A Theory of Labor Market Fluctuations and Sticky Wages*

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March 23, 2022

Abstract

We develop a continuous-time model of directed job search in a frictional labor market with idiosyncratic shocks to worker productivity, sticky wages, and two-sided lack of commitment. We formulate this problem as a nonzero-sum stochastic differential game with stopping times and prove the existence of a unique equilibrium. Two-sided limited commitment gives rise to endogenous match separations through both worker quits and firm layoffs. We fully characterize both sides' separation policies as a function of model fundamentals, including the opportunity flow costs as well as the drift and volatility of the stochastic process for workers' productivity. We use the model to provide an analytical mapping between (i) the observed rates of wage adjustments and employment transitions, and (ii) the unobserved distribution of the wage-to-productivity ratios. We show that the latter is a sufficient statistic that can be used to quantify the prevalence of inefficient job separations due to nominal and contracting frictions.

Keywords: D20, D31, E12, E32

JEL Classification: Directed search; nonzero-sum stochastic differential games; stopping times; sticky wages; earnings risk; aggregate fluctuations

*We thank Andreas Mueller as well as seminar participants at the University of Texas at Austin and the University of Michigan for helpful comments. Any errors are our own.

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1 Introduction

Inflation is back in the spotlight. Following the shock of the COVID-19 pandemic, economies worldwide are facing a surge in prices, with an average 5.2% annual inflation across OECD countries. At the same time, a record number of job quits (sometimes termed the "Great Resignation"), large unemployment fluctuations, and new social insurance policies have revived interest among policymakers and academics in labor market dynamics. For example, how do wages and employment adjust following a large increase in inflation? What economic mechanisms determine the relative magnitudes of flows into and out of jobs? And how are labor market fluctuations linked to sticky wages?

This paper develops a continuous-time search and matching theory of idiosyncratic and aggregate labor market risk with sticky wages and two-sided lack of commitment. The purpose is to develop a microfounded labor market model to answer the questions posed above. We consider an economy with a continuum of workers facing idiosyncratic labor productivity shocks. At any moment, a worker can be employed or unemployed. If a worker is employed, the match can be endogenously dissolved by each of the two sides unilaterally, either because the worker quits or because the firm fires the worker. Job matches also separate exogenously at some Poisson rate. As in Menzio and Shi (2010), unemployed workers search for jobs in frictional labor markets, which are segmented across workers' productivity and nominal wages. Finally, firms post vacancies to attract workers and, whenever a match between a firm and a worker is formed, generate output based on the worker's productivity, a part of which the worker gets paid according to a posted, rigid wage.

In the environment described above, we characterize: (1) The existence of a Block recursive equilibrium and the economic mechanisms determining firms' and workers' endogenous separation decisions; (2) a one-to-one mapping between labor market data and the consequences of sticky wages across workers; and (3) the labor market reaction to a monetary shock. Critical in our analysis, we show how the steady-state consequences of sticky wages for labor income risk provides sufficient statistics for the response of the labor market to aggregate shocks.

Existence of Block recursive equilibrium with endogenous separations. We study a continuous-time version of Menzio and Shi (2010), where a worker's productivity follows a diffusion process. Since at any point in time, workers or firms can terminate the match, there is a strategic consideration of *when* to dissolve the match. We recast the game as a nonzero-sum stochastic differential game with stopping time (Bensoussan and Friedman, 1977). This approach describes the games' equilibrium conditions across players with a set of quasi-variational inequalities over workers' and firms' value functions. There are

two payoffs of our approach.

First, we can show the existence of a unique block recurve equilibrium with endogenous separations and sticky wages. Theorems for the existence of block recursive equilibrium uses Schauder's fixed point theorem (see, for instance, Menzio and Shi, 2010; Schaal, 2017). Two conditions are critical to apply this theorem: Continuity in the value functions and continuity in the mapping between value functions characterizing the Block recursive equilibrium. Endogenous separations break these regularity conditions needed to show the equilibrium existence.

To understand how endogenous separations break the conditions to show equilibrium's existence, we focus on the continuity of the value functions—the same economic argument holds for the continuity of the functional equations. Assume a wage high enough that the firm is indifferent between firing or keeping the worker. As a result, if there is a marginal increment in the wage, then the firm fires the worker, but its value does not decrease—the firm was indifferent at the initial wage. Nevertheless, the worker's value is discontinuous at this wage-firing trigger. The reason is that the current wage is strictly higher than the flow value of unemployment, and the employment continuation value is at least higher than the unemployment value. Therefore, with any marginal increment in the wage, the worker losses a discrete amount of current payoffs during employment, and the worker's value function jumps at this point. This argument does not apply in the continuous-time version of the model since the current flow payoff has zero measure. While the strategy for proving the existence of a block recursive equilibrium in the continuous-time version of the model is different than in the discrete-time counterpart, this paper provides the first step in our simplified environment.

Second, we characterize the economic mechanisms determining the quit and the firing decisions of workers and firms as a function of the current wage-to-productivity ratio. Three economic mechanisms shape the separation policies across players. First, the flow opportunity cost of firms and workers determine the static motives for separating, which are similar to the mechanisms in the McCall model. Second, the variance of workers' productivity shocks introduces an option value for each player, thus delaying separations. We show that this mechanism is small and bounded. The premise for this result is that delaying separation is like an investment and since there is a lack of commitment across players, firms and workers are reluctant to invest in the match. Therefore, a larger variance of workers' productivity shocks induces more separations. Third, a larger expected growth rate in workers' productivity induces a larger entry wage since workers anticipate a higher future productivity. As the option value effect, we show that this mechanism is small and bounded. The premise for this result is that a higher entry wage can also be interpreted as investment in the match since it delays separations at the cost of deviating from the standard job finding-wage optimality condition. The worker's lack of commitment

across times makes this agent reluctant to invest in the match by increasing the entry wage.

Economic implications of sticky wages on idiosyncratic labor income risk. Since wages are sticky, the wage-to-productivity ratio fluctuates with idiosyncratic productivity shocks. Thus, total income equals a worker's productivity—the exogenous state in our model—times the wage-to-productivity ratio—the endogenous state's in our model. Moreover, since there is a lack of commitment across firms and workers, fluctuations in the wage-to-productivity ratios generate endogenous separations in the form of quits and fires. Thus, a proper theory of labor income risk needs to characterize the observable implications for: (1) how agents' policies shape the distribution of wage-to-productivity ratios and (2) the distribution of wage changes and employment flows. Additionally, this theory needs to show how the observed wage changes and employment flows identify the wage-to-productivity ratio distribution. Now, we summarize these results.

We characterize the observable implications of the model by recasting the model's endogenous state with minus cumulative idiosyncratic productivity shocks since the start of the employment or unemployment spells. This variable offers three advantages: (i) it is single-dimensional, (2) it is stationary, and (3) it is well-defined during unemployment spells. We show that the distribution of minus cumulative idiosyncratic productivity shocks during an employment spell characterizes multiple labor market variables of interest—i.e., the unemployment rate, the job finding and separation probabilities, the distribution of the duration of completed employment and unemployment spells, and distribution of wages growth rate across jobs.

We identify the moments in the distribution of wage changes that identifies the distribution of minus cumulative idiosyncratic productivity shocks during employment, i.e., the critical variable for the consequence of sticky wage and lack of commitment in labor income risk. With this task in mind, we proceed in three steps. First, we recover the parameters governing the stochastic process of idiosyncratic productivity. For example, the average wage change must compensate for the average drift between two employment spells. Thus, the average wage change divided by the sum of the average duration of completed employment and unemployment spells is equal to the drift of idiosyncratic productivity.

The critical challenge is how we can recover the distribution of productivity shocks during the employment state given that we only observe wage changes across jobs, which reflect the combination of productivity shocks experienced during the employment and unemployment spells. Here is where the theory guides the proper thinking about this problem. Suppose that there is a lack of selection in the entry margin into employment—i.e., the job finding rate is independent of the unemployed worker's productivity. Then, after imposing a functional form for the stochastic process of idiosyncratic

productivity—in our case, a random walk with drift—we can identify the distribution of cumulative productivity shocks at the time when workers find a job. That is, after we identifying the distribution of cumulative productivity shocks during unemployment, we can reverse engineer the distribution of cumulative productivity shocks during employment.

Related literature. [TBC]

Outline. The rest of the paper is organized as follows. Section 2 describes the model's environment, equilibrium definition, and characterize equilibrium policies. Section 3 characterizes the one-to-one mapping between the models' endogenous state and observable data.

2 A Model of Idiosyncratic Labor Income Risk and Sticky Wages

In this section, we develop and characterize a parsimonious search and matching model with the following features: a single good is used for consumption or hiring, workers face idiosyncratic productivity shocks, wages are sticky within the match, and workers and firms cannot commit to future actions at the moment of forming the match.

2.1 Environment

Time is continuous, and it is denoted by t. The economy is populated by a measure one of workers and an endogenous mass of firms who meet in a frictional labor market. The firms' and workers' objective is to maximize the net present value of profits and expected discounted utility, respectively. All agents discount the future at rate $\rho > 0$.

Preferences and Production Technology. Workers are risk-neutral. They maximize the expected discounted consumption stream:

$$\mathbb{E}\left[\int_{t=0}^{\infty} e^{-\rho t} c_t \, \mathrm{d}t\right], \tag{1}$$

where $\{c_t\}_{t=0}^{\infty}$ is the stochastic consumption. Without loss of generality due to the linearity of preferences, we assume that workers consume their flow income y_t of the single good, so that $c_t = y_t$.

A worker's income depends on the employment state S_t and the productivity level Z_t. A worker's employment state can be either employed (h) or unemployed (u). While employed, a worker produces

output equal to its idiosyncratic productivity and receives flow income equal to their wage level W_t. While unemployed, a worker receives a flow income equal to home production $B(Z_t)$.

Henceforth, we use lower-case letters to denote the natural logarithm of variables in upper-case letters. For example, z_t is log worker productivity and w_t is the log nominal wage.

Exogenous shocks. Workers' idiosyncratic productivity follows a Brownian motion in logs and can be written as

$$dz_t = \gamma \, \mathrm{d}t + \sigma \, \mathrm{d}\mathcal{W}_t, \tag{2}$$

where γ is the drift, σ is the volatility, and $\mathcal{W}_t$ is a Wiener process. For the time being, we assume that there are no other aggregate shocks—we remove this assumption in Section 4.

Search Frictions. Unemployed workers search for jobs in a frictional labor market. Search is directed, as in (Menzio and Shi, 2010), and segmented across $x = (w, z)$, that consists of the (log) wage w and workers' (log) productivity z. In each market, firms post vacancies $\mathcal{V}$ at cost $K(Z_t)$. Given $\mathcal{U}$ unemployed workers and $\mathcal{V}$ vacancies in a market, a Cobb-Douglas matching function with constant returns produces $m(\mathcal{U}, \mathcal{V}) = \mathcal{U}^{\alpha}\mathcal{V}^{1-\alpha}$ matches, where α is the elasticity of matches to the unemployed rate. Thus, the worker's and the firm's matching rates are denoted by $p(w, z) = \theta(\omega, z)^{1-\alpha}$ and $q(w, z) = \theta(\omega, z)^{-\alpha}$, respectively, where $\theta(w, z) = \mathcal{V}(w, z)/\mathcal{U}(w, z)$ denotes the market tightness in each submarket. Once the worker and the firm get matched, the firm pays the constant wage w to the worker. At any point in time, existing matches get exogenously dissolved, which we model as a Poisson process with the rate δ, or it can be endogenously dissolved either by the firm or the worker.

Agents' Choices. In each period, an unemployed worker chooses the submarket in which to search for a job. This corresponds to a choice of a stopping time τ^u, which is distributed according to a Poisson counter with arrival rate $p(w, z)$. On the other hand, a matched firm chooses the duration of the match before firing the worker, which is denoted by the stopping time τ^j. Similarly, a matched worker chooses the duration of the match before quitting, which is denoted by the stopping time τ^h. Given these two choices, and the exogenous stopping time τ^δ, the duration of a match is determined by the smallest stopping time in the vector $\vec{\tau}^m = (\tau^h, \tau^j, \tau^\delta)$, which we denote by $\tau^m = \min\{\tau^h, \tau^u, \tau^\delta\}$.

Value Functions. In what follows, we describe the value functions of each agent. These values are functions of a worker's and firm's idiosyncratic state w and z, potentially depending on aggregate vari-

ables, such as the joint distribution of workers' productivities, wages, and employment states. Below, we show that our model has a unique Block Recursive Equilibrium (see Menzio and Shi, 2010)—i.e., equilibrium objects do not depend on the distribution of workers' idiosyncratic states. For this reason, we omit aggregate endogenous state in the notation.

The value of an unemployed worker with productivity z is given by

$$U(z) = \max_{\{w_t\}_{t=0}^{\tau^u}} \mathbb{E}\left[\int_0^{\tau^u} e^{-\rho t} B(e^{z_t})\,\mathrm{d}t + e^{-\rho\tau^u}\left[H(w_{\tau^u}, z_{\tau_u}, \vec{\tau}^h(w_{\tau^u}, z_{\tau^u}))\right] \middle| z_0 = z\right]. \tag{3}$$

That is, the unemployed worker searches for a job in submarket (w_t, z_t) until period τ^u, when she becomes employed at wage w_{τ^u} and receives the value of employment $H(w_{\tau^u}, z_{\tau_u}, \vec{\tau}^m(w_{\tau^u}, z_{\tau^u}))$.

For a given vector of stopping times $\vec{\tau}^m$, the value of a worker with productivity z and employed at wage w is given by

$$H(w, z, \vec{\tau}^m) = \mathbb{E}_0\left[\int_0^{\tau^m} e^{-\rho t} e^w\,\mathrm{d}t + e^{-\rho\tau^m} U(z_{\tau^m})\right] \tag{4}$$

—i.e., the employed worker consumes the constant wage w until time τ^m, when she either endogenously or exogenously transitions to unemployment. Similarly, the value of the firm matched with a worker with productivity z is given by

$$J(w, z, \vec{\tau}^m) = \mathbb{E}\left[\int_0^{\tau^m} e^{-\rho t}\left[e^{z_t} - e^w\right]\mathrm{d}t\right] \tag{5}$$

—the match produces e^{z_t}, of which e^w is paid to the worker, until it gets dissolved at time τ^m.

Finally, firms choose the number of vacancies to post in each of the submarkets. The firm trades-off the benefit of posting the vacancy, which is given by the probability of matching with a worker and the value $J(w, z, \vec{\tau}^m)$, and the cost, which is given by the cost of posting the vacancy. Firms post vacancies in a given submarket as long as the expected benefit is at least as large as the cost. Thus, the free entry condition is given by:

$$\left[K(e^{z_t}) - q(w,z)J(w, z, \vec{\tau}^m(w,z))\right]^+ \theta(w,z) = 0 \quad \forall (w,z). \tag{6}$$

and $\theta(w,z) \geq 0$.[1]

[1] We use the notation $[x]^+ := \max\{0, x\}$.

Equilibrium. Having described the environment and agents' choices and values, we now define the equilibrium.

Definition 1. *An equilibrium consists of a set of value functions and market tightness* $\{H(w,z,\vec{\tau}^m), J(w,z,\vec{\tau}^m), U(z), \theta(w,z)$ *and policy functions* $\{\tau^{h*}, \tau^{j*}, w^*(z_t)\}$*, such that:*

1. *Given* $U(z)$*, for all* (w,z)*,* (τ^{h*}, τ^{j*}) *is a Nash Equilibrium; i.e. for all stopping times* (τ^w, τ^j)*,*

$$H(w,z,\tau^{h*},\tau^{j*},\tau^\delta) \geq H(w,z,\tau^h,\tau^{j*},\tau^\delta), \tag{7}$$

$$J(w,z,\tau^{h*},\tau^{j*},\tau^\delta) \geq J(w,z,\tau^{h*},\tau^j,\tau^\delta), \tag{8}$$

   with $Pr(\tau^{m*}=0)=0$ *for* $(w^*(z),z)$*.*

2. *Given* $H(w,z,\vec{\tau}^{m*}(w,z))$ *and* $\theta(w,z)$*,* $\{w(z_t)\}_{t=0}^{\tau^{u*}}$ *solves* (3) *with value* $U(z)$*.*

3. *Given* $J(w,z,\vec{\tau}^{m*}(w,z))$*,* $\theta(w,z)$ *solves the free entry condition* (10)*.*

The first condition defines the Nash equilibrium of the game played by a matched worker-firm pair, in which the worker decides when to quit, and the firm chooses when to fire the worker. It requires that τ^{h*} is the worker's best-response given firms' optimal stopping time τ^{j*}—similar analysis applies to the firm. It is easy to verify that this game always has a trivial Nash equilibrium in which both the firm and the worker choose to dissolve the match immediately. The equilibrium definition rules out this case for the optimal search policy of an unemployed worker.

Commitment. The equilibrium reflects two side lack of commitment. We can see this property because the stopping time functions depend on the full history of shocks. This property is equivalent to the discrete-time counterpart, where both agents decide to continue or stop at each period. The lack of commitment affects the unemployed worker's policies through $H(w,z,\vec{\tau}^{m*}(w,z))$ and the market tightness function through $J(w,z,\vec{\tau}^{m*}(w,z))$. In conclusion, lack of commitment affects the separation margin and the job finding rate.

Homothetic search frictions. Productivity shocks affect agents' decisions because they change: (i) the value of wages relative to productivity and (ii) the value of unemployment and the cost of hiring a worker relative to a worker's productivity. The latter mechanism is captured by the ratios $B(Z_t)/e^{z_t}$ and $K(Z_t)/e^z_t$. This paper focuses on the first mechanism and abstracts away from the second mechanism by assuming search costs and unemployment income that are homothetic in workers' productivity, i.e., $B(Z_t) = \tilde{B}Z_t$, with $\tilde{B} \in (0,1)$ and $K(Z_t) = \tilde{K}Z_t$.

Strategic interaction between firm and worker. The strategic interaction between the firm and the worker has three features. First, the agents play a nonzero-sum game since the per-period value of a match e^z exceeds the values of unemployment $\tilde{B}e^z$ when $\tilde{B} < 1$. Second, the agents' payoffs are stochastic and follow Ito's process. Third, agents' strategies consist of stopping times. Thus, this constitutes a nonzero-sum stochastic differential game with stopping times, whose solution has been previously studied by Bensoussan and Friedman (1977) and is explained in the next section.

2.2 Equilibrium Characterization

We characterize the sufficient equilibrium conditions. Let $j_w(z)$, $h_w(z)$, and $u(z)$ denote the agents' values evaluated at equilibrium policies, where the subscript w conditions on a constant (log) wage w. The HBJ of the unemployed worker is given by

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} + \max_w p(w,z)[h_w(z) - u(z)], \quad \forall z \in \mathbb{R} \tag{9}$$

and equilibrium condition for market tightness is given by

$$[K(e^{z_t}) - q(w,z)j_w(z)]^+ \theta(w,z) = 0 \quad \forall (w,z). \tag{10}$$

We describe the equilibrium conditions for the firm and the employed worker with the help of Figure 1. This figure describes the equilibrium policies, values, and opportunity cost for both agents.

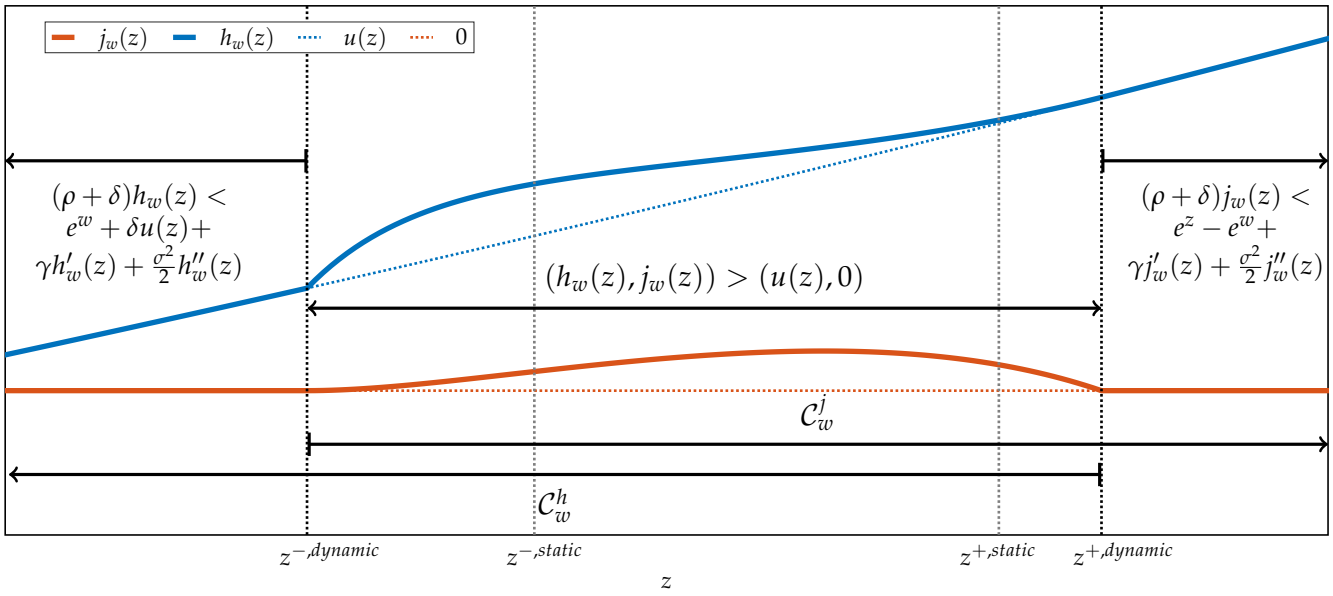
The fact the firms' and workers' can walk away from the match at any time imposes the game's lower bound for both players. This condition can be seen in Figure 1, where it shows that agents value functions bigger or equal than zero. Thus,

$$(h_w(z), j_w(z)) \geq (u(z), 0). \tag{11}$$

We focus in explaining the continuation set for the firm, *mutatis mutandis* for the worker. Let $\mathcal{C}_w^j$ denotes the interior of the set of productivities for which the firm prefers to continue the match at wage w. Importantly, this set includes productivity levels where the worker decides to quit when the firm would like to continue.

If flow profits are positive, the firm would never fire the worker—she has the option to fire the worker at any time in the future. Then, for all productivity levels $z \geq z^{-,static} = w$, the firm prefers not to fire the

FIGURE 1. EQUILIBRIUM VALUES AND OPTIMAL POLICIES



Notes: The figure plots the value functions of workers and firms for a given log wage. The red and blue solid lines show the values functions for the firm and the worker, respectively. The red and blue solid lines shows the opportunity cost for the firm and the worker, respectively. The optimal separation trigger with flow profits for the firm is $z^{-,static} := w$. The optimal trigger with flow profits for the worker satisfies $\tilde{B}e^{z^{+,static}} + \max_{w^*} p(w^*, z^{+,static})[h_{w^*}(z^{+,static}) - u(z^{+,static})] = e^w$. The optimal dynamic separation triggers for the firm and the worker are $z^{-,dynamic} := \max\{z : h_w(z) > u(z)\}$ and $z^{+,dynamic} := \min_z\{z : j_w(z) > 0\}$, respectively. $h_w(z)$ and $u(z)$ are normalized. The parameter values are $(\gamma, \sigma, \rho, \alpha, \tilde{K}, \delta, \tilde{B}) = (0.001, 0.007, 0.03, 0.5, 1, 0.005, 0.4)$.

worker..[2] Moreover, if the worker does not quit, then the equilibrium condition is included in all the z such that

$$j_w(z) > 0. \tag{13}$$

In Figure 1, the productivity levels with non-negative profits and continuation on the side of the worker are given for all $z \in (z^{-,static}, z^{+,dynamic})$. On the other end, if the worker quits, the equilibrium condition is described by

$$\rho j_w(z) < e^z - e^w + \gamma \frac{\partial j_w(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 j_w(z)}{\partial z^2} - \delta j_w(z). \tag{14}$$

The argument for this result is the simple. If the worker quits, then $j_w(z) = \frac{\partial j_w(z)}{\partial z} = \frac{\partial^2 j_w(z)}{\partial z^2} = 0$ and (14) becomes

$$z \geq w = z^{+,static}. \tag{15}$$

Thus, as Figure 1 shows, the productivity levels with non-negative profits and quitting on the side of the worker are given for all $z \in (z^{+,dynamic}, \infty)$. While until we focus with flow payoffs consideration to describe the firms' continuation set, condition (13) does not only describe static trade-off, but also dynamic considerations. If static profits are negative, it could be the case that firing the worker is suboptimal, e.g., if future productivity is high enough.

With this discussion, we can see that the continuation set is describe by two conditions. If both agents decide two continue, then condition (13) must holds; and if only the firms prefers to continue, then condition (14) must hold. Thus, the continuation set for the firm is

$$\mathcal{C}_w^j := int \left\{ z \in \mathbb{R} : j_w(z) > 0 \; or \; \rho j_w(z) \leq e^z - e^w + \gamma \frac{\partial j_w(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 j_w(z)}{\partial z^2} - \delta j_w(z) \right\} \tag{16}$$

and for the worker is

$$\mathcal{C}_w^h := int \left\{ z \in \mathbb{R} : h_w(z) > u(z) \; or \; \rho h_w(z) \leq e^w + \gamma \frac{\partial h_w(z)}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 h_w(z)}{\partial z^2} + \delta \left(u(z) - h_w(z) \right) \right\} \tag{17}$$

With the player's continuation set, we can describe the equilibrium conditions for the productivity levels inside and outside other agent continuation set.

[2] For the worker, the flow payoff of continue in the match relative to quitting is given by all productivity levels such that

$$z \leq z^{+,static}, \text{ where } e^w = \underbrace{\tilde{B} e^{z^{+,static}} + \max_{w^*} p(w^*, z^{+,static}) [h_{w^*}(z^{+,static}) - u(z^{+,static})]}_{\text{flow payoff of unemployment}}. \tag{12}$$

If $z \in \mathcal{C}_w^h$, then worker prefers to continue and the firm's value function satisfies

$$j_w(z) = \max\left\{(e^z - e^w)\,\mathrm{d}t + e^{-\rho\,\mathrm{d}t}\mathbb{E}_z[j_w(z')], 0\right\}, \tag{18}$$

in the limit,

$$\rho j_w(z) = \max\left\{e^z - e^w + \gamma\frac{\partial j_w(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j_w(z)}{\partial z^2} - \delta j_w(z), 0\right\}. \tag{19}$$

Similarly for the worker, we have that for all $z \in \mathcal{C}_w^j$,

$$\rho h_w(z) = \max\left\{e^w + \gamma\frac{\partial h_w(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h_w(z)}{\partial z^2} + \delta\left(u(z) - h_w(z)\right), \rho u(z)\right\}. \tag{20}$$

In opposite case, if any player stop the game, then the other player gets her opportunity value. Therefore, the worker's and the firm's values should satisfy the following conditions

$$\forall z \in (\mathcal{C}_w^j)^c : h_w(z) = u(z) \text{ and } \forall z \in (\mathcal{C}_w^h)^c : j_w(z) = 0. \tag{21}$$

Condition (21) defines the game's value matching conditions. Finally, the probability one of continuation for a new match is given by the non-emptiness of the game's continuation set given by at the optimal entry wage

$$\mathcal{C}_{w^*(z)}^h \cap \mathcal{C}_{w^*(z)}^j \neq \emptyset. \tag{22}$$

The worker quits the match and the firm fires the worker at times τ^h and τ^j, respectively, whenever the productivity is outside their continuation region. With the optimality condition for the entry wage, agents' policies are:

$$w^*(z) = \arg\max_w \theta(w,z)^{1-\alpha}\left(h_w(z) - u(z)\right), \tag{23}$$

$$\tau^h(w,z) = \inf\left\{t \geq 0 : z_t \notin \mathcal{C}_h^w\right\}, \tag{24}$$

$$\tau^j(w,z) = \inf\left\{t \geq 0 : z_t \notin \mathcal{C}_j^w\right\}. \tag{25}$$

Next proposition formalizes the sufficient equilibrium conditions.

Lemma 1. *Suppose that the values functions $(j_w(z), h_w(z), u(z), \theta(w,z))$ satisfy HBJ for the unemployed worker (9), free entry condition (10), the game's lower bound (11), game's value matching conditions (21) and variation inequalities for each player (19) and (20) given the continuation set (17) and (16). Then, the policy functions*

$\{\tau^{h*}, \tau^{j*}, w^*(z_t)\}$ *given by* (23)-(25) *with* $\{H(w,z,\vec{\tau}^m), J(w,z,\vec{\tau}^m), u(z), \theta(w,z)\}$ *are a block recursive equilibrium with*

$$h_w(z) = H(w,z,\tau^{h*}(w,z),\tau^{j*}(w,z),\tau^{\delta}) \tag{26}$$

$$j_w(z) = J(w,z,\tau^{h*}(w,z),\tau^{j*}(w,z),\tau^{\delta}) \tag{27}$$

$$u(z) = U(z) \tag{28}$$

With the equilibrium conditions, we can now prove the existence of a unique block recursive equilibrium.

Proposition 1. *There exists a unique Block Recursive Equilibrium, in which* $u(\cdot)$, $h_w(\cdot)$, $j_w(\cdot)$, $\theta(\cdot)$, *and the set* $\{\tau^{h*}, \tau^{j*}, \{w_t\}_{t=0}^{\tau^u}\}$.

Proposition 1 shows the existence and uniqueness of a Block Recursive equilibrium in our model with lack of commitment and fixed wage contracts. As far as we know, this proof had been elusive in the literature and it is core part of search and matching models with sticky wages and lack of commitment. Theorems for the existence of block recursive equilibrium with exogenous separation uses Schauder's fixed point theorem (see, for instance Menzio and Shi, 2010; Schaal, 2017). Two conditions are critical to apply this theorem: Continuity in the value functions and continuity in the mapping between value functions characterizing the Block recursive equilibrium. Sticky wages and lack of commitment jointly generate inefficient endogenous separations breaking these regularity conditions needed to show the equilibrium existence.

To understand how endogenous separations break the conditions to show equilibrium's existence, we focus on the continuity of the value functions—the same economic argument holds for the continuity of the functional equations. Assume a wage high enough that the firm is indifferent between firing or keeping the worker. As a result, if there is a marginal increment in the wage, then the firm fires the worker, but its value does not decrease—the firm was indifferent at the initial wage. Nevertheless, the worker's value is discontinuous at this wage-firing trigger. The reason is that the current wage is strictly higher than the flow value of unemployment, and the employment continuation value is at least higher than the unemployment value. Therefore, with any marginal increment in the wage, the worker losses a discrete amount of current payoffs during employment, and the worker's value function jumps at this point. This argument does not apply in the continuous-time version of the model since the current flow payoff has zero measure. While the strategy for proving the existence of a block recursive equilibrium in the continuous-time version of the model is different than in the discrete-time counterpart, this paper provides the first step in our simplified environment.

2.3 Economic mechanisms

We next characterize the optimal equilibrium policies. We first discuss the general properties of the equilibrium. Then, we provide a sharper characterization by analyzing the model with constant idiosyncratic productivity, followed by an independent analysis of the effects of the drift and the stochastic component of productivity.

Finding the State. To further describe equilibrium allocations, it is worth noting that we can recast equilibrium conditions in terms of a smaller state space. Since the flow income of the unemployed worker and the firm's vacancy cost are proportional to productivity e^{z_t}, the relevant state for a worker and a firm is given by the wage-to-productivity ratio $\hat{w} := w - z$. Then, we can normalize equilibrium values in the following way. Let $\tilde{U} = u(z)/e^z$ denote the productivity-normalized value of an unemployed worker with productivity z. Similarly, let $\tilde{J}(w-z) = j_w(z)/e^z$ denote the productivity-normalized value of a firm matched with a worker of productivity z at wage w, and let $\tilde{W}(w-z) = (h_w(z) - u(z))/e^z$ denote the worker's productivity-normalized *net* value of employment. The equilibrium policies can be similarly normalized by $\hat{w}^* = w^*(z) - z$, and $\tilde{\theta}(w-z) = \theta(w,z)$. Finally, to simplify notation, we redefine the drift $\nu := \gamma + \sigma^2$ and the discount factor $\tilde{\rho} =: \rho - \gamma - \sigma^2/2$. Appendix X shows the mapping between the original and the transformed equilibrium conditions.

The following Proposition 2 provides a general characterization of equilibrium allocations.

Proposition 2. *Let $\tilde{U}$, $\tilde{W}(\hat{w})$ and $\tilde{J}(\hat{w})$ denote equilibrium values, $\tilde{\mathcal{C}}^h$ and $\tilde{\mathcal{C}}^j$ denote equilibrium continuation regions, and $\hat{w}^*$ denote equilibrium wage-to-productivity ratio. Then,*

1. *the worker's opportunity cost of employment satisfies*

$$\tilde{B} \leq \tilde{\rho}\tilde{U} \leq 1$$

2. *the worker's and firm's continuation sets satisfy:*

$$\{\hat{w} : \hat{w} > \log(\tilde{\rho}\tilde{U})\} \subset \{\hat{w} : \hat{w} > \hat{w}^-\} = \tilde{\mathcal{C}}^h$$
$$\{\hat{w} : \hat{w} < 0\} \subset \{\hat{w} : \hat{w} < \hat{w}^+\} = \tilde{\mathcal{C}}^j,$$

3. *$\hat{w}^*$ satisfies the "Nash Bargaining" solution*

$$\hat{w}^* = \arg\max_{\hat{w}} \left\{ \tilde{J}(\hat{w})^{1-\alpha} \tilde{W}(\hat{w})^{\alpha} \right\}$$

4. values satisfy the smooth pasting conditions

$$\tilde{W}'(\hat{w}^-) = \tilde{J}'(\hat{w}^+) = 0.$$

The proposition first states that in equilibrium the productivity-normalized flow payoff of the unemployed worker is bounded between the values of the flow income when unemployed and output per efficiency unit of labor. Furthermore, it shows that optimal stopping times, and the corresponding continuation sets, satisfy threshold rules. Workers choose to stay in the match as long as the wage-to-productivity ratio $\hat{w}$ is higher than the threshold $\hat{w}^-$. Similarly, firms choose not to fire workers as long as $\hat{w}$ is lower than the threshold $\hat{w}^+$. Thus, the continuation set for the *match* is given by $\tilde{\mathcal{C}}^h \cap \tilde{\mathcal{C}}^j$. Importantly, these thresholds satisfy $\hat{w}^- < \log(\tilde{\rho}\tilde{U})$ and $\hat{w}^+ > 0$, implying that both the firm and the worker are willing to accept flow payoffs from the match that are lower than the flow payoff from their outside options. We refer to this result as the *option value effect*, which we discuss below.

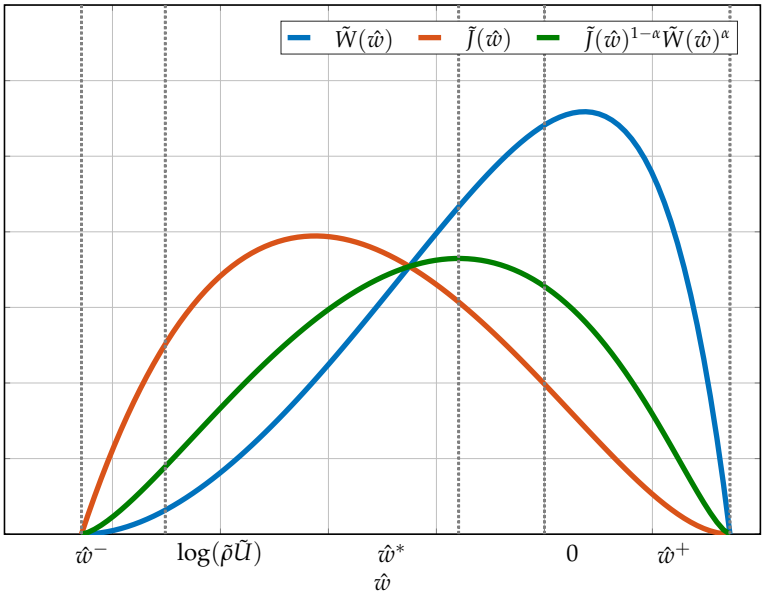
The next result is standard and states that the unemployed worker searches for a job in the submarket that offers a wage determined by the Nash bargaining solution, in which the firm's bargaining power is given by the elasticity of the job-finding rate. To understand this result, notice that in (9) the optimal wage $w^*(z)$ balances the trade-off between a higher employment value of high-paying jobs and a higher job-finding rate of low-paying jobs. The free entry condition imposes that the job-finding rate is proportional to the firm's value $j_w(z)$.

The last part of the proposition requires that equilibrium values satisfy the smooth pasting conditions at the boundaries of the respective continuation sets. The fact that the derivates of values are zero at the boundaries guarantees that in expectation, there are no gains from postponing or bringing forward the stopping time; i.e., the continuation regions are optimal.

Figure 2 condenses all the results in Lemma 1 and Proposition 2. For example, consider the value of the worker. At the separation trigger $\hat{w}^-$, the value satisfies the value matching and the smooth pasting conditions. As $\hat{w}$ increases, the value is affected by two forces: A higher normalized wage directly increases the flow utility of the worker and reduces the likelihood of a quit, but reduces the value of the firm and increases the likelihood of being laid-off. The former dominates for low $\hat{w}$ and the value $\tilde{W}(\hat{w})$ is increasing; at higher wages, the latter dominates and the value is decreasing. As the wage approaches the firing trigger $\hat{w}^+$, the value of the employed worker approaches the value of unemployment (i.e. $\tilde{W}(\hat{w}) \to 0$). A similar intuition applies to the firm.

To better characterize the forces behind the optimal policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ in isolation, we next analyze the equilibrium without idiosyncratic shocks. Then, we study the role played by the stochastic

FIGURE 2. EQUILIBRIUM VALUES AND OPTIMAL POLICIES



Notes: The Figure plots the equilibrium values of the firm $\tilde{J}(\hat{w})$ and worker $\tilde{W}(\hat{w})$, and the objective of the"Nash bargaining" problem. The vertical lines mark the boundaries of the continuation sets $\tilde{\mathcal{C}}^j$ and $\tilde{\mathcal{C}}^h$, the optimal wage-to-productivity ratio $\hat{w}^*$.

component of idiosyncratic productivity and its drift.

Non-stochastic equilibrium. We start by discussing the role played by the opportunity cost of the firm and the worker. We assume away any changes in idiosyncratic productivity—i.e., we set $\nu = \sigma = 0$. The following proposition fully characterizes the equilibrium.

Proposition 3. *Assume $\nu = \sigma = 0$. Then, in equilibrium,*

1. *optimal policies are given by*

$$\exp(\hat{w}^-,\ \hat{w}^*,\ \hat{w}^+) = (\tilde{\rho}\tilde{U},\ \alpha + (1-\alpha)\tilde{\rho}\tilde{U},\ 1),$$

2. *the flow value of unemployment is given by*

$$\tilde{\rho}\tilde{U} = \tilde{b} + \Theta\left(1 - \tilde{\rho}\tilde{U}\right)^{1/\alpha},$$

where $\Theta := \left(\frac{1}{\tilde{\rho}+\delta}\left(\frac{1-\alpha}{\tilde{K}}\right)^{1-\alpha}\right)^{\frac{1}{\alpha}}\alpha$.

The separation triggers of the worker and the firm are given by the flow payoffs of the unemployed worker and the match, respectively. Since productivity is constant, a firm is willing to pay the worker

at most the full value of the current output. Similarly, the worker is willing to participate only if she receives a wage higher than the flow value of unemployment. In equilibrium, the optimal reset wage-to-productivity ratio is given by the weighted average of these two. Given this optimal wage, neither the worker nor the firm wants to endogenously dissolve the match. The last part of Proposition 3 characterizes the flow value of unemployment.

Note that the lack of commitment and nominal frictions do not generate any inefficiencies. This is because the only matches created in equilibrium are those with a positive surplus. Since productivity is fixed, it is then efficient to keep such matches for as long as possible. Because all separations are exogenous, gains from trade are maximized and the equilibrium is efficient.

Role of Volatility. To understand the role played by the volatility of productivity, we assume away the drift component—i.e., $\nu = 0$. The following proposition characterizes the equilibrium allocation for the special case $\alpha = 1/2$ and makes use of a first-order approximation of flow payoffs.

Proposition 4. *Assume $\nu = 0$ and $\alpha = 1/2$. Up to a first-order approximation of the flow payoffs around $\hat{w}^*$,*

1. *optimal policies are given by*

$$\exp(\hat{w}^-,\ \hat{w}^*,\ \hat{w}^+) = \left(\exp(\hat{w}^* - \iota(\sigma)),\ \alpha + (1-\alpha)\tilde{\rho}\tilde{U},\ \exp(\hat{w}^* + \iota(\sigma))\right),$$

   where $l(\sigma) \in [\Phi, 3\Phi]$, $l'(\sigma) > 0$ and $\Phi = \frac{1-\tilde{\rho}\tilde{U}}{1+\tilde{\rho}\tilde{U}}$,

2. *the flow value of unemployment is given by*

$$\tilde{\rho}\tilde{U} = \tilde{b} + \xi(\sigma)\Theta\left(1 - \tilde{\rho}\tilde{U}\right)^{1/\alpha},$$

   where $\xi(\sigma)$ is a function of volatility σ that satisfies $\xi'(\sigma) < 0$, $\lim_{\sigma\to 0}\xi(\sigma) = 1$, and $\lim_{\sigma\to\infty}\xi(\sigma) = 1/4$.

The first part of the proposition states that idiosyncratic volatility does not affect the split of the *flow* surplus between the worker and the firm. In addition to spliting the flow surplus, the reset wage affects the duration of the match and, thus, the surplus. A large (small) $\hat{w}^*$ reduces the surplus by increasing the likelihood of a layoff (quit). The optimal reset wage results from the balance between the objective to maximize the duration of the match (and the total surplus) and to split the surplus according to the weight α. With $\alpha = 1/2$ and the first order approximation of flow payoffs around $\hat{w}^*$, these two objectives are consistent with each other.

The more novel results relate to the worker's and firm's choices to dissolve the match. First, the continuation region of the match $(\hat{w}^-,\ \hat{w}^+)$ is symmetric and centered around the optimal reset wage

$\hat{w}^*$. Second, the width of this region is increasing in the volatility σ. This is due to the canonical option value effect (see Dixit, 1991) Although the worker's productivity might be low today, the firm is willing to wait before firing the worker in case productivity improves in the future. In addition, a higher volatility increases the continuation region because now productivity can recover faster.

However, our model features a departure from canonical models of inaction (see, e.g., Dixit, 1991). In those models, the width of the continuation region typically grows unboundedly with the size of the volatility σ. Instead, in our model, the size of the continuation region has an upper bound. Intuitively, when the separation trigger $\hat{w}^-$ is below the flow value of unemployment $\tilde{\rho}\tilde{U}$, the worker is "investing" in the match by delaying the separation. The marginal benefit of this investment is that, with some probability, productivity experiences a small decrease and the wage-to-productivity ratio increases, making the worker more willing to stay at the prevailing constant wage. The risk of this investment is that there is a positive probability that productivity experiences a decrease large enough for the firm to decide to fire the worker, in which case the investment is worthless. As the volatility goes to infinity, the probability of this occurring tends to one, and neither the firm nor the worker has any incentive to further invest by widening the continuation region. This result is driven by the lack of commitment. To see this, suppose that the firm can commit to a given separation trigger $\hat{w}^+$, Then, it can be shown that the worker's policy $\hat{w}^-(\hat{w}^+, \sigma)$ satisfies $\frac{\partial^2 w^-(\hat{w}^+,\sigma)}{\partial \hat{w}^+ \partial \sigma} < 0$. That is, when the firm commits to a higher separation trigger, the worker is more willing to reduce the quit threshold $\hat{w}^-$ when volatility increases.

The inefficiency generated by the lack of commitment also manifests itself in the flow value of unemployment, which is illustrated by the presence of a "wedge" in the value of unemployment in Proposition 4. The function $\xi(\sigma) \in (1/4, 1]$ is decreasing in the volatility σ and captures the cost imposed by lack of commitment. A higher volatility reduces the expected duration of the match, which has two effects: (1) it directly reduces the net value of employment $\tilde{W}(\hat{w})$ and (2) it reduces $\tilde{J}(\hat{w})$, the firm's incentives to post vacancies and the job finding rate.

Role of the Drift. To understand the role played by productivity drift, we assume away the stochastic component—i.e., $\sigma = 0$ and focus on the case with positive drift—i.e., $\nu > 0$.

Proposition 5. *Assume $\sigma = 0$ and $\nu > 0$. Then, in equilibrium,*

1. *optimal policies are given by*

$$\exp(\hat{w}^-,\ \hat{w}^*,\ \hat{w}^+) = \left(\tilde{\rho}\tilde{U},\ \tilde{\rho}\tilde{U} \times \tilde{T}\Big(\frac{\tilde{\rho}+\delta}{\nu}, \tilde{U}\Big),\ \infty\right),$$

$$\text{with } \frac{\partial \tilde{T}\left(\frac{\tilde{\rho}+\delta}{\nu},\tilde{U}\right)}{\partial \frac{\tilde{\rho}+\delta}{\nu}} > 0, \lim_{\frac{\tilde{\rho}+\delta}{\nu}\to\infty} \exp(\hat{w}^*) = \alpha + (1-\alpha)\tilde{\rho}\tilde{U} \text{ and } \lim_{\frac{\tilde{\rho}+\delta}{\nu}\to 0} \exp(\hat{w}^*) = \exp(\hat{w}^{limit}), \text{ where}$$

$$\frac{e^{\hat{w}^{limit}} - e^{\hat{w}^-}}{\hat{w}^{limit} - \hat{w}^-} = \mathbb{E}[e^w] = \alpha + (1-\alpha)\tilde{\rho}\tilde{U} + \frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})}{\hat{w}^{limit} - \hat{w}^-}\frac{\mathbb{E}[e^w] - \tilde{\rho}\tilde{U}}{\mathbb{E}[e^w]}.$$

2. *the flow value of unemployment is given by*

$$\tilde{\rho}\tilde{U} = \tilde{b} + \varsigma(\nu)\Theta\left(1-\tilde{\rho}\tilde{U}\right)^{\frac{1}{\alpha}}$$

where $\varsigma(\nu)$ is a function of the drift.

When productivity grows at a constant rate, the separation trigger $\hat{w}^-$ equals the static opportunity cost of employment. Also, because productivity grows over time, it is never optimal for the firm to fire the worker; i.e., the firm's separation trigger is not active. More importantly, the optimal reset wage $\hat{w}^*$ is increasing in the drift, which we denote as the *anticipatory effect*. Intuitively, for a given wage, the duration of the match is decreasing in the drift as workers become more productive at a faster rate and choose to quit earlier in the match. In anticipation of this, the optimal reset wages increases to extend the duration of the match. In fact, the proposition shows that in the limiting case when the drift becomes infinitely large, the limiting wage is determined by two components. The first one is given by the Nash Bargaining solution, which balances the trade-off between the job finding rate and the wage. There is an additional second component, which arises because a higher wage extends the duration of match, which increases the surplus and further increases the wage.

Relative to the case with constant productivities, the agent's lack of commitment decreases the value of the match, which is again captured by the "wedge" $\varsigma(\nu)$ in the second part of the proposition.

3 From model to data: Testable implications and identification

This section has two objectives. First, it spells out the empirical predictions of the model outlined in the previous subsection. Specifically, we characterize the model's observable implications that can be measured in the data and delineate the consequences of wage rigidity for endogenous separations. Second, we answer the inverse question. Specifically, we characterize sufficient statistics in the data for the distribution of the wage-to-productivity ratio and the ratio of endogenous-to-total separations. In conclusion, this section shows a one-to-one correspondence between the model's state and the data.

Notation and plan of attack. The model has a set of observable implications. First, agents' policies imply a rate of unemployment U, and aggregate transition rates from employment to unemployment s and from unemployment to employment $p(\theta^*)$. Second, the model predicts a joint distribution for the duration of completed employment spells τ^m, the duration of completed unemployment spells τ^u, and the log nominal wage change between job spells. We denote the joint distribution of $(\tau^m, \tau^u, \Delta w)$ with $l(\tau^m, \tau^u, \Delta w)$ and the marginal distribution of each variable with $l(\tau^m)$, $l(\tau^u)$, and $l(\Delta w)$. Let $\mathcal{D} := \{U,\ s,\ p(\theta^*),\ l(\tau^m,\ \tau^u,\ \Delta w)\}$ summarize the model's observable implications.

To characterize $\mathcal{D}$, we need to find the state. While, in principle, we can characterize $\mathcal{D}$ as a function of the joint distribution of employment status, wages, and productivity; in practice, we characterize $\mathcal{D}$ as a function of the distribution of (the negative of) the cumulative idiosyncratic productivity shocks since the beginning of the employment status, which we denote with Δz and refer as cumulative worker shocks. Let $g_h(\Delta z)$ and $g_u(\Delta z)$ be the distributions of Δz across employed and unemployed workers, respectively. We use these cumulative worker shocks as the state to characterize $\mathcal{D}$ because it has two advantages: it is stationary and one-dimensional. Also, by definition, $\hat{w} = \hat{w}^* + \Delta z$ across workers. Observe that the support of $g_h(\Delta z)$ is given by $[-\Delta^-, \Delta^+]$, where $\Delta^- := \hat{w}^* - \hat{w}^-$ and $\Delta^+ := \hat{w}^+ - \hat{w}^*$. We use $\mathcal{M} = \{g_h(\Delta z), g_u(\Delta z)\}$, together with (γ, σ), as the model's state for $\mathcal{D}$.

Distribution of workers. The steady-state distributions $g_h(\Delta z)$ and $g_u(\Delta z)$ solve the Kolmogorov forward equations, the border conditions, and constant unemployment rate as a function of $(\delta, p(\theta^*), \gamma, \sigma, \Delta^-, \Delta^+)$. Since the law of motion of Δz is independent on the employment state $\Delta_t = -\gamma\,\mathrm{d}t + \mathrm{d}W_t$, the Kolmogorov forward equations for employed and unemployed workers are

$$\delta g_h(\Delta z) = \gamma g_h'(\Delta z) + \frac{\sigma^2}{2} g_h''(\Delta z) \quad \text{for all } \Delta z \in (-\Delta^-, \Delta^+)/\{0\}, \tag{29}$$

$$p(\theta^*) g_u(\Delta z) = \gamma g_u'(\Delta z) + \frac{\sigma^2}{2} g_u''(\Delta z) \quad \text{for all } \Delta z \in (-\infty, \infty)/\{0\}. \tag{30}$$

Here, $\delta\,\mathrm{d}t$ is the exogenous exit rate of employed workers and $p(\theta^*)\,\mathrm{d}t$ the finding rate of unemployed workers. Since the entry state for employment and unemployment states of cumulative shock during the state is $\Delta z = 0$, the Kolmogorov forward equations for $g_h(\Delta z)$ and $g_u(\Delta z)$ do not hold at zero.

The border conditions imply zero measure at the borders

$$g_h(-\Delta^-) = g_h(\Delta^+) = 0, \tag{31}$$

$$\lim_{\Delta z \to -\infty} g_u(\Delta z) = \lim_{\Delta z \to \infty} g_u(\Delta z) = 0, \tag{32}$$

and

$$1 = \int_{-\infty}^{\infty} g_u(\Delta z)\, \mathrm{d}\Delta z + \int_{-\Delta_{*-}}^{\Delta_{*+}} g_h(\Delta z)\, \mathrm{d}\Delta z, \tag{33}$$

$$\underbrace{p(\theta^*)U}_{\text{mass of u-to-h}} = \underbrace{\delta(1-U) + \frac{\sigma^2}{2}\left[g_h'(-\Delta^-) - g_h'(\Delta^+)\right]}_{\text{mass of h-to-u}}, \tag{34}$$

where $\int_{-\infty}^{\infty} g_u(\Delta z)\, \mathrm{d}\Delta z = U$. Condition (33) imposes a measure one of worker and condition (34) imposes that the mass of workers entering to employment (i.e., u-to-h) is equal to the mass of workers exiting employment (i.e., h-to-u); thus, the employment rate is constant.

Equation (29) to (34), together with continuity of $g_u(\Delta z)$ and $g_h(\Delta z)$, describe the equilibrium conditions for the steady-state distribution of cumulative idiosyncratic shocks. With the triggers for endogenous separations Δ^- and Δ^+, the drift and volatility of Δz, and transition rates $(\delta, p(\theta^*))$, we can characterize $g_u(\Delta z)$ and $g_h(\Delta z)$. Proposition 6 shows this result.

Proposition 6. *Assume $\delta > 0$. $g_h(\Delta z)$ and $g_u(\Delta z)$ are given by*

$$g_h(\Delta z) = (1-U)\mathcal{G}_h \begin{cases} \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)} - e^{\beta_2(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} & \text{if } \Delta z \in (-\Delta^-, 0] \\ \frac{e^{\beta_1(\delta)(\Delta z-\Delta^+)} - e^{\beta_2(\delta)(\Delta z-\Delta^+)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} & \text{if } \Delta z \in [0, \Delta^+) \end{cases} \tag{35}$$

$$g_u(\Delta z) = U\mathcal{G}_u \begin{cases} e^{\beta_2(p(\theta^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(p(\theta^*)\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases} \tag{36}$$

where

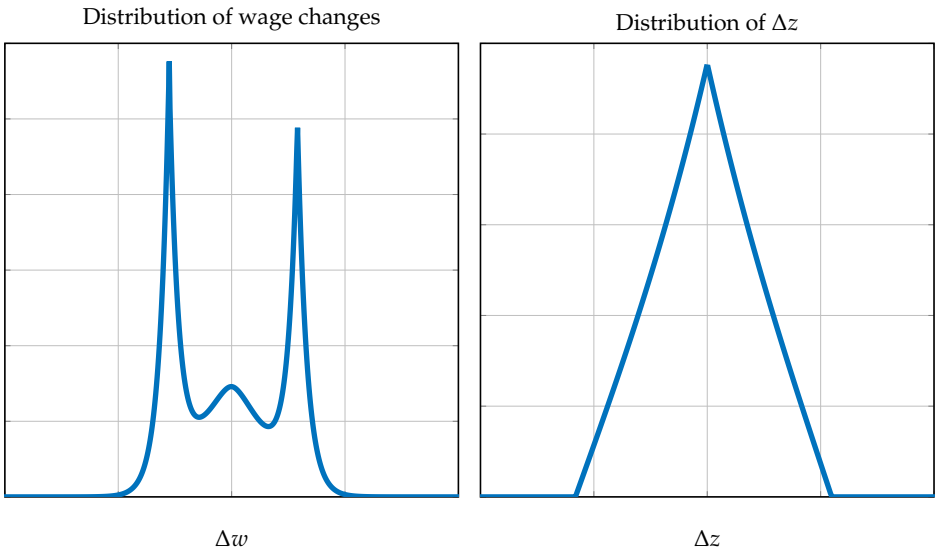
$$\beta_1(x) = \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}, \beta_2(x) = \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}, \tag{37}$$

and $\mathcal{G}_h$, $\mathcal{G}_u$, and U are a function of $(\Delta^-, \Delta^+, \gamma, \sigma, \delta, p(\theta^*))$ describe in equations (272), (272), and (272) in Online Appendix Section B.5

Figure 3 shows the distribution of $g_h(\Delta z)$ when $\delta, \gamma \to 0$.

From distribution of workers $\mathcal{M}$ to $\mathcal{D}$. Equipped with $\mathcal{M}$ and (γ, σ), we can characterize $\mathcal{D}$ uniquely. Trivially, $\int_{-\infty}^{\infty} g_u(\Delta z)\, \mathrm{d}\Delta z = U$. The finding and separation rates are given by the discontinuity of $g_h'(\Delta z)$ and $g_u'(\Delta z)$ at $\Delta = 0$. Intuitively, the jump in the marginal change is given by the entry rate at the entry point.

FIGURE 3. DISTRIBUTION OF WAGE CHANGES AND MINUS CUMULATED PROD. SHOCKS DURING UNEMPLOYEMENT



Notes: The Figure plots the distribution of Δz and Δw for $\gamma = 0$ and $\delta \downarrow 0$.

Proposition 7. *The finding rate $p(\theta^*)$ and separating rate s are given*

$$p(\theta^*) = \frac{\sigma^2}{2U}\left[\lim_{\Delta z \uparrow 0} g_h'(\Delta z) - \lim_{\Delta z \downarrow 0} g_h'(\Delta z)\right], \tag{38}$$

$$s = \frac{\sigma^2}{2(1-U)}\left[\lim_{\Delta z \uparrow 0} g_u'(\Delta z) - \lim_{\Delta z \downarrow 0} g_u'(\Delta z)\right]. \tag{39}$$

The ratio of endogenous s^{end} to total separations s is given by

$$\frac{s^{end}}{s^{exo}} = \frac{\frac{\sigma^2}{2(1-U)}\left[\lim_{\Delta z \downarrow -\Delta^-} g_h'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} g_h'(\Delta z)\right]}{s} \tag{40}$$

and $\delta = s^{exo} - s^{end}$.

We skip the proof, since it is easy to derive (288) and (289) by integrating (29) and (30) and using the border conditions.

Next, we characterize the marginal distribution of $l(\tau^m,\ \tau^u,\ \Delta w)$. The following proposition characterizes the distributions of completed employment and unemployment spells.

Proposition 8. *The distributions of τ^u and τ^h are given by*

$$l_{\tau^u}(\tau^u) = p(\tilde{\theta}^*)e^{-p(\tilde{\theta}^*)\tau^u}, \tag{41}$$

$$l_{\tau^m}(\tau^m) = e^{-\delta\tau^m}\left[\delta + \sum_{n=1}^{\infty}\mathcal{A}(n)\left(e^{-\mathcal{B}(n)\tau^m} - \frac{\delta}{\mathcal{B}(n)}\left(1 - e^{-\mathcal{B}(n)\tau^m}\right)\right)\right]. \tag{42}$$

with $\mathcal{A}(n)$ and $\mathcal{B}(n)$ are function of $(\gamma, \sigma, \Delta^+, \Delta^-)$ describe in equations (293) and (294) in Online Appendix Section (B.7).

The hazard rate for finding a job for an unemployed worker is constant and its level is given the the finding probability $\frac{l(\tau^u)}{\int_{\tau^u}^{\infty} l(s)\,\mathrm{d}s} = p(\tilde{\theta}^*)$. The hazard rate for lossing a job for an employed worker depends on three mechanisms. First, the rate of exogenous separations. If there are only exogenous separations, e.g., when $\Delta^+, \Delta^- \to \infty$, then the hazard rate is $\frac{l(\tau^h)}{\int_{\tau^h}^{\infty} l(s)\,\mathrm{d}s} = \delta$. In the other extreme case, when $\delta = \gamma = 0$ and $\Delta^+ = \Delta^-$, the hazard rate stars at zero $\lim_{\tau^h \downarrow 0} \frac{l(\tau^h)}{\int_{\tau^h}^{\infty} l(s)\,\mathrm{d}s} = 0$, an increases in a convex-concave to the asymptote $\lim_{\tau^h \to \infty} \frac{l(\tau^h)}{\int_{\tau^h}^{\infty} l(s)\,\mathrm{d}s} = \frac{\pi^2}{8\mathbb{E}^l[\tau^h]}$. Finally, whenever $\sigma = 0$ and $\gamma > 0$, the hazard rate is constant and equal to δ between $[0, \frac{\Delta^-}{\gamma}]$ and then jumps to infinity at $T = \frac{\Delta^-}{\gamma}$.

The last step is to characterize the marginal distribution of the log wage changes across jobs. To understand how to characterize this distribution, it is important analyze the dynamic of h-to-u and u-to-u. Assume a worker start the job at t_0. The worker starts with a log nominal wage equal to w_{t_0}. At date $t_0 + \tau^m$, the workers losses the job, and date $t_0 + \tau^m + \tau^u$ finds a new job with nominal wage $w_{t_0+\tau^m+\tau^u}$. The growth rate in wages is given by

$$\begin{aligned}
\Delta w &= w_{t_0+\tau^m+\tau^u} - w_{t_0}, \\
&=^{(1)} \underbrace{w_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^h+\tau^u}}_{=\hat{w}^*} - \underbrace{\left(w_{t_0} - z_{t_0}\right)}_{=\hat{w}^*} + \underbrace{z_{t_0+\tau^m+\tau^u} - z_{t_0}}_{\text{cumulate dz across h-u-h}}, \\
&=^{(2)} z_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m} + z_{t_0+\tau^m} - z_{t_0}, \\
&=^{(3)} -(\Delta z|\text{h-u transition} + \Delta z|\text{u-h transition and } z_{t_0+\tau^m}).
\end{aligned} \tag{43}$$

Step (1) sum and subtract $z_{t_0+\tau^h+\tau^u}$ and z_{t_0}. Under the observation that wage-to-productivity ratio is equal across at the entry job, the first two terms cancel and we have cumulated idiosyncratic productivity shocks between employment to employment following an unemployment transition. Step (1) sum and subtract the productivity level at the time of the separation and step (3) uses the definition of Δz. Equation (43) shows that the distribution of wage changes across job is minus the sum of Δz conditional to a separation and conditional to a finding given an initial level of productivity $z_{t_0+\tau^m}$. *By the assumption*

of homotheticity of search frictions, the finding probability is independent on the level of productivity. Thus,

$$\Delta w = -(\Delta z|\text{h-u transition} + \Delta z|\text{u-h transition}) \tag{44}$$

and $\Delta z|$u-h transition is independent of $\Delta z|$h-u transition. Define $\bar{G}_h(\Delta z)$ and $\bar{G}_u(\Delta z)$ the distributions of Δz conditional on a separation and finding a new job, respectively. Since the finding rate is independent of the state, we have that

$$\bar{G}_u(\Delta z) = \frac{\int_{-\infty}^{\Delta z} g_u(x)\,\mathrm{d}x}{U} \tag{45}$$

and by the Ss nature of separation policy

$$\bar{G}_h(\Delta z) = \begin{cases} 1 & \text{if } \Delta z \in [\Delta^+, \infty) \\ \frac{1}{s(1-U)}\left[\frac{\sigma^2}{2} g_h'(\Delta z) + \delta \int_{-\Delta^-}^{\Delta z} g_h(x)\,\mathrm{d}x\right] & \text{if } \Delta z \in [-\Delta^-, \Delta^+) \\ 0 & \text{if } \Delta z \in (-\infty, -\Delta^-) \end{cases} \tag{46}$$

With results (44), (45), and (46), we can characterize the distribution of wage changes.

Proposition 9. *The distribution of log nominal wage changes satisfies*

$$l_{\Delta w}(\Delta w) = \mathcal{G}_u\left[\beta_2(p(\tilde{\theta}^*))e^{-\beta_2(p(\tilde{\theta}^*))\Delta w}\Gamma_2(\Delta w) + \beta_1(p(\tilde{\theta}^*))e^{-\beta_1(p(\tilde{\theta}^*))\Delta w}\Gamma_1(\Delta w)\right] \tag{47}$$

with

$$(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(p(\tilde{\theta}^*)))x}\bar{G}_h(x)\,\mathrm{d}x, \int_{-c}^{\infty} e^{-\beta_2(p(\tilde{\theta}^*))x}\bar{G}_h(x)\,\mathrm{d}x\right). \tag{48}$$

Figure 3 shows the distribution of $l(\Delta w)$

From data $\mathcal{D}$ to distribution of workers $\mathcal{M}$. Now, we provide the inverse exercise. As economist we are interest in $\mathcal{M}$ since it is informative of the consequence of sticky wages and lack of comment in labor income risk. As empiricist, we are only equip with $l(\tau^m,\ \tau^u,\ \Delta w)$. Can we recover $\mathcal{M}$ from $\mathcal{D}$? The answer is yes.

We process in three steps. First, we recover (γ, σ). Then, from $(\gamma, \sigma, p(\theta^*), l_{\Delta w}(\Delta w))$, we recover $\bar{G}_u(\Delta z)$ and $\bar{G}_h(\Delta z)$. Finally, with $\bar{G}_u(\Delta z)$ and $\bar{G}_h(\Delta z)$, we can recover $g_u(\Delta z)$ and $g_h(\Delta z)$.

The following proposition characterizes γ and σ. From now on, we define $\tau := \tau^m + \tau^u$.

Lemma 2. *The drift γ and volatility σ satisfy*

$$\gamma = \frac{\bar{\mathbb{E}}[\Delta w]}{\bar{\mathbb{E}}[\tau]}, \tag{49}$$

$$\sigma^2 = \frac{\bar{\mathbb{E}}[(\Delta w - \gamma\tau)^2]}{\bar{\mathbb{E}}[\tau]} \tag{50}$$

Expressions in (49) and (50) provide a mapping to infer the parameters of the stochastic process. The first expression shows that in a stationary environment, the average wage change $\bar{\mathbb{E}}[\Delta w]$ must compensate for the average drift between two employments $\gamma\overline{\mathbb{E}}[\tau]$. Similarly, the dispersion of de-trended adjustment size $\bar{\mathbb{E}}[(\Delta w - \gamma\tau)^2]$ reflects the cumulative shocks received during a h-to-h transition $\sigma^2\overline{\mathbb{E}}[\tau]$. With (γ, σ) we can characterizes $\bar{G}_u(\Delta z)$ and $\bar{G}_h(\Delta z)$. The distribution of $\bar{G}_u(\Delta z)$ comes directly from equation (36) and the knowledge of $(\gamma, \sigma, p(\theta^*))$.

Proposition 10. *Define $\Sigma = \frac{\sigma^2 p(\theta^*)}{\gamma^2}$. The distribution of Δz conditional of separation is given by*

$$\bar{G}_h(\Delta z) = -2\frac{\sqrt{1+2\Sigma}}{\Sigma}\frac{\mathrm{d}l_{\Delta w}(-\Delta z)}{\mathrm{d}z} + \frac{p(\theta^*)}{\sqrt{1+2\Sigma}}l_{\Delta w}(-\Delta z) + \frac{\gamma^2\Sigma}{\sqrt{1+2\Sigma}}\left[1 - L_{\Delta w}(-\Delta z)\right]. \tag{51}$$

where $L_{\Delta w}(\Delta w)$ denotes the cumulative distribution of $l_{\Delta w}(\Delta w)$.

What are the critical assumptions to obtain proposition 10? The first assumption that we did is an extreme Ss policy, where the separation rate is equal to δ in $\Delta z \in [-\Delta z^-, \Delta z^+]$ and infinite in at $\Delta z \in \{-\Delta z^-, \Delta z^+\}$. This assumption is not important, and it can be relaxed with a general separation hazard (see, for instance, Álvarez *et al.*, 2020). The second assumption is the lack of another transition (e.g., job-to-job transition) or wage adjustment within the job spell. This assumption is not important and it can be relaxed with conditional distribution as in Baley and Blanco (2021).

The critical assumption for proposition (10) is that we have a parametric specification of $\bar{g}_u(\Delta z)$ that can be recover with data. In our model, lack of selection in the finding rate implies full identification of $\bar{g}_u(\Delta z)$ as a function on $(\gamma, \sigma, p(\theta^*))$. More general, since the distribution of wage changes is a function of the sum of shocks during employment and unemployment status, we need to have good information of one to get the other. Finally, with the distribution of minus cumulative shock following a separation $\bar{G}_h(\Delta z)$, we can obtain the distribution of wage-to-productivity across employed workers.

Proposition 11. *Assume $\gamma \neq 0$. The distribution of minus cumulative idiosyncratic shocks is given by*

$$g_h(\Delta z) = \frac{s(1-U)}{\gamma}\left[\int_{-\Delta^-}^{\Delta z}(e^{\frac{\gamma^2}{\sigma^2}(y+\Delta^-)} - 1)\bar{g}_h(y)\,\mathrm{d}y + \bar{G}_h(-\Delta^-)\left[1 - e^{-\frac{\gamma^2}{\sigma^2}(\Delta z - \Delta^-)}\right]\right]. \tag{52}$$

The fraction of inefficient to efficient separations is given by

$$\frac{s^{end}}{s^{exo}} = \frac{\sigma^2}{2s(1-U)} \left[g_h'(-\Delta^-) - g_h'(\Delta^+) \right]. \tag{53}$$

4 A Model of Aggregate Labor Income Risk and Sticky Wages [TBC]

5 Connecting Idiosyncratic and Aggregate Labor Income Risk [TBC]

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A Auxiliary Theorems

Proposition 12. *Let $\mathcal{A}$ be the characteristic operator of $\{X_t\}$ with $X_t \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differential function with compact (bounded and closed) support $(support(f) = \{x : f(x) \neq 0\})$, If τ is a stopping time with $\mathbb{E}^x[\tau] < \infty$, then*

$$\mathbb{E}^x[f(x_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau \mathcal{A}f(X_t)\,\mathrm{d}t\right] \tag{54}$$

Moreover, if τ is the first exit time of a bounded set, then (54) holds.

Proposition 13. *Let x_t be a strong Markov process, τ be a stopping time measurable with the filtration generated by x_t, and τ^δ a exponential random variable independent of τ^δ. Then*

$$\mathbb{E}\left[\int_0^{\tau\wedge\tau^\delta} e^{-\rho t}u(x_t)\,\mathrm{d}t + e^{-\rho(\tau\wedge\tau^\delta)}h(x_{\tau\wedge\tau^\delta})\middle| x_0 = x\right] = \mathbb{E}\left[\int_0^\tau e^{-(\rho+\delta)t}[u(x_t)+\delta h(x_t)]\,\mathrm{d}t + e^{-(\rho+\delta)\tau}h(x_\tau)\middle| x_0 = x\right] \tag{55}$$

Proposition 14. *Let V be a Hilbert space and P a closed convex cone of V satisfying*

$$P = \{x \in V : (x, y) \geq 0\ \forall y \in P\}. \tag{56}$$

Let T be an increasing map from V to itself s.t. there exist a $\underline{x}, \overline{x} \in V$

$$\underline{x} \leq \overline{x},\ \ \underline{x} \leq T(\underline{x}),\ \ T(\overline{x}) \leq \overline{x}. \tag{57}$$

Then the subset of fixed point x^* of T satisfy $\underline{x} \leq x^* \leq \overline{x}$ is non-empty and has a larger and smallest element.

Proof. For a proof, see Chapter 15, proposition 2, page 539 of Aubin (2007). □

B Appendix

B.1 Proof of Lemma 1

Define the sufficient conditions for the Nash Equilibrium as

$$h_w(z) \geq u(z), \ \ \forall z \in \mathbb{R} \tag{58}$$

$$j_w(z) \geq 0, \ \ \forall z \in \mathbb{R} \tag{59}$$

$$\text{If } z \in (\mathcal{C}_w^h)^c \Rightarrow j_w(z) = 0, \tag{60}$$

$$\text{If } z \in (\mathcal{C}_w^j)^c \Rightarrow h_w(z) = u(z), \tag{61}$$

$$0 = \max\{u(z) - h_w(z), \mathcal{A}h_w(z) + \delta u(z) + e^w\}, \ \forall z \in \mathcal{C}_w^j, \tag{62}$$

$$0 = \max\{-j_w(z), \mathcal{A}j_w(z) + e^z - e^w\}, \ \forall z \in \mathcal{C}_w^h, \tag{63}$$

$$\emptyset \neq \mathcal{C}_{w^*(z)}^h \cap \mathcal{C}_{w^*(z)}^j. \tag{64}$$

where the the continuation sets and the characteristic operator of the game are given by

$$\mathcal{C}_h^w := int\left\{z \in \mathbb{R} : h_w(z) \geq u(z) \text{ or } \mathcal{A}h_w(z) + \delta u(z) + e^w \geq 0\right\}, \tag{65}$$

$$\mathcal{C}_j^w := int\left\{z \in \mathbb{R} : j_w(z) \geq 0 \text{ or } \mathcal{A}j_w(z) + e^z - e^w \geq 0\right\}, \tag{66}$$

$$\mathcal{A}(f(z)) := -(\rho + \delta)f + \gamma\frac{f(z)}{\partial z} + \frac{\sigma^2}{2}\frac{f(z)}{\partial z^2} \tag{67}$$

Lemma 3. *If there exist two functions $h_w(z),\ j_w(z) \in C^2(\mathbb{R})$ satisfying (58) to (64), then*

$$\tau^{w*}(x) = \inf\left\{t \geq 0 : z_t^{w^*} \notin \mathcal{C}_h^w\right\} \tag{68}$$

$$\tau^{j*}(x) = \inf\left\{t \geq 0 : z_t^{w^*} \notin \mathcal{C}_j^w\right\} \tag{69}$$

form a Nash equilibrium. Moreover

$$h(x) = H(x, \tau^{w*}(x), \tau^{j*}(x), \tau^\delta), j(x) = J(x, \tau^{w*}(x), \tau^{j*}(x), \tau^\delta). \tag{70}$$

Proof. We prove that

$$h_w(z) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta) \geq H(x, \tau^h(x), \tau^{j*}(x), \tau^\delta). \tag{71}$$

for any $\tau^w \in \mathcal{T}$. The proof for

$$j_w(z) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta) \geq J(x, \tau^{h*}(x), \tau^j(x), \tau^\delta) \tag{72}$$

for any $\tau^j \in \mathcal{T}$ follows the same arguments. Steps one shows the equivalence between the (58) to (64) and the time-inhomogenous case (see Oksendal (2007), Chapter 10, pag. 220).

Step 1: Here, we announce (without any proof) that if $h_w(z)$ and $j_z(w)$ satisfies (58) to (64) if and only if $\tilde{h}_w(t,z) = e^{-(\rho+\delta)t}h_w(z)$ and $\tilde{j}_w(t,z) = e^{-(\rho+\delta)t}j_w(z)$ satisfy

$$\tilde{h}_w(t,z) \geq e^{-(\rho+\delta)t}u(z), \tag{73}$$

$$\tilde{j}_w(t,z) \geq 0, \tag{74}$$

$$\text{If } (t,z) \in (\tilde{\mathcal{C}}_w^h)^c \Rightarrow \tilde{j}_w(t,z) = 0, \tag{75}$$

$$\text{If } (t,z) \in (\tilde{\mathcal{C}}_w^j)^c \Rightarrow \tilde{h}_w(t,z) = e^{-(\rho+\delta)t}u(z), \tag{76}$$

$$\emptyset \neq \tilde{\mathcal{C}}_h^{w^*(z)} \cap \tilde{\mathcal{C}}_j^{w^*(z)}. \tag{77}$$

$$0 = \max\{e^{-(\rho+\delta)t}u(z) - \tilde{h}_w(t,z), \tilde{\mathcal{A}}\tilde{h}_w(t,z) + \delta e^{-(\rho+\delta)t}u(z) + e^{-(\rho+\delta)t+w}\},\ \forall z \in \mathcal{C}_j^w, \tag{78}$$

$$0 = \max\{-\tilde{j}_w(t,z), \tilde{\mathcal{A}}j_w(t,z) + e^{-(\rho+\delta)t}(e^z - e^w)\},\ \forall z \in \mathcal{C}_h^w, \tag{79}$$

where the the continuation sets and the second order differential operator $\tilde{\mathcal{A}}$ are given by

$$\tilde{\mathcal{C}}_w^h := \left\{(t,z) \in \mathbb{R}_+ \times \mathbb{R} : \tilde{h}_w(t,z) \geq e^{-(\rho+\delta)t}u(z) \text{ or } \tilde{\mathcal{A}}\tilde{h}_w(t,z) + \delta e^{-(\rho+\delta)t}u(z) + e^{-(\rho+\delta)t+w} \geq 0\right\}, \tag{80}$$

$$\tilde{\mathcal{C}}_w^j := \left\{(t,z) \in \mathbb{R}_+ \times \mathbb{R} : \tilde{j}_w(t,z) \geq 0 \text{ or } \tilde{\mathcal{A}}j_w(t,z) + e^{-(\rho+\delta)t}(e^z - e^w) \geq 0\right\}, \tag{81}$$

$$\tilde{\mathcal{A}}(f(t,z)) := \frac{\partial f(t,z)}{\partial t} + \gamma\frac{f((t,z))}{\partial z} + \frac{\sigma^2}{2}\frac{f((t,z))}{\partial z^2} \tag{82}$$

The proof immediately. The key observation from step one is that the second order differential operator $\tilde{\mathcal{A}}$ is the characteristic operator of $[t, z_t]$ with initial condition $(0, z_0)$. Thus, we can apply the Dynkin lemma to $\tilde{h}_w(t,z)$ and $\tilde{j}_w(t,z)$

Step 2: Now, we show that $h_w(z) \geq H(x, \tau^w(x), \tau^{j*}(x), \tau^\delta)$. Let τ^w be any stopping time (not necessary the optimal). Let $U_k \subset \mathbb{R}_+ \times \mathbb{R}$ be an increasing sequence of bounded sets s.t. $\cup_{k=1}^\infty U_k = \mathbb{R}_+ \times \mathbb{R}$. Since U_k are bounded, $\tau_k = \inf\{(t,z_t) : (t,z_t) \notin U_k\}$, we do not need compact support for applying

Lemma 12. Applying Lemma 12 to the stopping time $\tau_k^h = \tau^w \wedge \tau^{j*} \wedge \tau_k$

$$\mathbb{E}[\tilde{h}(\tau_k^h, x_{\tau_k^h})|z_0 = z] = \tilde{h}_w(0,z) + \mathbb{E}\left[\int_0^{\tau_k^h} \tilde{\mathcal{A}}\tilde{h}_w(t,z_t)\,\mathrm{d}t|z_0 = 0\right]. \tag{83}$$

Using condition (73), since $\tilde{h}_w(t,z) \geq e^{-(\rho+\delta)}u(z)$ for all (t,z), we have that $\mathbb{E}[\tilde{h}_w(\tau_k^h, z_{\tau_k^h})|z_0 = z] \geq \mathbb{E}[e^{-(\rho+\delta)\tau_k^h}u(z_{\tau_k^h})|z_0 = z]$. Thus,

$$\mathbb{E}[e^{-(\rho+\delta)\tau_k^h}u(z_{\tau_k^h})|z_0 = 0] - \mathbb{E}\left[\int_0^{\tau_k^h} \tilde{\mathcal{A}}\tilde{h}_w(t,z_t)\,\mathrm{d}t\right] \leq \tilde{h}_w(0,z). \tag{84}$$

From condition (78) , we have $\tilde{\mathcal{A}}\tilde{h}_w(t,z) + \delta e^{-\rho t}u(z) + e^{-(\rho+\delta)t+w} \leq 0$ for all (t,z). Thus,

$$\mathbb{E}\left[\int_0^{\tau_k^h} e^{-(\rho+\delta)t}\left(\delta u(z_t) + e^w\right)\mathrm{d}t\right] \leq -\mathbb{E}\left[\int_0^{\tau_k^h} \tilde{\mathcal{A}}\tilde{h}_w(t,z_t)\,\mathrm{d}t\right]. \tag{85}$$

Using this result

$$\mathbb{E}\left[e^{-(\rho+\delta)\tau_k^h}u(z_{\tau^h}) + \int_0^{\tau_k^h} e^{-(\rho+\delta)t}\left(\delta u(z_t) + e^w\right)\mathrm{d}t\right] \leq \tilde{h}_w(0,z) \tag{86}$$

Using Lemma 13 and since $\tilde{h}_w(0,z) = h_w(z)$, we have that

$$\tilde{h}_w(0,z) \geq \mathbb{E}\left[\int_0^{\tau^w \wedge \tau^{j*} \wedge \tau^\delta \wedge \tau_k} e^{-\rho t + w}\,\mathrm{d}t + e^{-\rho(\tau^w \wedge \tau^{j*} \wedge \tau^\delta \wedge \tau_k)}u(z_{\tau^w \wedge \tau^{j*} \wedge \tau^\delta \wedge \tau_k})|z_0 = z\right] \tag{87}$$

Taking the limit $k \to \infty$, we have that

$$h_w(z) = \tilde{h}_w(0,z) \geq H(x, \tau^w(x), \tau^{j*}(x), \tau^\delta) \tag{88}$$

Step 3: Now, we show that $h_w(z) = H(x, \tau^{w*}(x), \tau^{j*}(x), \tau^\delta)$. Applying Lemma 12 to the stopping time $\tau_k^h = \tau^{w*} \wedge \tau^{j*} \wedge \tau_k$

$$\mathbb{E}[\tilde{h}(\tau_k^{h*}, x_{\tau_k^{h*}})|z_0 = z] = \tilde{h}_w(0,z) + \mathbb{E}\left[\int_0^{\tau_k^{h*}} \tilde{\mathcal{A}}\tilde{h}_w(t,z_t)\,\mathrm{d}t|z_0 = 0\right]. \tag{89}$$

Since for all $t < \tau^{w*}$, we have that $e^{-(\rho+\delta)t}u(z) < \tilde{h}_w(t,z)$ and therefore $\tilde{\mathcal{A}}\tilde{h}_w(t,z) + \delta e^{-\rho t}u(z) + e^{-(\rho+\delta)t+w} =$

0 for all (t, z). Thus,

$$\mathbb{E}\left[e^{-(\rho+\delta)\tau_k^h} h_w(z_{\tau^h}) + \int_0^{\tau_k^h} e^{-(\rho+\delta)t} \left(\delta u(z_t) + e^w\right) \mathrm{d}t\right] = h_w(z). \tag{90}$$

Taking the limit $k \to \infty$,

$$\mathbb{E}\left[e^{-(\rho+\delta)(\tau^{w*}\wedge\tau^{j*})} h_w(z_{\tau^{w*}\wedge\tau^{j*}}) + \int_0^{\tau^{w*}\wedge\tau^{j*}} e^{-(\rho+\delta)t} \left(\delta u(z_t) + e^w\right) \mathrm{d}t\right] = h_w(z). \tag{91}$$

Since $z_{\tau^{w*}\wedge\tau^{j*}} \in \partial(\tilde{\mathcal{C}}_h^{w^*(z)} \cap \tilde{\mathcal{C}}_j^{w^*(z)})$, we have that

$$h_w(z_{\tau^{w*}\wedge\tau^{j*}}) = u(z_{\tau^{w*}\wedge\tau^{j*}}). \tag{92}$$

Moreover, using Lemma 13, we have

$$H(x, \tau^{w*}(x), \tau^{j*}(x), \tau^{\delta}) = h_w(z). \tag{93}$$

□

B.2 Proof of Proposition 1

For the next proof, it will be useful to define the normalized equilibrium conditions

$$0 = \left[J(\hat{w}) - \tilde{\kappa}\tilde{\theta}(\hat{w})^{\alpha}\right] \tilde{\theta}(\hat{w}), \tag{94}$$

$$\tilde{\rho}\tilde{U} = \tilde{b} + \max_{\hat{w}^*} \tilde{\theta}(\hat{w}^*)^{1-\alpha} \tilde{W}(\hat{w}^*), \tag{95}$$

$$\tilde{W}(\hat{w}) \geq 0, \tag{96}$$

$$\tilde{J}(\hat{w}) \geq 0, \tag{97}$$

$$if\ \hat{w} \in (\tilde{\mathcal{C}}_h)^c \Rightarrow \tilde{J}(\hat{w}) = 0, \tag{98}$$

$$if\ \hat{w} \in (\tilde{\mathcal{C}}_j)^c \Rightarrow \tilde{W}(\hat{w}) = 0, \tag{99}$$

$$0 = \max\{-\tilde{W}(\hat{w}), \mathcal{A}\tilde{W}(\hat{w}) + e^{\hat{w}} - \tilde{\rho}\tilde{U}\},\ \forall \hat{w} \in \tilde{\mathcal{C}}_j, \tag{100}$$

$$0 = \max\{-\tilde{J}(\hat{w}), \mathcal{A}\tilde{J}(\hat{w}) + 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \tilde{\mathcal{C}}_h, \tag{101}$$

$$\emptyset \neq \mathcal{C}_h \cap \mathcal{C}_j. \tag{102}$$

where the stopping sets and the characteristic operator of the game is given by

$$\mathcal{C}_h := int\left\{\hat{w} \in \mathbb{R} : \tilde{W}(\hat{w}) > 0, \tilde{\mathcal{A}}\tilde{W}(\hat{w}) + (e^{\hat{w}} - \tilde{\rho}\tilde{U}) \geq 0\right\}, \tag{103}$$

$$\mathcal{C}_j := int\left\{\hat{w} \in \mathbb{R} : \tilde{J}(\hat{w}) > 0, \tilde{\mathcal{A}}\tilde{J}(\hat{w}) + \left(1 - e^{\hat{w}}\right) \geq 0\right\}, \tag{104}$$

$$\tilde{\mathcal{A}}(f) := -(\tilde{\rho} + \delta)f - \nu\frac{f(\hat{w})}{\partial\hat{w}} + \frac{\sigma^2}{2}\frac{f(\hat{w})}{\partial\hat{w}^2}, \tag{105}$$

where $\tilde{\rho} = \rho - \gamma - \sigma^2/2$ and $\nu = \gamma + \sigma^2$.

Proposition 15. *Define* $\tilde{U} = \frac{u(z)}{e^z}$, $\hat{w}^* = \frac{w^*(z)}{e^z}$, $\tilde{W}(w-z) = \frac{h_w(z)-u(z)}{e^z}$, $\tilde{J}(w-z) = \frac{j_w(z)}{e^z}$, *and* $\theta(x) = \tilde{\theta}(w-z)$.

$$(u(z), w^*(z), h_w(z), j_w(z), \theta(x)) \tag{106}$$

satisfy (**??**) *to* (64) *if and only if*

$$(\tilde{U}, \hat{w}^*, \tilde{W}(\hat{w}), \tilde{J}(\hat{w}), \tilde{\theta}(\hat{w})) \tag{107}$$

satisfy (94) *to* (102).

Proof. Using $\tilde{J}(w-z) = \frac{j_w(z)}{e^z}$ and the assumption $k(z) = \tilde{\kappa}e^z$, we have that

$$\begin{aligned}\left[j_w(z) - \frac{\tilde{\kappa}e^z}{q(\theta(x))}\right]\theta(x) = 0 &\iff \left[\tilde{J}(w-z)e^z - \frac{\tilde{\kappa}e^z}{q(\theta(x))}\right]\theta(x) = 0 \\ &\iff \left[\tilde{J}(\hat{w}) - \frac{\tilde{\kappa}}{q(\hat{w})}\right]\theta(\hat{w}) = 0.\end{aligned} \tag{108}$$

Since $q(\hat{w}) = \tilde{\theta}(\hat{w})^\alpha$, we have that (94) if and only if (**??**). Now, we show that (**??**) implies (95). The fact that (95) implies (**??**) follows similar steps. Using that $\tilde{U} = \frac{u(z)}{e^z}$, we have that

$$\tilde{U}e^z = u'(z) \text{ and } \tilde{U}e^z = u''(z). \tag{109}$$

Using this result and the fact that $p(x) = \theta(x)^{1-\alpha}$

$$\begin{aligned}\rho u(z) = \tilde{b}e^z + \gamma_u u'(z) + \frac{\sigma_u^2}{2}u''(z) + \max_w \theta(x)^{1-\alpha}[h_w(z) - u(z)] &\iff \\ \rho\tilde{U}e^z = \tilde{b}e^z + \gamma\tilde{U}e^z + \frac{\sigma^2}{2}\tilde{U}e^z + \max_w \tilde{\theta}\,(w-z)^{1-\alpha}\left[\tilde{H}(w-z)e^z - \tilde{U}e^z\right] &\iff \\ \left(\rho - \gamma - \frac{\sigma^2}{2}\right)\tilde{U}e^z = \tilde{b}e^z + \max_{\hat{w}} \tilde{\theta}\,(\hat{w})^{1-\alpha}\,\tilde{W}\,(\hat{w}) \iff \tilde{\rho}\tilde{U} = \tilde{b} + \max_{\hat{w}} \theta\,(\hat{w})^{1-\alpha}\,\tilde{W}\,(\hat{w}).\end{aligned} \tag{110}$$

It is easy to show that conditions (96) to (99) hold if and only if (58) to (61) hold. Now, we show that condition (62) implies condition (100) (a similar argument holds for the inverse implication). We need to

show

$$u(z) - h(x) = 0 \text{ or } \mathcal{A}^w h(x) + e^w = 0 \iff \tilde{W}(w-z) = 0 \text{ or } \mathcal{A}\tilde{W}(w-z) + e^{w-z} - \tilde{\rho}\tilde{U} = 0. \tag{111}$$

Since $e^z > 0$, we have that $u(z) - h(x) = 0$ if and only if $\tilde{W}(w-z) = 0$. Observe the following property. Since

$$\tilde{W}(w-z) = \frac{h(x) - u(z)}{e^z}, \tag{112}$$

we have that

$$e^z\tilde{W}(w-z) - e^z\tilde{W}'(w-z) = \frac{\mathrm{d}h_w(z)}{\mathrm{d}z} - u'(z) \text{ and } e^z\tilde{W}(w-z) - 2e^z\tilde{W}'(w-z) + e^z\tilde{W}''(w-z) = \frac{d^2h_w(z)}{\mathrm{d}z^2} - u''(z). \tag{113}$$

Using these conditions and from condition (62), we have that

$$\begin{aligned}
0 &= \mathcal{A}^w h(x) + e^w \\
&= -(\rho+\delta)h(x) + \gamma\frac{h(x)}{\partial z} + \frac{\sigma^2}{2}\frac{h(x)}{\partial z^2} + \delta u(z) - e^w \\
&= -(\rho+\delta)e^z[\tilde{W}(w-z) + \tilde{U}] + \gamma e^z\left[-\tilde{W}'(w-z) + \tilde{W}(w-z) + \tilde{U}\right] + \delta\tilde{U}e^z + e^w \ldots \\
&\cdots + \frac{\sigma^2}{2}e^z\left[\tilde{W}''(w-z) - 2\tilde{W}'(w-z) + \tilde{W}(w-z) + \tilde{U}\right]e^z \\
&= -\tilde{\rho}\tilde{W}(w-z) + (e^{w-z} - (\rho - \gamma - \frac{\sigma^2}{2})\tilde{U}) - (\gamma+\sigma^2)\tilde{W}'(w-z) + \frac{\sigma^2}{2}\tilde{W}''(w-z)
\end{aligned} \tag{114}$$

Thus, we show that

$$0 = \mathcal{A}^w h(x) - e^w \text{ if and only if } 0 = \tilde{A}\tilde{W}(\hat{w}) - (e^{\hat{w}} + \tilde{\rho})\tilde{U} \tag{115}$$

The equivalence between (63), (65), and(66) with (101), (103), and(104) follow similar steps. □

Proposition 16. *Assume that $b < 1$. Then, there exist an unique solution to condition (94) to (102).*

Proof. We divide the proof in 5 steps. Steps 1 to 3 fix $\tilde{U}$ and shows the existence of an unique non-trivial Nash equilibrium. Step 1 shows the quasi-variation inequalities conditions and that if a solution solves the quasi-variation inequalities, then they solve the Non-Trivial Nash Equilibrium. Step 2 shows the exist of a unique Non-Trivial Nash Equation. Step 3 shows uniqueness, continuity, and decreasing solution for $\max_{\hat{w}} \tilde{J}(\hat{w}, \tilde{U})^{\frac{1-\alpha}{\alpha}}\tilde{W}(\hat{w}, \tilde{U})$

For step 1, it would be useful to define static optimal stopping time s.t. the player continue has

positive static flow payoffs.

$$\tau^{SO} = \inf_{t>0} \left\{ t : \hat{w}_t \notin (\log(\tilde{\rho}\tilde{U}), 0) \right\} \tag{116}$$

Then we can define

$$W^{SO}(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{SO} \wedge \tau^{\delta}} e^{-\rho t} (e_t^{\hat{w}} - \tilde{\rho} U) \, \mathrm{d}t \right], \tag{117}$$

$$J^{SO}(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{SO} \wedge \tau^{\delta}} e^{-\rho t} (1 - e^{\hat{w}_t}) \, \mathrm{d}t \right]. \tag{118}$$

Observe that $W^{SO}(\hat{w}) \geq 0$ and $J^{SO}(\hat{w}) \geq 0$, and with equality if $\hat{w} \notin (\log(\tilde{\rho}\tilde{U}), 0)$. Moreover, as we will show believe (see proposition **??**), if $\mathcal{C}_h = (\log(\tilde{\rho}\tilde{U}), 0)$ (resp. $\mathcal{C}_j = (\log(\tilde{\rho}\tilde{U}), 0)$), then $(\log(\tilde{\rho}\tilde{U}), 0) \subset \mathcal{C}_j$ ($(\log(\tilde{\rho}\tilde{U}), 0) \subset \mathcal{C}_h$).

For step 2, it would be useful to define static optimal stopping time s.t. the player continue has positive static flow payoffs.

$$\tau^{BSW} = \inf_{t>0} \left\{ t : \hat{w}_t \notin (\log(\tilde{\rho}\tilde{U}), \infty) \right\}, \quad \tau^{BSJ} = \inf_{t>0} \left\{ t : \hat{w}_t \notin (-\infty, 0) \right\}. \tag{119}$$

Then we can define

$$W^{BSW}(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{BSW} \wedge \tau^{\delta}} e^{-\rho t} (e_t^{\hat{w}} - \tilde{\rho} U) \, \mathrm{d}t \right], \tag{120}$$

$$J^{BSJ}(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{BSJ} \wedge \tau^{\delta}} e^{-\rho t} (1 - e^{\hat{w}_t}) \, \mathrm{d}t \right]. \tag{121}$$

Step 1: Let $V = H^1(\mathbb{R})$. We define the following function $a : V \times V \to \mathbb{R}$ and sets $K_h(\tilde{J})$ and $K_j(\tilde{W})$

$$a(\tilde{J}, \tilde{W}) := \frac{\sigma^2}{2} \int_{\mathbb{R}} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}\tilde{W}(\hat{w})}{\mathrm{d}\hat{w}} \, \mathrm{d}\hat{w} + \nu \int_{\mathbb{R}} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} \tilde{W}(\hat{w}) \, \mathrm{d}\hat{w} + (\tilde{\rho} + \delta) \int_{\mathbb{R}} \tilde{J}(\hat{w}) \tilde{W}(\hat{w}) \, \mathrm{d}\hat{w} \tag{122}$$

$$K_h(\tilde{J}) := \left\{ \tilde{W} \in V : \tilde{W}(\hat{w}) \geq W^{SO}(\hat{w}) \;\&\; if\ \hat{w} \in (\tilde{\mathcal{C}}_j)^c \Rightarrow \tilde{W}(\hat{w}) = 0 \right\}, \tag{123}$$

$$K_j(\tilde{W}) := \left\{ \tilde{W} \in V : \tilde{J}(\hat{w}) \geq J^{SO}(\hat{w}) \;\&\; if\ \hat{w} \in (\tilde{\mathcal{C}}_h)^c \Rightarrow \tilde{J}(\hat{w}) = 0 \right\}. \tag{124}$$

Now, we show some properties over $K_h(\tilde{J})$ and $K_j(\tilde{W})$. First, $K_h(\tilde{J})$ (resp. $K_j(\tilde{W})$) is closed. To see this property, take a sequence in $\{\tilde{W}_n(\hat{w})\}_{n=1}^{\infty}$ s.t. $\tilde{W}_n \in K_h(\tilde{J}) \in K_h(\tilde{J})$ and $\tilde{W}_n(\hat{w}) \to \tilde{W}^*(\hat{w})$. Then

$$\tilde{W}_n(\hat{w}) \geq W^{SO}(\hat{w}), if\ \hat{w} \in (\tilde{\mathcal{C}}_j)^c \Rightarrow \tilde{W}_n(\hat{w}) = 0 \tag{125}$$

By construction, $\hat{w} \in (\tilde{\mathcal{C}}_j)^c$ is independent of the sequence, thus taking the limit

$$\tilde{W}^*(\hat{w}) \geq W^{SO}(\hat{w}), if\ \hat{w} \in (\tilde{\mathcal{C}}_j)^c \Rightarrow \tilde{W}^*(\hat{w}) = 0 \tag{126}$$

and we have that $\tilde{W}^* \in K_h(\tilde{J}) \in K_h(\tilde{J})$. Second, $K_h(\tilde{J})$ (resp. $K_j(\tilde{W})$) is convex. To see this property, take $\tilde{W}_1 \in K_h(\tilde{J})$ and $\tilde{W}_2 \in K_h(\tilde{J})$, then

$$\tilde{W}_1(\hat{w}) \geq W^{SO}(\hat{w}), if\ \hat{w} \in (\tilde{\mathcal{C}}_j)^c \Rightarrow \tilde{W}_1(\hat{w}) = 0, \tag{127}$$

$$\tilde{W}_2(\hat{w}) \geq W^{SO}(\hat{w}), if\ \hat{w} \in (\tilde{\mathcal{C}}_j)^c \Rightarrow \tilde{W}_2(\hat{w}) = 0 \tag{128}$$

Then for any $\alpha \in [0, 1]$,

$$\alpha\tilde{W}_1(\hat{w}) + (1-\alpha)\tilde{W}_2(\hat{w}) \geq W^{SO}(\hat{w}),\ if\ \hat{w} \in (\tilde{\mathcal{C}}_j)^c \Rightarrow \alpha\tilde{W}_1(\hat{w}) + (1-\alpha)\tilde{W}_2(\hat{w}) = 0. \tag{129}$$

Thus, $\alpha\tilde{W}_1(\hat{w}) + (1-\alpha)\tilde{W}_2(\hat{w}) \in K_h(\tilde{J})$. Third, $K_h(\tilde{J})$ (resp. $K_j(\tilde{W})$) is non-empty, since $\tilde{W}_1(\hat{w}) = W^{SO}(\hat{w})$ is always in the set. With these observation, we can write the optimality quasi-variational inequalities

$$\tilde{W} \in K_h(\tilde{J}),\ \ \tilde{J} \in K_j(\tilde{W}) \tag{130}$$

$$a(\tilde{J}, v - \tilde{J}) \geq \int_{\mathbb{R}} \left(1 - e^{\hat{w}}\right)\left(v - \tilde{J}\right) \mathrm{d}\hat{w}, v \in K_j(\tilde{W}) \tag{131}$$

$$a(\tilde{W}, v - \tilde{W}) \geq \int_{\mathbb{R}} (e^{\hat{w}} - \tilde{\rho}\tilde{U})\left(v - \tilde{W}\right) \mathrm{d}\hat{w}, v \in K_h(\tilde{J}) \tag{132}$$

Now, we show the following theorem. Suppose $\tilde{J}, \tilde{W} \in C^2(\mathbb{R})$ satisfies (130), (131), (132) . Then $(\tilde{J}, \tilde{W})$ satisfy (96) and (105) and

If $\tilde{W} \in K_h(\tilde{J})$ and $\tilde{J} \in K_j(\tilde{W})$, then $(\tilde{W}(\hat{w}), \tilde{J}(\hat{w})) \geq (W^{SO}(\hat{w}), J^{SO}(\hat{w}))$. Since $(W^{SO}(\hat{w}), J^{SO}(\hat{w})) \geq (0, 0)$, we have that $(\tilde{W}(\hat{w}), \tilde{J}(\hat{w})) \geq (0, 0)$. So, we check (96) and (97). Similar argument for (98) and (99). Additionally, since $(W^{SO}(\hat{w}), J^{SO}(\hat{w})) > (0, 0)$ for all $\hat{w} \in (\log(\tilde{\rho}\tilde{U}), 0)$, we have that there is at least one element in $(\log(\tilde{\rho}\tilde{U}), 0) \subset \tilde{\mathcal{C}}_j \cap \tilde{\mathcal{C}}_h$, thus the intersection of the continuation set is non-empty.

Now, we show that (131) implies (101). Take any $v \in K_j(\tilde{W})$. Then if $\hat{w} \in (\tilde{\mathcal{C}}_h)^c$, we have that

$\tilde{J}(\hat{w}) = 0$. Therefore, we have that for every $v \in K_j(\tilde{W})$

$$a(\tilde{J}, v - \tilde{J}) \geq \int_{\mathbb{R}} (e^{\hat{w}} - \tilde{\rho}\tilde{U}) \left(v - \tilde{J}\right) \iff \tag{133}$$

$$\frac{\sigma^2}{2} \int_{\mathcal{C}_j^c} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(v(\hat{w}) - \tilde{J}(\hat{w}))}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \nu \int_{\mathcal{C}_j^c} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} (v(\hat{w}) - \tilde{J}(\hat{w})) \,\mathrm{d}\hat{w} + (\tilde{\rho} + \delta) \int_{\mathcal{C}_j^c} \tilde{J}(\hat{w})(v(\hat{w}) - \tilde{J}(\hat{w}))) \,\mathrm{d}\hat{w} +$$

$$\frac{\sigma^2}{2} \int_{\mathcal{C}_j} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(v(\hat{w}) - \tilde{J}(\hat{w}))}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \nu \int_{\mathcal{C}_j} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} (v(\hat{w}) - \tilde{J}(\hat{w})) \,\mathrm{d}\hat{w} + (\tilde{\rho} + \delta) \int_{\mathcal{C}_j} \tilde{J}(\hat{w})(v(\hat{w}) - \tilde{J}(\hat{w})) \,\mathrm{d}\hat{w} \geq$$

$$\int_{\mathcal{C}_j} (1 - e^{\hat{w}}) \left(v(\hat{w}) - \tilde{J}(\hat{w})\right) \mathrm{d}\hat{w} + \int_{\mathcal{C}_j^c} (1 - e^{\hat{w}}) \left(v(\hat{w}) - \tilde{J}(\hat{w})\right) \mathrm{d}\hat{w} \iff$$

$$\frac{\sigma^2}{2} \int_{\mathcal{C}_j} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(v(\hat{w}) - \tilde{J}(\hat{w}))}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \nu \int_{\mathcal{C}_j} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} (v(\hat{w}) - \tilde{J}(\hat{w})) \,\mathrm{d}\hat{w} + (\tilde{\rho} + \delta) \int_{\mathcal{C}_j} \tilde{J}(\hat{w})(v(\hat{w}) - \tilde{J}(\hat{w})) \,\mathrm{d}\hat{w} \geq$$

$$\int_{\mathcal{C}_j} (1 - e^{\hat{w}}) \left(v(\hat{w}) - \tilde{J}(\hat{w})\right) \mathrm{d}\hat{w}$$

Using integration by part and the property that if $\hat{w} \in \partial\mathcal{C}_j$ ($\mathcal{C}_j$ is an open set), then $\hat{w} \in \partial\mathcal{C}_j^c$, we have that

$$= \frac{\sigma^2}{2} \int_{\mathcal{C}_j} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(v(\hat{w}) - \tilde{J}(\hat{w}))}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} \tag{134}$$

$$= \underbrace{\frac{\sigma^2}{2} \frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} (v(\hat{w}) - \tilde{J}(\hat{w}))\Big|_{w \in \partial \in \mathcal{C}_j}}_{=0} - \frac{\sigma^2}{2} \int_{\mathcal{C}_j} \frac{d^2\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}} (v(\hat{w}) - \tilde{J}(\hat{w})) \,\mathrm{d}\hat{w}. \tag{135}$$

Using this result,

$$\int_{\mathcal{C}_j} \left(\tilde{\mathcal{A}}\tilde{J}(\hat{w}) + (1 - e^{\hat{w}})\right) (v(\hat{w}) - \tilde{J}(\hat{w})) \,\mathrm{d}\hat{w} \leq 0 \tag{136}$$

Let $\mathcal{O}$ be an open ball in $\mathcal{C}_j$. Then we can find a family of smooth functions $o_{\hat{w}}(\epsilon) \in [0, 1]$, s.t. $o_{\hat{w}}(\epsilon) = 0$ outside $\mathcal{C}_j$, $o_{\hat{w}}(\epsilon) \to 1$ in $\mathcal{O}$, and $o_{\hat{w}}(\epsilon) \to 0$ outside $\mathcal{O}$. Since $\tilde{J}(\hat{w}) + \theta_{\hat{w}}(\epsilon) \geq 0$, $\tilde{J}(\hat{w}) + \theta_{\hat{w}}(\epsilon) \in K_j(\tilde{W})$ and

$$\int_{\mathcal{O}} \left(\tilde{\mathcal{A}}\tilde{J}(\hat{w}) + (1 - e^{\hat{w}})\right) \theta_{\hat{w}}(\epsilon) \,\mathrm{d}\hat{w} + \int_{\mathcal{C}_j \cap \mathcal{O}} \left(\tilde{\mathcal{A}}\tilde{J}(\hat{w}) + (1 - e^{\hat{w}})\right) \theta_{\hat{w}}(\epsilon) \,\mathrm{d}\hat{w} \leq 0 \tag{137}$$

Taking the limit $\epsilon \to 0$, we have that

$$\int_{\mathcal{O}} \left(\tilde{\mathcal{A}}\tilde{J}(\hat{w}) + (1 - e^{\hat{w}})\right) \mathrm{d}\hat{w} \leq 0. \tag{138}$$

Since $\mathcal{O}$ is arbitrary, $\left(\tilde{\mathcal{A}}\tilde{J}(\hat{w}) + (1 - e^{\hat{w}}\right) \leq 0$ a.e. Since $\tilde{J}(\hat{w}) \in C^2(\mathbb{R})$, then $\left(\tilde{\mathcal{A}}\tilde{J}(\hat{w}) + (1 - e^{\hat{w}})\right) \leq 0$ for

all $\hat{w}$. To obtain the other inequality, take $\tilde{J}(\hat{w})(1-o_{\hat{w}}(\epsilon))+0o_{\hat{w}}(\epsilon)\in K_j(\tilde{W})$ and we have that

$$-\int_{\mathcal{O}}\left(\tilde{\mathcal{A}}\tilde{J}(\hat{w})+(1-e^{\hat{w}})\right)\tilde{J}(\hat{w})o_{\hat{w}}(\epsilon)\,\mathrm{d}\hat{w}-\int_{\mathcal{C}_j\cap\mathcal{O}}\left(\tilde{\mathcal{A}}\tilde{J}(\hat{w})+(1-e^{\hat{w}})\right)\tilde{J}(\hat{w})o_{\hat{w}}(\epsilon)\,\mathrm{d}\hat{w}\leq 0 \tag{139}$$

Taking the limit $\epsilon\downarrow 0$, we have that $\int_{\mathcal{O}}\left(\tilde{\mathcal{A}}\tilde{J}(\hat{w})+(1-e^{\hat{w}})\right)(-\tilde{J}(\hat{w}))\,\mathrm{d}\hat{w}\leq 0$ a.e.. Since $\tilde{J}(\hat{w})\in C^2(\mathbb{R})$, we have that for all $\hat{w}\in\mathcal{C}_j$

$$\left(\tilde{\mathcal{A}}\tilde{J}(\hat{w})+(1-e^{\hat{w}})\right)(-\tilde{J}(\hat{w}))\leq 0. \tag{140}$$

Since $\tilde{J}(\hat{w})\geq 0$ and $\left(\tilde{\mathcal{A}}\tilde{J}(\hat{w})+(1-e^{\hat{w}})\right)\leq 0$, we have that

$$0=\max\{-\tilde{J}(\hat{w}),\mathcal{A}\tilde{J}(\hat{w})+1-e^{\hat{w}}\},\ \forall\hat{w}\in\tilde{\mathcal{C}}_h. \tag{141}$$

The proof of (100) follows similar steps as before.

Step 2: Now, we show that there exist an unique solution to

$$\begin{aligned}
a(\tilde{J},\tilde{W}) &:= \frac{\sigma^2}{2}\int_{\mathbb{R}}\frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}\tilde{W}(\hat{w})}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w}+\nu\int_{\mathbb{R}}\frac{\mathrm{d}\tilde{J}(\hat{w})}{\mathrm{d}\hat{w}}\tilde{W}(\hat{w})\,\mathrm{d}\hat{w}+(\tilde{\rho}+\delta)\int_{\mathbb{R}}\tilde{J}(\hat{w})\tilde{W}(\hat{w})\,\mathrm{d}\hat{w} \qquad (142)\\
K_h(\tilde{J}) &:= \left\{\tilde{W}\in V:\tilde{W}(\hat{w})\geq W^{SO}(\hat{w})\ \&\ if\ \hat{w}\in(\tilde{\mathcal{C}}_j)^c\Rightarrow\tilde{W}(\hat{w})=0\right\},\\
K_j(\tilde{W}) &:= \left\{\tilde{J}\in V:\tilde{J}(\hat{w})\geq J^{SO}(\hat{w})\ \&\ if\ \hat{w}\in(\tilde{\mathcal{C}}_h)^c\Rightarrow\tilde{J}(\hat{w})=0\right\},\\
\mathcal{C}_h &:= int\left\{\hat{w}\in\mathbb{R}:\tilde{W}(\hat{w})>0,\tilde{\mathcal{A}}\tilde{W}(\hat{w})+(e^{\hat{w}}-\tilde{\rho}\tilde{U})\geq 0\right\},\\
\mathcal{C}_j &:= int\left\{\hat{w}\in\mathbb{R}:\tilde{J}(\hat{w})>0,\tilde{\mathcal{A}}\tilde{J}(\hat{w})+\left(1-e^{\hat{w}}\right)\geq 0\right\},\\
\tilde{W}\in K_h(\tilde{J}),\ \ \tilde{J}&\in K_j(\tilde{W}),\\
a(\tilde{J},v-\tilde{J}) &\geq \int_{\mathbb{R}}(e^{\hat{w}}-\tilde{\rho}\tilde{U})\left(v-\tilde{J}\right)\mathrm{d}\hat{w}, v\in K_j(\tilde{W}),\\
a(\tilde{W},v-\tilde{W}) &\geq \int_{\mathbb{R}}\left(1-e^{\hat{w}}\right)\left(v-\tilde{W}\right)\mathrm{d}\hat{w}, v\in K_h(\tilde{J}).
\end{aligned}$$

Define the following mapping

$$T_h(\tilde{J})=\left\{\tilde{W}\in L^2(\mathbb{R}):a(\tilde{W},v-\tilde{W})\geq\left(e^{\hat{w}}-\tilde{\rho}\tilde{U},v-\tilde{W}\right)\ \text{for any } v,\tilde{W}\in K_h(\tilde{J})\right\} \tag{143}$$

$$T_j(\tilde{W})=\left\{\tilde{J}\in L^2(\mathbb{R}):a(\tilde{J},v-\tilde{J})\geq\left(1-e^{\hat{w}},v-\tilde{J}\right)\ \text{for any } v,\tilde{J}\in K_j(\tilde{W})\right\} \tag{144}$$

where $\int_{\mathbb{R}}fg\,\mathrm{d}\hat{w}=(f,g)$. First, we show that $T_h(\cdot)$ and $T_j(\cdot)$ are functions. For this, we need to check the conditions for the Lemma **??**. Since $\sigma^2,\nu,\tilde{\rho}+\delta$ satisfies conditions (2.2) to (2.5) from Bensoussan and Friedman (1977) (see equation 2.7). Thus, $a(\cdot,\cdot)$ is coercive and since $K_h(\tilde{J})$ and $K_j(\tilde{W})$ are closed and

convex, we can apply **??**. Thus, $T_h(\cdot)$ and $T_j(\cdot)$ are functions.

Define $S_h = T_h \circ T_j$. Then if $\tilde{W} = S_h(\tilde{W})$, then it is easy to see that $(\tilde{W}, T_j(\tilde{W}))$ is a solution of (142). Thus, we only need to show the existence of a unique fixed point in S_h. For this, we will use the fixed point theorem in 14. For this, first we look for the lower bound and the upper bound of the mapping, and then we show the mapping is increasing.

Lower bound: The main idea here is that the stopping time after taking firms and worker best response has to be at least better than the static solution for any player. By construction, $T_h(J) \geq W^{SO}$ for any $T_h(J) \in K_h(\tilde{J})$, in particular, for $T_j(W^{SO})$. Thus, we have that $T_h(T_j(W^{SO})) \geq W^{SO}$.

Upper bound: The main idea here is that the stopping time after taking firms and worker best response has to be at least worst than the solution whenever the static payoff is positive. By construction, $T_h(J) \leq W^{SSW}$ for any $T_h(J) \in K_h(\tilde{J})$, in particular, for $T_j(W^{BSW})$. Thus, we have that $T_h(T_j(W^{BSW})) \leq W^{BSW}$.

The mapping is increasing: Take $X_h \geq X_h'$. Write $\tilde{J} = T_j(X_h)$ and $\tilde{W} = T_h(T_j(X_h))$ (resp. $\tilde{J}' = T_j(X_h')$ and $\tilde{W} = T_h(T_j(X_h'))$). First we will proof that

$$\tilde{J}' \geq \tilde{J} \tag{145}$$

Since $X_h \geq X_h'$, then the stopping set of X_h' is larger and $K_j(X_h') \subset K_j(X_h)$. Therefore $\tilde{J}, \tilde{J}' \in K_j(X_h)$ and also $\min\{\tilde{J}, \tilde{J}'\} \in K_j(X_h)$. With similar argument, we have that $\max\{\tilde{J}, \tilde{J}'\} \in K_j(X_h')$. Since we can write

$$\min\{\tilde{J}, \tilde{J}'\} = \tilde{J} - \max\{0, \tilde{J} - \tilde{J}'\} \ , \quad \max\{\tilde{J}, \tilde{J}'\} = \tilde{J}' + \max\{0, \tilde{J} - \tilde{J}'\} \tag{146}$$

we get

$$-a(\tilde{J}, \max\{0, \tilde{J} - \tilde{J}'\}) \geq -(1 - e^w, \max\{0, \tilde{J} - \tilde{J}'\}), \tag{147}$$

$$a(\tilde{J}', \max\{0, \tilde{J} - \tilde{J}'\}) \geq (1 - e^w, \max\{0, \tilde{J} - \tilde{J}'\}) \tag{148}$$

and, by addition, $a(\max\{0, \tilde{J} - \tilde{J}'\}, \max\{0, \tilde{J} - \tilde{J}'\}) \leq 0$. Since $a(\cdot, \cdot)$ is coercive, $a(\max\{0, \tilde{J} - \tilde{J}'\}, \max\{0, \tilde{J} - \tilde{J}'\}) \geq 0$, and we have that $a(\max\{0, \tilde{J} - \tilde{J}'\}, \max\{0, \tilde{J} - \tilde{J}'\}) = 0$, thus $\max\{0, \tilde{J} - \tilde{J}'\} = 0$. So, $\tilde{J} \leq \tilde{J}'$. With similar argument, we can show that $\tilde{W}' \leq \tilde{W}$. Thus, the operator is monotone.

With these properties verified, we can apply 14 and we have that there exist a non-empty set of fixed points s.t.

$$(W^{SO}, J^{SO}) \leq (\tilde{W}^*, \tilde{J}^*) \leq (W^{SWE}, J^{SWJ}). \tag{149}$$

Steps 3 *Uniqueness, continuity, bounded continuation set, and decreasing values in* $\tilde{\rho}\tilde{U}$.

Fixed point for $\tilde{U}$. Define the operator $T : [b, 1] \to [b, 1]$

$$T(\tilde{\rho}\tilde{U}) := b + \max_{\hat{w}} \frac{1}{\kappa^{\alpha}} \tilde{J}(\hat{w}, \tilde{\rho}\tilde{U})^{\frac{1-\alpha}{\alpha}} \tilde{W}(\hat{w}, \tilde{\rho}\tilde{U}). \tag{150}$$

Since $cl\{\mathcal{C}_j \cap \mathcal{C}_h\}$ is bounded and close, and $\tilde{J}(\cdot, \tilde{U}), \tilde{W}(\cdot, \tilde{U})$ are continuos, there exist a unique solution to $\max_{\hat{w}} \frac{1}{\kappa^{\alpha}} \tilde{J}(\hat{w}, \tilde{U})^{\frac{1-\alpha}{\alpha}} \tilde{W}(\hat{w}, \tilde{U})$. Since $\tilde{J}(\hat{w}, \cdot), \tilde{W}(\hat{w}, \cdot)$ are decreasing, we have that for all $\tilde{U}' \geq \tilde{U}$ with solution $\hat{w}(\tilde{U})$

$$\frac{1}{\kappa^{\alpha}} \tilde{J}(\hat{w}(\tilde{\rho}\tilde{U}), \tilde{\rho}\tilde{U})^{\frac{1-\alpha}{\alpha}} \tilde{W}(\hat{w}(\tilde{\rho}\tilde{U}), \tilde{\rho}\tilde{U}) \geq^{(1)} \frac{1}{\kappa^{\alpha}} \tilde{J}(\hat{w}(\tilde{U}'), \tilde{\rho}\tilde{U})^{\frac{1-\alpha}{\alpha}} \tilde{W}(\hat{w}(\tilde{\rho}\tilde{U}'), \tilde{\rho}\tilde{U}) \tag{151}$$

$$\geq^{(2)} \frac{1}{\kappa^{\alpha}} \tilde{J}(\hat{w}(\tilde{\rho}\tilde{U}'), \tilde{\rho}\tilde{U}')^{\frac{1-\alpha}{\alpha}} \tilde{W}(\hat{w}(\tilde{\rho}\tilde{U}'), \tilde{\rho}\tilde{U}') \tag{152}$$

Optimality implies step (1) and step (2) is implied since $\tilde{J}(\hat{w}, \cdot), \tilde{W}(\hat{w}, \cdot)$ are decreasing. Therefore, $T(\tilde{U})$ exist and is unique, it is continuous, and decreasing. Since $T(b) > b$,$T(1) < 1$, and it is decreasing, by the intermediate value theorem, there exist an unique $\tilde{U}$ s.t. $T(\tilde{\rho}\tilde{U}) = \tilde{\rho}\tilde{U}$. □

B.3 Proof of Proposition 2

Proof. Now, we proof each equilibrium property.

1. Since for all $\hat{w}$ we have that $\tilde{\theta}(\hat{w}), \tilde{W}(\hat{w}) \geq 0$, the product is also positive in $\hat{w}^*$

$$\tilde{\rho}\tilde{U} = \tilde{b} + \max_{\hat{w}^*} \tilde{\theta}(\hat{w}^*)^{1-\alpha} \tilde{W}(\hat{w}^*) \geq \tilde{b} \iff \tilde{U}\rho \geq \tilde{b} \tag{153}$$

Using the recurse definition of the value function, we have that

$$\tilde{W}(\hat{w}) = \mathbb{E}^{\hat{w}} \left[\int_0^{\tau^{h*}} e^{-\tilde{\rho}t} (e^{w_t} - \tilde{\rho}\tilde{U}) \, \mathrm{d}t \right] \tag{154}$$

$$\tilde{J}(\hat{w}) = \mathbb{E}^{\hat{w}} \left[\int_0^{\tau^{h*}} e^{-\tilde{\rho}t} (1 - e^{w_t}) \, \mathrm{d}t \right] \tag{155}$$

where τ^{*h} is the Nash equilibrium in the math with the exogenous separations. Summing up the previous two equations

$$\tilde{W}(\hat{w}) + \tilde{J}(\hat{w}) = \mathbb{E}^{\hat{w}} \left[\int_0^{\tau^{h*}} e^{-(\tilde{\rho}+\delta)t} (1 - \tilde{\rho}\tilde{U}) \, \mathrm{d}t \right] = (1 - \tilde{\rho}\tilde{U}) \mathbb{E}^{\hat{w}} \left[\int_0^{\tau^{h*}} e^{-(\tilde{\rho}+\delta)t} \, \mathrm{d}t \right]. \tag{156}$$

Solving the Riemman integral $\int_0^{\tau^{h*}} e^{-(\tilde{\rho}+\delta)t}\,\mathrm{d}t = \frac{1-e^{-(\tilde{\rho}+\delta)t}}{(\tilde{\rho}+\delta)}$, thus

$$\tilde{W}(\hat{w}) + \tilde{J}(\hat{w}) = (1-\tilde{\rho}\tilde{U})\frac{1-\mathbb{E}^{\hat{w}}\left[e^{-\tilde{\rho}\tau^{h*}}\right]}{\tilde{\rho}} \tag{157}$$

Since $\tilde{W}(\hat{w}), \tilde{J}(\hat{w}) \geq 0$ and $\frac{1-\mathbb{E}^{\hat{w}}\left[e^{-\tilde{\rho}\tau^{h*}}\right]}{\tilde{\rho}} \geq 0$ (remember that $\tau^{h*} = 0$ cannot be an equilibrium), from conditions (96), (97), and (102), we have that

$$0 \leq \tilde{W}(\hat{w}) + \tilde{J}(\hat{w}) = (1-\tilde{\rho}\tilde{U})\frac{1-\mathbb{E}^{\hat{w}}\left[e^{-\tilde{\rho}\tau^{h*}}\right]}{\tilde{\rho}} \iff (1-\tilde{\rho}\tilde{U}) \geq 0 \iff 1 \geq \tilde{\rho}\tilde{U}. \tag{158}$$

2. Now, we show the property

$$\{\hat{w} : w > \log(\tilde{\rho}\tilde{U})\} \subset S_h^c \text{ and} \tag{159}$$

$$\{\hat{w} : w < 0\} \subset S_j^c \tag{160}$$

following Oksendal (2007) Section 10.3.. The idea here is to show that the worker or the firm would never stop if the static payoff is positive and the terminal payoff is independent of the stopping decision. Let U_h and U_j be the set defined in the $(t, \hat{w})$ space, such that, the static payoff is positive for the worker and firm, respectively. From Oksendal (2007), it is easy to see that $U_h \subset [0,\infty) \times S_h$ and $U_j \subset [0,\infty) \times S_j$. Since the terminal payoff is zero, it is easy to show that

$$U_h = \{(t,\hat{w}) : e^{-(\tilde{\rho}+\delta)t}(e^{\hat{w}} - \tilde{\rho}\tilde{U}) > 0\} = \{(t,\hat{w}) : \hat{w} > \log(\tilde{\rho}\tilde{U})\} \tag{161}$$

and

$$U_j = \{(t,\hat{w}) : e^{-(\tilde{\rho}+\delta)t}(1 - e^{\hat{w}}) > 0\} = \{(t,\hat{w}) : \hat{w} < 0\}. \tag{162}$$

To show that continuation set is not-empty, we need to show that $\tilde{\rho}\tilde{U} < 1$. From the first property, we have $\tilde{\rho}\tilde{U} \leq 1$. By contradiction, assume that $\tilde{\rho}\tilde{U} = 1$. Then using equation (95), (97) ,and $\tilde{W}(\hat{w}) + \tilde{J}(\hat{w}) = (1-\tilde{\rho}\tilde{U})\frac{1-\mathbb{E}^{\hat{w}}\left[e^{-\tilde{\rho}\tau^{h*}}\right]}{\tilde{\rho}} \geq \tilde{W}(\hat{w})$, we have

$$\tilde{\rho}\tilde{U} = \tilde{b} + \max_{\hat{w}} \theta(\hat{w})^{1-\alpha}\tilde{W}(\hat{w}), \tag{163}$$

$$\leq \tilde{b} + \theta(\hat{w}^*)^{1-\alpha}(1-\tilde{\rho}\tilde{U})\frac{1-\mathbb{E}^{\hat{w}^*}\left[e^{-\tilde{\rho}\tau^{h*}}\right]}{\tilde{\rho}} = \tilde{b} < 1. \tag{164}$$

In conclusion, we have the contradiction. To show that S_h^c and S_j^c are connected, assume they are not. Without loss of generality, assume that $S_h^c = \{\hat{w} : \hat{w} > \hat{w}^-\} \cup (a, b)$ with $a < b < w^-$. Then, since $\hat{w}^- < \tilde{\rho}\tilde{U}$, it must be the case that for all $\hat{w} \in (a, b)$, we have $(e^{\hat{w}} - \tilde{\rho}\tilde{U}) < 0$ for all $\hat{w} \in (a, b)$, and $\tilde{W}(x) = \mathbb{E}^x \left[\int_0^{\tau_{S_h^c \cap S_j^c}} e^{-(\tilde{\rho}+\delta)t}(e^{w_t} - \tilde{\rho}\tilde{U})\, \mathrm{d}t \right] < 0$ for all $x \in (a, b)$ due to continuity of Brownian motions. Since $\tilde{W}(x) \geq 0$, we have a contradiction.

3. To show this property, first we show that $J(\hat{w}) > 0$ for all $\hat{w} \in (\tilde{\rho}\tilde{U}, 0)$. Define

$$\tau_{(w^-,0)} = \inf_t \{t : \hat{w}_t \notin (\tilde{\rho}\tilde{U}, 0)\}. \tag{165}$$

By optimality of the firm,

$$J(\hat{w}) = \mathbb{E}^{\hat{w}} \left[\int_0^{\tau^{h*}} e^{-(\tilde{\rho}+\delta)t}(1 - e^{w_t})\, \mathrm{d}t \right] \geq \mathbb{E}^{\hat{w}} \left[\int_0^{\min\{\tau_{(\tilde{\rho}\tilde{U},0)}, \tau^{*h}\}} e^{-(\tilde{\rho}+\delta)t}(1 - e^{w_t})\, \mathrm{d}t \right] > 0. \tag{166}$$

Thus, there is an open set s.t. $J(\hat{w}) > 0$, $\tilde{\theta}(\hat{w}) > 0$, and $J(\hat{w}) - \tilde{\kappa}\tilde{\theta}(\hat{w})^\alpha = 0$. Therefore,

$$\arg\max_{\hat{w}} \left\{ p(\tilde{\theta}(\hat{w}))\tilde{W}(\hat{w}) \right\} = \arg\max_{\hat{w}} \left\{ \left(\frac{\tilde{J}(\hat{w})}{\tilde{\kappa}} \right)^{\frac{1-\alpha}{\alpha}} \tilde{W}(\hat{w}) \right\} \propto \arg\max_{\hat{w}} \left\{ \tilde{J}(\hat{w})^{1-\alpha} \tilde{W}(\hat{w})^\alpha \right\}. \tag{167}$$

Taking first order conditions

$$(1 - \alpha) \frac{\mathrm{d}\log \tilde{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -\alpha \frac{\mathrm{d}\log \tilde{W}(\hat{w}^*)}{\mathrm{d}\hat{w}}. \tag{168}$$

4. The argument for the smooth pasting is standard. For a longer proof see, for example, Oksendal (2007). We show that $\tilde{W}'(\hat{w}^-) = 0$. The proof for $\tilde{J}'(\hat{w}^+) = 0$ is similar. Since $W(\hat{w}) = 0$ for all $\hat{w} < \hat{w}^-$, we have that $\lim_{\hat{w} \uparrow \hat{w}^-} \tilde{W}'(\hat{w}) = 0$. Now, we show that $\lim_{\hat{w} \downarrow \hat{w}^-} \tilde{W}'(\hat{w}) = 0$. If $\lim_{\hat{w} \downarrow \hat{w}^-} \tilde{W}'(\hat{w}) < 0$, since $W(\hat{w}^-) = 0$, it is easy to show that there exists an $\epsilon > 0$ s.t. $W(\hat{w}^- + \epsilon) < 0$ and therefore we violate the optimality condition $W(\hat{w}^- + \epsilon) \geq 0$. Intuitively, in this case, the worker prefers to decrease the duration of the match. Assume $\lim_{\hat{w} \downarrow \hat{w}^-} \tilde{W}'(\hat{w}) > 0$ with stopping region for the worker $(-\infty, w^-)$. Then, we can perturb the stopping region $(-\infty, \hat{w}^- - \epsilon)$ and it is easy to show that $W(\hat{w})$ increases for all $\hat{w}$ under this perturbation. Thus, $\lim_{\hat{w} \downarrow \hat{w}^-} \tilde{W}'(\hat{w}) > 0$ cannot be optimal. Intuitively, in this case, the worker prefers to increase the duration of the match.

□

B.4 Proof of Propositions 3, 4, and

Define $\tilde{\mathcal{R}} = (\hat{w}^-, \hat{w}^+)$. From proposition **??**, we can work with the following HJB conditions

$$(\tilde{\rho} + \delta)\tilde{W}(\hat{w}) = e^{\hat{w}} - \tilde{\rho}\tilde{U} - \nu\tilde{W}'(\hat{w}) + \frac{\sigma^2}{2}\tilde{W}''(\hat{w}) \quad \forall \hat{w} \in \tilde{\mathcal{R}} \tag{169}$$

$$(\tilde{\rho} + \delta)\tilde{J}(\hat{w}) = 1 - e^{\hat{w}} - \nu\tilde{J}'(\hat{w}) + \frac{\sigma^2}{2}\tilde{J}''(\hat{w}) \quad \forall \hat{w} \in \tilde{\mathcal{R}} \tag{170}$$

$$\tilde{\rho}\tilde{U} = \tilde{b} + \kappa^{1-\alpha^{-1}}\tilde{J}(\hat{w}^*)^{\frac{1-\alpha}{\alpha}}\tilde{W}(\hat{w}^*) \tag{171}$$

$$(1-\alpha)\frac{\mathrm{d}\log\tilde{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -\alpha\frac{\mathrm{d}\log\tilde{W}(\hat{w}^*)}{\mathrm{d}\hat{w}}, \tag{172}$$

with the value matching

$$\tilde{W}(\hat{w}^-) = \tilde{J}(\hat{w}^-) = \tilde{W}(\hat{w}^+) = \tilde{J}(\hat{w}^+) = 0; \tag{173}$$

and smooth pasting conditions

$$\tilde{W}'(\hat{w}^-) = \tilde{J}'(\hat{w}^+) = 0. \tag{174}$$

Proposition 17. *Assume $\nu, \sigma^2 \downarrow 0$. Then*

$$\hat{w}^+ = 0 \tag{175}$$

$$\hat{w}^- = \log\left(\tilde{\rho}\tilde{U}\right) \tag{176}$$

$$\hat{w}^* = \log(\alpha + (1-\alpha)\tilde{\rho}\tilde{U}) \tag{177}$$

where $\tilde{U}$ satisfies

$$\tilde{\rho}\tilde{U} - \tilde{b} = \Theta\left(1 - \tilde{\rho}\tilde{U}\right)^{1/\alpha} \tag{178}$$

$\tilde{\rho}\tilde{U}$ is increasing in $\Theta = \left(\kappa^{1-\alpha}(\tilde{\rho}+\delta)\right)^{-1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}}\alpha$ and $\tilde{b}$.

Proof. If $\nu = \sigma = 0$, conditions (169) and (170) imply

$$\tilde{W}(\hat{w}) = \frac{e^{\hat{w}} - e^{\hat{w}^-}}{\tilde{\rho} + \delta} \quad ; \quad \tilde{J}(\hat{w}) = \frac{e^{\hat{w}^+} - e^{\hat{w}}}{\tilde{\rho} + \delta}. \tag{179}$$

Therefore, conditions (173) imply

$$\hat{w}^+ = 0 \quad ; \quad \hat{w}^- = \log\left(\tilde{\rho}\tilde{U}\right). \tag{180}$$

Using the optimality condition for new hires wages (171)

$$(1-\alpha)\frac{e^{\hat{w}}}{1-e^{\hat{w}}} = \alpha\frac{e^{\hat{w}}}{e^{\hat{w}}-\tilde{\rho}\tilde{U}} \tag{181}$$

Thus,

$$w^* = \log(\alpha + (1-\alpha)(\tilde{\rho}_h - \delta)\tilde{U}). \tag{182}$$

With these result, we obtain the value of unemployment

$$\begin{aligned}
\tilde{\rho}\tilde{U} &= \tilde{b} + \kappa^{1-\alpha^{-1}}\left(J(\hat{w}^*)^{1-\alpha}\tilde{W}(\hat{w}^*)^{\alpha}\right)^{1/\alpha} \\
&= \tilde{b} + \kappa^{1-\alpha^{-1}}\left(\left(\frac{1-e^{\hat{w}^*}}{\tilde{\rho}+\delta}\right)^{1-\alpha}\left(\frac{e^{\hat{w}*}-\tilde{\rho}\tilde{U}}{\tilde{\rho}+\delta}\right)^{\alpha}\right)^{1/\alpha} \\
&= \tilde{b} + \kappa^{1-\alpha^{-1}}(\tilde{\rho}+\delta)^{-1/\alpha}\left(\left(1-(\alpha+(1-\alpha)\tilde{\rho}\tilde{U})\right)^{1-\alpha}\left((\alpha+(1-\alpha)\tilde{\rho}\tilde{U})-\tilde{\rho}\tilde{U}\right)^{\alpha}\right)^{1/\alpha} \\
&= \tilde{b} + \left(\kappa^{1-\alpha}(\tilde{\rho}+\delta)\right)^{-1/\alpha}\left(\left((1-\alpha)(1-\tilde{\rho}\tilde{U})\right)^{1-\alpha}\left(\alpha(1-\tilde{\rho}\tilde{U})\right)^{\alpha}\right)^{1/\alpha} \\
&= \tilde{b} + \underbrace{\left(\kappa^{1-\alpha}(\tilde{\rho}+\delta)\right)^{-1/\alpha}(1-\alpha)^{\frac{1-\alpha}{\alpha}}\alpha}_{=\Theta>0}\left(1-\tilde{\rho}\tilde{U}\right)^{1/\alpha}
\end{aligned} \tag{183}$$

Operating we have

$$\tilde{\rho}\tilde{U} - \tilde{b} = \Theta\left(1-\tilde{\rho}\tilde{U}\right)^{1/\alpha} \tag{184}$$

□

Proposition 18. *Assume $\nu \downarrow 0$, $\alpha = 1/2$, and a linear approximation over flow profits around w^*. Then $\hat{w}^- = \hat{w}^* - h$ and $\hat{w}^+ = \hat{w}^* + h$*

$$e^{\hat{w}^*} = \frac{1+\tilde{\rho}\tilde{U}}{2}, h(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$$

with $\varphi = \sqrt{2\rho+\delta}/\sigma$, $\Phi = \frac{1-\tilde{\rho}\tilde{U}}{1+\tilde{\rho}\tilde{U}}$, and $\omega(z)$ decreasing with $\lim_{z\to 0}\omega(z) = 3$ and $\lim_{z\to\infty}\omega(z) = 1$. The opportunity cost is given by

$$\tilde{\rho}\tilde{U} = \tilde{b} + (1-\tilde{\rho}\tilde{U})^2\Theta(1/2)\left(-\frac{\sinh(\varphi h(\varphi,\Phi))}{\sinh(\varphi 2h(\varphi,\Phi))}+1\right)^2$$

Proof. Let us guess and verify the following solution $w^* = \log\left(\frac{1+(\tilde{\rho}_u-\delta)\tilde{U}}{2}\right)$ and $\hat{w}^- = w^* - k$ and $\hat{w}^+ =$

$w^* + k$ for a given k. Using a Taylor approximation over the flow profits around w^*

$$e^{\hat{w}} - \tilde{\rho}\tilde{U} \approx e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) - \tilde{\rho}\tilde{U} = \frac{1 - \tilde{\rho}\tilde{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*), \tag{185}$$

$$1 - e^{\hat{w}} \approx 1 - e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) = \frac{1 - \tilde{\rho}\tilde{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*). \tag{186}$$

We can write the optimality conditions as

$$(\tilde{\rho} + \delta)\tilde{W}(\hat{w}) = \frac{1 - \tilde{\rho}\tilde{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{\sigma^2}{2}\tilde{W}''(\hat{w}), \quad \forall \hat{w} \in (w^* - k, w^* + k) \tag{187}$$

$$(\tilde{\rho} + \delta)\tilde{J}(\hat{w}) = \frac{1 - \tilde{\rho}\tilde{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{\sigma^2}{2}\tilde{J}''(\hat{w}), \quad \forall \hat{w} \in (w^* - k, w^* + k) \tag{188}$$

with the border conditions

$$\tilde{W}(\hat{w}^* - k) = \tilde{J}(\hat{w}^* - k) = \tilde{W}(\hat{w}^* + k) = \tilde{J}(\hat{w}^* + k) = 0, \tag{189}$$

$$\tilde{W}'(\hat{w}^* - k) = \tilde{J}'(\hat{w}^* + k) = 0. \tag{190}$$

Now, we show that we can transform $J(x) = \frac{\tilde{J}(x + \hat{w}^*) - \frac{1 - \tilde{\rho}\tilde{U}}{2(\tilde{\rho} + \delta)}}{e^{w^*}}$. A similar argument applies to value function of the worker. Making the following transformation $J(x) = \frac{\tilde{J}(x + \hat{w}^*) - \frac{1 - \tilde{\rho}\tilde{U}}{2(\tilde{\rho} + \delta)}}{e^{w^*}}$, and using (170)

$$\begin{aligned}(\tilde{\rho} + \delta)J(x) &= (\tilde{\rho} + \delta)\left(\frac{\tilde{J}(x + \hat{w}^*) - \frac{1 - \tilde{\rho}\tilde{U}}{2(\tilde{\rho} + \delta)}}{e^{w^*}}\right), \\ &= -x + \frac{\sigma^2}{2}\frac{1}{e^{w^*}}\tilde{J}''(x + \hat{w}^*), \\ &= -x + \frac{\sigma^2}{2}J''(x).\end{aligned} \tag{191}$$

Thus,

$$(\tilde{\rho} + \delta)W(x) = x + \frac{\sigma^2}{2}W''(x) \quad \forall x \in (-h, h) \tag{192}$$

$$(\tilde{\rho} + \delta)J(x) = -x + \frac{\sigma^2}{2}J''(x) \quad \forall x \in (-h, h) \tag{193}$$

Defining

$$\Phi = \frac{\frac{1 - \tilde{\rho}\tilde{U}}{2}}{e^{w^*}} = \frac{1 - \tilde{\rho}\tilde{U}}{1 + \tilde{\rho}\tilde{U}} > 0, \tag{194}$$

it is easy to shown

$$W(k) = J(k) = W(-k) = J(-k) = -\frac{\Phi}{\tilde{\rho}+\delta} \quad ; \quad W'(-k) = J'(k) = 0. \tag{195}$$

Thus, $W(x) = J(-x)$. Given that this problem, we verify the guess of symmetric Ss bands and $\frac{1}{2}W'(0) = -\frac{1}{2}J(-0)$. The latter property, implies that w^* satisfies the Nash bargaining solution.

Now, we show that $k = \omega(\varphi)\Phi$ with $\varphi = \sqrt{2\rho+\delta}/\sigma$. Note that $W(x) = J(-x)$. Thus, we can only focus in $W(x)$ using the smooth pasting condition over $-k$. The solution to this system of differential equations is given by

$$W(x) = Ae^{\varphi x} + Be^{-\varphi x} + \frac{x}{\tilde{\rho}+\delta} \tag{196}$$

$$W(k) = W(-k) = -\frac{\Phi}{\tilde{\rho}+\delta} \ \ and \ \ W'(-h) = 0 \tag{197}$$

with $\varphi = \sqrt{2(\tilde{\rho}+\delta)/\sigma^2}$. Writing the value matching conditions

$$Ae^{\varphi k} + Be^{-\varphi k} + \frac{k}{\tilde{\rho}+\delta} = -\frac{\Phi}{\tilde{\rho}+\delta} \tag{198}$$

$$Ae^{-\varphi k} + Be^{\varphi k} - \frac{k}{\tilde{\rho}+\delta} = -\frac{\Phi}{\tilde{\rho}+\delta} \tag{199}$$

Taking the difference and the sum

$$A\left(e^{\varphi k} + e^{-\varphi k}\right) + B\left(e^{-\varphi k} + e^{\varphi k}\right) = -2\frac{\Phi}{\tilde{\rho}+\delta} \tag{200}$$

$$A\left(e^{\varphi k} - e^{-\varphi k}\right) + B\left(e^{-\varphi k} - e^{\varphi kh}\right) = -2\frac{k}{\tilde{\rho}+\delta} \tag{201}$$

Therefore

$$A = \frac{-2\frac{\Phi}{\tilde{\rho}+\delta}\left(e^{-\varphi k}-e^{\varphi k}\right)+2\frac{h}{\tilde{\rho}+\delta}\left(e^{-\varphi k}+e^{\varphi k}\right)}{\left(e^{\varphi k}+e^{-\varphi k}\right)\left(e^{-\varphi k}-e^{\varphi k}\right)-\left(e^{-\varphi k}+e^{\varphi k}\right)\left(e^{\varphi k}-e^{-\varphi k}\right)} \tag{202}$$

$$= \frac{e^{-\varphi k}\left(-\frac{\Phi}{\tilde{\rho}+\delta}+\frac{h}{\tilde{\rho}+\delta}\right)+e^{\varphi k}\left(\frac{h}{\tilde{\rho}+\delta}+\frac{\Phi}{\tilde{\rho}+\delta}\right)}{\left(e^{\varphi k}+e^{-\varphi k}\right)\left(e^{-\varphi k}-e^{\varphi k}\right)} \tag{203}$$

$$= -\frac{1}{\tilde{\rho}+\delta}\frac{e^{-\varphi k}\left(-\Phi+k\right)+e^{\varphi k}\left(k+\Phi\right)}{e^{2\varphi k}-e^{-2\varphi k}} \tag{204}$$

$$B = \frac{-2\frac{h}{\tilde{\rho}+\delta}\left(e^{\varphi k}+e^{-\varphi k}\right)+2\frac{\Phi}{\tilde{\rho}+\delta}\left(e^{\varphi k}-e^{-\varphi k}\right)}{\left(e^{\varphi k}+e^{-\lambda k}\right)\left(e^{-\varphi k}-e^{\varphi k}\right)-\left(e^{-\varphi k}+e^{\varphi k}\right)\left(e^{\varphi k}-e^{-\varphi k}\right)} \tag{205}$$

$$= -\frac{e^{\varphi k}\left(-\frac{\Phi}{\tilde{\rho}+\delta}+\frac{h}{\tilde{\rho}+\delta}\right)+e^{-\varphi k}\left(\frac{h}{\tilde{\rho}+\delta}+\frac{\Phi}{\tilde{\rho}+\delta}\right)}{\left(e^{\varphi k}+e^{-\varphi k}\right)\left(e^{-\lambda k}-e^{\lambda k}\right)} \tag{206}$$

$$= \frac{1}{\tilde{\rho}+\delta}\frac{e^{\varphi k}\left(-\Phi+k\right)+e^{-\varphi k}\left(k+\Phi\right)}{e^{2\varphi k}-e^{-2\varphi k}} \tag{207}$$

Therefore

$$W(x) = -\frac{1}{\tilde{\rho}+\delta}\frac{e^{-\varphi k}\left(-\Phi+k\right)+e^{\varphi k}\left(k+\Phi\right)}{e^{2\varphi k}-e^{-2\varphi k}}e^{\lambda x}+\frac{1}{\tilde{\rho}+\delta}\frac{e^{\varphi k}\left(\Phi+k\right)+e^{-\varphi k}\left(k-\Phi\right)}{e^{2\varphi k}-e^{-2\varphi k}}e^{-\lambda x}+\frac{x}{\tilde{\rho}+\delta} \tag{208}$$

Taking the derivative and evaluating in $x = -k$

$$W'(-h) = -\frac{e^{-\varphi k}\left(-\Phi+k\right)+e^{\varphi k}\left(k+\Phi\right)}{e^{2\varphi k}-e^{-2\varphi k}}\varphi e^{-\varphi k}-\frac{e^{\varphi k}\left(-\Phi+k\right)+e^{-\varphi k}\left(k+\Phi\right)}{e^{2\varphi k}-e^{-2\varphi k}}\varphi e^{\varphi k}+1=0 \tag{209}$$

or equivalently

$$-\Phi(e^{-2\varphi k}+e^{2\lambda k}-2) = \frac{1}{\varphi}(e^{2\varphi k}-e^{-2\varphi k})-\frac{1}{2\varphi}2\varphi k\left(e^{2\varphi k}+e^{-2\varphi k}+2\right) \tag{210}$$

It would be useful to express equation (210) using $\sinh(x) = \frac{e^x-e^{-x}}{2}$ and $\cosh(x) = \frac{e^x+e^{-x}}{2}$. Using the hyperbolic functions

$$-\Phi 2\left(\cosh(2\varphi k)-1\right) = \frac{2\sinh(2\varphi k)}{\varphi}-2k\left(\cosh(2\varphi k)+1\right) \tag{211}$$

Multiplying both sides by φ

$$-\Phi 2\varphi\left(\cosh(2\varphi k)-1\right) = 2\sinh(2\varphi k)-\varphi 2k\left(\cosh(2\varphi k)+1\right) \tag{212}$$

While in principle $k(\varphi, \Phi)$, we can change variables with $x \equiv 2\varphi k$ and x as the implicit solution of

$$-\Phi 2\varphi\left(\cosh(x)-1\right)+x\left(\cosh(x)+1\right)=2\sinh(x). \tag{213}$$

Thus, $k=\frac{x(\Phi 2\varphi)}{2\varphi}$. Let $b=\Phi 2\lambda>0$, then we can express the function $x(\cdot)$ as the solution of

$$b=-\frac{2\sinh(x(b))-x(b)\left(\cosh(x(b))+1\right)}{\left(\cosh(x(b))-1\right)} \tag{214}$$

Notice that if we define

$$f(x)=-\frac{2\sinh(x)-x\left(\cosh(x)+1\right)}{\left(\cosh(x)-1\right)} \tag{215}$$

The following properties hold over $f(x)$

1. $\lim_{x\to 0} f(x)=0$ and $\lim_{x\to\infty} f(x)=\infty$.
2. $f(x)$ is increasing and convex, with $\lim_{x\to 0} f'(x)=1/3$ and $\lim_{x\to\infty} f(x)=1$.
3. $\frac{\mathrm{dlog}(f(x))}{\mathrm{dlog}(x)}>1$.

Given these properties, we can write $k(\varphi,\Phi)=\frac{f^{-1}(2\varphi\Phi)}{2\varphi}$ and show the following properties over $k(\varphi,\Phi)$

1. $k(\varphi,\Phi)$ is increasing in Φ: Since $f^{-1}(\cdot)$ is increasing ,we have the result.
2. $k(\varphi,\Phi)$ is decreasing in φ: Taking derivative of $k(\varphi,\Phi)=\frac{f^{-1}(2\varphi\Phi)}{2\varphi}$ with φ and operating

$$\begin{aligned}\frac{\partial k(\varphi,\Phi)}{\partial\varphi} &= \left.\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x}\right|_{x=2\varphi\Phi}\frac{2\Phi}{2\varphi}-\frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\\ &= \frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\left[\left.\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x}\right|_{x=2\varphi\Phi}\frac{2\varphi\Phi}{f^{-1}(2\varphi\Phi)}-1\right]\\ &= \frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\left[\left.\frac{\mathrm{dlog}(x)}{\mathrm{dlog}(f(x))}\right|_{x=2\varphi\Phi}\frac{2\varphi\Phi}{f^{-1}(2\varphi\Phi)}-1\right]\\ &< 0.\end{aligned} \tag{216}$$

3. $\lim_{\varphi\downarrow 0} k(\varphi,\Phi)=3\Phi$ and $\lim_{\varphi\to\infty} k(\varphi,\Phi)=\Phi$: Applying L'hospital and using the derivative of the inverse property

$$\lim_{\varphi\to\infty} k(\varphi,\Phi)=\lim_{\varphi\to\infty}\frac{f^{-1}(2\varphi\Phi)}{2\varphi}=\lim_{\varphi\to\infty}\frac{1}{f'(2\varphi\Phi)}\Phi=\Phi \tag{217}$$

$$\lim_{\varphi\downarrow 0} k(\varphi,\Phi)=\lim_{\varphi\downarrow 0}\frac{f^{-1}(2\varphi\Phi)}{2\varphi}=\lim_{\varphi\downarrow 0}\frac{1}{f'(2\varphi\Phi)}\Phi=3\Phi \tag{218}$$

4. $k(\varphi, \Phi) = \omega(2\varphi\Phi)\Phi$: Define $\omega(z) = \frac{f^{-1}(z)}{z}$, then it is easy to see that $h(\varphi, \Phi) = \omega(2\varphi\Phi)\Phi$. Moreover, from property 2 and 3, $\omega(z)$ is decreasing with $\lim_{z\downarrow 0} \omega(z) = 3$ and $\lim_{z\to\infty} \omega(z) = 1$.

Now, we compute condition for the value of unemployment. By definition of $W(x)$ and $J(x)$, we have that

$$\tilde{J}(w^*) = J(0)e^{\hat{w}^*} + \frac{1-\tilde{\rho}\tilde{U}}{2(\rho+\delta)}, \tag{219}$$

$$\tilde{W}(w^*) = W(0)e^{\hat{w}^*} + \frac{1-\tilde{\rho}\tilde{U}}{2(\rho+\delta)}. \tag{220}$$

Using the previous equations and since $W(x) = J(-x)$, and $\alpha = 1/2$, we have that

$$\begin{aligned}
\tilde{\rho}\tilde{U} &= \tilde{b} + \frac{1}{\kappa}\tilde{J}(\hat{w}^*)\tilde{W}(\hat{w}^*) \qquad (221)\\
&= \tilde{b} + \frac{1}{\kappa}\left(W(0)\frac{1+\tilde{\rho}\tilde{U}}{2} + \frac{1-\tilde{\rho}\tilde{U}}{2(\rho+\delta)}\right)^2\\
&= \tilde{b} + \frac{1}{\kappa}\times\\
&\cdots\left[\left[-\frac{1}{\tilde{\rho}+\delta}\frac{e^{-\varphi k}(-\Phi+k) + e^{\varphi k}(k+\Phi)}{e^{2\varphi k} - e^{-2\varphi k}} + \frac{1}{\tilde{\rho}+\delta}\frac{e^{\varphi k}(-\Phi+k) + e^{-\varphi k}(k+\Phi)}{e^{2\varphi k} - e^{-2\varphi k}}\right]\frac{1+\tilde{\rho}\tilde{U}}{2} + \frac{1-\tilde{\rho}\tilde{U}}{2(\rho+\delta)}\right]^2\\
&= \tilde{b} + \frac{(1-\tilde{\rho}\tilde{U})^2}{\kappa 4(\tilde{\rho}+\delta)^2}\left[\frac{-e^{-\varphi k}(-\Phi+k) - e^{\varphi k}(k+\Phi)}{\Phi(e^{2\varphi k} - e^{-2\varphi k})} + \frac{e^{\varphi k}(-\Phi+k) + e^{-\varphi k}(k+\Phi)}{\Phi(e^{2\varphi k} - e^{-2\varphi k})}) + 1\right]^2\\
&= \tilde{b} + \frac{(1-\tilde{\rho}\tilde{U})^2}{\kappa 4(\tilde{\rho}+\delta)^2}\left[\frac{(e^{\varphi k} - e^{-\varphi k})(-\Phi+k) + (e^{-\varphi k} - e^{\varphi k})(k+\Phi)}{\Phi(e^{2\varphi k} - e^{-2\varphi k})} + 1\right]^2\\
&= \tilde{b} + \frac{(1-\tilde{\rho}\tilde{U})^2}{\kappa 4(\tilde{\rho}+\delta)^2}\left[\frac{(e^{\varphi k} - e^{-\varphi k})(-\Phi+k) - (e^{\varphi k} - e^{-\varphi k})(\Phi+h)}{\Phi(e^{2\varphi k} - e^{-2\varphi k})} + 1\right]^2\\
&= \tilde{b} + \frac{(1-\tilde{\rho}\tilde{U})^2}{\kappa 4(\tilde{\rho}+\delta)^2}\left[-\frac{e^{\varphi k} - e^{-\varphi k}}{(e^{2\varphi k} - e^{-2\varphi k})} + 1\right]^2\\
&= \tilde{b} + \frac{(1-\tilde{\rho}\tilde{U})^2}{\kappa 4(\tilde{\rho}+\delta)^2}\left(-\frac{\sinh(\varphi k)}{\sinh(\varphi 2k)} + 1\right)^2
\end{aligned}$$

Evaluating in the equilibrium policy and using the definition of $\Theta(\alpha)$

$$\tilde{\rho}\tilde{U} = \tilde{b} + (1-\tilde{\rho}\tilde{U})^2\Theta(1/2)\left(-\frac{\sinh(\varphi k(\varphi,\Phi))}{\sinh(\varphi 2k(\varphi,\Phi))} + 1\right)^2. \tag{222}$$

It is easy to check that $\frac{\sinh(x)}{\sinh(2x)}$ is decreasing in x, with $lim_{x\to 0}\frac{\sinh(x)}{\sinh(2x)} = 1/2$ and $lim_{x\to\infty}\frac{\sinh(x)}{\sinh(2x)} = 0$.

□

Proposition 19. *Assume $\sigma \downarrow 0$ and $\nu > 0$. Then $\hat{w}^- = \log\left(\tilde{\rho}\tilde{U}\right)$ and*

$$w^* = \hat{w}^- + \tilde{T}\left(\frac{(1-\alpha)+\alpha\tilde{\rho}\tilde{U}}{\tilde{\rho}\tilde{U}}, \frac{\tilde{\rho}+\delta}{\nu}, \frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})}{\tilde{\rho}\tilde{U}}\right). \tag{223}$$

$\tilde{T}(\cdot)$ is defined as

$$a = e^{\tilde{T}(a,b,c)}\frac{1-e^{-(1+b)\tilde{T}(a,b,c)}}{1-e^{-b\tilde{T}(a,b,c)}}\frac{b}{b+1} - cb\frac{e^{-b\tilde{T}(a,b,c)}}{1-e^{-b\tilde{T}(a,b,c)}}\left[1-\frac{b+1}{b}\frac{1-a^{-b\tilde{T}(a,b,c)}}{e^{\tilde{T}(a,b,c)}-e^{-b\tilde{T}(a,b,c)}}\right] \tag{224}$$

where $\tilde{T}(\cdot)$ is increasing in the first argument and decreasing in the second argument. Moreover,

$$\exp\left(\lim_{(\tilde{\rho}+\delta)/\nu\to\infty}\hat{w}^*(\cdot)\right) = \alpha + (1-\alpha)\tilde{\rho}\tilde{U} \tag{225}$$

If $(\tilde{\rho}+\delta)/\nu \to 0$, then $\lim_{(\tilde{\rho}+\delta)/\nu\to 0}\hat{w}^*(\cdot) = \hat{w}^{limit}$

$$\mathbb{E}[e^w] = \alpha + (1-\alpha)\tilde{\rho}\tilde{U} + \frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})}{\hat{w}^{limit}-\hat{w}^-}\frac{\mathbb{E}[e^w]-\tilde{\rho}\tilde{U}}{\mathbb{E}[e^w]} \tag{226}$$

$$\mathbb{E}[e^w] = \frac{\exp(\hat{w}^{limit})-\exp(\hat{w}^-)}{\hat{w}^{limit}-\hat{w}^-} \tag{227}$$

The value of unemployment satisfies

$$\tilde{U} = \tilde{b} + \Theta(\alpha)\left(1-e^{-\frac{\tilde{\rho}+\delta}{\nu}\tilde{T}(\cdot)}\right)^{1/\alpha}(1-\tilde{\rho}\tilde{U})^{1/\alpha} \tag{228}$$

Proof. Now, we take the $\sigma \downarrow 0$. The equilibrium conditions in this case are

$$(\tilde{\rho}+\delta)\tilde{W}(\hat{w}) = e^{\hat{w}} - \tilde{\rho}\tilde{U} - \nu\tilde{W}'(\hat{w}) \quad \forall \hat{w} \in \tilde{\mathcal{R}} \tag{229}$$

$$(\tilde{\rho}+\delta)\tilde{J}(\hat{w}) = 1 - e^{\hat{w}} - \nu\tilde{J}'(\hat{w}) \quad \forall \hat{w} \in \tilde{\mathcal{R}} \tag{230}$$

$$\tilde{\rho}\tilde{U} = \tilde{b} + \kappa^{1-\alpha^{-1}}J(\hat{w}^*)^{\frac{1-\alpha}{\alpha}}\tilde{W}(\hat{w}^*) \tag{231}$$

$$(1-\alpha)\frac{\mathrm{dlog}\,\tilde{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -\alpha\frac{\mathrm{dlog}\,\tilde{W}(\hat{w}^*)}{\mathrm{d}\hat{w}} \tag{232}$$

with VM and SP

$$\tilde{W}(\hat{w}^-) = \tilde{J}(\hat{w}^-) = \tilde{W}(\hat{w}^+) = \tilde{J}(\hat{w}^+) = 0 \quad ; \quad \tilde{W}'(\hat{w}^-) = \tilde{J}'(\hat{w}^+) = 0. \tag{233}$$

Without idiosyncratic shocks and $\nu > 0$ the upper Ss band is not active. Thus, we discard the opti-

mality condition for this $\hat{w}^+$. The stopping time is a deterministic function in this case, hence, it is easier to work in the sequential formulation.

$$\tilde{W}(\hat{w}) = \max_{T} \int_0^T e^{-(\tilde{\rho}+\delta)s} \left(e^{\hat{w}-\nu s} - \tilde{\rho}\tilde{U}\right) \mathrm{d}s \tag{234}$$

$$\tilde{J}(\hat{w}) = \int_0^{T(w)} e^{-(\tilde{\rho}+\delta)s} \left(1 - e^{\hat{w}-\nu s}\right) \mathrm{d}s. \tag{235}$$

In equation (235), $T(\hat{w})$ is the optimal policy of the worker. Taking the first order conditions with $T(\hat{w})$

$$e^{\hat{w}-\nu T(\hat{w})} = \tilde{\rho}\tilde{U}. \tag{236}$$

Solving the previous equation

$$T(\hat{w}) = \frac{\hat{w} - \log\left(\tilde{\rho}\tilde{U}\right)}{\nu}. \tag{237}$$

Thus, if $\hat{w} = \hat{w}^*$ we have that $\hat{w}^* - \nu T(\hat{w}^*) = \hat{w}^-$ satisfies

$$\hat{w}^- = \log(\tilde{\rho}\tilde{U}). \tag{238}$$

Taking the derivatives of $\tilde{W}(\hat{w})$ and $\tilde{J}(\hat{w})$, and using the envelope condition for $\tilde{W}'(\hat{w})$, we have that

$$\tilde{W}'(\hat{w}) = \int_0^{T(w)} e^{-(\tilde{\rho}+\delta)s} \left(e^{\hat{w}-\nu s}\right) \mathrm{d}s, \tag{239}$$

$$\tilde{J}'(\hat{w}) = -\int_0^{T(w)} e^{-(\tilde{\rho}+\delta)s} \left(e^{\hat{w}-\nu s}\right) \mathrm{d}s + e^{-(\tilde{\rho}+\delta)T(\hat{w})} \left(1 - e^{\hat{w}-\nu T(\hat{w})}\right) \underbrace{T'(\hat{w})}_{=1/\nu}. \tag{240}$$

From equations (239) and (240), we get the Nash bargaining solution

$$-\alpha \frac{\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s} \left(e^{\hat{w}^*-\nu s}\right) \mathrm{d}s}{\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s} \left(e^{\hat{w}^*-\nu s} - \tilde{\rho}\tilde{U}\right) \mathrm{d}s} = (1-\alpha) \frac{\left[-\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s} \left(e^{\hat{w}^*-\nu s}\right) \mathrm{d}s + e^{-(\tilde{\rho}+\delta)T^*} \frac{(1-\tilde{\rho}\tilde{U})}{\nu}\right]}{\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s} \left(1 - e^{\hat{w}^*-\nu s}\right) \mathrm{d}s} \tag{241}$$

Define

$$\Omega^*(a) := \frac{1 - e^{-aT^*}}{a} \tag{242}$$

$$\mathcal{Z} := \frac{e^{-(\tilde{\rho}+\delta)T^*}(1 - \tilde{\rho}\tilde{U})}{\nu \int_0^{T^*} e^{-(\tilde{\rho}+\delta)s} \left(e^{\hat{w}^*-\nu s}\right) \mathrm{d}s} \tag{243}$$

Operating

$$\alpha \int_0^{T^*} e^{-(\tilde{\rho}+\delta)s}\left(1-e^{\hat{w}^*-\nu s}\right)\mathrm{d}s = (1-\alpha)\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s}\left(e^{\hat{w}^*-\nu s}-\tilde{\rho}\tilde{U}\right)\mathrm{d}s\left[1-\mathcal{Z}\right] \iff$$

$$\alpha\left[\Omega^*(\tilde{\rho}+\delta)-e^{\hat{w}^*}\Omega^*(\tilde{\rho}+\delta+\nu)\right] = (1-\alpha)\left[e^{\hat{w}^*}\Omega^*(\tilde{\rho}+\delta+\nu)-\tilde{\rho}\tilde{U}\Omega^*(\tilde{\rho}+\delta)\right]\times$$

$$\ldots\left[1-\frac{e^{-(\tilde{\rho}+\delta)T^*}(1-\tilde{\rho}\tilde{U})}{\nu e^{\hat{w}^*}\Omega(\tilde{\rho}+\delta+\nu)}\right] \iff$$

$$\left(\alpha+(1-\alpha)\tilde{\rho}\tilde{U}\right)\Omega^*(\tilde{\rho}+\delta) = e^{\hat{w}^*}\Omega^*(\tilde{\rho}+\delta+\nu)+\ldots$$

$$\ldots(1-\alpha)\left[e^{\hat{w}^*}\Omega^*(\tilde{\rho}+\delta+\nu)-\tilde{\rho}\tilde{U}\Omega^*(\tilde{\rho}+\delta)\right]\frac{e^{-(\tilde{\rho}+\delta)T^*}(1-\tilde{\rho}\tilde{U})}{\nu e^{\hat{w}^*}\Omega(\tilde{\rho}+\delta+\nu)} \iff$$

$$\left(\alpha+(1-\alpha)\tilde{\rho}\tilde{U}\right)\Omega^*(\tilde{\rho}+\delta) = e^{\hat{w}^*}\Omega^*(\tilde{\rho}+\delta+\nu)-\frac{(1-\alpha)e^{-(\tilde{\rho}+\delta)T^*}(1-\tilde{\rho}\tilde{U})}{\nu}\left[1-\tilde{\rho}\tilde{U}\frac{\Omega^*(\tilde{\rho}+\delta)}{e^{\hat{w}^*}\Omega^*(\tilde{\rho}+\delta+\nu)}\right]$$

The policy $(T^*, \hat{w}^*)$ solves

$$e^{\hat{w}^*-\nu T^*} = \tilde{\rho}\tilde{U} \tag{244}$$

$$\left(\alpha+(1-\alpha)\tilde{\rho}\tilde{U}\right)\Omega^*(\tilde{\rho}+\delta) = e^{\hat{w}^*}\Omega^*(\tilde{\rho}+\delta+\nu)-\frac{(1-\alpha)e^{-(\tilde{\rho}+\delta)T^*}(1-\tilde{\rho}\tilde{U})}{\nu}\left[1-\tilde{\rho}\tilde{U}\frac{\Omega^*(\tilde{\rho}+\delta)}{e^{\hat{w}^*}\Omega^*(\tilde{\rho}+\delta+\nu)}\right] \tag{245}$$

Define $\tilde{T} = \nu T^*$ and $\tilde{\Omega}(a) := \frac{1-e^{-a\tilde{T}}}{a}$. Then

$$e^{\hat{w}^*-\tilde{T}} = \tilde{\rho}\tilde{U} \tag{246}$$

$$\left(\alpha+(1-\alpha)\tilde{\rho}\tilde{U}\right)\nu^{-1}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right) = e^{\hat{w}^*}\nu^{-1}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}+1\right)-\frac{(1-\alpha)e^{-\frac{\tilde{\rho}+\delta}{\nu}\tilde{T}}(1-\tilde{\rho}\tilde{U})}{\nu}\left[1-\tilde{\rho}\tilde{U}\frac{\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right)}{e^{\hat{w}^*}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}+1\right)}\right] \tag{247}$$

Therefore, the optimal stopping is given by

$$\frac{\alpha+(1-\alpha)\tilde{\rho}\tilde{U}}{\tilde{\rho}\tilde{U}}\frac{\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right)}{\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}+1\right)} = e^{\tilde{T}}-\frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})\left[1-\frac{\tilde{\rho}+\delta}{\nu}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right)\right]}{\tilde{\rho}\tilde{U}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}+1\right)}\left[1-\frac{\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right)}{e^{\tilde{T}}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}+1\right)}\right] \tag{248}$$

or

$$\frac{\alpha+(1-\alpha)\tilde{\rho}\tilde{U}}{\tilde{\rho}\tilde{U}} = e^{\tilde{T}}\frac{\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}+1\right)}{\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right)}-\frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})\left[1-\frac{\tilde{\rho}+\delta}{\nu}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right)\right]}{\tilde{\rho}\tilde{U}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right)}\left[1-\frac{\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}\right)}{e^{\tilde{T}}\tilde{\Omega}\left(\frac{\tilde{\rho}+\delta}{\nu}+1\right)}\right] \tag{249}$$

Now, we show the property over the policy. Let us define the following function:

$$f(a,b,c) := e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - c\left[\frac{1-b\frac{1-e^{-ba}}{b}}{\frac{1-e^{-ba}}{b}}\right]\left[1-\frac{1-e^{-ba}}{1-e^{-(1+b)a}}\frac{b+1}{be^a}\right] \tag{250}$$

$$= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - c\left[\frac{b}{1-e^{-ba}} - b\right]\left[\frac{b(e^a-e^{-ba}) - (1-e^{-ba})(b+1)}{b(e^a-e^{-ba})}\right] \tag{251}$$

$$= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[\frac{be^a-(b+1)+e^{-ba} \pm be^{-ba}}{b(e^a-e^{-ba})}\right] \tag{252}$$

$$= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[\frac{b(e^a-e^{-ba}) - (b+1) + (1+b)e^{-ba}}{b(e^a-e^{-ba})}\right] \tag{253}$$

$$= e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-a^{-ba}}{e^a-e^{-ba}}\right] \tag{254}$$

The following properties are easy to show:

1. $f(a,b,c)$ is increasing in a :

2. If $a, c > 0$, $b \to \infty$, then $f(a,b,c) \to e^a$: To see this property, taking the limit

$$= \lim_{a>0, b\to\infty, c\propto b}\left[e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^a-e^{-ba}}\right]\right] \tag{255}$$

$$= e^a \underbrace{\lim_{a>0,b\to\infty}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}}_{=1}\underbrace{\lim_{a>0,b\to\infty}\frac{b}{b+1}}_{=1} - \underbrace{\lim_{a>0,b\to\infty} cb\frac{e^{-ba}}{1-e^{-ba}}}_{=0}\left[1-\underbrace{\lim_{b\to\infty}\frac{b+1}{b}}_{=1}\underbrace{\lim_{a>0,b\to\infty}\frac{1-e^{-ba}}{e^a-e^{-ba}}}_{=e^{-a}}\right] \tag{256}$$

$$= e^a \tag{257}$$

3. If $a, c > 0$ and $b \to 0$ then $f(a,b,c) \to \frac{e^a-1}{a}$: To see this property, taking the limit

$$= \lim_{a>0,b\to 0}\left[e^a \frac{1-e^{-(1+b)a}}{1-e^{-ba}} \frac{b}{b+1} - cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^a-e^{-ba}}\right]\right] \tag{258}$$

$$= e^a(1-e^{-a})\underbrace{\lim_{a>0,b\to 0}\frac{b}{1-e^{-ba}}}_{=1/a} - c\underbrace{\lim_{a>0,b\to 0}\frac{b}{1-e^{-ba}}}_{=1/a}\left[1-\frac{1}{e^a-1}\underbrace{\lim_{b\to\infty}\frac{1-e^{-ba}}{b}}_{=a}\right] \tag{259}$$

$$= \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a} \tag{260}$$

4. $e^a \geq f(a,b,c) \geq \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a}$ where the upper bound is reach when $b \to \infty$ and the lower bound

when $b \downarrow 0$.

With this properties, we can proof the equilibrium policies:

1. $\tilde{T}(\frac{\alpha+(1-\alpha)\tilde{\rho}\tilde{U}}{\tilde{\rho}\tilde{U}}, \frac{\tilde{\rho}+\delta}{\nu}, \frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})}{\nu\tilde{\rho}\tilde{U}})$ is increasing in the first argument.

2. If $\frac{\tilde{\rho}+\delta}{\nu}, \frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})}{\nu\tilde{\rho}\tilde{U}} \to \infty$
$$\alpha + (1-\alpha)\tilde{\rho}\tilde{U} = e^{\hat{w}^*}. \tag{261}$$

3. Showing the limit of $\tilde{\rho} + \delta \to 0$ is more involve, so we prove it below. Let us depart from equation (241)
$$-\alpha \frac{\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s}\left(e^{\hat{w}^*-\nu s}\right) \mathrm{d}s}{\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s}\left(e^{\hat{w}^*-\nu s} - \tilde{\rho}\tilde{U}\right) \mathrm{d}s} = (1-\alpha)\frac{\left[-\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s}\left(e^{\hat{w}^*-\nu s}\right) \mathrm{d}s + e^{-(\tilde{\rho}+\delta)T^*}\frac{(1-\tilde{\rho}\tilde{U})}{\nu}\right]}{\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s}\left(1 - e^{\hat{w}^*-\nu s}\right) \mathrm{d}s} \tag{262}$$

Taking the limit $\tilde{\rho} + \delta \to 0$
$$\alpha \int_0^{T^*} (1 - e^{w_t}) \, \mathrm{d}t = (1-\alpha)\int_0^{T^*} \left(e^{w_t} - \tilde{\rho}\tilde{U}\right) \mathrm{d}t - \frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})}{\nu}\frac{\int_0^{T^*}\left(e^{w_t} - \tilde{\rho}\tilde{U}\right) \mathrm{d}t}{\int_0^{T^*} e^{w_t} \, \mathrm{d}t}. \tag{263}$$

Operating and using the occupancy measure
$$\alpha + (1-\alpha)\tilde{\rho}\tilde{U} + \frac{(1-\alpha)(1-\tilde{\rho}\tilde{U})}{\nu T^*}\frac{\frac{\int_0^{T^*} e^{w_t} \, \mathrm{d}t}{T^*} - \tilde{\rho}\tilde{U}}{\frac{\int_0^{T^*} e^{w_t} \, \mathrm{d}t}{T^*}} = \frac{\int_0^{T^*} e^{w_t} \, \mathrm{d}t}{T^*} \tag{264}$$

If and only if
$$\alpha + (1-\alpha)\tilde{\rho}\tilde{U} + \frac{1-\alpha}{\nu T^*}\frac{\mathbb{E}[e^w] - \tilde{\rho}\tilde{U}}{\mathbb{E}[e^w]}(1 - \tilde{\rho}\tilde{U}) = \mathbb{E}[e^w]. \tag{265}$$

Since $\hat{w}^* = \hat{w}^- + \nu T^*$, we have that
$$\alpha + (1-\alpha)\tilde{\rho}\tilde{U} + \frac{1-\alpha}{\hat{w}^* - \hat{w}^-}\frac{\mathbb{E}[e^w] - \tilde{\rho}\tilde{U}}{\mathbb{E}[e^w]}(1 - \tilde{\rho}\tilde{U}) = \mathbb{E}[e^w] \tag{266}$$

Here, $\mathbb{E}[e^w] = \frac{\int_{\hat{w}^-}^{\hat{w}^*} e^w g_h(\hat{w}) \, \mathrm{d}\hat{w}}{\int_{\hat{w}^-}^{\hat{w}^*} g_h(\hat{w}) \, \mathrm{d}\hat{w}}$ is taken over the steady-state distribution of wage-productivity under $\frac{\delta}{\nu} \to 0$. It is easy to show, the distribution of $\hat{w}$ is an uniform distribution, thus
$$\mathbb{E}[e^w] = \frac{e^{w^*} - e^{\hat{w}^-}}{\hat{w}^* - \hat{w}^-}. \tag{267}$$

With these properties, we compute the optimal unemployment value

$$
\begin{aligned}
\tilde{\rho}\tilde{U} &= \tilde{b} + \kappa^{1-\alpha^{-1}} \left[J(\hat{w}^*)^{1-\alpha} \tilde{W}(\hat{w}^*)^{\alpha} \right]^{1/\alpha} \\
&= \tilde{b} + \kappa^{1-\alpha^{-1}} \left[\left[\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s} \left(1 - e^{\hat{w}^* - \nu s} \right) \mathrm{d}s \right]^{1-\alpha} \left[\int_0^{T^*} e^{-(\tilde{\rho}+\delta)s} \left(e^{\hat{w}^* - \nu s} - \tilde{\rho}\tilde{U} \right) \mathrm{d}s \right]^{\alpha} \right]^{1/\alpha} \\
&= \tilde{b} + \kappa^{1-\alpha^{-1}} \left[\left[\frac{1 - e^{-(\tilde{\rho}+\delta)T^*}}{\tilde{\rho}+\delta} - e^{\hat{w}^*} \frac{1 - e^{-(\nu+\tilde{\rho}+\delta)T^*}}{\nu+\tilde{\rho}+\delta} \right]^{1-\alpha} \left[e^{\hat{w}^*} \frac{1 - e^{-(\nu+\tilde{\rho}+\delta)T^*}}{\nu+\tilde{\rho}+\delta} - \tilde{\rho}\tilde{U} \frac{1 - e^{-(\tilde{\rho}+\delta)T^*}}{\tilde{\rho}+\delta} \right]^{\alpha} \right]^{1/\alpha} \\
&= \tilde{b} + \kappa^{1-\alpha^{-1}} \left(\frac{1 - e^{-\frac{\tilde{\rho}+\delta}{\nu}\tilde{T}}}{\tilde{\rho}+\delta} \right)^{1/\alpha} \cdots \\
&\times \left[\left[1 - e^{\hat{w}^*} \frac{1 - e^{-(1+\frac{\tilde{\rho}+\delta}{\nu})\tilde{T}}}{1 - e^{-\frac{\tilde{\rho}+\delta}{\nu}\tilde{T}}} \frac{\tilde{\rho}+\delta}{\nu+\tilde{\rho}+\delta} \right]^{1-\alpha} \left[e^{\hat{w}^*} \frac{1 - e^{-(1+\frac{\tilde{\rho}+\delta}{\nu})\tilde{T}}}{1 - e^{-\frac{\tilde{\rho}+\delta}{\nu}\tilde{T}}} \frac{\tilde{\rho}+\delta}{\nu+\tilde{\rho}+\delta} - \tilde{\rho}\tilde{U} \right]^{\alpha} \right]^{1/\alpha} \\
&= \tilde{b} + \kappa^{1-\alpha^{-1}} \left(\frac{1 - e^{-\frac{\tilde{\rho}+\delta}{\nu}\tilde{T}}}{\tilde{\rho}+\delta} \right)^{1/\alpha} \left[\left[1 - \left(\alpha + (1-\alpha)\tilde{\rho}\tilde{U} \right) \right]^{1-\alpha} \left[\left(\alpha + (1-\alpha)\tilde{\rho}\tilde{U} \right) - \tilde{\rho}\tilde{U} \right]^{\alpha} \right]^{1/\alpha} \\
&= \tilde{b} + \kappa^{1-\alpha^{-1}} \left(\frac{1 - e^{-\frac{\tilde{\rho}+\delta}{\nu}\tilde{T}}}{\tilde{\rho}+\delta} \right)^{1/\alpha} (1-\alpha)^{\frac{1-\alpha}{\alpha}} \alpha (1 - \tilde{\rho} \\
&= \tilde{b} + \Theta(\alpha) \left(1 - e^{-\frac{\tilde{\rho}+\delta}{\nu}\tilde{T}} \right)^{1/\alpha} (1 - \tilde{\rho}\tilde{U})^{1/\alpha}
\end{aligned} \tag{268}
$$

□

B.5 Proof of Proposition 6

Proposition 20. *Assume $\delta > 0$. $g_h(\Delta z)$ and $g_u(\Delta z)$ are given by*

$$
g_h(\Delta z) = (1-U)\mathcal{G}_h \begin{cases} \frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)} - e^{\beta_2(\delta)(\Delta z + \Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} & \text{if } \Delta z \in (-\Delta^-, 0] \\ \frac{e^{\beta_1(\delta)(\Delta z - \Delta^+)} - e^{\beta_2(\delta)(\Delta z - \Delta^+)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} & \text{if } \Delta z \in [0, \Delta^+) \end{cases} \tag{269}
$$

$$
g_u(\Delta z) = U\mathcal{G}_u \begin{cases} e^{\beta_2(p(\theta^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(p(\theta^*)\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases} \tag{270}
$$

where

$$\beta_1(x) = \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}, \beta_2(x) = \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}, \tag{271}$$

$$1 - U = \frac{p(\tilde{\theta}^*)}{p(\tilde{\theta}^*) + \delta + \frac{\sigma^2}{2}\mathcal{G}_h \left[\frac{\beta_1(\delta) - \beta_2(\delta)}{e^{\beta 1(\delta)\Delta_{*-}} - e^{\beta_2(\delta)\Delta_{*-}}} - \frac{\beta_1(\delta) - \beta_2(\delta)}{e^{-\beta 1(\delta)\Delta_{*+}} - e^{-\beta_2(\delta)\Delta_{*+}}}\right]}, \tag{272}$$

$$\mathcal{G}_h = \left[\frac{\frac{e^{\beta_1(\delta)\Delta_{*-}} - 1}{\beta_1(\delta)} - \frac{e^{\beta_2(\delta)\Delta_{*-}} - 1}{\beta_2(\delta)}}{e^{\beta_1(\delta)\Delta_{*-}} - e^{\beta_2(\delta)\Delta_{*-}}} + \frac{\frac{1 - e^{-\beta_1\Delta_{*+}}}{\beta 1(\delta)} - \frac{1 - e^{-\beta_2\Delta_{*+}}}{\beta_2(\delta)}}{e^{-\beta_1(\delta)\Delta_{*+}} - e^{-\beta_2(\delta)\Delta_{*+}}}\right]^{-1}, \tag{273}$$

$$\mathcal{G}_u = \left[-\beta_1(p(\theta^*))^{-1} + \beta_2(p(\theta^*))^{-1}\right]^{-1}. \tag{274}$$

modify below U and $\mathcal{G}$

Proof. Let us write the KFE and border conditions:

$$\delta g_h(\Delta z) = \gamma g_h'(\Delta z) + \frac{\sigma^2}{2} g_h''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta_+)/\{0\} \tag{275}$$

$$g_h(-\Delta^-) = g_h(\Delta^+) = 0, \tag{276}$$

$$p(\theta^*) g_u(\Delta z) = \gamma g_u'(\Delta z) + \frac{\sigma^2}{2} g_u''(\Delta z) \quad \forall \Delta z \in (-\infty, \infty)/\{0\}, \tag{277}$$

$$\lim_{\Delta z \to -\infty} g_u(\Delta z) = \lim_{\Delta z \to \infty} g_u(\Delta z) = 0, \tag{278}$$

$$1 = \int_{-\infty}^{\infty} g_u(\Delta z)\, \mathrm{d}\Delta z + \int_{-\Delta_{*-}}^{\Delta_{*+}} g_h(\Delta z)\, \mathrm{d}\Delta z, \tag{279}$$

$$p(\theta^*) U = \delta(1 - U) + \frac{\sigma^2}{2}\left[g_h'(-\Delta_{*-}) - g_h'(\Delta_{+*})\right], \tag{280}$$

$$g_h(\Delta z), g_u(\Delta z) \in \text{set of continuous functions} \tag{281}$$

We guess and verify the solution the following solution. Define $e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-} =: \mathcal{K}\mathcal{K}_-$, using the guess in $\Delta z < 0$ in (275), for any Δz we have

$$0 = -\delta\mathcal{H}\frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)}}{\mathcal{K}\mathcal{K}_-} + \gamma\beta_1(\delta)\mathcal{H}\frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)}}{\mathcal{K}\mathcal{K}_-} + \mathcal{H}\frac{\sigma^2}{2}\beta_1(\delta)^2\frac{e^{\beta_1(\delta)(\Delta z + \Delta^-)}}{\mathcal{K}\mathcal{K}_-} \iff \tag{282}$$

$$0 = -\delta + \gamma\beta_1(\delta) + \frac{\sigma^2}{2}\beta_1(\delta)^2 \tag{283}$$

Given the definition of $\beta_1(\delta)$, the guess satisfies (275). Similar argument applies when $\Delta z > 0$. It is easy to check that

$$g_h(-\Delta_{*-}) = g_h(\Delta_{+*}) = 0. \tag{284}$$

and continuity at $\Delta z = 0$. Repeating the same steps for $g_u(\Delta z)$, we verify conditions (277) and (278). Now, we verify the border condition (279).

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g_u(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta_{*-}}^{\Delta_{+*}} g_h(\Delta z)\,\mathrm{d}\Delta z, \\
&= U\mathcal{G}_u\left[\int_{-\infty}^{0} e^{\beta_2(p(\theta^*))\Delta z}\,\mathrm{d}\Delta z + \int_{0}^{\infty} e^{\beta_1(p(\theta^*))\Delta z}\,\mathrm{d}\Delta z\right] + \dots \\
&\dots (1-U)\mathcal{G}_h\left[\int_{-\Delta^-}^{0} \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)} - e^{\beta_2(\delta)(\Delta z+\Delta^-)}}{e^{\beta 1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}}\,\mathrm{d}\Delta z + \int_{0}^{\Delta^+} \frac{e^{\beta_1(\delta)(\Delta z-\Delta^+)} - e^{\beta_2(\delta)(\Delta z-\Delta^+)}}{e^{-\beta 1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}}\,\mathrm{d}\Delta z\right], \\
&= U\mathcal{G}_u\left[\frac{1-\lim_{\Delta z\to\infty} e^{\beta_2(p(\theta^*))\Delta z}}{\beta_2(p(\theta^*))} + \frac{\lim_{\Delta z\to\infty} e^{\beta_1(p(\theta^*))\Delta z} - 1}{\beta_1(p(\theta^*))}\right] + \dots \\
&\dots (1-U)\mathcal{G}_h\left[\frac{\frac{e^{\beta_1(\delta)\Delta^-}-1}{\beta_1(\delta)} - \frac{e^{\beta_2(\delta)\Delta^-}-1}{\beta_2(\delta)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \frac{\frac{1-e^{-\beta_1\Delta^+}}{\beta 1(\delta)} - \frac{1-e^{-\beta_2\Delta^+}}{\beta_2(\delta)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}}\right], \\
&= (1-U) + U = 1.
\end{aligned} \tag{285}
$$

Finally, from condition (280) the employment rate is

$$
1 - U = \frac{p(\tilde{\theta}^*)}{p(\tilde{\theta}^*) + \delta + \frac{\sigma^2}{2(1-U)}\left[g_h'(-\Delta^-) - g_h'(\Delta^+)\right]}, \tag{286}
$$

$$
= \frac{p(\tilde{\theta}^*)}{p(\tilde{\theta}^*) + \delta + \frac{\sigma^2}{2}\left[\frac{\beta_1(\delta)-\beta_2(\delta)}{e^{\beta 1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} - \frac{\beta_1(\delta)-\beta_2(\delta)}{e^{-\beta 1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}}\right]}. \tag{287}
$$

□

B.6 Proof of Proposition 7

Proposition 21. *The finding rate $p(\theta^*)$ and separating rate s are given*

$$
p(\theta^*) = \frac{\sigma^2}{2U}\left[\lim_{\Delta z\uparrow 0} g_h'(\Delta z) - \lim_{\Delta z\downarrow 0} g_h'(\Delta z)\right], \tag{288}
$$

$$
s = \frac{\sigma^2}{2(1-U)}\left[\lim_{\Delta z\uparrow 0} g_u'(\Delta z) - \lim_{\Delta z\downarrow 0} g_u'(\Delta z)\right]. \tag{289}
$$

The ratio of endogenous s^{end} to total separations s is given by

$$
\frac{s^{end}}{s^{exo}} = \frac{\frac{\sigma^2}{2(1-U)}\left[\lim_{\Delta z\downarrow -\Delta^-} g_h'(\Delta z) - \lim_{\Delta z\uparrow \Delta^+} g_h'(\Delta z)\right]}{s} \tag{290}
$$

and $\delta = s^{exo} - s^{end}$.

Add proof here

Proof. □

B.7 Proof of Proposition 8

Proposition 22. *The distributions of* τ^u *and* τ^h *are given by*

$$l(\tau^u) = p(\tilde{\theta}^*)e^{-p(\tilde{\theta}^*)\tau^u}, \tag{291}$$

$$l(\tau^h) = e^{-\delta\tau^h}\left[\delta + \sum_{n=1}^{\infty}\mathcal{A}(n)\left(e^{-\mathcal{B}(n)\tau^h} - \frac{\delta}{\mathcal{B}(n)}\left(1 - e^{-\mathcal{B}(n)\tau^h}\right)\right)\right]. \tag{292}$$

with

$$\mathcal{A}(n) = \frac{\pi\sigma^2 n(-1)^{n-1}}{\Delta_{+-}^2}\left[\sin\left(\pi n\frac{\Delta^-}{\Delta_{+-}}\right)e^{-\frac{\gamma}{\sigma^2}\Delta^+} + \sin\left(\frac{\pi n\Delta^+}{\Delta_{+-}}\right)e^{\frac{\gamma}{\sigma^2}\Delta^-}\right], \tag{293}$$

$$\mathcal{B}(n) = \frac{1}{2}\left(\frac{n^2\pi^2\sigma^2}{\Delta_{+-}^2} + \frac{\gamma^2}{\sigma^2}\right). \tag{294}$$

where $\Delta_{+-} = \Delta^+ - \Delta^-$.

Proof. The distribution of $h(\tau^u)$ is the standard formula for a Poisson counter with arrival rate $p(\tilde{\theta}^*)$. Next, we derive the formula for τ^h. Since $\tau^m = \min\{\tau^\delta, \tau^{h*}, \tau^{j*}\}$, we have that

$$\begin{aligned}
\mathbb{P}r(\tau^m \leq T) &= \mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \leq T | \tau^\delta \geq T)\mathbb{P}r(\tau^\delta \geq T) + \mathbb{P}r(\tau^\delta \leq T) \\
&= \mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \leq T | \tau^\delta \geq T)[1 - \mathbb{P}r(\tau^\delta \leq T)] + \mathbb{P}r(\tau \leq T) \qquad (295) \\
&= \mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \leq T)[e^{-\delta t}] + [1 - e^{-\delta t}] \qquad (296)
\end{aligned}$$

Taking derivative with respect to T

$$h(T) = e^{-\delta t}\left[\frac{\mathrm{d}\mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \leq T)}{\mathrm{d}T} - \delta\mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \leq T) + \delta\right]. \tag{297}$$

From Kolkiewicz (2002), we have that

$$
\begin{aligned}
&= \frac{\mathrm{d}\mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \leq T)}{\mathrm{d}T} \\
&= \frac{\pi\sigma^2}{(\hat{w}^+ - \hat{w}^-)^2} \sum_{i=1}^{\infty} n(-1)^{n-1} e^{-\frac{n^2\pi^2\sigma^2}{2(\hat{w}^+ - \hat{w}^-)^2}T} \times \dots \\
&\left[\sin\left(\pi n \frac{\hat{w}^* - \hat{w}^-}{\hat{w}^+ - \hat{w}^-}\right) e^{-\frac{\gamma}{2\sigma^2}(2(\hat{w}^+ - \hat{w}^*) + \gamma T)} + \sin\left(\pi n \frac{\hat{w}^+ - \hat{w}^*}{\hat{w}^+ - \hat{w}^-}\right) e^{-\frac{\gamma}{2\sigma^2}(2(\hat{w}^- - \hat{w}^*) + \gamma T)}\right] \\
&= \frac{\pi\sigma^2}{\Delta_{+-}^2} \sum_{i=1}^{\infty} n(-1)^{n-1} e^{-\left(\frac{n^2\pi^2\sigma^2}{2\Delta_{+-}^2} + \frac{\gamma^2}{2\sigma^2}\right)T} \left[\sin\left(\frac{\pi n \Delta^-}{\Delta_{+-}}\right) e^{-\frac{\gamma}{\sigma^2}\Delta^+} + \sin\left(\frac{\pi n \Delta^+}{\Delta_{+-}}\right) e^{\frac{\gamma}{\sigma^2}\Delta^-}\right] \\
&= \sum_{i=1}^{\infty} \mathcal{A}(n) e^{-\mathcal{B}(n)T}. \qquad (298)
\end{aligned}
$$

Combining (297) and (298), we obtain (292). □

B.8 Proof of Proposition 9

Proposition 23. *The distribution of log nominal wage changes satisfies*

$$
l_{\Delta w}(\Delta w) = \mathcal{G}_u \left[\beta_2(p(\tilde{\theta}^*)) e^{-\beta_2(p(\tilde{\theta}^*))\Delta w} \Gamma_2(\Delta w) + \beta_1(p(\tilde{\theta}^*)) e^{-\beta_1(p(\tilde{\theta}^*))\Delta w} \Gamma_1(\Delta w)\right] \qquad (299)
$$

with

$$
(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(p(\tilde{\theta}^*)))x} \bar{G}_h(x)\,\mathrm{d}x, \int_{-c}^{\infty} e^{-\beta_2(p(\tilde{\theta}^*))x} \bar{G}_h(x)\,\mathrm{d}x\right). \qquad (300)
$$

Proof. Fix a date t_0 and assume a new hire. Then the distribution of wage changes is given by

$$
\begin{aligned}
Pr(\Delta w \leq c) &= Pr(w_{t_0+\tau^m+\tau^u} - w_{t_0} \leq c) \qquad (301) \\
&=^{(1)} Pr(w_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^h+\tau^u} - (w_{t_0} - z_{t_0}) + (z_{t_0+\tau^m+\tau^u} - z_{t_0}) \leq c) \\
&=^{(2)} Pr(\hat{w}^* - \hat{w}^* + (z_{t_0+\tau^m+\tau^u} - z_{t_0}) \leq c) \\
&=^{(3)} Pr(-(\Delta z^h + \Delta z^u) \leq c)
\end{aligned}
$$

Here, step (1) subtracts and adds productivity at the begging of the h and u states. In step (2), we use the result that $\hat{w}^*$ is constant. Steps 3 uses that τ^u and the brownian motion increments are independent of

$\mathcal{F}_{\tau_u}$. For those reason, the distribution Δz^h and Δz^u conditional of adjustment are

$$\bar{G}_h(\Delta z) = \begin{cases} 1 & \text{if } \Delta z \in [\Delta^+, \infty) \\ \frac{1}{s(1-U)} \left[\frac{\sigma^2}{2} g_h'(\Delta z) + \delta \int_{-\Delta^-}^{\Delta z^h} g_h(x)\, \mathrm{d}x \right] & \text{if } \Delta z \in [-\Delta^-, \Delta^+) \\ 0 & \text{if } \Delta z \in (-\infty, -\Delta^-) \end{cases} \tag{302}$$

$$\bar{g}_u(\Delta z) = \mathcal{G}_u \begin{cases} e^{\beta_2(p(\theta^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(p(\theta^*)\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases} \tag{303}$$

Thus,

$$\begin{aligned}
&= \Pr(\Delta w \leq c), \\
&= \Pr(-(\Delta z^u + \Delta z^h) \leq c), \\
&= 1 - \Pr(\Delta z^u + \Delta z^h \leq -c), \\
&=^{(1)} 1 - \int_{-\infty}^{\infty} \bar{G}_h(-(c + \Delta z^u)) \bar{g}_u(\Delta z^u)\, \mathrm{d}z^u, \\
&=^{(2)} 1 - \mathcal{G}_u \left[\int_{-\infty}^{0} e^{\beta_2(p(\tilde{\theta}^*))\Delta z^u} \bar{G}_h(-(c + \Delta z^u))\, \mathrm{d}z^u + \int_{0}^{\infty} e^{\beta_1(p(\tilde{\theta}^*))\Delta z^u} \bar{G}_h(-(c + \Delta z^u))\, \mathrm{d}z^u \right], \\
&=^{(3)} 1 + \mathcal{G}_u \left[\int_{\infty}^{-c} e^{-\beta_2(p(\tilde{\theta}^*))(c+x)} \bar{G}_h(x)\, \mathrm{d}x + \int_{-c}^{-\infty} e^{-\beta_1(p(\tilde{\theta}^*))(c+x))} \bar{G}_h(x)\, \mathrm{d}x \right], \\
&=^{(4)} 1 - \mathcal{G}_u \left[e^{-\beta_2(p(\tilde{\theta}^*))c} \int_{-c}^{\infty} e^{-\beta_2(p(\tilde{\theta}^*))x} \bar{G}_h(x)\, \mathrm{d}x + e^{-\beta_1(p(\tilde{\theta}^*))c} \int_{-\infty}^{-c} e^{-\beta_1(p(\tilde{\theta}^*)))x} \bar{G}_h(x)\, \mathrm{d}x \right].
\end{aligned} \tag{304}$$

Here, in step (1), we use independence of Δz^u abd Δz^h. In step (2), we use the definition of $g_u(\Delta z)$. In step (3), we use integration by substitution using $x = -c - \Delta z$ and in step (4), we use properties of an integral. The last step comes by defining

$$(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(p(\tilde{\theta}^*)))x} \bar{G}_h(x)\, \mathrm{d}x, \int_{-c}^{\infty} e^{-\beta_2(p(\tilde{\theta}^*))x} \bar{G}_h(x)\, \mathrm{d}x \right). \tag{305}$$

□

B.9 Proof of Proposition 10

Proposition 24. *Define* $\Sigma = \frac{\sigma^2 p(\theta^*)}{\gamma^2}$. *The distribution of* Δz *conditional of separation is given by*

$$\bar{G}_h(\Delta z) = -2\frac{\sqrt{1+2\Sigma}}{\Sigma} \frac{\mathrm{d}l_{\Delta w}(-\Delta z)}{\mathrm{d}z} + \frac{p(\theta^*)}{\sqrt{1+2\Sigma}} l_{\Delta w}(-\Delta z) + \frac{\gamma^2 \Sigma}{\sqrt{1+2\Sigma}} \left[1 - L_{\Delta w}(-\Delta z)\right]. \tag{306}$$

where $-L_{\Delta w}(-\Delta z)$ *denotes the cumulative distribution of* $l_{\Delta w}(\Delta w)$.

Proof. The objective in this proof is to use non-differentiability of $\Delta z = 0$ in the distribution of $\bar{g}_s(\Delta z)$ for $s = h, u$ to write the distribution of Δz conditional to a separation. Observe that

$$\begin{aligned} L(a) &= Pr^l(\Delta w \leq a) \\ &=^{(1)} Pr^{\bar{G}^h, \bar{G}^u}(-(\Delta z^h + \Delta z^u) \leq a) \\ &=^{(2)} Pr^{\bar{G}^h, \bar{G}^u}(\Delta z^h + \Delta z^u \geq -a) \\ &=^{(3)} 1 - Pr(\Delta z^h + \Delta z^u \leq -a) \\ &=^{(4)} 1 - \int_{-\infty}^{\infty} \bar{G}_h(-a-y)\bar{g}_u(y)\,\mathrm{d}y. \end{aligned} \tag{307}$$

Here, in step (1) we use the definition of Δw, and steps (2) to (4) operate and use independent of $\bar{G}_h(\cdot)$ and $\bar{g}_u(\cdot)$. Since

$$g_u(\Delta z) = \mathcal{G}_u \begin{cases} e^{\beta_2(p(\theta^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(p(\theta^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases} \tag{308}$$

We have have that

$$L(\Delta w) = 1 - C_1(\Delta w) - C_2(\Delta w) \tag{309}$$

where

$$C_1(\Delta w) = \mathcal{G}_u \int_0^{\infty} \bar{G}_h(-\Delta w - u) e^{\beta_1(p(\theta^*))u}\,\mathrm{d}u, \tag{310}$$

$$C_2(\Delta w) = \mathcal{G}_u \int_{-\infty}^{0} \bar{G}_h(-\Delta w - u) e^{\beta_2(p(\theta^*))u}\,\mathrm{d}u. \tag{311}$$

Departing from $L(\Delta w) = 1 - \int_{-\infty}^{\infty} \bar{G}_h(-\Delta w - y)\bar{g}_u(y)\,\mathrm{d}y$ and doing a change of variables $x = -\Delta w - y$ with $dx = -dy$

$$L(\Delta w) = 1 - \int_{-\infty}^{\infty} \bar{G}_h(x)\bar{g}_u(-\Delta w - x)\,\mathrm{d}x. \tag{312}$$

Taking the derivative in the left hand side and right hand side

$$l(\Delta w) = \int_{-\infty}^{\infty} \bar{G}_h(x)\bar{g}'_u(-\Delta w - x)\,\mathrm{d}x. \tag{313}$$

Doing another change of variable and since $\bar{g}_u(-\Delta w - x)$ is non-differentiable

$$\begin{aligned} l(\Delta w) &= \mathcal{A}_u \int_{-\infty}^{\infty} \bar{G}_h(-\Delta w - y)\bar{g}'_u(y)\,\mathrm{d}u \\ &= \int_{-\infty}^{0} \bar{G}_h(-\Delta w - u)\mathcal{G}_u\beta_2(p(\theta^*))e^{\beta_2(p(\theta^*))u}\,\mathrm{d}u + \int_{0}^{\infty} \bar{G}_h(-\Delta w - u)\beta_1(p(\theta^*))\mathcal{G}_u e^{\beta_1(p(\theta^*))u}\,\mathrm{d}u \\ &= \beta_1(p(\theta^*))C_1(\Delta w) + \beta_2(p(\theta^*))C_2(\Delta w). \end{aligned} \tag{314}$$

Thus,

$$l(\Delta w) = \beta_1(p(\theta^*))C_1(\Delta w) + \beta_2(p(\theta^*))C_2(\Delta w). \tag{315}$$

To obtain the last condition, observe that

$$C_1(\Delta w) = \int_{0}^{\infty} \bar{G}_h(-\Delta w - u)\mathcal{G}_u e^{\beta_1(p(\theta^*))u}\,\mathrm{d}u, \tag{316}$$

$$= -\mathcal{G}_u \int_{-\Delta w}^{-\infty} \bar{G}_h(y)e^{\beta_1(p(\theta^*))(-\Delta w - y)}\,\mathrm{d}y, \tag{317}$$

$$= \mathcal{G}_u \int_{-\infty}^{-\Delta w} \bar{G}_h(y)e^{\beta_1(p(\theta^*))(-\Delta w - y)}\,\mathrm{d}y. \tag{318}$$

$$C_2(\Delta w) = \int_{-\infty}^{0} \bar{G}_h(-\Delta w - u)\mathcal{G}_u e^{\beta_2(p(\theta^*))u}\,\mathrm{d}u. \tag{319}$$

$$= -\mathcal{G}_u \int_{\infty}^{-\Delta w} \bar{G}_h(y)e^{\beta_2(p(\theta^*))(-\Delta w - y)}\,\mathrm{d}y, \tag{320}$$

$$= \mathcal{G}_u \int_{-\Delta w}^{\infty} \bar{G}_h(y)e^{\beta_2(p(\theta^*))(-\Delta w - y)}\,\mathrm{d}y. \tag{321}$$

Taking the derivative and using the Leibniz rule

$$C'_1(\Delta w) = -\mathcal{G}_u\bar{G}_h(-\Delta w) - \beta_1(p(\theta^*))\mathcal{G}_u \int_{-\infty}^{-\Delta w} \bar{G}_h(y)e^{\beta_1(p(\theta^*))(-\Delta w - y)}\,\mathrm{d}y, \tag{322}$$

$$= -\mathcal{G}_u\bar{G}_h(-\Delta w) - \beta_1(p(\theta^*))C_1(\Delta w) \tag{323}$$

$$C'_2(\Delta w) = \mathcal{G}_u\bar{G}_h(-\Delta w) - \beta_2(p(\theta^*))C_2(\Delta w) \tag{324}$$

Taking derivative over (315)

$$l'(\Delta w) = \beta_1(p(\theta^*))C'_1(\Delta w) + \beta_2(p(\theta^*))C'_2(\Delta w) \tag{325}$$

and using conditions (323) and (324),

$$l'(\Delta w) = \bar{G}_h(-\Delta w)\mathcal{G}_u[\beta_2(p(\theta^*)) - \beta_1(p(\theta^*))] - \beta_1(p(\theta^*))^2 C_1(\Delta w) - \beta_2(p(\theta^*))^2 C_2(\Delta w). \tag{326}$$

Equations (309), (315), and (326) provide a system of three functional equations with three unknowns

$$\begin{aligned} 1 - L(\Delta w) &= C_1(\Delta w) + C_2(\Delta w), \\ l(\Delta w) &= \beta_1(p^*)C_1(\Delta w) + \beta_2(p^*)C_2(\Delta w), \\ l'(\Delta w) &= \bar{G}_h(-\Delta w)\mathcal{G}_u[\beta_2(p(\theta^*)) - \beta_1(p(\theta^*))] - \beta_1(p(\theta^*))^2 C_1(\Delta w) - \beta_2(p(\theta^*))^2 C_2(\Delta w). \end{aligned} \tag{327}$$

Operating over the system of functional equations

$$l'(\Delta w) + (\beta_2(p(\theta^*)) + \beta_1(p(\theta^*)))\, l(\Delta w) + \beta_1(p^*)\beta_2(p^*)\,[1 - L(\Delta w)] = \bar{G}_h(-\Delta w)\mathcal{G}_u[\beta_2(p(\theta^*)) - \beta_1(p(\theta^*))], \tag{328}$$

with

$$\mathcal{G}_u = \left(\beta_2(p(\theta^*))^{-1} + \beta_1(p(\theta^*))^{-1}\right)^{-1} \tag{329}$$

$$\beta_1(p(\theta^*)) = \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}}{\sigma^2}, \tag{330}$$

$$\beta_2(p(\theta^*)) = \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}}{\sigma^2}. \tag{331}$$

Define $\Sigma = \frac{\sigma^2 p(\theta^*)}{\gamma^2}$. Then

$$\begin{aligned}
\mathcal{G}_u[\beta_2(p(\theta^*)) - \beta_1(p(\theta^*))] &= \frac{\beta_2(p(\theta^*)) - \beta_1(p(\theta^*))}{\beta_2(p(\theta^*))^{-1} + \beta_1(p(\theta^*))^{-1}}, \\
&= 2\frac{\sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}}{\sigma^2}\sigma^2 \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)} + (-\gamma - \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)})}{(-\gamma - \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)})(-\gamma + \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)})}, \\
&= 2\sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}\frac{-2\gamma}{\gamma^2 - \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}^2}, \\
&= -2\sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}\frac{\gamma}{\sigma^2 p(\theta^*)}, \\
&= -2\sqrt{\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)^2 + 2\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)}, \\
&= -2\frac{\sqrt{1 + 2\Sigma}}{\Sigma}. \\
\frac{\beta_2(p(\theta^*)) + \beta_1(p(\theta^*))}{\mathcal{G}_u[\beta_2(p(\theta^*)) - \beta_1(p(\theta^*))]} &= \frac{\frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}}{\sigma^2} + \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}}{\sigma^2}}{-2\sqrt{\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)^2 + 2\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)}}, \\
&= p(\theta^*)\frac{\frac{\gamma^2}{\sigma^2 p(\theta^*)}}{\sqrt{\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)^2 + 2\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)}}, \\
&= \frac{p(\theta^*)}{\sqrt{1 + 2\Sigma}}. \\
\frac{\beta_1(p(\theta^*))\beta_2(p(\theta^*))}{\mathcal{G}_u[\beta_2(p(\theta^*)) - \beta_1(p(\theta^*))]} &= \frac{\gamma^2 - \sqrt{\gamma^2 + 2\sigma^2 p(\theta^*)}^2}{-2\sqrt{\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)^2 + 2\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)}}, \\
&= \gamma^2 \frac{1}{\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)\sqrt{\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)^2 + 2\left(\frac{\gamma^2}{\sigma^2 p(\theta^*)}\right)}}, \\
&= \frac{\gamma^2 \Sigma}{\sqrt{1 + 2\Sigma}}.
\end{aligned}$$

Therefore the differential equation is given by (51).

□

B.10 Proof of Proposition 11

Proposition 25. *Assume $\gamma \neq 0$. The distribution of minus cumulative idiosyncratic shocks is given by*

$$g_h(\Delta z) = \frac{s(1-U)}{\gamma}\left[\int_{-\Delta^-}^{\Delta z}(e^{\frac{\gamma^2}{\sigma^2}(y+\Delta^-)}-1)\bar{g}_h(y)\,\mathrm{d}y + \bar{G}_h(-\Delta^-)\left[1-e^{-\frac{\gamma^2}{\sigma^2}(\Delta z-\Delta^-)}\right]\right]. \tag{332}$$

The fraction of inefficient to efficient separations is given by

$$\frac{s^{end}}{s^{exo}} = \frac{\sigma^2}{2s(1-U)}\left[g_h'(-\Delta^-) - g_h'(\Delta^+)\right]. \tag{333}$$

Proof.

$$\delta g_h(\Delta z) = \gamma g_h'(\Delta z) + \frac{\sigma^2}{2}g_h''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)/\{0\} \tag{334}$$

$$g_h(-\Delta^-) = g_h(\Delta^+) = 0, G_h(\Delta^+) = 1-U, \tag{335}$$

$$g_h(\Delta z) \in \mathbb{C} \tag{336}$$

and

$$\bar{G}_h(\Delta z) = \begin{cases} 1 & \text{if } \Delta z \in [\Delta_+, \infty) \\ \frac{1}{s}\left[\frac{\sigma^2}{2}\frac{g_h'(\Delta z)}{1-U} + \delta\frac{\int_{-\Delta_{*-}}^{\Delta z} g_h(z)\,\mathrm{d}x}{1-U}\right] & \text{if } \Delta z \in [-\Delta^-, \Delta^+) \\ 0 & \text{if } \Delta z \in (-\infty, -\Delta^-) \end{cases} \tag{337}$$

Using these two conditions, we have that

$$s(1-U)\bar{g}_h(\Delta z) = \gamma g_h'(\Delta z) + \frac{\sigma^2}{2}g_h''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)/\{0\} \tag{338}$$

$$g_h(-\Delta^-) = g_h(\Delta^+) = 0, G_h(\Delta^+) = 1-U. \tag{339}$$

Multiplying both sides by $e^{\frac{\gamma^2}{\sigma^2}\Delta z}$

$$s(1-U)e^{\frac{\gamma^2}{\sigma^2}\Delta z}\bar{g}_h(\Delta z) = \gamma e^{\frac{\gamma^2}{\sigma^2}\Delta z}g_h'(\Delta z) + \frac{\sigma^2}{2}e^{\frac{\gamma^2}{\sigma^2}\Delta z}g_h''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)/\{0\} \tag{340}$$

$$= \frac{\sigma^2}{2}\frac{\mathrm{d}(e^{\frac{\gamma^2}{\sigma^2}\Delta z}g_h'(\Delta z))}{\mathrm{d}\Delta z}. \tag{341}$$

Operating and taking the integral

$$s(1-U)\int_{-\Delta_{*-}}^{\Delta z} e^{\frac{\gamma_2^2}{\sigma^2}x}\bar{g}_h(x)\,\mathrm{d}x = \frac{\sigma^2}{2}\left[e^{\frac{\gamma_2^2}{\sigma^2}\Delta z}g_h'(\Delta z) - e^{-\frac{\gamma_2^2}{\sigma^2}\Delta^-}g_h'(-\Delta^-)\right],$$
$$= \frac{\sigma^2}{2}e^{\frac{\gamma_2^2}{\sigma^2}\Delta z}g_h'(\Delta z) - s(1-U)e^{-\frac{\gamma_2^2}{\sigma^2}\Delta_{*-}}\bar{G}_h(-\Delta^-). \tag{342}$$

Thus

$$\frac{2s(1-U)}{\sigma^2}\left[\int_{-\Delta_{*-}}^{\Delta z} e^{\frac{\gamma_2^2}{\sigma^2}(x-\Delta z)}\bar{g}_h(x)\,\mathrm{d}x + e^{-\frac{\gamma_2^2}{\sigma^2}(\Delta_{*-}+\Delta z)}\bar{G}_h(-\Delta_{*-})\right] = g_h'(\Delta z). \tag{343}$$

Taking integral in the left hand side and the right hand side

$$\int_{-\Delta_{*-}}^{\Delta z} g_h'(x)\,\mathrm{d}x = g_h(\Delta z) - \underbrace{g_h(-\Delta^-)}_{=0} \tag{344}$$
$$= \frac{2s(1-U)}{\sigma^2}\int_{-\Delta^-}^{\Delta z}\left[\int_{-\Delta^-}^{x} e^{\frac{\gamma_2^2}{\sigma^2}(y-x)}\bar{g}_h(y)\,\mathrm{d}y + e^{-\frac{\gamma_2^2}{\sigma^2}(\Delta^-+x)}\bar{G}_h(-\Delta^-)\right]\mathrm{d}x \tag{345}$$
$$= \frac{2s(1-U)}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\int_{-\Delta^-}^{x} e^{\frac{\gamma_2^2}{\sigma^2}(y-x)}\bar{g}_h(y)\,\mathrm{d}y\,\mathrm{d}x + \frac{\sigma^2}{2\gamma}\bar{G}_h(-\Delta^-)\left[1-e^{-\frac{\gamma_2^2}{\sigma^2}(\Delta z-\Delta_{*-})}\right]\right] \tag{346}$$
$$= \frac{2s(1-U)}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\int_{-\Delta^-}^{y} e^{\frac{\gamma_2^2}{\sigma^2}(y-x)}\bar{g}_h(y)\,\mathrm{d}x\,\mathrm{d}y + \frac{\sigma^2}{2\gamma}\bar{G}_h(-\Delta^-)\left[1-e^{-\frac{\gamma_2^2}{\sigma^2}(\Delta z-\Delta^-)}\right]\right] \tag{347}$$
$$= \frac{2s(1-U)}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\underbrace{\left[\int_{-\Delta^-}^{y} e^{\frac{\gamma_2^2}{\sigma^2}(y-x)}\,\mathrm{d}x\right]}_{=\frac{\sigma^2}{2\gamma}(e^{\frac{\gamma_2^2}{\sigma^2}(y+\Delta^-)}-1)}\bar{g}_h(y)\,\mathrm{d}y + \frac{\sigma^2}{2\gamma}\bar{G}_h(-\Delta^-)\left[1-e^{-\frac{\gamma_2^2}{\sigma^2}(\Delta z-\Delta^-)}\right]\right] \tag{348}$$
$$= \frac{s(1-U)}{\gamma}\left[\int_{-\Delta_{*-}}^{\Delta z}(e^{\frac{\gamma_2^2}{\sigma^2}(y+\Delta^-)}-1)\bar{g}_h(y)\,\mathrm{d}y + \bar{G}_h(-\Delta^-)\left[1-e^{-\frac{\gamma_2^2}{\sigma^2}(\Delta z-\Delta^-)}\right]\right] \tag{349}$$

□