$1 = \eta(\tilde{C}_2 - \tilde{C}_1) = \eta(\tilde{C}_2 e^{-\eta \bar{x}_{ss}} - \tilde{C}_1 e^{\eta \bar{x}_{ss}})$$

which has solution:

$$\tilde{C}_1 = \frac{1}{\eta}\frac{(1 - e^{-\eta \bar{x}_{ss}})}{(e^{-\eta \bar{x}_{ss}} - e^{\eta \bar{x}_{ss}})}$$
$$\tilde{C}_2 = \frac{1}{\eta}\frac{(1 - e^{\eta \bar{x}_{ss}})}{(e^{-\eta \bar{x}_{ss}} - e^{\eta \bar{x}_{ss}})}$$

Using value matching we get:

$$\eta \bar{x}_{ss} + \frac{\eta \theta_n}{\rho \tilde{\theta}_{ss}}\mathcal{E}_{ss} + \eta(\tilde{C}_1 e^{\eta \bar{x}_{ss}} + \tilde{C}_2 e^{-\eta \bar{x}_{ss}}) = \frac{\eta}{\tilde{\theta}_{ss}}c$$

Letting $y \equiv \eta \bar{x}_{ss}$ we can write

$$\tilde{\psi}(y) \equiv y + \eta(\tilde{C}_1 e^{y} + \tilde{C}_2 e^{-y}) + \eta\frac{\theta_n}{\rho \tilde{\theta}_{ss}}\mathcal{E}_{ss}$$

Using $\eta\tilde{C}_2 = 1 + \eta\tilde{C}_1$ and the definition of $\tilde{C}_1$ we get

$$\tilde{\psi}(y) \equiv y + e^{-y} - \frac{(1-e^{-y})}{(e^{y}-e^{-y})}(e^{y}+e^{-y}) + \eta\frac{\theta_n}{\rho\tilde{\theta}_{ss}}\mathcal{E}_{ss}$$

We have the following properties:

1. $\tilde{\psi}(0) = \eta\frac{\theta_n}{\rho\tilde{\theta}_{ss}}\mathcal{E}_{ss}$
2. $\tilde{\psi}'(y) = \frac{e^{2y}+1}{(e^{y}+1)^2}$ so $\tilde{\psi}'(0) = \frac{1}{2}, \tilde{\psi}'(\infty) = 1$, and $\tilde{\psi}''(y) > 0$,
3. $\tilde{\psi}(y) = \frac{y}{2} + \frac{y^3}{24} + o(y^4) + \eta\frac{\theta_n}{\rho\tilde{\theta}_{ss}}\mathcal{E}_{ss}$ and $\lim_{y\to\infty}\frac{\tilde{\psi}(y) - y - \eta\frac{\theta_n}{\rho\tilde{\theta}_{ss}}\mathcal{E}_{ss}}{y} = 0$

For fixed $0 < \eta < \infty$ and small c using the first order approximation:

$$\bar{x}_{ss} = 2\left(\frac{c}{\tilde{\theta}_{ss}} - \frac{\theta_n}{\rho\tilde{\theta}_{ss}}\mathcal{E}_{ss}\right)$$

For the case when σ is small (i.e. η is large) we find:

$$\bar{x}_{ss} = \frac{c}{\tilde{\theta}_{ss}} + \frac{\sigma}{\sqrt{2\rho}} - \frac{\theta_n}{\rho\tilde{\theta}_{ss}}\mathcal{E}_{ss}$$

Defining $\gamma = \sqrt{2\nu/\sigma^2}$, for the uniform case we have:

$$\begin{aligned}
N_{ss} &= 1 - \int_0^{\bar{x}_{ss}(N_{ss})} \tilde{m}(s; N_{ss})dx \\
&= 1 - \int_0^{\bar{x}_{ss}} \frac{1}{U}\left[1 - \frac{(e^{\gamma x} + e^{-\gamma x})}{(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})}\right]dx \\
&= 1 - \frac{\bar{x}_{ss}}{U} + \frac{(e^{-\gamma\bar{x}_{ss}} - e^{\gamma\bar{x}_{ss}})}{\gamma U\,(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{ss} &= U/2 - \int_0^{\bar{x}_{ss}(N_{ss})} x\tilde{m}(s; N_{ss})dx \\
&= U/2 - \int_0^{\bar{x}_{ss}} \frac{x}{U}\left[1 - \frac{(e^{\gamma x} + e^{-\gamma x})}{(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})}\right]dx \\
&= U/2 - \frac{\bar{x}_{ss}^2}{2U} + \frac{1}{U\,(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})}\int_0^{\bar{x}_{ss}} \left(xe^{\gamma x} + xe^{-\gamma x}\right)dx \\
&= U/2 - \frac{\bar{x}_{ss}^2}{2U} + \frac{\bar{x}}{\gamma U}\frac{(e^{\gamma\bar{x}_{ss}} - e^{-\gamma\bar{x}_{ss}})}{(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})} + \frac{1}{\gamma^2 U}\frac{2}{(e^{\gamma\bar{x}_{ss}} + e^{-\gamma\bar{x}_{ss}})} - \frac{1}{\gamma^2 U}
\end{aligned}$$

E Dynamics of N and Flow of Adoption Cost

Recall that

$$N(t) = 1 - \int_0^{\bar{x}(t)} m(x,t)dx.$$

Taking the derivative with respect to time

$$\begin{aligned} N_t(t) &= -\frac{d}{dt}\int_0^{\bar{x}(t)} m(x,t)dx \\ &= \underbrace{-m(\bar{x}(t),t)}_{=0}\frac{d\bar{x}(t)}{dt} - \int_0^{\bar{x}(t)} m_t(x,t)dx \end{aligned}$$

where the first term is zero from the exit point of the distribution of non-adopters. Using the law of motion of m

$$\begin{aligned} N_t(t) &= -\int_0^{\bar{x}(t)} \left(-\nu m(x,t) + \nu f(x) + \frac{\sigma^2}{2}m_{xx}(x,t)\right)dx \\ &= \nu\int_0^{\bar{x}(t)} m(x,t) - \frac{\nu\bar{x}(t)}{U} - \frac{\sigma^2}{2}\int_0^{\bar{x}(t)} m_{xx}(x,t)dx \\ &= \nu\left(1 - N(t)\right) - \frac{\nu\bar{x}(t)}{U} - \frac{\sigma^2}{2}\left(\underbrace{m_x(\bar{x}(t),t)}_{<0} - \underbrace{m_x(0,t)}_{=0}\right) \end{aligned}$$

where the last term is zero from our assumption of reflecting barriers. Let the adoption cost per unit of time $A(t)$ be defined as

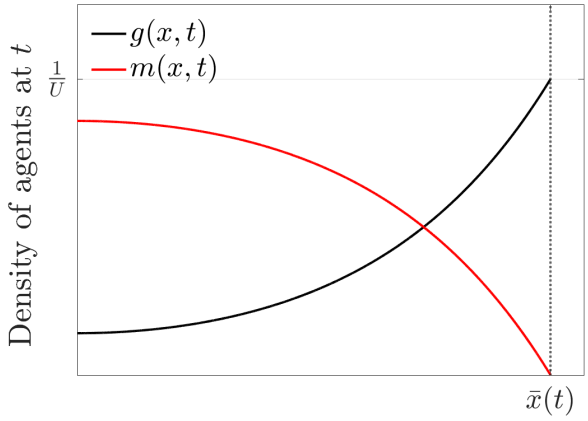
$$\begin{aligned} A(t) &\equiv c\left(N_t(t) + \nu N(t)\right) \\ &= c\left(\nu\left(1 - N(t)\right) - \frac{\nu\bar{x}(t)}{U} - \frac{\sigma^2}{2}m_x(\bar{x}(t),t) + \nu N(t)\right) \\ &= c\left(\nu\left(1 - \frac{\bar{x}(t)}{U}\right) - \frac{\sigma^2}{2}m_x(\bar{x}(t),t)\right) \end{aligned}$$

where the first term are the agents that are replaced with $x \geq \bar{x}(t)$. The second term are the agents that hit $\bar{x}(t)$ from below per unit of time so they pay c and adopt the technology.

Figure E1 shows the distribution of agents to the left of $\bar{x}(t)$, where we let $g(x,t)$ be the distribution of agents who have adopted the technology so that $m(x,t) + g(x,t) = \frac{1}{U}$ for

all $x \in [0, U]$. The figure shows both the distribution of adopters, $g(x,t)$ and non-adopters $m(x,t)$. It shows that, in the stochastic case, the distribution of agents that have adopted the technology is non-zero to the left of $\bar{x}(t)$. This is because, although these agents had adopted the technology, their values of x had moved to the left of $\bar{x}(t)$.

Figure E1: Density of Adopters and Non-Adopters



Note: The figure plots the distribution of adopters, $g(x,t)$, and non-adopters, , $m(x,t)$ for values of σ larger than zero (i.e .$\sigma = 0.002$). The parameters used in the example are $r = 0.03$, $\nu = 0.05$, $U = 0.05$, $c = 0.1$, $\theta_0 = 0.01$, and $\theta_n = 1.2$.

F Solution of Linearized KB Equation

PROPOSITION 6. The solution for the KBE equation for d is given by

$$d(x,t) = \sum_{j=0}^{\infty} \varphi_j(x)\hat{d}_j(t) \qquad \text{for } x \in [0, \bar{x}_{ss}] \text{ and } t \in [0, T]$$

where for all $j = 1, 2, ...$ we have:

$$\begin{aligned}
\hat{d}_j(t) &= \int_t^T e^{-\psi_j(\tau - t))} \hat{z}_j(\tau) d\tau && \text{for } t \in [0, T] \\
\hat{z}_j(t) &= -\theta_n n(t) \frac{\langle \varphi_j, x \rangle}{\langle \varphi_j, \varphi_j \rangle} && \text{for } t \in [0, T] \\
\varphi_j(x) &= \sin\left(\left(\frac{1}{2} + j\right)\pi\left(1 - \frac{x}{\bar{x}_{ss}}\right)\right) && \text{for } x \in [0, \bar{x}_{ss}] \\
\langle \varphi_j, h \rangle &\equiv \int_0^1 h(x)\varphi_j(x) dx \\
\hat{d}_j(T) &= 0 \\
\psi_j &= \rho + \frac{1}{2}\sigma^2 \left(\frac{\pi(\frac{1}{2} + j)}{\bar{x}_{ss}}\right)^2
\end{aligned}$$

The proof can be done by verifying that the equation hold at the boundaries, that for $t > 0$ the p.d.e holds in the interior since

$$\hat{d}_j'(t) = \psi_j \hat{d}_j(t) + \hat{z}_j(t) \qquad \text{for } t \in [0, T] \text{ and } j = 1, 2, ...$$

and since $\{\varphi_j(x)\}$ form an orthogonal bases for functions, and finally that the boundary holds at $t = 0$ for $x \in [0, \bar{x}_{ss}]$.

Note that the derivative of the solution for d is

$$d_x(\bar{x}_{ss}, t) = -\theta_n \int_t^T \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau - t)} n(\tau) d\tau$$

where $c_j = 2\left(1 - \frac{\cos(\pi j)}{\pi(j + \frac{1}{2})}\right)$.

G Solution of Linearized KF Equation

PROPOSITION 7. The solution for the KFE equation for p is given by

$$\begin{aligned}
p(x, t) &= \sum_{j=0}^{\infty} \varphi_j(x) \hat{p}_j(t) + r(t) && \text{for } x \in [0, \bar{x}_{ss}] \text{ and } t \in [0, T] \\
r(t) &\equiv -\bar{y}(t) \tilde{m}_x(\bar{x}_{ss}) && \text{for } t \in [0, T]
\end{aligned}$$

where for all $j = 1, 2, ...$ we have:

$$\begin{aligned}
\hat{p}_j(t) &= \hat{p}_j(0)e^{-\mu_j t} + \int_0^t e^{-\mu_j(t-\tau)} q_j(\tau) d\tau && \text{for } t \in [0, T] \\
\hat{q}_j(t) &= -(r'(t) + \nu r(t)) \frac{\langle 1, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle} && \text{for } t \in [0, T] \\
\varphi_j(x) &= \sin\left(\left(\frac{1}{2} + j\right)\pi\left(1 - \frac{x}{\bar{x}_{ss}}\right)\right) && \text{for } x \in [0, \bar{x}_{ss}] \\
\langle \varphi_j, h \rangle &\equiv \int_0^1 h(x)\varphi_j(x) dx \\
\hat{p}_j(0) &= \frac{\langle \varphi_j, \epsilon - r(0) \rangle}{\langle \varphi_j, \varphi_j \rangle} \\
\mu_j &= \nu + \frac{1}{2}\sigma^2 \left(\frac{\pi(\frac{1}{2} + j)}{\bar{x}_{ss}}\right)^2
\end{aligned}$$

The proof can be done by verifying that the equation hold at the boundaries, that for $t > 0$ the p.d.e holds in the interior since

$$\hat{p}_j'(t) = \mu_j \hat{p}_j(t) + \hat{q}_j(t) \qquad \text{for } t \in [0, T] \text{ and } j = 1, 2, ...$$

and since $\{\varphi_j(x)\}$ form an orthogonal bases for functions, and finally that the boundary holds at $t = 0$ for $x \in [0, \bar{x}_{ss}]$.

H Equilibrium Perturbed MFG

Recall that from equation (11), $\bar{y}(t)$ is equal to

$$\bar{y}(t) = \frac{\theta_n}{\tilde{D}_{xx}(\bar{x}_{ss})} \int_t^T G(\tau - t) n(\tau) d\tau$$

where $G(s) \equiv \sum_{j=0}^{\infty} c_j e^{-\psi_j s}$ for $s \geq 0$. From equation (15) we also know that $n(t)$ is

$$n(t) = n_0(t) + \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2}{\bar{x}_{ss}} \int_0^t J(t - \tau)\bar{y}(\tau) d\tau$$

where $J(s) = \sum_{j=0}^{\infty} e^{-\mu_j s}$ and $n_0(t) \equiv -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+j)} \frac{\langle \varphi_j, \epsilon \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$. Combining equation (11) and equation (15) we get

$$
\begin{aligned}
n(t) &= n_0(t) + \Theta(\bar{x}_{ss}) \int_0^t \int_\tau^T J(t-\tau)\bar{G}(s-\tau)n(s)dsd\tau \\
&= n_0(t) + \Theta(\bar{x}_{ss}) \int_0^T \int_0^{\min\{s,t\}} J(t-\tau)G(s-\tau)n(s)dsd\tau \\
&= n_0(t) + \Theta(\bar{x}_{ss}) \int_0^T K(t,s)n(s)ds
\end{aligned}
$$

where $K(t,s) = \int_0^{\min\{s,t\}} J(t-\tau)\bar{G}(s-\tau)d\tau$ and $\Theta(\bar{x}_{ss}) \equiv \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2\theta_n}{\bar{x}_{ss}\tilde{D}_{xx}(\bar{x}_{ss})}$. Using the definitions of $J(s)$ and $G(s)$ we find

$$
\begin{aligned}
K(t,s) &= \int_0^{\min\{s,t\}} J(t-\tau)G(s-\tau)d\tau \\
&= \int_0^{\min\{s,t\}} \left(\sum_{j=0}^{\infty} e^{-\mu_j(t-\tau)}\right)\left(\sum_{j=0}^{\infty} c_j e^{-\psi_j(s-\tau)}\right) d\tau \\
&= \sum_{j=0}^{\infty}\sum_{j=0}^{\infty} c_j e^{-\mu_i t - \psi_j s} \int_0^{\min\{s,t\}} e^{(\mu_i+\psi_j)\tau} d\tau \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j e^{-\mu_i t - \psi_j s} \left[\frac{e^{(\mu_i+\psi_j)\min\{t,s\}} - 1}{\mu_i + \psi_j}\right].
\end{aligned}
$$

Note that $K(t,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j \left[\frac{1-e^{-(\mu_i+\psi_j)t}}{\mu_i+\psi_j}\right]$.

To calculate the Lipschitz bound $\text{Lip}_K \equiv \sup_{t\in[0,T]} \int_0^T |K(t,s)|ds$, let

$$
\kappa_{ij}(t) \equiv \int_0^T e^{-\mu_i t - \psi_j s}(e^{(\mu_i+\psi_j)\min\{t,s\}} - 1)
$$

so that

$$
\int_0^T K(t,s)ds = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_j \frac{\kappa_{ij}(t)}{\mu_i + \psi_j}.
$$

Computing the integrals in $\kappa_{ij}(t)$ we get

$$\begin{aligned}\kappa_{ij}(t) &= \int_0^t e^{-\mu_i t-\mu_i s}ds + \int_t^T e^{-\psi_j t-\psi_j s}ds - \int_0^T e^{-\mu_i t-\psi_j s}ds \\ &= \frac{e^{-\mu_i t}(e^{\mu_i t}-1)}{\mu_i} + \frac{e^{\psi_j t}(e^{-\psi_j T}-e^{-\psi_j t})}{-\psi_j} - \frac{e^{-\mu_i t}(e^{-\psi_j T}-1)}{-\psi_j} \\ &= \left(\frac{\psi_j+\mu_i}{\psi_j\mu_j}\right)(1-e^{-\mu_i t}) + e^{-\psi_j T}(e^{-\mu_i t}-e^{\psi_j t})\end{aligned}$$

and as $T \to \infty$

$$\begin{aligned}\kappa_{ij}(t) &= \left(\frac{\psi_j+\mu_i}{\psi_j\mu_j}\right)(1-e^{-\mu_i t}) \\ &\leq \frac{\psi_j+\mu_i}{\psi_j\mu_i}.\end{aligned}$$

Using that $\int_0^T |K(t,s)|ds \leq \int_0^\infty |K(t,s)|ds$ we get

$$\begin{aligned}\int_0^T K(t,s)ds &= \sum_{i=0}^\infty \sum_{j=0}^\infty c_j \frac{\kappa_{ij}(t)}{\mu_i+\psi_j} \\ &\leq \sum_{i=0}^\infty \sum_{j=0}^\infty c_j \frac{1}{\mu_i\psi_j} \\ &= \left(\sum_{i=0}^\infty \frac{1}{\mu_i}\right)\left(\sum_{j=0}^\infty \frac{c_j}{\psi_j}\right).\end{aligned}$$

We can use the definitions of μ_j, ψ_j, and c_j to further simplify this expression. First note that

$$\begin{aligned}\sum_{i=0}^\infty \frac{1}{\mu_i} &= \sum_{i=0}^\infty \frac{1}{\nu + \frac{1}{2}\sigma^2\left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2} \\ &\leq \frac{2\bar{x}_{ss}^2}{\sigma^2}\sum_{i=0}^\infty \frac{1}{\left(\pi(\frac{1}{2}+j)\right)^2} \\ &= \frac{\bar{x}_{ss}^2}{\sigma^2}\end{aligned}$$

where we obtain the bound for $\nu = 0$. Notice also that

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{c_j}{\psi_j} &= \sum_{j=0}^{\infty} \frac{2\left(1 - \frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right)}{\rho + \frac{1}{2}\sigma^2 \left(\frac{\pi(\frac{1}{2}+j)}{\bar{x}_{ss}}\right)^2} \\
&\leq \frac{4\bar{x}_{ss}^2}{\sigma^2} \sum_{j=0}^{\infty} \frac{\left(1 - \frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right)}{\left(\pi(\frac{1}{2}+j)\right)^2} \\
&= \frac{4\bar{x}_{ss}^2}{\sigma^2} \sum_{j=0}^{\infty} \left(\frac{1}{\left(\pi(\frac{1}{2}+j)\right)^2} - \frac{(-1)^j}{\left(\pi(\frac{1}{2}+j)\right)^3}\right) \\
&= \frac{4\bar{x}_{ss}^2}{\sigma^2} \sum_{j=0}^{\infty} \left(\frac{1}{2} - \frac{1}{4}\right) \\
&= \frac{\bar{x}_{ss}^2}{\sigma^2}
\end{aligned}$$

where the bound is obtained for $\rho = 0$. Putting these together we find the Lipschitz bound

$$\begin{aligned}
\text{Lip}_K \equiv \sup_{t\in[0,T]} \int_0^T K(t,s)ds &\leq \left(\sum_{i=0}^{\infty} \frac{1}{\mu_i}\right)\left(\sum_{j=0}^{\infty} \frac{c_j}{\psi_j}\right) \\
&= \left(\frac{\bar{x}_{ss}^2}{\sigma^2}\right)^2.
\end{aligned}$$

A sufficient condition for the existence and uniqueness of the equilibrium IRF, i.e. of the uniqueness and existence of a solution to equation (16) is that $|\Theta(\bar{x}_{ss})|\text{Lip}_K < 1$. To establish a bound for $\Theta(\bar{x}_{ss})$, in terms of the fundamental model parameters, that ensures existence and uniqueness, we use the definition of $\Theta(\bar{x}_{ss})$ and the Lipschitz bound as follows:

$$\begin{aligned}
\Theta(\bar{x}_{ss})\left(\frac{\bar{x}_{ss}}{\sigma^2}\right)^2 &= \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2\theta_n}{\bar{x}_{ss}\tilde{D}_{xx}(\bar{x}_{ss})}\left(\frac{\bar{x}_{ss}^2}{\sigma^2}\right)^2 \\
&= \frac{\tilde{m}_x(\bar{x}_{ss})\theta_n\bar{x}_{ss}^3}{\tilde{D}_{xx}(\bar{x}_{ss})\sigma^2} \\
&= \frac{\theta_n(\gamma\bar{x}_{ss})^2}{2U} \frac{\tanh(\gamma\bar{x}_{ss})}{\left(\theta_0 + \theta_n\left(1 - \frac{\gamma\bar{x}_{ss}}{\gamma U} + \frac{\tanh(\gamma\bar{x}_{ss})}{\gamma U}\right)\right)\gamma\bar{x}_{ss} - \rho c\gamma}
\end{aligned}$$

where we obtained $D_{xx}(\bar{x}_{ss})$ evaluating equation (8) at $\bar{x}_{ss}$ and using equation (7), and we calculate $\tilde{m}_x(\bar{x}_{ss})$ from $\tilde{m}(x) = \frac{1}{U}\left(1 - \frac{\cosh(\gamma x)}{\cosh(\gamma\bar{x}_{ss})}\right)$.

I Solution of Linearized KB - Planning Problem

PROPOSITION 8. The solution for the KBE equation for ℓ is given by

$$\ell(x,t) = \sum_{j=0}^{\infty} \varphi_j(x)\hat{\ell}(t) \qquad \text{for } x \in [0, \bar{x}_{ss}] \text{ and } t \in [0,T]$$

where for all $j = 1, 2, ...$ we have:

$$\begin{aligned}
\hat{\ell}(t) &= \int_t^T e^{-\psi_j(\tau - t))} \hat{s}_j(\tau) d\tau && \text{for } t \in [0,T] \\
\hat{s}_j(t) &= -\theta_n n(t) \frac{\langle \varphi_j, x \rangle}{\langle \varphi_j, \varphi_j \rangle} - \theta_n \mathbf{e}(t) \frac{\langle \varphi_j, 1 \rangle}{\langle \varphi_j, \varphi_j \rangle} && \text{for } t \in [0,T] \\
\varphi_j(x) &= \sin\left(\left(\frac{1}{2} + j \right) \pi \left(1 - \frac{x}{\bar{x}_{ss}} \right) \right) && \text{for } x \in [0, \bar{x}_{ss}] \\
\langle \varphi_j, h \rangle &\equiv \int_0^1 h(x)\varphi_j(x) dx \\
\hat{\ell}(T) &= 0 \\
\psi_j &= \rho + \frac{1}{2}\sigma^2 \left(\frac{\pi(\frac{1}{2} + j)}{\bar{x}_{ss}} \right)^2
\end{aligned}$$

The proof can be done by verifying that the equation hold at the boundaries, that for $t > 0$ the p.d.e holds in the interior since

$$\hat{\ell}_j'(t) = \psi_j \hat{\ell}(t) + \hat{s}_j(t) \qquad \text{for } t \in [0,T] \text{ and } j = 1, 2, ...$$

and since $\{\varphi_j(x)\}$ form an orthogonal bases for functions, and finally that the boundary holds at $t = 0$ for $x \in [0, \bar{x}_{ss}]$.

Note that the derivative of the solution for λ is

$$\ell_x(\bar{x}_{ss}, t) = -\theta_n \int_t^T \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau - t)} n(\tau) d\tau - \theta_n \frac{2}{\bar{x}_{ss}} \int_t^T \sum_{j=0}^{\infty} e^{-\psi_j(\tau - t)} \mathbf{e}(\tau) d\tau$$

where $c_j = 2\left(1 - \frac{\cos(\pi j)}{\pi(j + \frac{1}{2})}\right)$.

J Equilibrium Perturbed MFG - Planning Problem

Recall that from equation (20), $\bar{y}(t)$ is equal to

$$\begin{aligned}\bar{y}(t) &= \frac{-\ell_x(\bar{x}_{ss}, t)}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \\ &= \int_t^T \frac{\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau - t)} n(\tau) d\tau + \int_t^T \frac{2\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})\bar{x}_{ss}} \sum_{j=0}^{\infty} c_j e^{-\psi_j(\tau-t)} \mathbf{e}(\tau) d\tau \\ &= \int_t^T G_{yn}(\tau - t) n(\tau) d\tau + \int_t^T G_{ye}(\tau - t)\mathbf{e}(\tau) d\tau \end{aligned} \tag{29}$$

The expression for $n(t)$ is given by equation (15) and can be written as

$$n(t) = n_0(t) + \int_0^t H_{ny}(t-s)\bar{y}(s) ds.$$

where as before $n_0(t) \equiv -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}}{\pi(\frac{1}{2}+j)} \frac{\langle \varphi_j, \epsilon \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$. We can obtain a similar expression for $\mathbf{e}(t)$ using the solution for $p(x,t)$ as

$$\begin{aligned}\mathbf{e}(t) &= -\int_0^{\bar{x}_{ss}} x p(x,t) dx \\ &= -\sum_{j=0}^{\infty} \hat{p}_j(t) \int_0^{\bar{x}_{ss}} x \varphi_j(x) dx \\ &= -\sum_{j=0}^{\infty} \frac{\bar{x}_{ss}^2 (\pi(j+\frac{1}{2}) - \cos(j\pi))}{(\pi(\frac{1}{2}+j))^2} \frac{\langle \varphi_j, \epsilon \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t} + \tilde{m}_x(\bar{x}_{ss})\sigma^2 \int_0^t \sum_{j=0}^{\infty} \frac{\pi(j+\frac{1}{2}) - \cos(j\pi))}{\pi(j+\frac{1}{2})} e^{-\mu_j(t-\tau)} \bar{y}(\tau) d\tau \\ &= \mathbf{e}_0(t) + \int_0^t H_{ey}(t-s)\bar{y}(s) ds \end{aligned}$$

where $\mathbf{e}_0(t) \equiv -\sum_{j=0}^{\infty} \frac{c_j}{2} \frac{\bar{x}_{ss}^2}{\pi(j+\frac{1}{2})} \frac{\langle \varphi_j, \epsilon \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\mu_j t}$ and $c_j \equiv \left(1 - \frac{\cos(\pi j)}{\pi(j+\frac{1}{2})}\right)$. Then, equation (29) can

be written as

$$\begin{aligned}
\bar{y}(t) &= \int_t^T G_{yn}(\tau - t)\left(n_0(\tau) + \int_0^t H_{ny}(\tau - s)\bar{y}(s)ds\right)d\tau \\
&+ \int_t^T G_{ye}(\tau - t)\left(\mathbf{e}_0(\tau) + \int_0^t H_{ey}(\tau - s)\bar{y}(s)ds\right)d\tau \\
&= \int_t^T G_{yn}(\tau - t)n_0(\tau)d\tau + \int_t^T \int_0^t G_{yn}(\tau - t)H_{ny}(\tau - s)\bar{y}(s)ds\, d\tau \\
&+ \int_t^T G_{ye}(\tau - t)\mathbf{e}_0(\tau)d\tau + \int_t^T \int_0^t G_{ye}(\tau - t)H_{ey}(\tau - s)\bar{y}(s)ds\, d\tau \\
&= \bar{y}_0(t) + \int_0^T M(t,s)\bar{y}(s)ds
\end{aligned}$$

where

$$\bar{y}_0(t) \equiv \int_t^T G_{yn}(\tau - t)n_0(\tau)d\tau + \int_t^T G_{ye}(\tau - t)\mathbf{e}_0(\tau)d\tau$$

and

$$\begin{aligned}
\int_0^T M(t,s)\bar{y}(s)ds &\equiv \int_t^T \int_0^t G_{yn}(\tau - t)H_{ny}(\tau - s)\bar{y}(s)ds\, d\tau + \int_t^T \int_0^t G_{ye}(\tau - t)H_{ey}(\tau - s)\bar{y}(s)ds\, d\tau \\
&= \int_0^T \int_{\max\{t,s\}}^T G_{yn}(\tau - t)H_{ny}(\tau - s)\bar{y}(s)ds\, d\tau + \int_0^T \int_{\max\{t,s\}}^T G_{ye}(\tau - t)H_{ey}(\tau - s)\bar{y}(s)ds\, d\tau
\end{aligned}$$

with

$$\begin{aligned}
G_{yn}(w) &= \frac{\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})}\sum_{j=0}^{\infty} c_j e^{-\psi_j(w)} \\
G_{ye}(w) &= \frac{2\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})\bar{x}_{ss}}\sum_{j=0}^{\infty} e^{-\psi_j(w)} \\
H_{ey}(q) &= \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2}{2}\sum_{j=0}^{\infty} c_j e^{-\mu_j(q)} \\
H_{ny}(q) &= \frac{\tilde{m}_x(\bar{x}_{ss})\sigma^2}{\bar{x}_{ss}}\sum_{j=0}^{\infty} e^{-\mu_j(q)}
\end{aligned}$$

where $e^{-rq}G_{yn}(w)H_{ny}(q) = G_{ye}(w)H_{ey}(q)e^{-rq}$. Using the definitions of $n_0(t)$ and $\mathbf{e}_0(t)$ we

first find the value of $\bar{y}_0(t)$ as

$$\begin{aligned}
\bar{y}_0(t) &\equiv \int_t^T G_{yn}(\tau - t) n_0(\tau) d\tau + \int_t^T G_{ye}(\tau - t) \mathbf{e}_0(\tau) d\tau \\
&= \frac{-\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \int_t^T \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_j \frac{\bar{x}_{ss}}{\pi(\frac{1}{2} + i)} \frac{\langle \varphi_i, \epsilon \rangle}{\langle \varphi_i, \varphi_i \rangle} e^{-\psi_j(\tau - t)} e^{-\mu_i \tau} d\tau \\
&+ \frac{-\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \int_t^T \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_i \frac{\bar{x}_{ss}}{\pi(\frac{1}{2} + i)} \frac{\langle \varphi_i, \epsilon \rangle}{\langle \varphi_i, \varphi_i \rangle} e^{\psi_j t} e^{-\psi_j(\tau - t)} e^{-\mu_i \tau} d\tau \\
&= \frac{-\theta_n}{\tilde{\lambda}_{xx}(\bar{x}_{ss})} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (c_j + c_i) \frac{\bar{x}_{ss}}{\pi(\frac{1}{2} + i)} \frac{\langle \varphi_i, \epsilon \rangle}{\langle \varphi_i, \varphi_i \rangle} e^{\psi_j t} \left(\frac{e^{-(\psi_j + \mu_i)t} - e^{-(\psi_j + \mu_i)T}}{\psi_j + \mu_i} \right) \qquad (30)
\end{aligned}$$

Then, we find

$$\begin{aligned}
\int_0^T M(t,s) \bar{y}(s) ds &= \int_0^T \left(\int_{\max\{t,s\}}^T G_{yn}(\tau - t) H_{ny}(\tau - s) \bar{y}(s) d\tau + \int_{\max\{t,s\}}^T G_{ye}(\tau - t) H_{ey}(\tau - s) \bar{y}(s) d\tau \right) ds \\
&= \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \int_{\max\{t,s\}}^T \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_j e^{-\psi_j(\tau - t)} e^{-\mu_i(\tau - t)} \bar{y}(s) d\tau \, ds \\
&+ \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \int_{\max\{t,s\}}^T \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_i e^{-\psi_j(\tau - t)} e^{-\mu_i(\tau - t)} \bar{y}(s) d\tau \, ds
\end{aligned}$$

where we let $\tilde{\Theta}(\bar{x}_{ss}) \equiv \frac{\theta_n \tilde{m}_x(\bar{x}_{ss}) \sigma^2}{\tilde{\lambda}_{xx}(\bar{x}_{ss}) \bar{x}_{ss}}$. Solving the integrals we get

$$\begin{aligned}
\int_0^T M(t,s) \bar{y}(s) ds &= \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \left(\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_j e^{\psi_j t + \mu_i s} \left(\frac{e^{-(\psi_j + \mu_i) \max\{t,s\}} - e^{-(\psi_j + \mu_i)T}}{\psi_j + \mu_i} \right) \right) ds \\
&+ \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \left(\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_i e^{\psi_j t + \mu_i s} \left(\frac{e^{-(\psi_j + \mu_i) \max\{t,s\}} - e^{-(\psi_j + \mu_i)T}}{\psi_j + \mu_i} \right) \right) ds \\
&= \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \left(\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} (c_j + c_i) e^{\psi_j t + \mu_i s} \left(\frac{e^{-(\psi_j + \mu_i) \max\{t,s\}} - e^{-(\psi_j + \mu_i)T}}{\psi_j + \mu_i} \right) \right) ds \\
&= \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \tilde{K}(t,s) ds.
\end{aligned}$$

Thus, equation (29) can be written as

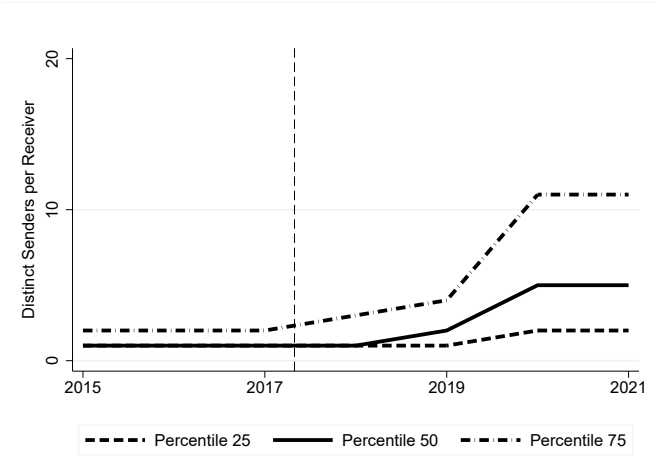
$$\bar{y}(t) = \bar{y}_0(t) + \tilde{\Theta}(\bar{x}_{ss}) \int_0^T \tilde{K}(t,s) \bar{y}(s) ds$$

Notice also that since $e^{-rt}M(t,s) = e^{-rs}M(t,s)$

$$\int_0^T e^{-rt}M(t,s)\bar{y}(s)ds = \tilde{\Theta}(\bar{x}_{ss})\int_0^T \left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(c_j+c_i)e^{\mu_j t+\mu_i s}\left(\frac{e^{-(r+\mu_j+\mu_i)\max\{t,s\}} - e^{-(r+\mu_j+\mu_i)T}}{\mu_j+\mu_i+r}\right)\right)ds$$

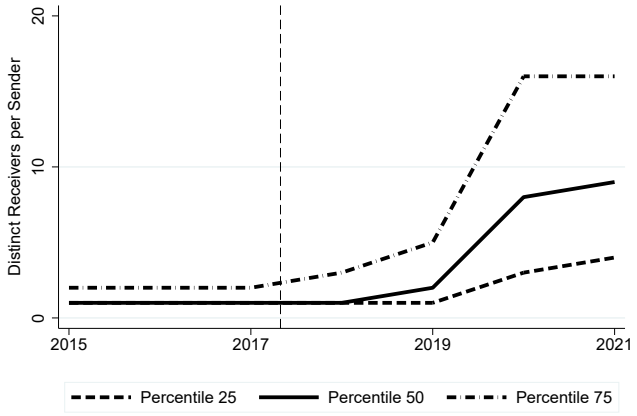
K Additional Figures

Figure K1: Distribution of Distinct Senders per Receiver (Yearly)



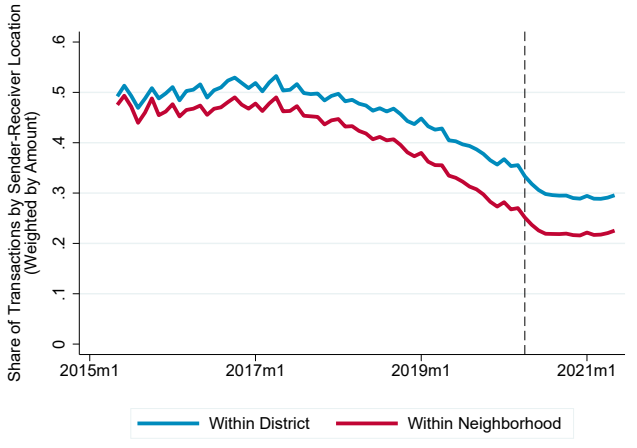
Notes: The figure shows distinct number of senders per receiver each year. Year 2021 only includes information up to June.

Figure K2: Distribution of Distinct Receivers per Sender (Yearly)



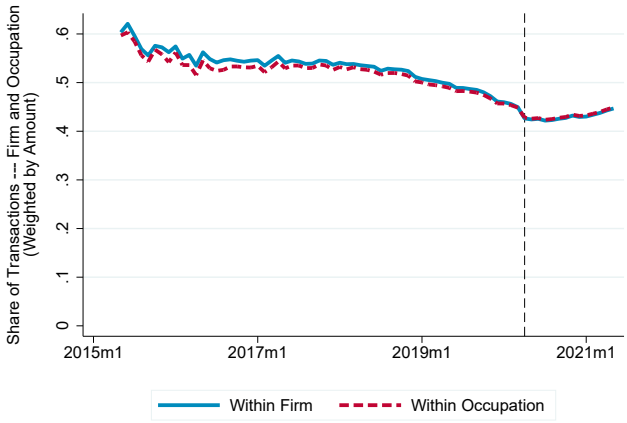
Notes: The figure shows distinct number of receivers per sender each year. Year 2021 only includes information up to June.

Figure K3: Number of Transactions Between Users Living in the Same Location (Share, Weighted by Amount)



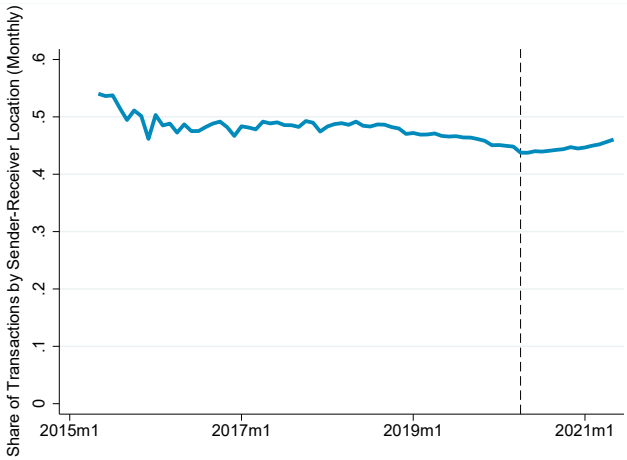
Notes: The figure shows total value of SINPE transactions between users living in the same location in 2018, as a share of the value of all transactions.

Figure K4: Number of Transactions Between Users Working in the Same Firm (Share, Weighted by Amount)



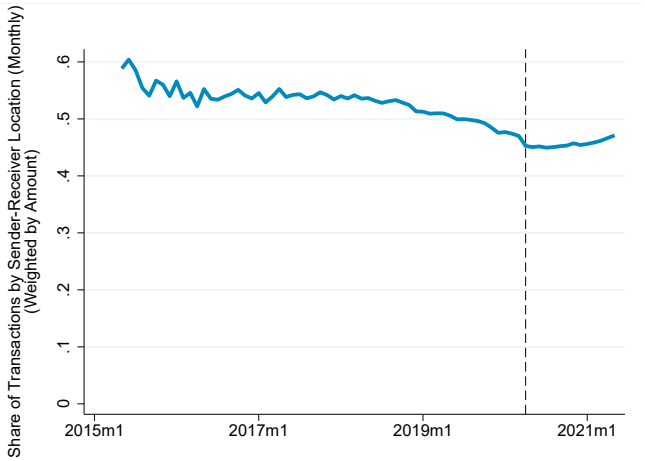
Notes: The figure shows total value of SINPE transactions between users working at the same firm in 2018, as a share of the value of all transactions.

Figure K5: Number of Transactions Between Users Employed and Living in the Same Location (Share)



Notes: The figure shows total number of SINPE transactions between employed users living at the same location in 2018, as a share of all transactions.

Figure K6: Number of Transactions Between Users Employed and Living in the Same Location (Share, Weighted by Amount)



Notes: The figure shows total amount of SINPE transactions between employed users living at the same location in 2018, as a share of the value of all transactions.