Medium term endogenous fluctuations in three-sector optimal growth models*

Kazuo NISHIMURA

RIEB, Kobe University

Florian PELGRIN

EDHEC Business School

and

Alain VENDITTI

Aix-Marseille Univ., CNRS, AMSE &

EDHEC Business School

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Abstract: *This paper provides some theoretical support to explain the existence of mid-term endogenous fluctuations for the main macroeconomic time series. In the spirit of Beaudry* et al. *[3, 5], we first provide a spectral analysis of US quarterly detrended data on gross domestic product, gross private investment and personal consumption expenditures in order to identify the peaks associated to a mid-term periodicity. We found an average periodicity between 8 and 12 years. Then, building on a multisector optimal growth model (without frictions) with a non-linear utility function, we give some clear conditions for the occurrence of endogenous cycles. We provide the key ingredients (3-sector specification with a pure consumption good and a mixed investment-consumption good) that lead to mid-term endogenous fluctuations under empirically realistic values for the main structural parameters. We prove indeed that if the share of the mixed consumption good is large enough and the utility function is not too concave, a Hopf bifurcation can occur under realistic values for the elasticity of intertemporal substitution in consumption. Endogenous fluctuations are then likely to occur through two mechanisms: The first one relies on sectoral technologies through the sign of the capital intensity differences between sectors, and the second one on intertemporal consumption allocations based on some substitution effects between the two consumption goods. We also provide a precise evaluation of the periodicity of the endogenous cycles that depends on the technological and preference parameters of the model. Using a numerical simulation based on a calibration in the US, we show that our results are in line with the empirical evidence for both the parameters' values and the medium term periodicities identified in the data.*

Keywords: *Three-sector optimal growth models, mid-term fluctuations, Hopf bifurcation, endogenous cycle, periodicity*

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1 Introduction

Most of the standard modern macroeconomic literature focuses on the idea that the market economy is inherently stable and characterized by market forces that direct the economy toward a regular steady state or a balanced-growth path. As a consequence, any fluctuations or dynamic adjustments have to occur through individual adjustments to stochastic shocks affecting preferences and / or the technology.

However, a recent trend of the literature, mainly driven by Beaudry *et al.* [3, 4, 5], has resuscitated an alternative explanation: fluctuations can result from internal forces of the economy generating alternate periods of booms and busts. Such a literature has been strongly developed in the 1970s and 1980s from the seminal contributions of Benhabib and Nishimura [6, 7]. It was shown that even in standard models featuring forward-looking agents and a competitive equilibrium structure, the steady state or balanced growth path was inherently unstable and deterministic fluctuations were easily obtained as soon as the fundamental nonlinear structure of the model was taken into account.[1]

The recent contributions of Beaudry *et al.* [3, 4, 5] provide important conclusions that put again the emphasis on structural instability and thus on endogenous cycles. First, Beaudry *et al.* [4] suggest that the local stability of the macroeconomic system should not be evaluated using linear models, even if nonlinearities are thought to be minor and relevant only far away from the steady state. Indeed, they show, when allowing for simple nonlinearities in estimation, that the macroeconomic system may be locally unstable. Second, Beaudry *et al.* [3, 5] provide extensive evidence regarding the existence of a recurrent peak in several spectral densities in many US trendless macroeconomic data suggesting the presence of periodicities around 9 to 10 years. Moreover, they argue that the existence of a limit cycle through a Hopf bifurcation is an appropriate theoretical support for these empirical conclusions.[2]

The existence of a Hopf bifurcation in models featuring forward-looking agents and a competitive equilibrium structure has been proved in Benhabib and Nishimura [6]. However, the consideration of at least three sectors and of dimension-four dynamical systems is necessary to be able to get such deterministic fluctuations based on the existence of complex characteristic roots. This structure appears indeed to be the simplest optimal intertemporal macroeconomic dynamic model (without any imperfection) able to endogenously generate periodic cycles through such a bifurcation. However as shown in many contributions, although sufficient

[1] Boldrin and Montrucchio [10] show more generally that any type a dynamic behavior, even complex chaotic fluctuations, can characterize the optimal solution of a standard optimal growth model.

[2] See also Dufourt *et al.* [15] where the Hopf bifurcation is shown to be relevant from an empirical perspective in two-sector models with productive externalities and sunspot fluctuations.

conditions for the existence of real roots are known,[3] there is no simple clear-cut sufficient conditions for the existence of complex characteristic roots in dynamical system larger than dimension 2.[4] Moreover, most of the available results are based on the extreme assumption of a linear utility function.[5]

Our main objective in this paper is to consider an optimal growth model (without frictions) with a non-linear utility function and to provide the key ingredients (3-sector specification with a pure consumption good and a mixed investment-consumption good) that lead to mid-term endogenous fluctuations under empirically realistic values for the main structural parameters. It is first necessary to precisely identify in the data the mid-term periodicities of the main macroeconomic data. While section 2 is based on spectral analysis in the same spirit as Beaudry *et al.* [5], one main difference is that we need to deal with nonstationary variables and thus to use a detrending transformation. This is done through the approach of Müller and Watson [28, 29, 30, 31]. Accordingly we provide evidence for new series regarding the medium term component. We show that on average, the peaks for all the variables correspond to a periodicity of between 8 and 12 years.

Based on these empirical results, the theoretical analysis is aimed at finding clear conditions for the existence of a Hopf bifurcation. It is then necessary to identify the conditions under which complex characteristic roots occur, and second to characterize the range of values of the elasticity of intertemporal substitution in consumption that yields to the existence of a pair of purely imaginary roots. In this respect, we consider two formulations for the model. The first formulation, which is commonly used in the literature, assumes that there is only one consumption good which is consumed by the household. For instance, this is the formulation of Benhabib and Nishimura [6]. Provided that the utility function is close to a linear specification, we propose precise conditions for the existence of complex characteristic roots and a Hopf bifurcation. The core mechanism is based on the fact that the sector of the pure consumption good is more intensive in at least one capital good than the sector of the capital good itself. These technological conditions generate through the Rybczinski effect oscillations of stocks and outputs. Moreover, the Stolper-Samuelson effect generates corresponding fluctuations of the relative prices of both capital stocks provided the discount

[3] Magill and Scheinkman [27] show that symmetric problems are necessarily characterized by real roots.

[4] Magill [24, 25, 26] shows that certain asymmetric stock-flow interaction terms arising in the local equations of motion about the equilibrium point are necessary to get complex roots.

[5] This assumption is usually justified by the conclusions of Rockafellar [35] who shows that the saddle-point property of the steady state is ensured as soon as the degree of concavity of the utility function is large enough. However, some results with a non-linear utility function are contained in Cartigny and Venditti [11] and Venditti [36], but the consideration of general multisector models prevents from getting precise conditions.

rate is large enough to prevent intertemporal arbitrage opportunities. However, from an empirical point of view, the close-to-linear utility function leads to an unrealistic elasticity of intertemporal substitution in consumption.

Building on this first set of results, the second formulation assumes that both the pure consumption good and a mixed consumption-investment good are consumed by the household. The utility function is then non-linear and we consider a Cobb-Douglas formulation which is non strictly concave (i.e. homogeneous of degree one) or strictly concave (i.e. homogeneous of degree lower than one). In such a case, the elasticity of intertemporal substitution in consumption remains finite even in the non strictly concave case. We then show that if the share of the mixed consumption good is large enough a Hopf bifurcation can occur. The consideration of a second consumption good in the utility function provides indeed an additional mechanism. Beside the technological mechanism previously mentioned, endogenous fluctuations are here also generated from intertemporal consumption allocations based on substitution effects between the two consumption goods.

In contrast to the first formulation, it turns out that the occurrence of a Hopf bifurcation is consistent with a plausible elasticity of intertemporal substitution even if it still requires a weakly concave utility function. We also provide a precise evaluation of the periodicity of the endogenous cycles generated by the Hopf bifurcation that depends on the technological and preference parameters of the model. Our results definitely prove the importance of a mixed good in the existence of endogenous fluctuations within a realistic framework. Indeed we show through a numerical simulation of our model first that our conclusions are compatible with empirical evidence on capital shares derived from input-output tables, and second that the periodic orbit is characterized by a periodicity almost equal to those identified in Section 2.

The rest of the paper is organized as follows: Section 2 provides some empirical evidence for the existence of mid-term fluctuations in macroeconomic data. In Section 3 we present the model and the intertemporal equilibrium, we prove the existence and uniqueness of the steady state and we provide the expression of the characteristic polynomial. In Section 4 we consider the standard formulation with one consumption good consumed by the household. Section 5 provides new results for the formulation with two consumption goods. In Section 6, we finally show with a simple numerical simulation that our results are in line with the empirical evidence for both the parameters' values and the medium term periodicities obtained in Section 2. Section 7 concludes the paper and all the proofs are contained in a final Appendix.

2 Medium term fluctuations in US data

This section explores some empirical properties of some main variables in our 3-sector model. Our treatment is notable in three dimensions. First, we make use of the recent low-frequency approach initiated by Müller and Watson ([30], [31]) to identify and estimate the long-run component of each variable, and thus to build the corresponding cyclical component. Second, following the recent contributions of Beaudry *et al.* [5], we test the presence of a significant peak (range) on the spectral density, which provides some support of recurrent cyclical fluctuations at medium term and thus characterizes the periodicity of such cycles. At the same time, in the spirit of Beaudry *et al.* [5], it is worth emphasizing that we only discuss *indirect* empirical evidence on the variables of interest: the presence of a peak range does not necessarily imply strong endogenous cyclical forces. It only suggest that data cannot, at least, contradict the existence of endogenous (stochastic) limit cycles. Third, armed with these empirical estimates of the peak range of medium-term fluctuations, we reconciliate them with an optimal 3-sector model in the sense we can compare the identified periodicity in our data with the one generated by our model.

2.1 Data

Using US quarterly data over the period 1970Q1-2019Q4, our aim is to study the spectral density of the cyclical component of the following macro variables: the gross domestic product, the gross private domestic investment, and personal consumption expenditures (and its decomposition durable, nondurables and services) as well as the corresponding price deflator series. Unsurprisingly, a quick eye inspection of Figure 1 clearly suggests that all of these variables are nonstationary as the existence of deterministic and/or stochastic trend can not be ruled out for the gross domestic product, the gross private domestic investment, and personal consumption expenditures. This is further confirmed by conventional unit root tests. Such a nonstationarity is challenging since spectral analysis requires the (weakly) stationarity of the series and thus a detrending procedure, which avoids spurious cycles, is necessary to isolate the long-run component.[6]

Assuming that initial series can be thought as the standard sum of a (weakly) stationary component and a nonstationary (say, of order 1) component, the cyclical component (respectively, the long-run component) is captured by the stationary component (respectively, the nonstationary component).[7] Accordingly, the use of the spectral density to identify a medium-term peak

[6] A short overview of spectral analysis is provided in Appendix 2.

[7] Note that we are interested in the cyclical component of the level variables and thus do not proceed with

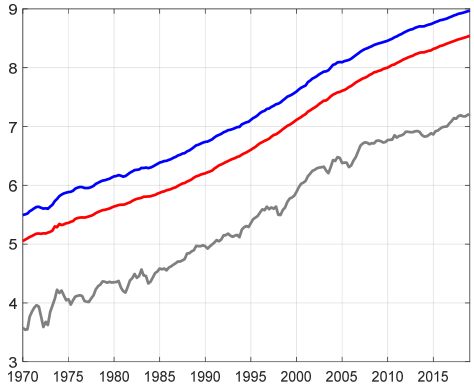


Figure 1: Gross domestic product, gross private domestic investment, and personal consumption expenditures

Note: The grey, blue and red lines depict the (log of) gross private domestic investment, gross domestic product, and personal consumption expenditures series, respectively.

range amounts of first extracting the long-run component and then computing the cyclical component. Before discussing the estimation of the two components, it is worth emphasizing that (stationary) time series are generally split into three ranges in the frequency domain. Indeed, the general consensus is that high-frequency or business cycle frequencies correspond to periodicity below 6 to 8 years, whereas medium-term frequencies are those of periodicity in between 6-8 and 40-50 years and low or long-run frequencies are those with periodicity above 40-50 years.[8]

2.2 The long-run and cyclical component

We now proceed with the estimation of the long-run component and cyclical component. As proposed by Müller and Watson ([30], [31]), we extract the long-run sample information after isolating a small number of low-frequency trigonometric weighted averages.[9]

Before presenting the regression-based filter of Müller and Watson [31], note that one key

a first-difference (log-) transformation.

[8]Comin and Gertler [14] define medium-term business cycles as all cyclical fluctuations between 2 and 50 years, and then decompose these cycles into a high-frequency component (below 8 years) and a low-frequency component (between 8 and 50 years).

[9]We also apply the Christiano and Fitzgerald ([13], [12]) approximation of the ideal band-pass filter since it rests on the assumption that data are generated by a random walk process. We also compute the long-run component using the HP filter. However, as shown by Phillips and Jin [34], the properties of the HP-filtered series heavily depend on the choice of the smoothing parameter (say λ) that trades off the cyclical and (stochastic) trend component. More specifically, when $\lambda = O(T^4)$, where T is the sample size, the HP filter does remove the stochastic trend in the limit (i.e. as $T \to \infty$) and thereby explaining some "spurious cycle" effects of the HP filter. On the other hand, when $\lambda = o(T)$, the HP filter eliminates the stochastic trend. Monte-Carlo simulations show that the $\lambda = O(T^4)$ limit theory is more likely in applied macroeconomics and thus the HP long-run component can be very noisy, especially in the presence of a unit root. Finally, we also transform our data using the first difference operator and apply the Baxter and King [2] filter. This requires in turn to cumulate the filtered series. However, the initial first-difference transformation generally emphasizes movements at higher frequencies (respectively, deemphasizes those at lower frequencies). All in all, the approach of Müller and Watson [30] is preferable as it remains valid in both the stationary and nonstationary case (using the appropriate limit theory).

feature of the low-frequency approach of Müller and Watson ([30], [31]) relative to bandpass and moving average filters, is that it is applicable beyond the I(0) assumption.[10]

Let $\{x_t, t = 1, \cdots, T\}$ denote a (scalar) time series. Let $\Psi(s) = [\Psi_1(s), \cdots, \Psi_q(s)]'$ denote a $\mathbb{R}^q$-valued function with $\Psi_j(s) = \sqrt{2}cos(js\pi)$, and let

$$\Psi_T = \left[\Psi\left(\tfrac{1-0.5}{T}\right), \Psi\left(\tfrac{2-0.5}{T}\right), \cdots, \Psi\left(\tfrac{T-0.5}{T}\right)\right]'$$

denote the $T \times q$ matrix after evaluating $\Psi(.)$ at $s = \frac{t-0.5}{T}$, for $t = 1, \cdots, T$. The low-frequency projection is the fitted series from the OLS regression of $[x_1, \cdots, x_T]$ onto a constant and Ψ_T. In so doing, we project the series into a constant and eight (q) cosine functions with periods $\frac{2T}{j}$ for $j = 1, \cdots, 14$ in order to capture the variability for periods longer than 40 years ($2T/q$).[11] This provides the long-run component, denoted $x_t^{\mathtt{lr}}$, and thus the cyclical component is $x_t - x_t^{\mathtt{lr}}$, for all $t = 1, \cdots, T$.

The left panel of Figure 2 displays the cyclical component of each variable (the gross domestic product, the domestic private investment and personal consumption expenditures) whereas the right panel features those of the corresponding implicit price deflators. As to be expected, all the filtered series look like mean-reverting processes and this is further confirmed by the two (local) point-optimal unit root tests of Müller and Watson [30], as well as the DF-GLS unit root test of Elliott, Rothenberg, and Stock [16], and the stationarity test of Nyblom [32].[12]

2.3 The medium-term cyclical component

We now study the spectral density of the cyclical component of each variable. More specifically, we highlight in light (resp., dark) grey the band of frequencies corresponding to periodicities from 8 to 32 quarters (resp., 32 to 60 quarters) in Figure 3.

Irrespective of the smoothing method (Hanning, Parzen or Blackman-Tukey), one dominant feature is the distinct hump in the spectral density surrounding the local peak at around 50 quarters (respectively, 40 quarters) for the gross domestic product and personal consumption expenditures (respectively, domestic private investment). This suggests that the three variables exhibit important recurrent cyclical phenomena at approximatively 10 to 12-year intervals. Unsurprisingly, the second feature of Figure 3 is that there are also some business fluctuations

[10] For an extensive discussion about the relationship of this approach with spectral analysis, the scarcity of low-frequency information, and the relevance of the approximation using a small q, see Müller and Watson [30].

[11] As a sensitivity analysis, we also experiment different upper bounds for q. All in all, our results remain robust. Detailed results are available upon request.

[12] We implement the so-called LFST and LFUR point-optimal tests, which respectively test the I(0) and I(1) null hypotheses. All detailed results and codes are available upon request.

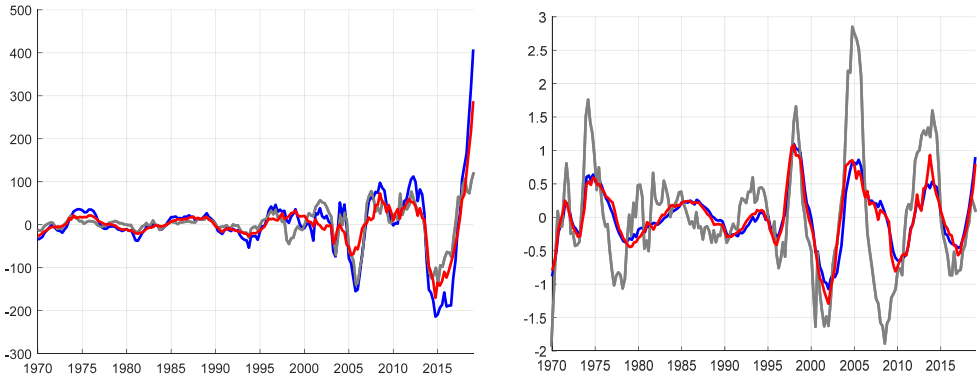


Figure 2: Cyclical component

Note: The grey, blue and red lines on the left panel (resp., right panel) depict the cyclical component of the gross domestic product (resp., GDP deflator), the gross private domestic investment (resp., the implicit investment price deflator), and personal consumption expenditures (resp., the corresponding price deflator), respectively. The cyclical component is the difference between the raw series and the long-run component. The long-run component is estimated using the cosine-based approach of Müller and Watson ([31]).

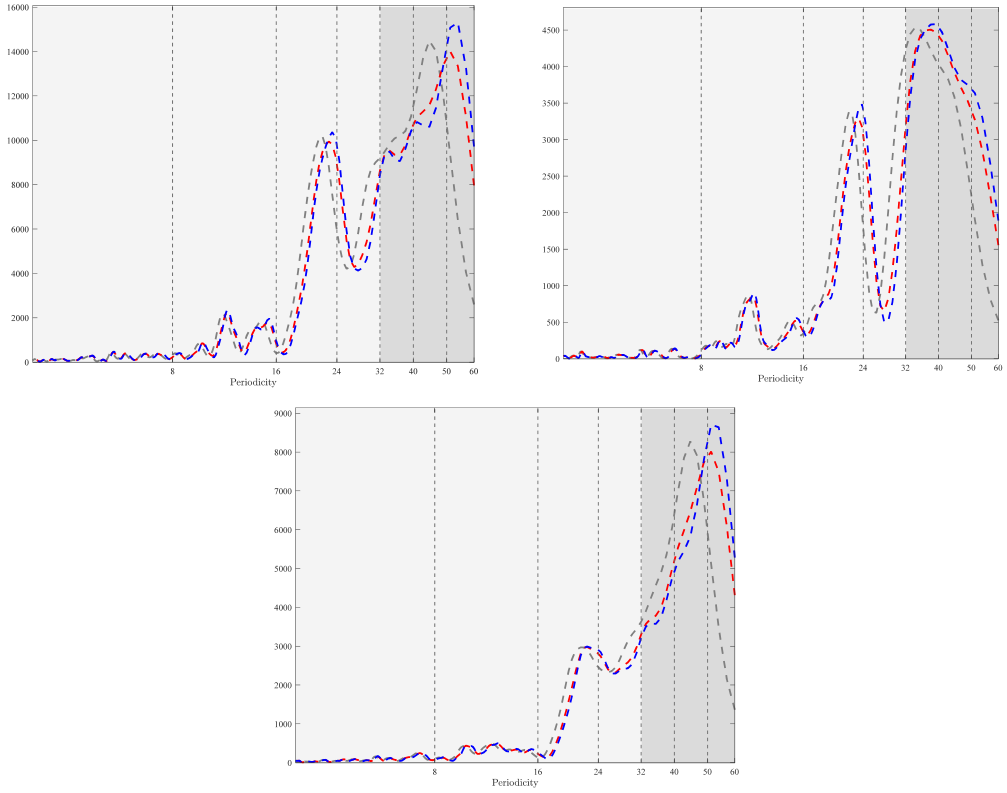


Figure 3: Periodogram

Note: The grey, red, and blue dotted lines represent the spectral density using, respectively the Hanning, Parsen and Blackman-Tukey smoothing method of the gross domestic product (top left panel), the domestic private investment (top right panel) and personal consumption expenditures (bottom panel).

(or higher frequency movements) since all spectral densities displays a *local* peak contained in the 16-24 quarter range. These results are fully consistent with the ones outlined by Beaudry *et al.* [5]. To go one step further, we also consider the spectral densities of the implicit price deflator for each (cyclical) level variable (Figure 4). Our main conclusions remain robust. Interestingly, our results shows that the medium-term characterization of most stationary level (macro) variables proposed by Beaudry *et al.* [5] has also some strong support when detrendring appropriately traditional nonstationary macro variables (e.g., the gross domestic product and its price deflator).

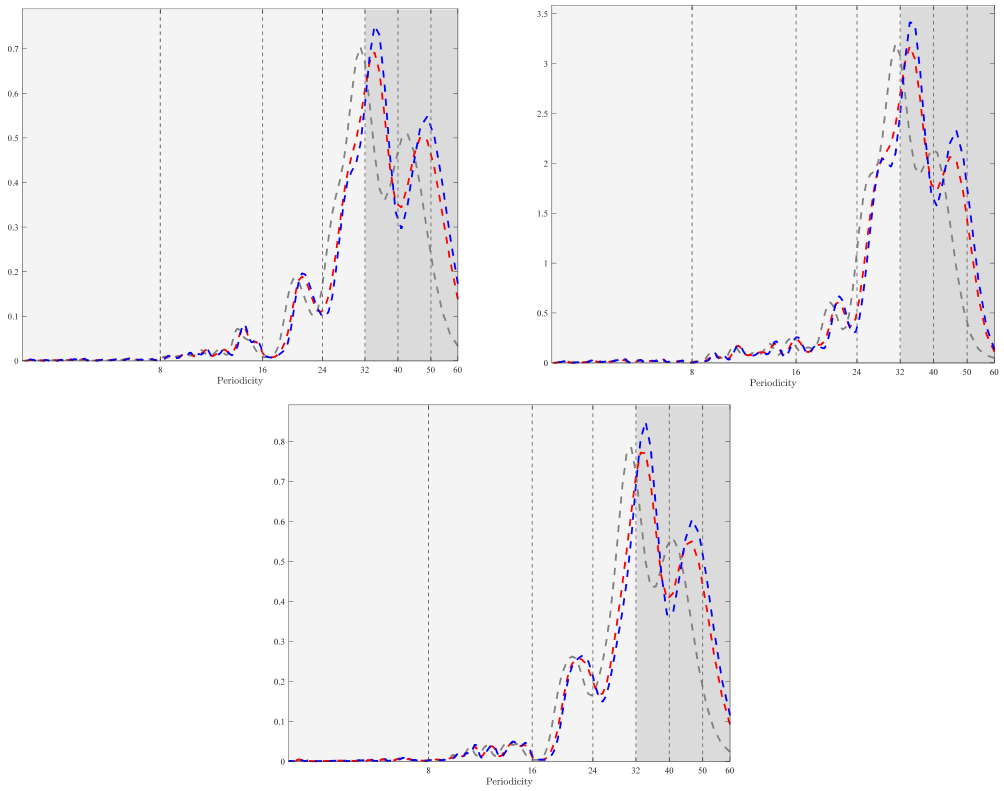


Figure 4: Periodogram

Note: The grey, red, and blue dotted lines represent the spectral density using, respectively the Hanning, Parsen and Blackman-Tukey smoothing method of price deflator of the gross domestic product (top left panel), the domestic private investment (top right panel) and personal consumption expenditures (bottom panel).

In the same spirit as Beaudry *et al.* [5], we formally test the presence of a shape restriction on the spectral density: we consider a "peak range" for 32-50 quarters and test the null hypothesis of a flat spectral density against a "peak range".[13] We strongly reject at 5 percent

[13]The null hypothesis can be view as an uninformative prior for the spectral density at that periodicity range. Following Beaudry *et al.* [5] and the traditional view of a bell-shaped spectral density of most macroeconomic variables at low frequencies (Granger, [18]), we also test the null hypothesis that the spectral density at a given frequency interval is inherited from the determination of the cyclical component of a persistent autoregressive process of order 1. At standard significance levels, we reject such a null hypothesis. Detailed results and codes are available upon request.

level that the spectrum is flat in the "peak range". This result is robust when considering a narrow "peak range". Accordingly, after filtering for the long-run component, the adjusted variables are predominantly dominated by medium-term fluctuations. While there are also other local peaks which correspond to traditional business cycles, we then conclude that on average, the peaks for all the variables correspond to a periodicity of between 8 and 12 years. This range is our benchmark and we will compare it with the predictions of our model in Section 6.

3 The model

3.1 The production structure

We consider an economy producing a pure consumption good y_0 and a mixed good y_1 which is both consumed and used as an investment good, and a pure investment good y_2. Each good is assumed to be produced by capital k_1^j, k_2^j and labor l^j, $j = 0, 1, 2$, through Cobb-Douglas production functions. The representative firms in each industry indeed face the following technologies:

$$\begin{aligned} y_0 &= A_0(k_1^0)^{\alpha_1}(k_2^0)^{\alpha_2}(l^0)^{1-\alpha_1-\alpha_2}, \\ y_1 &= A_1(k_1^1)^{\beta_1}(k_2^1)^{\beta_2}(l^1)^{1-\beta_1-\beta_2}, \\ y_2 &= A_2(k_1^2)^{\gamma_1}(k_2^2)^{\gamma_2}(l^2)^{1-\gamma_1-\gamma_2} \end{aligned} \tag{1}$$

with A_j the TFP of sector $j = 0, 1, 2$. Total labor is given by $1 = l^0 + l^1 + l^2$,[14] and the total stocks of capital are given by $k_1 = k_1^0 + k_1^1 + k_1^2$ and $k_2 = k_2^0 + k_2^1 + k_2^2$.

A firm in each industry maximizes its profit given the output prices p_0, p_1 and p_2, the rental rates of capital r_1 and r_2, and the wage rate w. Choosing the consumption good as the numéraire, i.e. $p_0 = 1$, the first order conditions subject to the technologies (1) give the following input coefficients

$$\begin{array}{ccccc} a_{00}(w) = \frac{l^0}{y_0} = \frac{1-\alpha_1-\alpha_2}{w}, & & a_{10}(r_1) = \frac{k_1^0}{y_0} = \frac{\alpha_1}{r_1}, & & a_{20}(r_2) = \frac{k_2^0}{y_0} = \frac{\alpha_2}{r_2} \\ a_{01}(w, p_1) = \frac{l^1}{y_1} = \frac{p_1(1-\beta_1-\beta_2)}{w}, & & a_{11}(r_1, p_1) = \frac{k_1^1}{y_1} = \frac{p_1\beta_1}{r_1}, & & a_{21}(r_2, p_1) = \frac{k_2^1}{y_1} = \frac{p_1\beta_2}{r_2} \\ a_{02}(w, p_2) = \frac{l^2}{y_2} = \frac{p_2(1-\gamma_1-\gamma_2)}{w}, & & a_{12}(r_1, p_2) = \frac{k_1^2}{y_2} = \frac{p_2\gamma_1}{r_1}, & & a_{22}(r_2, p_2) = \frac{k_2^2}{y_2} = \frac{p_2\gamma_2}{r_2} \end{array} \tag{2}$$

It may be easily shown that the factor-price frontier and the factor market clearing equation depends on the input coefficients:[15]

[14] We assume in our model that labor is exogenous. Considering endogenous labor would complexify significantly the analysis without providing different conclusions.

[15] See Benhabib and Nishimura [6].

Lemma 1. *Denote* $p = (1, p_1, p_2)'$, $\omega = (w, r_1, r_2)'$ *and*

$$\mathcal{A}(\omega, p) = \begin{pmatrix} a_{00}(w) & a_{01}(w, p_1) & a_{02}(w, p_2) \\ a_{10}(r_1) & a_{11}(r_1, p_1) & a_{12}(r_1, p_2) \\ a_{20}(r_2) & a_{21}(r_2, p_1) & a_{22}(r_2, p_2) \end{pmatrix}$$

Then $p = \mathcal{A}'(\omega, p)\omega$ *and* $dp = \mathcal{A}'(\omega, p)d\omega$.

Lemma 2. *Denote* $x = (1, k_1, k_2)'$ *and* $y = (y_0, y_1, y_2)'$. *Then* $\mathcal{A}(\omega, p)y = x$ *and*

$$\mathcal{A}(w, p)dy + \begin{pmatrix} \left(\frac{\partial a_{00}}{\partial w} + \frac{\partial a_{01}}{\partial w} + \frac{\partial a_{02}}{\partial w}\right) dw + \frac{\partial a_{01}}{\partial p_1} dp_1 + \frac{\partial a_{02}}{\partial p_2} dp_2 \\ \left(\frac{\partial a_{10}}{\partial w} + \frac{\partial a_{11}}{\partial w} + \frac{\partial a_{12}}{\partial w}\right) dw + \frac{\partial a_{11}}{\partial p_1} dp_1 + \frac{\partial a_{12}}{\partial p_2} dp_2 \\ \left(\frac{\partial a_{20}}{\partial w} + \frac{\partial a_{21}}{\partial w} + \frac{\partial a_{22}}{\partial w}\right) dw + \frac{\partial a_{21}}{\partial p_1} dp_1 + \frac{\partial a_{22}}{\partial p_2} dp_2 \end{pmatrix} = dx$$

We derive that at the equilibrium, the wage rate and the rental rates are functions of the output prices only, i.e. $w = w(p_1, p_2)$, $r_1 = r_1(p_1, p_2)$ and $r_2 = r_2(p_1, p_2)$, while outputs are functions of the capital stocks and the output prices, $y_j = y_j(k_1, k_2, p_1, p_2)$, $j = 0, 1, 2$.

As usual in multi-sector optimal growth models, there is a duality between the Rybczinski and Stolper-Samuelson effects. Indeed, denoting

$$\left[\frac{\partial y}{\partial k}\right] = \begin{pmatrix} \frac{\partial y_1}{\partial k_1} & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} \end{pmatrix} \text{ and } \left[\frac{\partial r}{\partial p}\right] = \begin{pmatrix} \frac{\partial r_1}{\partial p_1} & \frac{\partial r_1}{\partial p_2} \\ \frac{\partial r_2}{\partial p_1} & \frac{\partial r_2}{\partial p_2} \end{pmatrix}$$

we have

$$\left[\frac{\partial y}{\partial k}\right] = \left[\frac{\partial r}{\partial p}\right]^t \tag{3}$$

3.2 Intertemporal equilibrium and steady state

The economy is populated by a large number of identical infinitely-lived agents. We assume without loss of generality that the total population is constant and normalized to one. At each period a representative agent supplies inelastically one unit of labor and derives utility from consuming the pure consumption good c_0 and the mixed good c_1 according to the following function

$$u(c_0, c_1) = \frac{(c_0^\theta c_1^{1-\theta})^{1-\sigma}}{1-\sigma}$$

with $\sigma \geq 0$ and $\theta \in (0, 1]$. The parameter θ measures the share of good c_0 into total utility.[16] We can easily compute the following elasticities

[16] We could also consider a model with three goods consumed but this would introduce useless complexities that do not allow however to get additional conclusions.

$$\begin{aligned}
&\varepsilon_{00} = -\tfrac{u_1}{u_{11}c_0} = \tfrac{1}{1-\theta(1-\sigma)},\ \varepsilon_{01} = -\tfrac{u_1}{u_{12}c_1} = -\tfrac{1}{(1-\theta)(1-\sigma)},\\
&\varepsilon_{10} = -\tfrac{u_2}{u_{21}c_0} = -\tfrac{1}{\theta(1-\sigma)},\ \varepsilon_{11} = -\tfrac{u_2}{u_{22}c_1} = \tfrac{1}{1-(1-\theta)(1-\sigma)}
\end{aligned} \tag{4}$$

ε_{00} and ε_{11} are the elasticities of intertemporal substitution in consumption goods c_0 and c_1. It is important to notice that even in the case of a non strictly concave utility with $\sigma = 0$, ε_{00} and ε_{11} remain finite as long as $\theta < 1$. Of course, if $\theta = 1$ we are in the standard case of a unique consumption good and the elasticity of intertemporal substitution in consumption is given by $\varepsilon_{00} = 1/\sigma$ and is infinite when the utility if linear.

Profit maximization in both sectors described in Section 3.1 gives demands for capital and labor as functions of the capital stocks and the production levels of the investment goods, namely $l^j = l^j(k_1, k_2, y_1, y_2)$, $k_1^j = k_1^j(k_1, k_2, y_1, y_2)$ and $k_2^j = k_2^j(k_1, k_2, y_1, y_2)$, $j = 0, 1, 2$. The optimal amount of the pure consumption good is then defined as

$$\begin{aligned}
c_0 &= T(k_1, k_2, y_1, y_2)\\
&= A_0(k_1^0(k_1, k_2, y_1, y_2))^{\alpha_1}(k_2^0(k_1, k_2, y_1, y_2))^{\alpha_2}(l^0(k_1, k_2, y_1, y_2))^{1-\alpha_1-\alpha_2}
\end{aligned}$$

From the envelope theorem we easily get $r_1 = T_{k_1}(k_1, k_2, y_1, y_2)$, $r_2 = T_{k_2}(k_1, k_2, y_1, y_2)$, $p_1 = -T_{y_1}(k_1, k_2, y_1, y_2)$ and $p_2 = -T_{y_2}(k_1, k_2, y_1, y_2)$.

The intertemporal optimization problem of the representative agent can be described as:

$$\begin{aligned}
\max_{\{c_0(t),c_1(t),k_1(t),k_2(t),y_1(t),y_2(t)\}} \quad & \int_0^{+\infty} \frac{\left[c_0(t)^\theta c_1(t)^{1-\theta}\right]^{1-\sigma}}{1-\sigma} e^{-\delta t} dt\\
s.t. \quad & c_0(t) = T(k_1(t), k_2(t), y_1(t), y_2(t))\\
& \dot{k}_1(t) = y_1(t) - gk_1(t) - c_1(t)\\
& \dot{k}_2(t) = y_2(t) - gk_2(t)\\
& k_1(0), k_2(0)\ given
\end{aligned} \tag{5}$$

where $\delta \geq 0$ is the discount rate and $g > 0$ is the depreciation rate of the capital stock which is assumed to be the same for the two investment goods. Substituting the expression of the pure consumption good $c_0(t)$ into the utility function, we can write the modified Hamiltonian in current value as:

$$\begin{aligned}
\mathcal{H} &= \tfrac{\left[T(k_1(t),k_2(t),y_1(t),y_2(t))^\theta c_1(t)^{1-\theta}\right]^{1-\sigma}}{1-\sigma} + q_1(t)\left(y_1(t) - gk_1(t) - c_1(t)\right)\\
&+ q_2(t)\left(y_2(t) - gk_2(t)\right)
\end{aligned}$$

The necessary conditions which describe the solution to problem (5) are therefore given by the following equations:

$$
\begin{aligned}
q_1(t) &= p_1(t)\theta c_0(t)^{\theta(1-\sigma)-1}c_1(t)^{(1-\theta)(1-\sigma)} && (6)\\
q_2(t) &= p_2(t)\theta c_0(t)^{\theta(1-\sigma)-1}c_1(t)^{(1-\theta)(1-\sigma)} && (7)\\
q_1(t) &= (1-\theta)c_0(t)^{\theta(1-\sigma)}c_1(t)^{(1-\theta)(1-\sigma)-1} && (8)\\
\dot{k}_1(t) &= y_1(t) - gk_1(t) - c_1(t) && (9)\\
\dot{k}_2(t) &= y_2(t) - gk_2(t) && (10)\\
\dot{q}_1(t) &= (\delta+g)q_1(t) - r_1(t)\theta c_0(t)^{\theta(1-\sigma)-1}c_1(t)^{(1-\theta)(1-\sigma)} && (11)\\
\dot{q}_2(t) &= (\delta+g)q_2(t) - r_2(t)\theta c_0(t)^{\theta(1-\sigma)-1}c_1(t)^{(1-\theta)(1-\sigma)} && (12)
\end{aligned}
$$

As shown in Section 3.1, we have $r_1 = r_1(p_1,p_2)$, $r_2 = r_2(p_1,p_2)$, $y_j = y_j(k_1,k_2,p_1,p_2)$, $j=1,2$, and thus $c_0 = y_0 = y_0(k_1,k_2,p_1,p_2) = T(k_1,k_2,y_1(k_1,k_2,p_1,p_2),y_2(k_1,k_2,p_1,p_2)) = c_0(k_1,k_2,p_1,p_2)$. Using (6) and (8) we derive:

$$
c_1(t) = c_1(k_1(t),k_2(t),p_1(t),p_2(t)) = c_0(k_1(t),k_2(t),p_1(t),p_2(t))\tfrac{1-\theta}{\theta p_1(t)} \tag{13}
$$

Obviously, if $\theta = 1$, we get $c_1(t) = 0$ for any $t \geq 0$. Straightforward computations then yield:

$$
\tfrac{\partial c_1}{\partial k_1} = \tfrac{1-\theta}{\theta p_1}\tfrac{\partial c_0}{\partial k_1},\quad \tfrac{\partial c_1}{\partial k_2} = \tfrac{1-\theta}{\theta p_1}\tfrac{\partial c_0}{\partial k_2},\quad \tfrac{\partial c_1}{\partial p_1} = \tfrac{1-\theta}{\theta p_1}\tfrac{\partial c_0}{\partial p_1} - \tfrac{c_1}{p_1},\quad \tfrac{\partial c_1}{\partial p_2} = \tfrac{1-\theta}{\theta p_1}\tfrac{\partial c_0}{\partial p_2} \tag{14}
$$

Considering (6)-(12) and (14), and denoting

$$
\begin{aligned}
E(k_1,k_2,p_1,p_2) &\equiv 1 - \tfrac{[1-\theta(1-\sigma)]\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)}{c_0} + \tfrac{(1-\theta)(1-\sigma)\left(p_1\frac{\partial c_1}{\partial p_1}+p_2\frac{\partial c_1}{\partial p_2}\right)}{c_1}\\
&= \theta + \sigma\left[1-\theta-\tfrac{1}{c_0}\left(p_1\tfrac{\partial c_0}{\partial p_1}+p_2\tfrac{\partial c_0}{\partial p_2}\right)\right] \equiv E^\theta_\sigma
\end{aligned} \tag{15}
$$

the equations of motion are derived as

$$
\begin{aligned}
\dot{k}_1 &= y_1(k_1,k_2,p_1,p_2) - gk_1 - c_1(k_1,k_2,p_1,p_2)\\
\dot{k}_2 &= y_2(k_1,k_2,p_1,p_2) - gk_2\\
\dot{p}_1 &= \tfrac{1}{E(k_1,k_2,p_1,p_2)}\Big[(\delta+g)p_1 - r_1(p_1,p_2) + \sigma\tfrac{\partial c_0}{\partial p_2}\tfrac{r_1(p_1,p_2)p_2 - r_2(p_1,p_2)p_1}{c_0}\\
&+ \sigma\tfrac{p_1}{c_0}\Big(\tfrac{\partial c_0}{\partial k_1}\left(y_1(k_1,k_2,p_1,p_2) - gk_1 - c_1(k_1,k_2,p_1,p_2)\right) + \tfrac{\partial c_0}{\partial k_2}\left(y_2(k_1,k_2,p_1,p_2) - gk_2\right)\Big)\Big]\\
\dot{p}_2 &= \tfrac{1}{E(k_1,k_2,p_1,p_2)}\Big[(\delta+g)p_2 - r_2(p_1,p_2) + \tfrac{[r_2(p_1,p_2)p_1 - r_1(p_1,p_2)p_2]\left(\sigma\frac{\partial c_0}{\partial p_1}+(1-\theta)(1-\sigma)\frac{c_0}{p_1}\right)}{c_0}\\
&+ \sigma\tfrac{p_2}{c_0}\Big(\tfrac{\partial c_0}{\partial k_1}\left(y_1(k_1,k_2,p_1,p_2) - gk_1 - c_1(k_1,k_2,p_1,p_2)\right) + \tfrac{\partial c_0}{\partial k_2}\left(y_2(k_1,k_2,p_1,p_2) - gk_2\right)\Big)\Big]
\end{aligned} \tag{16}
$$

Any solution $\{k_1(t),k_2(t),p_1(t),p_2(t)\}_{t\geq 0}$ that also satisfies the transversality conditions

$$
\lim_{t\to+\infty} e^{-\delta t}q_1(t)k_1(t) = 0 \text{ and } \lim_{t\to+\infty} e^{-\delta t}q_2(t)k_2(t) = 0
$$

with $q_1(t)$ and $q_2(t)$ as given by (6) and (7), is called an equilibrium path.
A steady state is defined by a vector $(c_1^*,k_1^*,k_2^*,p_1^*,p_2^*)$ solution of

$$\begin{aligned} y_1(k_1,k_2,p_1,p_2) &= gk_1 + c_1 = gk_1 + c_0(k_1,k_2,p_1,p_2)\tfrac{1-\theta}{\theta p_1} \\ y_2(k_1,k_2,p_1,p_2) &= gk_2 \\ r_1(p_1,p_2) &= (\delta+g)p_1 \\ r_2(p_1,p_2) &= (\delta+g)p_2 \end{aligned} \tag{17}$$

We get the following result:

Proposition 1. *There exists a unique steady state* $(c_1^*,k_1^*,k_2^*,p_1^*,p_2^*)>0$ *with* $c_1^*=c_1(k_1^*,k_2^*,p_1^*,p_2^*)$ *as derived from (13).*

Proof. See Appendix 8.1

The value of the pure consumption level at the steady state is obviously obtained as $c_0^* = c_0(k_1^*,k_2^*,p_1^*,p_2^*)$.

3.3 Characteristic polynomial

Linearizing the dynamical system around $(c_1^*,k_1^*,k_2^*,p_1^*,p_2^*)$ gives a 4×4 Jacobian matrix $\mathcal{J}$ which is provided in Appendix 8.2. Let us denote $\mathcal{T}$ the sum of minors of order one, $\mathcal{S}_\sigma$ the sum of minors of order two, Σ_σ the sum of minors of order three and $\mathcal{D}_\sigma$ the determinant of $\mathcal{J}$. The following Proposition provides some properties of the eigenvalues of $\mathcal{J}$ and the expression of the characteristic polynomial:

Proposition 2. *Consider* E_σ^θ *as given by (15). If* λ *is an eigenvalue of the Jacobian matrix* $\mathcal{J}$*, then* $\bar{\lambda}$*,* $\delta-\lambda$ *and* $\delta-\bar{\lambda}$ *are also eigenvalues and thus*

$$\mathcal{T}=2\delta \text{ and } \Sigma_\sigma^\theta = \mathcal{T}\tfrac{S_\sigma^\theta-\delta^2}{2} = \delta\left(S_\sigma^\theta-\delta^2\right).$$

The degree-4 characteristic polynomial is given by

$$\mathcal{P}_\sigma^\theta(\lambda) = \lambda^4 - \lambda^3 2\delta + \lambda^2\mathcal{S}_\sigma^\theta - \lambda\delta\left(S_\sigma^\theta-\delta^2\right) + \mathcal{D}_\sigma^\theta \tag{18}$$

or equivalently

$$\begin{aligned} \mathcal{P}_\sigma^\theta(\lambda) &= \left[\lambda^2 - \lambda\tfrac{\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right)}{\theta} + \tfrac{\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}}{\theta}\right] \\ &\times \left[\lambda^2 - \lambda\left(\tfrac{\partial y_1}{\partial k_1}-g-\tfrac{\partial c_1}{\partial k_1}+\tfrac{\partial y_2}{\partial k_2}-g\right) + \left(\tfrac{\partial y_1}{\partial k_1}-g-\tfrac{\partial c_1}{\partial k_1}\right)\left(\tfrac{\partial y_2}{\partial k_2}-g\right) - \left(\tfrac{\partial y_1}{\partial k_2}-\tfrac{\partial c_1}{\partial k_2}\right)\tfrac{\partial y_2}{\partial k_1}\right] \\ &- \tfrac{\sigma}{E_\sigma^\theta c_0}\tilde{\mathcal{P}}(\lambda) \\ &\equiv \mathcal{Q}_1^\theta(\lambda)\mathcal{Q}_2^\theta(\lambda) - \tfrac{\sigma}{E_\sigma^\theta c_0}\tilde{\mathcal{P}}(\lambda) \end{aligned}$$

with

$$\begin{aligned}
\tilde{\mathcal{P}}(\lambda) &= \lambda^2\left[\frac{\Gamma_\theta\left[1-\frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]}{\theta}+\Theta_\theta\right]-\lambda\delta\left[\frac{\Gamma_\theta\left[1-\frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]}{\theta}+\Theta_\theta\right]\\
&+\frac{\mathcal{D}_\sigma^\theta E_\sigma^\theta\left[(1-\theta)c_0-\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]}{\theta}\\
\mathcal{D}_\sigma^\theta &= \frac{\left[\left(\frac{\partial y_1}{\partial k_1}-g-\frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2}-g\right)-\left(\frac{\partial y_1}{\partial k_2}-\frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}\right]\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]}{E_\sigma^\theta}\\
\mathcal{S}_\sigma^\theta &= \mathcal{S}_0^\theta-\sigma\frac{\Gamma_\theta\left[1-\frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]+\theta\Theta_\theta}{\theta E_\sigma^\theta c_0}
\end{aligned}\tag{19}$$

and $\mathcal{S}_0^\theta$, Γ_θ, Θ_θ *given in Appendix 8.2. Moreover, one of the following cases necessarily holds:*

i) the four roots are real and distincts,

ii) the four roots are given by two pairs of non-real complex conjugates,

iii) there are two real double roots.

Proof. See Appendix 8.2

The results about the structure of the characteristic roots are in line with the conclusions of Kurz [22] and Levhari and Liviatan [23]. We provide however more precise results related to the consideration of a non-linear utility function. It is worth emphasizing that $\tilde{\mathcal{P}}(\lambda)$ does not depend on σ.[17] This property will be crucial to study the case of a strictly concave utility function with $\sigma > 0$.

4 The standard model with one consumption good

We start with the standard formulation where the household consumes one pure consumption good, i.e. $\theta = 1$, $c_1 = 0$ and $\partial c_1/\partial k_1 = \partial c_1/\partial k_2 = 0$. We will study separately two configurations depending on whether the utility function is linear or strictly concave.

4.1 The case of a linear utility

In the case of a linear utility function such that $\sigma = 0$ and $u(c_0, c_1) = c_0$, the characteristic polynomial simplifies as follows

$$\begin{aligned}
\mathcal{P}_0^1(\lambda) &= \left[\lambda^2-\lambda\left[2(\delta+g)-\frac{\partial y_1}{\partial k_1}-\frac{\partial y_2}{\partial k_2}\right]+\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]\\
&\times\left[\lambda^2-\lambda\left(\frac{\partial y_1}{\partial k_1}+\frac{\partial y_1}{\partial k_1}-2g\right)+\left(\frac{\partial y_1}{\partial k_1}-g\right)\left(\frac{\partial y_2}{\partial k_2}-g\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]\\
&\equiv \mathcal{Q}_1^1(\lambda)\mathcal{Q}_2^1(\lambda)
\end{aligned}$$

Denoting Δ_i^1 the discriminant of polynomial $\mathcal{Q}_i^1(\lambda)$, $i = 1, 2$, we easily get

[17] Notice indeed that the product $\mathcal{D}_\sigma^\theta E_\sigma^\theta$ does not depend on σ.

$$\begin{aligned}\Delta_1^1 = \Delta_2^1 = \Delta^1 &= \left[\frac{\partial y_1}{\partial k_1} - \frac{\partial y_2}{\partial k_2}\right]^2 + 4\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} \\ &= \left[2(\delta+g) - \frac{\partial y_1}{\partial k_1} - \frac{\partial y_2}{\partial k_2}\right]^2 \\ &- 4\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]\end{aligned} \tag{20}$$

Complex characteristic roots are therefore obtained if and only if $\Delta^1 < 0$. Moreover, the four characteristic roots are given by:

$$\lambda_{1,2} = \frac{2(\delta+g) - \frac{\partial y_1}{\partial k_1} - \frac{\partial y_2}{\partial k_2} \pm i\sqrt{-\Delta^1}}{2},\ \lambda_{3,4} = \frac{\frac{\partial y_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - 2g \pm i\sqrt{-\Delta^1}}{2} \tag{21}$$

Building on the paper of Benhabib and Nishimura [6], we do not need to provide a complete and general stability analysis of the steady state. We rather focus on the case with complex characteristic roots as our aim is to provide more precise conditions for the occurrence of endogenous fluctuations and a Hopf bifurcation.

Proposition 3. *When $\sigma = 0$ and $\theta = 1$, the characteristic roots are complex if and only if*

$$\begin{aligned}&[\alpha_2(1-\gamma_1) - \gamma_2(1-\alpha_1) + \beta_1(1-\alpha_2) - \alpha_1(1-\beta_2)]^2 \\ <\ &4[\alpha_2(1-\beta_1) - \beta_2(1-\alpha_1)][\gamma_1(1-\alpha_2) - \alpha_1(1-\gamma_2)]\end{aligned} \tag{22}$$

Moreover, under condition (22), the following results hold:

***1** - The steady state is saddle-point stable with oscillating convergence for any $\delta \geq 0$ if and only if*

$$\left(\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}\right)^2 > 1$$

***2** - If the following condition holds:*

$$\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \in (-1,1) \tag{23}$$

then the steady state is saddle-point stable with oscillating convergence when $\delta \in [0, \delta^*)$ and totally unstable when $\delta > \delta^*$, with

$$\delta^* = g\frac{\beta_2(\gamma_1-\alpha_1)-(1-\beta_1)(\alpha_2-\gamma_2)+(1-\gamma_2)(\beta_1-\alpha_1)-\gamma_1(\alpha_2-\beta_2)}{\alpha_2(1-\gamma_1)-\gamma_2(1-\alpha_1)+\alpha_1(1-\beta_2)-\beta_1(1-\alpha_2)} > 0 \tag{24}$$

Moreover, δ^* is a Hopf bifurcation value generically giving rise to non constant saddle-point stable (or unstable) closed orbits around the steady state in a right (or left) neighborhood of δ^*. When δ is close to δ^*, the period of the closed orbit is approximately equal to $T^* = \frac{2\pi}{\sqrt{-\Delta^1}}\Big|_{\delta=\delta^*}$.

Proof. See Appendix 8.3.

Before discussing the conditions on the technological parameters to get complex eigenvalues and a Hopf bifurcation, it is worth noting that we do provide an approximation of the cyclical periodicity. It depends on the size of the imaginary part of the bifurcating eigenvalues and thus on the technological coefficients: the larger the value of $|\Delta^1|$, the lower the period. To the best of our knowledge, beside Benhabib and Nishimura [6] who provide a theoretical

discussion, there is no paper in the literature that gives such a precise approximation of the periodicity of the cycle in terms of the fundamental characteristic of the model.
Going back to conditions (22) and (23), one can show that the condition

$$\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \in (-1,1)$$

is satisfied if

$$\frac{\partial y_1}{\partial k_1}+\frac{\partial y_2}{\partial k_2}>0$$

but is not too large, while condition (22) will hold if

$$\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}<0 \text{ and } \frac{\partial y_1}{\partial k_1}-\frac{\partial y_2}{\partial k_2} \text{ is close enough to zero.}$$

Using the expressions of the input coefficient (2) together with Lemma 8.1 provided in Appendix 8.3, we can show that these conditions can be satisfied under various sets of conditions on sectoral relative capital intensity differences. For instance, the following two sets of conditions are of particular interest:

- Set 1: Sector 1 is more capital intensive in capital good 2 than sector 0 which is itself more capital intensive in capital good 2 that sector 2, i.e. $\frac{k_2^1}{l^1}>\frac{k_2^0}{l^0}>\frac{k_2^2}{l^2}$. Sector 0 is more capital intensive in capital good 1 than sectors 1 and 2, i.e. $\frac{k_1^0}{l^0}>\frac{k_1^1}{l^1},\frac{k_1^2}{l^2}$. Moreover, the difference $\frac{k_2^0}{l^0}-\frac{k_2^2}{l^2}-\left(\frac{k_1^0}{l^0}-\frac{k_1^1}{l^1}\right)$ and the share of labor in sector 2, i.e. $1-\gamma_1-\gamma_2$, are small enough.

- Set 2: Sector 2 is more capital intensive in capital good 2 than sector 0 which is itself more capital intensive in capital good 2 that sector 1, i.e. $\frac{k_2^2}{l^2}>\frac{k_2^0}{l^0}>\frac{k_2^1}{l^{21}}$. Sectors 1 and 2 are more capital intensive in capital good 1 than sector 0, i.e. $\frac{k_1^1}{l^1},\frac{k_1^2}{l^2}>\frac{k_1^0}{l^0}$. Moreover, the difference $\frac{k_2^2}{l^2}-\frac{k_2^0}{l^0}-\left(\frac{k_1^1}{l^1}-\frac{k_1^0}{l^0}\right)$ and the share of labor in sector 2, i.e. $1-\gamma_1-\gamma_2$, are small enough.[18]

In both cases, note that the sector 0 of the pure consumption good is always more intensive in at least one capital good j than the sector of the capital good j itself. As shown by Benhabib and Nishimura [7], such a property is at the core of the mechanism leading to the existence of endogenous fluctuations. The use of the Rybczinski and Stolper-Samuelson effects indeed provides a simple economic intuition for this result. For instance, in the case of Set 1 in which the pure consumption good sector is more intensive in both capital stocks than the respective capital good sectors, suppose that there is an instantaneous increase in the capital stocks $k_1(t)$ and $k_2(t)$. This results in two opposing mechanisms:

- On the one hand, the trade-off in production becomes more favorable to the consumption good, and the Rybczinsky effect implies a decrease of the outputs of the capital goods $y_1(t)$ and $y_2(t)$. This tends to lower both the investments and the capital stocks in the subsequent period.

- On the other hand, in the subsequent period the decreases of both capital stocks imply again through the Rybczinsky effect an increase of the outputs of the capital goods.

[18] Similar conditions, though thus more restrictive, have been identified by Nishimura and Takahashi [33].

Indeed the decreases of the capital stocks improve the trade-off in production in favor of the investment goods which are relatively less intensive in their own capital and this tends to increase the investments and the capital stocks in following period. We then obtain endogenous fluctuations of stocks and outputs.

Of course, under both mechanisms, the Stolper-Samuelson effect generates corresponding fluctuations of the relative prices of both capital stocks. But then, the existence of persistent fluctuations requires that the oscillations in consumption and relative prices must not present intertemporal arbitrage opportunities. Consequently, a minimum level of the discount rate δ is thus necessary.

Remark: As clearly shown by the expression of the bifurcation value of the discount rate δ^*, for a given depreciation rate of capital g, choosing appropriately the coefficients of the Cobb-Douglas technologies would push δ^* as close to zero as desired. In this respect, we provide here a new proof of the main conclusion of Benhabib and Rustichini [9] showing that, for any positive discount rate, possibly arbitrarily close to zero, there exists a large family of standard Cobb-Douglas technologies with three sectors which have optimal growth paths of persistent cycles.

4.2 The case of a strictly concave utility

Assuming now that $\sigma > 0$. Since $\theta = 1$ implies $\Gamma_1 = 0$ (see Appendix 8.2), our analysis is built on the property that the characteristic polynomial can be written as

$$\begin{aligned}
\mathcal{P}_\sigma^1(\lambda) &= \left[\lambda^2 - \lambda\left(\delta + g - \tfrac{\partial y_1}{\partial k_1} + \delta + g - \tfrac{\partial y_2}{\partial k_2}\right) + \left(\delta + g - \tfrac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \tfrac{\partial y_2}{\partial k_2}\right) - \tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1}\right] \\
&\times \left[\lambda^2 - \lambda\left(\tfrac{\partial y_1}{\partial k_1} - g + \tfrac{\partial y_2}{\partial k_2} - g\right) + \left(\tfrac{\partial y_1}{\partial k_1} - g\right)\left(\tfrac{\partial y_2}{\partial k_2} - g\right) - \tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1}\right] \\
&- \tfrac{\sigma}{E_\sigma^1 c_0}\tilde{\mathcal{P}}(\lambda) \\
&\equiv \mathcal{Q}_1^1(\lambda)\mathcal{Q}_2^1(\lambda) - \tfrac{\sigma}{E_\sigma^1 c_0}\tilde{\mathcal{P}}(\lambda)
\end{aligned}$$

with

$$\tilde{\mathcal{P}}(\lambda) = \lambda^2\Theta_1 - \lambda\delta\Theta_1 - \mathcal{D}_\sigma^1 E_\sigma^1\left(p_1\tfrac{\partial c_0}{\partial p_1} + p_2\tfrac{\partial c_0}{\partial p_2}\right)$$

and Θ_1 given in Appendix 8.2. Equivalently, any characteristic root λ must be a solution of

$$\mathcal{Q}_1^1(\lambda)\mathcal{Q}_2^1(\lambda) = \tfrac{\sigma}{E_\sigma^1 c_0}\tilde{\mathcal{P}}(\lambda) \text{ or } \mathcal{P}_0^1(\lambda) = \tfrac{\sigma}{E_\sigma^1 c_0}\tilde{\mathcal{P}}(\lambda)$$

Starting from the conditions established by Proposition 3 under which complex roots arise when $\sigma = 0$ as solutions of $\mathcal{P}_0^1(\lambda) = 0$, the strategy consists in analyzing the properties of $\tilde{\mathcal{P}}(\lambda)$ and how the right-hand-side $\sigma\tilde{\mathcal{P}}(\lambda)/(E_\sigma^1 c_0)$ evolves when σ increases from zero.

We first state an important technical property which holds for any $\theta \in (0, 1]$ and $\sigma \geq 0$:

Lemma 3. *Let $\theta \in (0, 1]$ and $\sigma \geq 0$. For any technological parameters $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2$, $p_1\frac{\partial c}{\partial p_1} + p_2\frac{\partial c}{\partial p_2} \leq 0$ and $E_\sigma^\theta > 0$ is an increasing function of σ.*

Proof. See Appendix 8.4.

We derive from this Lemma:

Proposition 4. *Under $\theta = 1$, let condition (23) hold and*

$$\begin{aligned} & [\beta_2(\gamma_1 - \alpha_1) + (1-\beta_1)(\gamma_2 - \alpha_2)]^2 + [\gamma_1(\beta_2 - \alpha_2) + (1-\gamma_2)(\beta_1 - \alpha_1)]^2 \\ < \quad & 4[\alpha_2(1-\beta_1) - \beta_2(1-\alpha_1)][\gamma_1(1-\alpha_2) - \alpha_1(1-\gamma_2)] \end{aligned} \tag{25}$$

Then, there exists $\bar{\sigma} > 0$ such that the characteristic roots are complex if and only if $\sigma \in [0, \bar{\sigma})$. Moreover, there exists $\bar{\delta} > \delta^*$ such that when $\delta \in [0, \bar{\delta})$ and $\sigma = \bar{\sigma}$, the four characteristic roots are given by two pairs of real double roots with

$$\lambda_1 = \frac{\delta + \sqrt{3\delta^2 - 2S^1_{\bar{\sigma}}}}{2} > 0 \text{ and } \lambda_2 = \frac{\delta - \sqrt{3\delta^2 - 2S^1_{\bar{\sigma}}}}{2} = \delta - \lambda_1 < 0$$

Proof. See Appendix 8.5.

Conditions (23) and (25) ensure that $\Theta_1 > 0$, while condition (25) also ensures that (22) is satisfied and that the characteristic roots are complex.

Starting from the case $\sigma = 0$ in which all characteristic roots are complex, this allows us to conclude that increasing σ conducts the degree-two polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E^1_\sigma c_0)$ to go up monotonically and to come closer and closer to $\mathcal{P}^1_0(\lambda)$ until it has two tangency points with $\mathcal{P}^1_0(\lambda)$. This occurs when $\sigma = \bar{\sigma}$, and thus the imaginary part of the four roots is equal to zero and there exist two pairs of double real roots, one being positive and the other negative. Figure 5 displays how the polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E^1_\sigma c_0)$ (red curve) moves as σ increases, as well as $\mathcal{P}^1_0(\lambda)$ (blue curve) and the two tangency points (right panel).

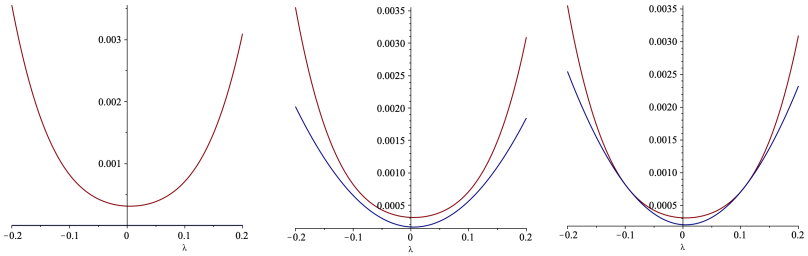


Figure 5: $\mathcal{P}^1_\sigma(\lambda)$ when $\sigma = 0$, $\sigma \in (0, \bar{\sigma})$ and $\sigma = \bar{\sigma}$

We can then state the following Theorem:

Theorem 1. *Let $\theta = 1$. Under conditions (23) and (25), the following two cases hold:*

i) if $\delta \in [0, \delta^*)$, the steady state is saddle-point stable with oscillating convergence for any $\sigma \in [0, \bar{\sigma})$;

ii) if $\delta > \delta^*$, there exists $\sigma^* \in (0, \bar{\sigma})$ such that the steady state is totally unstable when $\sigma \in [0, \sigma^*)$ and saddle-point stable with oscillating convergence when $\sigma \in (\sigma^*, \bar{\sigma})$. Moreover, σ^* is a Hopf bifurcation value giving rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a left (right) neighborhood of σ^*.

Proof. See Appendix 8.6.

Theorem 1 shows that a Hopf bifurcation value for the parameter σ may exist when we start from a configuration of total instability with $\sigma = 0$. While we cannot explicitly

compute the period of the cycles as in the case of a linear utility function, it is quite clear that this period will depend on the imaginary part of the associated bifurcating eigenvalues.

However, some numerical explorations show that the value of σ^* leading to the existence of periodic cycles is extremely close to zero, implying a large elasticity of intertemporal substitution in consumption which is not in line with empirical evidences. This means that the conclusion of Rockafellar [35] implies the saddle-point property of the steady state from a very weak degree of concavity for the utility function.[19] To get the existence of endogenous fluctuations in a realistic framework, it is therefore necessary to explore a more general formulation with two consumption goods.

5 The model with two consumption goods

We consider now the more general model with two consumption goods, i.e. $\theta \in (0,1)$. As shown by (4), even in the case of a non strictly concave utility with $\sigma = 0$, the elasticities of intertemporal substitution for the two goods ε_{00} and ε_{11} remain finite. We will proceed as in the previous Section considering first the case $\sigma = 0$ with a non strictly concave utility function, i.e. homogeneous of degree one.

5.1 The case of a non strictly concave utility

Assuming $\sigma = 0$, we get an homogeneous of degree one utility function $u(c_0, c_1) = c_0^{\theta} c_1^{1-\theta}$ which is concave non strictly. The characteristic polynomial becomes now

$$\begin{aligned}
\mathcal{P}_0^{\theta}(\lambda) &= \left[\lambda^2 - \lambda \frac{\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right)}{\theta} + \frac{\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}}{\theta}\right] \\
&\times \left[\lambda^2 - \lambda\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - g\right) + \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}\right] \\
&\equiv \mathcal{Q}_1^{\theta}(\lambda)\mathcal{Q}_2^{\theta}(\lambda)
\end{aligned}$$

with $\partial c_1/\partial k_1$ and $\partial c_1/\partial k_2$ as given by (14). Denoting Δ_i^{θ} the discriminant of polynomial $i = 1, 2$, we easily get as in the previous case

$$\begin{aligned}
\Delta_1^{\theta} = \Delta_2^{\theta} = \Delta^{\theta} &= \frac{\left[\theta\left(\frac{\partial y_1}{\partial k_1}-\frac{\partial y_2}{\partial k_2}\right)-\frac{1-\theta}{p_1}\frac{\partial c_0}{\partial k_1}\right]^2+4\theta\left(\theta\frac{\partial y_1}{\partial k_2}-\frac{1-\theta}{p_1}\frac{\partial c_0}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}}{\theta^2} \\
&= \frac{\left[\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right)\right]^2}{\theta^2} \\
&- \frac{4\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]}{\theta}
\end{aligned} \tag{26}$$

Complex characteristic roots are therefore obtained if and only if $\Delta^{\theta} < 0$. Moreover, we easily compute the four characteristic roots:

[19] Benhabib and Rustichini [9] also prove using a perturbation method that the existence of persistent cycles is compatible with a non-linear utility function. Our Theorem 1 provides a much more precise result but we confirm that a very weak degree of concavity of the utility function is enough to restore the saddle-point property.

$$\begin{aligned}\lambda_{1,2} &= \frac{\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right)\pm\sqrt{\Delta^\theta}}{2\theta} \\ \lambda_{3,4} &= \frac{\theta\left(\frac{\partial y_1}{\partial k_1}+\frac{\partial y_2}{\partial k_2}-2g\right)-\frac{1-\theta}{p_1}\frac{\partial c_0}{\partial k_1}\pm\sqrt{\Delta^\theta}}{2\theta}\end{aligned} \tag{27}$$

As in the previous case, we focus only on the existence of complex roots. We will build our analysis on the basis of the conditions identified in Proposition 3.

Proposition 5. *When $\sigma = 0$ and $\theta \in (0,1]$, there exists $\underline{\theta} \in (0,1)$ such that the characteristic roots are complex if and only if condition (22) holds and $\theta \in (\underline{\theta}, 1]$. Consider then the bound δ^* as given in Proposition 3. Under conditions (22) and (23), let $\theta \in (\underline{\theta}, 1]$ and*

$$\frac{\beta_2(1-\gamma_1)-\gamma_2(1-\beta_1)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} > 0 \tag{28}$$

Then the following results hold:

1 - *If $\delta > \delta^*$, the steady is saddle-point stable with oscillating convergence when $\theta \in (\underline{\theta}, \theta^*)$ and totally unstable when $\theta \in (\theta^*, 1]$, with*

$$\theta^* = \frac{\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)}{(\beta_1-\alpha_1)(1-\gamma_2)+(\beta_2-\alpha_2)(1-\gamma_1)+\beta_1\alpha_2-\alpha_1\beta_2} \in (\underline{\theta}, 1) \tag{29}$$

Moreover, θ^* is a Hopf bifurcation value generically giving rise to non constant saddle-point stable (or unstable) closed orbits around the steady state in a right (or left) neighborhood of θ^*. When θ is close to θ^*, the period of the closed orbit is approximately equal to $T^* = \frac{2\pi}{\sqrt{-\Delta^{\theta^*}}}$.

2 - *If $\delta \in [0, \delta^*)$, there exists $\hat{\theta} \in (\theta^*, 1)$ such that the steady is saddle-point stable with oscillating convergence when $\theta \in (\hat{\theta}, 1]$, totally unstable when $\theta \in (\theta^*, \hat{\theta})$ and again saddle-point stable when $\theta \in (\underline{\theta}, \theta^*)$. Moreover, θ^* and $\hat{\theta}$ are respectively Hopf bifurcation values generically giving rise to non constant saddle-point stable (or unstable) closed orbits around the steady state in a right (or left) neighborhood of θ^* and in a left (or right) neighborhood of $\hat{\theta}$. When θ is close to θ^* or $\hat{\theta}$, the periods of the closed orbits are approximately equal to $T^* = \frac{2\pi}{\sqrt{-\Delta^{\theta^*}}}$ or $\hat{T} = \frac{2\pi}{\sqrt{-\Delta^{\hat{\theta}}}}$.*

Proof. See Appendix 8.7.

This Proposition is important as it provides new results showing that the structure of the consumption bundle has a crucial impact on the possible existence of endogenous flutuations. We prove indeed that if the share of the mixed good into the utility function is large enough, a Hopf bifurcation giving rise to persistent fluctuations may arise. This requires that the additional condition (28) holds. Indeed, the inequality (28) implies that the bifurcation value θ^* exists and that $\partial c_0/\partial k_1 < 0$. On top of that, depending on the value of the discount rate δ, a second bifurcation values $\hat{\theta}$ may exist.

Beside the technological mechanism discussed in the case with a unique consumption good, the consideration of a second consumption good in the utility function provides an additional mechanism driving fluctuations. Let us for instance consider an instantaneous increase in the capital stock $k_1(t)$. Since $\partial c_0/\partial k_1 < 0$, it follows that $c_0(t)$ decreases, and thus, building of the substitution between $c_0(t)$ and $c_1(t)$, a constant utility level can be

obtained from an increase of $c_1(t)$. Using then the capital 1 accumulation equation (9) and taking $y_1(t)$ as given, we conclude that if the share of consumption of the mixed good is large enough, the increase of $c_1(t)$ is such that the capital stock will decrease in the subsequent period. But then, building again on $\partial c_0/\partial k_1 < 0$, we find that $c_0(t)$ now increases while $c_1(t)$ decreases from the substitutability properties, generating then an increase of the capital stock in the following period. Endogenous fluctuations are thus generated from intertemporal consumption allocations based on some substitution effects between the two consumption goods.

It is also important to note that if the discount rate δ is too small, the oscillations in consumption and relative prices will not present intertemporal arbitrage opportunities provided the above mechanism is not too strong, i.e. provided the share of consumption of the mixed good into total utility is not too large. This explains that in this case only intermediary values of θ are compatible with persistent fluctuations.

As shown by formulas (4), we also conclude that endogenous fluctuations can occur with a finite elasticity of intertemporal substitution in consumption even under a non strictly concave utility function. Moreover, as in the case with a unique consumption good, we provide an approximation of the period of the cycles in both cases covered by the Proposition. It depends of course on the size of the imaginary part of the bifurcating eigenvalues and thus both on the technological coefficients and on the share θ: the larger the value of $|\Delta^\theta|$, the lower the period. This result will allow us to compare the theoretical periodicity of the cycle with the one obtained from the data in Section 2.

We need however to study the robustness of these conclusions when $\sigma > 0$.

5.2 The case of a strictly concave utility

Let us assume finally a strictly concave utility function with $\sigma > 0$. We proceed as in the case with $\theta = 1$. Starting from the case $\sigma = 0$ and the appropriate value of θ in which the characteristic roots are all complex, if σ is increased, the degree-two polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E^\theta_\sigma c_0)$ is going up monotonically and comes closer and closer to $\mathcal{P}^1_\theta(\lambda)$ until it is characterized when $\sigma = \bar{\sigma}$ by two tangency points with $\mathcal{P}^1_\theta(\lambda)$ where the imaginary part of the four roots is equal to zero and there exist two pairs of double real roots, one being positive and the other negative (see Figure 1). We then get the following result:

Theorem 2. *Consider the bounds δ^*, $\underline{\theta}$, θ^* and $\hat{\theta}$ as given respectively in Propositions 3 and 5. Under conditions (23), (25) and (28), there exists $\tilde{\theta} \in [\underline{\theta}, \theta^*)$ and $\bar{\sigma}_\theta > 0$ such that for any given $\theta \in (\tilde{\theta}, 1]$, the characteristic roots are complex if and only if $\sigma \in [0, \bar{\sigma}_\theta)$. Moreover, the following cases hold:*

i) if $\delta \in [0, \delta^*)$, for any given $\theta \in (\tilde{\theta}, \theta^*) \cup (\hat{\theta}, 1]$, the steady state is saddle-point stable with oscillating convergence for any $\sigma \in [0, \bar{\sigma}_\theta)$. However, if $\theta \in (\theta^*, \hat{\theta})$, there exists $\sigma^*_\theta \in (0, \bar{\sigma}_\theta)$ such that the steady state is totally unstable when $\sigma \in [0, \sigma^*_\theta)$ and saddle-point stable with oscillating convergence when $\sigma \in (\sigma^*_\theta, \bar{\sigma}_\theta)$. Moreover, σ^*_θ is a Hopf bifurcation value giving rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a left (right) neighborhood of σ^*_θ.

ii) if $\delta > \delta^*$, for any given $\theta \in (\theta^*, 1]$, the steady state is saddle-point stable with oscillating convergence for any $\sigma \in [0, \bar{\sigma}_\theta)$. However, if $\theta \in (\tilde{\theta}, \theta^*)$, there exists $\sigma_\theta^* \in (0, \bar{\sigma}_\theta)$ such that the steady state is totally unstable when $\sigma \in [0, \sigma_\theta^*)$ and saddle-point stable with oscillating convergence when $\sigma \in (\sigma_\theta^*, \bar{\sigma}_\theta)$. Moreover, σ_θ^* is a Hopf bifurcation value giving rise to non-constant saddle-point stable (or unstable) closed orbits around the steady state in a left (right) neighborhood of σ_θ^*.

Proof. See Appendix 8.8.

As in the case with a unique consumption good, this Theorem shows that the conclusions of Proposition 5 still hold with a strictly concave utility function. We can find indeed a bifurcation value for σ associated to the existence of a Hopf bifurcation. However, as we will show with the following numerical illustration, this bifurcation value remains close to zero and thus associated to a weakly concave utility function. But contrary to the case with a unique consumption good, the more complex structure of the consumption bundle allows to keep a compatibility between endogenous business cycles and a realistic elasticity of intertemporal substitution in consumption. While we cannot explicitly compute the period of the cycles as in the case of an homogeneous of degree one utility function, it is obvious that this period will depend on the imaginary part of the associated bifurcating eigenvalues. We will numerically evaluate it in the numerical illustration.

6 A numerical illustration

The objective of this Section is to provide a numerical illustration with parameters values in line with the empirical evidence and which proves the existence of endogenous cycles with a periodicity compatible with the estimates provided in Section 2. We prove indeed the existence of a recurrent peak in several spectral densities in many US trendless macroeconomic data suggesting on average the presence of periodicities between 8 and 12 years.

The first step is to consider empirically realistic values for the technological coefficients. It is not easy to get empirical values for sectoral capital shares in a three-sector model. We know from the contribution of Baxter [1] that in a two-sector framework the share of capital in the non-durable (consumption) good sector is about 0.52, while it is about 0.32 in the durable (investment) good sector. We know also from aggregate data that the total capital share of the US economy is about 0.35. Such a value can be computed at the steady state in our model through the formula

$$S_K = \frac{r_1 k_1 + r_2 k_2}{y_0 + p_1 y_1 + p_2 y_2}$$

Focusing on yearly data, let us consider $g = 0.1$, $\alpha_1 = 0.01$ $\alpha_2 = 0.36$, $\beta_1 = 0.3$, $\beta_2 = 0.01$, $\gamma_1 = 0.375$ and $\gamma_2 = 0.05$. We get $\delta^* \approx 0.056$, $\underline{\theta} \approx 0.0116$, $\theta^* \approx 0.62$ and conditions (23), (25) and (28) are satisfied. If $\delta = 0.01$ we find $\hat{\theta} \approx 0.67$. In the case of non strictly concave utility function, θ^* and $\hat{\theta}$ are Hopf bifurcation values associated to elasticities of intertemporal substitution equal to $\varepsilon_{00} \approx 2.57$ and $\varepsilon_{11} \approx 1.636$ when $\theta = \theta^*$, and $\varepsilon_{00} \approx 3.1$ and $\varepsilon_{11} \approx 1.475$

when $\theta = \hat{\theta}$. These values are all in line with the recent estimates of Gruber [19]. We can also approximate the period of the corresponding cycles as follows

$$T^* = \frac{2\pi}{\sqrt{-\Delta^{\theta^*}}} \approx 8.7 \text{ and } \hat{T} = \frac{2\pi}{\sqrt{-\Delta^{\hat{\theta}}}} \approx 9.17$$

These numbers correspond respectively to periods of 8.7 and 9.17 years. Both periods are therefore in line with the periodicities obtained in Section 2. We can finally compute the aggregate shares of capital income $S_K \approx 0.368$ when $\theta = \theta^*$ and $S_K \approx 0.37$ when $\theta = \hat{\theta}$ which are also in line with US data.

Let us consider now the case of a strictly concave utility function. Choosing $\theta = 0.65$ allows to get $\bar{\sigma}_\theta \approx 0.1193$ and $\sigma_\theta^* \approx 0.00005$. It follows that the steady-state is totally unstable when $\sigma \in [0, \sigma_\theta^*)$ and saddle-point stable with damped oscillations when $\sigma \in (\sigma_\theta^*, \bar{\sigma}_\theta)$. σ_θ^* is a Hopf bifurcation value. We find then that $S_K = 0.369$ and the period of the cycle is equal to $T = 9$ years. Obviously, σ^* is extremely close to zero but is associated to elasticities of intertemporal substitution equal to $\varepsilon_{00} \approx 2.857$ and $\varepsilon_{11} \approx 1.538$. Although all these values remain fully in line with the data, we find again here that the Hopf bifurcation requires a very low value of σ. This means in this case also that the conclusion of Rockafellar [35] implies the saddle-point property of the steady state from a very weak degree of concavity for the utility function.

7 Concluding comments

This paper explores the existence of periodic limit cycles to explain the mid-term periodicities of the main macroeconomic variables. In the spirit of Beaudry *et al.* [3, 5], we have provided in Section 2 a spectral analysis of US quarterly detrended data on gross domestic product, gross private investment and personal consumption expenditures in order to identify the peaks associated to a mid-term periodicity. We have found an average periodicity between 8 and 12 years.

We have considered a 3-sector optimal growth model with a non-linear utility function and we have provided clear conditions for the existence of endogenous fluctuations through a Hopf bifurcation. We have used successively two formulations: in the first one, there is only one consumption good which is consumed by the household. We have provided precise conditions for the existence of complex characteristic roots and a Hopf bifurcation reaching however the conclusion that these require to consider a close to linear utility function characterized by a high elasticity of intertemporal substitution in consumption. In the second formulation where both the pure consumption good and a mixed consumption-investment good are consumed by the household, we have shown that if the share of the mixed consumption good is large enough a Hopf bifurcation can occur. Moreover, endogenous fluctuations are now compatible with realistic values for the elasticity of intertemporal substitution in consumption. We have also provided a precise evaluation of the periodicity of the endogenous cycles generated by the Hopf bifurcation that depends on the technological and preference parameters of our model. We have finally proved with a numerical simulation that our results are in line with the empirical evidence for both the parameters' values and the mid-term periodicities obtained

in Section 2.

As clearly suggested by all our conclusions, the consideration of the linear approximation of the dynamical system does not allow to ensure that the limit cycles are locally stable. Computing complex higher order Taylor approximations is necessary to reach such a solution.[20] Another possibility is to numerically simulate the model, in particular using the example of Section 6. It is however necessary to consider a discrete-time version of the three sector model and then to use the numerical procedure presented by Galizia [17] to locate the saddle-point limit cycle and to study its local stability. Building on the contribution of Benhabib and Nishimura [8], it is also possible to extend our analysis to stochastic models focusing then on the concept of stochastic limit cycles. Here also the technique of Galizia [17] can be used. This work is left for future research.

8 Appendix 1

8.1 Proof of Proposition 1

Maximizing profit in each of the three sectors yields to the following first order conditions

$$\begin{aligned}
w &= (1-\alpha_1-\alpha_2)A_0(k_1^0)^{\alpha_1}(k_2^0)^{\alpha_2}(l^0)^{-\alpha_1-\alpha_2} \\
&= p_1(1-\beta_1-\beta_2)A_1(k_1^1)^{\beta_1}(k_2^1)^{\beta_2}(l^1)^{-\beta_1-\beta_2} \\
&= p_2(1-\gamma_1-\gamma_2)A_2(k_1^2)^{\gamma_1}(k_2^2)^{\gamma_2}(l^2)^{-\gamma_1-\gamma_2} \\
r_1 &= \alpha_1 A_0(k_1^0)^{\alpha_1-1}(k_2^0)^{\alpha_2}(l^0)^{1-\alpha_1-\alpha_2} \\
&= p_1\beta_1 A_1(k_1^1)^{\beta_1-1}(k_2^1)^{\beta_2}(l^1)^{1-\beta_1-\beta_2} \\
&= p_2\gamma_1 A_2(k_1^2)^{\gamma_1-1}(k_2^2)^{\gamma_2}(l^2)^{1-\gamma_1-\gamma_2} \\
r_2 &= \alpha_2 A_0(k_1^0)^{\alpha_1}(k_2^0)^{\alpha_2-1}(l^0)^{1-\alpha_1-\alpha_2} \\
&= p_1\beta_2 A_1(k_1^1)^{\beta_1}(k_2^1)^{\beta_2-1}(l^1)^{1-\beta_1-\beta_2} \\
&= p_2\gamma_2 A_2(k_1^2)^{\gamma_1}(k_2^2)^{\gamma_2-1}(l^2)^{1-\gamma_1-\gamma_2}
\end{aligned} \tag{30}$$

Let us define:

$$\begin{aligned}
\omega_{10} &= \frac{r_1}{w} = \frac{\alpha_1}{1-\alpha_1-\alpha_2}\frac{l^0}{k_1^0} = \frac{\beta_1}{1-\beta_1-\beta_2}\frac{l^1}{k_1^1} = \frac{\gamma_1}{1-\gamma_1-\gamma_2}\frac{l^2}{k_1^2} \\
\omega_{12} &= \frac{r_1}{r_2} = \frac{\alpha_1}{\alpha_2}\frac{k_2^0}{k_1^0} = \frac{\beta_1}{\beta_2}\frac{k_2^1}{k_1^1} = \frac{\gamma_1}{\gamma_2}\frac{k_2^2}{k_1^2}
\end{aligned} \tag{31}$$

Substituting these expressions into (30) and using at the steady state $r_1 = (\delta+g)p_1$ and $r_2 = (\delta+g)p_2$ yield

$$\begin{aligned}
r_1 &= p_1\beta_1 A_1 \left(\frac{\beta_2}{\beta_1}\right)^{\beta_2}\left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{1-\beta_1-\beta_2}\omega_{10}^{1-\beta_1-\beta_2}\omega_{12}^{\beta_2} = (\delta+g)p_1 \\
r_2 &= p_2\gamma_2 A_2 \left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2-1}\left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{1-\gamma_1-\gamma_2}\omega_{10}^{1-\gamma_1-\gamma_2}\omega_{12}^{\gamma_2-1} = (\delta+g)p_2
\end{aligned} \tag{32}$$

[20] See Guckenheimer and Holmes [20] and the comments provided in Appendices 8.3 and 8.7.

Taking logs allows to get in matrix form

$$\begin{pmatrix} \ln(\delta+g) - \ln\left[\beta_1 A_1 \left(\frac{\beta_2}{\beta_1}\right)^{\beta_2} \left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{1-\beta_1-\beta_2}\right] \\ \ln(\delta+g) - \ln\left[\gamma_2 A_2 \left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2-1} \left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{1-\gamma_1-\gamma_2}\right] \end{pmatrix} = \begin{pmatrix} 1-\beta_1-\beta_2 & \beta_2 \\ 1-\gamma_1-\gamma_2 & \gamma_2-1 \end{pmatrix} \begin{pmatrix} \ln\omega_{10} \\ \ln\omega_{12} \end{pmatrix}$$

Solving this system leads to the following steady state values for ω_{10} and ω_{12}:

$$\begin{aligned} \omega_{10} &= (\delta+g)^{\frac{\gamma_2-1-\beta_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}} \frac{\left[\gamma_2 A_2 \left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2-1} \left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{1-\gamma_1-\gamma_2}\right]^{\frac{\beta_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}}{\left[\beta_1 A_1 \left(\frac{\beta_2}{\beta_1}\right)^{\beta_2} \left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{1-\beta_1-\beta_2}\right]^{\frac{\gamma_2-1}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}} \\ \omega_{12} &= (\delta+g)^{\frac{\gamma_1+\gamma_2-\beta_1-\beta_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}} \frac{\left[\beta_1 A_1 \left(\frac{\beta_2}{\beta_1}\right)^{\beta_2} \left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{1-\beta_1-\beta_2}\right]^{\frac{1-\gamma_1-\gamma_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}}{\left[\gamma_2 A_2 \left(\frac{\gamma_2}{\gamma_1}\right)^{\gamma_2-1} \left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{1-\gamma_1-\gamma_2}\right]^{\frac{1-\beta_1-\beta_2}{\beta_2\gamma_1-(1-\beta_1)(1-\gamma_2)}}} \end{aligned}$$

Consider now from (30):

$$\begin{aligned} \alpha_1 A_0 \left(\frac{k_2^0}{k_1^0}\right)^{\alpha_2} \left(\frac{l^0}{k_1^0}\right)^{1-\alpha_1-\alpha_2} &= p_1\beta_1 A_1 \left(\frac{k_2^1}{k_1^1}\right)^{\beta_2} \left(\frac{l^1}{k_1^1}\right)^{1-\beta_1-\beta_2} \\ &= p_2\gamma_1 A_2 \left(\frac{k_2^2}{k_1^2}\right)^{\gamma_2} \left(\frac{l^2}{k_1^2}\right)^{1-\gamma_1-\gamma_2} \end{aligned} \tag{33}$$

which can be written using (31)

$$\begin{aligned} \alpha_1 A_0 \left(\frac{\alpha_2}{\alpha_1}\omega_{12}\right)^{\alpha_2} \left(\frac{1-\alpha_1-\alpha_2}{\alpha_1}\omega_{10}\right)^{1-\alpha_1-\alpha_2} &= p_1\beta_1 A_1 \left(\frac{\beta_2}{\beta_1}\omega_{12}\right)^{\beta_2} \left(\frac{1-\beta_1-\beta_2}{\beta_1}\omega_{10}\right)^{1-\beta_1-\beta_2} \\ &= p_2\gamma_1 A_2 \left(\frac{\gamma_2}{\gamma_1}\omega_{12}\right)^{\gamma_2} \left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\omega_{10}\right)^{1-\gamma_1-\gamma_2} \end{aligned}$$

Solving these equations leads to the steady state values for p_1 and p_2:

$$\begin{aligned} p_1 &= \frac{\alpha_1 A_0}{\beta_1 A_1} \left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2} \left(\frac{1-\alpha_1-\alpha_2}{\alpha_1}\right)^{1-\alpha_1-\alpha_2} \left(\frac{\beta_2}{\beta_1}\right)^{-\beta_2} \left(\frac{1-\beta_1-\beta_2}{\beta_1}\right)^{\beta_1+\beta_2-1} \omega_{10}^{\beta_1+\beta_2-\alpha_1-\alpha_2} \omega_{12}^{\alpha_2-\beta_2} \\ p_2 &= \frac{\alpha_1 A_0}{\gamma_1 A_2} \left(\frac{\alpha_2}{\alpha_1}\right)^{\alpha_2} \left(\frac{1-\alpha_1-\alpha_2}{\alpha_1}\right)^{1-\alpha_1-\alpha_2} \left(\frac{\gamma_2}{\gamma_1}\right)^{-\gamma_2} \left(\frac{1-\gamma_1-\gamma_2}{\gamma_1}\right)^{\gamma_1+\gamma_2-1} \omega_{10}^{\gamma_1+\gamma_2-\alpha_1-\alpha_2} \omega_{12}^{\alpha_2-\gamma_2} \end{aligned}$$

and thus to the value of $w = (\delta+g)p_1/\omega_{10}$. Let us consider now Lemma 2. Solving the system $\mathcal{A}(\omega,p)y = x$ with respect to k_1 and k_2 using at the steady state $y_1 = gk_1$ and $y_2 = gk_2$ yield

$$\begin{aligned} k_1 &= \frac{a_{00}[a_{10}-g(a_{10}a_{22})-a_{12}a_{20})]}{[a_{00}-g(a_{22}a_{00}-ga_{20}a_{02})][a_{00}-g(a_{11}a_{00}-a_{10}a_{01})]-g^2(a_{12}a_{00}-a_{10}a_{02})(a_{21}a_{00}-a_{20}a_{01})} \\ &+ \frac{c_1[(a_{11}a_{00}-a_{10}a_{01})[a_{00}-g(a_{22}a_{00}-ga_{20}a_{02})]+g(a_{12}a_{00}-a_{10}a_{02})(a_{21}a_{00}-a_{20}a_{01})]}{[a_{00}-g(a_{22}a_{00}-ga_{20}a_{02})][a_{00}-g(a_{11}a_{00}-a_{10}a_{01})]-g^2(a_{12}a_{00}-a_{10}a_{02})(a_{21}a_{00}-a_{20}a_{01})} \\ k_2 &= \frac{a_{00}[a_{20}-g(a_{11}a_{20}-a_{21}a_{10})]+c_1a_{00}(a_{11}a_{00}-a_{10}a_{01})}{[a_{00}-g(a_{22}a_{00}-ga_{20}a_{02})][a_{00}-g(a_{11}a_{00}-a_{10}a_{01})]-g^2(a_{12}a_{00}-a_{10}a_{02})(a_{21}a_{00}-a_{20}a_{01})} \end{aligned}$$

The steady state values of k_1 and k_2 are then derived using the fact the input coefficients a_{ij} are functions of ω_{10}, p_1 and p_2 as follows

$$a_{00}=\frac{(1-\alpha_1-\alpha_2)\omega_{10}}{(\delta+g)p_1},\quad a_{10}=\frac{\alpha_1}{(\delta+g)p_1},\quad a_{20}=\frac{\alpha_2}{(\delta+g)p_2}$$
$$a_{01}=\frac{(1-\beta_1-\beta_2)\omega_{10}}{\delta+g},\quad a_{11}=\frac{\beta_1}{\delta+g},\quad a_{21}=\frac{p_1\beta_2}{(\delta+g)p_2}$$
$$a_{02}=\frac{p_2(1-\gamma_1-\gamma_2)\omega_{10}}{(\delta+g)p_1},\quad a_{12}=\frac{p_2\gamma_1}{(\delta+g)p_1},\quad a_{22}=\frac{\gamma_2}{\delta+g}$$

We also get the expression of c_0:

$$c_0=y_0=\frac{1-a_{01}(gk_1+c_1)-a_{02}gk_2}{a_{00}}$$

Recall then that $c_1=c_0(1-\theta)/\theta p_1$. Substituting this expression into k_1, k_2 and c_0 and solving for c_0 gives a unique solution. The existence and uniqueness of a steady state is then proved. □

8.2 Proof of Proposition 2

Linearizing the dynamical system around $(c_1^*,k_1^*,k_2^*,p_1^*,p_2^*)$ gives the Jacobian matrix $\mathcal{J}$:

$$\mathcal{J}=\begin{pmatrix} \frac{\partial y_1}{\partial k_1}-g-\frac{\partial c_1}{\partial k_1} & \frac{\partial y_1}{\partial k_2}-\frac{\partial c_1}{\partial k_2} & \frac{\partial y_1}{\partial p_1}-\frac{\partial c_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2}-\frac{\partial c_1}{\partial p_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2}-g & \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \\ \frac{\sigma}{E_\sigma^\theta}\frac{p_1}{c}\mathcal{B}_1 & \frac{\sigma}{E_\sigma^\theta}\frac{p_1}{c}\mathcal{B}_2 & J_{33} & J_{34} \\ \frac{\sigma}{E_\sigma^\theta}\frac{p_2}{c}\mathcal{B}_1 & \frac{\sigma}{E_\sigma^\theta}\frac{p_2}{c}\mathcal{B}_2 & J_{43} & J_{44} \end{pmatrix}\equiv\begin{pmatrix} \mathcal{J}_1 & \mathcal{J}_2 \\ \mathcal{J}_3 & \mathcal{J}_4 \end{pmatrix} \tag{34}$$

with

$$J_{33}=\frac{1}{E_\sigma^\theta}\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)E_\sigma^\theta-\sigma\frac{p_1\left(\mathcal{A}_1+\frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_1}\right)}{c_0}+(1-\theta)(1-\sigma)\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\right]$$
$$J_{44}=\frac{1}{E_\sigma^\theta}\left[\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)E_\sigma^\theta-\sigma\frac{p_2\left(\mathcal{A}_2+\frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_2}\right)}{c}-(1-\theta)(1-\sigma)\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right]$$
$$J_{34}=-\frac{1}{E_\sigma^\theta}\left[\frac{\partial y_2}{\partial k_1}E_\sigma^\theta+\sigma\frac{p_1\left(\mathcal{A}_2+\frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_2}\right)}{c}+(1-\theta)(1-\sigma)\frac{\partial y_2}{\partial k_1}\right]$$
$$J_{43}=-\frac{1}{E_\sigma^\theta}\left[\frac{\partial y_1}{\partial k_2}E_\sigma^\theta+\sigma\frac{p_2\left(\mathcal{A}_1+\frac{\partial c_0}{\partial k_1}\frac{\partial c_1}{\partial p_1}\right)}{c}-(1-\theta)(1-\sigma)\frac{p_2}{p_1}\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\right]$$

and

$$\mathcal{A}_1=\frac{\partial c_0}{\partial p_1}\left(\frac{\partial y_1}{\partial k_1}-g-\delta\right)+\frac{\partial c_0}{\partial p_2}\frac{\partial y_1}{\partial k_2}-\frac{\partial c_0}{\partial k_1}\frac{\partial y_1}{\partial p_1}-\frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial p_1}$$
$$\mathcal{A}_2=\frac{\partial c_0}{\partial p_1}\frac{\partial y_2}{\partial k_1}+\frac{\partial c_0}{\partial p_2}\left(\frac{\partial y_2}{\partial k_2}-g-\delta\right)-\frac{\partial c_0}{\partial k_1}\frac{\partial y_1}{\partial p_2}-\frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial p_2}$$
$$\mathcal{B}_1=\frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_1}-g-\frac{\partial c_1}{\partial k_1}\right)+\frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial k_1},\ \mathcal{B}_2=\frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_2}-\frac{\partial c_1}{\partial k_2}\right)+\frac{\partial c_0}{\partial k_2}\left(\frac{\partial y_2}{\partial k_2}-g\right)$$

Since the optimization program (5) has an Hamiltonian structure, and as initially proved by Kurz [22] and Levhari and Liviatan [23], if λ is a characteristic root then $\bar{\lambda}$, $\delta-\lambda$ and $\delta-\bar{\lambda}$ are also characteristic roots. This is confirmed by showing that $\mathcal{T}=2\delta$. Consider

indeed the fact that by definition

$$c_0(k_1,k_2,p_1,p_2) = T(k_1,k_2,y_1(k_1,k_2,p_1,p_2),y_2(k_1,k_2,p_1,p_2)) \tag{35}$$

It follows therefore

$$\begin{array}{rcl} \frac{\partial c_0}{\partial k_1} & = & T_{k_1} + T_{y_1}\frac{\partial y_1}{\partial k_1} + T_{y_2}\frac{\partial y_2}{\partial k_1} = r_1 - p_1\frac{\partial y_1}{\partial k_1} - p_2\frac{\partial y_2}{\partial k_1} \\ \frac{\partial c_0}{\partial k_2} & = & T_{k_2} + T_{y_1}\frac{\partial y_1}{\partial k_2} + T_{y_2}\frac{\partial y_2}{\partial k_2} = r_2 - p_1\frac{\partial y_1}{\partial k_2} - p_2\frac{\partial y_2}{\partial k_2} \end{array} \tag{36}$$

and

$$\begin{array}{rcl} \frac{\partial c_0}{\partial p_1} & = & T_{y_1}\frac{\partial y_1}{\partial p_1} + T_{y_2}\frac{\partial y_2}{\partial p_1} = -p_1\frac{\partial y_1}{\partial p_1} - p_2\frac{\partial y_2}{\partial p_1} = -p_1\frac{\partial y_1}{\partial p_1} - p_2\frac{\partial y_1}{\partial p_2} \\ \frac{\partial c_0}{\partial p_2} & = & T_{y_1}\frac{\partial y_1}{\partial p_2} + T_{y_2}\frac{\partial y_2}{\partial p_2} = -p_1\frac{\partial y_1}{\partial p_2} - p_2\frac{\partial y_2}{\partial p_2} = -p_1\frac{\partial y_2}{\partial p_1} - p_2\frac{\partial y_2}{\partial p_2} \end{array} \tag{37}$$

Evaluated at the steady state expressions (36) become:

$$\begin{array}{rcl} \frac{\partial c_0}{\partial k_1} & = & -\left[p_1\left(\frac{\partial y_1}{\partial k_1} - \delta - g\right) + p_2\frac{\partial y_2}{\partial k_1}\right] \\ \frac{\partial c_0}{\partial k_2} & = & -\left[p_1\frac{\partial y_1}{\partial k_2} + p_2\left(\frac{\partial y_2}{\partial k_2} - \delta - g\right)\right] \end{array} \tag{38}$$

From the Jacobian matrix (34), we derive

$$\mathcal{T} = 2\delta - \sigma\frac{p_1\mathcal{A}_1 + p_2\mathcal{A}_2}{E_\sigma^\theta c_0} - \frac{\partial c_0}{\partial k_1}\frac{\left[c_1 E_\sigma^\theta + \sigma\left(p_1\frac{\partial c_1}{\partial p_1} + p_2\frac{\partial c_1}{\partial p_2}\right)\right]}{E_\sigma^\theta c_0} + \frac{(1-\theta)(1-\sigma)\left[\delta + g - \frac{\partial y_1}{\partial k_1} - \frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right]}{E_\sigma^\theta}$$

From (37) and (38) we derive that $p_1\mathcal{A}_1 + p_2\mathcal{A}_2 = 0$ and using (14) we conclude that

$$\frac{\partial c_0}{\partial k_1}\frac{\left[c_1 E_\sigma^\theta + \sigma\left(p_1\frac{\partial c_1}{\partial p_1} + p_2\frac{\partial c_1}{\partial p_2}\right)\right]}{E_\sigma^\theta c_0} = \frac{(1-\theta)(1-\sigma)\left[\delta + g - \frac{\partial y_1}{\partial k_1} - \frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right]}{E_\sigma^\theta}$$

It follows therefore that $\mathcal{T} = 2\delta$. Tedious but straightforward computations also allow to compute the Determinant of the Jacobian matrix as:

$$\mathcal{D}_\sigma^\theta = \frac{\left[\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1}\right]\left[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]}{E_\sigma^\theta}$$

Denoting Σ_σ^θ the sum of minors of order three and $\mathcal{S}_\sigma^\theta$ the sum of minors of order two, let us consider the fact that if λ is a characteristic root of (18), then $\bar{\lambda}$, $\delta - \lambda$ and $\delta - \bar{\lambda}$ are also characteristic roots. In the case where the roots are complex, let us denote $\lambda = a + ib$. We easily derive that

$$\begin{array}{rcl} \mathcal{S}_\sigma^\theta & = & \lambda\bar{\lambda} + \lambda(\delta-\lambda) + \lambda(\delta-\bar{\lambda}) + \bar{\lambda}(\delta-\lambda) + \bar{\lambda}(\delta-\bar{\lambda}) + (\delta-\lambda)(\delta-\bar{\lambda}) \\ & = & \delta^2 + 2\left[b^2 - a^2 + \delta a\right] \\ \Sigma_\sigma^\theta & = & \lambda\bar{\lambda}(\delta-\lambda) + \lambda(\delta-\lambda)(\delta-\bar{\lambda}) + \bar{\lambda}(\delta-\lambda)(\delta-\bar{\lambda}) + \lambda\bar{\lambda}(\delta-\bar{\lambda}) \\ & = & 2\delta\left[b^2 - a^2 + \delta a\right] = \mathcal{T}\frac{S_\sigma^\theta - \delta^2}{2} = \delta\left(S_\sigma^\theta - \delta^2\right) \\ \mathcal{D}_\sigma & = & \lambda\bar{\lambda}(\delta-\lambda)(\delta-\bar{\lambda}) \\ & = & (a^2+b^2)\left[(\delta-a)^2 + b^2\right] = \left(\frac{S_\sigma - \delta^2}{2}\right)^2 + b^2\left[a - (\delta-a)\right]^2 \end{array} \tag{39}$$

In the case where the roots are real, let us denote $\lambda_1 = a_1$ and $\lambda_2 = a_2$. We get now:

$$
\begin{aligned}
\mathcal{S}_\sigma^\theta &= \lambda_1\lambda_2 + \lambda_1(\delta-\lambda_1) + \lambda_1(\delta-\lambda_2) + \lambda_2(\delta-\lambda_1) + \lambda_2(\delta-\lambda_2) + (\delta-\lambda_1)(\delta-\lambda_2) \\
&= \delta^2 + a_1(\delta-a_1) + a_2(\delta-a_2) \\
\Sigma_\sigma^\theta &= \lambda_1\lambda_2(\delta-\lambda_1) + \lambda_1(\delta-\lambda_1)(\delta-\lambda_2) + \lambda_2(\delta-\lambda_1)(\delta-\lambda_2) + \lambda_1\lambda_2(\delta-\lambda_2) \\
&= \delta\left[a_1(\delta-a_1) + a_2(\delta-a_2)\right] = \mathcal{T}\frac{S_\sigma^\theta-\delta^2}{2} = \delta\left(S_\sigma^\theta-\delta^2\right) \\
\mathcal{D}_\sigma^\theta &= \lambda_1\lambda_2(\delta-\lambda_1)(\delta-\lambda_2) \\
&= a_1a_2(\delta-a_1)(\delta-a_2) = \left(\frac{S_\sigma-\delta^2}{2}\right)^2 - \left[\frac{a_1(\delta-a_1)-a_2(\delta-a_2)}{2}\right]^2
\end{aligned}
$$

It follows therefore that for any set of characteristic roots we have

$$\Sigma_\sigma^\theta = \mathcal{T}\frac{S_\sigma^\theta\delta^2}{2} = \delta\left(S_\sigma^\theta-\delta^2\right)$$

Moreover, using the fact that $\mathcal{T}=2\delta$, we can also show that

$$\frac{\partial c_1}{\partial k_1} = \frac{1-\theta}{\theta}\left(\delta + g - \frac{\partial y_1}{\partial k_1} - \frac{p_2}{p_1}\frac{dy_2}{dk_1}\right)$$

or equivalently using (13) and (14)

$$\left(\delta + g - \frac{\partial y_1}{\partial k_1} - \frac{p_2}{p_1}\frac{dy_2}{dk_1}\right) = \frac{\partial c_0}{\partial k_1}\frac{1}{p_1}$$

The degree-4 characteristic polynomial can then be written as

$$\mathcal{P}_\sigma^\theta(\lambda) = \lambda^4 - \lambda^3 2\delta + \lambda^2\mathcal{S}_\sigma^\theta - \lambda\delta\left(S_\sigma^\theta-\delta^2\right) + \mathcal{D}_\sigma^\theta \tag{40}$$

with

$$\mathcal{S}_\sigma^\theta = \mathcal{S}_0^\theta - \sigma\frac{\Gamma_\theta\left[1-\frac{1}{c_0}\left(p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\right)\right]+\theta\Theta_\theta}{\theta E_\sigma^\theta c_0}$$

and

$$
\begin{aligned}
\mathcal{S}_0^\theta &= \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right)\left(\frac{\partial y_2}{\partial k_2} - g\right) - \left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right)\frac{\partial y_2}{\partial k_1} + \frac{\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}}{\theta} \\
&+ \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - g\right)\left[\delta + g - \frac{\partial y_1}{\partial k_1} + \delta + g - \frac{\partial y_2}{\partial k_2} + \frac{\partial c_1}{\partial k_1}\right] \\
\Gamma_\theta &= c_0(1-\theta)\left[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} + \frac{\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} + \frac{\partial y_2}{\partial k_2} - g\right)\frac{\partial c_0}{\partial k_1}}{p_1}\right] \\
\Theta_\theta &= \left[\frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right) + \frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]\left[p_1\left(\frac{\partial y_1}{\partial p_1} - \frac{\partial c_1}{\partial p_1}\right) + p_2\left(\frac{\partial y_1}{\partial p_2} - \frac{\partial c_1}{\partial p_2}\right)\right] \\
&+ \left[\frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right) + \frac{\partial c_0}{\partial k_2}\left(\frac{\partial y_2}{\partial k_2} - g\right)\right]\left[p_1\frac{\partial y_2}{\partial p_1} + p_2\frac{\partial y_2}{\partial p_2}\right] \\
&+ \mathcal{A}_1\left[p_1\left(\frac{\partial y_1}{\partial k_1} - g - \delta\right) + p_2\frac{\partial y_2}{\partial k_1}\right] + \mathcal{A}_2\left[p_1\frac{\partial y_1}{\partial k_2} + p_2\left(\frac{\partial y_2}{\partial k_2} - g - \delta\right)\right] \\
&+ \frac{\partial c_0}{\partial k_1}\left\{\frac{\partial c_1}{\partial p_1}\left[p_1\left(\frac{\partial y_1}{\partial k_1} - g + \delta - \frac{\partial c_1}{\partial k_1}\right) + p_2\frac{\partial y_2}{\partial k_1}\right] + \frac{\partial c_1}{\partial p_2}\left[p_1\frac{\partial y_1}{\partial k_2} + p_2\left(\frac{\partial y_2}{\partial k_2} - g + \delta - \frac{\partial c_1}{\partial k_1}\right)\right]\right\}
\end{aligned}
$$

We finally conclude that, because of the structure of the characteristic roots, one of the following cases necessarily hold:

i) the four roots are real and distincts,

ii) the four roots are given by two pairs of non-real complex conjugates,

iii) there are two real double roots. □

8.3 Proof of Proposition 3

Using Lemmas 1 and 2, and recalling (3), (13) and (14), we need first to compute explicitly all the partial derivatives that will affect the linearized dynamical system around the steady state:

Lemma 8.1. *At the steady state we have*

$$\begin{array}{rclcrcl}
\frac{\partial r_1}{\partial p_1} &=& \frac{\partial y_1}{\partial k_1} = \frac{(\delta+g)[\alpha_2(1-\gamma_1)-\gamma_2(1-\alpha_1)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}, & & \frac{\partial r_1}{\partial p_2} &=& \frac{\partial y_2}{\partial k_1} = \frac{(\delta+g)p_1[\beta_2(1-\alpha_1)-\alpha_2(1-\beta_1)]}{p_2[\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2]},\\
\frac{\partial r_2}{\partial p_1} &=& \frac{\partial y_1}{\partial k_2} = \frac{(\delta+g)p_2[\gamma_1(1-\alpha_2)-\alpha_1(1-\gamma_2)]}{p_1[\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2]}, & & \frac{\partial r_2}{\partial p_2} &=& \frac{\partial y_2}{\partial k_2} = \frac{(\delta+g)[\alpha_1(1-\beta_2)-\beta_1(1-\alpha_2)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2},\\
\frac{\partial c_0}{\partial k_1} &=& \frac{(\delta+g)p_1[\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}, & & \frac{\partial c_0}{\partial k_2} &=& \frac{(\delta+g)p_2[\beta_1(1-\gamma_2)-\gamma_1(1-\beta_2)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2},\\
\frac{\partial c_1}{\partial k_1} &=& \frac{1-\theta}{\theta}\frac{(\delta+g)[\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}, & & \frac{\partial c_1}{\partial k_2} &=& \frac{1-\theta}{\theta}\frac{p_2}{p_1}\frac{(\delta+g)[\beta_1(1-\gamma_2)-\gamma_1(1-\beta_2)]}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2},\\
\frac{\partial w}{\partial p_1} &=& \frac{w[\gamma_2\alpha_1-\gamma_1\alpha_2]}{p_1[\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2]}, & & \frac{\partial w}{\partial p_2} &=& \frac{w[\beta_1\alpha_2-\beta_2\alpha_1]}{p_2[\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2]},\\
\frac{\partial y_1}{\partial p_1} &=& \left(\frac{\partial w}{\partial p_1}\right)^2\frac{1}{w}+\left(\frac{\partial y_1}{\partial k_1}\right)^2\frac{k_1}{r_1}+\left(\frac{\partial y_1}{\partial k_2}\right)^2\frac{k_2}{r_2}-\frac{y_1}{p_1}, & & \frac{\partial y_2}{\partial p_2} &=& \left(\frac{\partial w}{\partial p_2}\right)^2\frac{1}{w}+\left(\frac{\partial y_2}{\partial k_1}\right)^2\frac{k_1}{r_1}+\left(\frac{\partial y_2}{\partial k_2}\right)^2\frac{k_2}{r_2}-\frac{y_2}{p_2},\\
\frac{\partial y_2}{\partial p_1} &=& \frac{\partial w}{\partial p_1}\frac{\partial w}{\partial p_2}\frac{1}{w}+\frac{\partial y_1}{\partial k_1}\frac{\partial y_2}{\partial k_1}\frac{k_1}{r_1}+\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_2}\frac{k_2}{r_2}, & & \frac{\partial y_1}{\partial p_2} &=& \frac{\partial y_2}{\partial p_1},\\
\frac{\partial c_0}{\partial p_1} &=& \frac{\partial w}{\partial p_1}\frac{\gamma_1\beta_2-\gamma_2\beta_1}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} & & \frac{\partial c_0}{\partial p_2} &=& \frac{\partial w}{\partial p_2}\frac{\gamma_1\beta_2-\gamma_2\beta_1}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}\\
&+& \frac{\partial c_0}{\partial k_1}\frac{\partial y_1}{\partial k_1}\frac{k_1}{r_1}+\frac{\partial c_0}{\partial k_2}\frac{\partial y_1}{\partial k_2}\frac{k_2}{r_2}, & & &+& \frac{\partial c_0}{\partial k_1}\frac{\partial y_2}{\partial k_1}\frac{k_1}{r_1}+\frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial k_2}\frac{k_2}{r_2},\\
\frac{\partial c_1}{\partial p_1} &=& \frac{1-\theta}{\theta p_1}\frac{\partial c_0}{\partial p_1}-\frac{c_0}{p_1} & & \frac{\partial c_0}{\partial p_2} &=& \frac{1-\theta}{\theta p_1}\frac{\partial c_0}{\partial p_2}
\end{array}$$

Proof. From Lemmas 1 and 2, we obtain the following derivatives:

$$\begin{array}{rclcrcl}
\frac{\partial r_1}{\partial p_1} &=& \frac{\partial y_1}{\partial k_1} = \frac{a_{02}a_{20}-a_{00}a_{22}}{X}, & & \frac{\partial r_1}{\partial p_2} &=& \frac{\partial y_2}{\partial k_1} = \frac{a_{00}a_{21}-a_{01}a_{20}}{X},\\
\frac{\partial r_2}{\partial p_1} &=& \frac{\partial y_1}{\partial k_2} = \frac{a_{00}a_{12}-a_{02}a_{10}}{X}, & & \frac{\partial r_2}{\partial p_2} &=& \frac{\partial y_2}{\partial k_2} = \frac{a_{01}a_{10}-a_{00}a_{11}}{X},\\
\frac{\partial c}{\partial k_1} &=& \frac{a_{01}a_{22}-a_{02}a_{21}}{X}, & & \frac{\partial c}{\partial k_2} &=& \frac{a_{02}a_{11}-a_{01}a_{12}}{X},\\
\frac{\partial w}{\partial p_1} &=& \frac{a_{10}a_{22}-a_{12}a_{20}}{X}, & & \frac{\partial w}{\partial p_2} &=& \frac{a_{11}a_{20}-a_{10}a_{21}}{X},\\
\frac{\partial y_1}{\partial p_1} &=& \left(\frac{\partial w}{\partial p_1}\right)^2\frac{1}{w}+\left(\frac{\partial y_1}{\partial k_1}\right)^2\frac{k_1}{r_1}+\left(\frac{\partial y_1}{\partial k_2}\right)^2\frac{k_2}{r_2}-\frac{y_1}{p_1}, & & \frac{\partial y_2}{\partial p_2} &=& \left(\frac{\partial w}{\partial p_2}\right)^2\frac{1}{w}+\left(\frac{\partial y_2}{\partial k_1}\right)^2\frac{k_1}{r_1}+\left(\frac{\partial y_2}{\partial k_2}\right)^2\frac{k_2}{r_2}-\frac{y_2}{p_2},\\
\frac{\partial y_2}{\partial p_1} &=& \frac{\partial w}{\partial p_1}\frac{\partial w}{\partial p_2}\frac{1}{w}+\frac{\partial y_1}{\partial k_1}\frac{\partial y_2}{\partial k_1}\frac{k_1}{r_1}+\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_2}\frac{k_2}{r_2}, & & \frac{\partial y_1}{\partial p_2} &=& \frac{\partial y_2}{\partial p_1},\\
\frac{\partial c}{\partial p_1} &=& \frac{\partial w}{\partial p_1}\frac{a_{12}a_{21}-a_{11}a_{22}}{X}+\frac{\partial c}{\partial k_1}\frac{\partial y_1}{\partial k_1}\frac{k_1}{r_1}+\frac{\partial c}{\partial k_2}\frac{\partial y_1}{\partial k_2}\frac{k_2}{r_2} & & \frac{\partial c}{\partial p_2} &=& \frac{\partial w}{\partial p_2}\frac{a_{12}a_{21}-a_{11}a_{22}}{X}+\frac{\partial c}{\partial k_1}\frac{\partial y_2}{\partial k_1}\frac{k_1}{r_1}+\frac{\partial c}{\partial k_2}\frac{\partial y_2}{\partial k_2}\frac{k_2}{r_2}
\end{array}$$

with $X = a_{10}(a_{01}a_{22} - a_{02}a_{21}) + a_{11}(a_{02}a_{20} - a_{00}a_{22}) + a_{12}(a_{00}a_{21} - a_{01}a_{20})$. The final expressions are obtained from the expressions of the input coefficient (2) and using (13) and (14). □

Using the expressions of the input coefficients (2) and Lemma 8.1, we easily derive that the discriminant (20) is negative if and only if condition (22) holds. Let us then assume that condition (22) holds. Using the expression (21), we derive the real parts of the complex characteristic roots such that

$$
\begin{aligned}
Re(\lambda_{1,2}) &= \frac{(\delta+g)}{2}\frac{\beta_2(\gamma_1-\alpha_1)-(1-\beta_1)(\alpha_2-\gamma_2)+(1-\gamma_2)(\beta_1-\alpha_1)-\gamma_1(\alpha_2-\beta_2)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \\
&= -\frac{\delta+g}{2}\left[\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}-1\right] \equiv -\frac{(\delta+g)\mathcal{X}}{2} \\
Re(\lambda_{3,4}) &= \delta\frac{\alpha_2(1-\gamma_1)-\gamma_2(1-\alpha_1)+\alpha_1(1-\beta_2)-\beta_1(1-\alpha_2)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \\
&+ g\frac{(1-\beta_1)(\alpha_2-\gamma_2)-\beta_2(\gamma_1-\alpha_1)+\gamma_1(\alpha_2-\beta_2)-(1-\gamma_2)(\beta_1-\alpha_1)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2} \\
&= \frac{\delta}{2}\left[\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}+1\right] \\
&+ \frac{g}{2}\left[\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}-1\right] \equiv \frac{\delta\mathcal{Y}+g\mathcal{X}}{2}
\end{aligned}
$$

with $\mathcal{X} = \mathcal{Z} - 1$, $\mathcal{Y} = \mathcal{Z} + 1$ and

$$
\mathcal{Z} = \frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}
$$

Since $\mathcal{Y} > \mathcal{X}$, we easily derive the following results:

1- If $\mathcal{Y} < 0$ and thus $\mathcal{X} < 0$ we derive that $Re(\lambda_{1,2}) > 0$ and $Re(\lambda_{3,4}) < 0$ for any $\delta \geq 0$. Similarly, if $\mathcal{X} > 0$ and thus $\mathcal{Y} > 0$ we derive that $Re(\lambda_{1,2}) < 0$ and $Re(\lambda_{3,4}) > 0$ for any $\delta \geq 0$. Therefore, if $\mathcal{X}\mathcal{Y} > 0$, or equivalently $\mathcal{Z}^2 > 1$, then the steady state is saddle-point stable for any $\delta \geq 0$.

2- If $\mathcal{X} < 0$ and $\mathcal{Y} > 0$, or equivalently $-1 < \mathcal{Z} < 1$, then $Re(\lambda_{1,2}) > 0$ and $Re(\lambda_{3,4}) < 0$ for any $\delta \in [0, \delta^*)$ with $\delta^* = -g\mathcal{X}/\mathcal{Y}$ such that $Re(\lambda_{3,4})|_{\delta=\delta^*} = 0$. Moreover, we obviously derive that

$$
\frac{dRe(\lambda_{3,4})}{d\delta}\Big|_{\delta=\delta^*} > 0 \tag{41}
$$

with $Re(\lambda_{3,4}) > 0$ for any $\delta > \delta^*$. It follows that δ^* is a Hopf bifurcation value generically giving rise to non constant saddle-point stable (or unstable) closed orbits around the steady state in a right (or left) neighborhood of δ^*. The stability properties of the periodic orbit depend on the sign of a parameter which is obtained through a Taylor expansion of degree 3 of the dynamical system (16) on the dimension-2 center manifold associated to the two bifurcating eigenvalues. If this parameter is negative the periodic orbit occurs on the right neighborhood of δ^* and is stable on the dimension-2 center manifold, while if this parameter is positive the periodic orbit occurs on the left neighborhood of δ^* and is unstable on the dimension-2 center manifold (see Guckenheimer and Holmes [20], Theorem 3.4.2, p.151-152).

There is also the possibility to discuss the periodicity on the cycle. Indeed, as shown initially by Hopf [21], and recalling that the bifurcating eigenvalues are such that $Im(\lambda_{3,4}) = \sqrt{-\Delta^1}|_{\delta=\delta^*}$, we derive from (41) that for every δ with $|\delta - \delta^*|$ sufficiently small, the periodic orbit is characterized by a period $T = T(\delta)$ such that

$$
\lim_{|\delta-\delta^*|\to 0} T(\delta) = \frac{2\pi}{\sqrt{-\Delta^1}}\Big|_{\delta=\delta^*} = T^*
$$

In other words, for δ very nearly equal to δ^*, the period of the (emergent) periodic orbits of (16) nearly equals the period of the concentric periodic orbits of the linearized system characterized by the Jacobiam matrix (34) with $\delta = \delta^*$, $\theta = 1$ and $\sigma = 0$.

□

8.4 Proof of Lemma 3

From (37) we derive that

$$\begin{aligned} p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2} &= -\left(p_1^2\frac{\partial y_1}{\partial p_1}+p_1p_2\frac{\partial y_2}{\partial p_1}+p_1p_2\frac{\partial y_1}{\partial p_2}+p_2^2\frac{\partial y_2}{\partial p_2}\right) \\ &= -\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \end{aligned}$$

Recalling that $c_0=c_0(k_1,k_2,p_1,p_2)$, $c_1=c_1(k_1,k_2,p_1,p_2)$ and $y_j=y_j(k_1,k_2,p_1,p_2)$, $j=1,2$, and using the first order conditions (6-12), we can define the maximized Hamiltonian $\bar{\mathcal{H}}$ as

$$\begin{aligned} \bar{\mathcal{H}}(k_1,k_2,p_1,p_2) &= \mathcal{H}(c_0,c_1,y_1,y_2,k_1,k_2,p_1,p_2) \\ &= \frac{(c_0^\theta c_1^{1-\theta})^{1-\sigma}}{1-\sigma}+c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}p_1\left(y_1-gk_1-c_1\right) \\ &+ c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}p_2\left(y_2-gk_2\right) \end{aligned}$$

Considering that at the optimum we have $\frac{\partial\mathcal{H}}{\partial c_0}=\frac{\partial\mathcal{H}}{\partial c_1}=\frac{\partial\mathcal{H}}{\partial y_1}=\frac{\partial\mathcal{H}}{\partial y_2}=0$, we find

$$\begin{aligned} \frac{\partial\bar{\mathcal{H}}}{\partial p_1} &= \frac{\partial\mathcal{H}}{\partial c_0}\frac{\partial c_0}{\partial p_1}+\frac{\partial\mathcal{H}}{\partial c_1}\frac{\partial c_1}{\partial p_1}+\frac{\partial\mathcal{H}}{\partial y_1}\frac{\partial y_1}{\partial p_1}+\frac{\partial\mathcal{H}}{\partial y_2}\frac{\partial y_2}{\partial p_1}+\frac{\partial\mathcal{H}}{\partial p_1} \\ &= \frac{\partial\mathcal{H}}{\partial p_1}=c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}\left(y_1-gk_1-c_1\right) \\ \frac{\partial\bar{\mathcal{H}}}{\partial p_2} &= \frac{\partial\mathcal{H}}{\partial c_0}\frac{\partial c_0}{\partial p_2}+\frac{\partial\mathcal{H}}{\partial c_1}\frac{\partial c_1}{\partial p_2}+\frac{\partial\mathcal{H}}{\partial y_1}\frac{\partial y_1}{\partial p_2}+\frac{\partial\mathcal{H}}{\partial y_2}\frac{\partial y_2}{\partial p_2}+\frac{\partial\mathcal{H}}{\partial p_2} \\ &= \frac{\partial\mathcal{H}}{\partial p_2}=c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}\left(y_2-gk_2\right) \end{aligned}$$

We then easily derive at the steady state with $y_1-gk_1-c_1=0$ and $y_2-gk_2=0$,

$$\begin{aligned} \begin{pmatrix} \frac{\partial^2\bar{\mathcal{H}}}{\partial p_1^2} & \frac{\partial^2\bar{\mathcal{H}}}{\partial p_1\partial p_2} \\ \frac{\partial^2\bar{\mathcal{H}}}{\partial p_2\partial p_1} & \frac{\partial^2\bar{\mathcal{H}}}{\partial p_2^2} \end{pmatrix} &= c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}\begin{pmatrix} \frac{\partial y_1}{\partial p_1}-\frac{\partial c_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2}-\frac{\partial c_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \\ &= c_0^{\theta(1-\sigma)-1}c_1^{(1-\theta)(1-\sigma)}\left(\left[\frac{\partial y}{\partial p}\right]-\begin{pmatrix} \frac{\partial c_1}{\partial p_1} & \frac{\partial c_1}{\partial p_2} \\ 0 & 0 \end{pmatrix}\right) \end{aligned}$$

Since the maximized Hamiltonian is convex in the prices, the matrix on the right-hand-side is quasi-positive definite. Using (14), we know that this property must be true for any $\theta\in(0,1]$. It follows that the matrix $\left[\frac{\partial y}{\partial p}\right]$ must quasi-positive definite. This implies therefore that $p_1\frac{\partial c_0}{\partial p_1}+p_2\frac{\partial c_0}{\partial p_2}\leq 0$ and thus that $E_\sigma^\theta>0$ is an increasing function of σ. □

8.5 Proof of Proposition 4

Let $\theta=1$ and condition (22) hold. Consider then the expression of the Determinant as given by (19). We know that $\mathcal{D}_0^1>0$. We derive from Lemma 3 that $\mathcal{D}_\sigma^1$ is a monotonous decreasing function of σ with

$$\lim_{\sigma\to+\infty}\mathcal{D}_\sigma^1=0$$

It follows that at least one characteristic root is equal to zero which implies that all characteristic roots are real. We then derive that there exists $\bar{\sigma} > 0$ such that the characteristic roots are complex if and only if $\sigma \in [0, \bar{\sigma})$. To prove such a claim, let us first study the sign of Θ_1. Using (37) and (38) into Θ_1 allows to simplify its expression as follows:

$$\begin{aligned}\Theta_1 &= -\frac{\partial c_0}{\partial p_1}\left[\frac{\partial c_0}{\partial k_1}\left(\frac{\partial y_1}{\partial k_1} - g\right) + \frac{\partial c_0}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right] - \frac{\partial c_0}{\partial p_2}\left[\frac{\partial c_0}{\partial k_1}\frac{\partial y_1}{\partial k_2} + \frac{\partial c_0}{\partial k_2}\left(\frac{\partial y_2}{\partial k_2} - g\right)\right] \\ &- \mathcal{A}_1\frac{\partial c_0}{\partial k_1} - \mathcal{A}_2\frac{\partial c_0}{\partial k_2}\end{aligned}$$

Straightforward computations using the expressions of $\mathcal{A}_1$ and $\mathcal{A}_2$ then give

$$\begin{aligned}\Theta_1 = &-\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ &-\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix}^2\begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ &+\begin{pmatrix} p_1 & p_2 \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix}\begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_2}{\partial k_1} \\ \frac{\partial y_1}{\partial k_2} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\end{aligned}$$

Denoting the vector $P = (p_1 \;\; p_2)$ and P^t its transpose, we can reformulate Θ_1 as a sum of three quadratic forms:

$$\begin{aligned}\Theta_1 &= -P\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]\left[\frac{\partial y}{\partial k} - gI\right]\left[\frac{\partial y}{\partial p}\right]P^t - P\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^2\left[\frac{\partial y}{\partial p}\right]P^t \\ &+ P\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]\left[\frac{\partial y}{\partial p}\right]\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^t P^t\end{aligned}$$

Since the matrix $\left[\frac{\partial y}{\partial p}\right]$ is positive definite, the third quadratic form is necessarily positive.

Consider now the matrix $\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]$. Under conditions (22) and (23) we get

$$\begin{aligned}\mathcal{D}et\left[\frac{\partial y}{\partial k} - (g+\delta)I\right] &= \left(\frac{\partial y_1}{\partial k_1} - g - \delta\right)\left(\frac{\partial y_2}{\partial k_2} - g - \delta\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} \geq 0 \\ \mathcal{T}r\left[\frac{\partial y}{\partial k} - (g+\delta)I\right] &= \frac{\partial y_1}{\partial k_1} - g - \delta + \frac{\partial y_2}{\partial k_2} - g - \delta < 0\end{aligned}$$

It follows that the matrix $\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]$ is quasi-negative definite with complex roots. Consider finally the matrix $\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^2$. We easily get

$$\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^2 = \begin{pmatrix} \left(\frac{\partial y_1}{\partial k_1} - g - \delta\right)^2 + \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} & \frac{\partial y_1}{\partial k_2}\left(\frac{\partial y_1}{\partial k_1} - g - \delta + \frac{\partial y_2}{\partial k_2} - g - \delta\right) \\ \frac{\partial y_2}{\partial k_1}\left(\frac{\partial y_1}{\partial k_1} - g - \delta + \frac{\partial y_2}{\partial k_2} - g - \delta\right) & \left(\frac{\partial y_2}{\partial k_2} - g - \delta\right)^2 + \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} \end{pmatrix}$$

and thus

$$\mathcal{D}et\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2 = \left[\left(\frac{\partial y_1}{\partial k_1}-g-\delta\right)\left(\frac{\partial y_2}{\partial k_2}-g-\delta\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]^2 \geq 0$$
$$\mathcal{T}r\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2 = \left(\frac{\partial y_1}{\partial k_1}-g-\delta\right)^2+\left(\frac{\partial y_2}{\partial k_2}-g-\delta\right)^2+2\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}$$

We then have $\mathcal{T}r\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2 < 0$ if and only if condition (25) holds. Moreover, considering the expression of the discriminant Δ^1 as given by (20), we easily show that

$$\begin{aligned}\Delta^1 &= \left[2(\delta+g)-\frac{\partial y_1}{\partial k_1}-\frac{\partial y_2}{\partial k_2}\right]^2-4\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right]\\ &= \mathcal{T}r\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2-2\mathcal{D}et\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]\end{aligned}$$

Therefore, under conditions (23) and (25), we have $\Delta^1 < 0$. It follows that under conditions (23) and (25), the matrix $\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2$ is quasi-negative definite with complex roots. But then tedious but straightforward computations allow to show that under conditions (23) and (25), the matrices $\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]\left[\frac{\partial y}{\partial k}-gI\right]\left[\frac{\partial y}{\partial p}\right]$ and $\left[\frac{\partial y}{\partial k}-(g+\delta)I\right]^2\left[\frac{\partial y}{\partial p}\right]$ are definite-negative with real roots. It follows that the first and second quadratic forms are necessarily negative and thus $\Theta_1 > 0$.

Recall now that when $\theta = 1$, the characteristic polynomial can be written as

$$\mathcal{P}^1_\sigma(\lambda) = \mathcal{P}^0_1(\lambda)-\tfrac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda) \tag{42}$$

with

$$\tilde{\mathcal{P}}(\lambda) = \lambda^2\Theta_1-\lambda\delta\Theta_1-\mathcal{D}^1_\sigma E^1_\sigma\left(p_1\tfrac{\partial c_0}{\partial p_1}+p_2\tfrac{\partial c_0}{\partial p_2}\right) \tag{43}$$

which does not depend on σ. Equivalently, any characteristic root λ must be a solution of

$$\mathcal{P}^1_0(\lambda) = \tfrac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda) \tag{44}$$

We know that $\mathcal{P}^1_0(\lambda)$ is a degree-four polynomial with $\mathcal{P}^1_0(0) = \mathcal{D}^1_0 > 0$ and $\lim_{\lambda\to\pm\infty}\mathcal{P}^1_0(\lambda) = +\infty$. Using Lemma 3, we also know, since $\Theta_1 > 0$, that $\tilde{\mathcal{P}}(\lambda)$ is a degree-two polynomial with $\tilde{\mathcal{P}}(0) > 0$ and $\lim_{\lambda\to\pm\infty}\tilde{\mathcal{P}}(\lambda) = +\infty$. We conclude that starting from the case $\sigma = 0$ in which the characteristic roots are all complex, if σ is increased, the degree-two polynomial $\sigma\tilde{\mathcal{P}}(\lambda)/(E^1_\sigma c_0)$, which is initially located below $\mathcal{P}^1_0(\lambda)$, is going up monotonically and comes closer and closer to $\mathcal{P}^1_0(\lambda)$ until $\sigma = \bar{\sigma}$ where it will be characterized by two tangency points with $\mathcal{P}^1_0(\lambda)$ where the imaginary part of the four roots is equal to zero and there exist two pairs of double real roots. Actually, when $\sigma \in (0, \bar{\sigma})$, the size of the imaginary part of the roots is proportional to the distance between the two polynomials and is then decreasing as σ is increased. Under conditions (23) and (25), it follows therefore that when $\sigma \in [0, \bar{\sigma})$, the characteristic roots are complex.

When $\sigma = \bar{\sigma}$, the four characteristic roots are given by two pairs of real double roots. In this case, we also derive from (39) with $b = 0$ that

$$\begin{aligned}
\mathcal{S}^1_{\bar\sigma} &= \delta^2 + 2a(\delta - a) \\
\Sigma^1_{\bar\sigma} &= 2\delta a(\delta - a) = \mathcal{T}\tfrac{S^1_{\bar\sigma} - \delta^2}{2} = \delta\left(S^1_{\bar\sigma} - \delta^2\right) \\
\mathcal{D}^1_{\bar\sigma} &= a^2(\delta - a)^2 = \left(\tfrac{S^1_{\bar\sigma} - \delta^2}{2}\right)^2
\end{aligned} \tag{45}$$

We can therefore write the characteristic polynomial as follows

$$\mathcal{P}^1_{\bar\sigma}(\lambda) = \lambda^4 - \lambda^3\mathcal{T} + \lambda^2\mathcal{S}^1_{\bar\sigma} - \lambda\mathcal{T}\left(\tfrac{S^1_{\bar\sigma} - \delta^2}{2}\right) + \left(\tfrac{S^1_{\bar\sigma} - \delta^2}{2}\right)^2 \tag{46}$$

We then observe that $\mathcal{P}^{\bar\sigma}(\lambda)$ is a quasi-palindromic polynomial such that

$$\mathcal{P}^1_{\bar\sigma}\left(\tfrac{S^1_{\bar\sigma} - \delta^2}{2}\lambda\right) = \tfrac{4\lambda^4}{(S^1_{\bar\sigma} - \delta^2)^2}\mathcal{P}^1_{\bar\sigma}\left(\tfrac{S^1_{\bar\sigma} - \delta^2}{2\lambda}\right)$$

Building on this property, let us consider the change of variable

$$z = \lambda + \tfrac{S^1_{\bar\sigma} - \delta^2}{2\lambda} \tag{47}$$

We then derive from (46)

$$\begin{aligned}
\tfrac{\mathcal{P}^1_{\bar\sigma}(\lambda)}{\lambda^2} &= \lambda^2 - \mathcal{T}\left(\lambda + \tfrac{S^1_{\bar\sigma} - \delta^2}{2\lambda}\right) + \mathcal{S}^1_{\bar\sigma} + \left(\tfrac{S^1_{\bar\sigma} - \delta^2}{2\lambda}\right)^2 \\
&= z^2 - \mathcal{T}z + \delta^2 = (z - \delta)^2 = 0
\end{aligned} \tag{48}$$

We can thus explicitly compute the two double real characteristic roots. Starting from $z = \delta$ we obtain indeed from (47) that the roots are solutions of the following polynomial:

$$\lambda^2 - \delta\lambda + \tfrac{S^1_{\bar\sigma} - \delta^2}{2} = 0$$

namely

$$\lambda_1 = \tfrac{\delta + \sqrt{3\delta^2 - 2S^1_{\bar\sigma}}}{2} \text{ and } \lambda_2 = \tfrac{\delta - \sqrt{3\delta^2 - 2S^1_{\bar\sigma}}}{2} = \delta - \lambda_1$$

Using (45), we derive $3\delta^2 - 2S^1_{\bar\sigma} = (\delta - 2a)^2 \geq 0$ and thus

$$\lambda_1 = \delta - a > 0 \text{ and } \lambda_2 = a$$

It follows that there exists $\bar\delta > 0$ such that when $\delta \in [0, \bar\delta)$ the two roots λ_1 and λ_2 have opposite sign. We need now to locate $\bar\delta$ with respect to the bound δ^* as given in Proposition 3.

From Proposition 2 with $\theta = 1$, we derive that

$$\begin{aligned}
\mathcal{S}^1_\sigma &= \left(\tfrac{\partial y_1}{\partial k_1} - g\right)\left(\tfrac{\partial y_2}{\partial k_2} - g\right) - \tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1} + \left(\delta + g - \tfrac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \tfrac{\partial y_2}{\partial k_2}\right) - \tfrac{\partial y_1}{\partial k_2}\tfrac{\partial y_2}{\partial k_1} \\
&+ \left(\delta + g - \tfrac{\partial y_1}{\partial k_1} + \delta + g - \tfrac{\partial y_2}{\partial k_2}\right)\left(\tfrac{\partial y_1}{\partial k_1} - g + \tfrac{\partial y_2}{\partial k_2} - g\right) - \tfrac{\sigma\Theta_1}{E^1_\sigma c_0}
\end{aligned}$$

with $\partial\mathcal{S}^1_\sigma/\partial\sigma = -\Theta_1/(E^1_\sigma c_0) < 0$, implying that $\mathcal{S}^1_\sigma$ is a monotone decreasing function of σ. The value δ^* is such that $Re(\lambda_{3,4}) = 0$, i.e.

$$\tfrac{\partial y_1}{\partial k_1} + \tfrac{\partial y_2}{\partial k_2} - 2g = 0$$

We then get

$$\mathcal{S}^1_\sigma\Big|_{\delta=\delta^*} = \delta^{*2} - 2\left[\left(\frac{\partial y_1}{\partial k_1} - g\right)^2 + \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right] - \frac{\sigma\Theta_1}{E^1_\sigma c_0}$$

and under conditions (23) and (25)

$$\Delta^1 = \left(\frac{\partial y_1}{\partial k_1} - g\right)^2 + \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} < 0$$

It follows therefore that $\mathcal{S}^1_0|_{\delta=\delta^*} > 0$. We then conclude that there exists $\tilde{\delta} > 0$ such that $\mathcal{S}^1_\sigma|_{\delta=\delta^*} = 0$. If we can show that when $\delta = \delta^*$ and $\sigma = \tilde{\sigma}$, the characteristic roots are still complex, then we can conclude that $\bar{\delta} > \delta^*$ and $\tilde{\sigma} < \bar{\sigma}$. We derive from (39) that $\mathcal{D}_{\tilde{\sigma}}|_{\delta=\delta^*} > (\delta^{*2}/2)^2$ and thus that

$$\mathcal{P}^\theta_{\tilde{\sigma}}(\lambda)\Big|_{\delta=\delta^*} > \lambda^4 - \lambda^3 2\delta^* - \lambda\delta^{*3} + \frac{\delta^{*4}}{4} = \frac{\left(\delta^{*2} + 2\lambda\delta^* - 2\lambda^2\right)^2}{4} \geq 0 \tag{49}$$

It is easy to see that the polynomial on the rhs of equation (49) is characterized by two double real roots of opposite sign. We then conclude that the polynomial $\mathcal{P}^\theta_{\tilde{\sigma}}(\lambda)$ has four complex roots and thus that when $\delta = \delta^*$, $\tilde{\sigma} < \bar{\sigma}$. Therefore, we derive that $\mathcal{S}^1_{\bar{\sigma}} < 0$ and the roots λ_1 and λ_2 have opposite sign as long as $\delta < \bar{\delta}$ with $\bar{\delta} > \delta^*$. □

8.6 Proof of Theorem 1

Recall from (42)-(44) that any characteristic root λ must be a solution of

$$\mathcal{P}^1_0(\lambda) = \frac{\sigma}{E^1_\sigma c_0}\tilde{\mathcal{P}}(\lambda)$$

As explained in the Proof of Proposition 4, starting from the case $\sigma = 0$ in which the characteristic roots are all complex, if σ is increased, $\sigma\tilde{\mathcal{P}}(\lambda)/(E^1_\sigma c_0)$, which is initially located below $\mathcal{P}^1_0(\lambda)$, is going up monotonically and comes closer and closer to $\mathcal{P}^1_0(\lambda)$ until it will be characterized by two tangency points with $\mathcal{P}^1_0(\lambda)$ where the imaginary part of the four roots is equal to zero and there exist two pairs of double real roots. When $\sigma \in (0, \bar{\sigma})$, the size of the real and imaginary parts of the roots are proportional to the distance between the two polynomials and are then decreasing as σ is increased.

Consider first the case where condition (23) of Proposition 3 holds with $\delta < \delta^*$. When $\sigma = 0$ the steady state is saddle-point stable with one pair of complex conjugate characteristic roots having positive real parts and one pair of complex conjugate characteristic roots having negative real parts. When σ is increased, the negative real part is decreasing and remains thus negative for any $\sigma < \bar{\sigma}$, since when $\sigma = \bar{\sigma}$, there is one double negative root and one double positive root. There is no Hopf bifucation and the steady state remains saddle-point stable for any $\sigma \in (0, \bar{\sigma})$.

Consider now the case where condition (23) of Proposition 3 holds with $\delta > \delta^*$. When $\sigma = 0$ the steady state is totally unstable with two pairs of complex conjugate characteristic roots having positive real parts. When σ is increased, the real part of one pair is decreasing until it reaches zero leading thus to a pair of purely imaginary roots. This case is obtained

for a value $\sigma^* < \bar{\sigma}$ and implies a Hopf bifurcation. When $\sigma \in (\sigma^*, \bar{\sigma})$, the steady state is saddle-point stable

□

8.7 Proof of Proposition 5

Using the expressions of the input coefficients (2) and Lemma 8.1, we easily derive that the discriminant Δ^θ as given by (26) is negative if and only if

$$\begin{aligned} & \Big[\theta\left[\alpha_2(1-\gamma_1)-\gamma_2(1-\alpha_1)+\beta_1(1-\alpha_2)-\alpha_1(1-\beta_2)\right] \\ - \quad & (1-\theta)[\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)]\Big]^2 \\ < \quad & 4\theta[\alpha_2(1-\beta_1)-\beta_2(1-\alpha_1)]\Big[\theta[\gamma_1(1-\alpha_2)-\alpha_1(1-\gamma_2)] \\ - \quad & (1-\theta)[\beta_1(1-\gamma_2)-\gamma_1(1-\beta_2)]\Big] \end{aligned}$$

Since this inequality cannot be satisfied when $\theta = 0$, we derive that there exists $\underline{\theta} \in (0,1)$ such that complex characteristic roots are obtained if and only if condition (22) holds and $\theta \in (\underline{\theta}, 1]$. We also have $\Delta^{\underline{\theta}} = 0$.

We need now to check whether there may exist a value $\theta^* \in (\underline{\theta}, 1)$ such that the real part of a pair of characteristic roots is equal to zero. We know that under condition (22) and $\theta \in (\underline{\theta}, 1]$, we have $\Delta^\theta < 0$ or equivalently

$$\begin{aligned} 0 < & \left[\theta\left(\delta+g-\frac{\partial y_1}{\partial k_1}+\delta+g-\frac{\partial y_2}{\partial k_2}\right)+(1-\theta)\left(\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}\right))\right]^2 \\ & < 4\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right] \end{aligned}$$

Considering $\lambda_{1,2}$ as given by (27), we derive that $Re(\lambda_{1,2}) = 0$ if and only if there is a value θ^* such that

$$\theta^* = -\frac{\delta+g-\frac{\partial y_1}{\partial k_1}-\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}}{\delta+g-\frac{\partial y_2}{\partial k_2}+\frac{p_2}{p_1}\frac{\partial y_2}{\partial k_1}} < 1$$

If such a value exists, then it must be larger than $\underline{\theta}$ since under condition (22) we have

$$\Delta^{\theta^*} = -4\left[\left(\delta+g-\frac{\partial y_1}{\partial k_1}\right)\left(\delta+g-\frac{\partial y_2}{\partial k_2}\right)-\frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1}\right] < 0$$

We need now to prove that such a value θ^* exists. Using the expression (27), we derive the real parts of the complex characteristic roots such that

$$
\begin{aligned}
Re(\lambda_{1,2}) &= -\frac{\delta+g}{2\theta}\left\{\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}-1\right. \\
&\quad \left.+\ (1-\theta)\left[\frac{\beta_2-\alpha_2+\beta_1-\alpha_1}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}+1\right]\right\} \\
&\equiv -\frac{(\delta+g)[\mathcal{X}+(1-\theta)\mathcal{V}]}{2\theta} = -\frac{(\delta+g)[\mathcal{X}+\mathcal{V}-\theta\mathcal{V}]}{2\theta} \\
Re(\lambda_{3,4}) &= \frac{1}{2\theta}\left\{\theta\delta\left[\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}+1\right]\right. \\
&\quad +\ \theta g\left[\frac{\alpha_2-\gamma_2+\alpha_1-\beta_1+\gamma_2\beta_1-\gamma_1\beta_2}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}-1\right] \\
&\quad \left.-\ (1-\theta)(\delta+g)\frac{\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}\right\} \\
&\equiv \frac{\theta(\delta\mathcal{Y}+g\mathcal{X})-(1-\theta)(\delta+g)\mathcal{W}}{2\theta} = \frac{\theta[\delta\mathcal{Y}+g\mathcal{X}+(\delta+g)\mathcal{W}]-(\delta+g)\mathcal{W}}{2\theta}
\end{aligned}
$$

with

$$
\mathcal{V} = \frac{\beta_2-\alpha_2+\beta_1-\alpha_1}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}+1 \text{ and } \mathcal{W} = \frac{\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)}{\beta_1(\alpha_2-\gamma_2)+\beta_2(\gamma_1-\alpha_1)+\gamma_2\alpha_1-\gamma_1\alpha_2}
$$

Note that $\mathcal{W} = -(\mathcal{X}+\mathcal{V})$. Let us then assume that $\mathcal{X} < 0$ and $\mathcal{Y} > 0$. We immediately derive that $Re(\lambda_{1,2}) > 0$ when $\theta = 1$, while the sign of $Re(\lambda_{1,2})$ is given by the sign of $-(\mathcal{X}+\mathcal{V})$ when θ is close to zero. Therefore, as $Re(\lambda_{1,2})$ is a monotone function of θ, it follows that there exists θ^* such that $Re(\lambda_{1,2})|_{\theta=\theta^*} = 0$ if and only if $\mathcal{X}+\mathcal{V} > 0$ or equivalently if and only if condition (28) holds. It is also worthwhile to notice that condition (28) is equivalent to $\partial c_0/\partial k_1 < 0$ and thus $\partial c_1/\partial k_1 < 0$, and implies $\mathcal{V} > 0$. The bifurcation value θ^* is then given by

$$
\theta^* = \frac{\mathcal{X}+\mathcal{V}}{\mathcal{V}} = \frac{\gamma_2(1-\beta_1)-\beta_2(1-\gamma_1)}{(\beta_1-\alpha_1)(1-\gamma_2)+(\beta_2-\alpha_2)(1-\gamma_1)+\beta_1\alpha_2-\alpha_1\beta_2} \in (\underline{\theta}, 1)
$$

We then get $Re(\lambda_{1,2}) > 0$ when $\theta \in (\theta^*, 1]$ and $Re(\lambda_{1,2}) < 0$ when $\theta \in (\underline{\theta}, \theta^*)$, with obviously

$$
\frac{dRe(\lambda_{1,2})}{d\theta}\Big|_{\theta=\theta^*} > 0 \tag{50}
$$

As explained in the proof of Proposition 3, the stability properties of the periodic orbit depend on the sign of a parameter which is obtained through a Taylor expansion of degree 3 of the dynamical system (16) on the dimension-2 center manifold associated to the two bifurcating eigenvalues. If this parameter is negative the periodic orbit occurs on the right neighborhood of θ^* and is stable on the dimension-2 center manifold, while if this parameter is positive the periodic orbit occurs on the left neighborhood of δ^* and is unstable on the dimension-2 center manifold (see Guckenheimer and Holmes [20], Theorem 3.4.2, p.151-152).

Since the bifurcating eigenvalues are such that $Im(\lambda_{1,2}) = \sqrt{-\Delta^{\theta^*}}$, we derive from (50) that for every θ with $|\theta-\theta^*|$ sufficiently small, the periodic orbit is characterized by a period $T = T(\theta)$ such that

$$
\lim_{|\theta-\theta^*|\to 0} T(\theta) = \frac{2\pi}{\sqrt{-\Delta^{\theta^*}}} = T^*
$$

In other words, for θ very nearly equal to θ^*, the period of the (emergent) periodic orbits

of (16) nearly equals the period of the concentric periodic orbits of the linearized system characterized by the Jacobiam matrix (34) with $\theta = \theta^*$ and $\sigma = 0$.

Let us consider now $Re(\lambda_{3,4})$. As proved by Proposition 3, there exists $\delta^* > 0$ such that $\delta\mathcal{Y} + g\mathcal{X} < 0$ when $\delta < \delta^*$ and $\delta\mathcal{Y} + g\mathcal{X} > 0$ when $\delta > \delta^*$. Since

$$Re(\lambda_{3,4}) = \frac{\theta(\delta\mathcal{Y}+g\mathcal{X})+(1-\theta)(\delta+g)(\mathcal{X}+\mathcal{V})}{2\theta}$$

we immediately conclude that when $\delta > \delta^*$ and condition (28) holds, $Re(\lambda_{3,4}) > 0$ for any $\theta \in (0, 1]$. If we consider instead that $\delta < \delta^*$ and that condition (28) still holds, then we can also find a bifurcation value $\hat{\theta}$ such that $Re(\lambda_{3,4}) = 0$. Indeed, we have in this case that $Re(\lambda_{3,4}) < 0$ when $\theta = 1$ while $Re(\lambda_{3,4}) > 0$ when θ is close to zero. Since $Re(\lambda_{3,4})$ is also a monotone function of θ, we conclude that there exists $\hat{\theta}$ such that $Re(\lambda_{3,4})|_{\theta=\hat{\theta}} = 0$ with

$$\hat{\theta} = \frac{(\delta+g)(\mathcal{X}+\mathcal{V})}{(\delta+g)(\mathcal{X}+\mathcal{V})-(\delta\mathcal{Y}+g\mathcal{X})}$$

and

$$\left.\frac{dRe(\lambda_{3,4})}{d\theta}\right|_{\theta=\hat{\theta}} < 0 \tag{51}$$

It is easy to derive that when $\delta < \delta^*$ and condition (28) holds, $\hat{\theta} \in (\theta^*, 1)$. As previously, if the parameter obtained through a Taylor expansion of degree 3 of the dynamical system (16) on the dimension-2 center manifold associated to the two bifurcating eigenvalues is negative the periodic orbit occurs on the left neighborhood of θ^* and is stable on the dimension-2 center manifold, while if it is positive the periodic orbit occurs on the right neighborhood of δ^* and is unstable on the dimension-2 center manifold (see Guckenheimer and Holmes [20], Theorem 3.4.2, p.151-152).

As the bifurcating eigenvalues are such that $Im(\lambda_{3,4}) = \sqrt{-\Delta^{\hat{\theta}}}$, we derive from (51) that for every θ with $|\theta - \hat{\theta}|$ sufficiently small, the periodic orbit is characterized by a period $T = T(\theta)$ such that

$$\lim_{|\theta-\hat{\theta}|\to 0} T(\theta) = \frac{2\pi}{\sqrt{-\Delta^{\hat{\theta}}}} = \hat{T}$$

In other words, for θ very nearly equal to $\hat{\theta}$, the period of the (emergent) periodic orbits of (16) nearly equals the period of the concentric periodic orbits of the linearized system characterized by the Jacobiam matrix (34) with $\theta = \hat{\theta}$ and $\sigma = 0$.

□

8.8 Proof of Theorem 2

Tedious but straightforward computations using (37)-(38) and the expressions of $\mathcal{A}_1$ and $\mathcal{A}_2$ give

$$
\begin{aligned}
\Theta_\theta &= -\begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1} & \frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g \end{pmatrix} \\
&\times \begin{pmatrix} \frac{\partial y_1}{\partial p_1} - \frac{\partial c_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} - \frac{\partial c_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\
&- \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix}^2 \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\
&+ \begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_1}{\partial k_2} \\ \frac{\partial y_2}{\partial k_1} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial p_1} & \frac{\partial y_1}{\partial p_2} \\ \frac{\partial y_2}{\partial p_1} & \frac{\partial y_2}{\partial p_2} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial k_1} - g - \delta & \frac{\partial y_2}{\partial k_1} \\ \frac{\partial y_1}{\partial k_2} & \frac{\partial y_2}{\partial k_2} - g - \delta \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\
&+ \frac{\partial c_0}{\partial k_1} \left\{ \left(2\delta - \frac{\partial c_1}{\partial k_1}\right) \left(p_1 \frac{\partial c_1}{\partial p_1} + p_2 \frac{\partial c_1}{\partial p_2}\right) - \frac{\partial c_0}{\partial k_1} \frac{\partial c_1}{\partial p_1} - \frac{\partial c_0}{\partial k_2} \frac{\partial c_1}{\partial p_2} \right\}
\end{aligned}
$$

Denoting the vector $P = (p_1 \;\; p_2)$ and P^t its transpose, we can reformulate Θ_θ as follows:

$$
\begin{aligned}
\Theta_\theta &= -P \left[\frac{\partial y}{\partial k} - (g+\delta)I\right] [\mathcal{J}_1] [\mathcal{J}_2] P^t - P \left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^2 \left[\frac{\partial y}{\partial p}\right] P^t \\
&+ P \left[\frac{\partial y}{\partial k} - (g+\delta)I\right] \left[\frac{\partial y}{\partial p}\right] \left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^t P^t \\
&+ \frac{\partial c_0}{\partial k_1} \left\{ \left(2\delta - \frac{\partial c_1}{\partial k_1}\right) \left(p_1 \frac{\partial c_1}{\partial p_1} + p_2 \frac{\partial c_1}{\partial p_2}\right) - P \left[\frac{\partial y}{\partial k} - (g+\delta)I\right] \left[\frac{\partial y}{\partial p}\right] P^t \right\}
\end{aligned}
$$

with $[\mathcal{J}_1]$ and $[\mathcal{J}_2]$ the 2×2 matrices as defined in (34). We know that the matrices $\left[\frac{\partial y}{\partial p}\right]$ and $[\mathcal{J}_2]$ are quasi-positive definite. As shown before, under conditions (23) and (25), the matrices $\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]$ and $\left[\frac{\partial y}{\partial k} - (g+\delta)I\right]^2$ are quasi-negative definite. Consider then matrix $[\mathcal{J}_1]$. We have

$$
\mathcal{D}et\,[\mathcal{J}_1] = \left(\frac{\partial y_1}{\partial k_1} - g - \frac{\partial c_1}{\partial k_1}\right) \left(\frac{\partial y_2}{\partial k_2} - g\right) - \frac{\partial y_2}{\partial k_1} \left(\frac{\partial y_1}{\partial k_2} - \frac{\partial c_1}{\partial k_2}\right) \geq 0
$$

and tedious computations show that under conditions (23) and (25) the matrices

$$
\left[\frac{\partial y}{\partial k} - (g+\delta)I\right] [\mathcal{J}_1] [\mathcal{J}_2] \;\text{ and }\; \left[\frac{\partial y}{\partial k} - (g+\delta)I\right] \left[\frac{\partial y}{\partial p}\right]
$$

are respectively negative definite and positive definite with real roots. Recalling that condition (28) implies $\partial c_0/\partial k_1 < 0$ and thus $\partial c_1/\partial k_1 < 0$. It follows therefore that Θ_θ is positive for any $\theta \in (0, 1]$.

Following the same lines as in the proof of Proposition 4, for any given $\theta \in (0, 1]$, there exists $\bar{\sigma}_\theta > 0$ such that the characteristic roots are complex if and only if $\sigma \in [0, \bar{\sigma}_\theta)$ with $\bar{\sigma}_\theta$ and increasing function of θ. Moreover, consider that the derivatives $\partial y_1/\partial k_1$, $\partial y_1/\partial k_2$, $\partial y_2/\partial k_1$, $\partial y_2/\partial k_2$ and $\partial c_0/\partial k_1$ do not depend on θ. Tedious but straightforward computations allow to show that under conditions (23), (25) and (28), Γ_θ as given by

$$
\begin{aligned}
\Gamma_\theta &= c_0(1-\theta) \Bigg[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right) \left(\delta + g - \frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2} \frac{\partial y_2}{\partial k_1} + \frac{\left(\frac{\partial y_1}{\partial k_1} - g + \frac{\partial y_2}{\partial k_2} - g\right) \frac{\partial c_0}{\partial k_1}}{p_1} \\
&- \frac{1-\theta}{\theta p_1^2} \left(\frac{\partial c_0}{\partial k_1}\right)^2 \Bigg]
\end{aligned}
$$

is a hump-shape function of θ over $[0,1]$ with $\Gamma_1 = 0$, $\partial\Gamma_\theta/\partial\theta|_{\theta=0} > 0$, $\lim_{\theta\to 0}\Gamma_\theta < 0$, and, when $\theta = \theta^*$,

$$\Gamma_{\theta^*} = c_0(1-\theta^*)\left[\left(\delta + g - \frac{\partial y_1}{\partial k_1}\right)\left(\delta + g - \frac{\partial y_2}{\partial k_2}\right) - \frac{\partial y_1}{\partial k_2}\frac{\partial y_2}{\partial k_1} + \frac{2\delta}{p_1}\frac{\partial c_0}{\partial k_1}\right] > 0$$

We conclude therefore that there exists $\tilde{\theta} \in [\underline{\theta}, \theta^*)$ such that $\Gamma_\theta \geq 0$ for any $\theta \in [\tilde{\theta}, 1]$. The rest of the proof, available upon request, follows exactly the same lines as the proof of Proposition 4 and Theorem 1.

□

9 Appendix 2

9.1 Spectral analysis

The second-order properties of a second-order stationary, zero mean time series $\{X_t\}_{-\infty<t<+\infty}$ may be described either by the autocovariances $\gamma_X(h) = E(X_{t+h}X_t)$ or, equivalently, by the spectral density $f(\omega)$ of the process, which is given by:

$$f_X(\omega) = \frac{1}{2\pi}\sum_{h=-\infty}^{+\infty}\gamma_X(h)e^{-i\omega h}, \quad \omega \in [-\pi;\pi].$$

provided that $\sum_{h=-\infty}^{+\infty}|\gamma_{ij}(h)| < \infty$. On the other hand, using the inverse of the Fourier transform, one can retrieves the autocovariances as:

$$\gamma_X(h) = \int_{-\pi}^{\pi} e^{i\omega h} f_X(\omega)d\omega.$$

In particular, for $h = 0$ and $\omega \in [-\pi, \pi]$, this gives the variance-covariance matrix of X_t,

$$\gamma_X(0) = \int_{-\pi}^{\pi} f_X(\omega)d\omega.$$

Thus, the spectrum decomposes the variance of X into components of different frequencies. More generally, for any ω between 0 and π, the expression

$$\int_{-w}^{w} f_X(\omega)d\omega$$

corresponds to the fluctuations of X_t associated with the frequency band $[-w;w]$. This allows to examine the fluctuations of the series for frequency intervals of interest as business cycle, medium-term or low frequencies.

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